ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 28, n° 6 (1995), p. 705-745

[<http://www.numdam.org/item?id=ASENS_1995_4_28_6_705_0>](http://www.numdam.org/item?id=ASENS_1995_4_28_6_705_0)

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BIFURCATION OF CONTRACTING SINGULAR CYCLES *

BY RAFAEL LABARCA

Dedicated to the memory of Professor R. Chuaqui (R.I.P.)

ABSTRACT. – The aim of this work is to continue the analysis of a new mechanism, the singular cycles, through which a vector field, depending on parameter, may evolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics. Specifically; if we start with a Morse - Smale vector field and move through a generic one - parameter family of vector fields to a contracting singular cycle and beyond, we reach a region filled up mostly with hyperbolic flows. In fact, the Lebesgue measure of parameter values corresponding to non Axiom A flows is zero. Moreover we provide a complete description of the bifurcation set that appear in these families.

1. Introduction

The aim of this work is to continue the analysis of a new mechanism, the singular cycles, introduced in [3] and [1] through which a vector field, depending on parameters, may evolve when the parameter varies from a vector field exhibiting simple dynamics into one having non-trivial dynamics.

Let M be a C^∞, m -dimensional, compact, connected, boundaryless, riemannian manifold. Let $X \in \mathcal{X}^r(M)$ be a C^r -vector field on M .

DEFINITION 1. – A cycle for the vector field X is a compact, invariant set $\Gamma \subset M$ formed by:

- (i) a finite number of singularities and periodic orbits $\Gamma_0 = \{\sigma_0, \dots, \sigma_n\}$;
- (ii) the complement $\Gamma_1 = (\Gamma \setminus \Gamma_0)$ is a set of non-periodic regular trajectories of the vector field X that satisfies:

$(CC)_1$ for any trajectory $\gamma \subset \Gamma_1$, there exists $0 \leq i \leq n$ such that $\omega(\gamma) \subset \sigma_{(i+1) \bmod (n+1)}$ and $\alpha(\gamma) \subset \sigma_i$;

$(CC)_2$ given $0 \leq i \leq n$ there exists a trajectory $\gamma \subset \Gamma_1$ such that $\omega(\gamma) \subset \sigma_{(i+1) \bmod (n+1)}$ and $\alpha(\gamma) \subset \sigma_i$.

Here $\omega(\gamma)$ (respectively $\alpha(\gamma)$) denotes the ω -limit set (respectively the α -limit set) of the trajectory γ .

* Partially supported by Fondecyt grants # 0449-91, # 1941080 and Dirección de Investigaciones (Dicyt) - USACH.

A cycle will be called *singular* if it contains a singularity; *hyperbolic* if all the critical elements in Γ are hyperbolic.

In this article we will deal with a 3-dimensional, hyperbolic, singular cycle, $\Gamma \subset M^3$, that contains a unique singularity, $\sigma_0(X)$, and periodic orbits $\sigma_1(X), \dots, \sigma_n(X)$, $n \geq 1$ (Fig. 1).

Fig. 1

We will assume the following regularity conditions:

(1) $\Gamma = \{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1^1(X), \gamma_1^2(X), \dots, \sigma_n(X), \gamma_n^1(X), \gamma_n^2(X)\}$, where $W_i^u = W_{\sigma_i(X)}^u$ intersects transversally $W_{(i+1) \bmod (n+1)}^s$ along the orbits $\gamma_i^1(X) \cup \gamma_i^2(X)$, $i = 1, \dots, n$.

We let $\sigma_0(Y), \sigma_1(Y), \dots, \sigma_n(Y)$ denote, respectively, the analytic continuation of $\sigma_0(X), \sigma_1(X), \dots, \sigma_n(X)$; for any $Y \in \mathcal{U}_X$. Here $\mathcal{U}_X$ denotes a small neighborhood of X in $\mathcal{X}^r(M^3)$ with the usual C^r -topology, $r \geq 3$;

(2) For any $Y \in \mathcal{U}_X$, the eigenvalues of $D_{\sigma_0(Y)}(Y) : T_{\sigma_0(Y)}(M^3) \rightarrow T_{\sigma_0(Y)}(M^3)$ are real numbers $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ and satisfy a k -Sternberg condition, k big enough to guarantee that we have C^3 -linearizing coordinates which depend C^2 on $Y \in \mathcal{U}_X$ in a neighborhood of $\sigma_0(Y)$;

(3) For every $p \in \gamma_0(X)$ and every invariant manifold of X , passing through $\sigma_0(X)$ and p , $W(\sigma_0(X))$, and tangent (at $\sigma_0(X)$) to the space spanned by the eigenvectors associated to $-\lambda_1(X)$ and $\lambda_2(X)$, we have $T_p(W(\sigma_0(X))) + T_p(W_{\sigma_1(X)}^s) = T_p M^3$;

(4) Γ is isolated: that is, there exists an open set $U \supset \Gamma$ such that $\cap_t X_t(U) = \Gamma$; here X_t denotes the flow defined by the vector field X ;

(5) Let $Q_i \subset M^3$, $1 \leq i \leq n$, be a transversal section at $q_i(Y) \in \sigma_i(Y)$. We let $P_i(Y) : V_i \subset Q_i \rightarrow Q_i$ denote the first return map defined in a neighborhood of $q_i(Y)$, any $Y \in \mathcal{U}_X$. We assume the eigenvalues of $D_{q_i} P_i : T_{q_i}(V_i) \rightarrow T_{q_i}(Q_i)$ are real numbers

and satisfy a k -Sternberg condition, k big enough to guarantee that we have C^3 -linearizing coordinates which depend C^2 on $Y \in \mathcal{U}_X$ in a neighborhood of $q_i(Y)$;

(6) The number $\alpha(Y) = \frac{\lambda_1(Y)}{\lambda_2(Y)}$ is greater than one and

$$\beta(Y) = \frac{\lambda_3(Y)}{\lambda_2(Y)} > \alpha(Y) + 2.$$

A cycle Γ as above is called a *contracting singular cycle*.

We let $\Gamma(Y, U) \subset M$ denote the set $\cap_t Y_t(U)$, for $Y \in \mathcal{U}_X$ (that is, the maximal invariant set in the neighborhood U for the vector field Y).

We let $\gamma_0(Y), \gamma_1^1(Y), \gamma_1^2(Y), \dots, \gamma_n^1(Y), \gamma_n^2(Y)$ denote, respectively, the analytic continuation of the trajectories $\gamma_0(X), \dots, \gamma_n^2(X)$ for any $Y \in \mathcal{U}_X$. These trajectories are included in the unstable manifolds $W^u(\sigma_0(Y)), \dots, W^u(\sigma_n(Y))$ respectively.

Comment: It is easy to see that there exists a codimension-one submanifold, $\mathcal{N} \subset \mathcal{X}^r(M)$, containing X such that:

- (i) $Y \in \mathcal{N}$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \gamma_0(Y), \dots, \gamma_n^2(Y)\}$;
- (ii) $(\mathcal{U}_X \setminus \mathcal{N})$ has two connected components and one of them, which is denoted $\mathcal{U}^-$, is such that $Y \in \mathcal{U}^-$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y), \dots, \sigma_n(Y), \gamma_n^1(Y), \gamma_n^2(Y)\}$; and
- (iii) Bifurcations for the maximal invariant set $\Gamma(Y, U)$ may appear only for $Y \in \mathcal{U}^+ = (\mathcal{U}_X \setminus (\mathcal{N} \cup \mathcal{U}^-))$.

$\mathcal{U}_H^+$ is defined to be the set of $Y \in \mathcal{U}^+$ such that $\Gamma(Y, U)$ consists of Γ_0 , a transitive hyperbolic set and a denumerable number of isolated hyperbolic periodic orbit, and $\mathcal{U}_A^+$ as the set of $Y \in \mathcal{U}^+$ such that $\Gamma(Y, U)$ consists of $\sigma_0(Y)$, a transitive hyperbolic set, a hyperbolic attracting periodic orbit (which is contained in the closure of the trajectory $\gamma_0(Y)$), and a denumerable number of isolated hyperbolic periodic orbit.

Under the above conditions we have the following :

THEOREM 1. – *a) $\mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)$ is laminated by codimension-one C^1 -submanifolds of the following type:*

- a₁) those laminas that present a saddle-node or a flip bifurcation for periodic orbits;*
- a₂) those laminas that present a contracting singular cycle;*
- a₃) those laminas that present a homoclinic behavior for the singularity; and*
- a₄) those laminas that present a recurrent behavior for the analytic continuation of the trajectory $\gamma_0(X)$.*

Moreover all elements in the same lamina have the same dynamics in the neighborhood U (that is, given a lamina $L \subset \mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)$ and $Y_1, Y_2 \in L$, there exists a homeomorphism $h : U \rightarrow U$ that is a topological equivalence between $Y_1|_U$ and $Y_2|_U$).

- b) Any $Y \in \mathcal{U}_H^+ \cup \mathcal{U}_A^+$ is structurally stable.*
- c) For any $Y \in (\mathcal{U}^+ \setminus (\mathcal{U}_H^+ \cup \mathcal{U}_A^+))$, $\Gamma(Y, U)$ decomposed into a chain recurrent expansive set, a denumerable number of isolated hyperbolic periodic orbits plus the closure of the trajectory $\gamma_0(Y)$.*

Now let $\{X_\mu\} \subset \mathcal{U}_X$ be a one-parameter family of vector fields such that $X_{\mu=0} \in \mathcal{N}$ and $\{X_\mu\}$ is transversal to $\mathcal{N}$ at $\mu = 0$.

THEOREM 2. – *There exists $\nu = \nu(X_\mu) > 0$ such that :*

$$m(\{\mu; 0 \leq \mu \leq \nu, X_\mu \notin (\mathcal{U}_H^+ \cup \mathcal{U}_A^+)\}) = 0$$

(here $m(A)$ denotes the Lebesgue measure of the set $A \subset \mathbf{R}$).

Following [3] we may now state a corollary for Theorem 1.

COROLLARY. – *Let $\{Y_\mu\}$ be another one-parameter family transversal to $\mathcal{N}$ at $\mu = 0$. There exists a reparametrization $\rho : [0, \nu(X_\mu)] \rightarrow [0, \nu(Y_\mu)]$ and, for each $\mu \in [0, \nu(X_\mu)]$, a homeomorphism $h_\mu : U \rightarrow U$ that is a topological equivalence between $X_\mu|_U$ and $Y_{\rho(\mu)}|_U$.*

Remark. – a) A particular case of Theorem 2 was proven by Pacifico and Rovella in [2]. In their case, Γ is given by $\{\sigma_0(X), \gamma_0(X), \sigma_1(X), \gamma_1(X)\}$ and the associated first return map preserves orientation. A more general case of the Pacifico-Rovella result was proven by San Martín in [8].

The techniques they use to prove their result do not apply in our case.

b) For the case $\alpha(X) < 1$ (an expanding singular cycle), theorems 1 and 2 and the above Corollary 1 were proven by Bamón, Labarca, Mañé and Pacifico in [1].

c) The main difference between the unfolding of expanding and contracting singular cycles is the following: the unfolding of contracting singular cycles must have saddle-node and flip bifurcations whereas the unfolding of the expanding singular cycles does not.

ACKNOWLEDGEMENTS

The author wishes to thank IMPA-Brazil and ICTP-Italy for their support and hospitality while preparing this paper.

2. Proof of Theorem 1

This Chapter is organized in the following way : In section 2.1 we make the necessary change of coordinates to obtain a simpler form of the First Return Map. Section 2.2 is devoted to give a characterization of the elements in $\mathcal{U}_H^+ \cup \mathcal{U}_A^+$. Sections 2.3 - 2.11 are devoted to the study of the one dimensional dynamics associated to a contracting singular cycle. In particular we obtain the proof of Theorem 1.

2.1. CHANGE OF COORDINATES AND THE FIRST RETURN MAP

Let $X \in \mathcal{X}^r(M^3)$ be a vector field having a contracting singular cycle, Γ , with isolated neighborhood $U \subset M$. For the sake of simplicity we will assume Γ contains a unique periodic orbit, and later on in Section III.5 we will make comments on the general case. Here Γ is the union of a singularity $\sigma_0 = \sigma_0(X)$, a periodic orbit $\sigma_1 = \sigma_1(X)$, an orbit $\gamma_0 = \gamma_0(X) \subset W_{\sigma_0}^u$ of nontransversal intersection between $W_{\sigma_0}^u$ and $W_{\sigma_1}^s$, and two orbits of transversal intersection between $W_{\sigma_1}^u$ and $W_{\sigma_0}^s$, $\gamma_1^1 = \gamma_1^1(X)$ and $\gamma_1^2 = \gamma_1^2(X)$.

Let Q be a cross section to the flow X at $q \in \sigma_1$ parametrized by $\{(x, y)/|x|, |y| \leq 1\}$ and satisfying $W_{\sigma_1}^s \supseteq \{(x, 0); |x| \leq 1\}$ and $W_{\sigma_1}^u \supseteq \{(0, y); |y| \leq 1\}$.

Let $p = p(X)$ be the first intersection between γ_0 and Q . Then $p = (x_0, 0) = (x_0(X), 0)$ and we assume $x_0 > 0$. It is clear that a first return map, $F = F(X)$, is defined on a subset of Q . Moreover if $q_1 = (0, y_1) = (0, y_1(X))$ and $q_2 = (0, y_2) = (0, y_2(X))$ are such that their ω -limit set is σ_0 , then there are horizontal strips $R_1 = R_1(X)$ and $R_2 = R_2(X)$ such that F is defined on $R_1 \cup R_2$. Here a horizontal strip is a closed set $C \subset Q$ bounded (in Q) by two disjoint continuous curves connecting the vertical sides of Q , $\{(-1, y)/|y| \leq 1\}$, and $\{(1, y)/|y| \leq 1\}$.

Since Γ is isolated, we have that $\Gamma \cap Q \subset \{(x, y)/y \geq 0\}$ and that :

$$F(R_1 \cup R_2) \subset \{(x, y)/y \leq 0\}$$

(See Fig. 2).

Fig. 2

If $Y \in \mathcal{X}^r$ is near X , then $W^s(\sigma_1(Y))$ intersects Q at a curve $c(Y)$, and the first intersection of $W^u(\sigma_0(Y))$ with Q is a point $p(Y)$. Note that both $c(Y)$ and $p(Y)$ vary smoothly with Y . The implicit function theorem on Banach spaces implies that the condition $p(Y) \in c(Y)$ defines a C^2 -codimension one submanifold, $\mathcal{N}$, in a neighborhood of X , $\mathcal{U} \subset \mathcal{X}^r$, such that $(\mathcal{U} \setminus \mathcal{N})$ has two connected components: one of them, which we denote by $\mathcal{U}^-$, is characterized by $p(Y) \in Q$ and lies below $c(Y)$; we let $\mathcal{U}^+$ denote the other component.

Clearly, $Y \in \mathcal{U}^-$ implies $\Gamma(Y, U) = \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y)\}$ and hence the dynamics of the vector field Y in U is simple.

If $Y \in \mathcal{U}^+$, then $\sigma_1(Y)$ has transversal homoclinic orbits and therefore Y does not have simple dynamics in U . As before we note that there exists a first return map F_Y defined on a subset of Q , every $Y \in \mathcal{U}^+$.

Since $\Gamma(Y, U)$ is the saturation of $\Gamma(Y, U) \cap Q$ by the flow Y_t , and $\Gamma(Y, U) \cap Q$ is the maximal invariant set of F_Y , it is necessary to describe the dynamics of F_Y to understand

the dynamics of Y on $\Gamma(Y, U)$. For this we choose coordinates (x, y) on Q , that depend C^2 on Y , such that:

- (i) $\{(x, 0)/|x| \leq 1\} \subset W^s(\sigma_1(Y))$;
- (ii) $\{(0, y)/|y| \leq 1\} \subset W^u(\sigma_1(Y))$;
- (iii) $\Gamma(Y, U) \cap Q \subset Q^+ = \{(x, y)/x \geq 0, y \geq 0\}$; and
- (iv) the analytic continuation of the point $p = p(X) = \gamma_0(X) \cap Q$ is a point $p(Y) = (x(Y), y(Y))$, with $0 < x(Y) < 1$.

Note that $Y \in \mathcal{U}^+$ if and only if $y(Y) > 0$.

Moreover $\Gamma(Y, U) \not\subseteq \{\sigma_0(Y), \sigma_1(Y), \gamma_1^1(Y), \gamma_1^2(Y)\}$ if and only if $y(Y) \geq 0$.

For $Y \in \mathcal{U}$ such that $y(Y) \geq 0$, let $q_1(Y) = (0, y_1(Y))$ (resp., $q_2(Y) = (0, y_2(Y))$) be the analytic continuation of the point q_1 (resp., q_2). Since $\omega(q_i(Y)) = \sigma_0(Y)$ and $\alpha(q_i(Y)) = \sigma_1(Y)$, $i = 1, 2$, there are horizontal strips $R_Y^i \ni q_i(Y)$ such that the positive orbits of points at R_Y^i first pass near $\sigma_0(Y)$ and afterwards return to Q . On the other hand, the positive orbits of points at a horizontal strip R_Y containing $W^s(\sigma_1(Y)) \cap Q$ goes around the closed orbit $\sigma_1(Y)$ and then return to Q (see Fig. 3).

Fig. 3

Therefore F_Y is defined on $R_Y \cup R_Y^1 \cup R_Y^2$, and the restriction of F_Y to R_Y coincides with the Poincaré map, P_Y , associated to $\sigma_1(Y)$. We further assume P_Y is linear on R_Y .

Let $\xi_Y > 1$ and $\tau_Y < 1$ be the eigenvalues of $DP_Y(0, 0)$. We have $R_Y^1 = \{(x, y)/x \geq 0, \Theta_Y^1(x) \leq y \leq \Theta_Y^2(x)\}$, $R_Y^2 = \{(x, y)/x \geq 0, \Theta_Y^2(x) \leq y \leq \Theta_Y^2(x)\}$, where $\Theta_Y^i(x) = \Theta^i(Y, x)$ is a smooth real function satisfying $\{(x, \Theta_Y^i(x)), 0 \leq x \leq 1\} \subset W^s(\sigma_0(Y))$ and $(0, \Theta_Y^i(0)) = q_i(Y)$, $i = 1, 2$. Moreover if $\delta_Y^i(x) = \delta^i(Y, x)$ is such that $\{(x, \Theta_Y^i(x) + (-1)^{i+1}\delta_Y^i(x)), 0 \leq x \leq 1\} \subset F_Y^{-1}(\{(x, 0); 0 \leq x \leq 1\}) \subset F_Y^{-1}(W^s(\sigma_1(Y)))$ $i = 1, 2$, then there is $\varepsilon > 0$ such that $\Theta^1 - \varepsilon > \Theta_Y^1(x) + \delta_Y^1(x)$ and $\Theta^2 + \varepsilon < \Theta_Y^2(x) - \delta_Y^2(x)$, every x .

