ALAN ADOLPHSON
STEVEN SPERBER

On the zeta function of a complete intersection

Annales scientifiques de l’É.N.S. 4e série, tome 29, n° 3 (1996), p. 287-328

<http://www.numdam.org/item?id=ASENS_1996_4_29_3_287_0>

ON THE ZETA FUNCTION OF
A COMPLETE INTERSECTION

BY ALAN ADOLPHSON* AND STEVEN SPERBER

ABSTRACT. - In this article, we compute the p-adic Dwork cohomology of a smooth complete intersection in $T^n \times A^n$ or $P^N$ over a finite field (where $T^n$ is the $m$-torus). As an application, we prove the “Katz Conjecture” (i.e., the assertion that the Newton polygon lies over the Hodge polygon) for such varieties. This result is new in the case of $T^n \times A^n$. (The case of $P^N$ is due to Mazur [14].)

1. Introduction

In [9], Dwork developed a $p$-adic cohomology theory for smooth projective hypersurfaces over finite fields. Given $f \in F_q[x_0, x_1, \ldots, x_N]$, a form of degree $d$ defined over the field of $q = p^a$ elements, Dwork constructed a complex $K_{Dw}(f)$ of $p$-adic Banach spaces. When the hypersurface $V(f)$ defined by the vanishing of $f$ in $P^N$ is nonsingular and has nonsingular intersection with every coordinate variety $H_A = \bigcap_{i \in A} \{x_i = 0\}$, where $A \subset S = \{0, \ldots, N\}$, $A \neq S$, then the complex $K_{Dw}(f)$ is acyclic except in degree 0. The characteristic polynomial of Frobenius acting on $H_0$ gives the primitive part of the middle-dimensional factor of the zeta function of $V(f)$. From this vantage point, there remained the problems of extending this work to varieties other than hypersurfaces, as well as to treat even in the hypersurface case open or singular varieties. Of course, the development of crystalline cohomology and rigid cohomology provided an excellent basis for these generalizations.

Our goal in the present paper is to use the approach of exponential modules or twisted de Rham theory pioneered by Dwork in the hypersurface case to treat complete intersections. In this we are continuing the early work of Ireland [11] and Barshay [4], who studied projective and multiprojective complete intersections from this point of view also. In their work, they constructed a complex of $p$-adic Banach spaces $K_{Dw}$ (related to the complex $K_{Dw}(S, S)$ of section 6 below), proved the acyclicity except in degree 0 of this complex in the smooth case, and related the characteristic polynomial of Frobenius acting on $H_0$ to the zeta function of the complete intersection defined by the simultaneous vanishing of forms $f_1, \ldots, f_r \in F_q[x_0, \ldots, x_N]$ in $P^N$. Specifically, they showed that this characteristic

* Partially supported by NSF grant no. DMS-9305514.
polynomial equals a product of certain factors (which the Weil conjectures imply are polynomials), from which they concluded this polynomial has the correct degree. They were unable to show the factors themselves are polynomials. In particular, they were unable to construct a finite-dimensional $p$-adic vector space with action of Frobenius whose characteristic polynomial is the interesting factor of the zeta function of a smooth projective complete intersection. We construct such a theory here.

The main application of our work that we give here is a proof of the “Katz Conjecture” (i.e., the assertion that the Newton polygon lies above the Hodge polygon: see [14]) for general smooth complete intersections in an affine space or a torus (as well as another proof in the projective case). Previously, such results were known only in the proper case. We note also that our approach eliminates the need to treat separately the case of hypersurfaces of degree divisible by $p$ (compare [9], [10]). In a future article, we plan to describe the relation between the theory developed here and classical de Rham cohomology. In particular, we believe that the description we give here of middle-dimensional cohomology and of a procedure for finding a basis for it should be useful in calculations involving the Gauss-Manin connection.

We describe our results more precisely. In the present work we study both open smooth complete intersections (in $T^m \times A^n$) in sections 2-5 and projective smooth complete intersections in sections 6-7. In the open case, we let $f_1, \ldots, f_r \in \mathbb{F}_q[x_1, \ldots, x_{m+n}, (x_1 \cdots x_m)^{-1}]$ be Laurent polynomials and $V$ be the variety in $T^m \times A^n$ defined by the simultaneous vanishing of the $f_i$’s. If we set $g = \sum_{j=1}^r x_{m+n+j} f_j(x_1, \ldots, x_{m+n})$, then it is well-known that

$$L(T^m \times A^n \times A^r, g; t) = Z(V/\mathbb{F}_q; q^r t),$$

where the right-hand side is the zeta function of $V$ and the left-hand side is the $L$-function of the exponential sum associated to $g$. It is also known from our earlier work [1] that, with $S = \{1, \ldots, m+n+r\}$, $S_{af} = \{m+1, \ldots, m+n+r\}$, there is a complex of $p$-adic Banach spaces $K(S, S_{af})$ which satisfies

$$L(T^m \times A^n \times A^r, g; t)(-1)^{m+n+r+1} = \det(I - t \text{Frob} | K(S, S_{af})), \tag{1.1}$$

where the right-hand side is shorthand for the alternating product of characteristic series of Frobenius acting on the complex $K(S, S_{af})$. While we have studied $L$-functions such as (1.1) in the past, our earlier results need further refinements here. Even if we make appropriate hypothesis on the $f_i$ to ensure that $g$ is nondegenerate (in the sense of Kouchnirenko [13]), the polynomial $g$ is not commode (in the sense of [1]) with respect to the set $S_{du} = \{m+n+1, \ldots, m+n+r\}$ (and therefore, a fortiori, not commode with respect to $S_{af}$). Setting $x_{m+n+1} = \cdots = x_{m+n+r} = 0$ in $g$ gives the zero polynomial, i.e., this substitution causes the dimension of its Newton polyhedron to drop by $m+n+r$, rather than simply $r$ which is what the condition of being commode requires. We are nevertheless able to calculate $H_*(K(S, S_{af}))$ in the open case of a smooth complete intersection $V$ in $T^m \times A^n$. Using this result, we are able to show that the Newton polygon of the primitive part of the middle-dimensional factor of the zeta function of $V$ lies over its corresponding Hodge polygon.
In the projective case we change our enumeration and notation, letting \( f_1, \ldots, f_r \) be forms in \( \mathbb{F}_q[x_0, \ldots, x_N] \). Let \( g = \sum_{j=1}^{r} x_{N+j} f_j(x_0, \ldots, x_N) \), \( S = \{0, 1, \ldots, N + r\} \), \( S_{sp} = \{0, 1, \ldots, N\} \), \( S_{du} = \{N + 1, \ldots, N + r\} \). The projective case is thus more complicated even when \( g \) is nondegenerate since the Newton polyhedron has dimension \( N + r \) rather than \( N + r + 1 \) (= \(|S|\)) and since \( g \) vanishes when one specializes all the variables in either \( S_{sp} \) or \( S_{du} \) to be 0. As we noted above, Ireland [11] was able to compute the homology of \( K^Dw \) in the smooth case. Here we compute the homology of \( K.(S, S) \) and show that the characteristic polynomial of Frobenius acting on \( H_0 \) gives the primitive part of the middle-dimensional factor of the zeta function of the smooth projective complete intersection defined by the simultaneous vanishing of the \( f_i \)'s in \( \mathbb{P}^N \). Our main technical tools in this analysis are some properties of Koszul complexes which we specify explicitly in the appendix. It is interesting that these properties are, in the case of a hypersurface defined by the vanishing of a form \( f \), sufficient to guarantee that

\[
\det(I - t \text{Frob} | K^Dw(f))^{(-1)^{N+1}} = Z(V(f)/\mathbb{F}_q;qt)(1 - qt) \cdots (1 - q^N t)
\]

even in the case \( f \) is singular. In the projective case, we also compare Newton and Hodge polygons giving another proof of Mazur’s theorem [14]. In the course of this, we specify a basis (valid for a Zariski open set of the moduli space of complete intersections with specified degrees) for the primitive middle-dimensional cohomology of a projective complete intersection, which may prove useful in explicit calculations.

The outline of the paper is as follows. In section 2, we compute the homology of the complex \( K.(S, S_{at}) \), which gives the zeta function of a smooth complete intersection in \( \mathbb{T}^m \times \mathbb{A}^n \) (Theorem 2.19). In order to estimate the Newton polygon of Frobenius acting on homology, we need a more precise description of a basis for these homology spaces. In section 3, we obtain such a description for \( \tilde{K}.(S, S_{at}) \), the “reduction mod \( p \)” of \( K.(S, S_{at}) \), by first obtaining such a description for its associated graded complex \( \tilde{K}.(S, S_{at}) \) (Theorems 3.26 and 3.37). We then explain in section 4 how this lifts to a basis for the homology of \( K.(S, S_{at}) \), which in turn leads to a lower bound for the Newton polygon of Frobenius acting on homology (Theorem 4.13). In section 5, we identify this lower bound with a Hodge polygon by using the ideas of [6] to explicitly compute the Hodge polygon of a general complete intersection in \( (\mathbb{T}^m \times \mathbb{A}^n)_C \). In sections 6 and 7, we repeat the above procedure for smooth complete intersections in \( \mathbb{P}^N \). Here some of our arguments are sketchier, because they are analogous to the case \( \mathbb{T}^m \times \mathbb{A}^n \) and because the result in the projective case is already known [14]. In section 8, we collect some general results on complexes that are useful in sections 6 and 7.

2. Cohomology of toric and affine complete intersections

Let \( p \) be a prime number, \( q = p^n \), and let \( \mathbb{F}_q \) be the finite field of \( q \) elements. Let \( \mathbb{T}^m \) be the \( m \)-torus over \( \mathbb{F}_q \) (i.e., \( \mathbb{T}^m \) is the product over \( \mathbb{F}_q \) of \( m \) copies of the multiplicative group) and let \( \mathbb{A}^n \) be affine \( n \)-space over \( \mathbb{F}_q \). Put \( N = m + n \). Take \( f_1, \ldots, f_r \in \mathbb{F}_q[x_1, \ldots, x_N, (x_1 \cdots x_m)^{-1}] \) and let \( V \subseteq \mathbb{T}^m \times \mathbb{A}^n \) be the variety
$f_1 = \cdots = f_r = 0$. We allow the possibility that $m$ or $n$ is zero. Let $V(F_{q^s})$ be the set of $F_{q^s}$-rational points of $V$ and let $N_s(V)$ be its cardinality. The zeta function of $V$ is defined to be

$$Z(V/F_q; t) = \exp\left(\sum_{s=1}^{\infty} N_s(V) \frac{t^s}{s}\right).$$

We begin by re-expressing $Z(V/F_q; t)$ in terms of $L$-functions of certain exponential sums.

Fix a nontrivial additive character $\Psi : F_q \to \mathbb{C}^\times$ and let $\Psi_s = \Psi \circ \text{Trace}_{F_{q^s}/F_q} : F_{q^s} \to \mathbb{C}^\times$. For an $F_q$-regular function $h$ on an $F_q$-variety $X$, define

$$S_s(X, h) = \sum_{x \in X(F_{q^s})} \Psi_s(h(x)),$$

$$L(X, h; t) = \exp\left(\sum_{s=1}^{\infty} S_s(X, h) \frac{t^s}{s}\right).$$

We introduce dummy variables $x_{N+1}, \ldots, x_{N+r}$ and put

$$g = x_{N+1}f_1(x_1, \ldots, x_N) + \cdots + x_{N+r}f_r(x_1, \ldots, x_N) \in F_q[x_1, \ldots, x_{N+r}, (x_1 \cdots x_m)^{-1}].$$

It is easily seen that

$$(2.1) \quad q^{-s}N_s(V) = \sum_{x_1, \ldots, x_m \in F_{q^s}} \sum_{x_{N+1}, \ldots, x_{N+r} \in F_{q^s}} \Psi_s(g(x_1, \ldots, x_{N+r})),$$

or equivalently,

$$(2.2) \quad Z(V/F_q; q^{-s}t) = L(T^m \times A^n \times A^r, g; t).$$

Put $A' = A^r \setminus \{(0, \ldots, 0)\}$. Since $g$ vanishes identically on $T^m \times A^n \times \{(0, \ldots, 0)\}$, an easy calculation gives

$$(2.3) \quad Z(V/F_q; q^{-s}t) = L(T^m \times A^n \times A', g; t) \prod_{j=0}^{m} (1 - q^{j+n}t)^{-\binom{m}{j}} (-1)^{m-j-1}.$$

We regroup terms as follows. Define a rational function $P(t)$ by

$$(2.4) \quad P(t) = \left(L(T^m \times A^n \times A', g; t) \prod_{j=0}^{r-n-1} \prod_{j=0}^{r-(n-j)} (1 - q^{j+n}t)^{-\binom{m}{j}} (-1)^{m-j-1}\right)^{(-1)^{N-r-1}},$$

so that (2.3) becomes

$$(2.5) \quad Z(V/F_q; q^{-s}t) = (P(t) (1 - q^{-s}t)^{-\binom{m}{j}} (-1)^{N-r-1} \prod_{j=1}^{N-r} (1 - q^{j+r}t)^{-\binom{m}{j+r-n}} (-1)^{N-r-j-1},$$

where we understand $\binom{a}{b} = 0$ if $b < 0$ or $b > a$. We shall identify the factors on the right-hand side with the action of Frobenius on $p$-adic cohomology spaces when $V$ is
sufficiently smooth. In particular, \( P(t) \) will be the polynomial corresponding to the action of Frobenius on the primitive part of middle-dimensional cohomology. For purposes of induction on \( r \), it will be convenient to allow \( r = 0 \). In this situation, we understand that \( V = T^m \times A^n \) and \( P(t) = 1 \).

We apply the theory of \([1]\) to the L-function \( L(T^m \times A^n \times A^r, g; t) \). For a Laurent polynomial \( h \) over any field in variables \( x_1, x_1^{-1}, \ldots, x_k, x_k^{-1} \), we denote by \( \text{supp}(h) \subseteq \mathbb{R}^k \) the set of exponents of the monomials appearing in \( h \), thought of as lattice points in \( \mathbb{R}^k \).

Let \( \Delta \subseteq \mathbb{R}^{N+r} \) be the convex hull of the origin and \( \text{supp}(g) \). Let \( C(\Delta) \) be the real cone generated by \( \Delta \), i.e., the collection of all nonnegative real multiples of points of \( \Delta \). Put \( M = \mathbb{Z}^{N+r} \cap C(\Delta) \). For \( u = (u_1, \ldots, u_{N+r}) \in C(\Delta) \), define its weight \( w(u) \) by \( w(u) = u_{N+1} + \cdots + u_{N+r} \).

Let \( \Omega_0 = \mathbb{Q}_p(\zeta_p, \zeta_{q-1}) \), where \( \mathbb{Q}_p \) denotes the \( p \)-adic numbers and \( \zeta_p \) and \( \zeta_{q-1} \) denote primitive \( p \)-th and \( (q-1) \)-st roots of unity, respectively. We normalize the \( p \)-adic valuation \( \text{ord} \) on \( \Omega_0 \) by setting \( \text{ord} \, p = 1 \). Let \( \mathcal{O}_0 \) be the ring of integers of \( \Omega_0 \) and let \( \pi \in \mathcal{O}_0 \) be a uniformizing parameter, so \( \text{ord} \, \pi = 1/(p-1) \). A key role will be played by the \( p \)-adic Banach space \( B \) and its unit ball \( B(\mathcal{O}_0) \):

\[
B = \left\{ \sum_{u \in M} A_u \pi^{w(u)} x^u \mid A_u \in \Omega_0, A_u \to 0 \text{ as } u \to \infty \right\},
\]

\[
B(\mathcal{O}_0) = \left\{ \sum_{u \in M} A_u \pi^{w(u)} x^u \in B \mid A_u \in \mathcal{O}_0 \text{ for all } u \right\}.
\]

The norm on \( B \) is defined by

\[
\| \sum_{u \in M} A_u \pi^{w(u)} x^u \| = \sup_{u \in M} |A_u|.
\]

It will be useful to consider some related spaces as well. For \( b, c \in \mathbb{R}, b > 0 \), define

\[
L(b, c) = \left\{ \sum_{u \in M} A_u x^u \mid A_u \in \Omega_0, \text{ord} A_u \geq bw(u) + c \right\},
\]

\[
L(b) = \bigcup_{c \in \mathbb{R}} L(b, c).
\]

Note that we have \( L(b) \subseteq B \subseteq L(1/(p-1)) \) for \( b > 1/(p-1) \).

Define an operator \( \psi \) on formal power series over \( M \) by

\[
\psi(\sum_{u \in M} A_u x^u) = \sum_{u \in M} A_{pu} x^u.
\]

Observe that \( \psi(L(b, c)) \subseteq L(pb, c) \). Associated to \( g \) is a series \( F_0(x) \in L(p/q(p-1), 0) \) (see \([1, \text{equation (1.15)}]\)) with the following property. Put \( \alpha = \psi^a o F_0 \), the composition of \( \psi^a \) with multiplication by \( F_0(x) \), where \( a = [F_q : F_p] \). Then \( \alpha \) is a completely continuous \( \Omega_0 \)-linear endomorphism of \( B \) and of \( L(b) \) for \( 0 < b \leq p/(p-1) \). Furthermore,

\[
(2.6) \quad L(T^{N+r}, g; t)^{(-1)^{N+r}-1} = \det(I - t\alpha)^{\delta^{N+r}},
\]
where $\det(I - t\alpha)$ is the Fredholm determinant of $\alpha$ as operator on $B$ or any of the $L(b)$, $0 < b < p/(p - 1)$, and $\delta$ is the operator on formal power series in $t$ with constant term $1$ defined by $h(t)^\delta = h(t)/h(qt)$.

We now explain how to modify this purely toric case to obtain $L(T^{m} \times \mathbb{A}^n \times \mathbb{A}^r, g; t)$. Different variables will play different roles in the argument, and we index them accordingly. The set of all variables is indexed by $S = \{1, \ldots, N + r\}$. Toric variables are indexed by the set $S_{\text{tor}} = \{1, \ldots, m\}$, affine variables by $S_{\text{af}} = \{m + 1, \ldots, N + r\}$, space variables by $S_{\text{sp}} = \{1, \ldots, N\}$, dummy variables by $S_{\text{du}} = \{N + 1, \ldots, N + r\}$. For any subset $I \subseteq S$, we use subscripts to denote its intersection with these subsets, e.g., $I_{\text{tor}} = I \cap S_{\text{tor}}$.

For any finite set $I$, we let $|I|$ denote its cardinality.

Fix $I \subseteq S$ with $S_{\text{tor}} \subseteq I$ and let $g_I \in \mathbb{F}_q[[x_1, \ldots, x_m]]$ be the polynomial obtained from $g$ by setting $x_j = 0$ for $j \in S_{\text{af}} \setminus I$. For $j \in S_{\text{af}}$, let $\theta_j : B \to B$ be the map "set $x_j = 0$" and let $\theta_I$ be the composition $\theta_I = \prod_{j \in S_{\text{af}} \setminus I} \theta_j$. We define $B_I = \theta_I(B)$, $B_I(\mathcal{O}_0) = \theta_I(B(\mathcal{O}_0))$, $L_I(b, c) = \theta_I(L(b, c))$, $L_I(b) = \theta_I(L(b))$, and $\alpha_I = \psi^\alpha \circ \theta_I(F_0)$.

