ADAM PARUSIŃSKI

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LIPSCHITZ STRATIFICATION OF SUBANALYTIC SETS

BY ADAM PARUSIŃSKI

ABSTRACT. – In this paper we show the existence of Lipschitz stratification (in the sense of Mostowski) for subanalytic sets. Such stratification ensures, in particular, bi-Lipschitz triviality of the stratified set along each stratum. In fact, our construction is more precise. We decompose each compact subanalytic set into pieces, called L-regular sets, distinguished by their simple Lipschitz properties. In a way our decomposition is similar to triangulation but technically more complicated. In the course of the proof we develop for subanalytic sets such techniques as: regular projection theorem, subanalytic sets in complex domain and an analog of the Weierstrass preparation theorem for subanalytic functions.

Introduction

Let (X, x), (X', x') be two germs of analytic subsets of $\mathbf{R}^n$. When can we say that the singularities of X at x and X' at x' are equivalent? Consider, for instance, two extreme approaches.

Analytic equivalence, that is given by analytic isomorphisms, certainly preserves all the interesting features of the singularity. However, one can easily produce examples of analytic families of analytically nonequivalent singularities (phenomenon of moduli). This is also the case for the weaker C^1-equivalence given by the restrictions of C^1 diffeomorphisms. This happens, for instance, for Whitney's example given by an equation $xy\,(x+y)\,(x-ty) = 0$ as a family of singularities in $\mathbf{R}^2$ parametrized by $t \in (0, \infty)$ (*see* [G], (2.1) Chapter II) although the singularities look very similar.

On the other hand, any analytic family of analytic singularities has locally only a finite number of nonhomeomorphic classes of singularities. But homeomorphisms seem to lose too much structure of the singularity.

In [SS] Siebenmann and Sullivan asked whether there are only countably many local Lipschitz structures (that is up to bi-Lipschitz homeomorphisms) on analytic spaces. Bi-Lipschitz homeomorphisms seem to have "good" properties of general interest. They preserve sets of measure zero, order of contact, and Łojasiewicz's exponents. In 1985 T. Mostowski [M1] introduced the notion of Lipschitz stratification and proved its existence for complex analytic sets. This stratification ensures the constancy of the Lipschitz type of the stratified set along each stratum and was used by Oshawa [O] in the proof of the Cheeger-Goresky-MacPherson Conjecture. The existence of Lipschitz stratification for real analytic sets was established in [P1], [P2]. For a review of results on Lipschitz stratification the reader can consult [P5].

The main purpose of this paper is to show the existence of Lipschitz stratification for subanalytic sets (Theorem 1.4 below).

Subanalytic sets, a category much wider than analytic sets, were studied in [H1], [DŁS], [BM1] for instance. Using their fundamental properties and the existence of Lipschitz stratification we show that each subanalytic family of compact subanalytic sets has locally only a finite number of nonequivalent global Lipschitz types (Theorem 1.6 below). Unfortunately, this does not give an answer to Siebenmann and Sullivan's question, since we cannot embed all analytic germs in countably many analytic families. Nevertheless, we can do it for polynomial singularities, so the answer is positive for semi-algebraic sets.

In Section 1 we recall the notion and properties of Lipschitz stratification and state the main result (Theorem 1.4) which we prove in Section 3. The proof is generally based on the ideas introduced in [M1] and [P1]. The main new ingredient, particular for the subanalytic geometry, is the use of the local flattening theorem, due to Hironaka, Lejeune-Jalabert, Teissier [HLT], which we recall in Section 4.

Section 2 presents the properties of L-regular set, which notion was introduced in [P1] in the semi-analytic set-up. These sets, distinguished by their simple metric properties (Lemma 2.2 below), are the pieces from which we build a Lipschitz stratification. Each compact subanalytic set can be decomposed (Proposition 2.13 below) into a finite union of L-regular sets in such a way that it is easy to glue Lipschitz stratifications of the pieces. Proposition 2.13 follows from the Regular Projection Theorem proven in Section 5.

Sections 6 and 7 are devoted to the proof of Proposition 3.1, the heart of the proof of the main theorem. An analogous result for semi-analytic functions was proven in [P1] by different methods. In Section 6 we develop a theory of subanalytic sets in complex domain. In Section 7 we show the product formula for locally blow-analytic functions (Theorem 7.5), a subanalytic version of the Weierstrass preparation theorem. This theorem allows us to complete the proof of Proposition 3.1 and construct Lipschitz stratifications of L-regular sets.

The estimates we obtain in Proposition 3.1 and Theorem 7.5 seem to be of independent interest.

Notation and conventions. – Let X be a subset of $\mathbf{R}^n$. By $\operatorname{dist}(\star, X)$ we mean the function of distance to X. If $X = \varnothing$ then we mean $\operatorname{dist}(\star, \varnothing) \equiv 1$. By $\operatorname{Fr}(X)$, $\operatorname{Int}(X)$ resp., we denote the topological frontier of X, the interior of X resp.

For subanalytic $X \subset \mathbf{R}^n$ by Reg (*X*) we mean the set of regular points (of the highest dimension) of X.

We call a homeomorphism φ *bi-Lipschitz* if both φ and φ^{-1} are Lipschitz mappings.

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1. Lipschitz stratification

In this section we first recall the definition of Lipschitz stratification and its basic properties ([M1], [P1]). Because of our interest we work in the subanalytic set-up. For more general review of Lipschitz stratification the reader can consult [P5]. Many of the facts presented in this section are proven in [M1], [P1] and [P2] so we omit the proofs. Then we state the main result of the paper (Theorem 1.4) and give some corollaries.

Let X be a subanalytic subset of $\mathbf{R}^n$. By a (subanalytic) stratification $\mathcal{S} = \{S_\alpha\}$ of X we mean a decomposition of X into a disjoint locally finite union

$$X = \bigcup S_\alpha,$$

where the subsets $S_\alpha \subseteq X$, called *strata*, are connected, subanalytic and nonsingular. For each S_α of $\mathcal{S}$ we require $\overline{S}_\alpha \setminus S_\alpha$ to be contained in the union of strata of dimension smaller than $\dim S_\alpha$. Therefore,

$$X^i = \bigcup_{\alpha,\, \dim S_\alpha \leqq i} S_\alpha$$

is a closed subanalytic subset of X and $\overset{\circ}{X}{}^i = X^i \setminus X^{i-1}$ is the union of strata of dimension i. We call X^i *the i-th skeleton of* $\mathcal{S}$. Thus, each stratification defines a filtration of X by its skeletons

$$X = X^d \supseteq X^{d-1} \supseteq \ldots \supseteq X^l \neq \varnothing. \tag{1.1}$$

Conversely, if we have a filtration $\mathcal{X} = \{X^i\}$ of X by closed (in X) subanalytic sets and such that, for each i, $\overset{\circ}{X}{}^i = X^i \setminus X^{i-1}$ is nonsingular of pure dimension i (we assume $X^{l-1} = \varnothing$), then this filtration defines a stratification by taking the connected components of $\overset{\circ}{X}{}^i$ $(i = l, \ldots, d)$ as strata.

Remark. – Usually one requires that a stratification satisfies *the frontier condition*, *i. e.* if $S_\alpha \cap \overline{S}_\beta \neq \varnothing$, then $S_\alpha \subseteq \overline{S}_\beta$, or equivalently that $\overline{S}_\beta \setminus S_\beta$ is the union of some strata of dimension smaller that $\dim S_\beta$. Since, in this paper, we use filtrations rather than stratifications, we do not consider this condition. If X is a locally closed subset of $\mathbf{R}^n$ then each Whitney stratification of X [and thus, each Lipschitz stratification, *see* Remark (ii) after Proposition 1.5] satisfies automatically the frontier condition ([G], Chapter II (5.7)).

