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SYMPLECTIC GEOMETRY ON MODULI SPACES OF STABLE PAIRS

BY FRANCESCO BOTTACIN

ABSTRACT. – In [H], Hitchin studied, from the point of view of symplectic geometry, the cotangent bundle $T^*\mathcal{U}_s(r,d)$ of the moduli space of stable vector bundles $\mathcal{U}_s(r,d)$ on a smooth irreducible projective curve C. He considered the map $H : T^*\mathcal{U}_s(r,d) \to \bigoplus_{i=1}^{r} H^0(C, K^i)$, which associates to a pair (E,ϕ) the coefficients of the characteristic polynomial of ϕ, and proved that this is an algebraically completely integrable Hamiltonian system.

Here we generalize such results by replacing the canonical line bundle K by any line bundle L for which $K^{-1} \otimes L$ has a non-zero section. We consider the moduli space $\mathcal{M}'(r,d,L)$ as constructed by Nitsure [N] and, in particular, the connected component $\mathcal{M}'_0$ of this space which contains the pairs (E,ϕ) for which E is stable; this component is a smooth quasi-projective variety. For each non-zero section s of $K^{-1} \otimes L$, we define a Poisson structure θ_s on $\mathcal{M}'_0$ and show that the Hitchin map $H : \mathcal{M}'_0 \to \bigoplus_{i=1}^{r} H^0(C, L^i)$ is again an algebraically completely integrable system (in a generalized sense). More precisely, H may be considered as a family of completely integrable systems on the symplectic leaves of $\mathcal{M}'_0$, parametrized by an affine space. This is a generalization of an analogous result proved by Beauville in [B1], in the special case $C = \mathbf{P}^1$.

Finally we shall describe the canonical symplectic structure of the cotangent bundle of the moduli space of stable parabolic vector bundles on C, and analyze the relationships with our previous results.

Introduction

Let us denote by $\mathcal{U}_s(r,d)$ the moduli space of stable vector bundles of rank r and degree d over a smooth irreducible projective curve C of genus $g \geq 2$, defined over the complex field $\mathbb{C}$. Let K be the canonical line bundle on C.

The cotangent bundle $T^*\mathcal{U}_s(r,d)$ to the moduli variety $\mathcal{U}_s(r,d)$ may be described as the set of isomorphism classes of pairs (E,ϕ), where E is a stable vector bundle and $\phi : E \to E \otimes K$ is a homomorphism of vector bundles.

Let us consider the map

$$H : T^*\mathcal{U}_s(r,d) \to W = \bigoplus_{i=1}^{r} H^0(C, K^i),$$

which associates to a pair (E,ϕ) the coefficients of the characteristic polynomial of ϕ. It happens that the dimension of the vector space W is equal to the dimension of the moduli variety $\mathcal{U}_s(r,d)$, hence $\dim T^*\mathcal{U}_s(r,d) = 2\dim W$. In [H], Hitchin proved that the component functions $H_1, \dots, H_N$ ($N = \dim W$) of H are functionally independent Poisson commuting functions, *i.e.*, $\{H_i, H_j\} = 0$, for every i, j, where $\{\cdot,\cdot\}$ is the Poisson bracket associated to the canonical symplectic structure of the cotangent bundle $T^*\mathcal{U}_s(r,d)$. Moreover the generic fiber of H is isomorphic to an open subset of an abelian variety, and the Hamiltonian vector fields corresponding to the functions $H_1, \dots, H_N$ give N commuting linear vector fields on these fibers. In other words, the map H is an algebraically completely integrable Hamiltonian system.

In this paper we generalize such results by replacing the canonical line bundle K by any line bundle L for which $K^{-1} \otimes L$ has a non-zero section.

In Section 1 we consider the moduli space $\mathcal{M}'(r,d,L)$ of stable pairs as constructed by Nitsure [N]. If $\mathcal{P}(r,d,L)$ denotes the open subset of $\mathcal{M}'(r,d,L)$ consisting of pairs (E,ϕ) for which E is a stable bundle, then the natural map $\pi : \mathcal{P}(r,d,L) \to \mathcal{U}_s(r,d)$, sending a pair (E,ϕ) to the vector bundle E, makes $\mathcal{P}(r,d,L)$ a vector bundle over $\mathcal{U}_s(r,d)$.

Then, in Section 2, we consider the analogue of the Hitchin map:

$$H : \mathcal{M}'(r,d,L) \to W = \bigoplus_{i=1}^{r} H^0(C, L^i).$$

In this case one proves that the dimension of $\mathcal{M}'$ is no more equal to twice the dimension of the vector space W. Even more importantly, the variety $\mathcal{M}'$ does not carry any canonically defined symplectic structure. This shows that our construction is not a trivial generalization of the situation described by Hitchin.

Actually, by the infinitesimal study of the variety $\mathcal{M}'$ carried out in Section 3, we are able to define, for any non-zero section s of $K^{-1} \otimes L$, a map

$$B_s : T^*\mathcal{M}'_0 \to T\mathcal{M}'_0,$$

which defines an antisymmetric contravariant 2-tensor $\theta_s \in H^0(\mathcal{M}'_0, \Lambda^2 T\mathcal{M}'_0)$.

In Section 4 we shall prove that this defines a Poisson structure on $\mathcal{M}'_0$. Needless to say, if $L = K$ and s is the identity section of $\mathcal{O}_C$, this Poisson structure is actually symplectic and coincides with the canonical symplectic structure of the cotangent bundle $T^*\mathcal{U}_s(r,d)$.

Then we shall see that the component functions $H_1, \dots, H_N$ of H still give N functionally independent holomorphic functions which are in involution, *i.e.*, $\{H_i, H_j\}_s = 0$, for every i, j, where $\{\cdot,\cdot\}_s$ is the Poisson bracket defined by θ_s. Again it may be seen that the generic fiber $H^{-1}(\sigma)$ is isomorphic to an open subset of an abelian variety (precisely the Jacobian variety of the spectral curve defined by σ), and that the Hamiltonian vector fields corresponding to the functions $H_1, \dots, H_N$ give N commuting vector fields on the fibers of H, which are linear on these fibers. Therefore we say that the map H defines an algebraically completely integrable Hamiltonian system (in a generalized sense).

More precisely, we shall see that H may be considered as a family of completely integrable systems on the symplectic leaves of the Poisson variety $\mathcal{M}'_0$, parametrized by an affine space. This generalizes an analogous result proved by Beauville in [B1], in the special case $C = \mathbf{P}^1$.

If we restrict to consider vector bundles with fixed determinant bundle, we get almost the same results as in the general case. The most relevant difference is that the Hamiltonian system defined by the Hitchin map H linearizes on the (generalized) Prym varieties of the coverings $\pi : X_\sigma \to C$, instead of on the Jacobian varieties of the spectral curves X_σ.

Finally, in Section 5, we shall consider the moduli spaces of stable parabolic vector bundles over C. By using our previous results, we are able to give an explicit description of the canonical symplectic structure of the cotangent bundle of these moduli varieties. We note here that an analogous result has been obtained by Biswas and Ramanan in [BR]. Their paper contains also a somewhat more general infinitesimal study of moduli functors in terms of hypercohomology.

Our construction will enable us to identify some special symplectic subvarieties of $\mathcal{P}$ with subvarieties of the cotangent bundle of the moduli space of stable parabolic bundles, with the induced canonical symplectic structure. This holds, in particular, for the cotangent bundle $T^*\mathcal{U}_s(r,d)$, which is embedded in $\mathcal{P}$ by the map sending a pair (E,ϕ) to $(E, s\phi)$.

Note: very recently we have been informed that a student of R. Donagi's, E. Markman, has obtained similar results in his PhD thesis [Ma].

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1. The moduli space of stable pairs

1.1 MODULI SPACES OF (SEMI)STABLE VECTOR BUNDLES.

Let C be a smooth irreducible projective curve of genus $g \geq 2$ over an algebraically closed field k. For a vector bundle E over C, we set $\mu(E) = \deg(E)/\mathrm{rank}(E)$, and we say that E is semistable (resp. stable) if, for every proper subbundle F of E, we have $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$).

Let $d, r \in \mathbb{N}$, with $r \geq 2$. We shall denote by $\mathcal{U}(r,d)$ the moduli space of S-equivalence classes of semistable vector bundles over C of rank r and degree d, and by $\mathcal{U}_s(r,d)$ the subvariety consisting of isomorphism classes of stable ones. We recall that if $(r,d) = 1$ then $\mathcal{U}(r,d) = \mathcal{U}_s(r,d)$ is a fine moduli space for isomorphism classes of stable vector bundles. As a consequence, we have the existence of a Poincaré vector bundle on $\mathcal{U}(r,d)$.

Remark 1.1.1. – If r and d are not coprime, it is known that there does not exist a Poincaré vector bundle on any (Zariski) open subset of $\mathcal{U}(r,d)$. However, Poincaré families of vector bundles do exist locally in the étale topology.

Remark 1.1.2. – We shall discuss here some problems connected with the existence of a Poincaré vector bundle on $\mathcal{U}(r,d)$. Using the notations of [S] or [Ne], let us denote by R the open subset of the Grothendieck 'quot' scheme $\mathcal{Q} = \mathrm{Quot}^P_{C/k}(\mathcal{O}_C \otimes k^p)$ consisting of points $F \in \mathcal{Q}$ such that F is a locally free sheaf and the natural morphism $H^0(C, \mathcal{O}_C \otimes k^p) \to H^0(C,F)$ is an isomorphism, and by R^{ss} (resp. R^s) the subset of R

consisting of semistable (resp. stable) vector bundles. These are $\mathrm{PGL}(p)$-invariant subsets of R, and we have $\mathcal{U}(r,d) = R^{ss}/\mathrm{PGL}(p)$ and $\mathcal{U}_s(r,d) = R^s/\mathrm{PGL}(p)$.

Let $\mathcal{F}$ be a universal quotient sheaf on R. The group $\mathrm{GL}(p)$ acts on $\mathcal{F}$, but this action does not factor through an action of $\mathrm{PGL}(p)$ because the action of $k^* \cdot I$ is not trivial, hence we cannot construct the quotient vector bundle of $\mathcal{F}$ by the action of $\mathrm{PGL}(p)$. In the special case when r and d are coprime, there exists a line bundle L on R^{ss}, such that the action of $k^* \cdot I$ on $\mathcal{F} \otimes p_R^*(L)$ is trivial, hence we can construct the vector bundle $\mathcal{E} = \mathcal{F} \otimes p_R^*(L)/\mathrm{PGL}(p)$ on $\mathcal{U}(r,d) \times C$. It follows easily that this is a Poincaré vector bundle on $\mathcal{U}(r,d)$.

On the other hand, the action of $k^* \cdot I$ on $\mathcal{F} \otimes \mathcal{F}^* \cong \mathcal{E}nd(\mathcal{F})$ is always trivial, hence we can always take the quotient bundle $\mathcal{E}nd(\mathcal{F})/\mathrm{PGL}(p)$, which will be denoted by $\mathcal{E}nd(\mathcal{E})$. Note that, when the Poincaré vector bundle $\mathcal{E}$ exists, $\mathcal{E}nd(\mathcal{E})$ is precisely the sheaf of endomorphisms of $\mathcal{E}$, but $\mathcal{E}nd(\mathcal{E})$ exists even if $\mathcal{E}$ does not. It is easy to see that the sheaf $\mathcal{E}nd(\mathcal{E})$ on $\mathcal{U}(r,d) \times C$ has the property that its restriction to $\{E\} \times C$ is isomorphic to $\mathcal{E}nd(E)$, for every $E \in \mathcal{U}(r,d)$.

In the sequel, for convenience of notation, we shall denote by E either a (semi)stable vector bundle on C, or the point of $\mathcal{U}(r,d)$ corresponding to the isomorphism class of the vector bundle E. We shall use indifferently the expressions 'vector bundle' or 'locally free sheaf'.

1.2 STABLE PAIRS.

Let L be a (fixed) line bundle on C and let us consider pairs (E,ϕ), where E is a vector bundle on C and $\phi : E \to E \otimes L$ is a homomorphism of $\mathcal{O}_C$-Modules. We say that (E,ϕ) is a semistable (resp. stable) pair if, for any ϕ-invariant proper subbundle F of E, we have $\mu(F) \le \mu(E)$ (resp. $\mu(F) < \mu(E)$).

Two pairs (E,ϕ) and (E',ϕ') are isomorphic if there exists an isomorphism $\lambda : E \to E'$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \otimes L \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda\otimes 1} \\ E' & \xrightarrow[\phi']{} & E' \otimes L. \end{array}$$

To any semistable pair (E,ϕ) there is associated its graded pair $\mathrm{gr}(E,\phi)$ (*see* [N] for the definition), and we have a notion of S-equivalent semistable pairs, analogous to the notion of S-equivalent semistable bundles.

In [N], Nitsure proved that there exists a coarse moduli space $\mathcal{M}(r,d,L)$ for S-equivalence classes of semistable pairs (E,ϕ) of rank r and degree d on C. The scheme $\mathcal{M}(r,d,L)$ is quasi-projective and has an open subscheme $\mathcal{M}'(r,d,L)$ which is the moduli scheme of stable pairs.

A family (E_Y, Φ_Y) of pairs parametrized by a scheme Y is a vector bundle E_Y on $C \times Y$ together with an element $\Phi_Y \in \Gamma(Y, (\pi_Y)_*(\mathcal{E}nd(E_Y) \otimes \pi_C^*(L)))$. Two families (E_Y, Φ_Y) and (E'_Y, Φ'_Y) are said to be equivalent if, for all points $y \in Y$, the pairs (E_y, Φ_y) and (E'_y, Φ'_y) are isomorphic.

DEFINITION 1.2.1. – Let $p' : \mathcal{M}' \times C \to C$ be the canonical projection. A Poincaré pair $(\mathcal{E}', \Phi)$ on $\mathcal{M}'$ consists of a vector bundle $\mathcal{E}'$ on $\mathcal{M}' \times C$ together with a morphism $\Phi : \mathcal{E}' \to \mathcal{E}' \otimes p'^*(L)$ such that for every noetherian scheme of finite type Y over $\mathbb{C}$ and for every pair $(\mathcal{F}, \Psi)$, where $\mathcal{F}$ is a locally free sheaf of finite rank over $Y \times C$ and $\Psi : \mathcal{F} \to \mathcal{F} \otimes p_C^*(L)$ is a homomorphism of $\mathcal{O}_{Y\times C}$-Modules, such that for every closed point $y \in Y$ the isomorphism class of the pair $(\mathcal{F}|_{\{y\}\times C}, \Psi|_{\{y\}\times C})$ belongs to $\mathcal{M}'$, there exists a unique morphism $\rho = \rho_{(\mathcal{F},\Psi)} : Y \to \mathcal{M}'$ such that $(\mathcal{F}, \Psi)$ is equivalent to $(\rho \times 1_C)^*(\mathcal{E}', \Phi)$.

Remark 1.2.2. – It is known that, when $(r, d) = 1$, a Poincaré pair exists on $\mathcal{M}'$. For general r and d one can prove that, locally in the étale topology on $\mathcal{M}'$, there exist Poincaré families of stable pairs. This will be enough to justify our reasoning involving infinitesimal deformations of a local Poincaré family.

Nitsure proved ([N, Proposition 3.6]) that there exists a local universal family for semistable pairs.

Remark 1.2.3. – The construction of the moduli space of stable pairs $\mathcal{M}'(r, d, L)$ made by Nitsure parallels the classical construction of the moduli space $\mathcal{U}(r, d)$. This may be summarized as follows. Keeping the notations introduced in Remark 1.1.2, it can be proved that there exists a scheme S^s together with a family of pairs $(\mathcal{F}_{S^s}, \Phi_{S^s})$ parametrized by S^s which is a local universal family for stable pairs ([N, Proposition 3.6]). The natural action of $\mathrm{GL}(p)$ on R^s extends canonically to an action on S^s. Note that the center of $\mathrm{GL}(p)$ acts trivially, hence this action goes down to an action of $\mathrm{PGL}(p)$. Finally we set $\mathcal{M}' = S^s/\mathrm{PGL}(p)$, and we can prove that this is a geometric quotient ([N, Section 5]).

The group $\mathrm{GL}(p)$ acts also on the local universal family $(\mathcal{F}_{S^s}, \Phi_{S^s})$, but this action does not factor through an action of $\mathrm{PGL}(p)$ since the center $k^* \cdot I$ of $\mathrm{GL}(p)$ acts on $\mathcal{F}_{S^s}$ by multiplication by constants, hence we cannot construct a universal family on $\mathcal{M}'$ by taking the quotient of the family $(\mathcal{F}_{S^s}, \Phi_{S^s})$. On the other hand the action of $k^* \cdot I$ on the element Φ_{S^s} is trivial, and is also trivial its action on $\mathcal{E}nd(\mathcal{F}_{S^s})$, hence we can take quotients by $\mathrm{PGL}(p)$. As a consequence we derive the existence of a sheaf, which we shall denote by $\mathcal{E}nd(\mathcal{E})$, and of a canonical section Φ of $\mathcal{E}nd(\mathcal{E}) \otimes p^*L$, with the property that, for every stable pair (E, ϕ) in $\mathcal{M}'$, the restriction of Φ to $\{(E, \phi)\} \times C$ is isomorphic to $\phi : E \to E \otimes L$.

In the sequel we shall denote by (E, ϕ) either a stable pair on C, or the point of $\mathcal{M}'(r, d, L)$ corresponding to the isomorphism class of (E, ϕ).

Remark 1.2.4. – Note that if E is a stable vector bundle then (E, ϕ) is a stable pair for all $\phi \in H^0(C, \mathcal{E}nd(E) \otimes L)$.

For an alternative construction of the moduli space of stable pairs we refer to the paper by Simpson [Si]. He considers stable pairs only in the case $L = K$, but his methods work in general to give an alternative proof of Nitsure's results. In particular the existence of étale Poincaré pairs follows from his Theorem 4.7.

1.3 THE VARIETY $\mathcal{P}(r, d, L)$.

Let L be a line bundle on C such that either $L \cong K$ or $\deg(L) > \deg(K)$. In this section we shall give an alternative description of the moduli space $\mathcal{P} = \mathcal{P}(r, d, L)$ of isomorphism classes of pairs (E, ϕ) with E stable. From the preceding considerations it follows immediately that this moduli space exists and is a proper open subset of $\mathcal{M}'(r,d,L)$. Here we shall give a direct construction of $\mathcal{P}(r,d,L)$ as a vector bundle on $\mathcal{U}_s(r,d)$.

From now on we shall work over the complex field $\mathbb{C}$. Let us suppose that $r \geq 2$, and set $\mathcal{U} = \mathcal{U}_s(r,d)$. Let $p : \mathcal{U} \times C \to C$ and $q : \mathcal{U} \times C \to \mathcal{U}$ be the canonical projections and denote by $\mathcal{E}nd(\mathcal{E})$ the sheaf on $\mathcal{U} \times C$ defined in Remark 1.1.2. Let us consider the quasi-coherent sheaf $\mathcal{H} = q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^*L)$ on $\mathcal{U}$. We have the following

LEMMA 1.3.1. – *If $L \cong K$ or $\deg(L) > \deg(K)$, then $\mathcal{H}$ is a locally free sheaf of finite rank on $\mathcal{U}$, and there is a canonical isomorphism $\mathcal{H}(\{E\}) \cong \mathrm{Hom}(E, E \otimes L)$.*

Proof. – The sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L) = \mathcal{E}nd(\mathcal{E}) \otimes p^*L$ is a locally free sheaf of finite rank on $\mathcal{U} \times C$, flat over $\mathcal{U}$. For each point $E \in \mathcal{U}$, let us denote by $j_E : \{E\} \to \mathcal{U}$ and $j'_E : \{E\} \times C \to \mathcal{U} \times C$ the canonical inclusions. We have:

$$\begin{aligned} h^0(\{E\}, \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L)) &\overset{\text{def}}{=} \dim H^0(\{E\} \times C, {j'}^*_E \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L)) \\ &= \dim \mathrm{Hom}(E, E \otimes L) \\ &= h^0(C, \mathcal{E}nd(E) \otimes L). \end{aligned}$$

The stability of E and the stated hypotheses on L imply that $h^0(C, \mathcal{E}nd(E) \otimes L)$ is constant as $\{E\}$ varies in $\mathcal{U}$. Thus we can apply the theorem of Grauert ([Ha2, Ch. 3, Cor. 12.9]), to prove that $\mathcal{H}$ is a locally free sheaf on $\mathcal{U}$ and that the natural map

$$\mathcal{H}(\{E\}) = j_E^* q_* \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L) \to H^0(\{E\} \times C, {j'}^*_E \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L)) = \mathrm{Hom}(E, E \otimes L)$$

is an isomorphism. □

We set $\mathcal{P} = \mathbf{Spec}(\mathrm{Sym}(\mathcal{H}^*))$, where $\mathrm{Sym}(\mathcal{H}^*)$ denotes the symmetric algebra of the dual sheaf of $\mathcal{H}$. $\mathcal{P}$ has a natural structure of vector bundle over $\mathcal{U}$, $\pi : \mathcal{P} \to \mathcal{U}$, and the preceding Lemma implies that the fiber $\pi^{-1}(E)$ is canonically isomorphic to the vector space $\mathrm{Hom}(E, E \otimes L)$. Hence the variety $\mathcal{P}$ may be described set-theoretically as the set of isomorphism classes of pairs (E, ϕ), with $E \in \mathcal{U}$ and $\phi \in \mathrm{Hom}(E, E \otimes L)$.

In general there does not exist a Poincaré pair on $\mathcal{P}$, since there does not even exist a Poincaré vector bundle on $\mathcal{U}$. When $(r,d) = 1$, however, a Poincaré pair on $\mathcal{P}$ may be obtained by restricting a Poincaré pair on $\mathcal{M}'$. In the following proposition we give an alternative construction of a Poincaré pair on $\mathcal{P}$.

PROPOSITION 1.3.2. – *If r and d are relatively prime, then there exists a Poincaré pair on $\mathcal{P}$.*

Proof. – Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{P} \times C & \xrightarrow{\pi'} & \mathcal{U} \times C \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \mathcal{P} & \xrightarrow[\pi]{} & \mathcal{U}, \end{array}$$

where p' and q' are the canonical projections and $\pi' = \pi \times 1$.

Let $\mathcal{E}$ be a Poincaré vector bundle on $\mathcal{U}$ and set $\mathcal{E}' = \pi'^*\mathcal{E}$. $\mathcal{E}'$ is a locally free sheaf on $\mathcal{P} \times C$ of rank equal to the rank of $\mathcal{E}$ and $\mathcal{E}'|_{\{(E,\phi)\}\times C} \cong \mathcal{E}|_{\{E\}\times C} \cong E$, for all points $(E,\phi) \in \mathcal{P}$.