We observe that by [1]

$$L(T^{|I|}, g_I; t)^{(-1)^{|I|}-1} = \det(I - t\alpha_I)^{\delta^{|I|}},$$

where $\alpha_I$ may be regarded as acting on $B_I$ or any of the $L_I(b)$, $0 < b < p/(p - 1)$.

Equation (2.6) may be interpreted as follows. In [1, section 2] it is shown that there exist elements $H_i \in L(p/(p - 1), -1)$ such that the (commuting) differential operators $D_i = x_i \partial/\partial x_i + H_i$, $i = 1, \ldots, N + r$, satisfy

$$\alpha \circ D_i = qD_i \circ \alpha$$

as operators on $B$ or $L(b)$, $0 < b < p/(p - 1)$. Let $K_r = K_r(B, \{D_i\}_{i=1}^{N+r})$ be the Koszul complex on $B$ defined by $D_1, \ldots, D_{N+r}$. For $0 \leq l \leq N + r$, its component of degree $l$ is

$$K_l = \bigoplus_{|A| = l} B e_A,$$

where the sum is over all subsets $A \subseteq S$ of cardinality $l$ and $e_A$ is a formal symbol. The boundary map $\partial_l : K_l \to K_{l-1}$ is given by: if $\xi \in B$ and $A = \{i_1, \ldots, i_l\}$ with $i_1 < \cdots < i_l$, then

$$\partial_l(\xi e_A) = \sum_{j=1}^l (-1)^{j-1}D_{i_j}(\xi) e_{A \setminus \{i_j\}}.$$

Define an endomorphism $\alpha_l : K_l \to K_l$ by

$$\alpha_l = \bigoplus_{|A| = l} q^l \alpha.$$

Then (2.8) implies that $\alpha$ is a chain map on $K_r$, hence by (2.6)

$$L(T^{N+r}, g; t)^{(-1)^{N+r}-1} = \prod_{l=0}^{N+r} \det(I - t\alpha_l | K_l)^{(-1)^l}.$$
Passing to homology, we have

\begin{equation}
L(T^{N+r}, g \cdot t)^{(-1)^{N+r-1}} = \prod_{l=0}^{N+r} \det(I-t\tilde{\alpha}_l | H_l(K))^{(-1)^l},
\end{equation}

where \(\tilde{\alpha}_l\) is the endomorphism of \(H_l(K)\) induced by \(\alpha_l\) on \(K_I\).

For \(I \subseteq S\) with \(S_{to} \subseteq I\), put \(D_{I,i} = x_i \partial / \partial x_i + \theta_I(H_i)\). We denote by \(K.(I)\) the Koszul complex on \(B_I\) formed by the operators \(D_{I,i}\) for \(i \in I\). In particular,

\[
K_I(I) = \bigoplus_{A \subseteq I, |A| = l} B_I e_A.
\]

We have in analogy with (2.10)

\begin{equation}
L(T^{[I], g_I \cdot t})^{(-1)^{|I|-1}} = \prod_{l=0}^{|I|} \det(I-t\tilde{\alpha}_{I,l} | H_l(K.(I)))^{(-1)^l}.
\end{equation}

Usually, no confusion will result if we denote \(\tilde{\alpha}_{I,l}\) simply by \(\tilde{\alpha}_l\) or even \(\tilde{\alpha}\).

The complexes \(K.(I)\) can be tied together by introducing some subcomplexes \(K.(I, I')\). For \(I \subseteq S\) with \(S_{to} \subseteq I\) and \(I' \subseteq I_{af}\), let

\[
B_I^{I'} = \bigcap_{i \in I'} \ker(\theta_i | B_I),
\]

i.e., \(B_I^{I'}\) consists of all elements of \(B_I\) that are divisible by \(x_i\) for all \(i \in I'\). Let \(K.(I, I')\) be the subcomplex of \(K.(I)\) defined by

\[
K_I(I, I') = \bigoplus_{A \subseteq I, |A| = l} B_I^{I' \setminus A} e_A.
\]

Note that for \(i \in I_{af}, i \not\in I'\), there is an exact sequence

\[
0 \rightarrow B_I^{I' \cup \{i\}} \rightarrow B_I^{I'} \rightarrow B_I^{I, \{i\}} \rightarrow 0,
\]

which induces an exact sequence of complexes

\begin{equation}
0 \rightarrow K.(I, I' \cup \{i\}) \rightarrow K.(I, I') \rightarrow K.(I \setminus \{i\}, I') \rightarrow 0.
\end{equation}

Using this exact sequence, equation (2.11), and induction on \(|I'|\), the natural toric decomposition of \(A^{[I']}\) gives

\begin{equation}
L(T^{[I] \setminus |I'|} \times A^{[I']}, g_I \cdot t)^{(-1)^{|I'|-1}} = \prod_{l=0}^{|I|} \det(I-t\tilde{\alpha} | H_l(K.(I, I')))^{(-1)^l}.
\end{equation}
To be precise, the left-hand side is the $L$-function corresponding to the exponential sum
\[\sum \Psi(g_1(x)),\]
where $x_i$ runs over $\mathbb{F}_q$ for $i \in I'$ and over $\mathbb{F}_q^\times$ for $i \in I, i \not\in I'$. In particular, taking $I = S$, $I' = S_{af}$, we have by (2.2) that
\[Z(V/\mathbb{F}_q, q^t) = \prod_{l=0}^{N+r} \det(I - t\bar{\alpha} | H_l(K,(S, S_{af})))^{-1}(1)^r.
\]

Let $\Delta_I \subseteq \mathbb{R}^{|I|}$ be the convex hull of the origin and $\text{supp}(g_I)$. In particular, $\Delta = \Delta_S$. We make two assumptions about $g$. We assume that for every subset $I \subseteq S$ such that $S_{to} \subseteq I$ and $I_{du} \neq \emptyset$ we have $\dim \Delta_I = |I|$. We call $g$ semi-convenient when this condition is satisfied. (When $g$ is semi-convenient, $g$ will be “commode” in the sense of [1] with respect to any subset $I \subseteq S_{af}$ such that $S_{du} \not\subseteq I$.) When $n = 0$, this is equivalent to requiring that the convex hull of $\text{supp}(f_i)$ have dimension $m$ for $i = 1, \ldots, r$. When $m = 0$, it is equivalent to requiring that each $f_i$ contain terms $a_{ij}x_j^{k_{ij}}$ for $j = 1, \ldots, n$, with $a_{ij} \neq 0$ and $k_{ij} > 0$, and that $f_i(0, \ldots, 0) \neq 0$ for $i = 1, \ldots, r$. We also assume that $g$ is nondegenerate (see [1, section 2]), which implies that $g_I$ is also nondegenerate for all $I \subseteq S$. Geometrically, this means that for all $I \subseteq S_{sp}$ with $S_{to} \subseteq I$ and $J \subseteq \{1, \ldots, r\}$, the equations $\theta_J(f_j) = 0$ for $j \in J$ define a smooth complete intersection $X$ in the torus $T^{|I|}$ and that there is a compactification $Y$ of the torus in which its closure $\bar{X}$ is smooth and meets all orbits transversally. This condition is generically satisfied ([3], [12]).

By [1, Theorems 2.9 and 3.13] we have the following.

**Theorem 2.15.** Suppose that $g$ is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{to} \subseteq I$ and $I' \subseteq I_{af}$ with $I_{du} \neq I_{du}$ we have $H_l(K,(I, I')) = 0$ for $l > 0$, $\bar{\alpha}$ is invertible on $H_0(K,(I, I'))$, and
\[\dim_{\mathbb{Q}_0} H_0(K,(I, I')) = \sum_{I', J \subseteq I} (-1)^{|I'| - |J|} (|J|)! \text{Vol}(\Delta_J),
\]
where $\text{Vol}(\Delta_J)$ denotes the volume of $\Delta_J$ relative to Lebesgue measure on $\mathbb{R}^{|J|}$. Thus by (2.13),
\[L(T^{|I'|-|I'|} \times A^{|I'|}, g_I; t)^{-1} = \det(I - t\bar{\alpha} | H_0(K,(I, I')))
\]
is a polynomial whose degree is given by (2.16).

Using this theorem and induction on $r$, we shall compute the homology of $K,(S, S_{af})$, which by (2.14) is the complex that gives the zeta function $Z(V/\mathbb{F}_q, q^t)$. First, for later application, we modify the formula (2.16) for $\dim_{\mathbb{Q}_0} H_0(K,(I, I'))$. For $S_{to} \subseteq I \subseteq S_{sp}$, let $\Delta_I \subseteq \mathbb{R}^{|I|}$ denote the convex hull of $\text{supp}(\theta_J(f_j))$, $i = 1, \ldots, r$. Regarding $\mathbb{R}^{|J|}$ as $\mathbb{R}^{|J_{sp}|} \times \mathbb{R}^{|J_{du}|}$, the projection of $\Delta_J \subseteq \mathbb{R}^{|J|}$ on $\mathbb{R}^{|J_{sp}|}$ is the simplex
\[\Lambda = \{ (\lambda_j)_{j \in J_{du}} \mid \sum_{j \in J_{du}} \lambda_j \leq 1, \lambda_j \geq 0 \text{ for all } j \}.
\]
The fiber of $\Delta_J$ over $(\lambda_j)_{j \in J_{du}} \in \Lambda$ is the Minkowski sum $\sum_{j \in J_{du}} \lambda_j \Delta_j^{J_{sp}}$, thus
\[\text{Vol}(\Delta_J) = \int_{\Lambda} \text{Vol}(\sum_{j \in J_{du}} \lambda_j \Delta_j^{J_{sp}}) \bigwedge_{j \in J_{du}} d\lambda_j,
\]
where \( \text{Vol}(\sum_{j \in J_{du}} \lambda_j \Delta_j^{J_{sp}}) \) denotes volume relative to Lebesgue measure on \( \mathbb{R}^{|J_{sp}|} \). It is a theorem of Minkowski that \( \text{Vol}(\sum_{j \in J_{du}} \lambda_j \Delta_j^{J_{sp}}) \) is a homogeneous polynomial of degree \( |J_{sp}| \) in \( \{\lambda_j\}_{j \in J_{du}} \), specifically,

\[
\text{Vol}(\sum_{j \in J_{du}} \lambda_j \Delta_j^{J_{sp}}) = \sum_{l_j = |J_{sp}|} \frac{(|J_{sp}|)!}{\prod_{j \in J_{du}} l_j!} \prod_{j \in J_{du}} \lambda_j^{l_j},
\]

where \( M(\{\Delta_j^{J_{sp}}, l_j\}_{j \in J_{du}}) \) denotes the Minkowski mixed volume of the collection of \( |J_{sp}| \) polytopes in \( \mathbb{R}^{|J_{du}|} \) obtained by taking \( \Delta_j^{J_{sp}} \) with multiplicity \( l_j \). Induction on \( |J_{du}| \) shows that

\[
\int \prod_{j \in J_{du}} \lambda_j^{l_j} d\lambda_j = \prod_{j \in J_{du}} \frac{l_j!}{(|J_{du}| + \sum_{j \in J_{du}} l_j)!} = \prod_{j \in J_{du}} \frac{l_j!}{(|J_{du}|)!},
\]

hence evaluation of the above integral for \( \text{Vol}(\Delta_j) \) gives

\[
(2.17) \quad (|J_{du}|)! \text{Vol}(\Delta_j) = (|J_{sp}|)! \sum_{\sum_{l_j = |J_{sp}|}} M(\{\Delta_j^{J_{sp}}, l_j\}_{j \in J_{du}}).
\]

When \( \Delta_j^{J_{sp}} \) is a single point and \( l_j = 0 \) for all \( j \in J_{du} \), we define \( M(\{\Delta_j^{J_{sp}}, l_j\}_{j \in J_{du}}) = 1 \), so that (2.17) remains valid even when \( J_{sp} = \emptyset \). (This case will arise when we try to generalize (2.16) to the case where \( I' = I_{du} \) when \( m = 0 \).)

Substituting the right-hand side of (2.17) into the right-hand side of (2.16) gives

\[
\dim_{\Theta_0} H_0(K,(I, I')) = \sum_{I \setminus I' \subseteq J \subseteq I} (-1)^{|I'|-|J|}(|J_{sp}|)! \sum_{\sum_{l_j = |J_{sp}|}} M(\{\Delta_j^{J_{sp}}, l_j\}_{j \in J_{du}}).
\]

To simplify this expression, we define an equivalence relation on the pairs \( (J, \{l_j\}_{j \in J_{du}}) \) with \( I \setminus I' \subseteq J \subseteq I \) and \( \sum_{j \in J_{du}} l_j = |J_{sp}| \) that appear in this sum. Let \( J_+ = \{j \in J_{du} | l_j > 0\} \). Define \( (J, \{l_j\}_{j \in J_{du}}) \sim (J', \{l'_j\}_{j \in J_{du}'} \) if \( J_{sp} = J_{sp}', J_+ = J_+', \) and \( l_j = l'_j \) for \( j \in J_+ \). When two pairs are equivalent, we have

\[
M(\{\Delta_j^{J_{sp}}, l_j\}_{j \in J_{du}}) = M(\{\Delta_j^{J_{sp}'}, l'_j\}_{j \in J_{du}'})
\]
since both collections consist of the same polytopes repeated with the same multiplicities.
Note that each equivalence class contains exactly one representative of the form
\((G \cup I_{du}, \{l_j\}_{j \in I_{du}})\), where \(I_{sp} \setminus I_{sp}' \subseteq G \subseteq I_{sp}\). It follows that
\[
\dim_{\Omega_0} H_0(K, (I, I'))
\]
\[
= \sum_{I_{sp} \setminus I_{sp}' \subseteq G \subseteq I_{sp}} (-1)^{|I_{sp}'|-|G|} |G|! \sum_{j \in I_{du}} M(\{\Delta_j^G, l_j\}_{j \in I_{du}}) \sum' (-1)^{|I_{du}'|-|I_{du}'|},
\]
where \(\sum'\) denotes a sum over pairs \((J', \{l'_j\}_{j \in J_{du}'})\) in the equivalence class of the
pair \((G \cup I_{du}, \{l_j\}_{j \in I_{du}})\). For every pair \((J', \{l'_j\}_{j \in J_{du}'})\) in this equivalence class, the
set \(J_{du}'\) can be represented in the form \(J_{du}' = I_{du} \cup F\) for a unique subset \(F\),
\(I_{du} \setminus (I_{du}' \cup I_{du}+) \subseteq F \subseteq I_{du} \setminus I_{du}+\) (namely, \(j \in F\) if and only if \(l'_j = 0\)). In
terms of \(F\), the innermost sum in the previous equation becomes
\[
\sum_{I_{du} \setminus (I_{du}' \cup I_{du}+) \subseteq F \subseteq I_{du} \setminus I_{du}+} (-1)^{|I_{du}'|-|I_{du}'|}.
\]
Putting \(F' = I_{du} \setminus (I_{du}+ \cup F)\), this sum becomes
\[
\sum_{0 \subseteq F' \subseteq I_{du}' \setminus I_{du}+} (-1)^{|F'|}.
\]
But this clearly vanishes unless \(I_{du}' \subseteq I_{du}+\) (i.e., \(l_j > 0\) for all \(j \in I_{du}'\)), in which case it
equals 1. We may thus restrict our sum to the classes of those pairs \((G \cup I_{du}, \{l_j\}_{j \in I_{du}})\)
for which \(l_j > 0\) for all \(j \in I_{du}'\) and take a single representative from each of these classes.
With a slight change in notation, the formula becomes
\[
(2.18) \quad \dim_{\Omega_0} H_0(K, (I, I'))
\]
\[
= \sum_{I_{sp} \setminus I_{sp}' \subseteq J \subseteq I_{sp}} (-1)^{|I_{sp}'|-|J|} |J|! \sum_{i \in I_{du}} M(\{\Delta_i^J, l_i\}_{i \in I_{du}}),
\]
\(i \geq 1\) for \(i \in I_{du}'\).

Let \(I \subseteq S\) with \(S_{to} \subseteq I\) and let \(I' \subseteq I_{af}\). When \(H_l(K, (I, I'))\) is finite-dimensional
for all \(l\), we define
\[
\chi(I, I') = \sum_{l=0}^{|I'|} (-1)^l \dim_{\Omega_0} H_l(K, (I, I')).
\]

**Theorem 2.19.** – Suppose that \(g\) is nondegenerate and semi-convenient. Then
(i) \(\dim_{\Omega_0} H_l(K, (S, S_{af})) < \infty\) for all \(l\).
(ii) \(H_l(K, (S, S_{af})) = 0\) for \(l > N - r\).
(iii) For $l = 1, \ldots, N - r$, $\dim_{\Omega_{0}} H_{l}(K_{l}(S, S_{af})) = \binom{m}{l + r - n}$ and Frobenius operates as multiplication by $q^{l+r}$. In particular, \( \det(I - t\alpha | H_{l}(K_{l}(S, S_{af}))) = (1 - q^{l+r}t)^{\binom{m}{l + r - n}} \) for $l = 1, \ldots, N - r$.

(iv) $\det(I - t\alpha | H_{0}(K_{l}(S, S_{af}))) = P(t)(1 - q^{r}t)^{\binom{m}{r-n}}$. 

(v) There is a subspace $H' \subseteq H_{0}(K_{l}(S, S_{af}))$ of dimension $\binom{m}{r-n}$ on which Frobenius operates as multiplication by $q^{r}$. In particular, \( \det(I - t\alpha | H') = (1 - q^{r}t)^{\binom{m}{r-n}} \) and $P(t)$ is a polynomial.

(vi) For $r \geq 1$,

$$
\chi(S, S_{af}) = \sum_{S_{o} \subseteq I \subseteq S_{ap}} (-1)^{|I| - |J|} \sum_{i_{1} + \cdots + i_{r} = |J|} M(\Delta_{1}^{J}, i_{1}; \cdots; \Delta_{r}^{J}, i_{r}).
$$

**Proof.** Note that assertion (iv) follows immediately from equations (2.5) and (2.14) and assertions (ii) and (iii). We prove statements (i), (ii), (iii), (v), and (vi) by induction on $r$. To simplify notation, we write $H_{l}(I, I')$ in place of $H_{l}(K_{l}(I, I'))$.