By abuse of notation we call a filtration $\mathcal{X}$ as above also *a stratification of* X. This coincides with the terminology in [M1], [P1], [P2] and we hope will cause no confusion.

For $q \in \overset{\circ}{X}{}^i$ let $P_q : \mathbf{R}^n \to T_q \overset{\circ}{X}{}^i$, $P_q^\perp = \mathrm{Id} - P_q : \mathbf{R}^n \to T_q^\perp \overset{\circ}{X}{}^i$ denote the orthogonal projections onto the tangent and the normal space to $\overset{\circ}{X}{}^i$ at q respectively. In terms of such orthogonal projections the condition (w) of Verdier [V] for a stratification $\mathcal{X} = \{X^i\}$ can be expressed as follows:

For each $p \in \overset{\circ}{X}{}^i$ there is a neighbourhood U_p of p in X and a constant C_p such that for each $j > i$ and each $q \in \overset{\circ}{X}{}^i \cap U_p$, $q' \in \overset{\circ}{X}{}^i \cap U_p$

$$|P_{q'}^{\perp} P_q| \leqq C_p |q - q'|. \tag{w}$$

Let U be an open subset of X. We call *a vector field* v *on* $X^k \cap U$ *rugose* if

(1) v is tangent to the strata of $\mathcal{X}$;

(2) v is smooth on each stratum;

(3) for each $i \leqq k$, each $p \in \overset{\circ}{X}{}^i \cap U$ and all $q \in \overset{\circ}{X}{}^i \cap U$ and $q' \in \overset{\circ}{X}{}^j \cap U$ close to p

$$|v(q) - v(q')| \leqq C_p |q - q'|.$$

As proven in [V], if $\mathcal{X}$ satisfies (w), each rugose vector field on $X^k \cap U$ can be extended to a rugose vector field on a neighbourhood of $X^k \cap U$ in X [and, by [B-T], this property is equivalent to the condition (w)].

Conditions for tangent spaces similar to (w), but more complicated, which imply the extension property of Lipschitz vector fields tangent to strata, were introduced by Mostowski in [M1]. Whereas the (w) condition has to be satisfied locally by all the pairs (q, q') such that q' is close to q, Mostowski's conditions are for sequences of points called chains. The chains can be defined in various ways (not necessarily equivalent) as in [M1], [P1], [P2], but all these definitions lead in fact to the same condition on stratification (*see* [P2] Remark 1 and [P1] Proposition 1.5).

Let $c > 1$ be a fixed constant and let $\mathcal{X} = \{X^i\}$ be as in (1.1). A *chain* (more exactly, a c-chain) for $q \in \overset{\circ}{X}{}^j$ is a strictly decreasing sequence of indices $j = j_1, j_2, \ldots, j_r = l$ and a sequence of points $q_{j_s} \in \overset{\circ}{X}{}^{j_s}$ such that $q_{j_1} = q$ and j_s is the greatest integer for which

$$\operatorname{dist}(q, X^k) \geqq 2c^2 \operatorname{dist}(q, X^{j_s}) \quad \text{for all} \quad k < j_s, \quad k \geqq l$$

and $|q - q_{j_s}| \leqq c \operatorname{dist}(q, X^{j_s})$.

The meaning of a chain is the following. Take $q \in X^j$ and compute the distances of q to the subsequent skeletons X^i. Mark those indices (j_s) where the distances increase rapidly. Next we choose points $\{q_{j_s}\}$ realizing (up to some constant) these distances.

DEFINITION 1.1 (Mostowski's Conditions for Lipschitz stratification). – We call a stratification $\mathcal{X} = \{X^i\}$ *a Lipschitz stratification* (in the sense of Mostowski) if for some constant $C > 0$, every chain $q = q_{j_1}, q_{j_2}, \ldots, q_{j_r}$ and every k, $1 \leqq k \leqq r$,

$$|P_q^{\perp} P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqq C\,|q - q_{j_2}|/\operatorname{dist}(q, X^{j_k - 1}). \tag{m1}$$

If, further, $q' \in \overset{\circ}{X}{}^{j_1}$, then

$$|(P_q - P_{q'}) P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqq C\,|q - q'|/\operatorname{dist}(q, X^{j_k - 1}), \tag{m2}$$

which means for $k = 1$

$$|P_q - P_{q'}| \leqq C\,|q - q'|/\operatorname{dist}(q, X^{j_1-1}).$$

(we set $\operatorname{dist}(q, X^{l-1}) \equiv 1$).

In Proposition 1.3 below we give a characterization of Lipschitz stratification in terms of extension of Lipschitz vector fields tangent to strata.

If a stratification is Lipschitz then conditions (m1) and (m2) are satisfied not only by chains but also by some other sequences of points [for (m1) it is enough to assume that $q_{j_2}, \ldots, q_{j_r}$ is a chain, [P2] Remarks 1 and 2]. But in general, as the example below shows, it is not possible to find a stratification satisfying (m1) and (m2) for all sequences of points (from the strata of decreasing dimensions).

Example 1.2. – Let $Y \subseteq \mathbf{R}^2$ be an ordinary cusp given by $(x, y) = (t^2, t^3)$, $t \in \mathbf{R}$. We take the product of three copies of such cusps $X = Y_1 \times Y_2 \times Y_3 \subseteq \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ parametrized by r, s and t respectively. Consider the standard stratification $\mathcal{X}$ of X (*i. e.* the product stratification of the standard stratifications of the cusps) and the points $q_3 = (-r, s, t)$, $q_2 = (r, s, 0)$ and $q_1 = (r, 0, 0)$. Then infinitesimally at the origin

$$|P_{q_3}^{\perp} P_{q_2} P_{q_1}| \sim |r|, \quad |q_3 - q_2| \sim |r|^3 + t^2 \quad \text{and} \quad \operatorname{dist}(q_3, X^0) \sim r^2 + s^2 + t^2,$$

and so (m1) is not satisfied if $s^2 \gg r^2 + t^2$. On the other hand, if $q_2, q_1, 0$ is a chain, then $s^2 \sim \min\{r^2, s^2\}$ and (m1) holds. If we have an arbitrary stratification of X, then we can move q_3, q_2, q_1 a little in such a way that they are in the strata of appropriate dimensions and (m1) is still not satisfied.

Let $\mathcal{X}$ be a stratification of X. A vector field v defined on a subset of X is called *compatible with* $\mathcal{X}$ *(or* $\mathcal{X}$*-compatible,* for short) if it is tangent to the strata of $\mathcal{X}$.

By [B], (7.5), p. 122, any Lipschitz function on a subset of X can be extended to a Lipschitz function on X with the same Lipschitz constant. Therefore, the same holds for Lipschitz vector fields (the Lipschitz constant can rise $\sqrt{n}$-times if the metric is euclidean). But there is no guarantee that such an extension of $\mathcal{X}$-compatible vector field is still $\mathcal{X}$-compatible.

From now, on unless otherwise stated, we assume the vector fields on stratified sets to be compatible with their stratifications.