The vector bundle $\pi^*\mathcal{H}$ on $\mathcal{P}$ has a canonical section and, by using the flatness of π and the fact that $\mathcal{E}$ is locally free of finite rank, we have:

$$\begin{aligned}\pi^*\mathcal{H} &= \pi^* q_* \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L)\\ &\cong q'_*\pi'^*\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*L)\\ &\cong q'_*\mathcal{H}om(\mathcal{E}', \mathcal{E}' \otimes p'^*L),\end{aligned}$$

hence the canonical section of $\pi^*\mathcal{H}$ determines a canonical section of $q'_*\mathcal{H}om(\mathcal{E}', \mathcal{E}'\otimes p'^*L)$, *i.e.*, a morphism $\Phi : \mathcal{E}' \to \mathcal{E}' \otimes p'^*(L)$.

By reasoning on the vector bundles associated to the corresponding locally free sheaves, it is easy to prove that the restriction of $(\mathcal{E}', \Phi)$ to $\{(E,\phi)\} \times C$ is isomorphic to the pair (E,ϕ). Now, by using the fact that $\mathcal{E}$ is a Poincaré bundle on $\mathcal{U}$, and recalling the definition of $\mathcal{P}$, it is not difficult to prove that $(\mathcal{E}', \Phi)$ is a Poincaré pair. □

Remark 1.3.3. – In the general case, *i.e.*, when r and d are not relatively prime, there does not exist a Poincaré vector bundle $\mathcal{E}$ on $\mathcal{U}$, but the sheaf $\mathcal{E}nd(\mathcal{E})$ is still defined (*see* Remark 1.1.2), and the same is true for the sheaf $\mathcal{H}$. Let us denote by $\mathcal{E}nd(\mathcal{E}')$ the pull-back $\pi'^*\mathcal{E}nd(\mathcal{E})$ of $\mathcal{E}nd(\mathcal{E})$ to $\mathcal{P} \times C$. By adapting the proof of Proposition 1.3.2, we can still prove that the sheaf $\pi^*\mathcal{H}$ is isomorphic to $q'_*(\mathcal{E}nd(\mathcal{E}') \otimes p'^*L)$, hence the canonical section of $\pi^*\mathcal{H}$ determines again a canonical section Φ of $\mathcal{E}nd(\mathcal{E}') \otimes p'^*L$.

Remark 1.3.4. – If $L = K$, the canonical bundle on C, the variety $\mathcal{P}$ is isomorphic to the cotangent bundle $T^*\mathcal{U}$ of $\mathcal{U}$. This is the situation considered by N. Hitchin in [H].

2. The Hitchin map

2.1 Twisted endomorphisms.

Let E be a vector bundle of rank r and L a line bundle on C, and let $\phi : E \to E \otimes L$ be a homomorphism of $\mathcal{O}_C$-Modules. By interpreting ϕ as a morphism from $E \otimes E^* \cong \mathcal{H}om(E,E)$ to L and taking the image of the identity section of $\mathcal{H}om(E,E)$, one can define the trace of ϕ, $\mathrm{Tr}(\phi) \in H^0(C,L)$. More generally, one can define the characteristic coefficients of ϕ, $a_i \in H^0(C,L^i)$, for $0 \le i \le r$, by setting $a_i = (-1)^{i-1}\mathrm{Tr} \wedge^i \phi$. The Cayley-Hamilton theorem then asserts that ϕ satisfies its characteristic equation, *i.e.*, that $\sum_{i=0}^{r} a_i\phi^{r-i} = 0$, interpreted as a homomorphism from E to $E \otimes L^r$.

The sections $\mathrm{Tr}(\wedge^n\phi)$ are related to $\mathrm{Tr}(\phi^n)$ by Newton's relations, of the form

$$\mathrm{Tr}(\wedge^n\phi) = \frac{(-1)^{n-1}}{n}\mathrm{Tr}(\phi^n) + Q_n(\mathrm{Tr}(\phi), \ldots, \mathrm{Tr}(\phi^{n-1})), \tag{2.1.1}$$

where Q_n is a universal polynomial in $n-1$ variables, with rational coefficients.

2.2 THE HITCHIN MAP.

By using the preceding result, we are able to define a morphism

$$H : \mathcal{M}'(r,d,L) \to \bigoplus_{i=1}^{r} H^0(C, L^i) \tag{2.2.1}$$

which associates to each pair $(E, \phi) \in \mathcal{M}'(r,d,L)$ the characteristic coefficients of ϕ. This morphism may be defined on the whole moduli space of semistable pairs $\mathcal{M}(r,d,L)$, in which case it is a proper morphism (*see* [N]).

As it is shown in [BNR], for every element $s = (s_i) \in \bigoplus_{i=1}^{r} H^0(C, L^i)$ we can construct a 1-dimensional scheme X_s and a finite morphism $\pi : X_s \to C$. The set of all s for which the scheme X_s is integral (*i.e.*, irreducible and reduced) and smooth is open and nonempty under general assumptions on L. X_s is called the *spectral curve* associated to s.

The principal result, proved in [BNR], is the following

THEOREM 2.2.1. – *Let $s = (s_i) \in \bigoplus_{i=1}^{r} H^0(C, L^i)$ be such that the corresponding scheme X_s is integral. Then there is a bijective correspondence between isomorphism classes of torsion free sheaves of rank 1 on X_s and isomorphism classes of pairs (E, ϕ), where E is a vector bundle of rank r on C and $\phi : E \to E \otimes L$ a homomorphism of $\mathcal{O}_C$-Modules with characteristic coefficients s_i.*

Remark 2.2.2. – When X_s is nonsingular we may replace 'torsion free sheaves of rank 1' by 'line bundles' in the preceding theorem.

This shows that the set of all pairs (E, ϕ), where E is a vector bundle of rank r on C and $H((E, \phi)) = s$, is isomorphic to the Jacobian variety $\mathrm{Jac}(X_s)$. Since we shall be interested only in pairs (E, ϕ) with E stable, it can be proved that the corresponding subset of $\mathrm{Jac}(X_s)$ is the complement of a closed subset of codimension $\geq 2g - 2$, if $r \geq 3$, and, in any case, the codimension is always ≥ 2, except for the case $g = r = 2$, which will be enough for us.

Finally we have seen that the inverse image $H^{-1}(s)$, for s generic, is isomorphic to an open subset of an abelian variety. It follows that $\dim H^{-1}(s) = \frac{1}{2} r(r-1)\deg(L) + r(g-1) + 1$.

3. Infinitesimal study of the variety $\mathcal{M}'$

3.1 INFINITESIMAL DEFORMATIONS OF PAIRS.

Let $\mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers over $\mathbb{C}$. By convenience of notations in the sequel it will be denoted simply by $\mathbb{C}[\epsilon]$. Let us denote by C_ϵ the fiber product $C \times \mathrm{Spec}(\mathbb{C}[\epsilon])$. If $p_\epsilon : C_\epsilon \to C$ is the natural morphism and F is a vector bundle on C, we shall denote by $F[\epsilon]$ its trivial infinitesimal deformation, *i.e.*, its pull-back to C_ϵ: $F[\epsilon] = p_\epsilon^*(F)$.

DEFINITION 3.1.1. – A (linear) infinitesimal deformation of a pair (E, ϕ) is a pair $(E_\epsilon, \phi_\epsilon)$, where E_ϵ is a locally free sheaf on C_ϵ and $\phi_\epsilon : E_\epsilon \to E_\epsilon \otimes L[\epsilon]$ is a morphism, together with isomorphisms $E \cong E_\epsilon \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}$ such that $\phi_\epsilon \otimes_{\mathbb{C}[\epsilon]} \mathbb{C}$ goes over into ϕ. We shall say, for short, that $(E_\epsilon, \phi_\epsilon)$ restricts to (E, ϕ). An isomorphism between two infinitesimal deformations $(E'_\epsilon, \phi'_\epsilon)$ and $(E''_\epsilon, \phi''_\epsilon)$ is defined as an isomorphism $\lambda_\epsilon : E'_\epsilon \to E''_\epsilon$ restricting to the identity on E, such that the following diagram is commutative:

$$\begin{array}{ccc} E'_\epsilon & \xrightarrow{\phi'_\epsilon} & E'_\epsilon \otimes L[\epsilon] \\ \downarrow \lambda_\epsilon & & \downarrow \lambda_\epsilon \otimes 1 \\ E''_\epsilon & \xrightarrow[\phi''_\epsilon]{} & E''_\epsilon \otimes L[\epsilon]. \end{array}$$

Let us consider the following complex of locally free sheaves on C, which we shall denote for short by $[\cdot, \phi]$:

$$0 \to \mathcal{E}nd(E) \xrightarrow{[\cdot,\phi]} \mathcal{E}nd(E) \otimes L \to 0,$$

where, for every section α of $\mathcal{E}nd(E)$ over an open subset $U \subset C$, $[\alpha, \phi]$ means $(\alpha \otimes 1_L) \circ \phi - \phi \circ \alpha$.

We have the following

PROPOSITION 3.1.2. – *The isomorphism classes of linear infinitesimal deformations of the pair (E, ϕ) are canonically parametrized by the first hypercohomology group $\mathbb{H}^1([\cdot, \phi])$ of the complex $[\cdot, \phi]$.*

Proof. – Let $(E_\epsilon, \phi_\epsilon)$ be such a deformation. Consider a covering $\mathcal{U} = (U_i)_{i \in I}$ of C by open affine subsets $U_i = \mathrm{Spec} A_i$, and write $U_{ij} = U_i \cap U_j = \mathrm{Spec} A_{ij}$. The open affine subsets $U_i[\epsilon] = \mathrm{Spec} A_i[\epsilon]$ constitute an open affine covering of the scheme C_ϵ. Writing $M_i = \Gamma(U_i, E)$, $M_{ij} = \Gamma(U_{ij}, E)$, $N_i = \Gamma(U_i, L)$, etc., for every U_i we have an isomorphism

$$E_\epsilon|_{U_i[\epsilon]} \overset{f_i}{\underset{\sim}{\longrightarrow}} \widetilde{M_i[\epsilon]}.$$

Then $f_j|_{U_{ij}[\epsilon]} \circ f_i^{-1}|_{U_{ij}[\epsilon]}$ induces an automorphism of $\widetilde{M_{ij}[\epsilon]}$ corresponding to an element of the form $1 + \epsilon\eta_{ij} \in \mathrm{End}(M_{ij}[\epsilon])$. The compatibility conditions on the intersection of three open subsets show that $\{\eta_{ij}\}$ is a 1-cocycle with values in $\mathcal{E}nd(E)$. It is easy to see that the isomorphism class of the infinitesimal deformation E_ϵ is uniquely determined by the element of $H^1(C, \mathcal{E}nd(E))$ defined by $\{\eta_{ij}\}$. This element is precisely the Kodaira-Spencer class of the infinitesimal deformation E_ϵ of E. Conversely, given a 1-cocycle $\{\eta_{ij}\}$ with values in $\mathcal{E}nd(E)$, the corresponding sheaf E_ϵ may be constructed by gluing the sheaves $\widetilde{M_i[\epsilon]}$ and $\widetilde{M_j[\epsilon]}$ along the open subset $U_{ij}[\epsilon]$ by means of the isomorphism $\widetilde{1 + \epsilon\eta_{ij}} : \widetilde{M_i[\epsilon]}|_{U_{ij}[\epsilon]} \to \widetilde{M_j[\epsilon]}|_{U_{ij}[\epsilon]}$.

Let us consider now the infinitesimal deformation $\phi_\epsilon : E_\epsilon \to E_\epsilon \otimes L[\epsilon]$ of $\phi : E \to E \otimes L$. By what we have seen, ϕ_ϵ may be described by giving homomorphisms $\phi_{\epsilon i} : M_i[\epsilon] \to M_i \otimes N_i[\epsilon]$ which restricts to ϕ modulo ϵ and which are compatible with the gluing isomorphisms $1 + \epsilon\eta_{ij}$. In other words, this means that $\phi_{\epsilon i} = \phi + \epsilon\alpha_i$, for some homomorphism $\alpha_i : M_i \to M_i \otimes N_i$, and, for each i, j, the following diagram is commutative:

$$(3.1.1)\qquad \begin{array}{ccc} M_{ij}[\epsilon] & \xrightarrow{\phi_{\epsilon_i}} & M_{ij} \otimes N_{ij}[\epsilon] \\ \downarrow {\scriptstyle 1+\epsilon\eta_{ij}} & & \downarrow {\scriptstyle (1+\epsilon\eta_{ij})\otimes 1} \\ M_{ij}[\epsilon] & \xrightarrow[\phi_{\epsilon_j}]{} & M_{ij} \otimes N_{ij}[\epsilon]. \end{array}$$

By replacing the expressions of ϕ_{ϵ_i} given above, it follows from (3.1.1) that

$$(\alpha_j - \alpha_i)|_{U_{ij}[\epsilon]} = [\eta_{ij}, \phi].$$

If $(E'_\epsilon, \phi'_\epsilon)$ is another infinitesimal deformation of (E, ϕ) isomorphic to $(E_\epsilon, \phi_\epsilon)$, and if $(\{\alpha'_i\}, \{\eta'_{ij}\})$ is the pair constructed from $(E'_\epsilon, \phi'_\epsilon)$ as $(\{\alpha_i\}, \{\eta_{ij}\})$ from $(E_\epsilon, \phi_\epsilon)$, then it follows that $\alpha'_i = \alpha_i + [\lambda_i, \phi]$ and $\eta'_{ij} = \eta_{ij} + (\lambda_j - \lambda_i)|_{U_{ij}[\epsilon]}$, where $\lambda_\epsilon = 1 + \epsilon\lambda$ is the isomorphism $E'_\epsilon \xrightarrow{\sim} E_\epsilon$.

Now, if we take the same open covering $\mathcal{U} = (U_i)_{i \in I}$ of C as a Čech covering to calculate cohomology, it is immediate to see that the pair $(\{\alpha_i\}, \{\eta_{ij}\})$ associated to $(E_\epsilon, \phi_\epsilon)$ defines an element of the first hypercohomology group $\mathbb{H}^1([\cdot, \phi])$ of the complex $[\cdot, \phi]$ which depends only on the isomorphism class of the infinitesimal deformation $(E_\epsilon, \phi_\epsilon)$. Moreover, to each element of $\mathbb{H}^1([\cdot, \phi])$ there corresponds an isomorphism class of infinitesimal deformations $(E_\epsilon, \phi_\epsilon)$ of (E, ϕ). This proves our assertion. □

By using the existence of a local universal family for stable pairs (*see* Remark 1.2.2), we get the following

PROPOSITION 3.1.3. – *The tangent space $T_{(E,\phi)}\mathcal{M}'$ to $\mathcal{M}'$ at the point (E, ϕ) is canonically isomorphic to $\mathbb{H}^1([\cdot, \phi])$.*

Let us consider now the following exact sequence of complexes (written vertically):

$$(3.1.2)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{E}nd(E) & \longrightarrow & \mathcal{E}nd(E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow {\scriptstyle [\cdot,\phi]} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}nd(E) \otimes L & \longrightarrow & \mathcal{E}nd(E) \otimes L & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Taking the associated long exact sequence of hypercohomology, we get

$$(3.1.3)\qquad \begin{aligned} 0 \to \mathbb{H}^0([\cdot, \phi]) &\to H^0(C, \mathcal{E}nd(E)) \xrightarrow{[\cdot,\phi]} H^0(C, \mathcal{E}nd(E) \otimes L) \to \mathbb{H}^1([\cdot, \phi]) \\ &\to H^1(C, \mathcal{E}nd(E)) \xrightarrow{[\cdot,\phi]} H^1(C, \mathcal{E}nd(E) \otimes L) \to \mathbb{H}^2([\cdot, \phi]) \to 0. \end{aligned}$$

Remark 3.1.4. – This exact sequence may also be deduced from the first spectral sequence of hypercohomology of the complex $[\cdot, \phi]$.

Now we need a result from the duality theory for the hypercohomology of a complex of locally free sheaves, which is the analogue of the classical Serre duality for ordinary cohomology (*see* [Ha1]).

Let $F^{\cdot}$ be a (bounded) complex of locally free sheaves on C,

$$F^{\cdot}: \quad 0 \to F^0 \to F^1 \to \cdots \to F^n \to 0.$$

Then the dual of the i-th hypercohomology group $\mathbb{H}^i(F^{\cdot})$ is canonically isomorphic to the $(n-i+1)$-th hypercohomology group $\mathbb{H}^{n-i+1}(\check{F}^{\cdot})$ of the dual complex $\check{F}^{\cdot}$ given by $\check{F}^j = (F^{n-j})^* \otimes K$, the coboundary morphisms being transposes of those in $F^{\cdot}$, tensored with the identity id_K.

By applying this result, we find that the dual complex of $[\cdot, \phi]$ is the complex

$$0 \to (\mathcal{E}nd(E))^* \otimes L^{-1} \otimes K \xrightarrow{[\cdot,\phi]^t \otimes 1_K} (\mathcal{E}nd(E))^* \otimes K \to 0,$$

and it is easy to prove that, under the canonical identification between $(\mathcal{E}nd(E))^*$ and $\mathcal{E}nd(E)$ given by the pairing trace, the above complex coincides with the following complex, which we shall denote by $[\phi, \cdot]$:

$$0 \to \mathcal{E}nd(E) \otimes L^{-1} \otimes K \xrightarrow{[\phi,\cdot]} \mathcal{E}nd(E) \otimes K \to 0.$$

Considering now the exact sequence (3.1.3), it is easy to see that $\mathbb{H}^0([\cdot,\phi]) = \{\alpha \in H^0(C, \mathcal{E}nd(E)) \mid [\alpha, \phi] = 0\}$. Assuming the stability of E, this gives $\mathbb{H}^0([\cdot,\phi]) = H^0(C, \mathcal{E}nd(E)) = \mathbb{C}$.

As for $\mathbb{H}^2([\cdot,\phi])$, it follows from the duality theory for hypercohomology that it is isomorphic to the dual of $\mathbb{H}^0([\phi,\cdot])$, hence $\mathbb{H}^2([\cdot,\phi])^* \cong \{\alpha \in H^0(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K) \mid [\phi, \alpha] = 0\}$. Again, for E stable, we have either $\dim \mathbb{H}^2([\cdot,\phi]) = 1$ or $\dim \mathbb{H}^2([\cdot,\phi]) = 0$, depending on whether $L \cong K$ or $\deg(L) > \deg(K)$. In both cases this implies that the morphism $\mathbb{H}^1([\cdot,\phi]) \to H^1(C, \mathcal{E}nd(E))$ is surjective. In conclusion, for $(E,\phi) \in \mathcal{P}$, *i.e.* for E stable, we derive from (3.1.3) the exact sequence

$$(3.1.4) \qquad 0 \to H^0(C, \mathcal{E}nd(E) \otimes L) \to \mathbb{H}^1([\cdot,\phi]) \to H^1(C, \mathcal{E}nd(E)) \to 0.$$

From the definition of $\mathcal{P}$ it follows that the sheaf of relative differentials $\Omega^1_{\mathcal{P}/\mathcal{U}}$ is isomorphic to $\pi^*(\mathcal{H}^*) = (\pi^*\mathcal{H})^*$ (*see* [EGA IV, Cor. 16.4.9]), and we get the exact sequence ([EGA IV, Cor. 16.4.19 and Remark 16.4.24])

$$(3.1.5) \qquad 0 \to \pi^*(\Omega^1_{\mathcal{U}}) \to \Omega^1_{\mathcal{P}} \to (\pi^*\mathcal{H})^* \to 0,$$

from which we derive, by duality,

$$(3.1.6) \qquad 0 \to \pi^*(\mathcal{H}) \to T\mathcal{P} \to \pi^*(T\mathcal{U}) \to 0.$$

Taking the fibers over a point $(E,\phi) \in \mathcal{P}$, we get the exact sequence

$$(3.1.7) \qquad 0 \to \mathrm{Hom}(E, E \otimes L) \to T_{(E,\phi)}\mathcal{P} \to T_E\mathcal{U} \to 0,$$

which coincides, under the natural identifications $T_E\mathcal{U} \cong H^1(C, \mathcal{E}nd(E))$ and $T_{(E,\phi)}\mathcal{P} \cong \mathbb{H}^1([\cdot,\phi])$, with the exact sequence (3.1.4).

Remark 3.1.5. – From the preceding considerations on tangent spaces, it follows immediately that $\mathcal{P}$ is a nonsingular variety of dimension $h^0(C, \mathcal{E}nd(E) \otimes L) + h^1(C, \mathcal{E}nd(E))$. For $L \cong K$, using the theorem of Riemann-Roch and the fact that, for E stable, $h^0(C, \mathcal{E}nd(E)) = 1$, it follows that $\dim \mathcal{P} = 2r^2(g-1) + 2 = 2 \dim \mathcal{U}$. In the general case, $\deg(L) > \deg(K)$, we have $\dim \mathcal{P} = r^2 \deg(L) + 1$.

The preceding considerations may be extended to the case of general stable pairs $(E, \phi) \in \mathcal{M}'(r,d,L)$, but first we need the following lemma, whose proof may be found in [N, Proof of Proposition 7.1]:

LEMMA 3.1.6. – *Let* (E, ϕ) *be a stable pair and* L' *a line bundle over* C *with* $\deg L' \leq 0$. *Then* $\dim\{\alpha \in H^0(C, \mathcal{E}nd(E) \otimes L') \mid [\alpha, \phi] = 0\} = 1$ *if* $L' \cong \mathcal{O}_C$, *and it is equal to* 0 *if* $\deg L' < 0$ *or* $\deg L' = 0$ *but* $L'^r \not\cong \mathcal{O}_C$.

As a consequence, we get

COROLLARY 3.1.7. – *If* (E, ϕ) *is a stable pair, then* $\mathbb{H}^0([\cdot, \phi]) \cong \mathbb{C}$.

COROLLARY 3.1.8. – *Let* (E, ϕ) *be a stable pair. Then* $\mathbb{H}^0([\phi, \cdot]) \cong \mathbb{C}$ *if* $L \cong K$, *and* $\mathbb{H}^0([\phi, \cdot]) = 0$ *if* $\deg L > \deg K$.

Now, by recalling the exact sequence (3.1.3) and using the theorem of Riemann-Roch, it follows that

$$\dim \mathbb{H}^1([\cdot, \phi]) = \begin{cases} 2r^2(g-1) + 2, & \text{if } L \cong K \\ r^2 \deg L + 1, & \text{if } \deg L > \deg K. \end{cases} \tag{3.1.8}$$

Remark 3.1.9. – We know that the moduli space $\mathcal{M}(2,d,L)$ is connected, for any d and L ([N, Theorem 7.5]). For $r > 2$ however, it is not known if $\mathcal{M}'(r,d,L)$ is in general connected, but it is evident that the variety $\mathcal{P}(r,d,L)$ is contained in a single connected component $\mathcal{M}'_0(r,d,L)$, which is an open subset of $\mathcal{M}'(r,d,L)$ It is not difficult to prove ([N, Proposition 7.4]) that $\mathcal{M}'_0(r,d,L)$, with the structure of an open subscheme of $\mathcal{M}'(r,d,L)$, is a smooth quasi-projective variety whose dimension is given by (3.1.8).