Making a linear change of coordinates we may also assume that

- (v) $|(\Theta_Y^i)'(x)| < \frac{1}{100}$ and that δ_Y goes to zero uniformly in the C^2 -topology when Y approaches $\mathcal{N}$.

Clearly $R_Y = \{(x, y)/x \geq 0, 0 \leq y \leq \xi_Y^{-1}\Theta_Y(x)\}$ and $F_Y(x, y) = (\tau_Y x, \xi_Y y)$, for $(x, y) \in R_Y$.

To obtain the expressions of $F_Y(x, y)$, for $(x, y) \in R_Y^1 \cup R_Y^2$, we proceed as follows:

Let $-\lambda_3(Y) < -\lambda_1(Y) < 0 < \lambda_2(Y)$ be the eigenvalues of $DY(\sigma_0(Y))$. We set $\alpha(Y) = \frac{\lambda_1(Y)}{\lambda_2(Y)}$ and $\beta(Y) = \frac{\lambda_3(Y)}{\lambda_2(Y)}$.

For $Y \in \mathcal{U}$, let (x_1, x_2, x_3) be C^3 -linearizing coordinates, in a neighborhood $U_0 \ni \sigma_0(Y)$, that depend C^2 on Y . We let L and $\tilde{L}$ denote the planes $x_1 = 1$ and $x_2 = 1$, respectively.

For $(x, y) \in R_Y^i$, we have $F_Y(x, y) = \pi_3 \circ \pi_2 \circ \pi_1^i(x, y) = (f_Y^i(x, y), g_Y^i(x, y))$ where:

(a) $\pi_1^i : V_i \subset Q^+ \rightarrow L$ is a diffeomorphism such that $\pi_1^i(x, \Theta_Y^i(x)) = (x_3, 0)$, for $0 \leq x \leq 1$, and $D\pi_1^i(x, y) = \begin{bmatrix} a_i(x, y) & b_i(x, y) \\ c_i(x, y) & d_i(x, y) \end{bmatrix}$ where $k_1 \leq |a_i(x, y)|, |d_i(x, y)| \leq K_1$, and k_1, K_1 are positive real constants. Up to replacing $\{(x, \Theta_Y^i(x)), x \in [0, 1]\}$ with some negative iterate of it (and shrinking $\mathcal{U}$) if necessary; we may assume that there are $0 < \eta \ll 1$ such that $\frac{|c_i(x, y)|}{|d_i(x, y)|} \leq \eta$, every $(x, y) \in R_Y^i$ and $Y \in \mathcal{U}^+$;

(b) $\pi_2 : L \rightarrow \tilde{L}$ is given by $\pi_2(x_3, x_2) = (\tilde{x}_3 = x_3 x_2^{\beta_Y}, \tilde{x}_1 = x_2^{\alpha_Y})$;

(c) $\pi_3 : \tilde{L} \rightarrow Q$ is a diffeomorphism such that

$$D\pi_3(\tilde{x}_3, \tilde{x}_1) = \begin{bmatrix} \tilde{a}(\tilde{x}_3, \tilde{x}_1) & \tilde{b}(\tilde{x}_3, \tilde{x}_1) \\ \tilde{c}(\tilde{x}_3, \tilde{x}_1) & \tilde{d}(\tilde{x}_3, \tilde{x}_1) \end{bmatrix}$$

with $k_2 \leq |\tilde{a}(\tilde{x}_3, \tilde{x}_1)|, |\tilde{d}(\tilde{x}_3, \tilde{x}_1)| \leq K_2$, some positive constants k_2, K_2 . Moreover, by replacing $p(Y)$ with some positive iterate of it (also contained in $W^u(\sigma_0(Y)) \cap S$), if necessary, we may assume that the quotient $|\tilde{b}|/|\tilde{d}|$ is small enough, and hence that $|\tilde{b}|/|\tilde{d}| \leq \eta$, some small $\eta > 0$.

We now state a very useful lemma that establishes the existence of a C^3 -invariant stable foliation for F_Y that depends C^2 on Y . The proof follows from the techniques in [4]; e.g. as may be found in [1] and [5].

LEMMA 1. – *For every $Y \in \mathcal{U}$, there exists an invariant C^3 stable foliation for F_Y , $\mathcal{F}_Y^s$, that depends C^2 on Y .*

After a C^3 change of coordinates, this lemma implies that $\Theta_Y^i(x)$, $\delta_Y^i(x)$ and $g_Y^i(x, y)$ are maps that do not depend on x .

For the sake of simplicity, we assume that $\Theta_Y^2(x) \equiv 1$ and that $\Theta_Y^1(x) = 1 - \delta$. We also have $c_i(x, y) \equiv 0$. Since $\pi_1^i(x, y)$ is a diffeomorphism, we have that $a_i(x, y) \neq 0$ and that $d_i(x, y) \neq 0$, every (x, y) . Thus we conclude that there are real positive constants C and K such that:

(d)

$$0 \leq \left| \frac{\partial}{\partial x} f_Y^i(x, y) \right| \leq K x_2^{\beta_Y} + r_1^i(x, y),$$

$$\left| \frac{\partial}{\partial y} f_Y^i(x, y) \right| = K x_2^{\alpha_Y - 1} + r_2^i(x, y)$$

and

$$\left| \frac{\partial}{\partial y} g_Y^i(x) \right| \leq C x_2^{\alpha_Y - 1} + r_3^i(y)$$

where, respectively, $|r_1^i(x, y)| \leq (\text{constant}) \cdot x_2^{\beta_Y - 1}$, $|r_2^i(x, y)| \leq (\text{constant}) \cdot x_2^{\beta_Y}$ and $|r_3^i(y)| \leq (\text{constant}) \cdot x_2^{\alpha_Y}$. In the above inequalities we replace x_2 with $y - (1 - \delta)$ or $1 - y$, according that $i = 1$ or 2 .

Moreover,

(e) $f_Y^1(x, 1 - \delta) = x_Y = f_Y^2(x, 1)$, for $x \in [0, 1]$, and $g_Y^1(1 - \delta) = y_Y = g_Y^2(1)$;

(f)

$$f_Y^1(x, 1 - \delta + \delta_Y^1) \subset \{(x, 0), x \in]0, 1[\},$$

$$f_Y^2(x, 1 - \delta_Y^2) \subset \{(x, 0); x \in]0, 1[\}, \text{ any } x \in [0, 1],$$

and $g_Y^1(1 - \delta + \delta_Y^1) = 0 = g_Y^2(1 - \delta_Y^2)$.

Conditions (d), (e) and (f) imply $\delta_Y^i = A_Y^i y_Y^{1/\alpha_Y}$, where A_Y^i is a positive constant for $i = 1, 2$.

Finally, by making another C^3 -change of coordinates, we obtain $F_Y(x, y) = (f_Y(x, y), g_Y(y))$, with

$$g_Y(y) = \begin{cases} \xi_Y y, & \text{for } y \in [0, \xi_Y^{-1}] \\ y_Y - J(Y, y)(y - (1 - \delta))^{\alpha_Y}, & \text{for } y \in [1 - \delta, 1 - \delta + \delta_Y^1] \\ y_Y - K(Y, y)(1 - y)^{\alpha_Y}, & \text{for } y \in [1 - \delta_Y^2, 1]. \end{cases}$$

Here $J(Y, y)$ and $K(Y, y)$ are C^2 -maps on Y , whereas C^3 -maps on y for $y \neq 1, 1 - \delta$. Furthermore using (d), (e) and (f), we obtain:

(g) $\left| \frac{\partial}{\partial y} g_Y(x) \right| \leq C|1 - y|^{\alpha_Y - 1}$ or $\left| \frac{\partial}{\partial y} g_Y(y) \right| \leq C|y - (1 - \delta)|^{\alpha_Y - 1}$ according, respectively, that $y \in [1 - \delta_Y^2, 1]$ or that $y \in [1 - \delta, 1 - \delta + \delta_Y^1]$.

Also

(i) $\left| \frac{\partial}{\partial y} K(Y, y) \right| \leq K_0$ and $\left\| \frac{\partial}{\partial Y} K(Y, y) \right\|$ is small;

(ii) $\left| \frac{\partial}{\partial y} J(Y, y) \right| \leq K_0$ and $\left\| \frac{\partial}{\partial Y} J(Y, y) \right\|$ is small;

(iii) $J(X, 1 - \delta) > 0$ and $K(X, 1) > 0$.

(h) $0 \leq \left| \frac{\partial}{\partial x} f_Y(x, y) \right| \leq K|1 - y|^{\beta_Y}$ or $0 \leq \left| \frac{\partial}{\partial x} f_Y(x, y) \right| \leq K|y - (1 - \delta)|^{\beta_Y}$, and $\left| \frac{\partial}{\partial y} f_Y(x, y) \right| \leq K|1 - y|^{\alpha_Y - 1}$ or $\left| \frac{\partial}{\partial y} f_Y(x, y) \right| \leq K|y - (1 - \delta)|^{\alpha_Y - 1}$; according, respectively, that $y \in [1 - \delta_Y^2, 1]$ or that $y \in [1 - \delta, 1 - \delta + \delta_Y^1]$.

We do not lose generality if, in the sequel, we assume that, for $Y \in \mathcal{U} : \alpha(Y) = \alpha, \beta(Y) = \beta, \xi_Y = \xi$ and $\tau_Y = \tau$.

Furthermore since the map $Y \rightarrow y_Y$ is a C^2 -submersion, we can find C^2 -coordinates (v, μ) in the neighborhood $\mathcal{U}(\mu \in \mathbf{R})$ such that:

(i) $\{(v, \mu)/\mu = 0\} \subset \mathcal{N} \cap \mathcal{U}$;

(ii) $F_{(v, \mu)}(x, y) = (\tau x, \xi y)$ if $0 \leq y \leq \xi^{-1}$,

$$F_{(v, \mu)}(x, y) = (x(\mu, v) + f^2(v, \mu; x, y), \mu - K(v, \mu; y)(1 - y)^\alpha) \\ \text{for } 1 - \delta^2(v, \mu) \leq y \leq 1,$$

$$F_{(v, \mu)}(x, y) = (x(v, \mu) + f^1(v, \mu; x, y), \mu - J(v, \mu; y)(y - (1 - \delta))^\alpha), \text{ for } \\ 1 - \delta \leq y \leq 1 - \delta + \delta^1(v, \mu).$$

Under these conditions we obtain $\delta^i(v, \mu) = A^i(v)\mu^{1/\alpha}$, with $\left\| \frac{\partial A^i}{\partial v} \right\|$ small numbers, for $i = 1, 2$.

We will use the notations $a(v, \mu) = 1 - \delta^2(v, \mu)$ and $b(v, \mu) = 1 - \delta + \delta^1(v, \mu)$.

2.2.

For a proof of Theorem 1 we first give a characterization of the elements in $\mathcal{U}_H^+ \cup \mathcal{U}_A^+$. Choose $\mu_1 > 0$ and $n_0 \in \mathbb{N}$ such that $\xi^{n_0} \mu_1 = 1, 1 > 1$.

LEMMA 2. – For $(v, \mu) \in \mathcal{U}$ such that $\xi^{-n_0} < \mu \leq \mu_1$, we have that

$$\Lambda(v, \mu) = \{(x, y)/F_{(v, \mu)}^n \in R(v, \mu) \cup R_1(v, \mu) \cup R_2(v, \mu), n \in \mathbb{Z}\}$$

is a hyperbolic transitive set.

Proof. – See Lemma 2 in [1].

We next assume $0 \leq \mu \leq \xi^{-n_0} = \mu_0$.

Set $I_0(v, \mu) = [0, \xi^{-1}]$, $I_{01}(v, \mu) =]\xi^{-1}, 1 - \delta[$,

$$\begin{aligned} I_1(v, \mu) &= [1 - \delta, b(v, \mu)], I_{12}(v, \mu) =]b(v, \mu), a(v, \mu)[\text{ and} \\ I_2(v, \mu) &= [a(v, \mu), 1]. \end{aligned}$$

For $(v, \mu) \in \mathcal{U}$, let $L(v, \mu, \cdot) : \cup_{i=0}^2 I_i(v, \mu) \rightarrow [0, 1]$ be the map $L(v, \mu; y) = \pi_y \circ F_{(v, \mu)}(x, y) =$ second component of the first return map $F_{(v, \mu)}(x, y)$.

Define $L_1(v, \mu; y) = L(v, \mu; y)$ and $L_{n+1}(v, \mu; y) = L(v, \mu; L_n(v, \mu; y))$ for $n \geq 1$.

Let

$$\begin{aligned} \Lambda(v, \mu) &= \{y \in [0, 1]/L_n(v, \mu; y) \in \cup_{i=0}^2 I_i(v, \mu), n \geq 0\} \\ \Gamma_0 &= \{(v, \mu) \in \mathcal{U} : 1 \notin \Lambda(v, \mu)\} \end{aligned}$$

and

$$\Gamma_1 = \{(v, \mu) \in \mathcal{U} : 1 \in \Lambda(v, \mu) \text{ and there exists a hyperbolic attracting periodic orbit for the map } L(v, \mu; \cdot)\}$$

LEMMA 3. – For $(v, \mu) \in \Gamma_0$ we have that $\Lambda(v, \mu)$ is a hyperbolic set for the map $L(v, \mu; \cdot)$.

Proof. – Let $(v, \mu) \in \Gamma_0$ and $n = n(v, \mu)$ be the integer such that $L_n(v, \mu; 1) \in I_{01}(v, \mu) \cup I_{12}(v, \mu)$. Due to the continuity of the map $(v, \mu; y) \mapsto L_n(v, \mu; y)$ we can find neighborhoods $U_{1-\delta} \subset I_1(v, \mu)$, $U_1 \subset I_2(v, \mu)$ of the points $1 - \delta$ and 1 , respectively, such that $y \in U_{1-\delta} \cup U_1$ implies $L_n(v, \mu; y) \in I_{01}(v, \mu) \cup I_{12}(v, \mu)$. This, in turn, implies that $\Lambda(v, \mu)$ is a compact invariant set with all its periodic points hyperbolic repelling and without critical points. Hence, by applying a result proved by Mañé [6] to the restriction map

$$L_{(v, \mu; \cdot)} / (I_0(v, \mu) \cup I_1(v, \mu) \cup I_2(v, \mu) \setminus U_{1-\delta} \cup U_1)$$

the result now follows. ■

DEFINITION 2. – Let $I \subset J$ be two intervals. We will say $f \in C^k(I, J)$, $k \geq 1$, satisfies Axiom A if:

- (i) f has a finite number of hyperbolic, attracting periodic orbits and no other attractors,
(ii) Let $B(f)$ denote the basin of attraction of the attracting periodic orbits for f . The set $\sum(f) = I \setminus B(f)$ is a hyperbolic set for f .

LEMMA 4. – For $(v, \mu) \in \Gamma_1$ we have that $L(v, \mu; \cdot)$ satisfies Axiom A.

Proof. – We note that $L(v, \mu; \cdot)|_{I_1(v, \mu) \cup I_2(v, \mu)}$ has negative Schwarzian derivative. By Singer's theorem we obtain that the attracting periodic orbit attracts all the critical points (since that all critical points eventually have the same orbit).

Since $L(v, \mu; \cdot)$ has a hyperbolic attracting periodic orbit, we have that it does not have saddle-node or attracting flip bifurcations. Since these are the only non-hyperbolic periodic orbits that appear in our family (see sections 2.3 through 2.14), we conclude that $\Lambda(v, \mu)$ does not contain non-hyperbolic periodic orbits. In particular, all the periodic points in $(\Lambda(v, \mu) \setminus B(L(v, \mu, \cdot)))$ are hyperbolic. This implies that $(\Lambda(v, \mu) \setminus B(L(v, \mu; \cdot)))$ is a hyperbolic set (see [dM, pg. 128]). ■

Using the techniques of [3] or [1], it is easy to see that $(v, \mu) \in \Gamma_0$ if and only if $(v, \mu) \in \mathcal{U}_H^+$ and $(v, \mu) \in \Gamma_1$ if and only if $(v, \mu) \in \mathcal{U}_A^+$. Part b) of Theorem 1 now follows.

2.3.

Since $X \in \mathcal{U}_X$ we have $X = (v_0, 0)$ some v_0 .

In the sequel we will deal with $(v, \mu) \in \mathcal{U}_X$ such that : $-\xi^{-(n_0-1)} \leq \mu \leq \xi^{-(n_0-1)}$; $\|v - v_0\| \leq r_0$, some $r_0 > 0$ small, and $n_0 \in \mathbb{N}$ chosen such that the number :

$$Q_0 = \inf \{ \alpha((A^1(v))^{-1} \xi^{\frac{n_0}{\alpha}} (1 - \delta - \xi^{-1}), \alpha(A^2(v))^{-1} \xi^{\frac{n_0}{\alpha}} (1 - \delta - \xi^{-1}); v \in V \}$$

satisfies $Q_0 > 2, \frac{2}{Q_0(1 - \xi^{-1})} < 1$ and, $\xi^{-1/\alpha} Q_0 > 1$.

Throughout, we will consider $k_0 \in \mathbb{N}$ such that $k_0 \geq n_0$.

Let $B(k_0)$ be the set $\{(v, \mu) \in \mathcal{U} / 1 - \delta \leq \xi^{k_0-1} \mu \leq 1; \|v - v_0\| \leq r_0\}$.