Proposition 1.3 ([P1] Proposition 1.5). – *Let* $\mathcal{X} = \{X^i\}$ *be a stratification of* X *as in (1.1).* $\mathcal{X}$ *is Lipschitz if and only if it satisfies the following extension property:*

(e1) *There exists* $C > 0$ *such that for every* $W \subset X$, *such that*

$$X^{j-1} \subseteq W \subseteq X^j \tag{1.2}$$

for some $j = l, \ldots, d$, *each Lipschitz* $\mathcal{X}$*-compatible vector field on W with Lipschitz constant L and bounded on* $W \cap X^l$ *by a constant K, can be extended to a Lipschitz* $\mathcal{X}$*-compatible vector field on X with a Lipschitz constant* $C(L + K)$.

Remarks. – (i) For simplicity, in this paper we consider mainly stratifications with nonempty zero dimensional skeleton ($l = 0$). Then $K = 0$ and the extension property is easier to check.

(ii) The above extension property is much stronger that simply saying: every Lipschitz $\mathcal{X}$-compatible vector field on a skeleton of X can be extended to a Lipschitz $\mathcal{X}$-compatible vector field on X. This seems to cause some trouble. For instance, there is no canonical Lipschitz stratification ([M2], §7, [P2], §7). It would be interesting to find local conditions equivalent exactly to the property of extending Lipschitz $\mathcal{X}$-compatible vector fields locally. Interesting steps in this direction were made, in the complex case, in [M2] and [M3].

Let $\{X_i\}$ be a family of subsets of X. We call *a stratification $\mathcal{X}$ of X compatible with* $\{X_i\}$ if each X_i is a union of some strata of $\mathcal{X}$. We say that $\mathcal{X}$ is *compatible with a stratification* $\mathcal{X}'$ of X (or that $\mathcal{X}$ refines $\mathcal{X}'$) if it is compatible with $\mathcal{X}'$ considered as a family of subsets of X.

THEOREM 1.4 (Main Theorem). – *Let X be a compact subanalytic subset of $\mathbf{R}^n$ and let $\{X_i\}$ be a finite family of subanalytic subsets of X. Then, there exists a Lipschitz stratification $\mathcal{X}$ of X, compatible with $\{X_i\}$ and such that*:

(1.3) *there is $C > 0$ such that for each stratum* S *of $\mathcal{X}$ and for each $q \in S$*

$$\operatorname{dist}(q, \overline{S} \setminus S) \leqq C \operatorname{dist}(q, X^{j-1}),$$

where $j = \dim S$ [*we set* $\operatorname{dist}(q, \varnothing) \equiv 1$].

We shall prove Theorem 1.4 in Section 3. Condition (1.3) is added mainly for the sake of inductive argument in the proof. A Lipschitz stratification satisfying (1.3) we call *a strong Lipschitz stratification.*

PROPOSITION 1.5. – *A stratification $\mathcal{X}$ as in (1.1) is strong Lipschitz if and only if it satisfies the following extension property*:

(e2) *There exists $c > 0$ such that for every $W \subset X$, such that*

(1.4) *for any stratum S of $\mathcal{X}$, if $W \cap S \neq \varnothing$, then $\overline{S} \setminus S \subset W$,*

each Lipschitz $\mathcal{X}$-compatible vector field on W with Lipschitz constant L and bounded on $W \cap X^l$ by a constant K, can be extended to a Lipschitz $\mathcal{X}$-compatible vector field on X with Lipschitz constant $C(L+K)$.

In particular, if $\mathcal{X}$ is strong Lipschitz then it induces a strong Lipschitz stratification on each closed union of strata.

Proof. – Assume that $\mathcal{X}$ is strong Lipschitz. Then, its restriction to any skeleton is Lipschitz. Let W satisfies (1.4), and let v be a Lipschitz vector field on W. Let $k \in \{l, ..., d\}$ be such that $X^{k-1} \subseteq W$ and $X^k \nsubseteq W$. By Proposition 1.3 we can extend v to a vector field $\tilde{v}$ on $X^k \cup S$ whose restriction to X^k is Lipschitz. Then, thanks to (1.3) and (1.4), we can show inductively on j that $\tilde{v}|_{X^j \cap W}$ is Lipschitz. Therefore $\tilde{v}$ is Lipschitz. Consequently, we can extend v skeleton by skeleton onto X.

Consequently, if $\mathcal{X}$ satisfies (e2), then by Proposition 1.3 it is Lipschitz. We show that it satisfies also (1.3). Take a point q of a stratum S and let $j = \dim S$. Let $q' \in X^{j-1}$ be such that $|q - q'| \leqq 2 \operatorname{dist}(q, X^{j-1})$. By dimension argument, there is a vector $\mathbf{v} \in T_q S$ such that $|\mathbf{v}| = 1$ and $P_{q'} \mathbf{v} = 0$. Define a vector field v on $\{q\} \cup (\overline{S} \setminus S)$ by: $v \equiv 0$ on

$\overline{S} \setminus S$ and $v(q) = \mathbf{v}$. It is Lipschitz with constant $(\operatorname{dist}(q, \overline{S} \setminus S))^{-1}$ and by (e2) can be extended to a Lipschitz vector field $\tilde{v}$ on X. Then

$$1 \leqq |\tilde{v}(q) - \tilde{v}(q')| \leqq C|q - q'|/\operatorname{dist}(q, \overline{S} \setminus S) \leqq 2C \operatorname{dist}(q, X^{j-1})/\operatorname{dist}(q, \overline{S} \setminus S),$$

which shows (1.3). □

We call a pair $(X, \mathcal{X})$ *a (strong) Lipschitz stratified set* if $\mathcal{X}$ is a (strong) Lipschitz stratification of X.

Remarks. – (i) If U is an open subset of a (strong) Lipschitz stratified set $X \subseteq \mathbf{R}^n$, then the induced stratification $\mathcal{X}'$ on $X \cap U$ is not necessarily (strong) Lipschitz. Although, after intersection with U, the left-hand sides of inequalities (m1), (m2) remain the same, the distances to skeletons are different. Nevertheless, then $\mathcal{X}'$ is locally Lipschitz in the sense of [P5].

If X is compact, $U \subseteq \mathbf{R}^n$ is a C^1 submanifold with boundary and this boundary is transverse to all strata of $\mathcal{X}$, then the induced stratifications of $X \cap U$, $X \cap \operatorname{Int}(U)$ are (strong) Lipschitz.

(ii) ([M1], Proposition 7.1, [P1], Corollary 1.6). If $\mathcal{X}$ is a Lipschitz stratification, then it satisfies the (w) condition.

(iii) The extensions of Lipschitz vector fields mentioned in Propositions 1.3 and 1.5 can be constructed skeleton by skeleton (as in the proof of Proposition 1.5) as follows: first we extend a given Lipschitz vector field to a Lipschitz vector field not necessarily tangent to the given stratum and then project onto the strata (*see* [M1], Section 2, [P1], Section 1).

Here are some consequences of Theorem 1.4.

THEOREM 1.6 (Lipschitz Isotopy Lemma). – (i) *Let $X \subset \mathbf{R}^n \times \mathbf{R}^m$ be subanalytic and let the projection $\pi : \overline{X} \to \mathbf{R}^n$ be proper. Then, there exists a closed subanalytic nowhere dense subset Z of $\pi(X)$ such that X is locally bi-Lipschitz trivial over $\pi(X) \setminus Z$, that is for each $y \in \pi(X) \setminus Z$ there is a neighbourhood $U_y \ni y$ in $\pi(X)$ and a bi-Lipschitz homeomorphism over U_y*

$$\pi^{-1}(U_y) \to U_y \times \pi^{-1}(y).$$

(ii) *Let $(X, \mathcal{X})$ be a Lipschitz stratified set and assume that X is locally closed in $\mathbf{R}^n$. Then $(X, \mathcal{X})$ is locally bi-Lipschitz trivial along each stratum, that is for each stratum S of $\mathcal{X}$ and each $x \in S$ there are: a neighbourhood $U_x \ni x$ in X, a Lipschitz stratified set F and a bi-Lipschitz stratified set and a bi-Lipschitz homeomorphism*

$$U_x \to (U_x \cap S) \times F,$$

which sends the induced stratification of U_x onto the product stratification of $(U_x \cap S) \times F$.