Now we turn to the study of the cotangent space $T^*_{(E,\phi)}\mathcal{M}'$ to $\mathcal{M}'$ at the point (E, ϕ). From our previous discussion on the duality theory for hypercohomology, we derive immediately the following

PROPOSITION 3.1.10. – *The cotangent space* $T^*_{(E,\phi)}\mathcal{M}'$ *is canonically identified with the first hypercohomology group* $\mathbb{H}^1([\phi, \cdot])$ *of the complex* $[\phi, \cdot]$.

Remark 3.1.11. – By computing cohomology using a Čech covering $\mathcal{V}$, the group $\mathbb{H}^1([\phi, \cdot])$ may be described explicitly as the set of pairs $(\{\alpha_i\}, \{\eta_{ij}\}) \in C^0(\mathcal{V}, \mathcal{E}nd(E) \otimes K) \times C^1(\mathcal{V}, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$, such that $\{\eta_{ij}\}$ is a 1-cocycle and $(\alpha_j - \alpha_i)|_{V_{ij}} = [\phi, \eta_{ij}]$, modulo the equivalence relation defined by $(\{\alpha'_i\}, \{\eta'_{ij}\}) \sim (\{\alpha_i\}, \{\eta_{ij}\})$ iff there exists an element $\{\theta_i\} \in C^0(\mathcal{V}, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$ such that $\alpha'_i = \alpha_i + [\phi, \theta_i]$ and $\eta'_{ij} = \eta_{ij} + (\theta_j - \theta_i)|_{V_{ij}}$.

Now, for $(E, \phi) \in \mathcal{P}$, applying the same reasoning as before to the complex $[\phi, \cdot]$, we get the exact sequence

$$0 \to H^0(C, \mathcal{E}nd(E) \otimes K) \to \mathbb{H}^1([\phi, \cdot]) \to H^1(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K) \to 0, \tag{3.1.9}$$

which, by Serre duality, coincides with the dual of (3.1.4).

Remark 3.1.12. – It is now easy to globalize this construction to the whole tangent and cotangent bundles to $\mathcal{M}'_0$ (we restrict to $\mathcal{M}'_0$ because it is not known if $\mathcal{M}'$ is smooth). For simplicity let us denote by $\mathcal{E}nd(\mathcal{E})$ the sheaf on $\mathcal{M}'_0 \times C$ which was previously denoted $\mathcal{E}nd(\mathcal{E}')$. Let Φ be the canonical section of $\mathcal{E}nd(\mathcal{E}) \otimes p^*L$ defined in Remark 1.3.3, and let us denote by $q : \mathcal{M}'_0 \times C \to \mathcal{M}'_0$ and $p : \mathcal{M}'_0 \times C \to C$ the canonical projections. If we denote by $[\cdot, \Phi]$ the complex of vector bundles over $\mathcal{M}'_0 \times C$

$$0 \to \mathcal{E}nd(\mathcal{E}) \xrightarrow{[\cdot,\Phi]} \mathcal{E}nd(\mathcal{E}) \otimes p^*(L) \to 0$$

and by $[\Phi, \cdot]$ the complex

$$0 \to \mathcal{E}nd(\mathcal{E}) \otimes p^*(L^{-1} \otimes K) \xrightarrow{[\Phi,\cdot]} \mathcal{E}nd(\mathcal{E}) \otimes p^*(K) \to 0,$$

we have canonical identifications $T\mathcal{M}'_0 \cong \mathbb{R}^1 q_*([\cdot, \Phi])$ and $T^*\mathcal{M}'_0 \cong \mathbb{R}^1 q_*([\Phi, \cdot])$, where $\mathbb{R}^1 q_*$ is the first hyperderived functor of q_*.

3.2 Explicit expression of the duality between $\mathbb{H}^1([\cdot, \phi])$ and $\mathbb{H}^1([\phi, \cdot])$

In the previous section we have found an explicit description of the hypercohomology groups $\mathbb{H}^1([\cdot, \phi])$ and $\mathbb{H}^1([\phi, \cdot])$ in terms of Čech cocycles. By Serre duality we have a nondegenerate pairing

$$\mathbb{H}^1([\cdot, \phi]) \times \mathbb{H}^1([\phi, \cdot]) \to H^1(C, K) \xrightarrow[\sim]{\text{res}} \mathbb{C}, \tag{3.2.1}$$

where res denotes the "residue" map.

We have the following

Proposition 3.2.1. – *In terms of Čech cocycles, the pairing* (3.2.1) *may be described as follows: to the elements of* $\mathbb{H}^1([\cdot, \phi])$ *and* $\mathbb{H}^1([\phi, \cdot])$ *represented respectively by cocycles* $(\{\alpha_i\}, \{\eta_{ij}\}) \in C^0(\mathcal{V}, \mathcal{E}nd(E) \otimes L) \times C^1(\mathcal{V}, \mathcal{E}nd(E))$ *and* $(\{\beta_i\}, \{\theta_{ij}\}) \in C^0(\mathcal{V}, \mathcal{E}nd(E) \otimes K) \times C^1(\mathcal{V}, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$, *we associate the element of* $H^1(C, K)$ *represented by the 1-cocycle defined, on the open subset* V_{ij}, *by* $\operatorname{Tr}(\alpha_i \theta_{ij} + \beta_j \eta_{ij})$.

Proof. – By using the explicit description of the hypercohomology groups previously found and elementary properties of the trace map, it is easy to prove that the collection of sections $\operatorname{Tr}(\alpha_i \theta_{ij} + \beta_j \eta_{ij}) \in \Gamma(V_{ij}, K)$ represents a well-defined element of $H^1(C, K)$. From this, our assertion follows immediately. □

3.3 Vector fields.

In this section we shall give an expression for vector fields on the variety $\mathcal{P}$ in terms of first order differential operators.

First let us recall some general facts. If X is a k-scheme, a tangent vector field on X is a k-linear map of sheaves $D : \mathcal{O}_X \to \mathcal{O}_X$ such that the induced map $D(U) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is a k-derivation, for every open subset U of X. Equivalently, a vector field on X may be expressed by an automorphism over $\operatorname{Spec}(k[\epsilon])$

$$\begin{array}{ccc} X \times \operatorname{Spec}(k[\epsilon]) & \xrightarrow{\tilde{D}} & X \times \operatorname{Spec}(k[\epsilon]) \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}(k[\epsilon]) & \end{array}$$

which restricts to the identity morphism of X when one looks at the fibers over $\mathrm{Spec}(k)$.

Over an open affine subset $U = \mathrm{Spec}A$ of X the tangent field $D : \mathcal{O}_X \to \mathcal{O}_X$ is given equivalently by a k-derivation $D(U) : A \to A$. In this situation the automorphism $\tilde{D}$ is determined by the k-algebra homomorphism $\tilde{D}(U) : A[\epsilon] \to A[\epsilon]$ given by $\tilde{D}(U) = 1 + \epsilon D(U)$.

We have the following result (*see* [M, pp. 100–101]):

Lemma 3.3.1. – *Let D_1 and D_2 be two vector fields on X and set $D_3 = [D_1, D_2]$. Let us denote by $\tilde{D}_1$, $\tilde{D}_2$ and $\tilde{D}_3$ the corresponding automorphisms of $X \times \mathrm{Spec}(k[\epsilon])$. Let $\sigma_i : k[\epsilon] \to k[\epsilon, \epsilon']$ be k-algebra homomorphisms defined by $\sigma_1(\epsilon) = \epsilon$, $\sigma_2(\epsilon) = \epsilon'$ and $\sigma_3(\epsilon) = \epsilon\epsilon'$. Then σ_i induces a morphism $\mathrm{Spec}(\sigma_i) : \mathrm{Spec}(k[\epsilon, \epsilon']) \to \mathrm{Spec}(k[\epsilon])$ and we get automorphisms*

$$\begin{array}{ccc} X \times \mathrm{Spec}(k[\epsilon,\epsilon']) & \xrightarrow{\tilde{D}'_i} & X \times \mathrm{Spec}(k[\epsilon,\epsilon']) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k[\epsilon,\epsilon']) & \end{array}$$

by taking fiber products with $\mathrm{Spec}(k[\epsilon, \epsilon'])$ over $\mathrm{Spec}(k[\epsilon])$ via $\mathrm{Spec}(\sigma_i)$.

Under these hypotheses it follows that $\tilde{D}'_3$ is equal to the commutator $[\tilde{D}'_2, \tilde{D}'_1] = \tilde{D}'_2\tilde{D}'_1\tilde{D}'^{-1}_2\tilde{D}'^{-1}_1$.

Let now $D : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$ be a tangent vector field on $\mathcal{P}$ and $\tilde{D}$ the corresponding automorphism of $\mathcal{P} \times \mathrm{Spec}(\mathbb{C}[\epsilon])$. Let $(\mathcal{E}, \Phi)$ be a local universal family for stable pairs in $\mathcal{P}$ (*see* Remark 1.2.2) and $(\mathcal{E}[\epsilon], \Phi[\epsilon])$ its pull-back to $\mathcal{P} \times \mathrm{Spec}(\mathbb{C}[\epsilon]) \times C$. The vector field D (or the automorphism $\tilde{D}$) may be described locally by giving the infinitesimal deformation $(\mathcal{E}_\epsilon, \Phi_\epsilon) = (\tilde{D} \times 1_C)^*(\mathcal{E}, \Phi)$ of the local universal family $(\mathcal{E}, \Phi)$. At a point $(E, \phi) \in \mathcal{P}$ the corresponding tangent vector is given by $(E_\epsilon, \phi_\epsilon) = (\mathcal{E}_\epsilon, \Phi_\epsilon)|_{\{(E,\phi)\} \times C_\epsilon}$, which is an infinitesimal deformation of the pair (E, ϕ).

From what we have previously seen (*see* Remark 3.1.12), the tangent field $(\mathcal{E}_\epsilon, \Phi_\epsilon)$ corresponds to a global section $(\alpha, \eta) = (\{\alpha_i\}, \{\eta_{ij}\})$ of $\mathbb{R}^1 q_*([\cdot, \Phi])$, which can be described in terms of first order differential operators.

First we need the following

Remark 3.3.2. – Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and $\mathcal{F}$, $\mathcal{G}$ two locally free sheaves on X. Let $\mathcal{D}\mathit{iff}^n_{X/Y}(\mathcal{F}, \mathcal{G})$ denote the sheaf of relative differential operators from $\mathcal{F}$ to $\mathcal{G}$ of order $\leq n$. In the sequel we shall be concerned only with first order differential operators. From general well-known results (*see* [EGA IV, §16.8]) it is easy to derive the following exact sequence

$$0 \to \mathcal{H}om_X(\mathcal{F}, \mathcal{G}) \to \mathcal{D}\mathit{iff}^1_{X/Y}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \otimes \mathcal{H}om_X(\mathcal{F}, \mathcal{G}) \to 0,$$

where σ is the *symbol* morphism. Then, if $\mathcal{G} = \mathcal{F}$ and if we restrict to differential operators with 'scalar symbol', written simply $\mathcal{D}^1_{X/Y}(\mathcal{F})$, we get the exact sequence

$$(3.3.1) \qquad 0 \to \mathcal{E}nd_X(\mathcal{F}) \to \mathcal{D}^1_{X/Y}(\mathcal{F}) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \to 0.$$

When r and d are relatively prime, *i.e.*, when there exists a Poincaré pair $(\mathcal{E}, \Phi)$ on $\mathcal{P}$, we may apply this result to the morphism $p : \mathcal{P} \times C \to C$, with $\mathcal{F} = \mathcal{E}$, finding the exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_C(\mathcal{E}) \to q^*T\mathcal{P} \to 0, \tag{3.3.2}$$

where $\mathcal{D}^1_C(\mathcal{E}) = \mathcal{D}^1_{\mathcal{P}\times C/C}(\mathcal{E})$ is the sheaf of first-order differential operators with scalar symbol on $\mathcal{E}$ which are $p^*(\mathcal{O}_C)$-linear, and $q : \mathcal{P} \times C \to \mathcal{P}$ is the canonical projection.

In the general case there does not exist a Poincaré pair on $\mathcal{P}$, but the sheaf $\mathcal{E}nd(\mathcal{E})$ is always defined, as we have seen in Remarks 1.1.2 and 1.3.3. By applying the same reasoning, we may prove that the sheaf $\mathcal{D}^1_C(\mathcal{E})$ is always defined, hence the exact sequence (3.3.2) exists even if r and d are not relatively prime.

We have the following

PROPOSITION 3.3.3. – *Let $D : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$ be a tangent vector field to $\mathcal{P}$ corresponding to the infinitesimal deformation $(\mathcal{E}_\epsilon, \Phi_\epsilon) = (\tilde{D} \times 1_C)^*(\mathcal{E}, \Phi)$, described by the global section $(\alpha, \eta) = (\{\alpha_i\}, \{\eta_{ij}\})$ of $\mathbb{R}^1 q_*([\cdot, \Phi])$. Then there exist an open affine covering $\mathcal{V} = (V_i)_{i\in I}$ of $\mathcal{P} \times C$ and first-order differential operators $\dot{D}_i \in \Gamma(V_i, \mathcal{D}^1_C(\mathcal{E}))$ such that $\alpha_i = [\dot{D}_i, \Phi]$ and $\eta_{ij} = (\dot{D}_j - \dot{D}_i)|_{V_{ij}}$.*

Proof. – We notice that if $\dot{D}$ is a section of $\mathcal{D}^1_C(\mathcal{E})$ over an open subset V, then $[\dot{D}, \Phi]$ is a section of $\mathcal{E}nd(\mathcal{E}) \otimes p^*(L)$, hence we have a homomorphism

$$[\cdot, \Phi] : \mathcal{D}^1_C(\mathcal{E}) \to \mathcal{E}nd(\mathcal{E}) \otimes p^*(L). \tag{3.3.3}$$

The complex $0 \to \mathcal{D}^1_C(\mathcal{E}) \xrightarrow{[\cdot,\Phi]} \mathcal{E}nd(\mathcal{E}) \otimes p^*(L) \to 0$ will be denoted by $[\cdot, \Phi]_{\mathcal{D}^1_C(\mathcal{E})}$.

By recalling (3.3.2) we can write the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}nd(\mathcal{E}) & \longrightarrow & \mathcal{D}^1_C(\mathcal{E}) & \longrightarrow & q^*T\mathcal{P} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle [\cdot,\Phi]} & & \big\downarrow{\scriptstyle [\cdot,\Phi]_{\mathcal{D}^1_C(\mathcal{E})}} & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{E}nd(\mathcal{E}) \otimes p^*(L) & = & \mathcal{E}nd(\mathcal{E}) \otimes p^*(L) & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Now, by applying the functor q_*, and noting that $q_* q^* T\mathcal{P} \cong T\mathcal{P}$, since q is a proper morphism, we get a long exact sequence of hyperderived functors, a piece of which is

$$\cdots \to T\mathcal{P} \to \mathbb{R}^1 q_*([\cdot, \Phi]) \to \mathbb{R}^1 q_*([\cdot, \Phi]_{\mathcal{D}^1_C(\mathcal{E})}) \to \cdots. \tag{3.3.4}$$

It is evident that the map $T\mathcal{P} \to \mathbb{R}^1 q_*([\cdot, \Phi])$ is the isomorphism described in Remark 3.1.12, hence the image of $\mathbb{R}^1 q_*([\cdot, \Phi])$ in $\mathbb{R}^1 q_*([\cdot, \Phi]_{\mathcal{D}^1_C(\mathcal{E})})$ is zero. This means that, for each section $(\alpha, \eta) = (\{\alpha_i\}, \{\eta_{ij}\})$ of $\mathbb{R}^1 q_*([\cdot, \Phi])$, there exist sections $\dot{D}_i$ of $\mathcal{D}^1_C(\mathcal{E})$ over suitable open subsets V_i, such that $\alpha_i = [\dot{D}_i, \Phi]$ and $\eta_{ij} = (\dot{D}_j - \dot{D}_i)|_{V_{ij}}$. □

Remark 3.3.4. – Now we give another interpretation of (the proof of) the preceding proposition, which will be useful in the sequel.

Let $D : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$ be the derivation corresponding to the infinitesimal deformation $(\mathcal{E}_\epsilon, \Phi_\epsilon) = (\tilde{D} \times 1_C)^*(\mathcal{E}, \Phi)$ determined by the global section $(\alpha, \eta) = (\{\alpha_i\}, \{\eta_{ij}\})$ of $\mathbb{R}^1 q_*([\cdot, \Phi])$. Let $(V_i)_{i\in I}$, $V_i = \mathrm{Spec}(A_i)$, be an open affine covering of $\mathcal{P} \times C$. The vector field D is locally described by giving, for each $i \in I$, a $\mathbb{C}[\epsilon]$-automorphism of $A_i[\epsilon]$ of the form $1 + \epsilon D_i$, where $D_i : A_i \to A_i$ is the $\mathbb{C}$-derivation determined by the restriction of D to V_i. Let $M_i = \Gamma(V_i, \mathcal{E})$ and $M_i[\epsilon] = \Gamma(V_i, \mathcal{E}[\epsilon])$. The infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times 1_C)^* \mathcal{E}[\epsilon]$ may be described as obtained by gluing the sheaves $\widetilde{M_i[\epsilon]}$ by means of suitable isomorphisms.

Let us denote by

$$\widetilde{1 + \epsilon \dot{D}_i} : \mathcal{E}_\epsilon|_{V_i \times \mathrm{Spec}(\mathbb{C}[\epsilon])} \xrightarrow{\sim} \widetilde{M_i[\epsilon]}$$

the trivialization isomorphisms, where $\dot{D}_i : M_i \to M_i$ is a first order differential operator with associated $\mathbb{C}$-derivation $D_i : A_i \to A_i$. By what we have previously seen, the gluing isomorphism on the intersection $V_i \cap V_j$ is given by $1 + \epsilon \eta_{ij} = (1 + \epsilon \dot{D}_j)(1 + \epsilon \dot{D}_i)^{-1} = 1 + \epsilon(\dot{D}_j - \dot{D}_i)$, hence $\eta_{ij} = \dot{D}_j - \dot{D}_i$.

Noticing that the pull-back of Φ to $\mathcal{E}_\epsilon = (\tilde{D} \times 1_C)^*(\mathcal{E})$ is locally given by

$$M_i[\epsilon] \otimes_{A_i[\epsilon]} A_i[\epsilon] \xrightarrow{\Phi \otimes 1} M_i \otimes L_i[\epsilon] \otimes_{A_i[\epsilon]} A_i[\epsilon],$$

we get the following commutative diagram defining α_i:

$$\text{(3.3.5)} \qquad \begin{array}{ccc} \mathcal{E}_\epsilon \otimes L[\epsilon]|_{V_i \times \mathrm{Spec}(\mathbb{C}[\epsilon])} & \xrightarrow{\widetilde{1+\epsilon \dot{D}_i} \otimes 1} & M_i \otimes \widetilde{L_i[\epsilon]} \\ \uparrow \Phi & & \uparrow \Phi + \epsilon \alpha_i \\ \mathcal{E}_\epsilon|_{V_i \times \mathrm{Spec}(\mathbb{C}[\epsilon])} & \xrightarrow[\widetilde{1+\epsilon \dot{D}_i}]{} & \widetilde{M_i[\epsilon]}. \end{array}$$

From this we deduce that $\alpha_i = [\dot{D}_i, \Phi]$.

Remark 3.3.5. – An analogue of Proposition 3.3.3 holds when we replace the variety $\mathcal{P}$ by $\mathcal{U}$. From the exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_C(\mathcal{E}) \to q^* T\mathcal{U} \to 0,$$

where $\mathcal{E}nd(\mathcal{E})$ and $\mathcal{D}^1_C(\mathcal{E})$ are now sheaves on $\mathcal{U} \times C$, one gets

$$\cdots \to T\mathcal{U} \xrightarrow{\delta} R^1 q_* \mathcal{E}nd(\mathcal{E}) \to R^1 q_* \mathcal{D}^1_C(\mathcal{E}) \to \cdots .$$

As in the preceding case, it is easy to verify that the connecting homomorphism $\delta : T\mathcal{U} \to R^1 q_* \mathcal{E}nd(\mathcal{E})$ is the isomorphism given by the Kodaira-Spencer map, hence the image of $R^1 q_* \mathcal{E}nd(\mathcal{E})$ in $R^1 q_* \mathcal{D}^1_C(\mathcal{E})$ is zero. This means that for every section of $R^1 q_* \mathcal{E}nd(\mathcal{E})$, represented by a 1-cocycle $\{\eta_{ij}\}$, there exist differential operators $\dot{D}_i$ such that $\eta_{ij} = \dot{D}_j - \dot{D}_i$.

4. Symplectic geometry

4.1 Symplectic and Poisson structures.

In this section we briefly recall some definitions and results of symplectic geometry which we shall need later.

Let X be a smooth algebraic variety over the complex field $\mathbb{C}$. A symplectic structure on X is a closed nondegenerate 2-form $\omega \in H^0(X, \Omega_X^2)$. Note that the existence of a symplectic structure on X implies that the dimension of X is even. Given a symplectic structure ω we define, for every $f \in \Gamma(U, \mathcal{O}_X)$, the Hamiltonian vector field H_f by requiring that $\omega(H_f, v) = \langle df, v\rangle$, for every tangent field v. Then, for $f, g \in \Gamma(U, \mathcal{O}_X)$, we define the Poisson bracket $\{f, g\}$ of f and g by setting $\{f, g\} = \langle H_f, dg\rangle = \omega(H_g, H_f)$. The map $g \mapsto \{f, g\}$ is a derivation of $\Gamma(U, \mathcal{O}_X)$ whose corresponding vector field is precisely H_f. The pairing $\{\cdot, \cdot\}$ on $\mathcal{O}_X$ is a bilinear antisymmetric map which is a derivation in each entry and satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{4.1.1}$$

for any $f, g, h \in \Gamma(U, \mathcal{O}_X)$. This implies that $[H_f, H_g] = H_{\{f,g\}}$, where $[u, v] = uv - vu$ is the commutator of the vector fields u and v.

Example 4.1.1. – Let $\pi : T^*X \to X$ be the cotangent bundle to X. The cotangent morphism to π is a morphism $T^*\pi : \pi^*T^*X = T^*X \times_X T^*X \to T^*T^*X$. If we restrict this map to the diagonal of $T^*X \times_X T^*X$, we get a map $T^*X \to T^*T^*X$, which is a section of the bundle $T^*T^*X \to T^*X$, *i.e.*, a differential form of degree 1. This is the canonical 1-form on T^*X, denoted by α_X. The closed 2-form $\omega = -d\alpha_X$ is the canonical symplectic form on T^*X.

A Poisson structure on X is defined as a Lie algebra structure $\{\cdot, \cdot\}$ on $\mathcal{O}_X$ satisfying the identity $\{f, gh\} = \{f, g\}h + g\{f, h\}$. Equivalently one may give an antisymmetric contravariant 2-tensor $\theta \in H^0(X, \Lambda^2 TX)$ and set $\{f, g\} = \langle\theta, df \wedge dg\rangle$. Then θ is a Poisson structure if the bracket it defines satisfies the Jacobi identity (4.1.1). For any $f \in \Gamma(U, \mathcal{O}_X)$, the map $g \mapsto \{f, g\}$ is a derivation of $\Gamma(U, \mathcal{O}_X)$, hence corresponds to a vector field H_f on U, called the hamiltonian vector field associated to f. When θ has maximal rank everywhere, we say that the Poisson structure is symplectic. In fact, in this case, to give θ is equivalent to giving its inverse 2-form $\omega \in H^0(X, \Omega_X^2)$, *i.e.*, a symplectic structure on X.