Suppose $r = 0$, so that $S_{af} = \{m + 1, \ldots, m + n\}$. In this case $g = 0$, so $\Delta = (0, \ldots, 0) \in \mathbb{R}^{N}$, $M = (0, \ldots, 0) \in \mathbb{R}^{N}$, $B = \Omega_{0}$, and the differential operators are $D_{i} = x_{i}\partial/\partial x_{i}$, which act trivially on $\Omega_{0}$. It follows that $K_{l}(S, S_{af})$ is the complex with $K_{l}(S, S_{af}) = (\Omega_{0})^{\binom{m}{r-n}}$ and all boundary maps trivial. Thus when $r = 0$, $H_{l}(S, S_{af})$ is a space of dimension $\binom{m}{r-n}$ with Frobenius acting as multiplication by $q^{r}$. This is exactly the assertion of the theorem. We observe that for $r = 0$,

$$
(2.20) \quad \chi(S, S_{af}) = \sum_{l=0}^{N} (-1)^{l} \binom{m}{l - n} = \begin{cases} 0 & \text{for } m > 0, \\ (-1)^{n} & \text{for } m = 0. \end{cases}
$$

Suppose the theorem true for $r - 1$. For notational convenience, put $S' = \{1, \ldots, N + r - 1\}$. From (2.12) we get a short exact sequence

$$
(2.21) \quad 0 \to K_{l}(S, S_{af}) \to K_{l}(S', S'_{af}) \to K_{l}(S, S'_{af}) \to 0,
$$

and by Theorem 2.15, $H_{l}(S, S'_{af}) = 0$ for $l > 0$. The associated long exact homology sequence then gives an exact sequence

$$
(2.22) \quad 0 \to H_{1}(S', S'_{af}) \to H_{0}(S, S_{af}) \to H_{0}(S, S'_{af}) \to H_{0}(S', S'_{af}) \to 0
$$

and isomorphisms for $l \geq 1$

$$
(2.23) \quad H_{l}(S, S_{af}) \simeq H_{l+1}(S', S'_{af}).
$$

Assertion (i) is now immediate from Theorem 2.15 and the induction hypothesis. We apply the induction hypothesis to compute the homology of $K_{l}(S', S'_{af})$. We have $H_{l}(S', S'_{af}) = 0$ for $l > N - r + 1$, so by (2.23) we have $H_{l}(S, S_{af}) = 0$ for $l > \max\{0, N - r\}$. This establishes (ii) when $N - r \geq 0$. (The proof of (ii) when $N - r < 0$, i.e., the proof that $H_{l}(S, S_{af}) = 0$ for all $l$ when $N - r < 0$, is given below.) For $1 \leq l \leq N - r + 1$,
$H_i(S', S'_a)$ is $(t + r - 1)_n$-dimensional with Frobenius acting as multiplication by $q^{t+r-1}$, hence for $1 \leq l \leq N - r$, it follows from (2.23) that $H_i(S, S_a)$ is $(t + m - n)_l$-dimensional with Frobenius acting as multiplication by $q^{t+r}$. This establishes assertion (iii). The space $H'$ of assertion (v) is the image of $H_i(S', S'_a)$ under the injection $H_i(S', S'_a) \hookrightarrow H_0(S, S_a)$ of (2.22).

To prove (vi), we observe that the short exact sequence (2.21) gives

$$\chi(S, S_a) = \chi(S, S'_a) - \chi(S', S'_a).$$

From (2.18) and Theorem 2.15 we have

$$(2.24) \quad \chi(S, S'_a) = \sum_{S_{i_0}, J \leq S_{i_1}} (-1)^{N-|J|} \prod_{i_1, \ldots, i_r \geq 1} M(\Delta^I, i_1; \ldots; \Delta^J, i_r).$$

When $r \geq 2$, the induction hypothesis gives

$$(2.25) \quad \chi(S', S'_a) = \sum_{S_{i_0}, J \leq S_{i_1}} (-1)^{N-|J|} \prod_{i_1, \ldots, i_r \geq 1} M(\Delta^I, i_1; \ldots; \Delta^J, i_r).$$

Subtracting (2.25) from (2.24) gives assertion (vi) when $r \geq 2$. When $r = 1$, we follow the same argument except that $\chi(S', S'_a)$ is no longer given by the induction hypothesis but rather by the right-hand side of (2.20).

It remains to prove (ii) and (vi) when $N - r < 0$. Since the expression for $\chi(S, S_a)$ in (vi) vanishes for $N - r < 0$, we see that (vi) follows from (ii) in this case. Suppose $N - r = -1$.

Then $\dim_{\mathbb{Q}_p} H_0(S, S'_a) = \chi(S, S'_a)$ by Theorem 2.15 and $\dim_{\mathbb{Q}_p} H_0(S', S'_a) = \chi(S', S'_a)$ since $H_i(S', S'_a) = 0$ for $l > 0$ by (ii) (in the case $N - r = 0$, which was already proved). But $\chi(S, S'_a) = \chi(S', S'_a)$ by evaluating (2.24) and (2.25) with $N - r = -1$, so the map $H_0(S, S'_a) \rightarrow H_0(S', S'_a)$ in (2.22) is an isomorphism. Since $H_1(S', S'_a) = 0$, it follows that $H_0(S, S_a) = 0$, thus (ii) and (vi) are established for $N - r = -1$. For $N - r < -1$, we have $H_0(S, S'_a) = 0$ by (2.24), hence $H_0(S, S_a) = 0$ by (2.22).

3. Complexes in characteristic $p$

From Theorem 2.19 we have

$$(3.1) \quad P(t) = \det(I - t\alpha | H_0(K(S, S_a)))/(1 - q^r t^{\frac{|m|}{r - n}}).$$

Our goal (Corollary 4.14 below) is to use this formula to give a lower bound for the Newton polygon of $P(t)$. Later, we shall identify this lower bound with the Hodge polygon of the primitive part of middle-dimensional cohomology of the complete intersection $F_1 = \cdots = F_r = 0$ in $(\mathbb{C}^n \times \mathbb{A}^n)_x$, where $F_i \in \mathbb{C}[x_1, \ldots, x_N, x_1 \cdots x_m]^{-1}$ is the generic polynomial with the property that the convex hull of $\text{supp}(F_i)$ coincides with the convex hull of $\text{supp}(f_i)$. 
The first step is to describe a basis for \( H_0(K.(S, S_{af})). \) In fact, it is no more difficult to do this for all \( H_1(K.(S, S_{af})). \) We begin by considering some related complexes in characteristic \( p. \) Let \( R \) be the ring \( R = \mathbb{F}_q[t^u \mid u \in M]. \) This ring is graded by the weight function defined earlier, namely, let \( R^{(k)} \), the homogeneous part of degree \( k \), be the span of all monomials \( t^u \) with \( w(u) = k. \) For \( i = 1, \ldots, N + r, \) put

\[
g_i = x_i \frac{\partial g}{\partial x_i} \in R^{(1)}.\]

Let \( \overline{K} = K.(R, \{g_i\}_{i=1}^{N+r}) \) be the Koszul complex on \( R \) defined by \( g_1, \ldots, g_{N+r}. \) Grade \( \overline{K}. \) so that the boundary maps \( \partial_i \) are homogeneous of degree 0, i.e.,

\[
\overline{K}_i^{(k)} = \bigoplus_{A \subseteq S, |A| = l} R^{(k-l)} e_A.
\]

For \( I \subseteq S \) with \( S_{to} \subseteq I, \) we may regard \( \theta_I \) as an endomorphism of \( R, \) homogeneous of degree 0, and define \( R_I = \theta_I(R). \) We also put

\[
g_{I,i} = x_i \frac{\partial g}{\partial x_i} \in R_I^{(1)}
\]

for \( i \in I \) and let \( \bar{K}._I = K.(R_I, \{g_{I,i}\}_{i \in I}) \) be the Koszul complex on \( R_I \) defined by the \( g_{I,i} \) for \( i \in I. \) For \( I' \subseteq I_{af}, \) let

\[
R_I^{I'} = \bigcap_{i \in I'} \ker(\theta_i \mid R_I),
\]

the elements of \( R_I \) divisible by \( x_i \) for all \( i \in I'. \) Let \( \bar{K}._I(I', I') \) be the subcomplex of \( \bar{K}._I(I) \) defined by

\[
\bar{K}._I(I, I') = \bigoplus_{A \subseteq I, |A| = l} R_I^{I' \setminus A} e_A.
\]

For \( i \in I_{af}, \) \( i \notin I', \) we have, as in (2.12), an exact sequence of (graded) complexes

\[
0 \rightarrow \bar{K}._I(I, I' \cup \{i\}) \rightarrow \bar{K}._I(I, I') \rightarrow \bar{K}._I(I \setminus \{i\}, I') \rightarrow 0.
\]

We shall compute the dimension (over \( \mathbb{F}_q \)) of \( H_1(\bar{K}._I(S, S_{af}))^{(k)} \), the homogeneous component of \( H_1(\bar{K}._I(S, S_{af})) \) of degree \( k, \) for all \( l \) and \( k \) and use this information to describe a basis for \( H_1(\bar{K}._I(S, S_{af})). \) The answer will be expressed in terms of certain invariants of certain polyhedra. As before, let \( \Delta_{I_{sp}} \) be the convex hull of \( \text{supp}(\theta_{I_{sp}}(f_i)), \) \( i = 1, \ldots, r. \) When \( m = 0 \) and \( I_{sp} = \emptyset, \) we assume \( f_i(0, \ldots, 0) \neq 0 \) for \( i = 1, \ldots, r, \) so that \( \Delta_i^0 \neq \emptyset. \) As before, \( \Delta_I \) will denote the convex hull of the origin and \( \text{supp}(\theta_I(g)). \)

For any set \( Y \subseteq \mathbb{R}^N, \) let \( \ell(Y) \) denote the cardinality of \( Y \cap \mathbb{Z}^N. \) For any subset \( I \subseteq S \) with \( S_{to} \subseteq I, \) set \( \delta(I) = \{j \in \{1, \ldots, r\} \mid N + j \in I_{du}\} \) and define a power series in the variables \( t_j \) for \( j \in \delta(I) \) by

\[
P_I(\{t_j\}_{j \in \delta(I)}) = \sum_{k_j=0}^{\infty} \ell(\bigcup_{j \in \delta(I)} k_j \Delta_{I_{sp}}^{j}) \prod_{j \in \delta(I)} t_j^{k_j} \in \mathbb{Z}[\{(t_j)_{j \in \delta(I)}\}].
\]
It is easily seen that

\[(3.7)\]

\[P_I = P_{L_{\text{sp}}} \big|_{t_j=0} \text{ for } j \notin \delta(I).\]

It is well-known that \(\ell(k_1 \Delta_{L_{\text{sp}}}^1 + \cdots + k_r \Delta_{L_{\text{sp}}}^r)\) is a rational polynomial of degree \(\leq |L_{\text{sp}}|\) in \(k_1, \ldots, k_r\), say,

\[\ell(k_1 \Delta_{L_{\text{sp}}}^1 + \cdots + k_r \Delta_{L_{\text{sp}}}^r) = \sum_{e_1 + \cdots + e_r \leq |L_{\text{sp}}|} a_{e_1 \ldots e_r} k_1^{e_1} \cdots k_r^{e_r} \in \mathbb{Q}[k_1, \ldots, k_r].\]

Thus

\[
P_{L_{\text{sp}}} \big|_{S_{\text{du}}} (t_1, \ldots, t_r) = \sum_{k_1, \ldots, k_r=0}^{\infty} \sum_{e_1 + \cdots + e_r \leq |L_{\text{sp}}|} a_{e_1 \ldots e_r} \prod_{j=1}^{r} \left( t_j \frac{\partial}{\partial t_j} \right)^{e_j} t_1^{k_1} \cdots t_r^{k_r}\]

\[
= \sum_{e_1 + \cdots + e_r \leq |L_{\text{sp}}|} a_{e_1 \ldots e_r} \prod_{j=1}^{r} \left( t_j \frac{\partial}{\partial t_j} \right)^{e_j} \left( \prod_{k=1}^{r} \frac{1}{1-t_k} \right)\]

\[
= \sum_{e_1 + \cdots + e_r \leq |L_{\text{sp}}|} p_{e_1 \ldots e_r}(t_1, \ldots, t_r) \prod_{j=1}^{r} (1-t_j)^{e_j+1},\]

for some polynomial \(p_{e_1 \ldots e_r}(t_1, \ldots, t_r) \in \mathbb{Q}[t_1, \ldots, t_r].\) Note that if \(e_j > 0\), then \(p_{e_1 \ldots e_r}(t_1, \ldots, t_r)\) is divisible by \(t_j\); and if \(e_j = 0\), then \(t_j\) does not appear in \(p_{e_1 \ldots e_r}(t_1, \ldots, t_r)\). Furthermore, \(\deg p_{e_1 \ldots e_r}(t_1, \ldots, t_r) = e_1 + \cdots + e_r\) unless \(a_{e_1 \ldots e_r} = 0\), in which case this polynomial vanishes. From (3.7) we have

\[(3.8)\]

\[P_I(\{t_j\}_{j \in \delta(I)}) = \sum_{e_1 + \cdots + e_r \leq |L_{\text{sp}}|, e_j=0 \text{ for } j \notin \delta(I)} p_{e_1 \ldots e_r}(t_1, \ldots, t_r) \prod_{j=1}^{r} (1-t_j)^{e_j+1}.\]

It is easily seen from the definitions of \(\Delta_I\) and the weight function \(w\) that

\[\dim_{\mathbb{F}_q} R^{(k)} = \ell(k \Delta_I) - \ell((k-1) \Delta_I),\]

i.e.,

\[(3.9)\]

\[\sum_{k=0}^{\infty} (\dim_{\mathbb{F}_q} R^{(k)}) t^k = (1-t) \sum_{k=0}^{\infty} \ell(k \Delta_I) t^k.\]

We regard \(R^{[I]}\) as being fibered over \(R^{\text{du}}\). The fiber of \(C(\Delta_I)\) over a point \((k_j)_{j \in \delta(I)} \in \mathbb{N}^{\text{du}}\) is \(\sum_{j \in \delta(I)} k_j \Delta_{L_{\text{sp}}}^j\). Hence

\[\ell(k \Delta_I) = \sum_{\sum k_j \leq k} \ell \left( \sum_{j \in \delta(I)} k_j \Delta_{L_{\text{sp}}}^j \right),\]
i.e., by (3.6)

\[(1 - t) \sum_{k=0}^{\infty} \ell(k \Delta_I) t^k = P_I(\{t_j\}_{j \in \delta(I)}) |t_j = t \text{ for } j \in \delta(I)|.\]

It now follows from (3.8) and (3.9) that

\[\sum_{k=0}^{\infty} (\dim_{F_q} R_I^{(k)}) t^k = \sum_{e_1 + \cdots + e_r \leq |I_{sp}|, e_j = 0 \text{ for } j \notin \delta(I)} \frac{p_{e_1 \cdots e_r}(t, \ldots, t)}{(1 - t)^{|I_{sp}| + \sum_{j=1}^{r} e_j}} \cdot\]

Define

\[(3.11) \quad q_{e_1 \cdots e_r}(t) = \frac{p_{e_1 \cdots e_r}(t, \ldots, t)}{(1 - t)^{|I_{sp}| + \sum_{j=1}^{r} e_j}} \in \mathbb{Q}[t].\]

Then

\[(3.12) \quad \sum_{k=0}^{\infty} (\dim_{F_q} R_I^{(k)}) t^k = (1 - t)^{-|I|} \sum_{e_1 + \cdots + e_r \leq |I_{sp}|, e_j = 0 \text{ for } j \notin \delta(I)} q_{e_1 \cdots e_r}(t).\]

Note that if \(A = \text{card}\{j \mid e_j \geq 1\}\), then \(q_{e_1 \cdots e_r}(t)\) is divisible by \(t^A\). Furthermore, \(\deg q_{e_1 \cdots e_r}(t) \leq |I_{sp}|\). It is clear from the definitions that \(p_{0 \cdots 0}(t_1, \ldots, t_r) = 1\), hence \(q_{0 \cdots 0}(t) = (1 - t)^{|I_{sp}|}\).

**Lemma 3.13.** Suppose that \(g\) is nondegenerate and semi-convenient. Then for \(I \subseteq S\) with \(S_{to} \subseteq I\) and \(I_{da} \neq \emptyset\), we have \(H_l(\bar{K}(I)) = 0\) for \(l > 0\) and \(\dim_{F_q} H_0(\bar{K}(I))^{(k)}\) is the coefficient of \(t^k\) in

\[\sum_{e_1 + \cdots + e_r \leq |I_{sp}|, e_j = 0 \text{ for } j \notin \delta(I)} q_{e_1 \cdots e_r}(t).\]

**Proof.** The complex \(\bar{K}(I)\) is acyclic in positive dimension by [13] (see also [1, Theorem 2.17]). The formula for \(\dim_{F_q} H_0(\bar{K}(I))^{(k)}\) follows from this acyclicity and (3.12).

Since \(\bar{K}_l(I, I')(k)\) and \(H_l(\bar{K}(I, I'))^{(k)}\) are always finite-dimensional, we may define

\[
\bar{\chi}^{(k)}(I, I') = \sum_{l=0}^{|I|} (-1)^l \dim_{F_q} \bar{K}_l(I, I')^{(k)} = \sum_{l=0}^{|I|} (-1)^l \dim_{F_q} H_l(\bar{K}(I, I'))^{(k)}.
\]
LEMMA 3.14. Suppose that \( g \) is nondegenerate and semi-convenient. Let \( I \subseteq S \) with \( S_{to} \subseteq I \) and let \( I' \subseteq I_{af} \). Then \( \bar{\chi}^{(k)}(I, I') \) is the coefficient of \( t^k \) in

\[
\sum_{I_{sp} \setminus I_{sp} \subseteq J \subseteq I_{sp}} (-1)^{|I_{sp}| - |J|} \sum_{e_1 + \cdots + e_r \leq |J| \atop e_j = 0 \text{ for } j \notin b(I) \atop e_j \geq 1 \text{ for } j \in b(I')} q^J_{e_1 \cdots e_r}(t).
\]

If in addition \( I'_{du} \neq I_{du} \), then \( H_1(\tilde{K}(I, I')) = 0 \) for \( l > 0 \), hence \( \dim_{F_q} H_0(\tilde{K}(I, I'))^{(k)} \) is the coefficient of \( t^k \) in (3.15).

Proof. When \( I'_{du} \neq I_{du} \), \( g_I \) is nondegenerate and commode (in the sense of [1]) with respect to \( I' \), so the complex \( \tilde{K}(I, I') \) is acyclic in positive dimension by [1, Theorem 2.17], and we need to prove only the formula (3.15) for \( \bar{\chi}^{(k)}(I, I') \).