Proof. – The theorem follows from the standard properties of subanalytic sets and the following lemma.

LEMMA 1.7. – *Let $\mathcal{X}$ be a Lipschitz stratification of $X \subset \mathbf{R}^n$ and let M be a C^1 manifold. Let $f : \mathbf{R}^n \to M$ be a C^1 map such that $f|X$ is proper and submersive to M on each stratum of $\mathcal{X}$. Then each locally Lipschitz vector field on M can be lifted to a locally Lipschitz vector field on X.*

In particular, $(X, \mathcal{X})$ *is locally bi-Lipschitz trivial over M.*

Proof. – We follow the proof of Proposition 1.1 of [M1].

Let $k = \dim M$ and let v be a locally Lipschitz vector field on *M*. By partition of unity, it suffices to lift v to a neighbourhood of each $x \in X$. Let S be the stratum containing x. We identify a neighbourhood of $f(x)$ with an open subset V of $\mathbf{R}^k$. Let $e_1, \ldots, e_k$ be the standard constant vector fields on *V*. Since $f|_S$ is a submersion, each e_i can be lifted to a locally Lipschitz vector fields v_i on *S*. By the extension property they can be extended to Lipschitz vector fields w_i on a neighbourhood of x. In general $f_*(w_i) \neq e_i$, but since they are linearly independent at every point of a neighbourhood of x, we can replace them by suitable linear combinations and get Lipschitz liftings $\tilde{e_i}$ of e_i.

If now $v = \sum_i v^i e_i$ is an arbitrary Lipschitz vector field, then $\tilde{v} = \sum_i v^i \tilde{e}_i$ is the required lifting.

The last statement of the lemma follows by integrating suitable liftings of Lipschitz vector fields (*see* [M1] proof of Proposition 1.1 or [G] proof of Thom's Isotopy Lemma (5.2) Chapter II).

This finishes the proofs of the lemma and of the theorem. □

2. L-regular sets

In this section we decompose subanalytic sets into *L*-regular sets. The notion of *L*-regular set was introduced in [P1] (in the semi-analytic set-up) in the proof of the existence of Lipschitz stratification of semi-analytic sets. The main advantage of *L*-regular sets are their simple metric properties (*see* Lemma 2.2 below) which allow us to construct inductively a Lipschitz stratification of them. The decomposition theorem that is Proposition 2.13 below says that we can decompose (in the sense of Definition 2.3) an arbitrary compact subanalytic set into *L*-regular sets in such a way that we will be later able to glue Lipschitz stratifications of them. This method differs from that of [P1] where we showed that each semi-analytic set can merely be covered with *L*-regular sets (a similar result for subanalytic sets was shown in [P3]). A decomposition of subanalytic sets into *L*-regular sets was obtained independently by Kurdyka [K2]. The methods used in [K2] are simpler than ours but the result is weaker and seems to be not sufficiently strong for our purpose. It would be interesting to extend Kurdyka's methods to obtain a simpler proof of existence of Lipschitz stratification.

Consider the compact subanalytic subsets of $\mathbf{R}^n$. We call a subanalytic subset of $\mathbf{R}^n$ *thick* if it is non-empty and equals the closure of its interior, in other words, if it is of pure dimension *n*. We define *L*-regular subsets $X \subseteq \mathbf{R}^n$ inductively on n and $k = \dim X$.

DEFINITION 2.1 (compare [P1], Definition 3.2) . – By an *L-regular set* $X \subseteq \mathbf{R}^n$ (with respect to the given linear coordinates on $\mathbf{R}^n$) and its *boundary* ∂X we mean:

(1) if $\dim X = 0$, then *X* is a point and $\partial X = \emptyset$;

(2) if $\dim X = n$, then X is thick and is of the form

$$X = \{(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R};\ f(x') \leqq x_n \leqq g(x'),\ x' \in Y\} \tag{2.1}$$

where $Y \subseteq \mathbf{R}^{n-1}$ is L-regular and $\dim Y = n - 1$, f and g are continuous subanalytic functions which on $\operatorname{Int}(Y)$ are: analytic, have bounded derivatives and satisfy $f < g$. ∂X is the (topological) frontier of X;

(3) if $\dim X = k < n$, then X is the graph of a continuous subanalytic map $\Phi : Y \to \mathbf{R}^{n-k}$, where $Y \subseteq \mathbf{R}^k$ is L-regular and thick, and Φ is analytic on $\operatorname{Int}(Y)$ and has bounded derivative. ∂X is the graph of Φ restricted to ∂Y.

We call $X \subset \mathbf{R}^n$ *L-regular* if it is L-regular after a linear change of coordinates in $\mathbf{R}^n$.

Note that one dimensional thick L-regular sets are just the intervals in $\mathbf{R}$.

Below we list some other simple properties of L-regular sets. We leave the proof to the reader.

LEMMA 2.2. – *Let X be an L-regular subanalytic subset of* $\mathbf{R}^n$. *Then*

(1) *X is pure dimensional;*

(2) *$X \setminus \partial X$ is homeomorphic to an open ball;*

(3) *for every $x, x' \in X$ there exists a subanalytic curve γ in X joining x and x' and such that length $(\gamma) \leqq C|x - x'|$, where C does not depend on the choice of x, x';*

(4) *if X is thick and $\varphi : \operatorname{Int}(X) \to \mathbf{R}^s$ is a C^1 map with bounded derivative, then φ is Lipschitz;*

(5) *if X is as in (3) of Definition 2.1, then the standard projection $\pi : \mathbf{R}^n \to \mathbf{R}^k$ induces a bi-Lipschitz homeomorphism between X and Y.*

DEFINITION 2.3. – Let $X \subset \mathbf{R}^n$ be a compact subanalytic set. By a *decomposition into L-regular sets of X* we mean finite a union $X = \bigcup X_i$ such that for each $i \neq j$

$$X_i \cap X_j = \partial X_i \cap \partial X_j.$$

In order to decompose a compact subanalytic set X into L-regular sets, we shall study linear projections restricted to X.

Fix $d \in \{1, \dots, n-1\}$. Let $G(n, d)$ denote the Grassmannian of d-dimensional linear subspaces of $\mathbf{R}^n$. Assume that we have fixed a metric $d(V, V')$ on $G(n, d)$. For $V \in G(n, d)$ we denote by $\pi_V : \mathbf{R}^n \to V^\perp$ the orthogonal projection (along V) onto the orthogonal complement $V^\perp$ of V.

DEFINITION 2.4. – Let $X \subseteq \mathbf{R}^n$. For $\varepsilon > 0$ we say that $\pi = \pi_V : \mathbf{R}^n \to V^\perp$ is *ε-semi-regular at* $x_0 \in \mathbf{R}^n$ (with respect to X) if:

(a) $\pi_{|X}$ is finite (set-theoretically);

(b) for each $x \in \pi^{-1}(\pi(x_0))$, the intersection of *X* with the open cone

$$\mathcal{C}_\varepsilon(x, V) = \{x + v;\, v \in V' \setminus 0,\, d(V', V) < \varepsilon\}$$

is empty near x.