Let us describe an important example of a Poisson structure which is not symplectic.

Example 4.1.2. – Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is endowed with a canonical Poisson structure, called the Kostant-Kirillov structure, defined as follows: for $\alpha \in \mathfrak{g}^*$ we define $\theta(\alpha) \in \Lambda^2\mathfrak{g}^*$ by requiring that $\theta(\alpha)(a, b) = \alpha([a, b])$, for all $a, b \in \mathfrak{g}$.

Let f and g be holomorphic functions over an open subset U of $\mathfrak{g}^*$. For every $\alpha \in U$, the linear forms $f'(\alpha)$ and $g'(\alpha)$ over $\mathfrak{g}^*$ can be regarded as elements of $\mathfrak{g}$. The Poisson bracket of f and g is then given by

$$\{f, g\}(\alpha) = \alpha([f'(\alpha), g'(\alpha)]),$$

and the hamiltonian vector field H_f over U satisfies $H_f(\alpha) = {}^t[\mathrm{ad} f'(\alpha)](\alpha)$.

If the Lie algebra $\mathfrak{g}$ is reductive, it can be given an invariant separating symmetric bilinear form, which gives a $\mathbb{C}$-linear isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$. By means of this isomorphism we can transfer to $\mathfrak{g}$ the canonical Poisson structure of $\mathfrak{g}^*$. The Poisson structure defined in this way on $\mathfrak{g}$ is not symplectic: in fact it is tangent to the orbits of G, where G is the Lie group associated to $\mathfrak{g}$, and it induces a symplectic structure on each of these orbits.

In the next section we shall discuss a generalization of both Examples 4.1.1 and 4.1.2 which will be needed in the sequel. This construction was suggested by A. Beauville.

4.2 Canonical Poisson structures on the dual of a vector bundle endowed with a Lie algebra structure.

Let X be a smooth variety and $\mathfrak{G}$ a locally free $\mathcal{O}_X$-Module endowed with a structure of a locally free sheaf of Lie algebras over $\mathbb{C}$. We shall denote by $\mathfrak{G} \to X$ the corresponding vector bundle. Let $u : \mathfrak{G} \to TX$ be a homomorphism for the structures of $\mathcal{O}_X$-Modules and of sheaves of Lie algebras, satisfying the following compatibility condition between the two structures:

$$[\xi, f\,\zeta] = f\,[\xi,\zeta] + u(\xi)(f)\,\zeta, \tag{4.2.1}$$

for any $f \in \Gamma(U, \mathcal{O}_X)$ and any $\xi, \zeta \in \Gamma(U, \mathfrak{G})$, where $[\cdot,\cdot]$ denotes the Lie bracket operation on $\mathfrak{G}$. Let $\mathfrak{G}^*$ be the dual of $\mathfrak{G}$.

In this situation we can define a Poisson structure on $\mathfrak{G}^*$, considered as a variety over X. First we note that $\mathcal{O}_{\mathfrak{G}^*} = \mathrm{Sym}_{\mathcal{O}_X}(\mathfrak{G})$, the symmetric algebra of $\mathfrak{G}$ over $\mathcal{O}_X$. Then, for any open subset $U \subset X$ and sections $\xi, \zeta \in \Gamma(U, \mathfrak{G})$ and $f, g \in \Gamma(U, \mathcal{O}_X)$, we set

$$\begin{aligned} \{\xi,\zeta\} &= [\xi,\zeta],\\ \{\xi, f\} &= u(\xi)(f),\\ \{f, g\} &= 0, \end{aligned} \tag{4.2.2}$$

and extend $\{\cdot,\cdot\}$ to all of $\mathcal{O}_{\mathfrak{G}^*}$ by linearity and by using Leibnitz rule for the product of two elements. We have the following result, whose proof consists in a straightforward computation:

Proposition 4.2.1. – *The bracket $\{\cdot,\cdot\}$ is well-defined and is a Poisson bracket. The corresponding Poisson structure on the vector bundle $\mathfrak{G}^*$ is called the canonical Poisson structure associated to the sheaf of Lie algebras $\mathfrak{G}$ and the homomorphism $u : \mathfrak{G} \to TX$.*

Remark 4.2.2. – Let us apply u to (4.2.1). On the left hand side we get $u([\xi, f\,\zeta]) = [u(\xi), f\,u(\zeta)] = u(\xi)(f\,u(\zeta)) - f\,u(\zeta)\,u(\xi) = u(\xi)(f)\,u(\zeta) + f\,[u(\xi), u(\zeta)]$, since $u(\xi)$ and $u(\zeta)$ are tangent fields on X, while on the right hand side we find $u(f\,[\xi,\zeta]) + u(u(\xi)(f)\,\zeta) = f\,[u(\xi), u(\zeta)] + u(\xi)(f)\,u(\zeta)$. From this we derive that in general

$$[\xi, f\,\zeta] - f\,[\xi,\zeta] - u(\xi)(f)\,\zeta \in \Gamma(X, \mathrm{Ker}(u)). \tag{4.2.3}$$

If, for example, u is an injective morphism, then the condition (4.2.1) is automatically satisfied.

Remark 4.2.3. – The idea leading to the definition (4.2.2) originated by studying the following situation. Let us consider a variety X and a vector bundle $\pi : E \to X$. The natural action of $\mathbb{C}^*$ on E determines a direct sum decomposition of $\mathcal{O}_E$ as $\bigoplus_{n=0}^{\infty} \mathcal{O}_E^{(n)}$, where $\mathcal{O}_E^{(n)}$ denotes the subsheaf of $\mathcal{O}_E$ of rational functions of degree n with respect to the action of $\mathbb{C}^*$. As an example, for every section f of $\mathcal{O}_X$, the rational function $\tilde{f} = f \circ \pi$ has degree 0, while a section ξ of the dual vector bundle E^*, considered as a rational function on E, has degree 1.

In a similar way we have a degree decomposition of the whole tensor algebra over E, with the degree map satisfying $\deg(\alpha \otimes \beta) = \deg(\alpha) + \deg(\beta)$.

Let $\theta \in H^0(E, \Lambda^2 TE)$ be a Poisson structure on E, and denote by $\{\cdot,\cdot\}$ the corresponding Poisson bracket. Let us suppose that $\deg(\theta) = -1$. For any $f, g \in H^0(U, \mathcal{O}_E)$, their Poisson bracket is given by $\{f, g\} = \langle \theta, df \wedge dg \rangle$, hence we have

$$\deg(\{f,g\}) = \deg(f) + \deg(g) - 1.$$

It follows that, if $f, g \in H^0(U, \mathcal{O}_X)$ and $\tilde{f}$ and $\tilde{g}$ are the corresponding rational functions on E, one has $\deg(\{f,g\}) = -1$, hence $\{f,g\} = 0$.

For a section ξ of the dual sheaf E^* we have $\deg(\{\xi, f\}) = 0$, hence $\{\xi, f\}$ is a section of $\mathcal{O}_X$. Moreover it follows from the definition of a Poisson structure that the morphism $u(\xi) = \{\xi, \cdot\} : \mathcal{O}_X \to \mathcal{O}_X$, $f \mapsto \{\xi, f\}$, is a $\mathbb{C}$-derivation of the sheaf $\mathcal{O}_X$, *i.e.*, a tangent field to X. Hence we get a map $u : E^* \to TX$, $\xi \mapsto u(\xi) = \{\xi, \cdot\}$. It is easy to see that this is a homomorphism of $\mathcal{O}_X$-Modules.

Finally, if ξ and ζ are two sections of E^*, it follows that $\{\xi, \zeta\}$ has degree 1, hence is again a section of E^*. Thus the Poisson bracket $\{\cdot,\cdot\}$ induces a map $[\cdot,\cdot] : E^* \times E^* \to E^*$ which is easily seen to determine a Lie algebra structure on E^*.

Now, from the Jacobi identity for $\{\cdot,\cdot\}$ it follows that u is a homomorphism of sheaves of Lie algebras, while the compatibility condition (4.2.1) derives from the fact that a Poisson bracket is a derivation in each entry.

In conclusion this shows that the definition (4.2.2) characterizes all Poisson structures of degree -1 on a vector bundle E over a variety X.

Remark 4.2.4. – If $\mathfrak{G} = TX$, the tangent bundle of X, and u is the identity morphism, the canonical Poisson structure on $\mathfrak{G}^* = T^*X$ defined above coincides with the canonical symplectic structure of the cotangent bundle of X defined in Example 4.1.1.

If the variety X is reduced to a point, then the sheaf of Lie algebras $\mathfrak{G}$ is identified to a Lie algebra $\mathfrak{g}$. In this situation the canonical Poisson structure on $\mathfrak{G}^* = \mathfrak{g}^*$ is precisely the Kostant-Kirillov Poisson structure defined in Example 4.1.2.

4.3 The Poisson structure on the variety $\mathcal{P}$: first approach.

From now on we shall assume that $H^0(C, K^{-1} \otimes L) \neq 0$. Let us choose a non-zero section $s \in H^0(C, K^{-1} \otimes L)$ and denote also by $s : K \to L$ the homomorphism given by multiplication by s. Let Φ be the canonical section of $\mathcal{E}nd(\mathcal{E}) \otimes p^*L$ defined in

Remark 1.3.3. We have a morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}nd(\mathcal{E}) & \xrightarrow{[\cdot,\Phi]} & \mathcal{E}nd(\mathcal{E}) \otimes p^*(L) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle -s} & & \uparrow{\scriptstyle s} & & \\ 0 & \longrightarrow & \mathcal{E}nd(\mathcal{E}) \otimes p^*(L^{-1} \otimes K) & \xrightarrow[{[\Phi,\cdot]}]{} & \mathcal{E}nd(\mathcal{E}) \otimes p^*(K) & \longrightarrow & 0, \end{array} \tag{4.3.1}$$

which induces on hypercohomology the morphism

$$B_s : \mathbb{R}^1 q_*([\Phi, \cdot]) \to \mathbb{R}^1 q_*([\cdot, \Phi]). \tag{4.3.2}$$

Precisely, for every point $(E, \phi) \in \mathcal{M}'$, we have the morphism of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}nd(E) & \xrightarrow{[\cdot,\phi]} & \mathcal{E}nd(E) \otimes L & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle -s} & & \uparrow{\scriptstyle s} & & \\ 0 & \longrightarrow & \mathcal{E}nd(E) \otimes L^{-1} \otimes K & \xrightarrow[{[\phi,\cdot]}]{} & \mathcal{E}nd(E) \otimes K & \longrightarrow & 0, \end{array} \tag{4.3.3}$$

which induces a morphism on hypercohomology groups

$$B_s : \mathbb{H}^1([\phi, \cdot]) \to \mathbb{H}^1([\cdot, \phi]). \tag{4.3.4}$$

By recalling the natural identifications $T\mathcal{M}'_0 \cong \mathbb{R}^1 q_*([\cdot, \Phi])$ and $T^*\mathcal{M}'_0 \cong \mathbb{R}^1 q_*([\Phi, \cdot])$, we can define a contravariant 2-tensor $\theta_s \in H^0(\mathcal{M}'_0, \bigotimes^2 T\mathcal{M}'_0)$ by setting $\langle \theta_s, \alpha \otimes \beta \rangle = \langle \alpha, B_s(\beta) \rangle$, for 1-forms α and β considered as sections of $\mathbb{R}^1 q_*([\Phi, \cdot])$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $T\mathcal{M}'_0$ and $T^*\mathcal{M}'_0$.

More explicitly, if we fix our attention to the tangent and cotangent spaces to $\mathcal{M}'_0$ at a point (E, ϕ) and recall the description of the hypercohomology groups given in terms of Čech cocycles, the map $B_s : \mathbb{H}^1([\phi, \cdot]) \to \mathbb{H}^1([\cdot, \phi])$ may be written explicitly as follows: for $(\{\alpha_i\}, \{\eta_{ij}\}) \in \mathbb{H}^1([\phi, \cdot])$, we have $B_s(\{\alpha_i\}, \{\eta_{ij}\}) = (\{s\alpha_i\}, \{-s\eta_{ij}\})$. It is now immediate to prove that B_s is skew-symmetric, hence θ_s is actually an antisymmetric contravariant 2-tensor, *i.e.*, $\theta_s \in H^0(\mathcal{M}'_0, \Lambda^2 T\mathcal{M}'_0)$. To prove that θ_s defines a Poisson structure on $\mathcal{M}'_0$ it remains only to show that the corresponding bracket, defined by setting $\{f, g\} = \langle \theta_s, df \wedge dg \rangle$, satisfies the Jacobi identity. Unfortunately this is not easy.

Remark 4.3.1. – In the sequel we shall see that θ_s defines a Poisson structure on $\mathcal{P}$. It follows that this is true also for the connected component $\mathcal{M}'_0$ of $\mathcal{M}'$ containing $\mathcal{P}$. In particular this holds for $\mathcal{M}(2, d, L)$, since it is known to be connected.

4.4 THE MAP B_s.

Let us study more closely the morphism $B_s : \mathbb{H}^1([\phi, \cdot]) \to \mathbb{H}^1([\cdot, \phi])$. The global section $s \in H^0(C, K^{-1} \otimes L)$ defines an effective divisor D_s on C, such that $\mathcal{O}_C(D_s) = K^{-1} \otimes L$. For any sheaf $\mathcal{F}$ on C let us denote by $\mathcal{F}_{D_s}$ the sheaf $j^*(\mathcal{F})$, where $j : D_s \to C$ is the natural inclusion.

We have an exact sequence of complexes (written vertically)

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{E}nd(E)\otimes K & \xrightarrow{s} & \mathcal{E}nd(E)\otimes L & \xrightarrow{r} & \mathcal{E}nd(E)\otimes L_{D_s} & \longrightarrow 0 \\ & \uparrow{\scriptstyle [\phi,\cdot]} & & \uparrow{\scriptstyle [\cdot,\phi]} & & \uparrow{\scriptstyle [\cdot,\phi]_{D_s}} & \\ 0 \longrightarrow & \mathcal{E}nd(E)\otimes L^{-1}\otimes K & \xrightarrow{s} & \mathcal{E}nd(E) & \xrightarrow{r} & \mathcal{E}nd(E)_{D_s} & \longrightarrow 0 \end{array}$$

giving rise to a long exact sequence of hypercohomology groups

$$0 \to \mathbb{H}^0([\phi,\cdot]) \xrightarrow{B_s} \mathbb{H}^0([\cdot,\phi]) \to \mathbb{H}^0([\cdot,\phi]_{D_s}) \xrightarrow{\delta} \mathbb{H}^1([\phi,\cdot]) \xrightarrow{B_s} \mathbb{H}^1([\cdot,\phi]) \to \cdots. \tag{4.4.1}$$

If (E,ϕ) is a stable pair, we have seen that $\mathbb{H}^0([\cdot,\phi]) \cong \mathbb{C}$, and $\mathbb{H}^0([\phi,\cdot]) \cong \mathbb{C}$ if $L \cong K$ or is equal to 0 if $\deg L > \deg K$. Moreover it follows from the definitions that $\mathbb{H}^0([\cdot,\phi]_{D_s}) = \{\alpha \in H^0(C,\mathcal{E}nd(E)_{D_s}) \mid [\alpha,\phi]_{D_s} = 0\}$.

If $\deg L > \deg K$ let us suppose for simplicity that $D_s = \sum_{i=1}^{m} P_i$, with $P_i \neq P_j$ if $i \neq j$, where $m = \deg L - \deg K$. Under this assumption we have natural identifications $H^0(C,\mathcal{E}nd(E)_{D_s}) \cong \bigoplus_{i=1}^{m} \mathcal{E}nd(E)_{P_i}$ and $[\cdot,\phi]_{D_s} \cong \bigoplus_{i=1}^{m} [\cdot,\phi_{P_i}]$, where $\phi_{P_i} : E_{P_i} \to E_{P_i} \otimes L_{P_i}$ is the homomorphism induced by ϕ on the fibers over P_i. From this we derive that $\mathbb{H}^0([\cdot,\phi]_{D_s}) = \bigoplus_{i=1}^{m} C(\phi_{P_i})$, where $C(\phi_{P_i}) = \{\alpha_{P_i} \in \mathcal{E}nd(E)_{P_i} \mid [\alpha_{P_i},\phi_{P_i}] = 0\}$.

From the exact sequence (4.4.1) it follows that $\dim(\ker B_s) = \sum_{i=1}^{m} \dim C(\phi_{P_i}) - 1$. By recalling (3.1.8), we find that the rank of B_s is given by

$$\operatorname{rank}(B_s) = r^2 \deg(L) - \sum_{i=1}^{m} \dim C(\phi_{P_i}) + 2. \tag{4.4.2}$$

At a generic point $(E,\phi) \in \mathcal{M}'(r,d,L)$ we have $\dim C(\phi_{P_i}) = r$ (in which case the map ϕ_{P_i} is called *regular*), hence we find

$$\operatorname{rank}(B_s) = r(r-1)\deg(L) + 2r(g-1) + 2. \tag{4.4.3}$$

This is the maximum value attained by the rank of B_s. On the contrary its minimum value, reached at points $(E,\phi) \in \mathcal{M}'(r,d,L)$ where $\dim C(\phi_{P_i}) = r^2$ (e.g., if ϕ is represented by a scalar matrix), is $r^2\deg(K) + 2$.

4.5 THE SYMPLECTIC CASE: $L = K$.

In this section we shall restrict to the case $L = K$, where the variety $\mathcal{P}$ is canonically isomorphic to the cotangent bundle $T^*\mathcal{U}$ of the moduli variety $\mathcal{U}$, hence has a canonical symplectic structure. This is the case considered by N. Hitchin in [H].

We shall prove the following result

THEOREM 4.5.1. – *Let s be the identity section of $H^0(C, K^{-1} \otimes K)$,* i.e., *the identity homomorphism $s = \mathrm{id} : K \to K$. Then the antisymmetric contravariant 2-tensor $\theta_s = \theta_1$ defines a Poisson structure on $\mathcal{P}$ which is symplectic and coincides with the canonical symplectic structure of $T^*\mathcal{U}$, via the natural identification $\mathcal{P} \cong T^*\mathcal{U}$.*

Proof. – We recall that the variety $\mathcal{P}$ is the total space of the vector bundle $\mathcal{H} = q_*\mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes p^*(L))$. We have denoted by $\pi : \mathcal{P} \to \mathcal{U}$ the natural projection and by Φ the canonical section of $\pi^*\mathcal{H}$.

Let us denote by $\alpha_{\mathcal{P}} : \mathcal{P} \to T^*\mathcal{P}$ the canonical 1-form on $\mathcal{P} \cong T^*\mathcal{U}$ defined in Example 4.1.1. By recalling the identification $T^*\mathcal{P} \cong \mathbb{R}^1 q_*([\Phi, \cdot])$, we find that $\alpha_{\mathcal{P}}$ is the global section $(\Phi, 0)$ of $\mathbb{R}^1 q_*([\Phi, \cdot])$ defined as the image of Φ by the natural map $\pi^*\mathcal{H} \to \mathbb{R}^1 q_*([\Phi, \cdot])$ derived from the dual exact sequence of (3.1.6). Precisely, for every point $(E, \phi) \in \mathcal{P}$, the element $\alpha_{\mathcal{P}}(E, \phi)$ of the cotangent space $\mathbb{H}^1([\phi, \cdot])$ to $\mathcal{P}$ at the point (E, ϕ) is the image of the global section ϕ of $\mathcal{E}nd(E) \otimes K$ by the natural injective map $H^0(C, \mathcal{E}nd(E) \otimes K) \to \mathbb{H}^1([\phi, \cdot])$ (*see* (3.1.9)), *i.e.*, in terms of Čech cocycles, we may write $\alpha_{\mathcal{P}}(E, \phi) = (\phi, 0)$. The canonical symplectic form on $\mathcal{P}$ is then $\omega = -d\alpha_{\mathcal{P}}$.

Let D^1, $D^2 : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$ be two tangent vector fields to $\mathcal{P}$ and $(\{\alpha_i^1\}, \{\eta_{ij}^1\})$, $(\{\alpha_i^2\}, \{\eta_{ij}^2\})$ the corresponding global sections of $\mathbb{R}^1 q_*([\cdot, \Phi])$. From Proposition 3.3.3 it follows that there exist first order differential operators $\dot{D}_i^1$ and $\dot{D}_i^2$ such that $\alpha_i^h = [\dot{D}_i^h, \Phi]$ and $\eta_{ij}^h = \dot{D}_j^h - \dot{D}_i^h$, for all i, j and for $h = 1, 2$.

By recalling the conventions of Section 3.3, let us denote by

$$\begin{array}{ccc} \mathcal{P} \times \mathrm{Spec}(\mathbb{C}[\epsilon, \epsilon']) \times C & \xrightarrow{\tilde{D}^h \times 1_C} & \mathcal{P} \times \mathrm{Spec}(\mathbb{C}[\epsilon, \epsilon']) \times C \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\mathbb{C}[\epsilon, \epsilon']) & \end{array}$$

the automorphism corresponding to D^h. From what we have seen in Remark 3.3.4, the vector field D^h is given equivalently by the infinitesimal deformation $(\mathcal{E}_\epsilon^h, \Phi_\epsilon^h) = (\tilde{D}^h \times 1_C)^*(\mathcal{E}, \Phi)$ of $(\mathcal{E}, \Phi)$, described by the global section $(\{\alpha_i^h\}, \{\eta_{ij}^h\})$ of $\mathbb{R}^1 q_*([\cdot, \Phi])$. Let us recall that if $f_i^h : \mathcal{E}_\epsilon^h|_{U_i} \to \widetilde{M_i^h[\epsilon]}$ are isomorphisms, then the sheaf $\mathcal{E}_\epsilon^h$ is constructed by gluing the sheaves $\widetilde{M_i^h[\epsilon]}$ and $\widetilde{M_j^h[\epsilon]}$ along the open sets U_{ij} by means of the isomorphisms $1 + \epsilon\eta_{ij}^h = f_j^h|_{U_{ij}} \circ {f_i^h}^{-1}|_{U_{ij}}$. In terms of the sheaves $\widetilde{M_i^h[\epsilon]}$, the deformation Φ_ϵ^h of Φ is given locally by $\Phi + \epsilon\alpha_i^h$.