For $(v, \mu) \in B(k_0)$ denote by $D\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)(v, \mu) \subset I_1(v, \mu)$ $\left(D\left(\begin{smallmatrix} 2 \\ j \end{smallmatrix}\right)(v, \mu) \subset I_2(v, \mu)\right)$ the interval satisfying :

$$L\left(v, \mu, D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)\right) = \xi^{-(k_0-1)} \xi^{-j} [1 - \delta, 1], \quad \text{for } j \geq 1, \quad i = 1, 2.$$

$D\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)(v, \mu) \subset I_i(v, \mu)$ will denote, the interval satisfying :

$$L\left(v, \mu; D\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)(v, \mu)\right) = \xi^{-(k_0-1)} [1 - \delta, \xi^{k_0-1} \mu], \quad i = 1, 2.$$

Note that

$$D\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(v, \xi^{-(k_0-1)}(1 - \delta)) = \{1 - \delta\} \text{ and that } D\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)(v, \xi^{-(k_0-1)}(1 - \delta)) = \{1\}.$$

For $j \geq 1$, we let $\left\{ z\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu), y\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu) \right\}$ denote the boundary points of the interval $D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)$. These two points are defined by the equations

$$\begin{aligned} L\left(v, \mu; z\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)\right) &= \xi^{-(k_0-1)}\xi^{-j}(1-\delta) \text{ and} \\ L\left(v, \mu; y\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)\right) &= \xi^{-(k_0-1)}\xi^{-j}. \end{aligned}$$

For $j = 0$, we have that $D\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(v, \mu) = \left[1 - \delta, z\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(v, \mu)\right]$ and that $D\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)(v, \mu) = \left[z\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)(v, \mu); 1\right]$ where $L\left(v, \mu; z\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right)(v, \mu)\right) = \xi^{-(k_0-1)}(1-\delta)$, $i = 1, 2$.

We note that :

$$\lim_{\mu \rightarrow \xi^{-(k_0-1)}(1-\delta)} \frac{\partial z\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}{\partial \mu}(v, \mu) = +\infty \quad \text{and} \quad \lim_{\mu \rightarrow \xi^{-(k_0-1)}(1-\delta)} \frac{\partial z\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)}{\partial \mu}(v, \mu) = -\infty$$

The proof of the following lemma is easy and left to the reader.

LEMMA 5. – Given $\varepsilon > 0$ we can find $j_0 \in \mathbf{N}$ such that

$$\begin{aligned} &\max \left\{ \sup \left\{ \left| b(v, \mu) - z\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)(v, \mu) \right|, \left| \frac{\partial b}{\partial \mu}(v, \mu) - \frac{\partial z\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)}{\partial \mu}(v, \mu) \right|, \right. \\ &\quad \left. \left\| \frac{\partial b}{\partial v}(v, \mu) - \frac{\partial z\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)}{\partial v}(v, \mu) \right\| \right\}, \\ &\sup \left\{ \left| b(v, \mu) - y\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)(v, \mu) \right|, \left| \frac{\partial b}{\partial \mu}(v, \mu) - \frac{\partial y\left(\begin{smallmatrix} j \\ 1 \end{smallmatrix}\right)}{\partial \mu}(v, \mu) \right|, \right. \\ &\quad \left. \left\| \frac{\partial b}{\partial v}(v, \mu) - \frac{\partial y\left(\begin{smallmatrix} 1 \\ j \end{smallmatrix}\right)}{\partial v}(v, \mu) \right\| \right\}, \\ &\sup \left\{ \left| a(v, \mu) - z\left(\begin{smallmatrix} 2 \\ j \end{smallmatrix}\right)(v, \mu) \right|, \left| \frac{\partial a}{\partial \mu}(v, \mu) - \frac{\partial z\left(\begin{smallmatrix} 2 \\ j \end{smallmatrix}\right)}{\partial \mu}(v, \mu) \right|, \right. \\ &\quad \left. \left\| \frac{\partial a}{\partial v}(v, \mu) - \frac{\partial z\left(\begin{smallmatrix} 2 \\ j \end{smallmatrix}\right)}{\partial v}(v, \mu) \right\| \right\}, \end{aligned}$$

$$\sup \left\{ \left| a(v, \mu) - y \binom{2}{j}(v, \mu) \right|, \left| \frac{\partial a}{\partial v}(v, \mu) - \frac{\partial y \binom{2}{j}}{\partial \mu}(v, \mu) \right|, \right. \\ \left. \left\| \frac{\partial a}{\partial v}(v, \mu) - \frac{\partial y \binom{2}{j}}{\partial v}(v, \mu) \right\| \right\}; (v, \mu) \in B(k_0) \Big\} < \varepsilon,$$

for any $j \geq j_0$: that is, the sequences of maps $\left(z \binom{1}{j} \right), \left(y \binom{1}{j} \right)$ (resp. $\left(z \binom{2}{j} \right), \left(y \binom{2}{j} \right)$) converge to $b(v, \mu)$ (resp. $a(v, \mu)$) in the uniform C^1 -topology in $B(k_0)$. ■

We also note the following fact: for any $j \geq 1, y \in D \binom{i}{j}(v, \mu)$ and $y' \in D \binom{i}{j+1}(v, \mu)$ we have

$$\left| \frac{\frac{\partial L}{\partial y}(v, \mu, y')}{\frac{\partial L}{\partial y}(v, \mu, y)} \right| \geq \lambda_j > 1,$$

where the sequence (λ_j) satisfies $\lim_{j \rightarrow \infty} \lambda_j = 1$

We now have the following result for $(v, \mu) \in B(k_0)$.

LEMMA 6.

$$\min \left\{ \left| \frac{\partial L_{k_0}}{\partial y} \left(v, \mu; y \binom{1}{1}(v, \mu) \right) \right|, \left| \frac{\partial L_{k_0}}{\partial y} \left(v, \mu; y \binom{2}{1}(v, \mu) \right) \right| \right\} \geq \xi^{\frac{k_0 - n_0 - 1}{\alpha}} Q_0$$

Proof. – Since $L_{k_0}(v, \mu; y) = \xi^{k_0-1} L(v, \mu; y)$, for $(v, \mu) \in B(k_0), y \binom{1}{1}(v, \mu) \leq y \leq b(v, \mu)$ or $a(v, \mu) \leq y \leq y \binom{2}{1}(v, \mu)$ we have

$$(\star) \quad \begin{aligned} & \frac{\partial L_{k_0}}{\partial y} \left(v, \mu, y \binom{1}{1}(v, \mu) \right) \\ &= -\xi^{k_0-1} \alpha J \left(v, \mu; y \binom{1}{1}(v, \mu) \right) \left(y \binom{1}{1}(v, \mu) - (1 - \delta) \right)^{\alpha-1} \\ & \quad \left[1 + \frac{y \binom{1}{1}(v, \mu) - (1 - \delta)}{\alpha J \left(v, \mu, y \binom{1}{1}(v, \mu) \right)} \frac{\partial J}{\partial y} \left(v, \mu, y \binom{1}{1}(v, \mu) \right) \right] \end{aligned}$$

For $y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)(v, \mu)$ we have : $\mu - J\left(v, \mu, y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right)\left(y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) - (1 - \delta)\right)^\alpha = \xi^{-k_0}$ and $1 - \delta < y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)(v, \mu) < 1 - \delta + A^1(v)\mu^{1/\alpha}$.

Since $\xi^{-(k_0-1)}(1 - \delta) \leq \mu \leq \xi^{-(k_0-1)}$, we obtain

$$\xi^{-(\frac{k_0-1}{\alpha})}(1 - \delta)^{1/\alpha} \leq \mu^{1/\alpha} \leq \xi^{-(\frac{k_0-1}{\alpha})}$$

and hence $(\mu^{1/\alpha})^{-1} \geq \xi^{\frac{k_0-1}{\alpha}}$.

Therefore

$$\left| \alpha \xi^{k_0-1} J\left(v, \mu; y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)\right) \left(y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) - (1 - \delta)\right)^{\alpha-1} \right| > \alpha (A^1(v))^{-1} \xi^{\frac{k_0-1}{\alpha}} (1 - \delta - \xi^{-1}).$$

Using this fact in equation (*) the result follows for $y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)(v, \mu)$. The proof for $\left| \frac{\partial L_{k_0}}{\partial y} \left(v, \mu, y\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)(v, \mu)\right) \right|$ is analogous. ■

COROLLARY 1. – For $(v, \mu) \in B(k_0)$ and $y \in D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)$, $j \geq 1$, we have that

$$\left| \frac{\partial L_{k_0}}{\partial y}(v, \mu, y) \right| \geq \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0, \text{ for } y \in D\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)(v, \mu),$$

and that

$$\left| \frac{\partial L_{k_0}}{\partial y}(v, \mu; y) \right| \geq \lambda_1 \cdots \lambda_{j-1} \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0, \text{ for } y \in D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)$$

and any $j \geq 2$.

2.4. Associated to $\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)$ we next define the one-dimensional map

$$g\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu, \cdot) : D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu) \rightarrow [1 - \delta, 1] \text{ by } g\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu; y) = L_{k_0+j}(v, \mu; y).$$

Applying Corollary 1 we have that

$$\left| \frac{\partial g\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)}{\partial y}(v, \mu, y) \right| \geq \xi \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0 = P_1, \text{ for } y \in D\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)(v, \mu)$$

and that

$$\left| \frac{\partial g\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)}{\partial y}(v, \mu; y) \right| \geq \xi^j \lambda_1 \cdots \lambda_{j-1} \xi^{\frac{k_0-n_0-1}{\alpha}} Q_0 = P_j, \text{ for } y \in D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(v, \mu)$$

any $j \geq 2$.

From these estimates we get that the maps $g\binom{i}{j}(v, \mu; y)$, $i = 1, 2, j \geq 1$, are C^∞ -expanding diffeomorphisms onto their images (that are $[1 - \delta, 1]$). Moreover, for $i = 1$ all the maps $g\binom{1}{j}(v, \mu)$ reverse orientation, and for $i = 2$ all the maps $g\binom{2}{j}(v, \mu)$ preserve orientation.

Now given any sequence of two symbols, $\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots\right)$, let us define a sequence of nested sets and maps:

$$D\binom{i_0}{j_0}(v, \mu) \supset D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu) \supset \dots \supset D\left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r}\right)(v, \mu) \supset \dots$$

and

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu; \cdot), \dots, g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots, \binom{1_r}{j_r}\right)(v, \mu; \cdot), \dots$$

as follows:

$$D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu) = \left\{ y \in D\binom{i_0}{j_0}(v, \mu) : g\binom{i_0}{j_0}(v, \mu; y) \in D\binom{i_1}{j_1}(v, \mu) \right\}.$$

For $D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu) \neq \emptyset$ we associate a map

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu; \cdot) : D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu) \rightarrow [1 - \delta, 1]$$

defined by

$$g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}\right)(v, \mu; y) = g\binom{i_1}{j_1}\left(v, \mu, g\binom{i_0}{j_0}(v, \mu; y)\right).$$

For $r \geq 2$ and $D\left(\binom{i_0}{j_0}, \dots, \binom{i_{r-1}}{j_{r-1}}\right)(v, \mu) \neq \emptyset$, we define

$$D\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots, \binom{1_r}{j_r}\right)(v, \mu) = \left\{ y \in D\left(\binom{i_0}{j_0}, \dots, \binom{i_{r-1}}{j_{r-1}}\right)(v, \mu) / \right. \\ \left. g\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots, \binom{i_{r-1}}{j_{r-1}}\right)(v, \mu; y) \in D\binom{1_r}{j_r}(v, \mu) \right\}.$$

Associated to those $D\left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r}\right)(v, \mu)$ that are non-empty define the map

$$g\left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r}\right)(v, \mu; \cdot) : D\left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r}\right)(v, \mu) \rightarrow [1 - \delta, 1]$$

by

$$g\left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r}\right)(v, \mu; y) = g\binom{1_r}{j_r}\left(v, \mu, g\left(\binom{i_0}{j_0}, \dots, \binom{i_{r-1}}{j_{r-1}}\right)(v, \mu; y)\right).$$

Remark 1. – Given any finite set of two symbols, $\left\{ \binom{i_0}{j_0}, \dots, \binom{1_r}{j_r} \right\}$, such that $j_k \geq 1$, for $k = 0, 1, \dots, r$, by Corollary 1 we have that:

$$\left| \frac{\partial}{\partial y} \left(g \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) \right) (v, \mu; y) \right| \geq P_{j_0} \cdots P_{j_r},$$

any $y \in D \left(\binom{i_0}{j_0}, \dots, \binom{1_r}{j_r} \right) (v, \mu)$. From this inequality we conclude

$$\left| D \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu) \right| \leq (P_{j_0} \cdots P_{j_{r-1}})^{-1} \left| D \left(\binom{i_r}{j_r} \right) (v, \mu) \right|$$

and hence

$$\sum_{\substack{(i_0, j_0) \\ j_0 \geq 1}} \left(\sum_{\substack{(i_1, j_1) \\ j_1 \geq 1}} \left(\cdots \left(\sum_{\substack{(i_r, j_r) \\ j_r \geq 1}} \left| D \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu) \right| \right) \cdots \right) \right) \leq \delta \cdot \left(\frac{2}{P_1(1 - \xi^{-1})} \right)^r;$$

that is, for any $(v, \mu) \in B(k_0)$ we have :

COROLLARY 2. – *The set of points*

$$y \in \left(I_1(v, \mu) \setminus D \left(\binom{1}{0} \right) (v, \mu) \right) \cup \left(I_2(v, \mu) \setminus D \left(\binom{2}{0} \right) (v, \mu) \right)$$

that satisfy

(i) $L_i(v, \mu; y)$ is defined, all $i \geq 1$, and

(ii) there is no $i_0 \in \mathbf{N}$ such that $L_{i_0}(v, \mu; y) \in D \left(\binom{1}{0} \right) (v, \mu) \cup D \left(\binom{2}{0} \right) (v, \mu)$.

is a hyperbolic set of zero Lebesgue measure. ■

Remark 2. – Let denote the set above by $C \left(\binom{1}{1}, \binom{2}{1} \right) (v, \mu)$. As a consequence we obtain that its closure is a Cantor set of zero Lebesgue measure.

2.5. Let us now consider any sequence of two symbols $\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots \right)$, where $i_k = 1, 2$ and $j_k \geq 1$, all $k \in \mathbf{N}$.

Let

$$z_r(v, \mu) = z \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu), \quad y_r(v, \mu) = y \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu)$$

denote the boundary points of the interval $D \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu)$ defined, respectively, by the conditions

$$\Delta_r(v, \mu, z_r(v, \mu)) = g \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu; z \left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r} \right) (v, \mu)) = 1 - \delta$$

and

$$\Delta_r(v, \mu, y_r(v, \mu)) = g\left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r}\right)\left(v, \mu; y\left(\binom{i_0}{j_0}, \dots, \binom{i_r}{j_r}\right)(v, \mu)\right) = 1$$

From these relations we obtain

$$\begin{aligned}\frac{\partial z_r}{\partial v}(v, \mu) &= \frac{-\frac{\partial \Delta_r}{\partial v}(v, \mu, z_r(v, \mu))}{\frac{\partial \Delta_r}{\partial y}(v, \mu, z_r(v, \mu))} \\ \frac{\partial z_r}{\partial \mu}(v, \mu) &= \frac{-\frac{\partial \Delta_r}{\partial \mu}(v, \mu, z_r(v, \mu))}{\frac{\partial \Delta_r}{\partial y}(v, \mu, z_r(v, \mu))}\end{aligned}$$

Let us compute inductively the derivatives in the right-hand side.

Since $\Delta_r(v, \mu; y) = g\left(\binom{i_r}{j_r}\right)(v, \mu; \Delta_{r-1}(v, \mu; y))$, we have

$$\begin{aligned}\frac{\partial \Delta_r}{\partial v}(v, \mu; y) &= \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial v}(v, \mu, \Delta_{r-1}(v, \mu; y)) \\ &\quad + \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial y}(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial \Delta_{r-1}}{\partial v}(v, \mu; y) = \\ &= \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial v}(y, \mu, \Delta_{r-1}(v, \mu; y)) \\ &\quad + \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial v}(v, \mu, \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial g\left(\binom{i_{r-1}}{j_{r-1}}\right)}{\partial v}(y, \mu, \Delta_{r-2}) \\ &\quad + \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial y}(v, \mu, \Delta_{r-1}(v, \mu; y)) \cdot \frac{\partial g\left(\binom{i_{r-1}}{j_{r-1}}\right)}{\partial y}(v, \mu, \Delta_{r-2}) \\ &\quad \cdot \frac{\partial g\left(\binom{i_{r-3}}{j_{r-3}}\right)}{\partial v}(v, \mu; \Delta_{r-3}(v, \mu; y)) \\ &\quad + \dots + \frac{\partial g\left(\binom{i_r}{j_r}\right)}{\partial y}(v, \mu; \Delta_{r-1}(v, \mu; y)) \\ &\quad \dots \frac{\partial g\left(\binom{i_1}{j_1}\right)}{\partial y}(v, \mu; \Delta_0(v, \mu; y)) \cdot \frac{\partial \Delta_0}{\partial v}(v, \mu; y)\end{aligned}$$

We have a similar relation for $\frac{\partial \Delta_r}{\partial \mu}(v, \mu; y)$ by replacing $\frac{\partial}{\partial \mu}$ for $\frac{\partial}{\partial v}$ wherever it corresponds in the above formulas.