Note that if π is ε-semi-regular at x_0, then it is ε-semi-regular at every point of $\pi^{-1}(\pi(x_0))$. If, furthermore, x_0 is a regular (of dimension $n - d$) point of *X*, then *X* near x_0 is the graph of an analytic map with derivative bounded by $C(\varepsilon)$ (that is by a constant depending only on ε). Semi-regularity is a much weaker property than regularity of projection (*see* Section 5) but has some advantages. For instance, if π is ε-semi-regular with respect to X_1 and X_2 at the same point x_0 it is also ε-semi-regular with respect to $X_1 \cup X_2$ or $X_1 \cap X_2$ at x_0. The following proposition follows easily from much stronger Regular Projection Theorem (Theorem 5.5) which we prove in Section 5.

PROPOSITION 2.5. – *Let* $X \subset \mathbf{R}^n$ *be compact subanalytic and* $\dim X \leqq n - d$. *Then there exists a finite number of* $V_1, \ldots, V_s$ *such that*:

(2.2) *For some constants* $\varepsilon, \varepsilon' > 0$ *and generic* $x \in \mathbf{R}^n$ *(that is from the complement of a subanalytic subset of dimension smaller that n) there is* $V(x) \in \{V_1, \ldots, V_s\}$ *such that* π_V *is* ε*-semi-regular at x (with respect to X) if only* $d(V, V(x)) < \varepsilon'$.

In fact, as it follows from Theorem 5.5 below, (2.2) is satisfied by generic choice of $V_1, \ldots, V_{n+1}$.

Assume that *X* is a compact subanalytic and nowhere dense subset of $\mathbf{R}^n$. Consider linear projections $\pi_\eta : \mathbf{R}^n \to \mathbf{R}^{n-1}$ restricted to *X* and parametrized by $\eta \in \mathbf{RP}(n-1)$. Define *the set of lines tangent to X as*

$$L(X) = \overline{\{(x, l) \in \operatorname{Reg}(X) \times \mathbf{RP}(n-1);\, l \subset T_x \operatorname{Reg}(X)\}}.$$

Let p_1, p_2 be the standard projections of $L(X)$ to $\mathbf{R}^n$, $\mathbf{RP}(n-1)$ respectively. By *the set of lines tangent at* $x \in X$ we mean

$$L_x(X) = p_1^{-1}(x).$$

By $TL(X)$ we denote *the total set of lines tangent to X,* that is

$$TL(X) = \bigcup_{x \in X} L_x(X) = p_2(L(X)).$$

The sets $L(X)$, $L_x(X)$ and $TL(X)$ are compact and subanalytic. If $\xi \notin TL(X)$, then π_ξ is ε-semi-regular (for some $\varepsilon > 0$) at every $x \in \mathbf{R}^n$. But, in general, even $L_x(X)$ can be equal to the whole $\mathbf{RP}(n-1)$ (for example for Whitney's umbrella $\{x^2 = zy^2\} \subset \mathbf{R}^3$ at the origin). Then, locally near such point, *X* can not be written as a union of graphs of Lipschitz maps in any fixed system of coordinates. Therefore, we decompose X into the union of sets with sufficiently small total set of tangent lines. For this purpose we shall use *L*-regular sets.

LEMMA 2.6. – *If* $X' \subset \mathbf{R}^n$ *is nowhere dense and L-regular, then* $TL(X') \neq \mathbf{RP}(n-1)$. *If* $X \subseteq \mathbf{R}^n$ *is L-regular and* $\dim X = n$, *then* $TL(\partial X) \neq \mathbf{RP}(n-1)$.

Proof. – Since $\mathbf{e}_n = (0 : \ldots : 0 : \mathbf{1}) \in \mathbf{RP}(n-1)$ is not contained in $TL(X')$ the first statement follows. To prove the second one we use induction on n. Assume that X

is given by (2.1). Then $\partial X = X_1 \cup X_2$, where X_1 is the union of graphs of f and g, and X_2 is a subset of the cylinder $\partial Y \times \mathbf{R}$. It is easy to see that $\mathbf{e}_n \notin TL(X_1)$ and $TL(X_2) \subset \Pi^{-1}(TL(Y)) \cup \{\mathbf{e}_n\}$, where $\Pi : \mathbf{RP}(n-1) \setminus \{\mathbf{e}_n\} \to \mathbf{RP}(n-2)$ is the standard projection. By the inductive hypothesis, $TL(Y) \neq \mathbf{RP}(n-2)$ and since $TL(X) = TL(X_1) \cup TL(X_2)$, some points of $\mathbf{RP}(n-1)$ near $\mathbf{e}_n$ are not in $TL(X)$. This ends the proof. □

Even if $TL(\partial X)$ is a proper subset of $\mathbf{RP}(n-1)$, it is (in general) of dimension $n-1$. To estimate its size we use the ordinary volume form on $\mathbf{RP}(n-1)$ [*i. e.* induced by the standard volume on the unit sphere S^{n-1} and normalized to $\mathrm{Vol}(\mathrm{RP}(n-1)) = 1$]. Note that for every subanalytic subset T' of $\mathbf{RP}(n-2)$ the inverse image $\Pi^{-1}(T')$ by the standard projection $\Pi : \mathbf{RP}(n-1)\setminus\{\mathbf{e}_n\} \to \mathbf{RP}(n-2)$ satisfies $\mathrm{Vol}_{n-1}(\Pi^{-1}(T')) = \mathrm{Vol}_{n-2}(T')$.

LEMMA 2.7. – *Let $X \subseteq \mathbf{R}^n$ be compact subanalytic and thick. Then, for each $\delta > 0$ there exists a decomposition of X into thick L-regular sets X_i such that for each i*

$$\mathrm{Vol}(TL(\partial X_i)) < \delta.$$

Proof. – Induction on $n = \dim X$.

Let $\xi_1, \ldots, \xi_s$ satisfy the statement of Proposition 2.5 for $Fr(X)$ (with some ε, $\varepsilon' > 0$). Let $\pi = \pi_{\xi_s}$. Consider compact subanalytic sets

$$Y_s = \overline{\mathrm{Int}(\pi(\{x \in X;\ \pi \text{ is } \varepsilon\text{-semi-regular}\}))}$$

and $Z = \overline{\pi(X) \setminus Y_s}$. They are both thick, provided they are not empty, and their intersection is nowhere dense in $\mathbf{R}^{n-1}$. By the inductive assumption we can decompose Y_s into L-regular thick sets. Then, using semi-regularity of π, we can also decompose $X_s = \overline{\pi^{-1}(\mathrm{Int}(Y_s)) \cap X}$ into the union of thick L-regular sets.

Take one of these sets X' and assume that it is given by (2.1) with $Y = \pi(X')$ and defining functions f and g. Divide Y into small pieces Y_j' such that the volume of total sets of lines tangent to the graphs of f and g restricted to each Y_j' is sufficiently small (say smaller than $\delta/2$). Next we apply again the inductive hypothesis to each of Y_j' with $\delta/2$ in place of δ, and the decomposition obtained of X_s satisfies the required properties.