By applying the same reasoning, the second order differential operator D^1D^2 is equivalent to the pair $(\tilde{D}^1 \times 1_C)^*(\tilde{D}^2 \times 1_C)^*(\mathcal{E}, \Phi)$. We have isomorphisms

$$(\tilde{D}^1 \times 1_C)^*(\tilde{D}^2 \times 1_C)^*\mathcal{E}|_{U_i \times \mathrm{Spec}(\mathbb{C}[\epsilon, \epsilon'])} \xrightarrow{(1+\epsilon\dot{D}_i^1)\circ(1+\epsilon'\dot{D}_i^2)} \widetilde{M_i[\epsilon, \epsilon']},$$

hence the gluing isomorphisms are given by $((1 + \epsilon\dot{D}_j^1) \circ (1 + \epsilon'\dot{D}_j^2)) \circ ((1 + \epsilon\dot{D}_i^1) \circ (1 + \epsilon'\dot{D}_i^2))^{-1} = 1 + \epsilon(\dot{D}_j^1 - \dot{D}_i^1) + \epsilon'(\dot{D}_j^2 - \dot{D}_i^2) + \epsilon\epsilon'(\dot{D}_j^1\dot{D}_j^2 - \dot{D}_i^2\dot{D}_j^1 - \dot{D}_j^1\dot{D}_i^2 + \dot{D}_i^2\dot{D}_i^1)$. Note that this can be written in a simpler form as $1 + \epsilon\eta_{ij}^1 + \epsilon'\eta_{ij}^2 + \epsilon\epsilon'(\dot{D}_j^1\eta_{ij}^2 - \eta_{ij}^2\dot{D}_i^1)$.

For what concerns the pull-back $(\tilde{D}^1 \times 1_C)^*(\tilde{D}^2 \times 1_C)^*(\Phi)$, we have the following commutative diagram, analogous to (3.3.5):

$$\begin{array}{ccc} (\tilde{D}^1 \times 1_C)^*(\tilde{D}^2 \times 1_C)^*\mathcal{E} \otimes L[\epsilon,\epsilon']|_{U_i \times \mathrm{Spec}(\mathbb{C}[\epsilon,\epsilon'])} & \xrightarrow{(1+\epsilon\dot{D}^1_i)\circ(1+\epsilon'\dot{D}^2_i)} & M_i \otimes \widetilde{L_i[\epsilon,\epsilon']} \\ {\scriptstyle \Phi}\Big\uparrow & & \Big\uparrow{\scriptstyle \Phi+\epsilon\vartheta_i+\epsilon'\zeta_i+\epsilon\epsilon'\xi_i} \\ (\tilde{D}^1 \times 1_C)^*(\tilde{D}^2 \times 1_C)^*\mathcal{E}|_{U_i \times \mathrm{Spec}(\mathbb{C}[\epsilon,\epsilon'])} & \xrightarrow[(1+\epsilon\dot{D}^1_i)\circ(1+\epsilon'\dot{D}^2_i)]{} & \widetilde{M_i[\epsilon,\epsilon']}. \end{array}$$

From this it follows that $\vartheta_i = [\dot{D}^1_i, \Phi] = \alpha^1_i$, $\zeta_i = [\dot{D}^2_i, \Phi] = \alpha^2_i$ and $\xi_i = [\dot{D}^1_i, [\dot{D}^2_i, \Phi]] = [\dot{D}^1_i, \alpha^2_i]$.

In conclusion we have proved that the second order differential operator $D^1 D^2$ is described by giving gluing isomorphisms of the form

$$1 + \epsilon\eta^1_{ij} + \epsilon'\eta^2_{ij} + \epsilon\epsilon'(\dot{D}^1_j \eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i), \tag{4.5.1}$$

in terms of which the infinitesimal deformation of Φ is locally written as

$$\Phi + \epsilon\alpha^1_i + \epsilon'\alpha^2_i + \epsilon\epsilon'[\dot{D}^1_i, \alpha^2_i], \tag{4.5.2}$$

Analogously, we find that $D^2 D^1$ is equivalent to the data of

$$1 + \epsilon\eta^1_{ij} + \epsilon'\eta^2_{ij} + \epsilon\epsilon'(\dot{D}^2_j \eta^1_{ij} - \eta^1_{ij}\dot{D}^2_i), \tag{4.5.3}$$

and

$$\Phi + \epsilon\alpha^1_i + \epsilon'\alpha^2_i + \epsilon\epsilon'[\dot{D}^2_i, \alpha^1_i]. \tag{4.5.4}$$

Then, by Lemma 3.3.1, the vector field $[D^1, D^2]$ is given by

$$1 + \epsilon\epsilon'([\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i)$$

and

$$\Phi + \epsilon\epsilon'[[\dot{D}^1, \dot{D}^2]_i, \Phi].$$

Let us denote by ξ^1 and ξ^2 the vector fields corresponding to D^1 and D^2. We shall use the preceding results to compute $d\alpha_{\mathcal{P}}(\xi^1, \xi^2) = \xi^1(\langle \alpha_{\mathcal{P}}, \xi^2 \rangle) - \xi^2(\langle \alpha_{\mathcal{P}}, \xi^1 \rangle) - \langle \alpha_{\mathcal{P}}, [\xi^1, \xi^2] \rangle$.

The 1-form $\alpha_{\mathcal{P}}$ corresponds to the global section $(\Phi, 0)$ of $\mathbb{R}^1 q_*([\Phi, \cdot])$, hence, by the explicit expression of the duality given in Section 3.2, we have $\langle \alpha_{\mathcal{P}}, \xi^2 \rangle = \mathrm{res} \circ \mathrm{Tr}(\Phi \eta^2_{ij})$. To compute $\xi^1(\langle \alpha_{\mathcal{P}}, \xi^2 \rangle)$, *i.e.*, the derivative of the function $\langle \alpha_{\mathcal{P}}, \xi^2 \rangle$ with respect to the vector field ξ^1, we shall use first order Taylor series expansions of $\alpha_{\mathcal{P}}$ and ξ^2, *i.e.*, we shall compute $\langle \alpha_{\mathcal{P}_\epsilon}, \xi^2_\epsilon \rangle$, where $\alpha_{\mathcal{P}_\epsilon}$ and ξ^2_ϵ are "infinitesimal deformations along the vector field ξ^1" of $\alpha_{\mathcal{P}}$ and ξ^2 respectively.

By what we have previously seen we have:

$$\begin{aligned} \langle \alpha_{\mathcal{P}_\epsilon}, \xi^2_\epsilon \rangle &= \mathrm{res} \circ \mathrm{Tr}(\Phi_\epsilon \eta^2_{\epsilon\, ij}) \\ &= \mathrm{res} \circ \mathrm{Tr}((\Phi + \epsilon[\dot{D}^1_i, \Phi]) \cdot (\dot{D}^2_j - \dot{D}^2_i + \epsilon(\dot{D}^1_j \dot{D}^2_j - \dot{D}^2_j \dot{D}^1_i - \dot{D}^1_j \dot{D}^2_i + \dot{D}^2_i \dot{D}^1_i))) \\ &= \mathrm{res} \circ \mathrm{Tr}(\Phi \eta^2_{ij}) + \epsilon\, \mathrm{res} \circ \mathrm{Tr}(\Phi \dot{D}^1_j \dot{D}^2_j - \Phi \dot{D}^2_j \dot{D}^1_i - \Phi \dot{D}^1_j \dot{D}^2_i + \Phi \dot{D}^2_i \dot{D}^1_i \\ &\qquad - \Phi \dot{D}^1_i \dot{D}^2_j + \Phi \dot{D}^1_i \dot{D}^2_i + \dot{D}^1_i \Phi \dot{D}^2_j - \dot{D}^1_i \Phi \dot{D}^2_i), \end{aligned}$$

hence

$$\xi^1(\langle\alpha_{\mathcal{P}},\xi^2\rangle) = \text{res}\circ\text{Tr}(\Phi\dot{D}_j^1\dot{D}_j^2 - \Phi\dot{D}_j^2\dot{D}_i^1 - \Phi\dot{D}_j^1\dot{D}_i^2 + \Phi\dot{D}_i^2\dot{D}_i^1 - \Phi\dot{D}_i^1\dot{D}_j^2 + \Phi\dot{D}_i^1\dot{D}_i^2 + \dot{D}_i^1\Phi\dot{D}_j^2 - \dot{D}_i^1\Phi\dot{D}_i^2).$$

With similar computations for $\xi^2(\langle\alpha_{\mathcal{P}},\xi^1\rangle)$, we find that

$$\xi^2(\langle\alpha_{\mathcal{P}},\xi^1\rangle) = \text{res}\circ\text{Tr}(\Phi\dot{D}_j^2\dot{D}_j^1 - \Phi\dot{D}_j^1\dot{D}_i^2 - \Phi\dot{D}_j^2\dot{D}_i^1 + \Phi\dot{D}_i^1\dot{D}_i^2 - \Phi\dot{D}_j^2\dot{D}_j^1 + \Phi\dot{D}_j^2\dot{D}_i^1 + \dot{D}_j^2\Phi\dot{D}_j^1 - \dot{D}_j^2\Phi\dot{D}_i^1).$$

Finally we have:

$$\begin{aligned}\langle\alpha_{\mathcal{P}},[\xi^1,\xi^2]\rangle &= \text{res}\circ\text{Tr}(\Phi\cdot([\dot{D}^1,\dot{D}^2]_j - [\dot{D}^1,\dot{D}^2]_i))\\ &= \text{res}\circ\text{Tr}(\Phi\dot{D}_j^1\dot{D}_j^2 - \Phi\dot{D}_j^2\dot{D}_j^1 - \Phi\dot{D}_i^1\dot{D}_i^2 + \Phi\dot{D}_i^2\dot{D}_i^1).\end{aligned}$$

From this we derive that:

$$d\alpha_{\mathcal{P}}(\xi^1,\xi^2) = \text{res}\circ\text{Tr}(\dot{D}_i^1\Phi\dot{D}_j^2 - \dot{D}_i^1\Phi\dot{D}_i^2 - \Phi\dot{D}_i^1\dot{D}_j^2 + \Phi\dot{D}_i^1\dot{D}_i^2 - \dot{D}_j^2\Phi\dot{D}_j^1 + \dot{D}_j^2\Phi\dot{D}_i^1 + \Phi\dot{D}_j^2\dot{D}_j^1 - \Phi\dot{D}_j^2\dot{D}_i^1).$$

Now we recall that, by the choice of the identity section of $\mathcal{H}om(K,K)$, we have defined an antisymmetric contravariant 2-tensor θ_1 on $\mathcal{P}$. Its inverse 2-form ω_1 is defined by $\omega_1(\xi^1,\xi^2) = \langle\xi^1, B_1^{-1}(\xi^2)\rangle$.

We have:

$$\begin{aligned}\omega_1(\xi^1,\xi^2) &= \langle(\{\alpha_i^1\},\{\eta_{ij}^1\}), B_1^{-1}(\{\alpha_i^2\},\{\eta_{ij}^2\})\rangle\\ &= \langle(\{\alpha_i^1\},\{\eta_{ij}^1\}),(\{\alpha_i^2\},\{-\eta_{ij}^2\})\rangle\\ &= \text{res}\circ\text{Tr}(-\alpha_i^1\eta_{ij}^2 + \alpha_j^2\eta_{ij}^1),\end{aligned}$$

and, by replacing the expressions of α_i^h and η_{ij}^h in terms of $\dot{D}_i^h$, we find

$$\begin{aligned}\omega_1(\xi^1,\xi^2) &= \text{res}\circ\text{Tr}(-[\dot{D}_i^1,\Phi]\cdot(\dot{D}_j^2 - \dot{D}_i^2) + [\dot{D}_j^2,\Phi]\cdot(\dot{D}_j^1 - \dot{D}_i^1))\\ &= \text{res}\circ\text{Tr}(-\dot{D}_i^1\Phi\dot{D}_j^2 + \dot{D}_i^1\Phi\dot{D}_i^2 + \Phi\dot{D}_i^1\dot{D}_j^2 - \Phi\dot{D}_i^1\dot{D}_i^2\\ &\quad + \dot{D}_j^2\Phi\dot{D}_j^1 - \dot{D}_j^2\Phi\dot{D}_i^1 - \Phi\dot{D}_j^2\dot{D}_j^1 + \Phi\dot{D}_j^2\dot{D}_i^1),\end{aligned}$$

which proves that $\omega_1 = -d\alpha_{\mathcal{P}}$. Hence ω_1 is precisely the canonical symplectic structure on $\mathcal{P} \cong T^*\mathcal{U}$. □

4.6 THE POISSON STRUCTURE ON THE VARIETY $\mathcal{P}$: SECOND APPROACH.

In this section we shall make use of the construction discussed in Section 4.2 to define a Poisson structure on the variety $\mathcal{P}$.

Let us fix some notations: $p : \mathcal{U}\times C \to C$ and $q : \mathcal{U}\times C \to \mathcal{U}$ are the canonical projections, s is a fixed global section of $K^{-1}\otimes L$ and D_s is its divisor. We have $\mathcal{O}_C(D_s) = K^{-1}\otimes L$. The variety $\mathcal{P}$ is the total space of the sheaf $\mathcal{H} = q_*\mathcal{H}om(\mathcal{E},\mathcal{E}\otimes p^*L)$, whose dual is $\mathcal{H}^* = R^1q_*\mathcal{H}om(\mathcal{E},\mathcal{E}\otimes p^*(L^{-1}\otimes K))$, which we shall denote by $R^1q_*(\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_C(-D_s))$.

As we have seen, if we define a structure of sheaf of Lie algebras on $\mathcal{H}^*$ and a homomorphism $u : \mathcal{H}^* \to T\mathcal{U}$ satisfying certain conditions, we derive a Poisson structure on the variety $\mathcal{P}$ corresponding to the sheaf $\mathcal{H} \cong \mathcal{H}^{**}$.

If $L = K$ this is easy to define. In fact, in this case, we have $\mathcal{H}^* = R^1 q_* \mathcal{E}nd(\mathcal{E})$, and we have seen in Remark 3.3.5 that for every section of $R^1 q_* \mathcal{E}nd(\mathcal{E})$, represented by a 1-cocycle $\{\eta_{ij}\}$, there exist differential operators $\dot{D}_i$ such that $\eta_{ij} = \dot{D}_j - \dot{D}_i$. In view of the isomorphism $T\mathcal{U} \xrightarrow{\delta} R^1 q_* \mathcal{E}nd(\mathcal{E})$, the Lie algebra structure of $\mathcal{H}^*$ may be read on $T\mathcal{U}$ (and the homomorphism u is simply the inverse of the isomorphism δ). This implies that, if $\eta^1_{ij} = \dot{D}^1_j - \dot{D}^1_i$ and $\eta^2_{ij} = \dot{D}^2_j - \dot{D}^2_i$, their Lie bracket is given by

$$[\{\eta^1_{ij}\}, \{\eta^2_{ij}\}] = \{[\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i\} = \{[\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, \eta^2_{ij}]\}.$$

We have seen in Section 4.2 that the corresponding Poisson structure on $\mathcal{P}$ is symplectic and coincides with the canonical symplectic structure on $T^*\mathcal{U}$, via the natural isomorphism $\mathcal{P} \cong T^*\mathcal{U}$, hence coincides with the Poisson structure defined by the antisymmetric contravariant 2-tensor θ_1.

Now we turn to the general situation.

From the exact sequences

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_C(\mathcal{E}) \to q^* T\mathcal{U} \to 0$$

and

$$0 \to \mathcal{O}_C(-D_s) \xrightarrow{s} \mathcal{O}_C \xrightarrow{r} \mathcal{O}_{D_s} \to 0,$$

we derive

(4.6.1) $$0 \to \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s) \to \mathcal{D}^1_C(\mathcal{E}) \otimes \mathcal{O}_C(-D_s) \to q^* T\mathcal{U} \otimes \mathcal{O}_C(-D_s) \to 0$$

and

(4.6.2) $$0 \to \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s) \xrightarrow{s} \mathcal{E}nd(\mathcal{E}) \xrightarrow{r} \mathcal{E}nd(\mathcal{E})_{D_s} \to 0.$$

By applying the direct image functor q_*, we get the following commutative diagram:

(4.6.3) $$\begin{array}{ccccc}
 & & q_*\mathcal{E}nd(\mathcal{E}) & & \\
 & & \downarrow{\scriptstyle r} & & \\
0 & & q_*\mathcal{E}nd(\mathcal{E})_{D_s} & & \\
\downarrow & & \downarrow{\scriptstyle \delta} & & \\
T\mathcal{U} \otimes \mathcal{O}_C(-D_s) & \xrightarrow{\delta} & R^1 q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s)) & & \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
T\mathcal{U} & \underset{\delta}{\xrightarrow{\sim}} & R^1 q_* \mathcal{E}nd(\mathcal{E}) & \longrightarrow & R^1 q_* \mathcal{D}^1_C(\mathcal{E}) \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}$$

Note that the map $s : R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s)) \to R^1q_*\mathcal{E}nd(\mathcal{E})$ is surjective, but at the level of 1-cocycles is injective.

If $\{\eta^1_{ij}\}$ and $\{\eta^2_{ij}\}$ are 1-cocycles with values in $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$, we have $s\eta^1_{ij} = \dot{D}^1_j - \dot{D}^1_i$ and $s\eta^2_{ij} = \dot{D}^2_j - \dot{D}^2_i$, for some differential operators $\dot{D}^1_i$ and $\dot{D}^2_i$. Hence we can define the 1-cocycle $\{[s\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, s\eta^2_{ij}]\}_{ij}$ with values in $R^1q_*\mathcal{E}nd(\mathcal{E})$.

Let us recall that $\dot{D}^1_i$ and $\dot{D}^2_i$ are sections of $\mathcal{D}^1_C(\mathcal{E})$, hence they act as first order differential operators on functions on $\mathcal{U}$, but are linear with respect to functions defined on C. It follows that $\{[s\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, s\eta^2_{ij}]\} = s\{[\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, \eta^2_{ij}]\}$, where $\{[\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, \eta^2_{ij}]\}$ is a well-defined 1-cocycle with values in $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$.

We define the Lie bracket of $\{\eta^1_{ij}\}$ and $\{\eta^2_{ij}\}$ by setting

$$[\{\eta^1_{ij}\}, \{\eta^2_{ij}\}] = \{[\eta^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, \eta^2_{ij}]\}. \tag{4.6.4}$$

This is a well-defined antisymmetric bilinear map on $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$. By using the injectivity of the multiplication by s on 1-cocycles and the fact that the analogous bracket previously defined on $R^1q_*\mathcal{E}nd(\mathcal{E})$ is equivalent to the Lie algebra structure of the tangent bundle $T\mathcal{U}$, it follows that (4.6.4) defines a Lie algebra structure on $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$, which is exactly what we wanted. Now we take as $u : R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s)) \to T\mathcal{U}$ the composition of $s : R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s)) \to R^1q_*\mathcal{E}nd(\mathcal{E})$ with the canonical isomorphism $R^1q_*\mathcal{E}nd(\mathcal{E}) \cong T\mathcal{U}$. It is trivial to verify that u is a homomorphism of sheaves of Lie algebras and satisfies the compatibility condition (4.2.1).

Let us describe the induced Lie algebra structure on $q_*\mathcal{E}nd(\mathcal{E})_{D_s}/r(q_*\mathcal{E}nd(\mathcal{E}))$.

THEOREM 4.6.1. – *The Lie algebra structure defined on $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ by (4.6.4) induces on $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$ the usual Lie algebra structure,* i.e., *the usual commutator of endomorphisms. This structure passes to the quotient modulo $r(q_*\mathcal{E}nd(\mathcal{E}))$.*

More explicitly, on the fiber over a point $E \in \mathcal{U}$ we have the usual Lie algebra structure on $H^0(C, \mathcal{E}nd(E)_{D_s})/r(H^0(C, \mathcal{E}nd(E)))$. Note that the stability of E implies that $H^0(C, \mathcal{E}nd(E)) = \mathbb{C}$.

Proof. – Let us begin by giving an explicit description of the connecting homomorphism $\delta : q_*\mathcal{E}nd(\mathcal{E})_{D_s} \to R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$. On the fiber over the point E it is given by the connecting homomorphism $\delta : H^0(C, \mathcal{E}nd(E)_{D_s}) \to H^1(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$.

Let us suppose, for simplicity, that $D_s = \sum_{i=1}^{m} P_i$, with $P_i \neq P_j$ if $i \neq j$, and let $\psi = (\psi_i)_{i=1,\dots,m} \in H^0(C, \mathcal{E}nd(E)_{D_s}) \cong \bigoplus_{i=1}^{m} \mathcal{E}nd(E)_{P_i}$. Let $\bar{\psi} = \{\bar{\psi}_j\}$ be a 0-cochain with values in $\mathcal{E}nd(E)$, such that $\bar{\psi}_j(P_i) = \psi_i$, where $\bar{\psi}_j(P_i)$ denotes the endomorphism induced by $\bar{\psi}_j$ on the fiber of E over P_i (if P_i belongs to the open set where $\bar{\psi}_j$ is defined). We have $\delta(\bar{\psi})_{ij} = (\bar{\psi}_j - \bar{\psi}_i) = (s\sigma_{ij})$, for some 1-cocycle $\sigma = \{\sigma_{ij}\}$ with values in $\mathcal{E}nd(E) \otimes L^{-1} \otimes K$. The image $\delta(\psi)$ of ψ is the element of $H^1(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$ defined by the 1-cocycle σ.

Now let $\psi^1 = (\psi^1_i)$ and $\psi^2 = (\psi^2_i)$ be two global sections of $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$. As we have seen, we can find 0-cochains $\bar{\psi}^1 = (\bar{\psi}^1_j)$ and $\bar{\psi}^2 = (\bar{\psi}^2_j)$ with values in $q_*\mathcal{E}nd(\mathcal{E})$ such that $\bar{\psi}^h_j(P_i) = \psi^h_i$, for $h = 1, 2$. Then the equalities $\bar{\psi}^h_j - \bar{\psi}^h_i = s\sigma^h_{ij}$ define two 1-cocycles $\{\sigma^1_{ij}\}$ and $\{\sigma^2_{ij}\}$ with values in $q_*\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s)$, and we have $\delta(\psi^1) = \{\sigma^1_{ij}\}$ and $\delta(\psi^2) = \{\sigma^2_{ij}\}$.

Now we compute the Lie bracket of $\{\sigma^1_{ij}\}$ and $\{\sigma^2_{ij}\}$ in $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$.

Since we have $s\sigma^h_{ij} = \bar{\psi}^h_j - \bar{\psi}^h_i = \dot{D}^h_j - \dot{D}^h_i$, this implies that $\dot{D}^h_i = \bar{\psi}^h_i$, considered as a first order differential operator, hence it follows that

$$\begin{aligned}[s\sigma^1_{ij}, \dot{D}^2_j] + [\dot{D}^1_i, s\sigma^2_{ij}] &= [\bar{\psi}^1_j - \bar{\psi}^1_i, \bar{\psi}^2_j] + [\bar{\psi}^1_i, \bar{\psi}^2_j - \bar{\psi}^2_i]\\ &= [\bar{\psi}^1_j, \bar{\psi}^2_j] - [\bar{\psi}^1_i, \bar{\psi}^2_i]\\ &= s\tau_{ij},\end{aligned}$$

for a uniquely determined 1-cocycle $\{\tau_{ij}\}$. By definition, we have $[\{\sigma^1_{ij}\}, \{\sigma^2_{ij}\}] = \{\tau_{ij}\}$.