From the definitions,

\[
\tilde{K}_I(I, I')^{(k)} = \bigoplus_{A \subseteq I, |A| = l} R^I_{I' \setminus A, (k-l)} e^A.
\]

We fix a set \( T \subseteq I' \) and ask for which \( A \subseteq I, |A| = l \), we have \( I' \setminus A = T \). Clearly, we must have \( A = (I' \setminus T) \cup \tilde{T} \), where \( \tilde{T} \subseteq I \setminus I' \) has cardinality \( l + |T| - |I'| \). Thus

\[
\bar{\chi}^{(k)}(I, I') = \sum_{l=0}^{|I|} (-1)^l \sum_{T \subseteq I'} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) \dim_{F_q} R^I_{I', (k-l)}.
\]

A formula for \( \dim_{F_q} R^I_{I', (k-l)} \) can be obtained from (3.12). A standard inclusion-exclusion argument shows that

\[
\sum_{k=0}^\infty (\dim_{F_q} R^I_{I', (k-l)}) t^k = \sum_{I \setminus T \subseteq J \subseteq I} (-1)^{|I| - |J|} \sum_{k=0}^\infty (\dim_{F_q} R^I_{J}^{(k)}) t^k.
\]

Substituting from (3.12) into this formula and using (3.16) we see that \( \bar{\chi}^{(k)}(I, I') \) is the coefficient of \( t^k \) in

\[
\sum_{l=0}^{|I|} (-1)^l \sum_{T \subseteq I'} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) t^l \
\times \sum_{I \setminus T \subseteq J \subseteq I} (-1)^{|I| - |J|} (1 - t)^{-|J|} \sum_{e_1 + \cdots + e_r \leq |J_{sp}| \atop e_j = 0 \text{ for } j \notin b(J)} q^J_{e_1 \cdots e_r}(t).
\]

Fix a subset \( K \subseteq I_{sp} \) and fix \( e_1, \ldots, e_r \) with \( e_1 + \cdots + e_r \leq |K| \). We ask for the coefficient of \( q^K_{e_1 \cdots e_r}(t) \) in (3.17). Let \( A \subseteq S_{du} \) be the set of all indices \( N + j \) such that \( e_j \neq 0 \). The equality of (3.15) and (3.17) is equivalent to the assertion that the coefficient of \( q^K_{e_1 \cdots e_r}(t) \) in (3.17) is

\[
(-1)^{|I_{sp}| - |K|} \text{ if } I_{sp} \setminus I_{sp}' \subseteq K \subseteq I_{sp} \text{ and } I'_{du} \subseteq A \subseteq I_{du},
\]

otherwise.

4e série - TOME 29 - 1996 - N° 3
It is clear that \( q_{K,...,e}^J(t) \) does not appear in (3.17) unless \( I_{sp} \setminus I_{sp} \subseteq K \subseteq I_{sp} \) and \( A \subseteq I_{du} \).

Assume from now on that these conditions are satisfied. Then \( q_{K,...,e}^J(t) \) will appear in (3.17) for each \( J, I \setminus T \subseteq J \subseteq I \), such that \( J_{sp} = K \) and \( J_{du} \supseteq A \), thus the coefficient of \( q_{K,...,e}^J(t) \) in (3.17) is

\[
(3.19) \quad \sum_{t=0}^{\lfloor I \rfloor} (-1)^t \sum_{T \subseteq I'} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) t^l \sum_{I \setminus T \subseteq J \subseteq I} (-1)^{|I| - |J|} (1 - t)^{-|J|}.
\]

We show this expression equals \((-1)^{|I_{sp}| - |K|}\) if \( I_{du} \subseteq A \) and equals 0 otherwise.

We evaluate the innermost sum in (3.19) first. Note that it is the empty sum unless \( I_{sp} \setminus I_{sp} \subseteq K \). When this condition is satisfied, it equals

\[
\frac{(-1)^{|I_{sp}| - |K|}}{(1 - t)^{|K|}} \sum_{A \cup (I_{du} \setminus T_{du}) \subseteq L \subseteq I_{du}} (-1)^{|I_{du}| - |L|} (1 - t)^{-|L|}
\]

\[
= \frac{(-1)^{|I_{sp}| - |K|}}{(1 - t)^{|K|}} \sum_{j = |A \cup (I_{du} \setminus T_{du})|} (-1)^{|I_{du}| - j} (1 - t)^{-j},
\]

where we have set \( j = |L| \) in the second expression. Replacing \( j \) by \( j + |A \cup (I_{du} \setminus T_{du})| \) and using the fact that \( |I_{du}| - |A \cup (I_{du} \setminus T_{du})| = |T_{du} \setminus A| \), this simplifies to

\[
\frac{(-1)^{|I_{sp}| - |K|} |T_{du} \setminus A|}{(1 - t)^{|K| + |I_{du}|}}.
\]

Substitution into (3.19) transforms that expression into

\[
(3.20) \quad \sum_{t=0}^{\lfloor I \rfloor} (-1)^t \sum_{T \subseteq I'} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) t^l \cdot \left\{ \frac{(-1)^{|I_{sp}| - |K|} |T_{du} \setminus A|}{(1 - t)^{|K| + |I_{du}|}} \right\} \text{ if } I_{sp} \setminus I_{sp} \subseteq K,
\]

\[
\text{otherwise.}
\]

Interchanging the order of summation transforms this into

\[
(3.21) \quad \frac{(-1)^{|I_{sp}| - |K|}}{(1 - t)^{|K| + |I_{du}|}} \sum_{T \subseteq I'} \sum_{I \setminus T \subseteq K} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) t^l.
\]

The inner sum in (3.21) equals

\[
\sum_{t=0}^{\lfloor I \rfloor - |T|} \left( \frac{|I| - |I'|}{l + |T| - |I'|} \right) t^l = \sum_{t=0}^{\lfloor I \rfloor - |I'|} \left( \frac{|I| - |I'|}{l} \right) (-t)^{l + |I'| - |T|}
\]

\[
= (-t)^{|I'| - |T|} (1 - t)^{|I| - |I'|}.
\]

Substituting this last expression into (3.21) gives

\[
(3.22) \quad \frac{(-1)^{|I_{sp}| - |K|}}{(1 - t)^{|I'| + |K| - |I_{sp}|}} \sum_{T \subseteq I'} \sum_{I \setminus T \subseteq K} \left( \frac{|I| - |I'|}{l} \right) (-t)^{|I'| - |T|}.
\]
The decomposition $T = T_{sp} \cup T_{du}$ induces a decomposition of the sum appearing in (3.22) as a product of two other sums, i.e., (3.22) equals

$$\frac{(-1)^{|I_{sp}|-|K|}}{(1- t)^{|I'_{sp}|+|K|-|I_{sp}|}} \sum_{I_{sp} \subseteq T \subseteq I_{sp}'} (-t)^{|I_{sp}'|} \sum_{T_2 \subseteq T_{du}} \text{e}^{T_2 \setminus A} (-t)^{|I_{du}'|}$$

where we are implicitly using the assumption made earlier that $I_{sp} \setminus I_{sp}' \subseteq K$, otherwise the first of these sums is empty. Evaluating the first sum we get

$$\sum_{I_{sp} \subseteq T \subseteq I_{sp}'} (-t)^{|I_{sp}'|} = \sum_{j=|I_{sp}'|-|K|}^{|I_{sp}'|-|I_{sp}|+|K|} \left( (|I_{sp}'| - |I_{sp}| + |K|) \frac{(-t)^{|I_{sp}'|}}{j - |I_{sp}| + |K|} \right)$$

$$= \sum_{j=0}^{|I_{sp}'|-|I_{sp}|+|K|} \left( (|I_{sp}'| - |I_{sp}| + |K|) \frac{(-t)^{|I_{sp}'|}-|I_{sp}|+|K|}{j} \right)$$

$$= (1 - t)^{|I_{sp}'|} \frac{|I_{sp}'|-|I_{sp}|+|K|}$$

hence (3.23) becomes

$$\frac{(-1)^{|I_{sp}|-|K|}}{(1- t)^{|I_{du}'|}} \sum_{T_2 \subseteq T_{du}} \text{e}^{T_2 \setminus A} (-t)^{|I_{du}'|}$$

To complete the proof of Lemma 3.14, we need to show that the sum in (3.24) equals $(1 - t)^{|I_{du}'|}$ if $I_{du}' \subseteq A$ and is 0 otherwise. Putting $B = T_2 \setminus A$ we have

$$\sum_{T_2 \subseteq T_{du}'} \text{e}^{T_2 \setminus A} (-t)^{|I_{du}'|-|T_2|} = \sum_{B \subseteq I_{du}' \setminus A} \sum_{T_2 \subseteq T_{du}'} (-1)^{|I_{du}'|-|T_2|} \text{e}^{T_2 \setminus A} (-t)^{|I_{du}'|}$$

Putting $j = |T_2|$ and writing $T_2 = B \cup (T_2 \cap A)$, a disjoint union with $T_2 \cap A \subseteq T_{du} \cap A$, we see that the inner sum equals

$$\sum_{j=|B|}^{(|I_{du}' \cap A|)} \left( \frac{|I_{du}' \cap A|}{j - |B|} \right) (-1)^{|I_{du}'|} \sum_{j=0}^{(|I_{du}' \cap A|)} \left( \frac{|I_{du}' \cap A|}{j} \right) (-1)^{|I_{du}'|-j+|B|}$$

$$= (-1)^{|B|} \left( (-t)^{|I_{du}' \cap A|} \right) (1 - t)^{|I_{du}' \cap A|}$$

Substituting this result into the right-hand side of (3.25), that expression becomes

$$(-t)^{|I_{du}' \setminus A|} (1 - t)^{|I_{du}' \cap A|}$$

But this sum is clearly 0 unless $I_{du}' \subseteq A$, in which case the entire expression clearly equals $(1 - t)^{|I_{du}'|}$. 

4e série - tome 29 - 1996 - n° 3
THEOREM 3.26. - Suppose that $g$ is nondegenerate and semi-convenient. Then

(i) $H_l(\tilde{K}(S, S_{af})) = 0$ for $l > N - r$.

(ii) For $l = 1, \ldots, N - r$, $\dim_{\mathbb{F}_q} H_l(\tilde{K}(S, S_{af})) = (\binom{m}{l + r - n})$ if $k = l + r$, otherwise.

(iii) $\dim_{\mathbb{F}_q} H_0(\tilde{K}(S, S_{af}))^{(k)}$ is the coefficient of $t^k$ in

\[
\sum_{j=r+1}^{N} (-1)^{j-r-1} \binom{m}{j-n} t^{j} + \sum_{S_{af} \subseteq J \subseteq S_p} (-1)^{|J|} \sum_{e_1 + \cdots + e_{|J|} \leq |J|} q^{e_1 \cdots e_r}(t).
\]

In particular, $H_0(\tilde{K}(S, S_{af}))^{(k)} = 0$ for $k < r$ or $k > N$.

Remark. - When $r = 0$, i.e., $S_{af} = \{m + 1, \ldots, m + n\}$, $S_{du} = \emptyset$, we understand the second sum in Theorem 3.26 (iii) to be simply $\sum_{S_{af} \subseteq J \subseteq S_p} (-1)^{|J|} q_0^{e_1 \cdots e_r}(t) = (-t)^n (1 - t)^m$. The whole expression in (iii) is then just equal to 1 if $n = 0$ and equal to 0 if $n > 0$.

Proof. - To simplify notation, we write $H_l(I, I')$ in place of $H_l(\tilde{K}(I, I'))$. We note first that by Lemma 3.14, (iii) follows from (i) and (ii).

The proof of (i) and (ii) is by induction on $r$. Suppose $r = 0$. Then $\tilde{K}(S, S_{af})$ is the complex $\tilde{K}_l(S, S_{af}) = (\mathbb{F}_q)^{(m-n)}$ with all boundary maps trivial. Hence $H_l(S, S_{af}) = 0$ for $l > N$, and for $0 \leq l \leq N$ we have

\[
\dim_{\mathbb{F}_q} H_l(S, S_{af})^{(k)} = \binom{m}{l - n} \text{ if } k = l,
\]

otherwise.

This is exactly the assertion of the theorem when $r = 0$.

Suppose the theorem true for $r - 1$. For notational convenience, put $S' = \{1, \ldots, N + r - 1\}$. From (3.5) we get an exact sequence

\[
0 \to \tilde{K}(S, S_{af}) \to \tilde{K}(S, S_{af}') \to \tilde{K}(S', S_{af}') \to 0,
\]

and by Lemma 3.14, $H_l(S, S_{af}') = 0$ for $l > 0$. The associated long exact homology sequence then gives an exact sequence

\[
0 \to H_1(S', S_{af}') \to H_0(S, S_{af}) \to H_0(S, S_{af}') \to H_0(S', S_{af}') \to 0
\]

and isomorphisms for $l \geq 1$

\[
H_l(S, S_{af}) \cong H_{l+1}(S', S_{af}).
\]

By induction we have $H_l(S', S_{af}') = 0$ for $l > N - r + 1$, and for $1 \leq l \leq N - r + 1$

\[
\dim_{\mathbb{F}_q} H_l(S', S_{af}')^{(k)} = \binom{m}{l + r - 1 - n} \text{ if } k = l + r - 1,
\]

otherwise.

Parts (i) and (ii) of the theorem now follow from (3.29).
Let the $g_{I,i}$ be as in (3.3) and put

$$D_{I,i}' = x_i \frac{\partial}{\partial x_i} + g_{I,i},$$

an operator on $R_I$. Let $\bar{K}.'(I) = K.(R_I, \{D_{I,i}'\}_{i \in I})$ be the Koszul complex on $R_I$ defined by the $D_{I,i}'$ for $i \in I$. Let $\bar{K}.'(I, I')$ be the subcomplex of $\bar{K}.'(I)$ defined by taking $\bar{K}._{I}^{'}(I, I') = \bar{K}_i(I, I')$. The rings $R_{I}'$ have an increasing filtration $F.R_{I}'$ defined by taking $F_kR_{I}'$ to be the span of monomials $x^u$, $u \in M$, with $w(u) \leq k$. This induces a filtration $F.(\bar{K}.'(I, I'))$ on the complexes $\bar{K}.'(I, I')$, namely,

$$F_k\bar{K}._{I}^{'}(I, I') = \bigoplus_{A \subseteq I, |A| = l} F_{k-l}R_{I}'^{\text{span} e_A},$$

which in turn induces a filtration $F.(H.(\bar{K}.'(I, I')))$. We shall compute $\text{gr}^r(H.(\bar{K}.'(I, I'))) = \text{gr}^r_{H.\bar{K}.'(I, I'))}$, the associated graded of this filtration. It is clear from the definitions that $\text{gr}^r(\bar{K}.'(I, I'))$, the graded complex associated to the filtered complex $\bar{K}.'(I, I')$, is identified with the complex $\bar{K}.'(I, I')$ considered previously.

Associated to the filtered complex $\bar{K}.'(I, I')$ is a convergent $E_1$ spectral sequence [17, Chapter 9] with

$$E^{1}_{k,l} = H_{k+l}(\bar{K}.'(I, I'))^{(k)},$$
$$E^{\infty}_{k,l} = \text{gr}^r_{H_{k+l}(\bar{K}.'(I, I'))}. $$

Suppose we are in the situation of Lemma 3.14. Then $E^{1}_{k,l} = 0$ for $k + l > 0$ or $k + l < 0$, from which it is easily seen that all the differentials $d_{k,l}^{s} : E_{k,l}^{s} \rightarrow E_{k-s,l+s-1}^{s}$ of the spectral sequence are 0. Hence $E^{1}_{k,l} \simeq E^{2}_{k,l} \simeq \cdots \simeq E^{\infty}_{k,l}$ for all $k, l$. By Lemma 3.14 we have the following.

**Lemma 3.31.** – Suppose that $g$ is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{to} \subseteq I$ and $I' \subseteq S_{af}$ with $I'_{du} \neq I_{du}$, we have $H_l(\bar{K}.'(I, I')) = 0$ for $l > 0$ and $\dim F_{q} \text{gr}^r_{H_0(\bar{K}.'(I, I'))}$ is the coefficient of $t^k$ in

$$\sum_{I_{sp} \subseteq J \subseteq I_{sp}} (-1)^{|I_{sp}| - |J|} \sum_{e_1 + \cdots + e_r \leq |J| \atop e_j = 0 \text{ for } j \notin S(I)} q_{e_1 \cdots e_r}(t).$$

Now consider the case $I = S$, $I' = S_{af}$, so that

$$E^{1}_{k,l} \simeq H_{k+l}(\bar{K}.(S, S_{af}))^{(k)},$$
$$E^{\infty}_{k,l} \simeq \text{gr}^r_{H_{k+l}(\bar{K}.(S, S_{af}))}. $$

We shall again show that $E^{1}_{k,l} \simeq E^{\infty}_{k,l}$ for all $k, l$. We begin by computing $\dim F_{q} H_l(\bar{K}.(S, S_{af}))$ for all $l$. 

4$^e$ série – tome 29 – 1996 – n° 3
LEMMA 3.34. – Suppose that $g$ is nondegenerate and semi-convenient. Then for all $l$,
\[
\dim_{F_q} H_l(\bar{K}.(S, S_{af})) = \dim_{F_q} H_l(\bar{K}.(S, S_{af})),
\]

hence $\dim_{F_q} H_l(\bar{K}.(S, S_{af}))$ can be computed from Theorem 3.26.

Proof. – The existence of the spectral sequence implies that
\[
\sum_{l=0}^{N+r} (-1)^l \dim_{F_q} H_l(\bar{K}.(S, S'_{af})) = \sum_{l=0}^{N+r} (-1)^l \dim_{F_q} H_l(\bar{K}.(S, S_{af})),
\]

hence it suffices to prove the stated equality for $l \geq 1$. The proof is by induction on $r$. The case $r = 0$ is trivial since $\bar{K}.(S, S_{af}) = \bar{K}.(S, S_{af})$ in that case. Put
\[
S' = \{1, \ldots, N + r - 1\}.\]

As in (3.5), we have an exact sequence of complexes
\[
0 \to \bar{K}.(S, S_{af}) \to \bar{K}.(S', S'_{af}) \to \bar{K}.(S', S'_{af}) \to 0.
\]

By Lemma 3.31, $H_l(\bar{K}.(S, S_{af})) = 0$ for $l > 0$, hence the long exact homology sequence gives isomorphisms
\[
H_l(\bar{K}.(S, S_{af})) \simeq H_{l+1}(\bar{K}.(S', S'_{af}))
\]
for $l \geq 1$. By induction we have $\dim_{F_q} H_l(\bar{K}.(S', S'_{af})) = \dim_{F_q} H_l(\bar{K}.(S, S_{af}))$ and by (3.29) we have $H_l(\bar{K}.(S, S_{af})) \simeq H_{l+1}(\bar{K}.(S', S'_{af}))$ for $l \geq 1$. It follows that $\dim_{F_q} H_l(\bar{K}.(S, S_{af})) = \dim_{F_q} H_l(\bar{K}.(S, S_{af}))$ for $l \geq 1$. This proves Lemma 3.34.

From (3.32) and Theorem 3.26, we see that $E_{k,l}^1 = 0$ for all $k, l$ except possibly $E_{1+r,-r}^1$ for $l = 1, \ldots, N - r$ and $E_{k,-k}^1$ for $k = r, \ldots, N$. Therefore $\text{gr}^{(k)} H_{k+l}(\bar{K}.(S, S_{af})) = 0$ except possibly for $\text{gr}^{(l+r)} H_l(\bar{K}.(S, S_{af}))$, $l = 1, \ldots, N - r$, and $\text{gr}^{(k)} H_0(\bar{K}.(S, S_{af}))$, $k = r, \ldots, N$. If $l \in \{1, \ldots, N - r\}$, it now follows from Lemma 3.34 and Theorem 3.26 that
\[
\dim_{F_q} \text{gr}^{(l+r)} H_l(\bar{K}.(S, S_{af})) = \dim_{F_q} H_l(\bar{K}.(S, S_{af}))^{(l+r)}.
\]

Examining the differentials of the spectral sequence shows that $E_{1+r,-r}^1 \simeq E_{2}^{2-r,-r} \simeq \cdots \simeq E_{k,-k}^1$ if $k \neq r$, i.e.,
\[
gr^{(k)} H_0(\bar{K}.(S, S_{af})) \simeq H_0(\bar{K}.(S, S_{af}))^{(k)}
\]
for $k = r + 1, \ldots, N$. For $k = r$, one sees that $E_{1-r,-r}^\infty$ is a quotient of $E_{1-r,-r}^1$ (in fact, $E_{1-r,-r}^\infty \simeq E_{1-r,-r}^2$), hence $\dim_{F_q} E_{1-r,-r}^\infty \leq \dim_{F_q} E_{1-r,-r}^1$. But by Lemma 3.34 and equation (3.36), we see that we must have $\dim_{F_q} E_{1-r,-r}^\infty = \dim_{F_q} E_{1-r,-r}^1$, also. By Theorem 3.26 we now have the following.