The other derivative yields

$$\frac{\partial \Delta_r}{\partial y}(v, \mu; y) = \frac{\partial g \begin{pmatrix} i_r \\ j_r \end{pmatrix}}{\partial y}(v, \mu; \Delta_{r-1}(v, \mu; y)) \cdots \frac{\partial g \begin{pmatrix} i_0 \\ j_0 \end{pmatrix}}{\partial y}(v, \mu; y).$$

Denoting by g_r the map $g \begin{pmatrix} i_r \\ j_r \end{pmatrix}$, we have:

$$\frac{\partial z_r}{\partial v}(v, \mu) = \frac{\left\{ - \left[\frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) + \cdots + \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_1}{\partial y}(v, \mu; \Delta_0(z_r)) \cdot \frac{\partial \Delta_0}{\partial v}(v, \mu; z_r) \right] \right\}}{\frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_0}{\partial y}(v, \mu; z_r)}$$

and

$$\frac{\partial z_r}{\partial \mu}(v, \mu) = \frac{\left\{ - \left[\frac{\partial g_r}{\partial \mu}(v, \mu; \Delta_{r-1}(z_r)) + \cdots + \frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_1}{\partial y}(v, \mu; \Delta_0(z_r)) \frac{\partial \Delta_0}{\partial \mu}(v, \mu; z_r) \right] \right\}}{\frac{\partial g_r}{\partial y}(v, \mu; \Delta_{r-1}(z_r)) \cdots \frac{\partial g_0}{\partial y}(v, \mu; z_r)}.$$

Now, for any $\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}$, we have

$$\left| \frac{\frac{\partial g \begin{pmatrix} i_0 \\ j_0 \end{pmatrix}}{\partial v}(v, \mu; y)}{\frac{\partial g \begin{pmatrix} i_0 \\ j_0 \end{pmatrix}}{\partial y}(v, \mu; y)} \right| = \left| \frac{\frac{\partial L}{\partial v}(v, \mu; y)}{\frac{\partial L}{\partial y}(v, \mu; y)} \right|$$

and

$$\left| \frac{\frac{\partial g \begin{pmatrix} i_0 \\ j_0 \end{pmatrix}}{\partial \mu}(v, \mu; y)}{\frac{\partial g \begin{pmatrix} i_0 \\ j_0 \end{pmatrix}}{\partial y}(v, \mu; y)} \right| = \left| \frac{\frac{\partial L}{\partial \mu}(v, \mu; y)}{\frac{\partial L}{\partial y}(v, \mu; y)} \right|.$$

We note that the sequence $(z_r(v, \mu))$ converges uniformly in the C^0 -topology to

$$z_\infty(v, \mu) = z\left(\binom{i_0}{j_0}, \binom{i_1}{j_1}, \dots\right)(v, \mu)$$

i.e.,

$$\lim_{r \rightarrow \infty} \sup\{|z_\infty(v, \mu) - z_r(v, \mu)|; (v, \mu) \in B(k_0)\} = 0.$$

From this fact and the above computation for the derivatives of the maps $z_r(v, \mu)$, and since all the $g\left(\binom{i_r}{j_r}\right)$, $j_r \geq 1$ are C^∞ -diffeomorphisms, after a cumbersome computation, we obtain

LEMMA 7. – *The sequence $(z_r(v, \mu))$ satisfies the following property: Given $\varepsilon > 0$ there is an $r_0 \in \mathbf{N}$ such that*

$$\sup\{|z_{r+p}(v, \mu) - z_r(v, \mu)|, \left\|\frac{\partial z_{r+p}}{\partial v}(v, \mu) - \frac{\partial z_r}{\partial v}(v, \mu)\right\|,$$

$$\left\|\frac{\partial z_{r+p}}{\partial \mu}(v, \mu) - \frac{\partial z_r}{\partial \mu}(v, \mu)\right\|; (v, \mu) \in B(k_0)\} < \varepsilon \text{ for } r \geq r_0, \quad p \in \mathbf{N};$$

that is, the sequence $(z_r(v, \mu))$ is a Cauchy sequence of maps in the uniform C^1 -topology. ■

In particular we have that the map $(v, \mu) \mapsto z_\infty(v, \mu)$ is a C^1 -map on $B(k_0)$.

Let us now denote by

$$G(v, \mu, \cdot) : \cup_{i=1}^2 \left(\cup_{j \geq 1} D\left(\binom{i}{j}\right)(v, \mu) \right) \rightarrow [1 - \delta, 1]$$

the map defined by $G(v, \mu, y) = g\left(\binom{i}{j}\right)(v, \mu, y)$, for $y \in D\left(\binom{i}{j}\right)(v, \mu)$.

Let

$$C\left(\binom{1}{1}, \binom{2}{1}\right)(v, \mu)$$

denote the set of points $y \in \left[y\binom{1}{1}(v, \mu), y\binom{2}{1}(v, \mu)\right]$ such that it is defined $G_k(v, \mu, y)(G_{k+1}(v, \mu, y) = G(v, \mu, G_k(v, \mu, y)), G_1(v, \mu, y) = G(v, \mu, y))$ for all $k \in \mathbf{N}$ and $G_k(v, \mu, y) \in \left[y\binom{1}{1}(v, \mu), y\binom{2}{1}(v, \mu)\right]$.

Associated with any point $y \in C\left(\binom{1}{1}, \binom{2}{1}\right)(v, \mu)$ we may define a sequence $\Gamma(v, \mu) : \mathbf{N} \rightarrow \left\{\binom{i}{j}; i = 1, 2; j \geq 1\right\}$ by

$$\Gamma(v, \mu)(k) = \binom{i_s}{j_s} \iff G_k(v, \mu)(y) \in D\left(\binom{i_s}{j_s}\right)(v, \mu).$$

This defines a map $C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)(v, \mu) \xrightarrow{\Gamma(v, \mu)} \Sigma_1$,

$$\Sigma_1 = \left\{ \Gamma : \mathbf{N} \rightarrow \left\{ \begin{pmatrix} i \\ j \end{pmatrix}; i = 1, 2; j \geq 1 \right\} \right\}$$

which is, as usual, a homeomorphism and satisfies

$$\Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu),$$

where $\Sigma_1 \xrightarrow{\sigma_1} \Sigma_1$ denotes the shift map $\sigma_1(\Gamma)(k) = \Gamma(k+1)$.

For $\Gamma \in \Sigma_1$ we denote $p_\Gamma(v, \mu) = (\Gamma(v, \mu))^{-1}(\Gamma)$. As in Lemma 7 we may prove the following:

COROLLARY 3. – *The map $B(k_0) \xrightarrow{p_\Gamma} [1 - \delta, 1], (v, \mu) \mapsto p_\Gamma(v, \mu)$ is C^1 .* ■

We observe that the closure of the set $C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu)$ contains the points $b(v, \mu), a(v, \mu)$ and all their preimages under the map $G(v, \mu, \cdot)$ which are contained in the interval $\left[y\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)(v, \mu), y\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu)\right]$.

Denoting by $s(v, \mu)$ any of these preimages it is clear that the map $B(k_0) \rightarrow [1 - \delta, 1], (v, \mu) \mapsto s(v, \mu)$ is a C^1 map and can be approximated, in the C^1 -uniform topology, by a sequence of maps $z\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \dots, \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v, \mu)$ (or $y\left(\begin{pmatrix} i_0 \\ j_0 \end{pmatrix}, \dots, \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v, \mu)$) as in lemma 5.

In this sense we will say that the closure of the set $C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu)$ is a C^1 -Cantor set of Lebesgue measure zero for any $(v, \mu) \in B(k_0)$.

2.6. Let us now consider the surface

$$S_0 = \{(v, \mu; \xi^{k_0-1}\mu); (v, \mu) \in B(k_0)\} \subset \mathcal{U} \times [1 - \delta, 1].$$

Since S_0 is transversal to $Y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right) = \left\{ (v, \mu; y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v, \mu); (v, \mu) \in B(k_0) \right\}$, we have that the intersection $S_0 \cap Y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)$ defines a C^1 -surface, $\bar{Y}\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)$, parametrized by

$$\left\{ \left(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v), \xi^{k_0-1}C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v) \right); \|v - v_0\| \leq r_0 \right\}.$$

This defines a C^1 -map $C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right) : V \rightarrow [0, \mu_0], v \mapsto C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)$ that satisfies

$$G_0\left(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)\right)\left(y\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)\left(v, C\left(\begin{pmatrix} i \\ j \end{pmatrix}\right)(v)\right)\right) = 1.$$

This implies that the vector field $X\begin{pmatrix} i \\ j \end{pmatrix}(v)$, associated to the point $(v, C\begin{pmatrix} i \\ j \end{pmatrix}(v)) \in B(k_0) \subset \mathcal{U}$, will satisfy the homoclinic condition

$$\gamma_0\left(\sigma_0\left(X\begin{pmatrix} i \\ j \end{pmatrix}(v)\right)\right) \subset W^s\left(\sigma_1\left(X\begin{pmatrix} i \\ j \end{pmatrix}(v)\right)\right).$$

The same will apply to the intersection $S_0 \cap Z\begin{pmatrix} i \\ j \end{pmatrix}$ where

$$Z\begin{pmatrix} i \\ j \end{pmatrix} = \left\{ \left(v, \mu; Z\begin{pmatrix} i \\ j \end{pmatrix}(v, \mu) \right); (v, \mu) \in B(k_0) \right\}.$$

Next we consider

$$\begin{aligned} C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) &= \left\{ \left(v, \mu; C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)(v, \mu) \right); (v, \mu) \in B(k_0) \right\} \\ &= \{(v, \mu; p_\Gamma(v, \mu)); (v, \mu) \in B(k_0), \Gamma \in \Sigma_1\}. \end{aligned}$$

For any given C^1 -surface $\{(v, \mu; p_\Gamma(v, \mu)); (v, \mu) \in B(k_0)\} = P_\Gamma$, we have that P_Γ is transversal to S_0 and hence the intersection $S_0 \cap P_\Gamma$ will define a C^1 -surface, C_Γ , parametrized by $\{(v, C_\Gamma(v); P_\Gamma(v, C_\Gamma(v))); v \in V\}$. We denote by $X_\Gamma(v)$ the vector field associated to $(v, C_\Gamma(v)) \in B(k_0) \subset \mathcal{U}$. This vector field must satisfy one of the following conditions:

(i) the point $p_\Gamma(v, C_\Gamma(v))$ represents a periodic point of the map $G(v, C_\Gamma(v))$. In this case denote by $\sigma(p_\Gamma(v, C_\Gamma(v)))$ the hyperbolic periodic orbit of the vector field $X_\Gamma(v)$ associated to $p_\Gamma(v, C_\Gamma(v))$. Under these conditions we must have $\gamma_0(\sigma_0(X_\Gamma(v))) \subset W^s(\sigma(p_\Gamma(v, C_\Gamma(v))))$, that is, the vector field $X_\Gamma(v)$ presents a contracting singular cycle or

(ii) the point $p_\Gamma(v, C_\Gamma(v))$ has recurrent behavior with respect to the set $C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)(v, C_\Gamma(v))$ under the map $G(v, C_\Gamma(v))$. In this case the trajectory $\gamma_0(\sigma_0(X_\Gamma(v)))$ has recurrent behavior in the neighborhood U ; or

(iii) the point $p_\Gamma(v, C_\Gamma(v))$ is eventually periodic under the map $G(v, C_\Gamma(v), \cdot)$ (that is there is $s \in \mathbb{N}$ such that $G_{s_0}(v, C_\Gamma(v), p_\Gamma(v, C_\Gamma(v)))$ is a periodic point of the map $G(v, C_\Gamma(v), \cdot)$). In this case the situation for the vector field $X_\Gamma(v)$ is analogous to (i) above.

Now take any preimage, $s(v, \mu)$, of the points $b(v, \mu)$ or $a(v, \mu)$, in the closure of the set $C\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)(v, \mu)$. Since the C^1 surface $S = \{(v, \mu, s(v, \mu)); (v, \mu) \in B(k_0)\}$ is transversal to S_0 then the intersection $S \cap S_0$ define a C^1 surface S_b (resp S_a) parametrized by $\{(v, \bar{b}(v), s(v, \bar{b}(v))); v \in V\}$ (resp. $\{(v, \bar{a}(v), s(v, \bar{a}(v))); v \in V\}$). Let denote by $X_{\bar{b}}(v)$ (resp. $X_{\bar{a}}(v)$) the vector field associated to $(v, \bar{b}) \in B(k_0)$ (resp. $(v, \bar{a}) \in B(k_0)$). This vector field satisfies that :

$$\gamma_0(\sigma_0(X_{\bar{b}}(v))) \subset W^s(\sigma_1(X_{\bar{b}}(v))).$$

(resp. $\gamma_0(\sigma_0(X_{\bar{a}}(v))) \subset W^s(\sigma_1(X_{\bar{a}}(v)))$).

2.7.

In general let us consider the set of bisequences

$$\Sigma_0 = \left\{ \Gamma : \mathbf{N} \rightarrow \left\{ \binom{i}{j}, i = 1, 2; j \geq 0 \right\} \right\}$$

and the map

$$G(v, \mu, \cdot) : \bigcup_{i=1}^2 \left(\bigcup_{j \geq 0} D \binom{i}{j} (v, \mu) \right) \rightarrow [1 - \delta, 1]$$

given by

$$G(v, \mu, y) = g \binom{i}{j} (v, \mu; y), y \in D \binom{i}{j} (v, \mu)$$

and $(v, \mu) \in B(k_0)$.

Denote by $M(v, \mu)$ the set of points $y \in [1 - \delta, 1]$ such that it is defined $G_k(v, \mu, y)$ for all $k \in \mathbf{N}$.

Associated with any $y \in M(v, \mu)$ we can define a bisequence $\Gamma(v, \mu)(y) \in \Sigma_0$ by:

$$(\Gamma(v, \mu)(y))(k) = \binom{i_s}{j_s} \iff G_k(v, \mu, y) \in D \binom{i_s}{j_s} (v, \mu)$$

Clearly $\Gamma(v, \mu) : M(v, \mu) \rightarrow \Sigma_1$ is continuous and satisfies $\Gamma(v, \mu) \circ G(v, \mu) = \sigma_1 \circ \Gamma(v, \mu)$. Here $\sigma_0 : \Sigma_0 \rightarrow \Sigma_0$ is the shift map $\sigma_0(\Gamma)(k) = \Gamma(k + 1)$.

DEFINITION 3. — We will say that the bisequence $\Gamma \in \Sigma_0$ is admissible at the level (v, μ) if $\Gamma(v, \mu)^{-1}(\Gamma) \neq \emptyset$.

Remark 3. — 1) We note that $\Gamma(v, \xi^{-(k_0-1)})$ is a surjective map, for any $(v, \xi^{-(k_0-1)}) \in B(k_0)$.

2) From 1) we conclude that, given $\Gamma \in \Sigma_0$, we can find a first parameter value $\mu_\Gamma(v); \xi^{-(k_0-1)}(1 - \delta) \leq \mu_\Gamma(v) \leq \xi^{-(k_0-1)}$ such that Γ is admissible at the level (v, μ) , any $\mu \geq \mu_\Gamma(v)$ [for instance $\mu_\Gamma(v) = \xi^{-(k_0-1)}(1 - \delta)$, any $\Gamma \in \Sigma_1$].

DEFINITION 4. — Assume $(v, \mu) \in B(k_0)$ is a parameter value that satisfies $\{1 - \delta, 1\} \subset M(v, \mu)$. In this case we will call the bisequence $\sigma_0(\Gamma(v, \mu))(1) = \sigma_0(\Gamma(v, \mu))(1 - \delta)$ the itinerary of the map $G(v, \mu, \cdot)$, and we will denote it by $\Theta(v, \mu)$. We will say a bisequence $\Gamma \in \Sigma_0$ is realizable if there is a parameter value $(v, \mu) \in B(k_0)$ such that $\Theta(v, \mu) = \Gamma$. We will denote the bisequence $\Gamma(v, \mu)(1)$ (resp. $\Gamma(v, \mu)(1 - \delta)$) by $\Gamma_1(v, \mu)$ (resp. $\Gamma_{1-\delta}(v, \mu)$).

Remark 4. – The only bisequence that satisfies $\Gamma = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots\right)$ and is realizable is the bisequence $\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots\right)$. From here we conclude that there are bisequences which are not realizable.

Denote by $Per(\sigma_0) \subset \Sigma_0$ the set of all periodic bisequences $\Gamma \in \Sigma_0$. It is clear that $Per(\sigma_0)$ is a dense subset of Σ_0 . Let $\Sigma_2 \subset Per(\sigma_0)$ be the set of all periodic bisequences $\Gamma \in (Per(\sigma_0) \setminus \Sigma_1)$ such that $\Gamma = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots\right)$ or $\Gamma = \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \dots\right)$.

Given $\Gamma \in \Sigma_2$ we let Γ_0 denote its period (i.e., $\Gamma = (\Gamma_0, \Gamma_0, \Gamma_0, \dots)$). We have the following proposition:

PROPOSITION 1. – *For those $\Gamma \in \Sigma_2$ which satisfy that $\sigma_0(\Gamma)$ is realizable and the number of $\begin{pmatrix} 1 \\ j \end{pmatrix}$ that appears in Γ_0 is odd, we can find values of the parameter*

$\mu_{\Gamma_0}(v) < \mu_{\Gamma_0}^f(v) < \mu_{2\Gamma_0}(v)$ such that:

i) *for any $(v, \mu) \in B(k_0)$, $\mu_{\Gamma_0}(v) < \mu < \mu_{\Gamma_0}^f(v)$, the associated one-dimensional map $G(v, \mu, \cdot)$ has an attracting, hyperbolic, periodic orbit whose period is $\sharp(\Gamma_0)$. Moreover, one point of this orbit is contained in $D(\sigma_0^k(\Gamma_0))(v, \mu)$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$.*

ii) *for any $(v, \mu) \in B(k_0)$, $\mu_{\Gamma_0}^f(v) < \mu < \mu_{2\Gamma_0}(v)$, the associated one-dimensional map $G(v, \mu, \cdot)$ has an attracting, hyperbolic, periodic orbit whose period is $2\sharp(\Gamma_0)$. Moreover, two points of this orbit are contained in $D(\sigma_0^k(\Gamma_0))(v, \mu)$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$.*

iii) *for $(v, \mu_{\Gamma_0}(v)) \in B(k_0)$ we have that $D(\sigma_1^k(\Gamma_0))(v, \mu)$ is a single point, and the associated one-dimensional map $G(v, \mu, \cdot)$ satisfies*

$$G_{\sharp(\Gamma_0)}(v, \mu)(D(\sigma_0^k(\Gamma_0))(v, \mu)) = D(\sigma_0^k(\Gamma_0))(v, \mu),$$

any $0 \leq k \leq \sharp(\Gamma_0) - 1$, $\mu = \mu_{\Gamma_0}(v)$.

iv) *for $(v, \mu_{\Gamma_0}^f(v)) \in B(k_0)$ the associated one-dimensional map $G(v, \mu, \cdot)$ has a flip bifurcation of the attracting periodic orbit. Moreover, one point of this orbit is contained in the interior of $D(\sigma_0^k(\Gamma_0))(v, \mu_{\Gamma_0}^f(v))$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$.*

v) *for $(v, \mu_{2\Gamma_0}(v)) \in B(k_0)$ the associated one-dimensional map $G(v, \mu_{2\Gamma_0}(v), \cdot)$ satisfies*

$$G_{\sharp(\Gamma_0)}(\partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0})) = \partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0})$$

and interchanges the points in $\partial D(\sigma_0^k(\Gamma_0))(v, \mu_{2\Gamma_0})$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$. [in particular for $\Gamma_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots\right)$. (resp. $\Gamma_0 = \left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \dots\right)$) we have that $G_{2\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1 - \delta$ (resp. $G_{2\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1$), $\mu = \mu_{2\Gamma_0}$].

vi) *for $\mu_{\Gamma_0}(v) \leq \mu \leq \mu_{2\Gamma_0}(v)$, the pre-image $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is the interval $D(\sigma_0^k(\Gamma))(v, \mu)$.*

vii) *for any $(v, \mu) \in B(k_0)$ such that $\mu > \mu_{2\Gamma_0}(v)$, we have that $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is a hyperbolic repelling fixed point of the map $G_{\sharp(\Gamma_0)}(v, \mu, \cdot)$. Moreover $D(\sigma_0^k(\Gamma))(v, \mu)$ is exactly this repelling fixed point and*

viii) *all the maps $v \rightarrow \mu_{\Gamma_0}(v)$, $v \mapsto \mu_{\Gamma_0}^f(v)$, and $v \mapsto \mu_{2\Gamma_0}(v)$ are C^1 .*

Proof. – Without loss assume $\Gamma = (\Gamma_0, \Gamma_0, \dots)$ where $\Gamma_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Later we will make some comments on the general case.