On the other hand, starting from the global section $\xi = [\psi^1, \psi^2]$ of $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$, we find that the 0-cochain $\bar{\xi}$ is equal to $[\bar{\psi}^1, \bar{\psi}^2]$. Hence $\bar{\xi}_j - \bar{\xi}_i = [\bar{\psi}^1_j, \bar{\psi}^2_j] - [\bar{\psi}^1_i, \bar{\psi}^2_i] = s\tau_{ij}$, which implies that the image $\delta(\xi)$ of ξ is equal to the Lie bracket of $\{\sigma^1_{ij}\}$ and $\{\sigma^2_{ij}\}$.

In other words we have

$$\delta([\psi^1, \psi^2]) = [\delta(\psi^1), \delta(\psi^2)],$$

i.e., $\delta : q_*\mathcal{E}nd(\mathcal{E})_{D_s} \to R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ is a homomorphism of sheaves of Lie algebras, where $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$ is endowed with the natural Lie algebra structure given by the usual commutator of endomorphisms. It is obvious that this Lie algebra structure passes to the quotient modulo the image of $q_*\mathcal{E}nd(\mathcal{E})$ by r. □

Remark 4.6.2. – The Lie algebra structure of $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$ allows us to define a Poisson structure on the dual sheaf (this is the analogue of the classical Poisson structure of Kostant-Kirillov). The dual sheaf to $q_*\mathcal{E}nd(\mathcal{E})_{D_s}$ is canonically identified with $q_*(\mathcal{E}nd(\mathcal{E}) \otimes p^*(L))$, the duality pairing being given by the trace map. Hence the Poisson structure defined on $\mathcal{P}$ by the Lie algebra structure of $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ induces on the fiber $H^0(C, \mathcal{E}nd(E) \otimes L)$ of $\mathcal{P}$ over the point $E \in \mathcal{U}$ the usual Poisson structure of Kostant-Kirillov.

Let us turn back to the study of the Poisson structure of the variety $\mathcal{P}$.

We denote by $\{\cdot,\cdot\}_s$ the bracket defined on $\Gamma(U, \mathcal{O}_\mathcal{P})$ by the antisymmetric contravariant 2-tensor θ_s defined in Section 4.3, *i.e.*, $\{f, g\}_s = \langle \theta_s, df \wedge dg\rangle = \langle df, B_s(dg)\rangle$, for any $f, g \in \Gamma(U, \mathcal{O}_\mathcal{P})$, and by $\{\cdot,\cdot\}$ the Poisson bracket associated to the Lie algebra structure of $\mathcal{H}^*$ and to the homomorphism $s : \mathcal{H}^* \to T\mathcal{U}$. We recall that this is uniquely determined by setting $\{f, g\} = 0$, for any two sections f and g of $\mathcal{O}_\mathcal{U}$ (thought of as functions on $\mathcal{P}$ by composing with the canonical projection $\pi : \mathcal{P} \to \mathcal{U}$); $\{\xi, f\} = s(\xi)(f)$, for any section ξ of $\mathcal{H}^*$ and any section f of $\mathcal{O}_\mathcal{U}$, where $s(\xi) = s\,\xi$ is identified with a tangent vector field to $\mathcal{U}$, and $\{\xi, \zeta\} = [\xi, \zeta]$, for any two sections ξ and ζ of $\mathcal{H}^*$, where $[\cdot,\cdot]$ is the Lie bracket defined on $\mathcal{H}^*$.

We have the following result:

THEOREM 4.6.3. – *The bracket $\{\cdot,\cdot\}_s$ is equal to the Poisson bracket $\{\cdot,\cdot\}$, hence defines a Poisson structure on the variety $\mathcal{P}$.*

Proof. – We shall prove the equality of $\{\cdot,\cdot\}_s$ and $\{\cdot,\cdot\}$ in the three cases.

Case 1. – Let $f, g \in \Gamma(U, \mathcal{O}_{\mathcal{U}})$ and denote by $\tilde{f}, \tilde{g}$ the corresponding rational functions on $\mathcal{P}$. We have seen that, for any point $(E, \phi) \in \mathcal{P}$, we have an exact sequence

$$0 \to T^*_E\mathcal{U} \to T^*_{(E,\phi)}\mathcal{P} \to H^1(C, \mathcal{E}ndE \otimes L^{-1} \otimes K) \to 0.$$

It follows that $d\tilde{f}(E, \phi) \in T^*_{(E,\phi)}\mathcal{P}$ is the image of $df(E) \in T^*_E\mathcal{U}$, *i.e.*, if $df(E) = \alpha_f \in H^0(C, \mathcal{E}nd(E) \otimes K)$, then $d\tilde{f}(E, \phi) = (\alpha_f, 0) \in \mathbb{H}^1([\phi, \cdot])$. Then we have:

$$\{\tilde{f}, \tilde{g}\}_s(E, \phi) = \langle d\tilde{f}(E, \phi), B_s(d\tilde{g}(E, \phi))\rangle = \langle(\alpha_f, 0), B_s(\alpha_g, 0)\rangle = 0.$$

On the other hand $\{\tilde{f}, \tilde{g}\} = 0$ by definition, whence the equality of the two brackets.

Case 2. – Now let ξ be a section of $R^1q_*(\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_C(-D_s))$ represented by a 1-cocycle $\{\eta_{ij}\}$. The function $\tilde{\xi}$ corresponding to ξ is defined by setting $\tilde{\xi}(E, \phi) = \langle\{\eta_{ij}(E)\}, \phi\rangle$, where $\{\eta_{ij}(E)\}$ is the element of $H^1(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$ defined by restricting the global section $\{\eta_{ij}\}$ of $R^1q_*(\mathcal{E}nd(\mathcal{E})\otimes\mathcal{O}_C(-D_s))$ to the fiber over the point E, and $\langle\cdot, \cdot\rangle$ is the canonical duality pairing between $H^1(C, \mathcal{E}nd(E)\otimes L^{-1}\otimes K)$ and $H^0(C, \mathcal{E}nd(E)\otimes L)$.

The function $\tilde{\xi}$ is linear on the fibers of $\pi : \mathcal{P} \to \mathcal{U}$, hence its differential $d\tilde{\xi}$ is a section of $T^*\mathcal{P}$ whose image in $\pi^*R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ corresponds to the section ξ. This means that $d\tilde{\xi}(E, \phi)$ is represented by an element of $\mathbb{H}^1([\phi, \cdot])$ of the form $(?, \{\eta_{ij}\})$.

It follows that

$$\begin{aligned}\{\tilde{\xi}, \tilde{f}\}_s(E, \phi) &= \langle d\tilde{\xi}(E, \phi), B_s(d\tilde{f}(E, \phi))\rangle \\ &= \langle(?, \{\eta_{ij}\}), B_s(\alpha_f, 0)\rangle \\ &= \langle\{\eta_{ij}\}, s\,\alpha_f\rangle \\ &= \langle\{s\,\eta_{ij}\}, \alpha_f\rangle \\ &= s\xi(f).\end{aligned}$$

But this is precisely the definition of the Poisson bracket $\{\tilde{\xi}, \tilde{f}\}$.

Case 3. – This is the last and most difficult case.

Let ξ^1 and ξ^2 be two sections of $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ represented by 1-cocycles $\{\eta^1_{ij}\}$ and $\{\eta^2_{ij}\}$ respectively, and denote by $\tilde{\xi}^1$, $\tilde{\xi}^2$ the corresponding functions on $\mathcal{P}$. We recall that the Poisson bracket $\{\tilde{\xi}^1, \tilde{\xi}^2\}$ is equal to the function corresponding to the Lie bracket $[\xi^1, \xi^2]$, *i.e.*, we have $\{\tilde{\xi}^1, \tilde{\xi}^2\} = \widetilde{[\xi^1, \xi^2]}$. On the other hand, to compute $\{\tilde{\xi}^1, \tilde{\xi}^2\}_s$ we need an explicit expression for the differentials $d\tilde{\xi}^1$ and $d\tilde{\xi}^2$, so let us start by finding $d\tilde{\xi}$ for a section ξ of $R^1q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$.

Let ξ be represented by a 1-cocycle $\{\eta_{ij}\}$. We know that there exist first order differential operators $\dot{D}^{s\xi}_i$ (on the variety $\mathcal{U}$) such that $s\eta_{ij} = \dot{D}^{s\xi}_j - \dot{D}^{s\xi}_i$.

Let $\tau = (\{\alpha_i\}, \{\mu_{ij}\})$ be a tangent field to $\mathcal{P}$, corresponding to a derivation D^τ of $\mathcal{O}_{\mathcal{P}}$. We know that there exist first order differential operators $\dot{D}^\tau_i$ (this time on the variety $\mathcal{P}$), such that we have $\alpha_i = [\dot{D}^\tau_i, \Phi]$ and $\mu_{ij} = \dot{D}^\tau_j - \dot{D}^\tau_i$. Let us denote by $\tau(E, \phi) = (\{\alpha_i(E, \phi)\}, \{\mu_{ij}(E, \phi)\})$ the tangent vector to $\mathcal{P}$ at the point (E, ϕ) determined by the vector field τ. The tangent vector $\tau(E, \phi)$ projects to a tangent vector $\bar{\tau}$ to $\mathcal{U}$ at the point E, given by the element $\{\mu_{ij}(E, \phi)\} \in H^1(C, \mathcal{E}nd(E))$.

In terms of infinitesimal deformations, the vector field $s\xi$ is given equivalently by an automorphism

$$\mathcal{U} \times \operatorname{Spec}(\mathbb{C}[\epsilon]) \times C \xrightarrow{(1+\widetilde{\epsilon D^{s\xi}})\times 1_C} \mathcal{U} \times \operatorname{Spec}(\mathbb{C}[\epsilon]) \times C,$$

and is represented (locally) by the infinitesimal deformation $\mathcal{E}_\epsilon = ((1+\widetilde{\epsilon D^{s\xi}}) \times 1_C)^*\mathcal{E}[\epsilon]$ of the local universal family $\mathcal{E}$ on $\mathcal{U}$.

The tangent vector $\bar{\tau}$ is equivalent to a morphism $\bar{\tau} : \operatorname{Spec}(\mathbb{C}[\epsilon']) \times C \to \mathcal{U} \times C$, corresponding to an infinitesimal deformation $E_{\epsilon'} = \bar{\tau}^*\mathcal{E}[\epsilon']$, represented by the element $\{\mu_{ij}(E,\phi)\} \in H^1(C, \mathcal{E}nd E)$.

Now we can compute $\tilde{\xi}(E_\epsilon, \phi_\epsilon)$, where $(E_\epsilon, \phi_\epsilon)$ is the infinitesimal deformation of (E,ϕ) corresponding to the tangent vector $\tau(E,\phi)$.

We have $\tilde{\xi}(E_\epsilon, \phi_\epsilon) = \langle (\eta_{ij})_\epsilon, \phi_\epsilon \rangle$, where $(\eta_{ij})_\epsilon$ is the "infinitesimal deformation" of η_{ij} in the direction of τ at the point (E,ϕ).

Let us consider the tangent field $s\xi$ on $\mathcal{U}$. The situation may be summarized by the following diagram:

$$\begin{array}{ccc}
\mathcal{U} \times \operatorname{Spec}(\mathbb{C}[\epsilon,\epsilon']) \times C & \xrightarrow{(1+\widetilde{\epsilon D^{s\xi}})\times 1_C} & \mathcal{U} \times \operatorname{Spec}(\mathbb{C}[\epsilon,\epsilon']) \times C \\
\uparrow {\scriptstyle (1+\widetilde{\epsilon' D^{\tau}})\times 1_C} & & \\
\operatorname{Spec}(\mathbb{C}[\epsilon,\epsilon']) \times C & &
\end{array}$$

By pulling back we get the sheaf $((1+\widetilde{\epsilon' D^\tau}) \times 1_C)^*((1+\widetilde{\epsilon D^{s\xi}}) \times 1_C)^*\mathcal{E}[\epsilon,\epsilon']$, and we have already seen in Section 4.5 that this is described by giving gluing isomorphisms of the form $1+\epsilon(s\eta_{ij}) + \epsilon'(\mu_{ij}) + \epsilon\epsilon'(\dot{D}_j^\tau(s\eta_{ij}) - s\eta_{ij}\dot{D}_i^\tau)$. To simplify the notations we have not explicitly written the restrictions, but all differential operators are intended to be restricted to $\{E\} \times C$.

It follows that $s(\eta_{ij})_\epsilon = s\eta_{ij} + \epsilon(\dot{D}_j^\tau(s\eta_{ij}) - s\eta_{ij}\dot{D}_i^\tau) = s\eta_{ij} + s\epsilon(\dot{D}_j^\tau\eta_{ij} - \eta_{ij}\dot{D}_i^\tau)$, by the $\mathcal{O}_C$-linearity of the differential operators $\dot{D}_i^\tau$. Now, by the injectivity of the multiplication by s on cocycles, we derive that

$$(\eta_{ij})_\epsilon = \eta_{ij} + \epsilon(\dot{D}_j^\tau\eta_{ij} - \eta_{ij}\dot{D}_i^\tau). \tag{4.6.5}$$

It follows that

$$\begin{aligned}
\tilde{\xi}(E_\epsilon, \phi_\epsilon) &= \langle \{\eta_{ij}(E) + \epsilon(\dot{D}_j^\tau\eta_{ij} - \eta_{ij}\dot{D}_i^\tau)\}, \{\phi + \epsilon\alpha_i\} \rangle \\
&= \operatorname{res} \circ \operatorname{Tr}(\eta_{ij}\phi) + \epsilon \operatorname{res} \circ \operatorname{Tr}(\alpha_i\eta_{ij} + (\dot{D}_j^\tau\eta_{ij} - \eta_{ij}\dot{D}_i^\tau)\phi) \\
&= \tilde{\xi}(E,\phi) + \epsilon\tilde{\xi}'(E,\phi),
\end{aligned} \tag{4.6.6}$$

where $\tilde{\xi}'(E,\phi)$ denotes the derivative of $\tilde{\xi}$, with respect to the tangent vector τ, at the point (E,ϕ).

We have already seen in Case 2 that the differential of $\tilde{\xi}$ at the point (E,ϕ) is given by an element of $\mathbb{H}^1([\phi,\cdot])$ of the form $(\{\sigma_i(E,\phi)\}, \{\eta_{ij}(E,\phi)\})$, where $\{\eta_{ij}(E,\phi)\}$ is the element of $H^1(C, \mathcal{E}nd(E) \otimes L^{-1} \otimes K)$ determined by the global section $\{\eta_{ij}\}$ of $R^1 q_*(\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(-D_s))$ corresponding to ξ, and $\{\sigma_i(E, \phi)\}$ is unknown. Now, using the fact that $\langle d\tilde{\xi}(E, \phi), \tau(E, \phi)\rangle = \tilde{\xi}'(E, \phi)$, we have:

$$\begin{aligned}\tilde{\xi}'(E, \phi) &= \langle d\tilde{\xi}(E, \phi), \tau(E, \phi)\rangle \\ &= \langle(\{\sigma_i\}, \{\eta_{ij}\}), (\{\alpha_i\}, \{\mu_{ij}\})\rangle \\ &= \operatorname{res} \circ \operatorname{Tr}(\alpha_i \eta_{ij} + \sigma_j \mu_{ij}) \\ &= \operatorname{res} \circ \operatorname{Tr}(\alpha_i \eta_{ij} + (\dot{D}_j^\tau \eta_{ij} - \eta_{ij} \dot{D}_i^\tau)\phi).\end{aligned}$$

This shows that $\{\sigma_i\}$ is determined by requiring that

$$\operatorname{res} \circ \operatorname{Tr}(\sigma_j \mu_{ij}) = \operatorname{res} \circ \operatorname{Tr}((\dot{D}_j^\tau \eta_{ij} - \eta_{ij} \dot{D}_i^\tau)\phi). \tag{4.6.7}$$

Note that if in the formula (4.6.6) we had used $\phi + \epsilon\alpha_j$ in place of $\phi + \epsilon\alpha_i$, we would have found

$$\operatorname{res} \circ \operatorname{Tr}(\sigma_i \mu_{ij}) = \operatorname{res} \circ \operatorname{Tr}((\dot{D}_j^\tau \eta_{ij} - \eta_{ij} \dot{D}_i^\tau)\phi). \tag{4.6.8}$$

Now we are able to compute $\{\tilde{\xi}^1, \tilde{\xi}^2\}_s$.

$$\begin{aligned}\{\tilde{\xi}^1, \tilde{\xi}^2\}_s(E, \phi) &= \langle d\tilde{\xi}^1(E, \phi), B_s(d\tilde{\xi}^2(E, \phi))\rangle \\ &= \langle(\{\sigma_i^1\}, \{\eta_{ij}^1\}), B_s(\{\sigma_i^2\}, \{\eta_{ij}^2\})\rangle \\ &= \langle(\{\sigma_i^1\}, \{\eta_{ij}^1\}), (\{s\sigma_i^2\}, \{-s\eta_{ij}^2\})\rangle \\ &= \operatorname{res} \circ \operatorname{Tr}(-\sigma_i^1(s\eta_{ij}^2) + \sigma_j^2(s\eta_{ij}^1)) \\ &= \operatorname{res} \circ \operatorname{Tr}((-(\dot{D}_j^2 \eta_{ij}^1 - \eta_{ij}^1 \dot{D}_i^2)\phi + (\dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1)\phi) \\ &= \operatorname{res} \circ \operatorname{Tr}((-\dot{D}_j^2 \eta_{ij}^1 + \eta_{ij}^1 \dot{D}_i^2 + \dot{D}_j^1 \eta_{ij}^2 - \eta_{ij}^2 \dot{D}_i^1)\phi).\end{aligned}$$

From this, using the identities $\dot{D}_i^2 = \dot{D}_j^2 - s\eta_{ij}^2$ and $\dot{D}_j^1 = \dot{D}_i^1 + s\eta_{ij}^1$, one gets

$$\begin{aligned}\{\tilde{\xi}^1, \tilde{\xi}^2\}_s(E, \phi) &= \operatorname{res} \circ \operatorname{Tr}(([\eta_{ij}^1, \dot{D}_j^2] - \eta_{ij}^1 s\eta_{ij}^2 + [\dot{D}_i^1, \eta_{ij}^2] + s\eta_{ij}^1 \eta_{ij}^2)\phi) \\ &= \operatorname{res} \circ \operatorname{Tr}(([\eta_{ij}^1, \dot{D}_j^2] + [\dot{D}_i^1, \eta_{ij}^2])\phi) \\ &= \widetilde{[\xi^1, \xi^2]}(E, \phi),\end{aligned}$$

which proves the equality of the two brackets and completes the proof of the theorem. □

Remark 4.6.4. – From what we have seen, it follows immediately that the morphism from $\mathcal{P}(r, d, K)$ to $P(r, d, L)$ sending the point (E, ϕ) to $(E, s\phi)$ is a morphism of Poisson varieties.

4.7 Completely integrable systems.

In the preceding section we have seen that the antisymmetric contravariant 2-tensor θ_s defines a Poisson structure on $\mathcal{P}$. By what we have seen in Section 4.3, it follows that θ_s defines a Poisson structure at least on $\mathcal{M}'_0$, the connected component of $\mathcal{M}'$ containing $\mathcal{P}$ (*see* Remark 4.3.1). In the sequel we shall assume that $\deg(L) > \deg(K)$.

Let us consider the morphism $H : \mathcal{M}'_0 \to W = \bigoplus_{i=1}^{r} H^0(C, L^i)$ defined in Section 2.2, which associates to each point $(E,\phi) \in \mathcal{M}'_0$ the coefficients of the characteristic polynomial of ϕ. We have seen that the inverse image $H^{-1}(\sigma)$, for σ generic, is canonically isomorphic to an open subset of an abelian variety, precisely the Jacobian variety of the spectral curve X_σ defined by the section σ. The complement of $H^{-1}(\sigma)$ in $\mathrm{Jac}(X_\sigma)$ is a closed subset of codimension at least 2.

Let us choose a coordinate system on W. The morphism H is then given by N polynomial functions $H_1, \ldots, H_N$ (with $N = \dim W = \frac{1}{2}r(r+1)\deg(L) - r(g-1)$). Since $\dim \mathcal{M}'_0 = r^2 \deg(L) + 1$ and $\dim H^{-1}(\sigma) = \dim \mathrm{Jac}(X_\sigma) = \frac{1}{2}r(r-1)\deg(L) + r(g-1) + 1$, it follows, by a dimensional count, that $H_1, \ldots, H_N$ are functionally independent, *i.e.*, $dH_1 \wedge \ldots \wedge dH_N \neq 0$.

Let us denote by X_{H_i} the hamiltonian vector field associated to the function H_i. If the functions H_i are in involution, *i.e.*, if $\{H_i, H_j\} = 0$ for all i, j, then we have $[X_{H_i}, X_{H_j}] = X_{\{H_i,H_j\}} = 0$. It follows that the hamiltonian vector field X_{H_i} defines a holomorphic vector field on the generic fiber $H^{-1}(\sigma)$. By what we have previously seen, it extends as a holomorphic vector field to the whole Jacobian variety $\mathrm{Jac}(X_\sigma)$, hence is linear. In other words the hamiltonian vector fields X_{H_i} are linear on the fibers of H and span, on the generic fiber, the space of translation invariant vector fields.

PROPOSITION 4.7.1. – *The functions H_i are in involution,* i.e., $\{H_i, H_j\} = 0$ *for all* i, j.

Proof. – Let $P \in C$ and $U \subset C$ be an open subset containing P. Let us choose a local coordinate ζ on U centered at P and a trivialization $\lambda : L|_U \xrightarrow{\sim} \mathcal{O}_U$. By taking the germ of a section at P and composing with $\lambda(P)^{\otimes i}$, we get a map $v(P) : H^0(C, L^i) \to \mathbb{C}$. Let us denote by $\Lambda^i\mathrm{Tr}(P) : \mathcal{M}'_0 \to \mathbb{C}$ the map given by the composition of $(E,\phi) \mapsto \mathrm{Tr}(\wedge^i \phi)$ with $v(P)$, *i.e.*, $\Lambda^i\mathrm{Tr}(P)(E,\phi) = v(P)\mathrm{Tr}(\wedge^i\phi)$.

The space of functions H_i is generated by the functions $\Lambda^j\mathrm{Tr}(P)$, for $j \geq 0$ and P generic in C. These may be expressed in terms of the functions $\mathrm{Tr}^i(P) : (E,\phi) \mapsto v(P)\mathrm{Tr}(\phi^i)$ by means of Newton's relations (2.1.1):

$$v(P)\mathrm{Tr}(\wedge^j\phi) = \frac{(-1)^{j-1}}{j} v(P)\mathrm{Tr}(\phi^j) + Q_j(v(P)\mathrm{Tr}(\phi), \ldots, v(P)\mathrm{Tr}(\phi^{j-1})),$$

where Q_j is a universal polynomial in $j-1$ variables with rational coefficients. Hence the space of functions H_i is also generated by the functions $\mathrm{Tr}^j(P)$, and it suffices to prove that the Poisson bracket of any two of these is zero.