THEOREM 3.37. – Suppose that $g$ is nondegenerate and semi-convenient. Then
(i) $H_l(\bar{K}.(S, S_{af})) = 0$ for $l > N - r$.
(ii) For $l = 1, \ldots, N - r$,
\[
\dim_{F_q} \text{gr}^{(k)} H_l(\bar{K}.(S, S_{af})) = \begin{cases} m & \text{if } k = l + r, \\ 0 & \text{otherwise}. \end{cases}
\]
(iii) \( \dim F \cdot \text{gr}^{(k)} H_0(\tilde{K}, (S, S_a)) \) is the coefficient of \( t^k \) in

\[
\sum_{j=r+1}^{N} (-1)^{j-r-1} \binom{m}{j-n} t^j + \sum_{S_0 \subseteq J \subseteq S_{(p)}} (-1)^{N-|J|} \sum_{e_1 + \ldots + e_r \leq |J| \atop \tau_j \geq 1 \text{ for all } j} q_{e_1 \ldots e_r}^J(t).
\]

In particular, \( \text{gr}^{(k)} H_0(\tilde{K}, (S, S_a)) = 0 \) for \( k < r \) or \( k > N \).

4. \( p \)-Adic estimates

We now prove a lemma on lifting homology from characteristic \( p \) to characteristic 0, which will allow us to use the results of section 3 to obtain information about the complexes of section 2. Let \( \mathcal{O} \) be a complete discrete valuation ring with uniformizer \( \pi \) and let \( K \) be an \( \mathcal{O} \)-module. We call \( K \) flat if multiplication by \( \pi \) is injective and separated if \( \bigcap_{j=1}^\infty \pi^j K = 0 \). A separated \( \mathcal{O} \)-module \( K \) has an obvious metric space structure with the \( \{\pi^j K\} \) forming a fundamental system of neighborhoods of 0. We call \( K \) \( \mathcal{O} \)-complete if it is complete in this metric.

**Lemma 4.1.** Let \( K = \{ \ldots \xrightarrow{\partial_2} K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\partial_0} 0 \} \) be a complex of flat, separated, \( \mathcal{O} \)-complete \( \mathcal{O} \)-modules with \( \mathcal{O} \)-linear boundary maps. Let \( \tilde{K} \) be the complex obtained by reducing \( K \) modulo \( \pi \). If \( H_i(\tilde{K}) \) has dimension \( d \) over \( \mathcal{O}/(\pi) \) and multiplication by \( \pi \) is injective on \( H_i(\tilde{K}) \) and \( H_{i-1}(\tilde{K}) \), then \( H_i(K) \) is a finite, free \( \mathcal{O} \)-module of rank \( d \). Furthermore, any lifting of any basis for \( H_i(\tilde{K}) \) is a basis for \( H_i(K) \).

**Proof.** Consider \( H_i(K) = \ker \partial_i/\im \partial_{i+1} \). We claim \( \im \partial_{i+1} \) is complete. Let \( \{\partial_{i+1} z_s\}_{s=1}^\infty \) be a Cauchy sequence in \( \im \partial_{i+1} \), say,

\[
\partial_{i+1} z_{s+1} - \partial_{i+1} z_s = \pi^A(s) w_s,
\]

where \( A(s) \to \infty \) as \( s \to \infty \). Suppose we have found \( \tilde{z}_1, \ldots, \tilde{z}_s \in K_{i+1} \) such that \( \partial_{i+1} \tilde{z}_j = \partial_{i+1} z_j \) and \( \tilde{z}_{j+1} \equiv \tilde{z}_j \pmod{\pi^A(j)} \). Then \( \partial_{i+1} \tilde{z}_{s+1} - \partial_{i+1} \tilde{z}_s = \pi^A(s) w_s \), so \( \pi^A(s) w_s = 0 \) in \( H_i(K) \). By the injectivity of \( \pi \) on \( H_i(K) \), \( w_s = 0 \) in \( H_i(K) \), i.e., \( w_s = \partial_{i+1} y_s \) for some \( y_s \in K_{i+1} \). Hence \( \partial_{i+1} (z_{s+1} - \tilde{z}_s - \pi^A(s) y_s) = 0 \). Put \( \tilde{z}_{s+1} = \tilde{z}_s + \pi^A(s) y_s \). Then \( \tilde{z}_{s+1} \equiv \tilde{z}_s \pmod{\pi^A(s)} \) and \( \partial_{i+1} \tilde{z}_{s+1} = \partial_{i+1} \tilde{z}_{s+1} \). Since \( K_{i+1} \) is complete, the Cauchy sequence \( \{\tilde{z}_s\} \) has a limit \( \tilde{z} \). It is then clear that \( \{\partial_{i+1} z_s\} \) converges to \( \partial_{i+1} \tilde{z} \).

There is a short exact sequence of complexes

\[
0 \to K \xrightarrow{\pi} K \to \tilde{K} \to 0,
\]

where the second arrow means multiplication by \( \pi \). The associated long exact homology sequence and the hypothesis that multiplication by \( \pi \) is injective on \( H_{i-1}(K) \) imply that we have an isomorphism \( H_i(\tilde{K})/\pi H_i(K) \cong H_i(K) \). The completeness of \( \im \partial_{i+1} \) implies
that $H_1(K_\cdot)$ is separated. It is then straightforward to check that if $\xi_1, \ldots, \xi_d$ is a basis for $H_1(K_\cdot)$, then $\tilde{\xi}_1, \ldots, \tilde{\xi}_d$ is an $\mathcal{O}$-basis for $H_1(K_\cdot)$, where $\tilde{\xi}_i$ is any lifting of $\xi_i$ to $H_1(K_\cdot)$.

We shall apply this result in the following setting. In the definitions of our complexes in section 2, we may replace $B$ by its unit ball $B(0_0)$, thus obtaining complexes $K_\cdot(I, I'; \mathcal{O}_0)$ of $\mathcal{O}_0$-modules. We define a reduction map $\rho : B(0_0) \to R$ as follows. If $\xi = \sum_{u \in M} A_u \pi^{w(u)} x^u \in B(0_0)$, then $\rho(\xi) = \sum_{u \in M} \tilde{A}_u x^u \in R$, where $\tilde{A}_u \in \mathbb{F}_q$ is the reduction of $A_u$ modulo $\pi$. There are induced maps $\rho : B(I_0')(\mathcal{O}_0) \to R(I')$ for all $I \subseteq S$ with $S_{t_0} \subseteq I$ and $I' \subseteq I_{af}$. By [1, section 2], the image of the complex $K_\cdot(I, I'; \mathcal{O}_0)$ under $\rho$ is the complex $K_\cdot'(I, I')$. Thus we have short exact sequences of complexes

$$0 \to K_\cdot(I, I'; \mathcal{O}_0) \overset{\pi}{\to} K_\cdot(I, I'; \mathcal{O}_0) \overset{\rho}{\to} K_\cdot'(I, I') \to 0.$$  

Before we can apply Lemma 4.1, we must check that multiplication by $\pi$ is injective on all $H_l(K_\cdot(I, I'; \mathcal{O}_0))$.

**Lemma 4.3.** Suppose that $g$ is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{t_0} \subseteq I$ and $I' \subseteq I_{af}$ with $I_{du} \neq I_{du}$, multiplication by $\pi$ is injective on $H_l(K_\cdot(I, I'; \mathcal{O}_0))$ for all $l$.

**Proof.** The long exact homology sequence associated to (4.2) is

$$\cdots \to H_{l+1}(K_\cdot'(I, I')) \to H_l(K_\cdot(I, I'; \mathcal{O}_0)) \overset{\pi}{\to} H_l(K_\cdot(I, I'; \mathcal{O}_0)) \to H_l(K_\cdot'(I, I')) \to \cdots.$$  

The result now follows immediately from Lemma 3.31.

Applying Lemma 4.1 and Lemma 3.31 gives the following.

**Corollary 4.5.** Suppose that $g$ is nondegenerate and semi-convenient. For $I \subseteq S$ with $S_{t_0} \subseteq I$ and $I' \subseteq I_{af}$ with $I_{du} \neq I_{du}$, one has $H_l(K_\cdot(I, I'; \mathcal{O}_0)) = 0$ for $l > 0$ and $H_0(K_\cdot(I, I'; \mathcal{O}_0))$ is a free $\mathcal{O}_0$-module of finite rank.

**Lemma 4.6.** Suppose that $g$ is nondegenerate and semi-convenient. Then for all $I \subseteq S$ with $S_{t_0} \subseteq I$ and for all $l$, multiplication by $\pi$ is injective on $H_l(K_\cdot(I, I_{af}; \mathcal{O}_0))$.

**Proof.** The proof is by induction on $|I_{du}|$. When $I_{du} = \emptyset$, $K_\cdot(I, I_{af}; \mathcal{O}_0)$ is the complex with $K_\cdot(I, I_{af}; \mathcal{O}_0) = (\mathcal{O}_0)^{(m-|I_{du}|)}$ and all boundary maps trivial. Thus $H_l(K_\cdot(I, I_{af}; \mathcal{O}_0))$ is a free $\mathcal{O}_0$-module of rank $(m-|I_{du}|)$, so multiplication by $\pi$ is injective for all $l$. Now suppose we know the result for all subsets $I \subseteq S$ with $I_{du}$ of a given cardinality. Let $j \in S_{du}$, $j \not\in I_{du}$, and put $J = I \cup \{j\}$. There is a short exact sequence of complexes

$$0 \to K_\cdot(J, J_{af}; \mathcal{O}_0) \to K_\cdot(J, J_{af}; \mathcal{O}_0) \overset{\theta}{\to} K_\cdot(I, I_{af}; \mathcal{O}_0) \to 0.$$  

By Corollary 4.5, the associated long exact homology sequence gives an exact sequence

$$0 \to H_1(I, I_{af}) \to H_0(J, J_{af}) \to H_0(J, J_{af}) \to H_0(I, I_{af}) \to 0.$$
and isomorphisms for $l \geq 1$

$$H_l(J, J_{af}) \simeq H_{l+1}(I, I_{af}).$$

The induction hypothesis then implies that multiplication by $\pi$ is injective on $H_l(J, J_{af})$ for $l \geq 1$. By the induction hypothesis (resp. Lemma 4.3), multiplication by $\pi$ is injective on $H_l(1, I_{af})$ (resp. $H_0(J, I_{af})$). The injectivity on $H_0(J, J_{af})$ then follows from the exact sequence (4.7).

By Lemma 4.6, we may apply Lemma 4.1 to obtain a basis for $H_0(K(\mathcal{S}, S_{af}; \mathcal{O}_0))$ by lifting a basis for $H_0(\tilde{K}, (S, S_{af}))$. We choose a basis for $H_0(\tilde{K}, (S, S_{af}))$ as follows. For $k = r, \ldots, N$, let $B^{(k)} = \{\xi^{(k)}_1, \ldots, \xi^{(k)}_d(k)\}$ be a set of monomials of weight $k$ in $\{x^u \mid u \in M\}$ such that their images in $\text{gr}^{(k)}(H_0(\tilde{K}, (S, S_{af})))$ form a basis. By Theorem 3.37,

$$d(k) = \text{coefficient of } t^k \text{ in } \sum_{j=r+1}^{N} (-1)^{j-r-1} \binom{m}{j-n} t^j + \sum_{S_i \subseteq J \subseteq S_{af}} (-1)^{N-|J|} \sum_{e_1 + \cdots + e_e \leq |J|, e_j \geq 1 \text{ for all } j} q_j^{e_1 \cdots e_e}(t).$$

Clearly then, the set $B = \bigcup_{k=r}^{N} B^{(k)}$ is a basis for $H_0(\tilde{K}, (S, S_{af}))$, so by Lemma 4.1, $B$ is also a basis for $H_0(K, (S, S_{af}; \mathcal{O}_0))$.

Let $W$ be the $\mathcal{O}_0$-span of $B$ and set $W(b, c) = W \cap L(b, c)$. From the above remarks, we have

$$B_{S_{af}}^S(\mathcal{O}_0) = W \left(\frac{1}{p-1}, 0\right) \oplus \sum_{i=1}^{N+r} D_i B_{S_{af}}^S(\mathcal{O}_0),$$

$$B_{S_{af}}^S = W \oplus \sum_{i=1}^{N+r} D_i B_{S_{af}}^S(\mathcal{O}_0).$$

Let $L_I^r(b, c) = \bigcap_{i \in I} \ker(\theta_i \mid L_I(b, c))$. Set $e = b - 1/(p - 1)$. Repeating the proofs of [1, Proposition 3.6 and Theorem 3.8] (for a somewhat improved version of these arguments, see [2, Lemma 3.3 and Theorem 3.14]), we get the following.

**Theorem 4.11.** -- For $b, c \in \mathbb{R}$ with $1/(p - 1) \leq b \leq p/(p - 1)$,

$$L_{S_{af}}^S(b, c) = W(b, c) + \sum_{i=1}^{N+r} D_i L_{S_{af}}^S(\mathcal{O}_0),$$

$$L_{S_{af}}^S(b) = W  \oplus \sum_{i=1}^{N+r} D_i L_{S_{af}}^S(\mathcal{O}_0).$$

We now follow the method of [10, section 7]. Put $\Omega_1 = Q_p(\zeta_p)$ and let $\tau \in \text{Gal}(\Omega_0/\Omega_1)$ be a lifting of the Frobenius automorphism of $\text{Gal}(F_q/F_p)$. By [1, section 1], there exists
$F(x) \in L(1/(p-1), 0)$ such that the operator $\beta = \tau^{-1} \circ \psi \circ F$, an $\Omega_1$-linear endomorphism of $B$ and of $L(b)$ for $0 < b \leq p/(p-1)$, satisfies $\beta^n = \alpha$. Let $\{\sigma_i\}_{i=1}^a$ be an integral basis for $\Omega_0$ over $\Omega_1$. For $\xi^{(k')}_{j'} \in B^{(k')}$, write

$$\beta(\pi^{k'} \sigma_{j'} \xi^{(k')}_{j'}) = \sum_{l=1}^a \sum_{k=r}^b \sum_{j=1}^N A(j, k, l; j', k', l') \pi^{k} \sigma_{j} \xi^{(k)}_{j} \left( \mod \sum_{i=1}^{N+r} D_i B^{S \setminus \{i\}} \right).$$

By (4.10), the $A(j, k, l; j', k', l')$ are uniquely determined by this relation.

**Lemma 4.12.** $\text{ord}_{q} A(j, k, l; j', k', l') \geq k$.

**Proof.** Since $\pi^{k'} \sigma_{j'} \xi^{(k')}_{j'} \in L(1/(p-1), 0)$, it follows that $\beta(\pi^{k'} \sigma_{j'} \xi^{(k')}_{j'}) \in L(p/(p-1), 0)$. The lemma is then an immediate consequence of Theorem 4.11.

Let $\text{ord}_{q}$ be the $p$-adic valuation normalized by requiring $\text{ord}_{q} q = 1$. We compute all Newton polygons with respect to this valuation. By the argument that establishes [10, equation (7.7)], we now have the following. (See the two paragraphs preceding [10, Theorem 7.1] for the assertion about the endpoints.)

**Theorem 4.13.** Suppose that $g$ is nondegenerate and semi-convenient. Then the Newton polygon of the polynomial $\det \left( I - t \bar{\alpha} | B^{S\setminus\{i\}} / \sum_{i=1}^{N+r} D_i B^{S \setminus \{i\}} \right)$ lies on or above the Newton polygon of the polynomial $\prod_{k=r}^N (1 - q^k t)^{d(k)}$ and their endpoints coincide.

By Theorem 2.19 (iv), we finally have:

**Corollary 4.14.** The Newton polygon of the polynomial $P(t)$ lies on or above the Newton polygon of the polynomial $(1 - q^r t)^{N-r} \prod_{k=r}^N (1 - q^k t)^{d(k)}$ and their endpoints coincide.

**Remark.** Note that, in view of (2.5), the corresponding factor of $Z(V/F_g; t)$ is $P(q^{-r} t)$, which has lower bound determined by the polynomial $(1 - t)^{-N-r} \prod_{k=0}^N (1 - q^k t)^{d(k+r)}$.

### 5. Hodge polygon

The purpose of this section is to identify the lower bounds of Theorem 4.13 and Corollary 4.14 with certain Hodge polygons of toric and affine complete intersections. This is a rather straightforward calculation, based on the results of [6]. We begin by recalling the definition of Hodge polygon.