By the inductive hypothesis we may also decompose $Z = \bigcup Z_j$ in such a way that Z_j are L-regular and the volumes $\mathrm{Vol}(TL(\partial Z_j))$ are very small (in comparison to ε'). Fix one of $\overline{\pi^{-1}(\mathrm{Int}(Z_j)) \cap X}$ and denote it by X'. Then $Fr(X') = X_1' \cup X_2'$, where $X_1' \subseteq \overline{Fr(X) \setminus X_s}$ and $X_2' \subseteq \pi^{-1}(\partial Z_j)$. Therefore, $\xi_1, \ldots, \xi_{s-1}$ satisfy (2.2) with respect to X_1' and, after moving them a little, any of them satisfies (2.2) with respect X_2'. Therefore, they satisfy (2.2) with respect to $Fr(X')$ and the lemma follows by induction on s. □

DEFINITION 2.8. – Let X be an L-regular subanalytic set and let Z be another subanalytic subset of $\mathbf{R}^n$. We say that Z is *L-separated from* X if there exists $C > 0$ such that for every $x \in X$

$$\mathrm{dist}(x, \partial X) \leqq C\, \mathrm{dist}(x, Z). \tag{2.3}$$

We call two L-regular sets X_1, X_2 *L-biseparated* if they are L-separated from each other, *i. e.* there exists $C > 0$ such that for every $x_1 \in X_1$, $x_2 \in X_2$

$$\text{(B1)} \qquad \operatorname{dist}(x_1, \partial X_1) \leqq C \operatorname{dist}(x_1, X_2)$$

$$\text{(B2)} \qquad \operatorname{dist}(x_2, \partial X_2) \leqq C \operatorname{dist}(x_2, X_1).$$

Then, in particular $X_1 \cap X_2 = \partial X_1 \cap \partial X_2$.

The following lemma follows easily from (5) of Lemma 2.2.

LEMMA 2.9. – *Let $X \subset \mathbf{R}^n$ be an L-regular set given as the graph of $\Phi : Y \to \mathbf{R}^{n-k}$ as in (3) of Definition 2.1 and let $\pi : \mathbf{R}^n \to \mathbf{R}^k$ be the standard projection. Then any $Z \subseteq \pi^{-1}(\mathbf{R}^k \setminus \operatorname{Int}(Y))$ is L-separated from X.*

L-biseparation does not imply regular separation with exponent 1 in the sense of Łojasiewicz [Ł], §18 which says:

$$\text{(Ł1)} \qquad \operatorname{dist}(x_1, X_1 \cap X_2) \leqq C \operatorname{dist}(x_1, X_2)$$

$$\text{(Ł2)} \qquad \operatorname{dist}(x_2, X_1 \cap X_2) \leqq C \operatorname{dist}(x_2, X_1)$$

for every $x_1 \in X_1$, $x_2 \in X_2$ and universal $C > 0$. Also, in general, (B1) does not imply (B2), though this property is enjoyed by Łojasiewicz's regular separation, that is (Ł1) implies (Ł2) (the constant changes). This follows from the following general fact.

LEMMA 2.10. – *Let X_1, X_2, Z be arbitrary subsets of a metric space and let for each $x_1 \in X_1$*

$$\operatorname{dist}(x_1, Z) \leqq C \operatorname{dist}(x_1, X_2).$$

Then, for each $x_2 \in X_2$

$$\operatorname{dist}(x_2, Z) \leqq (C+1) \operatorname{dist}(x_2, X_1).$$

Proof. – We may ssume that the sets in question are closed. Take an arbitrary $x_2 \in X_2$ and let $x_1 \in X_1$ be such that $\operatorname{dist}(x_2, X_1) = |x_1 - x_2|$. Then

$$\begin{aligned} \operatorname{dist}(x_2, Z) &\leqq |x_1 - x_2| + \operatorname{dist}(x_1, Z) \leqq |x_1 - x_2| + C \operatorname{dist}(x_1, X_2) \\ &\leqq (C+1)|x_1 - x_2| = (C+1) \operatorname{dist}(x_2, X_1), \end{aligned}$$

which shows the lemma. □

In particular, if we put $Z = \partial X_1$, we see that (B1) for all $x_1 \in X_1$ implies

$$\operatorname{dist}(x_2, \partial X_1) \leqq (C+1) \operatorname{dist}(x_2, X_1)$$

for all $x_2 \in X_2$. This gives the following

LEMMA 2.11. – *Let X_1, X_2 be L-regular sets satisfying (B1) for each $x_1 \in X_1$ and*

$$\text{(B2}'\text{)} \qquad \operatorname{dist}(x_2, \partial X_2) \leqq C \operatorname{dist}(x_2, \partial X_1)$$

for every $x_2 \in X_2$ and some $C > 0$. Then, they are L-biseparated [*that is shortly speaking (B1) and (B2′) imply (B2)*].

DEFINITION 2.12. – We call a finite family $\{X_i\}$ of L-regular sets *nicely situated* if they are of the same dimension, $X_i \cap X_j = \partial X_i \cap \partial X_j$ for $i \neq j$, they are L-regular in the same system of coordinates and in description (3) Definition 2.1 they are the graphs of maps $\Phi_i : Y \to \mathbf{R}^{n-k}$ with the same L-regular source Y.

PROPOSITION 2.13. – *Let X be a compact subanalytic subset of $\mathbf{R}^n$ and $\dim X = k$. Then, there exists a finite number of finite nicely situated families $\{X_{\alpha, i}\}$ of k-dimensional L-regular sets such that*

$$X = X' \cup \bigcup_{\alpha, i} X_{\alpha, i},$$

where $\dim X' < k$ and for each α, i the set $\overline{X \setminus \bigcup_j X_{\alpha, j}}$ is L-separated from $X_{\alpha, i}$. In particular, $X_{\alpha, i}$, $X_{\beta, j}$ from different families are L-biseparated.

First we show a special case.

LEMMA 2.14. – *Let X be as in Proposition 2.13. Assume that there exists a linear projection $\pi : \mathbf{R}^n \to \mathbf{R}^k$ which is ε-semi-regular with respect to X at generic points of $\mathbf{R}^n$ (that is from the complement of a subanalytic set of dimension smaller than n). Then, Proposition 2.13 holds for X.*

Proof. – We may assume that π is the standard projection. Let $\operatorname{Reg}(X)$ be the set of regular points of X (of dimension k). It is a subanalytic subset of $\mathbf{R}^n$ (by [K1], for instance). We call $x \in X$ a critical point of $\pi|_X$ if either $x \in \operatorname{Sing}(X) = X \setminus \operatorname{Reg}(X)$ or it is a critical point of $\pi|_{\operatorname{Reg}(X)}$. The set of critical points of $\pi|_X$ is compact and subanalytic and so is its image T in $\mathbf{R}^k$.

Consider any connected component $\mathring{Y}$ of $\mathbf{R}^k \setminus T$ such that $\pi^{-1}(\mathring{Y}) \cap X$ is nonempty. Then $\mathring{Y}$ is relatively compact and $\pi^{-1}(\mathring{Y}) \cap X \to \mathring{Y}$ is a finite analytic covering. By Lemma 2.7 the closure of $\mathring{Y}$ can be a decomposed into the union of L-regular sets. Fix one of them Y_α. By (2) of Lemma 2.2, the analytic covering $\pi^{-1}(\operatorname{Int}(Y_\alpha)) \cap X \to \operatorname{Int}(Y_\alpha)$ is trivial. By assumption on π, the family $\{X_{\alpha, i}\}$ of the closures of the sheets of this covering is a family of nicely situated L-regular sets. Such families $\{X_{\alpha, i}\}$ constructed for all Y_α and all connected relatively compact components of $\mathbf{R}^k \setminus T$ satisfy the statement.

Proof of Proposition 2.13 in the general case. – For $k = n$ the statement follows from Lemma 2.7.