Let us consider a tangent vector $\tau = (\{\alpha_i\}, \{\mu_{ij}\}) \in \mathbb{H}^1([\cdot,\phi])$ to $\mathcal{M}'_0$ at the point (E,ϕ), corresponding to the infinitesimal deformation $(E_\epsilon, \phi_\epsilon)$. By using first order Taylor series expansions, we have

$$\mathrm{Tr}^i(P)(E_\epsilon, \phi_\epsilon) = \mathrm{Tr}^i(P)(E,\phi) + \epsilon \mathrm{Tr}^i(P)'(E,\phi),$$

where $\mathrm{Tr}^i(P)'(E,\phi) = \langle d\mathrm{Tr}^i(P)(E,\phi), \tau\rangle$ is the derivative of $\mathrm{Tr}^i(P)$ in the direction of τ at (E,ϕ).

By what we have previously seen, we have

$$\begin{aligned}\mathrm{Tr}^i(P)(E_\epsilon,\phi_\epsilon) &= v(P)\mathrm{Tr}(\phi_\epsilon^i)\\ &= v(P)\mathrm{Tr}((\phi+\epsilon\alpha_j)^i)\\ &= v(P)\mathrm{Tr}(\phi^i)+\epsilon\, i\, v(P)\mathrm{Tr}(\phi^{i-1}\alpha_j).\end{aligned}$$

Note that this is well-defined because we have $\alpha_k - \alpha_h = [\mu_{hk},\phi]$, hence $\mathrm{Tr}(\phi^{i-1}\alpha_h) = \mathrm{Tr}(\phi^{i-1}\alpha_k)$. From this we derive that $\mathrm{Tr}^i(P)'(E,\phi) = i\, v(P)\mathrm{Tr}(\phi^{i-1}\alpha_j)$.

Now we look for the differential of $\mathrm{Tr}^i(P)$ at the point (E,ϕ).

Let us set $d\mathrm{Tr}^i(P)(E,\phi) = (\{\sigma_i\},\{\nu_{ij}\}) \in \mathbb{H}^1([\phi,\cdot])$. We have:

$$\begin{aligned}\mathrm{Tr}^i(P)'(E,\phi) &= \langle d\mathrm{Tr}^i(P)(E,\phi),\tau\rangle\\ &= \langle(\{\sigma_i\},\{\nu_{ij}\}),(\{\alpha_i\},\{\mu_{ij}\})\rangle\\ &= \mathrm{res}\circ\mathrm{Tr}(\sigma_i\mu_{ij}+\alpha_j\nu_{ij}).\end{aligned}$$

Let us choose an open affine covering $(U_i)_{i=1,2}$ of the curve C, of the form $U_1 = U$ and $U_2 = C\setminus\{P\}$, such that $U_{12} = U\setminus\{P\}$. In this situation the trace isomorphism $\mathrm{res} : H^1(C,K)\to\mathbb{C}$ is given by taking residues at P:

$$\begin{aligned}\mathrm{res}_P : H^1(C,K) &\to \mathbb{C}\\ \{\eta_{ij}\} &\mapsto \mathrm{res}_P(\eta_{ij}).\end{aligned}$$

It follows that the differential of $\mathrm{Tr}^i(P)$ at the point (E,ϕ) is determined by the equation

(4.7.1) $$\mathrm{res}_P(\mathrm{Tr}(\sigma_i\mu_{ij}+\alpha_j\nu_{ij})) = i\, v(P)\mathrm{Tr}(\phi^{i-1}\alpha_j).$$

Let us consider the element $d\zeta/\zeta$ of $H^1(C,K)$. This gives a normalized basis of $H^1(C,K)$, in the sense that $\mathrm{res}_P(d\zeta/\zeta) = 1$.

From now on we use the trivialization λ to identify sections of L over U with sections of $\mathcal{O}_U$. For convenience of notation the composition with λ is not explicitly written. Then we have automorphisms $\phi^i : E|_U \to E|_U$, for each $i\geq 0$, and it follows that $\phi^i\, d\zeta/\zeta$ may be considered as an element of $\Gamma(U_{12},\mathcal{E}nd(E)\otimes L^{-1}\otimes K)$. We set $\nu_{12} = i\,\phi^{i-1}\,d\zeta/\zeta$, $\sigma_1 = 0$ and $\sigma_2 = 0$. This is consistent, in fact we have $\sigma_2-\sigma_1 = [\phi,\nu_{12}] = [\phi, i\,\phi^{i-1}\,d\zeta/\zeta] = i\,[\phi,\phi^{i-1}]\,d\zeta/\zeta = 0$.

To show that this is actually the differential of $\mathrm{Tr}^i(P)$ at the point (E,ϕ), we have only to check that (4.7.1) is satisfied.

We have

$$\begin{aligned}\mathrm{res}_P(\mathrm{Tr}(\sigma_i\mu_{ij}+\alpha_j\nu_{ij})) &= i\mathrm{res}_P(\mathrm{Tr}(\alpha_j\cdot\phi^{i-1}\,d\zeta/\zeta))\\ &= i\, v(P)\mathrm{Tr}(\phi^{i-1}\alpha_j),\end{aligned}$$

which proves our assertion.

In conclusion we have found that $d\mathrm{Tr}^i(P)(E,\phi) = (0, i\,\phi^{i-1}\,d\zeta/\zeta) \in \mathbb{H}^1([\phi,\cdot])$.

If $P'\neq P$ is another point of C, we may choose U such that $P'\in U$. If ζ is the local coordinate on U centered at P previously chosen, then $\zeta' = \zeta - \zeta(P)$ is a local coordinate centered at P'. By repeating the above reasoning for the point P', we get another isomorphism $\mathrm{res}_{P'} : H^1(C, K) \to \mathbb{C}$, given by taking residues at P'. It follows that there is a non-zero constant $c_{PP'}$ such that $\mathrm{res}_{P'} = c_{PP'}\mathrm{res}_P$.

Now, if we consider the function $\mathrm{Tr}^i(P')$, we find that its derivative in the direction of τ at (E, ϕ) is given by $\mathrm{Tr}^i(P')'(E, \phi) = i\, v(P')\mathrm{Tr}(\phi^{i-1}\alpha_j)$. Let us denote by $(\{\sigma_i\}, \{\nu_{ij}\}) \in \mathbb{H}^1([\phi, \cdot])$ the differential of $\mathrm{Tr}^i(P')$ at the point (E, ϕ). We claim that we may choose $\sigma_1 = \sigma_2 = 0$ and $\nu_{12} = c_{PP'}\, i\, \phi^{i-1}\, d\zeta'/\zeta'$. In fact we have

$$\begin{aligned}\mathrm{res}_P(\mathrm{Tr}(\sigma_i\mu_{ij} + \alpha_j\nu_{ij})) &= c_{PP'}\, i\mathrm{res}_P(\mathrm{Tr}(\alpha_j \cdot \phi^{i-1}\, d\zeta'/\zeta')) \\ &= i\mathrm{res}_{P'}(\mathrm{Tr}(\alpha_j \cdot \phi^{i-1}\, d\zeta'/\zeta')) \\ &= i\, v(P')\mathrm{Tr}(\phi^{i-1}\alpha_j),\end{aligned}$$

which proves our assertion. It is now immediate to prove that the functions $\mathrm{Tr}^i(P)$ are in involution, in fact we have:

$$\begin{aligned}\{\mathrm{Tr}^i(P), \mathrm{Tr}^j(P')\}(E, \phi) &= \langle d\mathrm{Tr}^i(P)(E, \phi), B_s(d\mathrm{Tr}^j(P')(E, \phi))\rangle \\ &= \langle(0, i\, \phi^{i-1}\, d\zeta/\zeta), (0, -s\, c_{PP'}\, j\, \phi^{j-1}\, d\zeta'/\zeta')\rangle \\ &= 0.\end{aligned}$$

□

From what we have previously seen, we derive immediately the following

THEOREM 4.7.2. – *The morphism $H : \mathcal{M}'_0 \to W$ defines an algebraically completely integrable hamiltonian system on the Poisson variety $\mathcal{M}'_0$.*

Note that, since $\mathcal{M}'_0$ is not a symplectic variety, the hamiltonian system H is completely integrable only in a generalized sense. More precisely, the Poisson variety $\mathcal{M}'_0$ is foliated, over the open subset where the Poisson structure has maximal rank, by symplectic varieties of dimension equal to the (maximum) rank of the Poisson structure. Then the restriction of H to each symplectic leaf gives an algebraically completely integrable system in the classical sense. The hamiltonian system H may thus be considered as a family of completely integrable systems on the symplectic leaves of $\mathcal{M}'_0$. In the sequel we shall make precise this remark.

First let us recall some basic results on Poisson varieties, which may be found for example in [W]. A function f on a Poisson variety M is called a Casimir function (or invariant function) if the hamiltonian vector field H_f is identically zero, *i.e.*, if $\{f, g\} = 0$, for any function g. If $f_1, \ldots, f_n$ span the vector space of Casimir functions in a neighborhood of a point $x \in M$, then the symplectic leaf through x is given locally by the zero set of the functions $f_1, \ldots, f_n$.

Let us come back now to our situation. We prove the following

LEMMA 4.7.3. – *If $P \in D_s$ then the function* $\mathrm{Tr}^i(P)$ *is a Casimir function on $\mathcal{M}'_0$.*

Proof. – We have seen in the proof of Proposition 4.7.1 that the differential of the function $\mathrm{Tr}^i(P) : \mathcal{M}'_0 \to \mathbb{C}$ at the point (E, ϕ) may be expressed as $d\mathrm{Tr}^i(P)(E, \phi) = (0, i\, \phi^{i-1}\, d\zeta/\zeta) \in \mathbb{H}^1([\phi, \cdot])$. If f is another function on $\mathcal{M}'_0$, and if we write $df(E, \phi) = (\{\alpha_i\}, \{\eta_{ij}\})$, the Poisson bracket of $\mathrm{Tr}^i(P)$ and f is given by

$$\begin{aligned}\{\mathrm{Tr}^i(P), f\} &= \langle(0, i\, \phi^{i-1}\, d\zeta/\zeta), B_s(\{\alpha_i\}, \{\eta_{ij}\})\rangle \\ &= \mathrm{res}_P(\mathrm{Tr}(i\, s\, \alpha_j\phi^{i-1}\, d\zeta/\zeta)),\end{aligned}$$

which is equal to zero, if $s(P) = 0$. $\square$

Let us set $W = \bigoplus_{i=1}^{r} H^0(C, L^i)$ and $W_{D_s} = \bigoplus_{i=1}^{r} H^0(C, L^i_{D_s})$, and consider the natural map $\rho : W \to W_{D_s}$, given by evaluation of sections at the points of the divisor D_s. We have

LEMMA 4.7.4. – *The image of $\rho : W \to W_{D_s}$ is a hyperplane $\bar{W}_{D_s}$ in the vector space W_{D_s}.*

Proof. – From the exact sequence

$$0 \to K \xrightarrow{s} L \xrightarrow{\rho} L_{D_s} \to 0,$$

we derive

$$0 \to K \otimes L^{i-1} \xrightarrow{s} L^i \xrightarrow{\rho} L^i_{D_s} \to 0, \tag{4.7.2}$$

for $i = 1, \ldots, r$. Taking the corresponding long exact cohomology sequence gives

$$H^0(C, L^i) \xrightarrow{\rho} H^0(C, L^i_{D_s}) \to H^1(C, K \otimes L^{i-1}) \to H^1(C, L^i).$$

Now, by using Serre duality and recalling that $\deg(L) > \deg(K)$, it follows that $H^1(C, L^i) = 0$ for $i = 1, \ldots, r$, and $\dim H^1(C, K \otimes L^{i-1}) = 1$ if $i = 1$ and is zero otherwise. From this we get the following exact sequence

$$\bigoplus_{i=1}^{r} H^0(C, L^i) \xrightarrow{\rho} \bigoplus_{i=1}^{r} H^0(C, L^i_{D_s}) \to \mathbb{C} \to 0,$$

which proves the lemma. $\square$

Now we can consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}'_0 & \xrightarrow{H} & W \\ & {\scriptstyle C}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \bar{W}_{D_s} & \end{array} \tag{4.7.3}$$

where the map C associates to a pair $(E, \phi) \in \mathcal{M}'_0$ the coefficients of the characteristic polynomial of ϕ_{P_i}, for each $P_i \in D_s$ (we are still assuming that $D_s = \sum_{i=1}^{m} P_i$, with $P_i \neq P_j$ if $i \neq j$).

By choosing a base for $\bar{W}_{D_s}$, the function C may be given by its N component functions $C_1, \ldots, C_N$, and it is easy to see that the vector space generated by these functions is spanned equivalently by the functions $C_{ij} = \mathrm{Tr}^i(P_j)$, for $i = 1, \ldots, r$ and $j = 1, \ldots, m$.

Now we are able to prove the following

THEOREM 4.7.5. – *The vector space spanned by the functions $C_{ij} = \mathrm{Tr}^i(P_j)$, for $i = 1, \ldots, r$ and $j = 1, \ldots, m$ (or, equivalently, by the functions $C_1, \ldots, C_N$) coincides, at a generic point of $\mathcal{M}'_0$, with the space of all Casimir functions on the Poisson variety $\mathcal{M}'_0$. Hence, for w generic in $\bar{W}_{D_s}$, the fibers $C^{-1}(w)$ of C are the symplectic leaves of $\mathcal{M}'_0$.*

Proof. – We have already proved in Lemma 4.7.3 that C_{ij} are Casimir functions. It remains only to show that these functions span, at a generic point of $\mathcal{M}'_0$, the vector space of all Casimir functions. Since $\dim \mathcal{M}'_0 = r^2 \deg(L) + 1$ and $\dim \bar{W}_{D_s} = mr - 1 = r(\deg(L) - \deg(K)) - 1$, it follows that $\dim C^{-1}(w) \geq r(r-1)\deg(L) + 2r(g-1) + 2$, where equality holds if and only if the functions $C_1, \ldots, C_N$ are functionally independent, *i.e.*, $dC_1 \wedge \ldots \wedge dC_N \neq 0$. From the exact sequence (4.7.2) we get

$$0 \to \bigoplus_{i=1}^{r} H^0(C, K \otimes L^{i-1}) \to \bigoplus_{i=1}^{r} H^0(C, L^i) \overset{\rho}{\to} \bar{W}_{D_s} \to 0,$$

which shows that

$$\dim \rho^{-1}(w) = \frac{r(r-1)}{2} \deg(L) + r(g-1) + 1.$$

We know that, for a generic $\sigma \in W$ we have $\dim H^{-1}(\sigma) = \dim \mathrm{Jac}(X_\sigma) = \frac{1}{2} r(r-1)\deg(L) + r(g-1) + 1$. It follows that, for a generic $w \in \bar{W}_{D_s}$, $\dim C^{-1}(w) = r(r-1)\deg(L) + 2r(g-1) + 2$, which proves that $C_1, \ldots, C_N$ are functionally independent.

Now we recall from Section 4.4 that the generic (and maximum) rank of the Poisson structure of $\mathcal{M}'_0$ is equal to $r(r-1)\deg(L) + 2r(g-1) + 2$, and this is precisely the dimension of the symplectic leaves of $\mathcal{M}'_0$, on the open subset where the Poisson structure has maximum rank. By comparing dimensions it follows immediately that the fibers of C are precisely the generic symplectic leaves of $\mathcal{M}'_0$, which proves that the functions $C_1, \ldots, C_N$ span the vector space of all Casimir functions. □

We have thus seen that the algebraically completely integrable hamiltonian system $H : \mathcal{M}'_0 \to W$ can be thought of, at least generically, as a family of algebraically completely integrable hamiltonian systems on the symplectic leaves of $\mathcal{M}'_0$, parametrized by the vector space $\bar{W}_{D_s}$.

Remark 4.7.6. – Now we want to discuss how our construction generalizes previous results obtained by A. Beauville in [B1] in the case $C = \mathbf{P}^1$.

It is well known that the moduli space of semistable vector bundles of rank r and degree d over $\mathbf{P}^1$ is either empty, if r does not divide d, or is reduced to a single point, namely the isomorphism class of a vector bundle E. We have $\mathrm{Aut}(E) \cong \mathrm{GL}(r, \mathbb{C})$. Then, if $r \mid d$, the variety $\mathcal{M}(r, d, L)$ is equal to $H^0(\mathbf{P}^1, \mathcal{E}nd(E) \otimes L)/\mathrm{Aut}(E)$.

By choosing an affine coordinate x on $\mathbf{P}^1$ and setting $L = \mathcal{O}_{\mathbf{P}^1}(d \cdot \infty)$, the vector space $H^0(\mathbf{P}^1, \mathcal{E}nd(E) \otimes L)$ can be identified with the set of $r \times r$ polynomial matrices with entries of degree $\leq d$. This is precisely the situation studied in [B1]. It is now easy to see that our previous results reduce, in this case, to the ones obtained by Beauville.

In the next section we shall see how our results may be restated, with only some minor changes, for the moduli space of stable pairs with fixed determinant bundle.

4.8 Stable pairs with fixed determinant bundle.

In this section we shall see that the results we have obtained so far for general vector bundles may be restated, with minor changes, for vector bundles with fixed determinant.

Let us denote by $J_C^{(d)}$ the space of isomorphism classes of line bundles of degree d on C. For $\zeta \in J_C^{(d)}$, we denote by $\mathcal{U}_s(r, \zeta)$ the moduli space of stable vector bundles of rank r with determinant isomorphic to ζ. If $(r, d) = 1$ this is a fine moduli space.

Now let E be a stable vector bundle. We have a direct sum decomposition of the sheaf of endomorphisms of E:

$$\mathcal{E}nd(E) \cong \mathcal{E}nd^0(E) \oplus \mathcal{O}_C, \tag{4.8.1}$$

where $\mathcal{E}nd^0(E)$ denotes the sheaf of trace-free endomorphisms. We point out that the stability of E implies that $H^0(C, \mathcal{E}nd^0(E)) = 0$.

From deformation theory it follows that the tangent space $T_E\mathcal{U}_s(r,\zeta)$ is canonically isomorphic to the vector space $H^1(C, \mathcal{E}nd^0(E))$. By using Riemann-Roch we can compute the dimension of $\mathcal{U}_s(r,\zeta)$, which turns out to be $(r^2-1)(g-1)$.

Let L be another fixed line bundle on C, and assume that either $L \cong K$ or $\deg(L) > \deg(K)$. Let us denote by $\mathcal{M}'(r,\zeta,L)$ the subset of $\mathcal{M}'(r,d,L)$ consisting of (isomorphism classes of) pairs (E,ϕ) with $\det(E) \cong \zeta$ and $\phi \in H^0(C, \mathcal{E}nd^0(E) \otimes L)$, by $\mathcal{P}(r,\zeta,L)$ the subset of $\mathcal{M}'(r,\zeta,L)$ consisting of pairs (E,ϕ) with E stable, and by $\mathcal{M}'_0(r,\zeta,L)$ the connected component of $\mathcal{M}'(r,\zeta,L)$ containing $\mathcal{P}(r,\zeta,L)$. It is immediate to prove that $\mathcal{P}(r,\zeta,L)$ is a vector bundle over $\mathcal{U}_s(r,\zeta)$, the fiber over E being the vector space $H^0(C, \mathcal{E}nd^0(E) \otimes L)$. Again, by using Riemann-Roch, we find that $\dim \mathcal{P}(r,\zeta,L) = (r^2-1)\deg(L)$.

Let $(E,\phi) \in \mathcal{M}'_0(r,\zeta,L)$ and consider the following complex:

$$[\cdot,\phi]^0 : \quad 0 \to \mathcal{E}nd^0(E) \xrightarrow{[\cdot,\phi]} \mathcal{E}nd^0(E) \otimes L \to 0.$$

By adapting the proof of Proposition 3.1.2 to the present situation, we can prove the following

PROPOSITION 4.8.1. – *The tangent space $T_{(E,\phi)}\mathcal{M}'_0(r,\zeta,L)$ to $\mathcal{M}'_0(r,\zeta,L)$ at the point (E,ϕ) is canonically isomorphic to the first hypercohomology group $\mathbb{H}^1([\cdot,\phi]^0)$.*

Noting that the sheaf $\mathcal{E}nd^0(E)$ is autodual under the pairing trace, it is easy to see that the dual complex to $[\cdot,\phi]^0$ is canonically identified to the complex

$$[\phi,\cdot]^0 : \quad 0 \to \mathcal{E}nd^0(E) \otimes L^{-1} \otimes K \xrightarrow{[\phi,\cdot]} \mathcal{E}nd^0(E) \otimes K \to 0.$$

By recalling Serre duality for hypercohomology, we get

PROPOSITION 4.8.2. – *The cotangent space $T^*_{(E,\phi)}\mathcal{M}'_0(r,\zeta,L)$ to $\mathcal{M}'_0(r,\zeta,L)$ at the point (E,ϕ) is canonically isomorphic to the first hypercohomology group $\mathbb{H}^1([\phi,\cdot]^0)$.*

Assume now that $H^0(C, K^{-1} \otimes L) \neq 0$. Given a non-zero section s of $K^{-1} \otimes L$, we define a morphism of complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{E}nd^0(E) & \xrightarrow{[\cdot,\phi]} & \mathcal{E}nd^0(E) \otimes L & \longrightarrow 0 \\ & \uparrow{\scriptstyle -s} & & \uparrow{\scriptstyle s} & \\ 0 \longrightarrow & \mathcal{E}nd^0(E) \otimes L^{-1} \otimes K & \xrightarrow[{[\phi,\cdot]}]{} & \mathcal{E}nd^0(E) \otimes K & \longrightarrow 0, \end{array}$$

which induces a homomorphism of hypercohomology groups:

$$B_s : \mathbb{H}^1([\phi,\cdot]^0) \to \mathbb{H}^1([\cdot,\phi]^0).$$

These maps give a homomorphism $B_s : T^*\mathcal{M}'_0(r,\zeta,L) \to T\mathcal{M}'_0(r,\zeta,L)$, which defines a contravariant 2-tensor θ_s (*see* Section 4.3). All the reasoning we have made to prove Theorem 4.6.3 can be repeated, with only some minor changes, to prove the following

THEOREM 4.8.3. – *The contravariant 2-tensor θ_s defines a Poisson structure on the variety $\mathcal{M}'_0(r,\zeta,L)$.*

Now we turn to the study of the completely integrable system on the Poisson variety $\mathcal{M}'_0(r,\zeta,L)$ defined by the Hitchin map.

First we note that if M is a line bundle on C then a vector bundle E is stable if and only if $E \otimes M$ is, hence the tensorization by M gives an isomorphism of $\mathcal{U}_s(r,\zeta)$ with $\mathcal{U}_s(r,\zeta \otimes M^r)$. Therefore it is not restrictive to assume that $\zeta = \mathcal{O}_C$.