Let $X$ be a complex variety and let $H^i_c(X, \mathbb{Q})$ denote rational cohomology with compact supports. By Deligne [7, 8], there is a mixed Hodge structure on $H^i_c(X, \mathbb{Q})$, in particular, there is a decreasing (Hodge) filtration $F^i$ on $H^i_c(X, \mathbb{C})$. The Hodge numbers $h^k(H^i_c(X, \mathbb{C}))$ are defined by

$$h^k(H^i_c(X, \mathbb{C})) = \dim_{\mathbb{C}} F^k H^i_c(X, \mathbb{C}) / F^{k+1} H^i_c(X, \mathbb{C}),$$

where $F^i H^i_c(X, \mathbb{C})$ is the $i$th graded piece of the filtration.
and the Hodge polygon of $H^j_c(X,\mathbb{C})$ is defined to be the Newton polygon of the polynomial $\prod_{k \geq 0} (1 - q^k t)^h_k((H^j_c(X,\mathbb{C}))).$ Alternatively, there is also an increasing (weight) filtration $W_\ast$ on $H^\ast_c(X,\mathbb{Q})$ such that the Hodge filtration $F^\ast$ induces a pure Hodge structure of weight $s$ on $W_s H^\ast_c(X,\mathbb{C})/W_{s-1} H^\ast_c(X,\mathbb{C}),$ i.e.,

$$W_s H^\ast_c(X,\mathbb{C})/W_{s-1} H^\ast_c(X,\mathbb{C}) \simeq \bigoplus_{a+b=s} H^{a,b},$$

where $H^{a,b} = \overline{H^{a,b}}$ and $\overline{H^{a,b}}$ is the complex conjugate of $H^{a,b}.$ If we put $h^{a,b}((H^j_c(X,\mathbb{C}))) = \dim_{\mathbb{C}} H^{a,b},$ then

$$h^k(H^j_c(X,\mathbb{C})) = \sum_{b \geq 0} h^{k,b}((H^j_c(X,\mathbb{C}))).$$

Consider the case where $F_i \in \mathbb{C}[x_1,\ldots,x_N,(x_1 \cdots x_m)^{-1}], i = 1,\ldots,r,$ is the generic polynomial with the property that the convex hull of $\text{supp}(F_i)$ equals $\Delta_{\mathbb{R}^m}$ and let $X \subset (\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C}$ be the smooth complete intersection $F_1 = \cdots = F_r = 0.$ Since $X$ is affine, of complex dimension $N - r,$ we have $H^l_c(X,\mathbb{C}) = 0$ for $l < N - r$ or $l > 2N - 2r.$ Furthermore, the Gysin map $H^{N-r+i}_c((\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C}) \to H^{N+r+i}_c(((\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C})$ is an isomorphism for $i = 1,\ldots,N - r$ and is surjective for $i = 0.$ It is a morphism of mixed Hodge structures sending $H^{a,b}(X)$ to $H^{a+r,b+r}((\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C}),$ and by (5) we have for $i = 0,\ldots,N$

$$h^{a,b}(H^{N+i}_c((\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C},\mathbb{C})) = \left\{ \begin{array}{ll} \frac{m}{i-n} & \text{if } a = b = i, \\
0 & \text{otherwise.} \end{array} \right.$$  

Hence for $i = 1,\ldots,N - r$

$$h^{a,b}(H^{N-r+i}_c(X,\mathbb{C})) = \left\{ \begin{array}{ll} \frac{m}{r+i-n} & \text{if } a = b = i, \\
0 & \text{otherwise,} \end{array} \right.$$  

in particular,

$$h^k(H^{N-r+i}_c(X,\mathbb{C})) = \left\{ \begin{array}{ll} \frac{m}{r+i-n} & \text{if } k = i, \\
0 & \text{otherwise.} \end{array} \right.$$  

Thus the only nontrivial case is the middle-dimensional cohomology $H^{N-r}_c(X,\mathbb{C}).$ The primitive part $PH^{N-r}_c(X,\mathbb{C})$ of middle-dimensional cohomology is defined to be the kernel of the (surjective) Gysin map $H^{N-r}_c(X,\mathbb{C}) \to H^{N+r}_c(((\mathbb{T}^m \times \mathbb{A}^n)_\mathbb{C}).$ By (5), $PH^{N-r}_c(X,\mathbb{C})$ has codimension $\left( \frac{m}{r-n} \right)$ in $H^{N-r}_c(X,\mathbb{C}).$ More precisely,

$$h^k(PH^{N-r}_c(X,\mathbb{C})) = \left\{ \begin{array}{ll} h^k(H^{N-r}_c(X,\mathbb{C})) & \text{if } k \neq 0, \\
h^0(H^{N-r}_c(X,\mathbb{C})) - \left( \frac{m}{r-n} \right) & \text{if } k = 0. \end{array} \right.$$

4$^\text{e}$ sèrie - TOME 29 - 1996 - N° 3
In [6] is described a procedure for computing the Hodge-Deligne numbers of $X$ in terms of invariants of the polytopes $\Delta^i_I$, $S_{s_0} \subseteq I \subseteq S_{s_{sp}}$, $i = 1, \ldots, r$. We carry out part of this procedure in order to verify the following.

**Theorem 5.5.** – The lower bound for the Newton polygon of the polynomial $P(q^{-1}t)$ given by Corollary 4.14 is the Hodge polygon of $PH_{c_{N-r}}(X, \mathbb{C})$.

**Proof.** – In view of (5.4) and the remark following Corollary 4.14, the theorem is equivalent to the assertion that

\[ h^k(H_{c_{N-r}}^r(X, \mathbb{C})) = d(k + r) \]

for all $k$, where $d(k)$ is given by (4.8).

Following [6], we set

\[ e^k(X) = \sum_{l \geq 0} (-1)^l h^k(H_{c_l}^r(X, \mathbb{C})). \]

Consider first the case $r = 1$, $n = 0$, i.e., $X$ is the hypersurface $F_1 = 0$ in $T^n_{\mathbb{C}}$. From section 3, we have

\[ \sum_{k=0}^{\infty} \ell(k\Delta_{S_{s_0}})^t^k = (1 - t)^{-m-1} \sum_{e_{s_0} = 0}^{m} q_{e_{s_0}0 \cdots 0}(t). \]

One of the basic results of [6] is the following.

**Theorem 5.8 ([6, Remark 4.6]).** – If $X$ is the hypersurface $F_1 = 0$ in $T^n_{\mathbb{C}}$, then

\[ (-1)^{m-1} e^k(X) = (-1)^k \left( \binom{m}{k+1} + \text{coefficient of } t^{k+1} \text{ in } \sum_{e_{s_0} = 0}^{m} q_{e_{s_0}0 \cdots 0}(t). \right) \]

This result may be generalized to complete intersections.

**Proposition 5.9.** – If $X$ is the complete intersection $F_1 = \cdots = F_r = 0$ in $(T^m \times \mathbb{A}^n)_C$, then

\[ (-1)^{N-r} e^k(X) = \text{coefficient of } t^{k+r} \text{ in } \sum_{S_{s_0} \subseteq J \subseteq S_{s_{sp}}} (-1)^{|J|} \sum_{e_{1} + \cdots + e_{s_{sp}} = |J|} q_{e_{1} \cdots e_{s_{sp}}}(t). \]

**Remark.** – This proposition implies Theorem 5.5, because it is now straightforward to prove (5.6) using (5.3) and (4.8).

**Proof of Proposition 5.9.** – We follow the outline given in [6, section 6]. Put

\[ G = x_{N+1}F_1 + \cdots + x_{N+r}F_r - 1 \]

and let $Y$ be the hypersurface $G = 0$ in $(T^m \times \mathbb{A}^n \times \mathbb{A}^r)_C$. By [6, section 6.2],

\[ e^k(X) = e^k((T^m \times \mathbb{A}^n)_C) - e^{k+r-1}(Y), \]
and by (5.1),

\[(5.11) \quad e^k((T^m \times \mathbb{A}^n)_C) = (-1)^{N-k} \binom{m}{k-n}. \]

We cannot apply Theorem 5.8 directly to compute \(e^{k+r-1}(Y)\) because \(Y\) is not a hypersurface in a torus but rather a hypersurface in \((T^m \times \mathbb{A}^n \times \mathbb{A}^r)_C\). So for \(I \subseteq S\) with \(S_{to} \subseteq I\) we put \(G_I = \sum_{i \in I_{d_u}} x_i \theta_{I_{sp}}(F_{i-N}) - 1\) and let \(Y_I\) be the hypersurface \(G_I = 0\) in \(T^{|I|}_C\). (Note that \(Y_I = \emptyset\) if \(I_{d_u} = \emptyset\).) Then \(Y = \bigcup_{S_{to} \subseteq I \subseteq S} Y_I\) (a disjoint union) and by [6, Proposition 1.6],

\[(5.12) \quad e^k(Y) = \sum_{S_{to} \subseteq I \subseteq S} e^k(Y_I). \]

We shall apply Theorem 5.8 to each \(Y_I\) to compute \(e^k(Y_I)\), then substitute into (5.12) to find \(e^k(Y)\) and substitute into (5.10) to find \(e^k(X)\).

The convex hull of \(\text{supp}(G_I)\) is \(\Delta_I\). By (3.9) and (3.12),

\[
\sum_{k=0}^{\infty} \ell(k\Delta_I)t^k = (1 - t)^{-|I|-1} \sum_{e_1 + \cdots + e_r \leq |I_{sp}|} q_{e_1 \cdots e_r}(t). \]

So by Theorem 5.8,

\[
(-1)^{|I|-1} e^k(Y_I) = (-1)^k \binom{|I|}{k+1} + \text{coefficient of } t^{k+1} \text{ in } \sum_{e_1 + \cdots + e_r \leq |I_{sp}|} q_{e_1 \cdots e_r}(t). \]

From (5.12),

\[(5.13) \quad e^k(Y) = \sum_{S_{to} \subseteq I \subseteq S} (-1)^{|I|+k-1} \binom{|I|}{k+1} + \text{coefficient of } t^{k+1} \text{ in } \sum_{S_{to} \subseteq I \subseteq S} (-1)^{|I|-1} \sum_{e_1 + \cdots + e_r \leq |I_{sp}|} q_{e_1 \cdots e_r}(t). \]

Using the decomposition \(S = S_{sp} \cup S_{du}\) we have

\[
\sum_{S_{to} \subseteq I \subseteq S} (-1)^{|I|-1} \sum_{e_1 + \cdots + e_r \leq |I_{sp}|} q_{e_1 \cdots e_r}(t)
\]

\[
= \sum_{S_{to} \subseteq I_1 \subseteq S_{sp}} \sum_{I_2 \subseteq S_{du}} (-1)^{|I_1|+|I_2|-1} \sum_{e_1 + \cdots + e_r \leq |I_1|} q_{e_1 \cdots e_r}(t)
\]

\[
= \sum_{S_{to} \subseteq I_1 \subseteq S_{sp}} (-1)^{|I_1|+r-1} \sum_{e_1 + \cdots + e_r \leq |I_1|} q_{e_1 \cdots e_r}(t)
\]

\[
\quad \quad e_{j \geq 1 \text{ for all } j}
\]
by a standard inclusion-exclusion argument. We also have
\[ \sum_{S \subseteq I \subseteq S} (-1)^{|I|+k-1} \binom{|I|}{k+1} = \sum_{j=m}^{N+r} (-1)^{j+k-1} \binom{n+r}{j} \binom{j}{k+1} \]
\[ = (-1)^{N-r+k-1} \binom{m}{k+1-r-n} \]
by a straightforward combinatorial argument. Thus (5.13) becomes
\[ e^k(Y) = (-1)^{N-r+k+1} \binom{m}{k+1-r-n} + \text{coefficient of } t^{k+1} \]
\[ + \sum_{S \subseteq I \subseteq S_{ap}} (-1)^{|I_{1}|+r-1} \sum_{\epsilon_1 + \cdots + \epsilon_r \leq |I_{1}|, \epsilon_j \geq 1 \text{ for all } j} q_{\epsilon_1} \cdots q_{\epsilon_r}(t). \]
Combining this with (5.10) and (5.11) yields Proposition 5.9.

6. Cohomology of projective complete intersections

In this section we change notation slightly. Let \( f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_N] \) be homogeneous polynomials with \( \deg f_i = d_i \) for \( i = 1, \ldots, r \). Let \( V \subseteq \mathbb{P}^N \) be the projective variety \( f_1 = \cdots = f_r = 0 \). Put \( g = x_{N+1}f_1 + \cdots + x_{N+r}f_r \in \mathbb{F}_q[x_0, \ldots, x_{N+r}] \). Let \( V^* \subseteq \mathbb{A}^{N+1} \) be the affine variety \( f_1 = \cdots = f_r = 0 \). As in (2.2),
\[ Z(V^*/\mathbb{F}_q; q^r t) = L(\mathbb{A}^{N+1} \times \mathbb{A}^r, g; t). \]
On the other hand, \( N_\ast(V^*) = (q^r - 1)N_\ast(V) + 1 \), hence
\[ Z(V^*/\mathbb{F}_q; t) = Z(V/\mathbb{F}_q; qt)Z(V/\mathbb{F}_q; t)^{-1}(1 - t)^{-1}. \]
Write
\[ Z(V/\mathbb{F}_q; t) = \frac{P(t)^{-1}N-r^{-1}}{(1-t)(1-qt) \cdots (1-q^{N-r}t)}. \]
Then (6.1) and (6.2) imply that
\[ L(\mathbb{A}^{N+1} \times \mathbb{A}^r, g; t)^{(-1)^{N-r}} = \frac{P(q^r t)(1 - q^{N+1}t)^{(-1)^{N-r}-1}}{P(q^r t)}. \]
Let \( S = \{0, 1, \ldots, N + r\}, S_{ap} = \{0, \ldots, N\}, S_{du} = \{N + 1, \ldots, N + r\} \). By (2.13),
\[ L(\mathbb{A}^{N+1} \times \mathbb{A}^r, g; t)^{(-1)^{N-r}} = \prod_{l=0}^{N+1+r} \det(I - t\alpha_l | H_l(K.(S,S)))^{(-1)^{\ell}}. \]
For purposes of induction, we shall have occasion to consider cases where \( S_{ap} = \emptyset \) or \( S_{du} = \emptyset \). Although such cases have no geometric interpretation in terms of complete
intersections, the $L$-function $L(A^{N+1} \times A^r, g; t)$ is still defined. Recall [1] that we say $f_i$ is commode if it contains each of the monomials $x_0^{a_i}, \ldots, x_N^{a_i}$ with nonzero coefficient.

**Theorem 6.5.** Suppose that $r \leq N$, $g$ is nondegenerate, and $f_i$ is commode for $i = 1, \ldots, r$. Then

(i) $\dim_{\mathbb{Q}} H_i(K.(S,S)) < \infty$ for all $l$. 

(ii) $H_i(K.(S,S)) = 0$ for $l \neq 0, 1, N + 1 - r$.

(iii) Suppose $N - r > 0$. Then $\dim_{\mathbb{Q}} H_{N+1-r}(K.(S,S)) = 1$ and Frobenius acts on $H_{N+1-r}(K.(S,S))$ as multiplication by $q^{N+1}$. The space $H_1(K.(S,S))$ with Frobenius $\tilde{\alpha}_1$ is isomorphic (as Frobenius module) to $H_0(K.(S,S))$ with Frobenius $q \tilde{\alpha}_0$.

(iv) Suppose $N - r = 0$. Then $H_1(K.(S,S))$ with Frobenius $\tilde{\alpha}_1$ is isomorphic to a direct sum $H_0(K.(S,S)) \oplus H'$, $\dim_{\mathbb{Q}} H' = 1$, with Frobenius acting on $H_0(K.(S,S))$ by $q \tilde{\alpha}_0$ and acting on $H'$ by multiplication by $q^{N+1}$.

(v) The Frobenius endomorphism $\tilde{\alpha}_0$ is invertible on $H_0(K.(S,S))$ and
\[
\dim_{\mathbb{Q}} H_0(K.(S,S)) = (-1)^{N-r+1}(N-r+1) + (-1)^{N+1} \sum_{l=r+1}^{N+1} (-1)^l \binom{N+1}{l} \sum_{i_1 + \cdots + i_r = l-1, i_j \geq 1 \text{ for all } j} d_1^{i_1} \cdots d_r^{i_r}.
\]

Comparing the right-hand sides of (6.3) and (6.4), we have immediately the following.

**Corollary 6.6.** Suppose that $r \leq N$, $g$ is nondegenerate, and $f_i$ is commode for $i = 1, \ldots, r$. Then
\[
P(q^t) = \det(I - t\tilde{\alpha}_0 | H_0(K.(S,S))),
\]
in particular, $P(t)$ is a polynomial. Its degree is given by Theorem 6.5 (v).

**Remark.** If $f_i = 0$, $i = 1, \ldots, r$, are smooth hypersurfaces in $P^N$ in general position, then one can always make a coordinate change on $P^N$ (defined over a finite extension of $F_q$) so that $g$ is nondegenerate and all $f_i$'s are commode. (See [3])

The proof of Theorem 6.5 will require several steps. First observe that the homogeneity condition on the $f_i$'s implies that every monomial $x_0^{a_0} \cdots x_N^{a_N} x_0^{b_1} \cdots x_N^{b_r}$ appearing in $g$ satisfies
\[
a_0 + \cdots + a_N = d_1 b_1 + \cdots + d_r b_r.
\]
This in turn implies that every $u \in M = \mathbb{Z}^{N+1+r} \cap C(\Delta)$ satisfies the same condition. It follows that
\[
D_0 + \cdots + D_N - d_1 D_{N+1} - \cdots - d_r D_{N+r} = 0
\]
as operator on $B$ or $L(b)$. More generally, for $I \subseteq S$ we have
\[
\sum_{i \in I_{ap}} D_{I,i} - \sum_{i \in I_{du}} d_{i-N} D_{I,i} = 0
\]
as operator on $B_1$ or $L_1(b)$. This relation implies that for $I' \subseteq I$, the map from $K_0(I, I') = B_1^{I'} e_0$ to $K_1(I, I') = \bigoplus_{i \in I} B_1^{I \setminus \{i\}} e_{(i)}$ defined by

\begin{equation}
(6.9) \quad \xi e_0 \mapsto \sum_{i \in I_{sp}} \xi e_{(i)} + \sum_{i \in I_{du}} (-d_{i-N}) \xi e_{(i)}
\end{equation}

has image lying in the set of 1-cycles. Suppose $\xi e_0$ is a 0-boundary, i.e., there exist $\xi_i \in B_1^{I \setminus \{i\}}$ such that $\xi = \sum D_{I,i}(\xi_i)$. Put

\[ e_i = \begin{cases} 1 & \text{if } i \in I_{sp}, \\ -d_{i-N} & \text{if } i \in I_{du}, \end{cases} \]

and define $\eta_{ij} = e_i \xi_j - e_j \xi_i \in B_1^{I \setminus \{i,j\}}$ for $i, j \in I$. Then $\{\eta_{ij}\}_{i,j \in I}$ is a skew-symmetric set and $\sum_{i \in J} D_{I,j}(\eta_{ij}) = e_i \xi_j$, i.e., the image of $\xi e_0$ under the map (6.9) is a 1-boundary. Thus (6.9) induces a map

\[ \phi_{I'}^f : H_0(K.(I, I')) \to H_1(K.(I, I')). \]

It is clear that $\phi_{I'}^f \circ q\bar{\alpha}_0 = \bar{\alpha}_1 \circ \phi_{I'}^f$, i.e., $\phi_{I'}^f$ respects the Frobenius structure.

**Proposition 6.10.** Suppose that $g$ is nondegenerate and $f_i$ is commode for $i = 1, \ldots, r$. Then for $I' \subseteq I \subseteq S$ with $I_{sp} \neq I_{sp}'$ and $I_{du} \neq I_{du}'$, we have $H_l(K.(I, I')) = 0$ for $l > 1$,

\begin{equation}
(6.11) \quad \dim_{\Omega_0} H_0(K.(I, I')) = \sum_{I_{sp} \subseteq J \subseteq I_{sp}} (-1)^{|I_{sp}'|} \sum_{i \in I_{du}} \left( \prod_{j \geq 1 \text{ for } i \in I_{du}'} d_i^j \right),
\end{equation}

$\bar{\alpha}_0$ is invertible on $H_0(K.(I, I'))$, and $\phi_{I'}^f$ is an isomorphism.

**Proof.** The proof is by induction on $|I'|$. When $I' = \emptyset$, Proposition 8.3 of the appendix says that $H_l(K.(I, I')) = 0$ for $l > 1$ and $\phi_{I'}^f$ is an isomorphism. The invertibility of $\bar{\alpha}_0$ is given by [1, Theorem 3.13] and the formula for $\dim_{\Omega_0} H_0(K.(I, I'))$ follows from [1, Theorem 2.9] and a calculation as in the derivation of (2.18).