In this situation $\mu_{\Gamma_0} = \xi^{-(k_0-1)}(1-\delta)$. For $(v, \mu) \in B(k_0)$ and $y \in D\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v, \mu)$ define :

$$E(v, \mu; y) = G(v, \mu, y) - y.$$

We have ; $E(v, \xi^{-(k_0-1)}(1-\delta), 1-\delta) = 0$ and

$$\left. \frac{\partial E}{\partial y}(v, \mu; y) \right|_{\substack{y=1-\delta \\ \mu=\xi^{-(k_0-1)}(1-\delta)}} = -1$$

By applying the implicit function theorem we can find a C^2 -map $y = y(v, \mu) \in D\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ such that $E(v, \mu; y(v, \mu)) = 0$.

That is, $G(v, \mu, y(v, \mu)) = y(v, \mu)$.

For fixed v such that $(v, \mu) \in B(k_0)$ we have

$$\frac{\partial y}{\partial \mu}(v, \mu) = \frac{\frac{\partial G}{\partial \mu}(v, \mu; y(v, \mu))}{1 - \frac{\partial G}{\partial y}(v, \mu; y(v, \mu))}.$$

Since $\frac{\partial G}{\partial \mu}(v, \mu; y) > 0$; $(v, \mu) \in B(k_0)$, $y \in D\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v, \mu)$, and $\frac{\partial G}{\partial y}(v, \mu; y) \leq 0$, for $(v, \mu) \in B(k_0)$, $y \in D\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v, \mu)$, we conclude that $\frac{\partial y}{\partial \mu}(v, \mu) > 0$, $(v, \mu) \in B(k_0)$ and

$$\left. \frac{\partial y}{\partial \mu}(v, \mu) \right|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = \xi^{k_0-1}$$

Since

$$\left. \frac{\partial G}{\partial y}(v, \mu, y(v, \mu)) \right|_{\mu=\xi^{-(k_0-1)}(1-\delta)} = 0$$

we conclude , for μ near $\xi^{-(k_0-1)}(1-\delta)$ such that $(v, \mu) \in B(k_0)$ that $y = y(v, \mu)$ is an attracting fixed point for the map $G(v, \mu, \cdot)$.

Now a cumbersome computation will show that

$$\left. \frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial y}(G(v, \mu, y)) \right) \right|_{y=y(v, \mu)} \leq 0.$$

Moreover, for $\mu > \xi^{-(k_0-1)}(1-\delta)$ we have :

$$\frac{\partial G}{\partial \mu}(v, \mu, y(v, \mu)) = -\frac{\xi^{k_0-1}\mu - y(v, \mu)}{J(v, \mu, y(v, \mu))} \left[\frac{\partial J}{\partial y}(v, \mu, y) + \frac{\alpha J^{1+\frac{1}{\alpha}} \xi^{\frac{k_0-1}{\alpha}}}{\xi^{k_0-1}\mu - y^{\frac{1}{\alpha}}} \right].$$

So, there exist a unique value $\mu = \mu_{\Gamma_0}^f(v)$ such that

$$\frac{\partial G}{\partial y}(v, \mu, y) \Big|_{\mu=\mu_{\Gamma_0}^f(v)} = -1.$$

Now it is not hard to see that :

$$\frac{\partial^3}{\partial y^3}(G(v, \mu, y(v, \mu))) \Big|_{\substack{y=y(v, \mu_{\Gamma_0}^f(v)) \\ \mu=\mu_{\Gamma_0}^f(v)}} < 0.$$

Under these circumstances we may consider the C^2 -map

$$H(v, \mu; y) = \begin{cases} \frac{G_2(v, \mu, y) - y}{y - y(v, \mu)}, & y \neq y(v, \mu) \\ \frac{\partial}{\partial y}(G_2(v, \mu, y)) - 1, & y = y(v, \mu) \end{cases}.$$

Clearly $H(v, \mu_{\Gamma_0}^f(v), y(v, \mu_{\Gamma_0}^f(v))) = 0$ and

$$\frac{\partial H}{\partial \mu}(v, \mu; y) \Big|_{\substack{\mu=\mu_{\Gamma_0}^f(v) \\ y=y(v, \mu_{\Gamma_0}^f(v))}} = \frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial y}(G_2(v, \mu, y)) \right) \Big|_{\substack{\mu=\mu_{\Gamma_0}^f(v) \\ y=y(v, \mu_{\Gamma_0}^f(v))}} \neq 0.$$

In this case there is a smooth map $\mu = \mu(v, y)$ such that $H(v, \mu(v, y), y) = 0$.

For $y \neq y(v, \mu)$ we have $G_2(v, \mu, y) = y$ which is a period two point for the map $G(v, \mu, \cdot)$.

It is easy to see that

$$\frac{\partial \mu}{\partial y}(v, y) \Big|_{\substack{y=y(v, \mu) \\ \mu=\mu_{\Gamma_0}^f(v)}} = 0$$

and that

$$\frac{\partial^2 \mu}{\partial y^2} \Big|_{\substack{y=y(v, \mu) \\ \mu=\mu_{\Gamma_0}^f(v)}} > 0.$$

We note that, whenever defined, the interval $\{(v, \mu)\} \times [0, 1]$ intersects the graph of the map $\mu = \mu(v, y)$ into two points: $(v, \mu; y_1), (v, \mu; y_2)$. These two points satisfy $G(v, \mu(v, y_1), y_1) = y_2, G(v, \mu(v, y_2), y_2) = y_1$, and $y_1 \leq y(v, \mu) \leq y_2$. Since

$$\left| \frac{\partial G_2}{\partial y}(v, \mu, y(v, \mu)) \right| \geq 1,$$

for $\mu \geq \mu_{\Gamma_0}^f(v)$, and since this absolute value is equal to one only for $\mu = \mu_{\Gamma_0}^f(v)$, we have that

$$\left| \frac{\partial G_2}{\partial y}(v, \mu(v, y_2), y_2) \right| < 1,$$

any $\mu > \mu_{\Gamma_0}^f(v)$ wherever y_2 is defined.

Since the graph of the map $\mu = \mu(v, y)$ intersects transversally the graph of the map $(v, \mu) \mapsto G(v, \mu, 1 - \delta)$, their intersection defines a C^1 -map $\mu = \mu_{\Gamma_0}(v)$ and thus the proof of Proposition 1 is now complete in the case $\Gamma_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots \right)$. ■

In the general case we can proceed as follows:

Let $\Gamma_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \dots, \begin{pmatrix} i_r \\ j_r \end{pmatrix} \right)$ here $r = \sharp(\Gamma_0) - 1$, and consider

$$\begin{aligned} D\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \dots, \begin{pmatrix} i_r \\ j_r \end{pmatrix}\right)(v, \mu) \\ = D(\Gamma_0)(v, \mu) \subset D\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i_1 \\ j_1 \end{pmatrix}, \dots, \begin{pmatrix} i_{r-1} \\ j_{r-1} \end{pmatrix}\right)(v, \mu) \subset \dots \subset D\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)(v, \mu). \end{aligned}$$

Clearly we have $G_{\sharp(\Gamma_0)}(v, \mu, 1 - \delta) \in D(\Gamma_0)(v, \mu)$.

Let $\mu_{\Gamma_0}(v) = \inf\{\mu; (v, \mu) \in B(k_0), \Theta(v, \mu) = \sigma_0(\Gamma)\}$. For $\mu = \mu_{\Gamma_0}(v)$ we must have $G_{\sharp(\Gamma_0)}(v, \mu, 1 - \delta) = 1 - \delta$ (and therefore $D(\Gamma_0)(v, \mu_{\Gamma_0}(v)) = 1 - \delta$).

Now we define the map $E(v, \mu, y), y \in D(\Gamma_0)(v, \mu), (v, \mu) \in B(k_0)$ such that $\mu \geq \mu_{\Gamma_0}(v)$ by:

$$E(v, \mu, y) = G_{\sharp(\Gamma_0)}(v, \mu, y) - y$$

Now the proof of the proposition 1 follows as in the previous case.

2.8.

Let $\Gamma \in \Sigma_2$ and denote by Γ_0 its period.

PROPOSITION 2. – For those $\Gamma \in \Sigma_2$ such that $\sigma_0(\Gamma)$ is realizable and the number of $\begin{pmatrix} 1 \\ j \end{pmatrix}$ that appears in Γ_0 is even, we can find values of the parameter $\mu_{\Gamma}(v) = \mu_{\Gamma_0}^{sn}(v) < \mu_{\Gamma_0}(v)$ such that:

i) for $(v, \mu_{\Gamma_0}^{sn}(v)) \in B(k_0)$, the associated one-dimensional map $G(v, \mu_{\Gamma_0}^{sn}(v), \cdot)$ has a saddle-node bifurcation whose period is $\sharp(\Gamma_0)$. Moreover, one point of this orbit is contained in the boundary of the interval $D(\sigma_0^k(\Gamma))(v, \mu)$, any $0 \leq k \leq \sharp(\Gamma_0) - 1$.

ii) for $(v, \mu) \in B(k_0)$; $\mu_{\Gamma_0}^{sn}(v) < \mu < \mu_{\Gamma_0}(v)$, the associated one-dimensional map $G(v, \mu, \cdot)$ has an attracting, hyperbolic, periodic orbit and a repelling, hyperbolic, periodic orbit contained in the interior of $D(\Gamma)(v, \mu) \cup D(\sigma_0(\Gamma))(v, \mu) \cup \dots \cup D(\sigma_0^{\sharp(\Gamma_0)-1}(\Gamma))(v, \mu)$.

Moreover one point, of any of the two periodic orbits, is contained in $D(\sigma_0^k(\Gamma))(v, \mu)$, any $0 \leq k \leq (\sharp(\Gamma_0) - 1)$.

iii) for $(v, \mu = \mu_{\Gamma_0}(v)) \in B(k_0)$, the associated one-dimensional map satisfies

$$G_{\sharp(\Gamma_0)}(v, \mu, \partial D(\sigma_0^k(\Gamma))(v, \mu)) = \partial D(\sigma_0^k(\Gamma))(v, \mu).$$

Under these circumstances the points in the boundary are fixed points for the map $G_{\sharp(\Gamma_0)}$. Note that the boundary $\partial D(\Gamma)(v, \mu)$ contains $1 - \delta$ or 1 depending on $\Gamma_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \dots \right)$ or $\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \dots \right)$, respectively.

iv) for $(v, \mu) \in B(k_0); \mu_{\Gamma_0}^{sn}(v) \leq \mu \leq \mu_{2\Gamma_0}(v)$ the pre-image $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is the interval $D(\sigma_0^k(\Gamma))(v, \mu)$.

v) for any $(v, \mu) \in B(k_0)$ such that $\mu > \mu_{\Gamma_0}(v)$ we have that $\Gamma(v, \mu)^{-1}(\sigma_0^k(\Gamma))$ is a hyperbolic, repelling fixed point of the map $G^{\sharp(\Gamma_0)}(v, \mu)(\cdot)$. Moreover $D(\sigma_0^k(\Gamma))(v, \mu)$ is exactly this repelling fixed point.

vi) The maps $V \longrightarrow [1 - \delta, 1]; v \longmapsto \mu_{\Gamma_0}^{sn}(v)$, and $v \longmapsto \mu_{\Gamma_0}(v)$ are C^1 .

Proof. – Assume $\Gamma_0 = \left(\binom{2}{0}, \binom{2}{0}, \dots \right)$. Later we will comments on the general case.

In this situation $\mu_{\Gamma_0}(v) = \xi^{-(k_0-1)}$.

For $(v, \mu) \in B(k_0)$ and $y \in D\left(\binom{2}{0}\right)(v, \mu)$ define the map: $E(v, \mu; y) = G(v, \mu; y) - y$.

We have :

$$E(v, \mu; y) = \xi^{k_0-1}[\mu - K(v, \mu; y)(1 - y)^\alpha] - y.$$

and, hence, $\frac{\partial E}{\partial \mu}(v, \mu; y)|_{y=1} = \xi^{k_0-1} \neq 0$, for any $(v, \mu) \in B(k_0)$. Therefore, by the implicit function theorem we obtain a C^1 -map, twice differentiable in the y -variable $\mu = \mu(v, y)$ such that: We solve the equation $E(v, \mu; y) = 0$ for $(v, \mu) \in B(k_0), y \in D\left(\binom{2}{0}\right)(v, \mu)$ if and only if $\mu = \mu(v, y)$.

From the relation $E(v, \mu(v, y); y) = 0$ we obtain

$$\frac{\partial \mu}{\partial y}(v, y) = \frac{\xi^{k_0-1} \left[-\frac{\partial K}{\partial y}(v, \mu; y)(1 - y)^\alpha - \alpha K(v, \mu; y)(1 - y)^{\alpha-1} \right] - 1}{\xi^{k_0} - \frac{\partial K}{\partial \mu}(v, \mu; y)(1 - y)^\alpha},$$

and from this relation we have that: $\frac{\partial \mu}{\partial y}(v, y) = 0$ if and only if

$$H(v, y) = -\frac{\partial K}{\partial y}(v, \mu(v, y); y)(1 - y)^\alpha + \alpha K(v, \mu(v, y); y)(1 - y)^{\alpha-1} - \xi^{-(k_0-1)} = 0.$$

Since $|1 - y|$ is small, $K(v, \mu; y) \neq 0$ and

$$\begin{aligned} \frac{\partial H}{\partial y}(v, y) = & (1 - y)^{\alpha-2} \left[\frac{\partial^2 K}{\partial y^2}(v, \mu(v, y); y)(1 - y)^2 + \right. \\ & \left. + 2\alpha \frac{\partial K}{\partial y}(v, \mu(v, y); y)(1 - y) - \alpha(\alpha - 1)K(v, \mu(v, y), y) \right], \end{aligned}$$

we have $\frac{\partial H}{\partial y}(v, y) \neq 0$, any (v, y) such that $H(v, y) = 0$.

Hence by the implicit function theorem we find a C^1 -map, $y = y(v)$, that simultaneously satisfies equations $E(v, \mu(v, y(v)); y(v)) = 0$ and $\frac{\partial \mu}{\partial y}(v, y(v)) = 0$.

Figure 4 shows the above relations obtained for the maps $\mu(v, y)$ and $y(v)$.

Fig. 4

Denote by $\mu_{\Gamma_0}^{sn} = \mu(v, y(v))$. For this map we have:

$$G(v, \mu_{\Gamma_0}^{sn}, y(v)) = y(v); \frac{\partial G}{\partial y}(v, \mu_{\Gamma_0}^{sn}, y(v)) \equiv 1$$

and

$$\frac{\partial^2 G}{\partial y^2}(v, \mu_{\Gamma_0}^{sn}, y) \neq 0$$

That is ; the one dimensional map $G(v, \mu_{\Gamma_0}^{sn}, \cdot)$, has a saddle-node at the point $y = y(v) \in D_0^{(2)}(v, \mu_{\Gamma_0}^{sn})$.

Now assume $(v, \mu) \in B(k_0)$ satisfies $\mu_{\Gamma_0}^{sn} < \mu < \mu_{\Gamma_0}(v)$. In this case the interval $\{(v, \mu)\} \times [1 - \delta, 1]$ intersects the graph of the map $\mu(v, y)$ into two points $(v, \mu; y_1)$ and $(v, \mu; y_2)$. These two points satisfy $G(v, \mu; y_1) = y_1$ and $G(v, \mu; y_2) = y_2$ with $y_1 < y_2$. Again, an easy computation shows $\frac{\partial G}{\partial y}(v, \mu; y_1) > 1 > \frac{\partial G}{\partial y}(v, \mu; y_2)$: that is the map $G(v, \mu; \cdot)$ has a hyperbolic, attracting periodic orbit whose period is k_0 , at $y = y_2$; and a hyperbolic repelling, fixed point at $y = y_1$.

Observe that, for $(v, \mu) \in B(k_0)$, $\mu \leq \mu_{\Gamma_0}^{sn}$, the one dimensional map $G(v, \mu \cdot)$; does not have fixed points in $D_0^{(2)}$. This complete the proof of proposition 2 in this particular case.

In the general case we can proceed as follows :

Let $\Gamma_0 = \left(\binom{2}{0}, \binom{i_1}{j_1}, \dots, \binom{i_r}{j_r} \right)$, here $r = \#(\Gamma_0) - 1$. Let us consider $D\left(\binom{2}{0}, \binom{i_1}{j_1}, \dots, \binom{i_r}{j_r}\right)(v, \mu) \subset D\left(\binom{2}{0}, \binom{i_1}{j_1}, \dots, \binom{i_{r-1}}{j_{r-1}}\right)(v, \mu) \subset \dots \subset D\left(\binom{2}{0}\right)(v, \mu)$. Clearly we have $G_{\#(\Gamma_0)}(v, \mu, 1) \in D(\Gamma)(v, \mu)$. Let $\mu_{\Gamma_0}(v) = \sup\{\mu; (v, \mu) \in B(k_0), \Theta(v, \mu) = \sigma_0(\Gamma)\}$. For $\mu = \mu_{\Gamma_0}(v)$ we must have

$G_{\# \Gamma_0}(v, \mu; 1) \equiv 1$. Now we define the map $E(v, \mu; y)$, $y \in D(\Gamma_0)(v, \mu)$, $(v, \mu) \in B(k_0)$ such that $\mu \leq \mu_{\Gamma_0}(v)$ by:

$$E(v, \mu; y) = G_{\#(\Gamma_0)}(v, \mu; y) - y.$$

Now the proof follows as in the previous case. ■

As a consequence of proposition 1 and 2 we get the following :

Remark 5. – Assume $\Gamma_1(v, \mu)$ or $\Gamma_{1-\delta}(v, \mu)$ is a periodic itinerary. In this situation the associated one dimensional map $G(v, \mu, \cdot)$ satisfies one of the following:

(i) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval which contains, in its interior, a hyperbolic, attracting periodic orbit or

(ii) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval which contains a flip or a saddle-node periodic orbit or

(iii) $D(\Gamma_1(v, \mu))(v, \mu)$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu)$) is an interval and $y = 1$ (or $y = 1 - \delta$) is an attracting periodic orbit or

(iv) $D(\Gamma_1(v, \mu))(v, \mu) = \{1\}$ (or $D(\Gamma_{1-\delta}(v, \mu))(v, \mu) = \{1 - \delta\}$).