By recalling the definition of $\mathcal{M}'_0(r,\zeta,L)$, it is immediate to see that the Hitchin map defined in Section 2.2 is given, in this case, by

$$H : \mathcal{M}'_0(r,\zeta,L) \to W = \bigoplus_{i=2}^{r} H^0(C,L^i). \tag{4.8.2}$$

If $\sigma' = (\sigma_2,\ldots,\sigma_r) \in W$, we set $\sigma = (0,\sigma') = (0,\sigma_2,\ldots,\sigma_r) \in \bigoplus_{i=1}^{r} H^0(C,L^i)$ and denote by X_σ the spectral curve defined by σ. We have a morphism $\pi : X_\sigma \to C$ making X_σ a ramified r-sheeted covering of C.

Let us denote by $\mathfrak{d}$ the line bundle $\det(\pi_*\mathcal{O}_{X_\sigma})^{-1}$ on C, and set $\delta = \deg(\mathfrak{d})$. From [BNR, Proposition 3.6] it follows that the intersection of the fiber $H^{-1}(\sigma')$ with $\mathcal{P}(r,\zeta,L)$, for a generic $\sigma' \in W$, is isomorphic to the subset of $J^{(\delta)}_{X_\sigma}$ consisting of isomorphism classes of line bundles M such that π_*M is stable and has determinant isomorphic to ζ.

Let us denote by $\mathrm{Nm} : \mathrm{Jac}(X_\sigma) \to \mathrm{Jac}(C)$ the norm map. Since $\deg(L) \neq 0$ it follows from [BNR, Remark 3.10] that the morphism $\pi^* : \mathrm{Jac}(C) \to \mathrm{Jac}(X_\sigma)$ is injective. As a consequence of this we derive that the norm map has a connected kernel, which is called the Prym variety of the covering $\pi : X_\sigma \to C$ and will be denoted $\mathrm{Prym}(X_\sigma/C)$. Under the isomorphisms $\mathrm{Jac}(C) \cong J^{(\delta)}_C$ and $\mathrm{Jac}(X_\sigma) \cong J^{(\delta)}_{X_\sigma}$, the variety $\mathrm{Prym}(X_\sigma/C)$ corresponds to the inverse image of $\mathfrak{d}$ by the norm map $\mathrm{Nm} : J^{(\delta)}_{X_\sigma} \to J^{(\delta)}_C$. It is now immediate to see that the intersection of the fiber $H^{-1}(\sigma')$ with $\mathcal{P}(r,\zeta,L)$ is isomorphic to the open subset of $\mathrm{Prym}(X_\sigma/C)$ consisting of isomorphism classes of line bundles M such that π_*M is a stable vector bundle.

It is known ([BNR, Proposition 5.7]) that the complement of this open set is of codimension at least 2 in $\mathrm{Prym}(X_\sigma/C)$, hence we are in a situation analogue to the one already studied for the map $H : \mathcal{M}'_0(r,d,L) \to \bigoplus_{i=1}^{r} H^0(C,L^i)$.

By repeating the considerations made in Section 4.7, we can prove the following

THEOREM 4.8.4. – *The morphism $H : \mathcal{M}'_0(r,\zeta,L) \to W$ defines an algebraically completely integrable hamiltonian system on the Poisson variety $\mathcal{M}'_0(r,\zeta,L)$. This system* *linearizes on the Prym varieties of coverings $\pi : X_\sigma \to C$, where X_σ is the spectral curve defined by an element $\sigma \in W$.*

Considerations on invariant (Casimir) functions analogous to those following Theorem 4.7.2 also hold in this situation.

5. Parabolic vector bundles

5.1 The moduli space of parabolic vector bundles.

Let $r \in \mathbb{N}$ with $r \geq 2$ and $d \in \mathbb{R}$, and let $S = \{P_1, \ldots, P_m\}$ be a finite set of points of C, called 'parabolic points'. Let $\alpha = (\alpha_{P,i})_{P \in S,\, 1 \leq i \leq n_P}$ be a sequence of real numbers such that, for each $P \in S$ one has $0 \leq \alpha_{P,1} < \ldots < \alpha_{P,n_P} < 1$, and $\kappa = (k_{P,i})_{P \in S,\, 1 \leq i \leq n_P}$ an increasing sequence of strictly positive integers such that, for every $P \in S$ one has $\sum_{i=1}^{n_P} k_{P,i} = r$. We denote by $\mathcal{U}_s(\kappa, \alpha, d)$ the moduli space of stable parabolic vector bundles of rank r, degree d, weights α and sequence of multiplicities κ. Let us denote by $\mathcal{P}ar\,\mathcal{H}om(E, F)$ the sheaf of homomorphisms of parabolic vector bundles of the parabolic vector bundles E and F, and set $\mathcal{P}ar\,\mathcal{E}nd(E) = \mathcal{P}ar\,\mathcal{H}om(E, E)$.

We want to study the tangent and cotangent spaces to the moduli variety $\mathcal{U}_s(\kappa, \alpha, d)$. The basic technique is the use of linear infinitesimal deformations of a stable parabolic bundle, and the results are similar to those given in Section 3 for stable vector bundles.

Let E be a parabolic vector bundle on C with parabolic structure at the points $P_1, \ldots, P_m$, and let us denote by $D = \sum_{i=1}^{m} P_i$ the effective divisor defined by the points P_i. We have the following exact sequence of sheaves

$$0 \to \mathcal{P}ar\,\mathcal{E}nd(E) \to \mathcal{E}nd(E) \to \mathcal{F}_D \to 0, \tag{5.1.1}$$

where $\mathcal{F}_D = \bigoplus_{i=1}^{m} \mathcal{F}_{P_i}$ is a skyscraper sheaf supported at the points $P_i \in S$.

Now we consider infinitesimal deformations of the parabolic bundle E fixing the parabolic structure. By slightly modifying the proof of Proposition 3.1.2 to keep track of the fact that the infinitesimal deformation of E must preserve the parabolic structure, it is not difficult to prove the following

Proposition 5.1.1. – *The isomorphism classes of linear infinitesimal deformations of the parabolic bundle E are canonically parametrized by the first cohomology group $H^1(C, \mathcal{P}ar\,\mathcal{E}nd(E))$.*

It follows immediately that

Corollary 5.1.2. – *We have a canonical identification of the tangent space $T_E\mathcal{U}_s(\kappa, \alpha, d)$ to $\mathcal{U}_s(\kappa, \alpha, d)$ at the point E with the vector space $H^1(C, \mathcal{P}ar\,\mathcal{E}nd(E))$.*

Remark 5.1.3. – From the exact sequence (5.1.1) we get a surjective morphism

$$H^1(C, \mathcal{P}ar\,\mathcal{E}nd(E)) \to H^1(C, \mathcal{E}nd(E)) \to 0.$$

It is evident from the preceding considerations that this is precisely the forgetful morphism sending an infinitesimal deformation of the parabolic bundle E to the corresponding infinitesimal deformation of the underlying vector bundle.

Now we turn to the study of the cotangent bundle to the moduli variety $\mathcal{U}_s(\kappa, \alpha, d)$.

By Corollary 5.1.2 and Serre duality it follows that the cotangent space $T^*_E\mathcal{U}_s(\kappa, \alpha, d)$ is canonically identified with $H^0(C, \mathcal{P}ar\,\mathcal{E}nd(E)^* \otimes K)$. Let $\mathcal{O}(D)$ be the invertible sheaf defined by the divisor D and s a global section of $\mathcal{O}(D)$ defining D, *i.e.*, such that $D = (s)$. By Serre duality, and by using Lemma 5.1.5, it is not difficult to prove the following

PROPOSITION 5.1.4. – *There is a canonical isomorphism*

$$T^*_E\mathcal{U}_s(\kappa, \alpha, d) \xrightarrow{\sim} H^0(C, \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D)),$$

where $\mathcal{E}nd_n(E)$ is the subsheaf of $\mathcal{E}nd(E)$ consisting of sections ψ which are nilpotent with respect to the parabolic structure of E, i.e., *such that $\psi_P(F_i(E)_P) \subset F_{i+1}(E)_P$, for $1 \leq i \leq n_P$ and for all $P \in D$.*

The following is a simple lemma of linear algebra, whose proof is left to the reader:

LEMMA 5.1.5. – *Let V be an r-dimensional vector space and*

$$V = F_1 \supset F_2 \supset \ldots \supset F_n \supset F_{n+1} = 0$$

a filtration of V by vector subspaces. Let $\phi : V \to V$ be an endomorphism of V. Then the following conditions are equivalent

(1) $\phi(F_i) \subset F_{i+1}$, *for* $i = 1, \ldots, n$;

(2) $\mathrm{Tr}(\phi\psi) = 0$ *for every* $\psi \in \mathrm{End}(V)$ *such that* $\psi(F_i) \subset F_i$ *for* $i = 1, \ldots, n$.

5.2 THE CANONICAL SYMPLECTIC STRUCTURE OF $T^*\mathcal{U}_s(\kappa, \alpha, d)$

In the preceding section we have seen that the cotangent bundle $\mathcal{M} = T^*\mathcal{U}_s(\kappa, \alpha, d)$ may be described as the set of isomorphism classes of pairs (E, ϕ), where E is a stable parabolic bundle and $\phi \in H^0(C, \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D))$. This situation is similar to the one we have studied in Section 3, for the variety $\mathcal{P}(r, d, L)$. By applying the same kind of reasoning, we get the following

PROPOSITION 5.2.1. – *Let $(E, \phi) \in \mathcal{M}$ and denote by $[\cdot, \phi]_{\mathcal{M}}$ the complex*

$$[\cdot, \phi]_{\mathcal{M}}: \quad 0 \to \mathcal{P}ar\,\mathcal{E}nd(E) \xrightarrow{[\cdot,\phi]} \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) \to 0.$$

Then the set of isomorphism classes of infinitesimal deformations of the pair (E, ϕ) in $\mathcal{M}$ is canonically identified with the first hypercohomology group $\mathbb{H}^1([\cdot, \phi]_{\mathcal{M}})$.

Proof. – We can repeat almost unchanged the proof of Proposition 3.1.2. Note that, if $(E_\epsilon, \phi_\epsilon)$ is an infinitesimal deformation of (E, ϕ) corresponding to $(\{\alpha_i\}, \{\eta_{ij}\})$, then the hypothesis that the parabolic structure of E is fixed implies that the 1-cocycle $\{\eta_{ij}\}$ defines an element of $H^1(C, \mathcal{P}ar\,\mathcal{E}nd(E))$ and α_i is a section of $\mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D)$, for each i. □

COROLLARY 5.2.2. – *The tangent space $T_{(E,\phi)}\mathcal{M}$ to $\mathcal{M}$ at the point (E, ϕ) is canonically isomorphic to the vector space $\mathbb{H}^1([\cdot, \phi]_{\mathcal{M}})$.*

Now we come to the study of the cotangent bundle to $\mathcal{M}$. We have already seen that there is a canonical isomorphism of $\mathcal{P}ar\,\mathcal{E}nd(E)^* \otimes K$ with $\mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D)$. From this it follows easily that the complex

$$0 \to \mathcal{E}nd_n(E)^* \otimes \mathcal{O}(-D) \xrightarrow{[\cdot,\phi]^*} \mathcal{P}ar\,\mathcal{E}nd(E)^* \otimes K \to 0,$$

dual to $[\cdot,\phi]_{\mathcal{M}}$, is canonically identified with the complex

$$[\phi,\cdot]_{\mathcal{M}}: \quad 0 \to \mathcal{P}ar\,\mathcal{E}nd(E) \xrightarrow{[\phi,\cdot]} \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) \to 0.$$

By using Serre duality for hypercohomology, we derive the following

PROPOSITION 5.2.3. – *The cotangent space $T^*_{(E,\phi)}\mathcal{M}$ to $\mathcal{M}$ at the point (E,ϕ) is canonically isomorphic to the first hypercohomology group $\mathbb{H}^1([\phi,\cdot]_{\mathcal{M}})$.*

Let us consider now the following isomorphism of complexes:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{P}ar\,\mathcal{E}nd(E) & \xrightarrow{[\cdot,\phi]} & \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) & \longrightarrow 0 \\ & \uparrow{\scriptstyle -1} & & \uparrow{\scriptstyle 1} & \\ 0 \longrightarrow & \mathcal{P}ar\,\mathcal{E}nd(E) & \xrightarrow[{[\phi,\cdot]}]{} & \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) & \longrightarrow 0, \end{array} \tag{5.2.1}$$

It induces an isomorphism on hypercohomology,

$$B_1 : \mathbb{H}^1([\phi,\cdot]_{\mathcal{M}}) \xrightarrow{\sim} \mathbb{H}^1([\cdot,\phi]_{\mathcal{M}}), \tag{5.2.2}$$

which, in terms of Čech cocycles, may be described as the map sending an element $(\{\alpha_i\},\{\eta_{ij}\}) \in \mathbb{H}^1([\phi,\cdot]_{\mathcal{M}})$ to $(\{\alpha_i\},\{-\eta_{ij}\})$. By recalling the natural identifications $T_{(E,\phi)}\mathcal{M} \cong \mathbb{H}^1([\cdot,\phi]_{\mathcal{M}})$ and $T^*_{(E,\phi)}\mathcal{M} \cong \mathbb{H}^1([\phi,\cdot]_{\mathcal{M}})$, we can define a contravariant 2-tensor $\theta_1 \in H^0(\mathcal{M}, \bigotimes^2 T\mathcal{M})$ by setting, for each point $(E,\phi) \in \mathcal{M}$, $\langle \theta_1(E,\phi), \alpha \otimes \beta \rangle = \langle \alpha, B_1(\beta) \rangle$, for all $\alpha,\beta \in T^*_{(E,\phi)}\mathcal{M}$, where $\langle \cdot,\cdot \rangle$ denotes the duality between $T_{(E,\phi)}\mathcal{M}$ and $T^*_{(E,\phi)}\mathcal{M}$. It is now immediate to see that θ_1 is actually an antisymmetric contravariant 2-tensor, *i.e.*, $\theta_1 \in H^0(\mathcal{M}, \Lambda^2 T\mathcal{M})$. The fundamental result is the following

THEOREM 5.2.4. – *The antisymmetric contravariant 2-tensor θ_1 defines a symplectic structure on the variety $\mathcal{M}$, which is precisely the canonical symplectic structure of $\mathcal{M}$, considered as the cotangent bundle to $\mathcal{U}_s(\kappa,\alpha,d)$.*

Proof. – The proof is the same of Theorem 4.5.1, with the obvious modifications. □.

Remark 5.2.5. – The same result is proved, by a different method, in [BR].

Remark 5.2.6. – In the sequel we shall be interested in the following situation: s is a global section of the line bundle $K^{-1} \otimes L$ and D is the effective divisor defined by s, so that we have an isomorphism $s : \mathcal{O}(D) \to K^{-1} \otimes L$ from the sheaf of meromorphic functions on C with poles at D to $K^{-1} \otimes L$, given by multiplication by s. Under these hypotheses we have canonical isomorphisms

$$T_{(E,\phi)}\mathcal{M} \cong \mathbb{H}^1([\cdot,\phi]_n), \quad T^*_{(E,\phi)}\mathcal{M} \cong \mathbb{H}^1([\phi,\cdot]_n), \tag{5.2.3}$$

where $[\cdot,\phi]_n$ and $[\phi,\cdot]_n$ denote respectively the complexes

$$[\cdot,\phi]_n : \quad 0 \to \mathcal{P}ar\,\mathcal{E}nd(E) \xrightarrow{[\cdot,\phi]} \mathcal{E}nd_n(E) \otimes L \to 0$$

and

$$[\phi,\cdot]_n : \quad 0 \to \mathcal{P}ar\,\mathcal{E}nd(E) \otimes L^{-1} \otimes K \otimes \mathcal{O}(D) \xrightarrow{[\phi,\cdot]} \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) \to 0.$$

The isomorphism of complexes (5.2.1) may be rewritten as

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{P}ar\,\mathcal{E}nd(E) & \xrightarrow{[\cdot,\phi]} & \mathcal{E}nd_n(E) \otimes L & \longrightarrow 0 \\ & \uparrow{\scriptstyle -s} & & \uparrow{\scriptstyle s} & \\ 0 \longrightarrow & \mathcal{P}ar\,\mathcal{E}nd(E) \otimes L^{-1} \otimes K \otimes \mathcal{O}(D) & \xrightarrow[{[\phi,\cdot]}]{} & \mathcal{E}nd_n(E) \otimes K \otimes \mathcal{O}(D) & \longrightarrow 0, \end{array}$$

hence the isomorphism (5.2.2) is given, in terms of the identifications (5.2.3), by

$$B_s : \mathbb{H}^1([\phi,\cdot]_n) \xrightarrow{\sim} \mathbb{H}^1([\cdot,\phi]_n), \tag{5.2.4}$$

sending an element $(\{\alpha_i\},\{\eta_{ij}\})$ to $(\{s\alpha_i\},\{-s\eta_{ij}\})$.

Note the similarity with the expression of the Poisson structure of the variety $\mathcal{P}$. The relationships between these two structures will be made more precise in the following section.

5.3 The structure of the fiber $C^{-1}(0)$.

Let us assume that $D_s = \sum_{i=1}^{m} P_i$, with $P_i \neq P_j$ if $i \neq j$, where $m = \deg(L) - \deg(K) > 0$. Let $C : \mathcal{P} \to \bar{W}_{D_s}$ be the restriction to $\mathcal{P}$ of the map defined in (4.7.3), and let us denote by $\mathcal{X}$ the fiber $C^{-1}(0)$. We have

$$\mathcal{X} = \{\,(E,\phi) \in \mathcal{P} \mid \operatorname{Tr}(\phi^i_{P_j}) = 0, \quad 1 \le i \le r, \quad 1 \le j \le m\,\}. \tag{5.3.1}$$

This shows that if $(E,\phi) \in \mathcal{X}$ then ϕ_{P_j} is nilpotent of order $n_{P_j}(E) \le r$, for every point P_j in the divisor D_s. Given such a pair (E,ϕ) we can associate to the vector bundle E a quasi-parabolic structure defined as follows: for each $P \in D_s$ consider the filtration of the fiber E_P given by

$$\begin{aligned} E_P = F_1(E)_P = \operatorname{Ker}(\phi_P^{n_P(E)}) \supset F_2(E)_P = \operatorname{Ker}(\phi_P^{n_P(E)-1}) \supset \\ \cdots \supset F_{n_P(E)}(E)_P = \operatorname{Ker}(\phi_P) \supset F_{n_P(E)+1}(E)_P = 0, \end{aligned} \tag{5.3.2}$$

i.e., we set $F_i(E)_P = \operatorname{Ker}(\phi_P^{n_P(E)-i+1})$, for $i = 1, \ldots, n_P(E)$. It is easy to show that $F_i(E)_P \neq F_{i+1}(E)_P$ for $1 \leq i \leq n_P$, hence this is a good definition.

Let $\alpha = (\alpha_{P_j,i})$ be an arbitrarily fixed system of weights, so that E becomes a parabolic vector bundle. It is easy to see that, for any choice of the weight system α, the parabolic vector bundle E is parabolic stable.

Let us consider now the moduli space $\mathcal{U}_s(\kappa, \alpha, d)$ of stable parabolic bundles with parabolic structure at the divisor D_s, and let $\mathcal{X}(\kappa, \alpha, d)$ denote the subset of $\mathcal{X}$ consisting of all pairs (E, ϕ) such that the vector bundle E, with the parabolic structure defined in (5.3.2), belongs to $\mathcal{U}_s(\kappa, \alpha, d)$. We have a natural map

$$\pi : \mathcal{X}(\kappa, \alpha, d) \to \mathcal{U}_s(\kappa, \alpha, d), \tag{5.3.3}$$

whose fiber over a point $E \in \mathcal{U}_s(\kappa, \alpha, d)$, consisting of a vector bundle E with parabolic structure $F_i(E)_{P_j}$, is

$$\pi^{-1}(E) = \{ (E, \phi) \in \mathcal{X} \mid \operatorname{Ker}(\phi_{P_j}^{n_{P_j}(E)-i+1}) = F_i(E)_{P_j}, \quad i = 1, \ldots, n_{P_j}(E), \quad j = 1, \ldots, m \}.$$

Now we need a result from linear algebra, whose proof is left to the reader.

LEMMA 5.3.1. – *Let V be an r dimensional vector space and $\phi : V \to V$ an endomorphism of V. The following conditions are equivalent:*

(1) $\operatorname{Tr}(\phi^i) = 0$, *for* $i = 1, \ldots, r$;

(2) $\operatorname{Tr}(\phi\psi) = 0$, *for every* $\psi \in \operatorname{End}(V)$ *such that* $\psi(\operatorname{Ker}\phi^i) \subset \operatorname{Ker}\phi^i$, *for* $i = 1, \ldots, r$.

By recalling the description of $T^*_{(E,\phi)}\mathcal{U}_s(\kappa, \alpha, d)$ given by Proposition 5.1.4, the definition of the quasi-parabolic structure of E given in (5.3.2), by using Lemma 5.3.1 and Lemma 5.1.5, it follows easily that there is a natural inclusion of $\mathcal{X}(\kappa, \alpha, d)$ in the cotangent bundle $T^*\mathcal{U}_s(\kappa, \alpha, d)$.

From what we have seen, we derive the following

THEOREM 5.3.2. – *The variety $\mathcal{X} = C^{-1}(0)$ is a Poisson variety foliated by the symplectic varieties $\mathcal{X}(\kappa, \alpha, d)$. Each symplectic leaf $\mathcal{X}(\kappa, \alpha, d)$ is canonically isomorphic to a subvariety of the cotangent bundle to $\mathcal{U}_s(\kappa, \alpha, d)$ with the induced canonical symplectic structure.*

As a special case let us consider the subvariety $\bar{\mathcal{X}} \subset \mathcal{X}$ consisting of pairs (E, ϕ) such that $\phi_{P_j} = 0$, for $j = 1, \ldots, m$. For each pair $(E, \phi) \in \bar{\mathcal{X}}$, the quasi-parabolic structure of the vector bundle E defined in (5.3.2) is trivial, *i.e.*, for each $P_j \in D_s$ we have the trivial filtration of E_{P_j} given by $E_{P_j} \supset 0$. In this situation the moduli space of stable parabolic bundles (with this trivial parabolic structure) coincides with the ordinary moduli space of stable vector bundles $\mathcal{U}_s(r, d)$, and, from (5.3.3), we derive a map

$$\pi : \bar{\mathcal{X}} \to \mathcal{U}_s(r, d).$$

Note that the hypothesis $\phi_{P_j} = 0$, for each $P_j \in D_s$, is equivalent to assuming that ϕ is in the image of the map $s : H^0(C, \mathcal{E}nd(E) \otimes K) \to H^0(C, \mathcal{E}nd(E) \otimes L)$. This shows that $\bar{\mathcal{X}}$ coincides with the image of the map $\mathcal{P}(r, d, K) \to \mathcal{P}(r, d, L)$, sending a pair (E, ϕ) to $(E, s\phi)$. With this identification we see that $\pi : \bar{\mathcal{X}} \to \mathcal{U}_s(r, d)$ is precisely the cotangent bundle to the moduli space $\mathcal{U}_s(r, d)$, and the Poisson structure of $\bar{\mathcal{X}}$ coincides with the canonical symplectic structure of $T^*\mathcal{U}_s(r, d)$.

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