Suppose $|I'| > 0$ and let $i \in I'$. Then there is a short exact sequence of complexes

\[ 0 \to K.(I, I') \to K.(I, I' \setminus \{i\}) \to K.(I \setminus \{i\}, I' \setminus \{i\}) \to 0. \]

The vanishing of $H_l(K.(I, I'))$ for $l > 1$ follows immediately from the associated long exact homology sequence and application of the induction hypothesis to $K.(I, I' \setminus \{i\})$ and $K.(I \setminus \{i\}, I' \setminus \{i\})$. The maps $\phi_{I' \setminus \{i\}}^f$ and $\phi_{I' \setminus \{i\}}^f$ are isomorphisms by the induction hypothesis, so Proposition 8.6 of the appendix implies that $\phi_{I'}^f$ is an isomorphism and that the sequence

\[ 0 \to H_0(K.(I, I')) \to H_0(K.(I, I' \setminus \{i\})) \to H_0(K.(I \setminus \{i\}, I' \setminus \{i\})) \to 0 \]

is exact.

**Proof (Continued).** The short exact sequence of complexes

\[ 0 \to K.(I, I') \to K.(I, I' \setminus \{i\}) \to K.(I \setminus \{i\}, I' \setminus \{i\}) \to 0. \]

The vanishing of $H_l(K.(I, I'))$ for $l > 1$ follows immediately from the associated long exact homology sequence and application of the induction hypothesis to $K.(I, I' \setminus \{i\})$ and $K.(I \setminus \{i\}, I' \setminus \{i\})$. The maps $\phi_{I' \setminus \{i\}}^f$ and $\phi_{I' \setminus \{i\}}^f$ are isomorphisms by the induction hypothesis, so Proposition 8.6 of the appendix implies that $\phi_{I'}^f$ is an isomorphism and that the sequence

\[ 0 \to H_0(K.(I, I')) \to H_0(K.(I, I' \setminus \{i\})) \to H_0(K.(I \setminus \{i\}, I' \setminus \{i\})) \to 0 \]

is exact.
is exact. By induction, \( \hat{\alpha}_0 \) on \( H_0(K.(I, I') \setminus \{i\}) \) and on \( H_0(K.(I \setminus \{i\}, I' \setminus \{i\})) \) is an isomorphism, hence \( \hat{\alpha}_0 \) on \( H_0(K.(I, I')) \) is also an isomorphism. We also get

\[
\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_0(K.(I, I' \setminus \{i\})) - \dim_{\Omega_0} H_0(K.(I \setminus \{i\}, I' \setminus \{i\})),
\]

so the desired formula for \( \dim_{\Omega_0} H_0(K.(I, I')) \) follows from the induction hypothesis.

**Remark.** – Note that in the case \( |I'_d| \geq |I_{sp}| \), the proposition implies all homology vanishes since the index set for the inner sum on the right-hand side of (6.11) is empty.

**Proposition 6.12.** – Suppose that \( g \) is nondegenerate and \( f_i \) is commode for \( i = 1, \ldots, r \). Let \( I' \subseteq I \subseteq S \) with \( I'_{sp} = I_{sp} \) but \( I'_{du} \neq I_{du} \). For notational convenience, set \( |I'_{du}| - |I_{sp}| = k \) and \( |I_{du}| - |I_{sp}| = \hat{k} \). Then \( H_l(K.(I, I')) = 0 \) if

\[
l \notin \{0, 1\} \cup \{k, k + 1, \ldots, \hat{k}\}.
\]

If in addition \( k \geq 0 \), then

\[
(6.13) \quad H_l(K.(I, I')) = 0 \quad \text{if} \ l \notin \{k, k + 1, \ldots, \hat{k}\},
\]

\[
(6.14) \quad H_l(K.(I, I')) \simeq (\Omega_0)^{(l-k)} \quad \text{if} \ l \in \{k, k + 1, \ldots, \hat{k}\}.
\]

**Proof.** – The proof is by induction on \( |I_{sp}| \). When \( I_{sp} = \emptyset \), one has

\[
B^l_{I' \setminus A} = \begin{cases} 
\Omega_0 & \text{if} \ I' \subseteq A, \\
0 & \text{if} \ I' \not\subseteq A,
\end{cases}
\]

hence

\[
K_l(I, I') = \bigoplus_{I' \subseteq A \subseteq I, |A| = l} \Omega_0 e_A
\]

and all boundary maps of the complex \( K.(I, I') \) are zero. Thus

\[
(6.15) \quad H_l(K.(I, I')) \simeq (\Omega_0)^{(l-k)}
\]

which implies the proposition when \( |I_{sp}| = 0 \).

If \( |I_{sp}| > 0 \), let \( i \in I_{sp} \) and consider the short exact sequence

\[
0 \to K.(I, I') \to K.(I, I' \setminus \{i\}) \to K.(I \setminus \{i\}, I' \setminus \{i\}) \to 0.
\]

The complex \( K.(I, I' \setminus \{i\}) \) satisfies the hypotheses of Proposition 6.10, so \( H_0(K.(I, I' \setminus \{i\})) = 0 \) if \( l \neq 0, 1 \). Thus the long exact homology sequence associated to (6.16) gives isomorphisms

\[
(6.17) \quad H_{l+1}(K.(I \setminus \{i\}, I' \setminus \{i\})) \simeq H_l(K.(I, I'))
\]

for \( l > 1 \). The assertion for \( H_l(K.(I, I')) \) now follows by applying the induction hypothesis to \( H_{l+1}(K.(I \setminus \{i\}, I' \setminus \{i\})) \). Suppose in addition that \( |I'_{du}| \geq |I_{sp}| \). Then
Hi{K.{I, I' \{%\})) = 0 for all; by equation (6.11), so (6.17) holds for all; \( \phi^I_J \) is an isomorphism. To fix ideas, suppose \( I_{du} = I_{du} \cup \{N + r\} \) and let \( \Xi = \sum_{i \in J} \xi_i e_{\{i\}} \in K_I(I, I') \) be a 1-cycle. Then \( \xi_i \in B_I^{I \setminus \{i\}} \), in particular, \( \xi_{N+r} \in B_I^{I'} \). If the homology class represented by this 1-cycle lies in the image of \( \phi^I_J \), then there exists \( \xi \in B_I^{I'} \) such that \( \Xi \) is homologous to the 1-cycle \( \sum_{i \in I_{sp}} \xi e_{\{i\}} + \sum_{i \in I_{du}} (-d_i - N) \xi e_{\{i\}} \). Thus there is a skew-symmetric set \( \eta_{ij} \in B_I^{I \setminus \{i,j\}} \) such that

\[
\sum_{j \in J} D_j(\eta_{ij}) = \xi_i - \begin{cases} 
\xi & \text{if } i \in I_{sp}, \\
(-d_i - N) \xi & \text{if } i \in I_{du}.
\end{cases}
\]

Taking \( i = N + r \) we get \( \eta_{N+r,j} \in B_I^{I \setminus \{j\}} \) such that

\[
\xi_{N+r} + d_r \xi = \sum_{j \in J} D_j(\eta_{N+r,j}).
\]

But this equation says that \( \xi_{N+r} \) and \( -d_r \xi \) are homologous 0-cycles in \( H_0(K.(I, I')) \). In other words, the map defined by sending \( \Xi \) to \( (-1/d_r) \xi_{N+r} \) is, by the argument above, defined on homology classes in the image of \( \phi^I_J \) and is a left inverse to \( \phi^I_J \):

\[
\phi^I_J \circ \phi^I_J = \text{identity} : H_0(K.(I, I')) \to H_0(K.(I, I')).
\]

Thus \( \phi^I_J \) is an isomorphism whenever

\[
\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_1(K.(I, I')) < \infty.
\]

**Corollary 6.18.** - Suppose that \( g \) is nondegenerate and \( f_i \) is commode for \( i = 1, \ldots, r \). Let \( I' \subseteq I \subseteq S \) with \( I_{sp} = I_{sp} \) and \( |I_{du}| = |I'_{du}| + 1 = |I_{sp}| + 1 \). Then \( H_l(K.(I, I')) = 0 \) for \( l > 1 \), \( \phi^I_J \) is an isomorphism, and \( \dim_{\Omega_0} H_0(K.(I, I')) = 1 \).

**Proof.** - It follows from (6.13) that \( H_l(K.(I, I')) = 0 \) for \( l > 1 \). To prove the remaining assertions, we see by the above discussion that it suffices to check that

\[
\dim_{\Omega_0} H_0(K.(I, I')) = \dim_{\Omega_0} H_1(K.(I, I')) = 1.
\]

But this follows from (6.14).

We now analyze the case \( I = S \), \( I' = S \setminus \{N + r\} \) in more detail.

**Corollary 6.19.** - Suppose that \( 1 \leq r \leq N + 2 \), \( g \) is nondegenerate, and \( f_i \) is commode for \( i = 1, \ldots, r \). Then \( H_l(K.(S, S \setminus \{N + r\})) = 0 \) if \( l > 1 \) and \( \phi^S_S \) is an isomorphism. Furthermore,

\[
(6.20) \quad \dim_{\Omega_0} H_0(K.(S, S \setminus \{N + r\})) = (-1)^{N+r} + (-1)^{N+1} \sum_{l = r}^{N+1} \binom{N+1}{l} \sum_{i_1 + \cdots + i_r = l-1} \left( d_{i_1} \cdots d_{i_r} \right).
\]
Proof. – For notational convenience, we put \( S' = S \setminus \{N + r\} \). The vanishing of \( H_l(K(S, S')) \) for \( l > 1 \) is an immediate consequence of Proposition 6.12 and the hypothesis that \( r \leq N + 2 \). We prove that \( \phi_S^{S'} \) is an isomorphism by induction on \( N + 2 - r \). Suppose that \( r < N + 2 \) and consider the short exact sequence

\[
0 \to K.(S, S') \to K.(S, S' \setminus \{N\}) \to K.(S \setminus \{N\}, S' \setminus \{N\}) \to 0.
\]

By Proposition 6.10, \( \phi_{S \setminus \{N\}}^{S'} \) is an isomorphism, so by Proposition 8.6 of the appendix, \( \phi_S^{S'} \) will be an isomorphism provided \( \phi_{S \setminus \{N\}}^{S'} \) is. Thus by induction it suffices to establish the result when \( r = N + 2 \). But \( I = S, I' = S', r = N + 2 \), is exactly the case covered by Corollary 6.18. Note that Corollary 6.18 also establishes (6.20) when \( r = N + 2 \). We also prove (6.20) by induction on \( N + 2 - r \). Since \( \phi_{S \setminus \{N\}}^{S'} \) is an isomorphism, Proposition 8.6 of the appendix implies that the connecting homomorphism in the exact homology sequence associated to (6.21) is zero, hence

\[
\dim_{\Omega_0} H_0(K.(S, S')) = \dim_{\Omega_0} H_0(K.(S, S' \setminus \{N\})) = \dim_{\Omega_0} H_0(K.(S \setminus \{N\}, S' \setminus \{N\})).
\]

The value of the first term on the right-hand side is given by (6.11). The desired expression for the left-hand side then follows by applying the induction hypothesis to the second term on the right-hand side.

Proof of Theorem 6.5. – Note that Theorem 6.5 (i) is a consequence of Theorem 6.5 (ii)-(v), so we need only prove parts (ii)-(v). We proceed by induction on \( r \), proving additionally that when \( N - r > 0 \), \( \phi_S^S \) is an isomorphism. When \( r = 0 \), i.e., \( S = \{0, 1, \ldots, N\} \), it is straightforward to check that \( K(S, S) \) is the complex with \( K_{N+1}(S, S) = \Omega_0 \) and \( K_l(S, S) = 0 \) for \( l \neq N + 1 \), where all boundary maps are zero and Frobenius acts on \( K_{N+1}(S, S) \) as multiplication by \( q^{N+1} \). This establishes Theorem 6.5 in that case (when one interprets appropriately the sum appearing in Theorem 6.5 (v)).

Suppose \( r \geq 1 \). We again put \( S' = S \setminus \{N + r\} \). There is a short exact sequence of complexes

\[
0 \to K.(S, S) \to K.(S, S') \to K.(S', S') \to 0.
\]

By Corollary 6.19, \( H_l(K.(S, S')) = 0 \) if \( l \neq 0, 1 \), and by the induction hypothesis, \( H_l(K.(S', S')) = 0 \) if \( l \neq 0, 1, N+2-r \). It follows from the associated long exact homology sequence that \( H_l(K.(S, S)) = 0 \) if \( l \neq 0, 1, N + 1 - r \), establishing Theorem 6.5 (ii).

If \( N - r > 0 \), then the long exact homology sequence gives an isomorphism

\[
H_{N+2-r}(K.(S', S')) \cong H_{N+1-r}(K.(S, S)),
\]

hence, by the induction hypothesis, \( \dim_{\Omega_0} H_{N+1-r}(K.(S, S)) = 1 \) and Frobenius acts as multiplication by \( q^{N+1} \). When \( N - r > 0 \), we also have by the induction hypothesis that \( H_2(K.(S', S')) = 0 \) and \( \phi_S^{S'} \) is an isomorphism. By Corollary 6.19, \( \phi_S^S \) is also an isomorphism, hence \( \phi_S^{S'} \) is an isomorphism by Proposition 8.6 of the appendix. This establishes Theorem 6.5 (iii).

Now suppose \( N - r = 0 \). By the induction hypothesis, \( \phi_S^{S'} \) is an isomorphism, and by Corollary 6.19, \( \phi_S^S \) is an isomorphism. Hence by Proposition 8.6 of the appendix, the
connecting homomorphism $H_1(K.(S', S')) \to H_0(K.(S, S))$ of the long exact homology sequence associated to (6.22) is zero. Thus the nontrivial terms of the long exact homology sequence give us a commutative diagram with exact rows (For typographical convenience, in this diagram we denote $H_i(K.(I, I'))$ by $H_i(K_{I, I'}).$):

$$
0 \to H_2(K_{S')^2} \xrightarrow{\tau} H_1(K_{S')^2} \to H_1(K_{S, S}) \to H_0(K_{S')^2} \to H_0(K_{S, S}) \to 0.
$$

(6.23)


By the induction hypothesis, $\dim_{\Omega_0} H_2(K.(S', S')) = 1$ and Frobenius acts on it as multiplication by $q^{N+1}$. Thus $\tau(H_2(K.(S', S'))) is a one-dimensional subspace of $H_1(K.(S, S))$ with Frobenius acting as multiplication by $q^{N+1}$. A diagram chase, using the above observation that the two right-most vertical arrows in (6.23) are isomorphisms, now shows that

$$
H_1(K.(S, S)) \cong \tau(H_2(K.(S', S'))) \oplus \phi_S(H_0(K.(S, S))),
$$

which establishes Theorem 6.5(iv).

What we have done so far implies that

$$
\dim_{\Omega_0} H_0(K.(S, S)) = \dim_{\Omega_0} H_0(K.(S, S')) - \dim_{\Omega_0} H_0(K.(S, S')).
$$

Evaluating the first term on the right-hand side by Corollary 6.19 and the second term on the right-hand side by the induction hypothesis establishes Theorem 6.5 (v).

### 7. Newton and Hodge polygons in the projective case

The calculation of a lower bound for the Newton polygon of $P(t)$ proceeds exactly as in sections 3 and 4, therefore we just summarize results while pointing out some differences in the projective case.

Let the ring $R$ be defined as in section 3 and let $\bar{K} = K.(R, \{g_i\}_{i=0}^{N+r})$ be the Koszul complex on $R$ defined by $g_0, \ldots, g_{N+r}$. As in section 3, one can define the related complexes $\bar{K}.(I), \bar{K}.(I, I')$ for $I' \subseteq I \subseteq S$. All these complexes are graded as in section 3.

Let $I \subseteq S$ with $I_{sp} \neq \emptyset$. The homogeneity condition on $f_1, \ldots, f_r$ implies that the polytopes $\Delta_i^{I_1}, \ldots, \Delta_i^{I_r}$ are $(|I_{sp}|-1)$-dimensional (instead of $|I_{sp}|$-dimensional). Thus $\ell(k_1 \Delta_i^{I_1} + \cdots + k_r \Delta_i^{I_r})$ is a rational polynomial of degree $\leq |I_{sp}|-1$ in $k_1, \ldots, k_r$, say,

$$
\ell(k_1 \Delta_i^{I_1} + \cdots + k_r \Delta_i^{I_r}) = \sum_{e_1 + \cdots + e_r \leq |I_{sp}|-1} a_{e_1, \ldots, e_r} k_1^{e_1} \cdots k_r^{e_r} \in \mathbb{Q}[k_1, \ldots, k_r].
$$

The hypothesis that $g$ is commode implies that for each $i$, $\Delta_i^{I_v}$ is the $(|I_{sp}|-1)$-simplex in $\mathbb{R}^{|I_{sp}|}$ with vertices $(d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_r)$, so, in fact,

$$
\ell(k_1 \Delta_i^{I_1} + \cdots + k_r \Delta_i^{I_r}) = \left(\frac{k_1 d_1 + \cdots + k_r d_r + |I_{sp}|-1}{|I_{sp}|-1}\right).
$$
This formula shows that the lattice point function on the left-hand side depends only on \(|I_{sp}|\), not on \(I_{sp}\) itself, thus the same is true for the formulas that follow.

The polynomials \(p_{e_1 \ldots e_r}(t_1, \ldots, t_r)\) can be defined as before and we have

\[
\sum_{k=0}^{\infty} (\dim_{F_q} R_{I}^{(k)}) t^k = \sum_{e_1 + \cdots + e_r \leq |I_{sp}| - 1} \frac{p_{e_1 \ldots e_r}(t_1, \ldots, t)}{(1 - t)^{|I_{da}| + \sum_{j=1}^{r} e_j}}.
\]

Define

\[
q_{e_1 \ldots e_r}(t) = p_{e_1 \ldots e_r}(t_1, \ldots, t)(1-t)^{|I_{sp}| - 1 - \sum_{j=1}^{r} e_j} \in \mathbb{Q}[t].
\]

Then

\[
(7.1) \sum_{k=0}^{\infty} (\dim_{F_q} R_{I}^{(k)}) t^k = (1-t)^{-|I|+1} \sum_{e_1 + \cdots + e_r \leq |I_{sp}| - 1} q_{e_1 \ldots e_r}(t).
\]

Using induction on \(|I'|\), as in the proof of Proposition 6.10, gives the following. (The case \(|I'| = 0\) is Proposition 8.4 of the appendix.)

**Proposition 7.2.** Suppose that \(g\) is nondegenerate and \(f_i\) is commode for \(i = 1, \ldots, r\). Then for \(I' \subseteq I \subseteq S\) with \(I'_{sp} \neq I_{sp}\) and \(I'_{du} \neq I_{du}\) we have \(H_l(\bar{K}.(I, I')) = 0\) for \(l > 1\),

\[
\dim_{F_q} H_1(\bar{K}.(I, I'))^{(k)} = \dim_{F_q} H_0(\bar{K}.(I, I'))^{(k-1)},
\]

and \(\dim_{F_q} H_0(\bar{K}.(I, I'))^{(k)}\) is the coefficient of \(t^k\) in

\[
\sum_{I_{sp} \setminus I'_{sp} \subseteq J \subseteq I_{sp}} (-1)^{|I_{sp}| - |J|} \sum_{e_1 + \cdots + e_r = |J| - 1} q_{e_1 \ldots e_r}(t).
\]

Arguing as in the proof of Proposition 6.12 gives the following.