Assume $k < n$. Let $d = n - k$ and let $V_1, \ldots, V_s$ satisfy (2.2) with respect to X. Let $\pi = \pi_{V_s}$ and

$$Y_1 = \overline{\operatorname{Int}(\pi(\{x \in X;\, \pi \text{ is } \varepsilon\text{-semi-regular at } x\}))}.$$

By Lemma 2.14 we can decompose $X_1 = \pi^{-1}(Y_1) \cap X$ as in the statement of the proposition

$$X_1 = X_1' \cup \bigcup_{\beta, i} X_{1, \beta, i},$$

and by Lemma 2.9, for each β, i, $X \setminus \bigcup_j X_{1, \beta, j}$ is L-separated from $X_{1, \beta, i}$. Let

$$X_2 = \overline{X \setminus X_1} \cup (X_1' \cup \bigcup_{\beta, i} \partial X_{1, \beta, i}).$$

Then $V_1, \ldots, V_{s-1}$ satisfy (2.2) with respect to X_2. By induction on s [the number of projections satisfying (2.2)] there is a decomposition

$$X_2 = X_2' \cup \bigcup_{\gamma, j} X_{2, \gamma, j}$$

as in the statement of the proposition. Then

$$X = X_1' \cup X_2' \cup \bigcup_{\beta, i} X_{1, \beta, i} \cup \bigcup_{\gamma, j} X_{2, \gamma, j}$$

has the desired properties.

Indeed, take any $\tilde{X}_1 = X_{1, \beta, i}$, $\tilde{X}_2 = X_{2, \gamma, j}$. By Lemma 2.9, $\tilde{X}_2$ is L-separated from $\tilde{X}_1$, and by construction, $\partial \tilde{X}_1$ is L-separated from $\tilde{X}_2$. Thus, by Lemma 2.11, $\tilde{X}_1$ and $\tilde{X}_2$ are L-biseparated. This ends the proof. □

3. Proof of the main theorem

This section contains the proof of Theorem 1.4. The proof is by induction on $k = \dim X$.

Let X be as in the assumption of Theorem 1.4. We shall construct a strong Lipschitz stratification of X. It suffices to construct a stratification $\mathcal{X}$ of X satisfying the properties (e1) (*see* Proposition 1.3) and (1.3) only for $j = k$. Indeed, let $\mathcal{X} = \{X^i\}$ be such a stratification. By the inductive hypothesis, there is a strong Lipschitz stratification $\mathcal{X}'$ of X^{k-1} compatible with $\{X^i\}_{i<k}$. Then, $\mathcal{X}'$ and the connected components of $X \setminus X^{k-1}$ give a strong Lipschitz stratification.

If we drop the condition of tangency to strata, then a Lipschitz extension of a Lipschitz vector field always exists [B], (7.5), p. 122. If $j = k = n$, then the tangency to k dimensional strata is trivially satisfied. In this case also (1.3) holds. Thus, for $k = n$ the existence of strong Lipschitz stratification follows easily from the inductive assumption.

Case 1. – Assume that X is a finite union of nicely situated L-regular sets $\{X_i\}$.

Let X_i be the graphs of $\Phi_i : Y \to \mathbf{R}^{n-k}$ as in (3) of Definition 2.1. The coordinate functions $\Phi_{i,j}$ of Φ_i $(j = 1, \ldots, n-k)$ are subanalytic and belong to the class $SSUB(\mathbf{R}^n)$

(*see* Definition 7.1 below). This class of functions was introduced by Kurdyka in [K1] where he showed that the partial derivatives of such functions are also in the class [K1], Théorème (2.4). In Section 7 we show the following result.

PROPOSITION 3.1 (compare [P1] Lemmas 4.4 and 4.5). – *Let U be an open, relatively compact and subanalytic subset of* $\mathbf{R}^k$ *and let a function* $f \in SSUB(\mathbf{R}^k)$ *be analytic in U. Then, there exist a stratification* $\mathcal{S}$ *of* $\overline{U}$ *and a constant* $C > 0$ *such that for every Lipschitz* $\mathcal{S}$*-compatible vector field* v *with Lipschitz constant L*

$$(3.1) \qquad |Df(x)\, v(x)| \leqq CL\, |f(x)| \quad \textit{for every } x \in U.$$

Remark. – It suffices to show the existence of a stratification of a big closed ball $\mathbf{B} \subset \mathbf{R}^k$ satisfying the above property. Indeed, let $\mathcal{B}$ be such a stratification. As we already have shown, by the inductive assumption of the proof of Theorem 1.4, we can find a strong Lipschitz stratification $\mathcal{B}'$ of $\mathbf{B}$ compatible with $\mathcal{B}$ and $\overline{U}$. Then, $\mathcal{B}'$ restricted to $\overline{U}$ satisfies the statement, since each Lipschitz vector field on $\overline{U}$ and compatible with $\mathcal{B}'$ can be extended on $\mathbf{B}$.

Let $\mathcal{Y}$ be a stratification of Y such that:

(i) each Φ_i is analytic on the strata of $\mathcal{Y}$;

(ii) $\mathcal{Y}$ is compatible with the zero sets of every $\Phi_{i_1} - \Phi_{i_2}$;

(iii) $\mathcal{Y}$ satisfies the statement of Proposition 3.1 for all $\Phi_{i_1,j} - \Phi_{i_2,j}$ on $\operatorname{Int}(Y)$;

(iv) $\mathcal{Y}$ satisfies the statement of Proposition 3.1 for all partial derivatives $\partial\Phi_{i,j}/\partial x_s$.

Let $\pi : \mathbf{R}^n \to \mathbf{R}^k$ be the standard projection. By (i) the inverse images (by $\pi|_{X_i}$) of the strata of $\mathcal{Y}$ form a stratification $\mathcal{X}_i$ of X_i. By (ii) they glue to a stratification $\mathcal{X}$ of X.

Before we show that $\mathcal{X}$ has the required properties we need some preparation. For $x \in \mathbf{R}^n$ we write $x = (x', x'')$, where $x' \in \mathbf{R}^k$ and $x'' \in \mathbf{R}^{n-k}$. Similarly, for a vector field v on $\mathbf{R}^n$, we write $v = (v', v'')$, where v' are the first k coordinates and v'' are the last $n-k$ ones. The following lemma, follows from the proof of Fact. 4.1 of [P1].

LEMMA 3.2. – *Let* $\Phi : Y \to \mathbf{R}^{n-k}$ *be like in (3) of Definition 2.1 and let* $\mathcal{Y}$ *be a stratification of Y satisfying the statement of Proposition 3.1 for all partial derivatives of coordinate functions of* Φ*. Assume also that* Φ *is analytic on the strata of* $\mathcal{Y}$*. Then, there exists a constant* $C > 0$ *such that if* $v'(x')$ *is a Lipschitz* $\mathcal{Y}$*-compatible vector field on Y with constant L, then* $v(x) = (v'(x'), D\Phi(x')\, v'(x'))$ *is a Lipschitz vector field on X with constant CL.*

By Lemma 3.2

$$v(x) = (v'(x'), D\Phi_i(x')\, v'(x')) \leftrightarrow v'(x')$$

establishes one to one correspondence between Lipschitz $\mathcal{Y}$-compatible vector fields on Y and Lipschitz $\mathcal{X}_i$-compatible vector fields on X_i. Assume that $X^{k-1} \subseteq W \subseteq X^k$ and v is a Lipschitz vector field on W. Let $\tilde{v}$ be a Lipschitz extension of v on X (not necessarily tangent to strata). Then, by Lemma 3.2,

$$w_i(x) = (\tilde{v}'(x), D\Phi_i(x')\, \tilde{v}'(x))$$

is a Lipschitz vector field on X_i. The w_i's glue to a Lipschitz vector field w on X. Indeed, we only need to check the Lipschitz condition for the pairs of points of the form: $p = (x', \Phi_{i_1}(x'))$, $q = (x', \Phi_{i_2}(x'))$. This follows from (iii)

$$|w(p) - w(q)| = |D(\Phi_{i_1}(x') - \Phi_{i_2}(x'))\, w'(x)| \leqq CL|\Phi_{i_1}(x') - \Phi_{i_2}(x')| \leqq CL|p - q|.$$

This finishes the proof in Case 1.