2.9.

Let us now define an order relation among the elements of Σ_0 .

We initially define

$$\binom{1}{0} < \binom{1}{1} < \dots < \binom{1}{n} < \binom{1}{n+1} < \dots < \binom{2}{n+1} < \binom{2}{n} < \dots < \binom{2}{0}.$$

Let $\Gamma_1 \neq \Gamma_2$ be any two bisequences. Assume that

$$\left(\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1} \right) = \left(\binom{i_0^2}{j_0^2}, \dots, \binom{i_k^2}{j_k^2} \right) \text{ and that } \binom{i_{k+1}^1}{j_{k+1}^1} \neq \binom{i_{k+1}^2}{j_{k+1}^2}.$$

– If there is an even number of $\binom{1}{j}$ among $\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1}$ and $\binom{i_{k+1}^1}{j_{k+1}^1} > \binom{i_{k+1}^2}{j_{k+1}^2}$, we will say Γ_1 is greater than Γ_2 and we will denote $\Gamma_1 > \Gamma_2$.

– If there is an odd number of $\binom{1}{j}$ among $\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1}$ and $\binom{i_{k+1}^1}{j_{k+1}^1} < \binom{i_{k+1}^2}{j_{k+1}^2}$, we will say Γ_1 is greater than Γ_2 and we will denote $\Gamma_1 > \Gamma_2$.

LEMMA 8. – The map $\Gamma(v, \mu) : M(v, \mu) \rightarrow \Sigma_0$ is order-preserving.

Proof. – Let $x_1, x_2 \in M(v, \mu)$ be two points such that $x_1 \leq x_2$. If $x_1 \in D\left(\binom{i_0}{j_0}\right)(v, \mu)$

and $x_2 \in D\left(\binom{i_1}{j_1}\right)$ with $\binom{i_0}{j_0} \neq \binom{i_1}{j_1}$, the result follows.

Assume $\Gamma(v, \mu)(x_1) = \Gamma_1$, and $\Gamma(v, \mu)(x_2) = \Gamma_2$ are such that

$$\left(\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1} \right) = \left(\binom{i_0^2}{j_0^2}, \dots, \binom{i_k^2}{j_k^2} \right) \quad \text{and} \quad \binom{i_{k+1}^1}{j_{k+1}^1} \neq \binom{i_{k+1}^2}{j_{k+1}^2}.$$

If there is an even number of $\binom{i}{j}$'s among the $\left(\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1}\right)$, then the restriction of the map $G_k(v, \mu)$ to the interval that contains $[x_1, x_2]$ preserves orientation. This implies that $G_k(v, \mu)(x_1) \leq G_k(v, \mu)(x_2)$ and therefore $\binom{i_{k+1}^1}{j_{k+1}^1} < \binom{i_{k+1}^2}{j_{k+1}^2}$. By the definition of the order relation in Σ_0 this implies $\Gamma_1 < \Gamma_2$.

If there is an odd number of $\binom{i}{j}$ among the $\left(\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1}\right)$, then the restriction map $G_k(v, \mu)(\cdot)$ to the interval $D\left(\binom{i_0^1}{j_0^1}, \dots, \binom{i_k^1}{j_k^1}\right)(v, \mu)$, which contains $[x_1, x_2]$, reverses orientation. This implies that $G_k(v, \mu)(x_1) > G_k(v, \mu)(x_2)$ and therefore $\binom{i_{k+1}^1}{j_{k+1}^1} > \binom{i_{k+1}^2}{j_{k+1}^2}$. By the definition of the order relation in Σ_0 we obtain $\Gamma_1 < \Gamma_2$. ■

Let us now consider two bisequences $\Gamma_1, \Gamma_2 \in \Sigma_0$ such that $\Gamma(v, \mu)(x_1) = \Gamma_1, \Gamma(v, \mu)(x_2) = \Gamma_2$, some $x_1, x_2 \in M(v, \mu)$.

LEMMA 9. – If $\Gamma_1 < \Gamma_2$, then $x_1 < x_2$.

Proof. – The proof is easy and left to the reader. ■

Let $\Gamma \in \Sigma_0$ be any realizable sequence and denote by $\mu_\Gamma = \inf\{\mu; \Theta(v, \mu) = \Gamma\}$. Let $\Gamma_2 \in \Sigma_0$ be any admissible bisequence at the level $(v, \mu_\Gamma(v))$ such that $\Gamma_2 > \Gamma$.

LEMMA 10. – Γ_2 is realizable.

Proof. – Denote by $x_1(v, \mu) \in M(v, \mu), x_2(v, \mu) \in M(v, \mu)$ two points which satisfy $\Gamma(v, \mu)(x_1(v, \mu)) = \Gamma$ and $\Gamma(v, \mu)(x_2(v, \mu)) = \Gamma_2$. We have $x_1(v, \mu) < x_2(v, \mu)$ and $x_1(v, \mu_\Gamma(v)) = \xi^{k_0-1}\mu_\Gamma(v)$. Since $\mu \mapsto \xi^{k_0-1}\mu$ is an increasing map we can find a parameter value $\mu_2 > \mu_\Gamma(v)$ such that $x_2(v, \mu_2) = \xi^{k_0-1}\mu_2$. This implies $x_2(v, \mu_2) = \Gamma(v, \mu)(G(v, \mu_2, 1 - \delta)) = \sigma_0 \circ (\Gamma(v, \mu_2)(1 - \delta)) = \Theta(v, \mu_2)$. That is Γ_2 is realizable. ■

Remark 6. – 1) Let $\Gamma \in \Sigma_0$ be any realizable sequence and $\mu_\Gamma(v) = \inf\{\mu, \Theta(v, \mu) = \Gamma\}$. Let $\Gamma_2 \in \Sigma_0; \Gamma_2 \leq \Gamma$ be any bisequence which is not realizable for $\xi^{-(k_0-1)}(1 - \delta) \leq \mu \leq \mu_\Gamma(v)$ then Γ_2 is not realizable at all, that is there no exists $\xi^{-(k_0-1)}(1 - \delta) \leq \mu \leq \xi^{-(k_0-1)}$ such that $\Theta(v, \mu) = \Gamma_2$.

2) Assume $(v, \mu_1), (v, \mu_2) \in B(k_0)$ satisfy $\xi^{k_0-1}\mu_1 \in M(v, \mu_1), \xi^{k_0-1}\mu_2 \in M(v, \mu_2)$. If $\mu_1 < \mu_2$ then we have $\Theta(v, \mu_1) = \Gamma(v, \mu_1)(\xi^{k_0-1}\mu_1) \leq \Theta(v, \mu_2) = \Gamma(v, \mu_2)(\xi^{k_0-1}\mu_2)$

3) Assume $(v, \mu_1), (v, \mu_2) \in B(k_0)$ satisfy $\xi^{k_0-1}\mu_1 \in M(v, \mu_1), \xi^{k_0-1}\mu_2 \in M(v, \mu_2)$ and $\Theta(v, \mu_1) < \Theta(v, \mu_2)$ then we have $\mu_1 < \mu_2$.

2.10.

Let $\Gamma \in \Sigma_2$ be any periodic bisequence which is realizable.

Assume $\mu_\Gamma(v) = \inf\{\mu; \Theta(v, \mu) = \Gamma\}$.

(A) Let $\Gamma_k = \sigma_0^k(\Gamma)$, for $1 \leq k \leq \sharp(\Gamma_0) - 1$. Suppose $\Gamma_j > \Gamma$, for some j . By Lemma 21 we have that Γ_j is realizable. In fact denote by $x_j(v, \mu) \in M(v, \mu)$ a point which satisfies $\Gamma(v, \mu)(x_j(v, \mu)) = \Gamma_j$. By (2.11) we know that $D(\Gamma_j)(v, \mu)$ is a hyperbolic, repelling, fixed point of the map $G_{\sharp(\Gamma_0)}(v, \mu)$, for $\mu > \mu_{2\Gamma_0}(v)$ or $\mu > \mu_{\Gamma_0}(v)$. Since the C^1 -surface $C_{\Gamma_j} = \{(v, \mu; x_j(v, \mu))/\mu \geq \mu_{\Gamma_0}(v) \text{ or } \mu \geq \mu_{2\Gamma_0}(v), (v, \mu) \in B(k_0)\}$ is transversal to $S_0 = \{(v, \mu; \xi^{k_0-1}\mu)/(v, \mu \in B(k_0))\}$ we have that $S_0 \cap C_{\Gamma_j}$, define a C^1 surface contained in $\mathcal{U} \times [1 - \delta, 1]$ and parametrized by $\{(v, C_{\Gamma_j}(v), x_j(v, C_{\Gamma_j}(v))) ; v \in V\}$.

Let us denote by $X_{\Gamma_j}(v)$ the vector field associated to $(v, C_{\Gamma_j}(v)) \in B(k_0)$.

Let $\sigma(x_j(v, C_{\Gamma_j}(v))) \subset U$ be the hyperbolic, periodic orbit associated to the point $x_j(v, C_{\Gamma_j}(v))$. We have

$$\gamma_0(\sigma_0(X_{\Gamma_j}(v))) \subset W^s(\sigma(x_j(v, C_{\Gamma_j}(v)))),$$

that is, the associated vector field $X_{\Gamma_j}(v)$ represents a contracting singular cycle.

(B) Let $\mathcal{X} \in \sum_0, \mathcal{X} > \Gamma$ be any admissible bisequence, at the level $(v, \mu_{\Gamma}(v))$, such that $\sigma_0^k(\mathcal{X}) = \Gamma$, some $k \in \mathbf{N}$.

Let us denote by $x_{\mathcal{X}}(v, \mu) \in M(v, \mu)$ a point which satisfies $\Gamma(v, \mu)(x_{\mathcal{X}}(v, \mu)) = \mathcal{X}$. We have: $\sigma_0^k \circ \Gamma(v, \mu)(x_{\mathcal{X}}(v, \mu)) = \sigma_0^k(\mathcal{X}) = \Gamma$. That is: $\Gamma(v, \mu)G_k(v, \mu)(x_{\mathcal{X}}(v, \mu)) = \Gamma(v, \mu)(p_{\Gamma}(v, \mu))$ (here $p_{\Gamma}(v, \mu)$ denotes the fixed point of the map $G_{\sharp\Gamma_0}(v, \mu)$ which satisfies $p_{\Gamma}(v, \mu) \in D(\Gamma)(v, \mu)$. In particular, $G_k(v, \mu)(x_{\mathcal{X}}(v, \mu)) \in D(\Gamma)(v, \mu)$. That is $x_{\mathcal{X}}(v, \mu) \in G^{-k}(v, \mu)(D(\Gamma)(v, \mu))$. From here we conclude that, for $\mu > \mu_{2\Gamma_0}(v)$ or $\mu > \mu_{\Gamma_0}(v)$, the point $x_{\mathcal{X}}(v, \mu)$ is a pre-image of the hyperbolic, repelling, fixed point $p_{\Gamma}(v, \mu)$. So in particular

$$C_{\mathcal{X}} = \{(v, \mu; x_{\mathcal{X}}(v, \mu)); (v, \mu) \in B(k_0), \mu > \mu_{2\Gamma_0}(v), \mu > \mu_{\Gamma_0}(v)\}$$

is a C^1 -surface transversal to S_0 . Therefore the intersection $S_0 \cap C_{\mathcal{X}}$ defines a C^1 -surface, $C_{\mathcal{X}}^0$, contained in $\mathcal{U} \times [1 - \delta, 1]$ and parametrized by

$$\{(v, C_{\mathcal{X}}^0(v), \mathcal{X}_{\mathcal{X}}(v, C_{\mathcal{X}}^0(v))) ; v \in V\}.$$

Denote by $X_{\mathcal{X}}(v)$ the vector field associated to $(v, C_{\mathcal{X}}^0(v)) \in B(k_0)$.

Let $\sigma(p_{\Gamma}(v, C_{\mathcal{X}}^0(v))) \subset U$ be the hyperbolic, periodic orbit associated to the point $p_{\Gamma}(v, C_{\mathcal{X}}^0(v)) \in M(v, C_{\mathcal{X}}^0(v))$. We have

$$\gamma_0(\sigma_0(X_{\mathcal{X}}(v))) \subset W^s(\sigma(p_{\Gamma}(v, C_{\mathcal{X}}^0(v)))),$$

that is, the vector field $X_{\mathcal{X}}(v)$ has a contracting singular cycle.

2.11.

Let $\Gamma \in \Sigma_0$ be any realizable bisequence. Assume $\mu_{\Gamma} = \mu_{\Gamma}(v)$ is the parameter value which satisfies $\Theta(v, \mu_{\Gamma}(v)) = \Gamma$ and $x_{\Gamma} = x_{\Gamma}(v, \mu) \in M(v, \mu)$ be a point which satisfies

$$\Gamma(v, \mu)(x_{\Gamma}(v, \mu)) = \Gamma.$$

(A) Assume $\Gamma \in Per(\sigma)$. In this case we have $\Gamma \in \Sigma_1$ or $\Gamma \in \Sigma_2$ or there is $k \in \mathbf{N}$ such that $\sigma_0^k(\Gamma) \in \Sigma_2$. In all the cases, as we have seen in (2.6), (2.7) (2.8) and (2.10), we

known that associated to Γ we can find a C^1 -surface $C_\Gamma^0 = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C_\Gamma^0$, presents a contracting singular cycle or a homoclinic orbit for the singularity $\sigma_0(X_\Gamma(v))$ or a saddle-node or a flip bifurcation.

(B) Suppose that $\Gamma \notin \text{Per}(\sigma)$ and that there is $k \in \mathbb{N}$ such that $\sigma_0^k(\Gamma) \in \text{Per}(\sigma)$. In this situation, as we have seen in (2.6) and (2.10), we know that associated to Γ , we can find a C^1 -surface $C_\Gamma^0 = \{(v, C_\Gamma(v)); v \in V\} \subset B(k_0)$ such that: the vector field $X_\Gamma(v)$, which represents the point $(v, C_\Gamma(v)) \in C_\Gamma^0$, presents a contracting singular cycle.

(C) Suppose $\Gamma \notin \text{Per}(\sigma)$ and $\sigma_0^k(\Gamma) \notin \text{Per}(\sigma)$, for any $k \in \mathbb{N}$. In this case we can find a sequence of realizable sequences $\Gamma_k \in \text{Per}(\sigma_0)$, $\Gamma_k < \Gamma$, such that

- (i) $\lim_{k \rightarrow \infty} \Gamma_k = \Gamma$
- (ii) $\mu_{\Gamma_i}(v) \rightarrow \mu_\Gamma(v)$, $\mu_{\Gamma_i}(v) < \mu_\Gamma(v)$ and
- (iii) $(\mu_{\Gamma_i}(\cdot))$ is a Cauchy sequence of maps in the C^1 -uniform topology (this can be proved as in (2.9)).

In this case, associated to Γ , we find a C^1 -surface $\{(v, C_\Gamma(v)); v \in V\}$ such that the vector field which represents the point $(v, C_\Gamma(v)) \in C_\Gamma^0$ satisfies that the trajectory $\gamma_0(X_\Gamma(v))$ has recurrent behavior in the neighborhood U .

(D) Let now $s(v, \mu)$ be any pre image of the points $b(v, \mu)$ or $a(v, \mu)$ in the closure of the set $M(v, \mu)$, such that $s(v, \mu) \geq \xi^{k_0-1}\mu$ for some $\xi^{-(k_0-1)}(1-\delta) \leq \mu \leq \xi^{-(k_0-1)}$. In this situation the C^1 -surface $\{(v, \mu, s(v, \mu))\} = S$ is transversal to S_0 and, therefore, the intersection $S \cap S_0$ define a C^1 -surface S_b (resp S_a) parametrized by $\{(v, \bar{b}(v), S(v, \bar{b}(v))); v \in V\}$ (resp. $\{(v, \bar{a}(v), S(v, \bar{a}(v))); v \in V\}$). Let $X_{\bar{b}}(v)$ (resp. $X_{\bar{a}}(v)$) denote the vector field associated to $(v, \bar{b}(v)) \in B(k_0)$ (resp. $v, \bar{a}(v)) \in B(k_0)$). This vector field satisfies that

$$\gamma_0(\sigma_0(X_{\bar{b}}(v))) \subset W^s(\sigma_1(X_{\bar{b}}(v)))$$

(resp. $\gamma_0(\sigma_0(X_{\bar{a}}(v))) \subset W^s(\sigma_1(X_{\bar{a}}(v)))$). That is presents a contracting singular cycles.

This completes the proof of Theorem 1. ■

An easy consequence of the results in (2.7) through (2.11) is

COROLLARY 4. – $\Gamma_0 \cup \Gamma_1$ is a dense subset of $B(k_0)$, any $k_0 \geq n_0$. ■

3. Proof of Theorem 2

Without loss of generality, we may assume that the family $\{X_\mu\}$ such that $X_{\mu=0} \in \mathcal{N}$ is given by $\{(\bar{v}, \mu); -\varepsilon_0 < \mu < \varepsilon_0\}$ for some $\bar{v} \in V$ and $\varepsilon_0 > 0$ small.

We let $L(\mu; y)$ denote the map $L(\bar{v}, \mu; y)$ given by

$$L(\mu; y) = \begin{cases} \xi y, & 0 \leq y \leq \xi^{-1} \\ \mu - J(\mu; y)(y - (1 - \delta))^\alpha, & 1 - \delta \leq y \leq b(\mu) \\ \mu - K(\mu; y)(1 - y)^\alpha, & a(\mu) \leq y \leq 1; \end{cases}$$

where $a(\mu) = 1 - \delta^2(\mu)$, $b(\mu) = 1 - \delta + \delta^1(\mu)$, $\delta^i(\mu) = A^i(\mu)\mu^{1/\alpha}$, $i = 1, 2$; J and K are C^2 -map in the μ -variable, C^3 in the y -variable for $y \neq 1 - \delta, 1$ and whose derivatives are small with μ small.

Also $J(\mu, y) > 0$ and $K(\mu; y) > 0$ for any $(\mu; y), 0 \leq \mu \leq \mu_0 = \xi^{-n_0}; y \in I_1(\mu) \cup I_2(\mu)$.