**Proposition 7.3.** Suppose that \(g\) is nondegenerate and \(f_i\) is commode for \(i = 1, \ldots, r\). Let \(I' \subseteq I \subseteq S\) with \(I'_{sp} = I_{sp}\) but \(I'_{du} \neq I_{du}\). Then \(H_l(\bar{K}.(I, I')) = 0\) if

\[
l \notin \{0, 1\} \cup \{|I'_{du}| - |I_{sp}|, |I'_{du}| - |I_{sp}| + 1, \ldots, |I_{du}| - |I_{sp}|\}.
\]

If in addition \(|I'_{du}| \geq |I_{sp}|\), then \(H_l(\bar{K}.(I, I')) = 0\) for \(l \notin \{|I'_{du}| - |I_{sp}|, \ldots, |I_{du}| - |I_{sp}|\}\) and

\[
\dim_{F_q} H_l(\bar{K}.(I, I'))^{(k)} = \begin{cases} (|I_{du}| - |I'_{du}|) & \text{if } l \in \{|I'_{du}| - |I_{sp}|, \ldots, |I_{du}| - |I_{sp}|\} \text{ and } k = l + |I_{sp}|, \\ 0 & \text{otherwise.} \end{cases}
\]

We record a special case of this result, an analogue of Corollary 6.18.
Corollary 7.4. Suppose that $g$ is nondegenerate and $f_i$ is commode for $i = 1, \ldots, r$. Let $I' \subseteq I \subseteq S$ with $I'_{sp} = I_{sp}$ and $|I|_{sa} = |I'|_{sa} + 1 = |I_{sp}| + 1$. Then $H_l(\bar{K}(I, I')) = 0$ if $l > 1$, $\dim_{F_q} H_1(\bar{K}(I, I'))^{(k)} = \dim_{F_q} H_0(\bar{K}(I, I'))^{(k-1)}$, and
\[
\dim_{F_q} H_0(\bar{K}(I, I'))^{(k)} = \begin{cases} 1 & \text{if } k = |I_{sp}|, \\ 0 & \text{if } k \neq |I_{sp}|.
\end{cases}
\]

Following the proof of Corollary 6.19, we arrive at the following.

Corollary 7.5. Suppose that $1 \leq r \leq N + 2$, $g$ is nondegenerate, and $f_i$ is commode for $i = 1, \ldots, r$. Then $H_l(\bar{K}(S, S \{N + r\})) = 0$ if $l > 1$,
\[
\dim_{F_q} H_1(\bar{K}(S, S \{N + r\}))^{(k)} = \dim_{F_q} H_0(\bar{K}(S, S \{N + r\}))^{(k-1)}.
\]

We finally arrive at the analogue of Theorem 6.5.

Theorem 7.6. Suppose that $1 \leq r \leq N$, $g$ is nondegenerate, and $f_i$ is commode for $i = 1, \ldots, r$. Then

(i) $\dim_{F_q} H_1(\bar{K}(S, S)) = 0$ for $l \neq 0, 1, N + 1 - r$.

(ii) $\dim_{F_q} H_0(\bar{K}(S, S))^{(k)}$ is the coefficient of $t^k$ in
\[
(7.7) \quad (-1)^{N-r+1}(t^r + t^{r+1} + \cdots + t^N) + \sum_{J \subseteq S_{sp}} (-1)^{N+1-|J|} \sum_{e_1 + \cdots + e_r \leq |J|-1 \atop e_j \geq 1 \text{ for all } j} q^{e_1 \cdots e_r}(t).
\]

(iii) If $N - r > 0$, then $\dim_{F_q} H_1(\bar{K}(S, S))^{(k)} = \dim_{F_q} H_0(\bar{K}(S, S))^{(k-1)}$ and
\[
\dim_{F_q} H_{N+1-r}(\bar{K}(S, S))^{(k)} = \begin{cases} 1 & \text{if } k = N + 1, \\ 0 & \text{otherwise}.
\end{cases}
\]

(iv) If $N - r = 0$, then
\[
\dim_{F_q} H_1(\bar{K}(S, S))^{(k)} = \dim_{F_q} H_0(\bar{K}(S, S))^{(k-1)} + \begin{cases} 1 & \text{if } k = N + 1, \\ 0 & \text{otherwise}.
\end{cases}
\]

The generalization of the rest of sections 3 and 4 is straightforward. Let $d(k)$ be the coefficient of $t^k$ in expression (7.7).

Theorem 7.8. Suppose that $1 \leq r \leq N$, $g$ is nondegenerate, and $f_i$ is commode for $i = 1, \ldots, r$. Then the Newton polygon of the polynomial $P(t) = \det \left( I - t \tilde{\alpha}_0 \mid B_S^{N+r} / \sum_{i=0}^{N+r} D_i P_S^{N}(i) \right)$ lies on or above the Newton polygon of the polynomial $\prod_{k=r}^N (1 - q^k t)^{d(k)}$ and their endpoints coincide.
Remark. – The corresponding factor of \( Z(V/F_q; t) \) is \( P(q^{-r}t) \), which has lower bound determined by the polynomial \( \prod_{k=0}^{N-r} (1 - q^k t)^{d(k+r)} \).

We now describe the Hodge polygon in the projective case. Let \( F_1, \ldots, F_r \in \mathbb{C}[x_0, \ldots, x_N] \) be the generic homogeneous polynomials of degrees \( d_1, \ldots, d_r \), respectively, and let \( X \subseteq \mathbb{P}^N_{\mathbb{C}} \) be the smooth complete intersection \( F_1 = \cdots = F_r = 0 \). For \( J \subseteq S_{sp} \), let \( U_J \subseteq \mathbb{P}^N_{\mathbb{C}} \) be the subset consisting of those points whose homogeneous coordinates \( (x_0, \ldots, x_N) \) satisfy \( x_j \neq 0 \) if and only if \( j \in J \). Thus \( U_J \simeq T^{\mid J\mid-1}_{\mathbb{C}} \) and there is a decomposition of \( \mathbb{P}^N_{\mathbb{C}} \) as \( \mathbb{P}^N_{\mathbb{C}} = \bigcup_{\emptyset \neq J \subseteq S_{sp}} U_J \). Putting \( X_J = X \cap U_J \), we get a decomposition

\[
X = \bigcup_{\emptyset \neq J \subseteq S_{sp}} X_J.
\]

(7.9)

For each \( J \subseteq S_{sp} \), \( J \neq \emptyset \), let \( \hat{F}_i^J \) be the dehomogenization of \( F_i \) with respect to any \( x_j, j \in J \), and let \( \hat{\Delta}_i^J \subseteq \mathbb{R}^{\mid J\mid-1} \) be the polytope and \( \hat{q}_{e_1,\ldots,e_r}^J(t) \) be the polynomials associated to \( \hat{F}_i^J \) as in section 3. The genericity of \( F_i \) implies that these are independent of the choice of \( x_j \). In fact, \( \hat{\Delta}_i^J \) is the \((\mid J\mid - 1)\)-simplex with vertices at the origin and \((d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_i)\). By Proposition 5.9, \( (-1)^{\mid J\mid-1} e^k(X_J) \) is the coefficient of \( t^k \) in

\[
\sum_{\mathclap{e_1 + \cdots + e_r \leq \mid J\mid - 1 \atop e_j \geq 1 \text{ for all } j}} q_{e_1,\ldots,e_r}^J(t).
\]

It is easily checked that \( q_{e_1,\ldots,e_r}^J(t) = q_{e_1,\ldots,e_r}^J(t) \) for all \( J \) and all \( e_1, \ldots, e_r \). Furthermore, the decomposition (7.9) implies that \( e^k(X) = \sum_{\emptyset \neq J \subseteq S_{sp}} e^k(X_J) \), hence \( (-1)^{N-r} e^k(X) \) is the coefficient of \( t^k \) in

\[
\sum_{\emptyset \neq J \subseteq S_{sp}} (-1)^{N+1-\mid J\mid} \sum_{\mathclap{e_1 + \cdots + e_r \leq \mid J\mid - 1 \atop e_j \geq 1 \text{ for all } j}} q_{e_1,\ldots,e_r}^J(t).
\]

Using well-known facts about the cohomology of smooth complete intersections in \( \mathbb{P}^N_{\mathbb{C}} \), we conclude that

\[
h^k(\mathbb{P}^N_{\mathbb{C}}(X, \mathbb{C})) = d(k + r).
\]

**Corollary 7.10.** – The lower bound for the Newton polygon of the polynomial \( P(q^{-r}t) \) given by Theorem 7.8 is the Hodge polygon of \( \mathbb{P}^N_{\mathbb{C}}(X, \mathbb{C}) \).
8. Appendix

We begin by proving an elementary result about Koszul complexes for which we do not know a reference. Let $A$ be a commutative ring, $M$ an $A$-module, and $\sigma_1, \ldots, \sigma_n$ commuting endomorphisms of $M$ as $A$-module. We compare the homology of the Koszul complexes $K(M, \{\sigma_i\}_{i=1}^n)$ and $K(M, \{\sigma_i^{-1}\}_{i=1}^{n-1})$ under the assumption that there is a relation

\begin{equation}
\sigma_n = \sum_{i=1}^{n-1} a_i \sigma_i, \quad a_i \in A.
\end{equation}

Clearly this implies that $H_0(K(M, \{\sigma_i\}_{i=1}^n)) = H_0(K(M, \{\sigma_i^{-1}\}_{i=1}^{n-1})$. It also implies that the map

\begin{equation}
m e_T \mapsto m e_{\{n\}} + \sum_{i=1}^{n-1} (-a_i) m e_{\{i\}}
\end{equation}

induces a homomorphism $\phi : H_0(K(M, \{\sigma_i\}_{i=1}^n)) \to H_1(K(M, \{\sigma_i^{-1}\}_{i=1}^{n-1}))$.

**Proposition 8.2.** Suppose $H_i(K(M, \{\sigma_i\}_{i=1}^n)) = 0$ for $l > 0$. Then $H_i(K(M, \{\sigma_i^{-1}\}_{i=1}^{n-1})) = 0$ for $l > 1$ and $\phi$ is an isomorphism.

**Proof.** Let $K^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1})$ be the complex $K(M, \{\sigma_i\}_{i=1}^{n-1})$ shifted one position to the left, i.e.,

\begin{equation}
K^{(1)}_i(M, \{\sigma_i\}_{i=1}^{n-1}) = K_{i-1}(M, \{\sigma_i\}_{i=1}^{n-1}).
\end{equation}

There is a short exact sequence of complexes

\begin{equation}
0 \to K(M, \{\sigma_i\}_{i=1}^n) \to K(M, \{\sigma_i\}_{i=1}^n) \to K^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1}) \to 0,
\end{equation}

where the map $K_l(M, \{\sigma_i\}_{i=1}^{n-1}) \to K_l(M, \{\sigma_i\}_{i=1}^n)$ is the natural inclusion defined by $m e_T \mapsto m e_T$ for $T \subseteq \{1, \ldots, n-1\}$, $|T| = l$, and the map

\begin{equation}
K_l(M, \{\sigma_i\}_{i=1}^n) \to K^{(1)}_l(M, \{\sigma_i\}_{i=1}^{n-1}) = K_{l-1}(M, \{\sigma_i\}_{i=1}^{n-1})
\end{equation}

is defined by

\begin{equation}
m e_T \mapsto \begin{cases} 0 & \text{if } n \not\in T, \\ m e_{T \setminus \{n\}} & \text{if } n \in T,
\end{cases}
\end{equation}

for $T \subseteq \{1, \ldots, n\}$, $|T| = l$. Since $H_l(K(M, \{\sigma_i\}_{i=1}^n)) = 0$ for $l > 0$, the associated long exact homology sequence immediately implies that $H_l(K(M, \{\sigma_i\}_{i=1}^n)) = 0$ for $l > 1$. The connecting homomorphism

\begin{equation}
H_1(K^{(1)}(M, \{\sigma_i\}_{i=1}^{n-1})) = H_0(K(M, \{\sigma_i\}_{i=1}^n)) \to H_0(K(M, \{\sigma_i\}_{i=1}^{n-1}))
\end{equation}

is an isomorphism.
sends the homology class of \( m \in M \) to the homology class of \( \sigma_n(m) \in M \). But 
\[
\sigma_n(m) = \sum_{i=1}^{n-1} r_i \sigma_i(m),
\]
so this homology class is trivial, i.e., the connecting homomorphism is zero. Thus the long exact homology sequence gives isomorphisms

\[
H_1(K(M, \{\sigma_i\}_{i=1}^n)) \cong H_1(K(M, \{\sigma_i\}_{i=1}^{n-1})),
\]
\[
H_0(K(M, \{\sigma_i\}_{i=1}^n)) \cong H_0(K(M, \{\sigma_i\}_{i=1}^{n-1})),
\]
hence an isomorphism

\[
H_1(K(M, \{\sigma_i\}_{i=1}^n)) \cong H_0(K(M, \{\sigma_i\}_{i=1}^n)).
\]

From the definitions, one checks that this isomorphism sends a 1-cycle \( \sum_{i=1}^n m_i e_{\{i\}} \) to the 0-cycle \( m_n e_\emptyset \). The map \( \phi \) defined above is inverse to this isomorphism.

We apply this result in the setting of section 6. Assume that \( g \) is nondegenerate and that \( f_i \) is homogeneous of degree \( d_i \) and commode, \( i = 1, \ldots, r \). Consider the Koszul complex \( K_.(S, \emptyset) = K_.(B, \{D_i\}_{i=0}^{N+r}) \). The homogeneity condition implies that \( \dim \Delta = N + r \), hence by [1, Theorem 2.9] there is a subset \( S' \subset S \) of cardinality \( N + r \), say, \( S' = S \setminus \{i_0\} \), such that \( K_.(B, \{D_i\}_{i \in S'}) \) is acyclic in positive dimension and \( D_{i_0} \) is a linear combination of the \( D_i \)'s, \( i \in S' \), say \( D_{i_0} = \sum_{i \in S'} a_i D_i, a_i \in \Omega_0 \). Thus by Proposition 8.2, the map (where \( \xi \in B \))

\[
\xi e_\emptyset \mapsto \xi_{i_0} e_{\{i_0\}} + \sum_{i \in S'} (-a_i) \xi_i e_{\{i\}}
\]
is an isomorphism of \( H_0(K_.(S, \emptyset)) \) and \( H_1(K_.(S, \emptyset)) \). From (6.7) we see that the map \( \phi_{S}^\emptyset \) defined by (6.9) is just a nonzero scalar multiple of this isomorphism (in fact, the scalar is \( e_{i_0} \), where \( e_i \) is defined following (6.9)), hence \( \phi_{S}^\emptyset \) is also an isomorphism. More generally, one has the following.

**Proposition 8.3.** – Suppose that \( g \) is nondegenerate and \( f_i \) is commode for \( i = 1, \ldots, r \). Then for \( I \subset S \) with \( I_{sp} \neq \emptyset \) and \( I_{du} \neq \emptyset \) we have \( H_l(K_.(I, \emptyset)) = 0 \) for \( l > 1 \) and \( \phi_{I}^\emptyset \) is an isomorphism.

**Proof.** – The hypothesis on \( I \) implies that \( \dim \Delta_I = |I| - 1 \), hence one can repeat the argument given above for the case \( I = S \).

There is a modification for the Koszul complex \( \tilde{K}. = K_.(R, \{g_i\}_{i=0}^{N+r}) \) of section 7. The results of [1], together with Proposition 8.2, imply that there is an isomorphism (homogeneous of degree 1) from \( H_0(\tilde{K}.(I_0)) \) onto \( H_1(\tilde{K}.(I)) \). The dimension of \( H_0(\tilde{K}.(I))^k \) can then be calculated from (7.1). However, note that the map \( \phi_S^\emptyset \) defined by (6.9) need not be an isomorphism unless \( d_1, \ldots, d_r \) are all prime to \( p \). So in general, we have the following weaker version of Proposition 8.3.

**Proposition 8.4.** – Suppose that \( g \) is nondegenerate and \( f_i \) is commode for \( i = 1, \ldots, r \). Then for \( I \subset S \) with \( I_{sp} \neq \emptyset \) and \( I_{du} \neq \emptyset \) we have \( H_l(K_.(I, \emptyset)) = 0 \) for \( l > 1 \),
ON THE ZETA FUNCTION OF A COMPLETE INTERSECTION

\[ \dim_F \chi_1(K(I,\emptyset))^{(k)} = \dim F \chi_0(K(I,\emptyset))^{(k-1)}, \text{ and } \dim F \chi_0(K(I,\emptyset))^{(k)} \text{ is the coefficient of } t^k \text{ in} \]

\[ \sum_{e_1+\ldots+e_r=|I_\mu|-1, e_i=0 \text{ if } N+i \notin I_\mu} q_{e_1\ldots e_r}(t). \]

We prove another simple lemma on complexes. Let

\[ 0 \to L \to M \to N \to 0 \]

be a short exact sequence of complexes of \( A \)-modules. Suppose that \( \rho : L_0 \to L_1, \phi : M_0 \to M_1, \) and \( \psi : N_0 \to N_1 \) are \( A \)-module homomorphisms with image contained in the space of 1-cycles and sending 0-boundaries to 1-boundaries (hence they induce homomorphisms \( \bar{\rho} : H_0(L) \to H_1(L), \phi : H_0(M) \to H_1(M), \) and \( \bar{\psi} : H_0(N) \to H_1(N) \)) such that the diagram

\[
\begin{array}{ccc}
L_1 & \to & M_1 \\
\rho & \uparrow & \phi \\
L_0 & \to & M_0
\end{array}
\]

alpha (8.5)

commutes.

**Proposition 8.6.** If \( \bar{\psi} \) is surjective, then the connecting homomorphism \( H_1(N) \to H_0(L) \) is zero. If in addition \( H_2(N) = 0 \) and \( \bar{\phi} \) and \( \bar{\psi} \) are isomorphisms, then \( \bar{\rho} \) is also an isomorphism.

**Proof.** We first show that the connecting homomorphism is zero. Let \( \xi \in N_1 \) be a 1-cycle. Since \( \bar{\psi} \) is surjective, we may choose \( \eta \in N_0 \) such that \( \psi(\eta) \) is homologous to \( \xi \). Choose \( \zeta \in M_0 \) such that \( \zeta \) maps to \( \eta \) under the surjection \( M_0 \to N_0 \). By the commutativity of (8.5), \( \phi(\zeta) \in M_1 \) maps to \( \psi(\eta) \in N_1 \) under the surjection \( M_1 \to N_1 \). But \( \phi(\zeta) \) is a 1-cycle in \( M_1 \), so by definition of the connecting homomorphism, \( \xi \) maps to 0 in \( H_0(L) \). If \( H_2(N) = 0 \), one then has a commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \to & H_1(L) \\
\ & \uparrow & \phi \\
\ & \bar{\rho} & \psi \\
0 & \to & H_0(L)
\end{array}
\]

Since \( \phi \) and \( \bar{\psi} \) are both isomorphisms, \( \bar{\rho} \) must also be an isomorphism.

REFERENCES


(Manuscript received February 28, 1995; revised September 30, 1995.)

A. ADOLPHSON
Department of Mathematics,
Oklahoma State University,
Stillwater, Oklahoma 74078
adolphs@math.okstate.edu

S. SPERBER
School of Mathematics,
University of Minnesota,
Minneapolis, Minnesota 55455
sperber@vx.cis.umn.edu