Given $0 \leq \mu \leq \mu_0$ we define $\Lambda(\mu) = \{y \in [0, 1] / L_\mu^n(y) \in \cup_{i=0}^2 I_i(\mu), \text{ for all } n \geq 0\}$. Let $\Gamma_0 = \{\mu \in [0, \mu_0] / 1 \notin \Lambda(\mu)\}$ and $\Gamma_1 = \{\mu \in [0, \mu_0] / 1 \in \Lambda(\mu) \text{ and there exists an hyperbolic attracting periodic orbit for the map } L_\mu(\cdot)\}$. Here $L_\mu(y) = L(\mu; y)$.

As we have seen in Chapter II, $\mu \in \Gamma_0 \cup \Gamma_1$ implies that the associated vector field $X(\bar{v}, \mu)$ is structurally stable in U . Let $H = \Gamma_0 \cup \Gamma_1$ and $B = [0, \mu_0] \setminus H$.

Theorem 2 will follow from the following

THEOREM 2'. — $m(H \cap [0, \mu_0]) = \mu_0$. (Here m denotes the Lebesgue measure.)

Using the Lebesgue density theorem it is enough to prove that given $0 \leq \mu \leq \mu_0$ we have

$$(*) \lim_{\varepsilon \rightarrow 0} \frac{m(B \cap [\mu - \varepsilon, \mu + \varepsilon])}{2\varepsilon} < 1.$$

3.1.

For $\mu \in [0, \mu_0]$, define $L_1(\mu) = L(\mu; 1)$ and $L_{n+1}(\mu) = L(\mu; L_n(\mu))$.

We have $L_{i+1}(\mu) = \xi L_i(\mu)$, for any $1 \leq i \leq n_0$ and $L_{n_0+1}(\mu) = \xi^{n_0} \mu$. Hence these maps satisfy:

a) $L'_i(\mu) > 0$ and $L''_i(\mu) = 0, \mu \in [0, \mu_0], 1 \leq i \leq n_0 + 1$,

b) $L'_i(\mu) \leq L'_i(0), 0 \leq \mu \leq \mu_0, 1 \leq i \leq n_0 + 1$.

For any $k \geq n_0 + 2$, let $I_k = I_k^1 \cup \dots \cup I_k^{m_k}$ be the domain of definition of the map L_k .

Let $I_k^j = [\nu_0, \nu_1]$ be a component of the domain I_k that satisfies $L'_i(\mu) \neq 0$, for $1 \leq i \leq k - 1$ and any $\mu \in I_k^j$.

LEMMA 11. — The map L_k satisfies one and only one of the following possibilities:

- (i) there exists a unique $\bar{\nu} \in I_k^j$ such that $L'_k(\bar{\nu}) = 0$ and $L''_k(\bar{\nu}) < 0$ or
- (ii) $L'_k(\mu) \neq 0$ and $L''_k(\mu) = 0$ for any $\mu \in I_k^j$ or
- (iii) $L'_k(\mu) \neq 0$ and $L''_k(\mu) < 0$ for any $\mu \in I_k^j$.

Proof. — See the appendix. ■

COROLLARY 5. — Let $I = [\nu_0, \nu_1] \subset I_k^j$ be an interval and assume $L'_i(\mu) \neq 0$ for $\mu \in I, 1 \leq i \leq k$. Then for any $\alpha, \beta, \nu_0 \leq \alpha \leq \beta \leq \nu_1$ we have $L'_k(\alpha) \geq L'_k(\beta)$.

Proof. — Let $\mathcal{X}(\mu) = \frac{L'_k(\mu)}{L'_k(\nu_0)}, \mu \in I$. We have $\mathcal{X}(\nu_0) = 1$.

If $L'_k(\mu) < 0$, then $\mathcal{X}'(\mu) = \frac{L''_k(\mu)}{L'_k(\nu_0)} > 0$ and $\mathcal{X}$ is an increasing map. So $\mathcal{X}(\alpha) \leq \mathcal{X}(\beta)$ and hence $L'_k(\alpha) \geq L'_k(\beta)$.

If $L'_k(\mu) > 0$, then $\mathcal{X}'(\mu) = \frac{L''_k(\mu)}{L'_k(\nu_0)} \leq 0$ and $\mathcal{X}$ is a decreasing map. In particular, $\mathcal{X}(\alpha) \leq \mathcal{X}(\beta)$ and hence $L'_k(\alpha) \geq L'_k(\beta)$. ■

3.2.

We note that $[0, \mu_0] = \{0\} \cup \cup_{k=n_0}^{\infty} \xi^{-k}]\xi^{-1}, 1]$.

Let $k \geq n_0$ be a given number and $I_k = \xi^{-k}]\xi^{-1}, 1]$. For any given $\mu \in I_k$ we have $\xi^{-1} < \xi^k \mu \leq 1$. Clearly that it is enough to prove that $m(B \cap I_k) = 0$, for any $k \geq n_0$.

Given $\mu \in I_k$ let $D_{(j)}^i(\mu)$ and $G_\mu(\cdot)$ denote the interval $D_{(j)}^i(\bar{v}, \mu)$ and the map $G(\bar{v}, \mu)$ as defined in (2.11).

Let $J_0 = \xi^{-k}[1 - \delta, 1]$ and $g_0 : J_0 \rightarrow [1 - \delta, 1]$ be the map $g_0(\mu) = \xi^k \mu$.

Let us define, inductively,

$$J_r = \left\{ \mu \in J_{r-1} / g_{r-1}(\mu) \in \cup_{j=0}^{\infty} \cup_{i=1,2} D_{(j)}^i(\mu) \right\}$$

and $g_r : J_r \rightarrow [1 - \delta, 1]$ by $g_r(\mu) = G_\mu(g_{r-1}(\mu))$, $r \geq 1$.

Let $J_r^t = [\nu_0, \nu_1]$ be a component of the domain J_r such that $g'_i(\mu) \neq 0$, for $0 \leq i \leq r-1$ and any $\mu \in J_r^t$.

COROLLARY 6. – *For the map $g_r|_{J_r^t}$ we have one and only one of the following possibilities:*

- (i) *there exists a unique $\bar{v} \in J_r^t$ such that $g'_r(\bar{v}) = 0$ and $g''_r(\mu) < 0$, any $\mu \in J_r^t$ or*
- (ii) *$g'_r(\mu) \neq 0$ and $g''_r(\mu) < 0$ for any $\mu \in J_r^t$.*

Proof. – The proof follows from Lemma 11. ■

COROLLARY 7. – *Let $J = [\nu_0, \nu_1] \subset J_r^t \subset J_r$ be an interval such that $g'_i(\mu) \neq 0$, for $0 \leq i \leq r$ and $\mu \in J_r^t$. Let α, β be the parameter values such that $\nu_0 \leq \alpha \leq \beta \leq \nu_1$ we have $g'_r(\alpha) \geq g'_r(\beta)$.*

Proof. – Similar to Corollary 5. ■

3.3.

Let us now consider a parameter value $\mu \in J_r$ that satisfies: there is an interval $[\alpha, \beta] \subset J_r$ such that $\mu \in]\alpha, \beta[$ and $g'_i(\nu) \neq 0$, $0 \leq i \leq r$, $\nu \in [\alpha, \beta]$.

(A₁) Let us assume $g'_r(\nu) > 0$, $\nu \in [\alpha, \beta]$;

$$[b(\beta), a(\beta)] \subset]g_r(\alpha), g_r(\beta)[\text{ and } g_r(\mu) \in I_1(\mu)$$

PROPOSITION 3. – *There exists $\bar{\mu} \in]\mu, \beta[$ such that $\frac{m(B \cap [\mu, \bar{\mu}])}{\bar{\mu} - \mu} \leq 1/3$, for k big enough.*

Proof. – Denote by $\mu \leq \mu_1 \leq \mu_2 \leq \beta$ the parameter values which satisfy $g_r(\mu_1) = b(\beta)$, and $g_r(\mu_2) = a(\beta)$. We have $g_r(\mu_2) - g_r(\mu_1) = \int_{\mu_1}^{\mu_2} g'_r(\nu) d\nu \leq g'_r(\mu_1)(\mu_2 - \mu_1)$ and

$$g_r(\mu_1) - g_r(\mu) = \int_{\mu}^{\mu_1} g'_r(\nu) d\nu \geq g'_r(\mu_1)(\mu_1 - \mu).$$

Since $g_r(\mu_1) - g_r(\mu) \leq b(\mu_1) - (1 - \delta)$ we have

$$\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq \frac{\mu_1 - \mu}{\mu_2 - \mu_1} \leq \frac{b(\mu_1) - (1 - \delta)}{a(\beta) - b(\beta)},$$

which can be taken smaller or equal to $1/3$ for k big. ■

(A₂) Assume $g'_r(\nu) < 0$, $\nu \in [\alpha, \beta]$; $[b(\beta), a(\beta)] \subset]g_r(\beta), g_r(\alpha)[$ and $g_r(\mu) \in I_1(\mu)$.

PROPOSITION 4. – *There exists $\bar{\mu} \in [\alpha, \mu]$ such that $\frac{m(B \cap [\bar{\mu}, \mu])}{\mu - \bar{\mu}} \leq 1/3$, for k big enough.*

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₁) Assume there is $\binom{i}{j}, j \neq 0$, such that $D\left(\binom{i}{j}\right)(\nu) \subset [g_r(\alpha), g_r(\beta)]$.

Given $\nu \in [\alpha, \beta]$ denote by $I_1\left(\binom{i}{j}\right)(\nu)$ the interval contained in $D\left(\binom{i}{j}\right)(\nu)$ such that $G\left(\nu, I_t\left(\binom{i}{j}\right)(\nu)\right) = I_t(\nu)$, for $t = 1, 2$.

(A₃₁) Assume that $g_r(\mu) \in I_1\left(\binom{i}{j}\right)(\mu); i = 2$ and $g'_r(\nu) > 0$, for $\nu \in [\alpha, \beta]$. Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = b(\beta)$ and $G(\mu_2, g_r(\mu_2)) = a(\beta)$, respectively. We have

PROPOSITION 5. – $\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq \frac{1}{3}$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₂) Assume that $g_r(\mu) \in I_1\left(\binom{i}{j}\right)(\mu); i = 2$ and that $g'_r(\nu) < 0$, for $\nu \in [\alpha, \beta]$.

Let denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = b(\beta)$, $G(\mu_2, g_r(\mu_2)) = a(\beta)$, respectively.

We have:

PROPOSITION 6. – $\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$ for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₃) Assume that $g_r(\mu) \in I_2\left(\binom{i}{j}\right)(\mu), i = 1$ and that $g'_r(\nu) > 0$, for $\nu \in [\alpha, \beta]$. Denote

by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $G(\mu_1, g_r(\mu_1)) = a(\beta)$ and $G(\mu_2, g_r(\mu_2)) = b(\beta)$, respectively. We have

PROPOSITION 7. – $\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₄) Assume that $g_r(\mu) \in I_2\left(\binom{i}{j}\right)(\mu), i = 1$ and that $g'_r(\nu) < 0$ for $\nu \in [\alpha, \beta]$. Let denote

by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $G(\mu_2, g_r(\mu_2)) = b(\beta)$ and $G(\mu_1, g_r(\mu_1)) = a(\beta)$, respectively.

We have:

PROPOSITION 8. – $\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₅) Assume that $g_r(\mu) \in I_2\left(\binom{i}{j}\right)(\mu), i = 2$ and that $g'_r(\nu) > 0$, for $\nu \in [\alpha, \beta]$ and,

additionally, $\left[y\left(\binom{2}{j}\right)(\beta), z\left(\binom{2}{j-1}\right)(\beta)\right] \subset]g_r(\alpha), g_r(\beta)[$.

Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = y\binom{2}{j}(\beta)$, $g_r(\mu_2) = z\binom{2}{j-1}(\beta)$, respectively.

We have

PROPOSITION 9. – $\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₆) Assume that $i = 2$; $g_r(\mu) \in I_2\binom{i}{j}(\mu)$ and that $g'_r(\nu) < 0$, for $\nu \in [\alpha, \beta]$ and

$$\left[y\binom{2}{j}(\beta), z\binom{2}{j-1}(\beta) \right] \subset]g_r(\alpha), g_r(\beta)[.$$

Denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $g_r(\mu_2) = z\binom{2}{j-1}(\beta)$, $g_r(\mu_1) = y\binom{2}{j}(\beta)$, respectively.

We have

PROPOSITION 10. – $\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₃₇) Assume that $i = 1$; $g_r(\mu) \in I_1\binom{i}{j}(\mu)$; $g'_r(\nu) > 0$, for $\nu \in [\alpha, \beta]$ and

$$\left[z\binom{1}{j}(\beta), y\binom{1}{j+1}(\beta) \right] \subset]g_r(\alpha), g_r(\beta)[.$$

Denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = z\binom{1}{j}(\beta)$ and $g_r(\mu_2) = y\binom{1}{j+1}(\beta)$, respectively.

We have

PROPOSITION 11. – $\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, for k big enough. ■

(A₃₈) Assume that $i = 1$; $g_r(\mu) \in I_1\binom{i}{j}(\mu)$; $g'_r(\nu) < 0$ for $\nu \in [\alpha, \beta]$ and

$$\left[z\binom{1}{j}(\beta), y\binom{1}{j+1}(\beta) \right] \subset]g_r(\alpha), g_r(\beta)[.$$

Let denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values which satisfy $g_r(\mu_2) = y\binom{1}{j+1}(\beta)$ and $g_r(\mu_1) = z\binom{1}{j}(\beta)$, respectively.

We have

PROPOSITION 12. – $\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$ for k big enough. ■

(A₄) Assume that $\left[z\binom{1}{0}(\beta), y\binom{1}{1}(\beta) \right] \subset]g_r(\alpha), g_r(\beta)[$ and $g_r(\mu) \in D\binom{1}{0}(\mu)$.

(A₄₁) Assume that $g'_r(\nu) > 0$ for $\nu \in [\alpha, \beta]$.

Let denote by $\mu < \mu_1 < \mu_2 < \beta$ the parameter values which satisfy $g_r(\mu_1) = z\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(\beta)$ and $g_r(\mu_2) = y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)(\beta)$, respectively.

We have

PROPOSITION 13. – $\frac{m(B \cap [\mu, \mu_2])}{\mu_2 - \mu} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(A₄₂) Assume that $g'_r(\nu) < 0$, for $\nu \in [\alpha, \beta]$.

Denote by $\alpha < \mu_2 < \mu_1 < \mu$ the parameter values that satisfy $g_r(\mu_2) = y\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)(\beta)$ and $g_r(\mu_1) = z\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)(\beta)$, respectively.

We have

PROPOSITION 14. – $\frac{m(B \cap [\mu_2, \mu])}{\mu - \mu_2} \leq 1/3$, for k big enough.

Proof. – The proof is similar to that of Proposition 3. ■

(3.4). Consider a parameter value $\mu \in J_0$ which satisfies: there exists $r_0 \in \mathbb{N}$ that

$$G_\mu^{r_0}(\xi^k \mu) \in \left([1 - \delta, 1] \setminus \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(\mu) \right)$$

In this case we have $\mu \in \Gamma_0$ or $G_\mu^{r_0}(\xi^k \mu) = b(\mu)$ or $G_\mu^{r_0}(\xi^k \mu) = a(\mu)$. It is clear that assertion (*) is true in any of the cases above. Let

$$T = \left\{ \mu \in J_0 / g_r(\mu) \in \bigcup_{j=0}^{\infty} \bigcup_{i=1,2} D\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right)(\mu), \text{ for any } r \geq 0 \right\}$$

For a given $\mu \in T$ we have three possibilities for the itinerary Γ_μ :

- (1) Γ_μ is a periodic itinerary;
- (2) Γ_μ is an itinerary which is eventually periodic and
- (3) Γ_μ do not satisfies (1) and (2) above.

Assume Γ_μ is periodic. In this case we know (see (2.11)) that there is an interval $[\alpha, \beta] \subset T$ such that $\Gamma_\nu = \Gamma_\mu$, for any $\nu \in [\alpha, \beta]$; $\mu \in [\alpha, \beta]$ and $B \cap [\alpha, \beta]$ is a finite number of points. So for these parameter values assertion (*) is true.

Assume Γ_μ is eventually periodic. Under these circumstances it is easy to see that we can find an interval $[\alpha, \beta] \subset J_0$ and an index $r \in \mathbb{N}$ such that

- (i) $\mu \in]\alpha, \beta[$;
- (ii) $g'_r(\nu) \neq 0$, $0 \leq i \leq r$ for any $\nu \in [\alpha, \beta]$ and
- (iii) $g_r/[\alpha, \beta]$ satisfies the conditions of one of the Propositions specified in (3.3) above.

It is clear that we can find a sequence of intervals $[\alpha_n, \beta_n] \subset]\alpha_{n-1}, \beta_{n-1}[$ and a sequence of indexes $r_n > r_{n-1}$ such that (i), (ii) and (iii) hold for any of the given $n \in \mathbb{N}$.

Therefore we can conclude the following

LEMMA 12. – *There exists a sequence $\mu_n \rightarrow \mu$ such that*

$$\frac{m(B \cap [\mu, \mu_n])}{\mu_n - \mu} \leq \frac{1}{3} \text{ or } \frac{m(B \cap [\mu_n, \mu])}{\mu - \mu_n} \leq \frac{1}{3},$$

for k big enough. ■

In particular, for any of these parameter values assertion (*) is true.

Assume Γ_μ satisfies (3) above. In this case we can find a sequence $\mu_n \rightarrow \mu$ such that Γ_{μ_n} satisfies (1) or (2) above. For these parameter values assertion (*) holds, therefore we conclude that it (*) is true for μ .

This completes the proof of Theorem 2. ■

(3.5) Comments on the general case

Let us now consider the general case for contracting singular cycles. In his paper San Martin [8] introduces a nice idea with which to work in this case. Let us consider the periodic orbits $\sigma_1(X), \dots, \sigma_r(X)$ that belong to the singular cycle Γ . Let $q_i(X) \in \sigma_i(X)$ be a point and $Q_i \subset M$ be a transversal section associated to $q_i(X)$, $i = 1, \dots, n$. Assume this cross section is parametrized by $\{(x_i, y_i); |x_i|, |y_i| \leq 1\}$ satisfying $W_{\sigma_i}^s \supseteq \{(x_i, 0); |x_i| \leq 1\}$ and $W_{\sigma_i}^u \supseteq \{(0, y_i); |y_i| \leq 1\}$.

Let $p_i^j = p_i^j(X)$ be the first intersection between $\gamma_i^j(X)$ and Q_{i+1} , $i = 1, 2, \dots, n-1$; $j = 1, 2$. We have $p_i^j = (x_{i+1}^j(X), 0)$ and assume $x_{i+1}^j > 0$. Denote by $q_i^j = q_i^j(X) = (0, y_i^j(X))$ the first intersection of the backward orbit of p_i^j with Q_i .

We will assume $y_i^j(X) > 0$, $i = 1, \dots, n-1$; $j = 1, 2$.

Since p_i^j and q_i^j are in the same orbit we can find horizontal strips $R_j^i(X) \ni q_i^j$ and neighborhoods $U_i^j \ni p_i^j$, so that the positive orbits of points at R_j^i intersect U_i^j . This procedure define Poincaré maps $P_i^j : R_j^i \rightarrow U_i^j$; $i = 1, 2, \dots, n-1$; $j = 1, 2$.

On the other hand, the positive orbit of points at a horizontal strip $R_i(X)$, containing $W^s(\sigma_i(X)) \cap Q_i$, turns around the closed orbit $\sigma_i(X)$ and then returns to Q_i . This define a return map $P_i : R_i \rightarrow Q_i$, $i = 1, \dots, n$.

Denote by $q_n^j = q_n^j(X)$ the last intersection of the orbit $\gamma_n^j(X)$ with Q_n , $j = 1, 2$. Since $w(q_n^j) = \sigma_0(X)$ and $\alpha(q_n^j) = \sigma_n(X)$, there are horizontal strips $R_j^n(X) \ni q_n^j$ such that the positive orbit of points at R_j^n pass first near $\sigma_0(X)$ and afterwards intersect Q_1 . This define maps $P_j^n : R_j^n \rightarrow Q_1$, $j = 1, 2$.

Therefore the first return map F_X is defined on $\cup_{i=1}^n (R_i \cup R_i^1 \cup R_i^2)$ with values on $\cup_{i=1}^n Q_i$ and its restriction to R_i coincides with the Poincaré map associated to $\sigma_i(X)$.

The same construction applies to vector field Y , near enough to X in the C^r -topology, $r \geq 3$.

From now and on the proof follows as in chapters II and III (3.1)-(3.4), that is: Give an explicit formula to the map F_Y ; show that there is an invariant stable foliation for F_Y ; change coordinates in the neighborhood $\mathcal{U}$ and prove the result for the one-dimensional map associated to F_Y .

4. Appendix

In this paragraph we prove Lemma 13. Let $L(\mu; y)$ denote the map given by

$$L(\mu; y) = \begin{cases} \xi y, & 0 \leq y \leq \xi^{-1} \\ \mu - J(\mu; y)(y - (1 - \delta))^\alpha, & 1 - \delta \leq y \leq b(\mu) \\ \mu - K(\mu; y)(1 - y)^\alpha, & a(\mu) \leq y \leq 1, \end{cases}$$

where $a(\mu) = 1 - \delta^2(\mu)$, $b(\mu) = 1 - \delta + \delta^1(\mu)$; $\delta^i(\mu) = A^i \mu^{1/\alpha}$, $A^i > 0$, for $i = 1, 2$; J and K are C^2 -maps in the μ -variable, C^3 in the y -variable $y \neq 1 - \delta, 1$ and whose derivatives $\frac{\partial J}{\partial y}, \frac{\partial J}{\partial \mu}, \frac{\partial^2 J}{\partial \mu \partial y}, \frac{\partial^2 J}{\partial y^2}, \frac{\partial^2 J}{\partial \mu^2}, \frac{\partial K}{\partial \mu}, \frac{\partial K}{\partial y}, \frac{\partial^2 K}{\partial \mu \partial y}, \frac{\partial^2 K}{\partial y^2}, \frac{\partial^2 K}{\partial \mu^2}$ are small numbers, with μ small. Moreover $J(\mu; y) > 0$ and $K(\mu; y) > 0$, any $(\mu; y)$, $0 \leq \mu \leq \mu_0 = \xi^{-n_0}$.

Define $L_1(\mu) = L(\mu; 1) = \mu$ and $L_{n+1}(\mu) = L(\mu; L_n(\mu))$, $n \geq 1$.

We have $L_{i+1}(\mu) = \xi L_i(\mu)$, $1 \leq i \leq n_0$ and $L_{n+1}(\mu) = \xi^{n_0} \mu$. Hence these maps satisfy:

(a) $L'_i(\mu) > 0$ and $L''_i(\mu) = 0$, $\mu \in [0, \mu_0]$, $0 \leq i \leq n_0 + 1$ and

(b) $L'_i(\mu) \leq L'_i(0)$, $0 \leq \mu \leq \mu_0$.

For any $k \geq n_0 + 2$, let $I_k = I_k^1 \cup I_k^2 \cup \dots \cup I_k^{n_k}$ be the domain of definition of the map L_k .

Let $I_k^j = [\nu_0, \nu_1]$ be a component of the domain I_k that satisfies $L'_i(\mu) \neq 0$, for $0 \leq i \leq k - 1$ and $\mu \in I_k^j$.

LEMMA 13. – For the map L_k we have one and only one of the following:

(i) there exists only one $\bar{\nu} \in I_k^j$ such that $L'_k(\bar{\nu}) = 0$ and $L''_k(\bar{\nu}) < 0$ or

(ii) $L'_k(\mu) \neq 0$ and $L''_k(\mu) = 0$ for $\mu \in I_k^j$, or

(iii) $L'_k(\mu) \neq 0$ and $L''_k(\mu) < 0$ for $\mu \in I_k^j$.

Proof. – For $L_{k-1}(\mu) \leq \xi^{-1}$, $\mu \in I_k^j$, we have $L_k(\mu) = \xi L_{k-1}(\mu)$ and the result follows by the induction hypothesis. Otherwise let us consider $A = \bigcup_{\mu \in [0, \mu_0]} [\{\mu\} \times I_1(\mu)]$

and $B = \bigcup_{\mu \in [0, \mu_0]} (\{\mu\} \times I_2(\mu))$.

We must have $A \cap (\text{Graph}(L_k/I_k^j)) \neq \emptyset$ or $B \cap (\text{Graph}(L_k/I_k^j)) \neq \emptyset$ (only one of these intersections in non-empty).

I) Assume $L'_{k-1}(\mu) < 0$ for $\mu \in I_k^j$.

(i) We have $L_{k-1}(\nu_0) = 1$ and $L_{k-1}(\nu_1) = a(\nu_1)$.

Under these conditions $L_k(\mu) = L(\mu; L_{k-1}(\mu)) = \mu - K(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^\alpha$.

So

$$\begin{aligned} L'_k(\mu) = & 1 - \frac{\partial K}{\partial \mu}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^\alpha \\ & + \left[-\frac{\partial K}{\partial y}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^\alpha \right. \\ & \left. + \alpha K(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^{\alpha-1} \right] \cdot L'_{k-1}(\mu) \end{aligned}$$

and

$$\begin{aligned}
 L_k''(\mu) = & (1 - L_{k-1}(\mu))^{\alpha-2} [-\alpha(\alpha-1)K(\cdot, \cdot)(L_{k-1}'(\mu))^2 \\
 & - K_{\mu\mu}(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2 + 2\alpha K_{\mu}(\cdot, \cdot)L_{k-1}'(\cdot)(1 - L_{k-1}(\cdot)) \\
 & - K_{yy}(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2(L_{k-1}'(\cdot))^2 \\
 & + \alpha K(\cdot, \cdot)(1 - L_{k-1}(\cdot)) \cdot L_{k-1}''(\cdot) \\
 & - 2K_{\mu y}(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2 \cdot L_{k-1}'(\cdot) \\
 & + 2\alpha K_y(\cdot, \cdot)(1 - L_{k-1}(\cdot))(L_{k-1}'(\cdot))^2 \\
 & - K_y(\cdot, \cdot)(1 - L_{k-1}(\cdot))^2 \cdot L_{k-1}''(\cdot)].
 \end{aligned}$$

Since

$$\begin{aligned}
 L_{k-1}(\mu) &= \xi L_{k-2}(\mu) = \dots = \xi^{j-1} L_{k-j}(\mu) \\
 &= \xi^{j-1} [\mu - K(\mu; L_{k-j-1}(\cdot))(1 - L_{k-j-1}(\mu))^\alpha] \\
 &\quad \text{if } a(\mu) \leq L_{k-j-1}(\mu) \leq 1 \\
 &= \xi^{j-1} [\mu - J(\mu; L_{k-j-1}(\cdot))(L_{k-j-1}(\mu) - 1 - \delta)^\alpha] \\
 &\quad \text{if } 1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu).
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 L_{k-1}'(\mu) = & \xi^{j-1} [1 - J_{\mu}(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha \\
 & - J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha \cdot L_{k-j-1}'(\mu) \\
 & - \alpha J(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^{\alpha-1} L_{k-j-1}'(\cdot)]
 \end{aligned}$$

or

$$\begin{aligned}
 L_{k-1}'(\mu) = & \xi^{j-1} [1 - K_{\mu}(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha \\
 & - K_y(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^\alpha L_{k-j-1}'(\cdot) \\
 & + \alpha K(\cdot, \cdot)(1 - L_{k-j-1}(\cdot))^{\alpha-1} L_{k-j-1}'(\cdot)],
 \end{aligned}$$

depending on $1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu)$ or $a(\mu) \leq L_{k-j-1}(\mu) \leq 1$, respectively. Since $L_{k-1}'(\mu) < 0$ we have

$$L_{k-j-1}'(\mu) > \frac{1 - J_{\mu}(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))^\alpha}{[L_{k-j-1}(\mu) - (1 - \delta)]^{\alpha-1} [\alpha J(\cdot, \cdot) + J_y(\cdot, \cdot)(L_{k-j-1}(\mu) - (1 - \delta))]}$$

or

$$-L_{k-j-1}'(\mu) > \frac{1 - K_{\mu}(\cdot, \cdot)(1 - L_{k-j-1}(\mu))^\alpha}{(1 - L_{k-j-1}(\mu))^\alpha [\alpha K(\cdot, \cdot) - K_y(\cdot, \cdot)(1 - L_{k-j-1}(\mu))]}$$

depending on $1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu)$ or $a(\mu) \leq L_{k-j-1}(\mu) \leq 1$, respectively. In any case we get $|L_{k-j-1}'(\mu)| \gg 20$, for $\mu \in I_k^j$.

Now consider the map $\rho(\mu)$ given by

$$\begin{aligned}
 \rho(\mu) = & J_{\mu}(\mu; L_{k-j-1}(\mu))(L_{k-j-1}(\mu) - (1 - \delta))^\alpha \\
 & + [J_y(\mu; L_{k-j-1}(\mu))(L_{k-j-1}(\mu) - (1 - \delta))^\alpha + \alpha J(\cdot, \cdot)(L_{k-j-1}(\mu) \\
 & - (1 - \delta))^{\alpha-1}] \times L_{k-j-1}'(\mu)
 \end{aligned}$$

or

$$\begin{aligned}\rho(\mu) = & K_\mu(\mu; L_{k-j-1}(\mu))(1 - L_{k-j-1}(\mu))^\alpha \\ & + [K_y(\mu; L_{k-j-1}(\mu))(1 - L_{k-j-1}(\mu))^\alpha \\ & - \alpha K(\mu; L_{k-j-1}(\mu))(1 - L_{k-j-1}(\mu))^{\alpha-1}] \times L'_{k-j-1}(\mu),\end{aligned}$$

depending on whether $1 - \delta \leq L_{k-j-1}(\mu) \leq b(\mu)$ or $a(\mu) \leq L_{k-j-1}(\mu) \leq 1$, respectively. In the first case an easy computation, using the facts that $L'_{k-j-1}(\mu) \gg 20$; $L''_{k-j-1}(\mu) < 0$ and $L_{k-j-1}(\mu) - (1 - \delta) > 0$ gives $\rho'(\mu) > 0$, for $\nu_0 \leq \mu \leq \nu_1$.

Similarly in the second case we get $\rho'(\mu) > 0$.

Since $L'_{k-1}(\mu) = \xi^{j-1}[1 - \rho(\mu)]$, we have:

$$\begin{aligned}L''_k(\mu) = & [1 - L_{k-1}(\mu)]^{\alpha-2} [-\alpha(\alpha-1)(K(\mu; L_{k-1}(\mu))[\xi^{j-1}(1 - \rho(\mu))])^2 \\ & - K_{\mu\mu}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^2 \\ & + 2\alpha K_\mu(\mu; L_{k-1}(\mu))\xi^{j-1}(1 - \rho(\mu))(1 - L_{k-1}(\mu)) \\ & - K_{yy}(\mu; L_{k-1}(\mu))(1 - L_{k-1}(\mu))^2 - (\xi^{j-1}(1 - \rho(\mu)))^2 \\ & - \alpha K(\cdot, \cdot)(1 - L_{k-1}(\mu))\xi^{j-1}\rho'(\mu) \\ & - 2K_{\mu y}(\cdot, \cdot)(1 - L_{k-1}(\mu))^2\xi^{j-1}(1 - \rho(\mu)) \\ & + 2\alpha K_y(\cdot, \cdot)(1 - L_{k-1}(\mu))(\xi^{j-1}(1 - \rho(\mu)))^2 \\ & + K_y(\cdot, \cdot)(1 - L_{k-1}(\mu))^2\xi^{j-1}\rho'(\mu)];\end{aligned}$$

which is clearly a negative number.

We note that $L'_k(\nu_0) = 1$. Let us compute $L'_k(\nu_1)$.

We have

$$L'_k(\nu_1) = 1 + \nu_1^{1-1/\alpha} \left[\alpha K^{1/\alpha} L'_{k-1}(\nu_1) - \frac{K_y}{K} L'_{k-1}(\nu_1) \nu_1^{1/\alpha} - \frac{K_\mu}{K} \nu_1^{1/\alpha} \right].$$

Since $L'_{k-1}(\nu_1) < 0$ and $L_{k-1}(\nu_1) = a(\nu_1)$, we get $L'_k(\nu_1) < 0$.

Since $L''_k(\mu) < 0$, we find only one $\bar{\nu} \in [\nu_0, \nu_1]$ such that $L'_k(\bar{\nu}) = 0$.

(ii) Assume $L_{k-1}(\nu_0) < 1$ and $L_{k-1}(\nu_1) = a(\nu_1)$.

Similarly, as in (i) of above, we obtain $L''_k(\mu) < 0$ for $\mu \in I_k^j$. If $L'_k(\nu_1) \geq 0$ then there exists only one $\bar{\nu} \in I_k^j$ such that $L'_k(\bar{\nu}) = 0$. If $L'_k(\nu_1) < 0$, we have $L'_k(\mu) < 0$ for $\mu \in I_k^j$.

(iii) Assume $L_{k-1}(\nu_0) = 1$ and $L_{k-1}(\nu_1) > a(\nu_1)$.

As before we get $L''_k(\mu) < 0$ for $\mu \in I_k^j$. If $L'_k(\nu_1) > 0$ then $L'_k(\mu) > 0$ for $\mu \in I_k^j$. If $L'_k(\nu_1) \leq 0$ then there is only one $\bar{\nu} \in I_k^j$ such that $L'_k(\bar{\nu}) = 0$.

(iv) Assume $L_{k-1}(\nu_0) < 1$ and $L_{k-1}(\nu_1) > a(\nu_1)$.

As before we prove that $L'_k(\mu)$ is a decreasing map and we get the result.

(v) Assume $L_{k-1}(\nu_0) = b(\nu_0)$ and $L_{k-1}(\nu_1) = 1 - \delta$.

We proceed as in (i) to prove $L''_k(\mu) < 0$ and hence we obtain $L'_k(\mu) \geq L'_k(\nu_1) = 1$, any $\mu \in I_k^j$.

(vi) Assume $L_{k-1}(\nu_0) < b(\nu_0)$ and $L_{k-1}(\nu_1) = 1 - \delta$.

In a similar way as in (i) we get $L''_k(\mu) < 0$ and then $L'_k(\mu) \geq L'_k(\nu_1) = 1$, any $\mu \in I_k^j$.

(vii) Assume $L_{k-1}(\nu_0) < b(\nu_0)$ and $L_{k-1}(\nu_1) > 1 - \delta$.

As before we get $L''_k(\mu) < 0$ and $L'_k(\mu) \geq 1$, any $\mu \in I_k^j$.

(viii) Assume $L_{k-1}(\nu_0) = b(\nu_0)$ and $L_{k-1}(\nu_1) > 1 - \delta$.

As before we get the result.

II) Assume $L'_{k-1}(\mu) > 0$ (non-constant) for $\mu \in I_k^j$.

As in Case (I) we have eight possibilities. We proceed as in (I)(i) to get the result in all of the cases.

III) The case $L'_{k-1}(\mu) = \text{constant}$, i.e., $\nu_0 = 0 \in I_k^j$ satisfies $L'_{k-1}(\mu) > 0$ and $L''_k(\mu) = 0$, for $\mu \in I_k^j$. ■

REFERENCES

- [1] R. BAMON, R. LABARCA, R. MAÑE and M. J. PACIFICO, *The explosion of singular cycles (Publications Mathématiques I.H.E.S. 78 pg. 1993, pp. 207-232).*
- [2] M. J. PACIFICO and A. ROVELLA, *Unfolding contracting singular cycles (Ann. Scient. Ec. Norm. Sup., 4^e série, T. 26, 1993, pp. 691-700).*
- [3] R. LABARCA and M. J. PACIFICO, *Stability of singular Horseshoes (Topology, Vol. 25, No. 3, 1986, pp. 337-352).*
- [4] M. HIRSCH, C. PUGH and M. SHUB, *Invariant manifolds (Lecture Notes in Mathematics, 583, Springer-Verlag, 1977).*
- [5] A. ROVELLA, *The dynamics of perturbations of the contracting Lorentz attractor (Bol. Soc. Bras. Mat., Vol. 24, No. 2, 1993, pp. 233-259).*
- [6] R. MAÑE., *Hyperbolicity, sinks and measure of one dimensional dynamics (Comm. in Math. Phys., Vol. 100, 1985, pp. 495-524 and Erratum Comm. in Math. Phys., Vol. 112, 1987, pp. 721-724).*
- [7] W. de MELO, *Lectures on one-dimensional dynamics (17o. Coloquio Brasileiro de Matemática, IMPA, 1989).*
- [8] B. SAN MARTIN, *Contracting singular cycles*, Pre-print IMPA. 1995.

(Manuscript received September 15, 1994;
revised April 15, 1995.)

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