General Case. – The general case follows from Case 1 and Proposition 2.13.

Similarly we show the existence of a strong Lipschitz stratification of X compatible with a finite family of subsets of X. Thus, to complete the proof of Theorem 1.5 it remains to show Proposition 2.5 and Proposition 3.1, which will be proven in Section 5 and Section 7 respectively. □

4. Local flattening theorem

The major technique we shall use in the next three chapters is the local flattening theorem. For the reader convenience we recall in the section its statement and give some corollaries (for the details the reader can consult [HLT], [H1]). For our purpose it suffices to consider only the case of nonsingular target. In this case the local flattening theorem can be stated as follows (compare [HLT], Théorème 4 or [H1], Theorem 4).

THEOREM 4.1. – *Let $f : X \to M$ be a morphism of complex analytic spaces and assume that M is nonsingular. Let L and K be compact subsets of X and M respectively. Then, there exists a finite number of analytic morphisms $s_\alpha : W_\alpha \to M$, such that:*

(1) *each s_α is the composition of a finite sequence of local blowings-up with smooth nowhere dense centers;*

(2) *for each α there is a compact subanalytic subset K_α of W_α and*

$$\bigcup_\alpha s_\alpha(K_\alpha) = K;$$

(3) *the strict transforms $f_\alpha : X_\alpha \to W_\alpha$ of f by s_α are flat at every point $x \in X_\alpha$ corresponding to L.*

The real analytic version can be expressed as follows (a stronger result can be obtained from [H1], 4.17 or [HLT], Theorem 4).

PROPOSITION 4.2. – *Let $f : X \to M$ be a morphism of real analytic spaces and assume that M is nonsingular. Let L and K be compact subsets of X and M respectively. Then, there exists a finite number of real analytic morphisms $s_\alpha : W_\alpha \to M$, such that:*

(1) *each s_α is the composition of a finite sequence of local blowings-up with smooth nowhere dense centers;*

(2) *for each α there is a compact subanalytic subset K_α of W_α and*

$$\bigcup_\alpha s_\alpha(K_\alpha) = K;$$

(3) *the strict transforms* $\tilde{f}_\alpha : \tilde{X}_\alpha \to \tilde{W}_\alpha$ *of a complexification of f by complexifications of* s_α *are flat at every point* $x \in \tilde{X}_\alpha$ *corresponding to L.*

Remarks (to Theorem 4.1 and Proposition 4.2). – (i) To choose the centers of local blowings-up nonsingular we use, for instance, [H2], Main Theorem II$''$ or [BM1], Theorem 4.4.

(ii) Let $\dim M = n$. In the complex case can always assume that W_α are small neighbourhoods of the origin in $\mathbf{C}^n$, and in the real one that each W_α is isomorphic to $\mathbf{R}^n$.

(iii) Let $E_\alpha \subset W_\alpha$ denote the exceptional divisor (that is the union of the inverse images of the exceptional divisors of subsequent local blowings-up). By the same argument as in (i) we can assume that E_α is normal crossings.

(iv) Let $q(t)$ be a germ of analytic curve and $q(0) \in K$. Then, there exist an α and a lifting q_α of q by s_α such that $q_\alpha(0) \in K_\alpha$.

(v) In this paper we only need the equidimensionality of the strict transforms which is a condition strictly weaker than flatness. *See* [P4] for a simple proof of such weaker versions of Theorem 4.1 and Proposition 4.2.

The following corollary follows from Proposition 4.2 and the fibre-cutting lemma ([H1], (7.3.5) or [BM1], Lemma 3.6).

COROLLARY 4.3. – *Let M, N be real analytic manifolds and let X be a compact nowhere dense subanalytic subset of* $M \times N$. *Let* $\varphi : X \to M$ *denote the map induced by the standard projection. Then, there exist a finite number of real analytic morphisms* $s_\alpha : W_\alpha \to M$ *and compacts* $K_\alpha \subseteq W_\alpha$ *satisfying (1) and (2) of Proposition 4.2 for given compact* $K \subseteq M$ *and such that all the fibres of*

$$\varphi_\alpha : \quad X_\alpha = \overline{X \times_M (W_\alpha \setminus E_\alpha)} \to W_\alpha,$$

where E_α *denotes the exceptional divisor of* s_α, *are nowhere dense in N.*

5. Regular projections theorem

Regular Projections Theorem was introduced by Mostowski [M1] in the course of proof of existence of Lipschitz stratification for (germs of) complex analytic sets. The subanalytic version of this theorem was proven in [P3]. In this section, we extend the method from [P3] to generalize Regular Projections Theorem to the case of projections of any codimension. Unlike the proof of Mostowski our proof allows to estimate the number of projections needed by the dimension of the ambient space plus one.

First we recall the classical notion of regular projection ([M1], Section 4, [P3]). Let $X \subseteq \mathbf{K}^n$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$). For $\xi \in \mathbf{K}^{n-1}$, we denote by $\pi_\xi : \mathbf{K}^n \to \mathbf{K}^{n-1}$ the linear projection parallel to the vector $(\xi, 1) \in \mathbf{K}^{n-1} \times \mathbf{K}$.

DEFINITION 5.1. – For $\xi \in \mathbf{K}^{n-1}$ we say that $\pi = \pi_\xi$ is *regular at* $x_0 \in \mathbf{K}^n$ *(with respect to X)* if there exist positive constants C, ε such that:

(a) $\pi_{|X}$ is finite (set theoretically);

(*b*) the intersection of X with the open cone

$$\mathcal{C}_\varepsilon (x_0, \xi) = \{x_0 + \lambda (\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbf{K} - 0\}$$

is an empty set or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i (\eta) (\eta, 1); |\eta - \xi| < \varepsilon\},$$

where λ_i are $\mathbf{K}$-analytic nowhere vanishing functions defined for $|\eta - \xi| < \varepsilon$;

(*c*) the functions λ_i from *(b)* satisfy for all $|\eta - \xi| < \varepsilon$

$$|\mathrm{grad}\, \lambda_i (\eta)| \leqq C |\lambda_i (\eta)|.$$

The following lemma explains the geometric meaning of regularity of projection.

LEMMA 5.2 (compare [M1], Proposition 4.2). – *Let the standard projection* $\pi = \pi_0 : \mathbf{K}^n \to \mathbf{K}^{n-1}$ *be regular (with constants* C, ε*) at* $x_0 \in \mathbf{K}^n$ *with respect to X. Then, there exist positive constants* ε_1, δ, M, *depending only on* ε, C, n *(and not on X or* x_0*) such that* $\mathcal{C}_{\varepsilon_1} (x_0, 0) \cap X$ *is contained in the disjoint finite union of the graphs of* $\mathbf{K}$*-analytic functions*

$$\varphi_i : \quad B (\pi (x_0), |\lambda_i (0)| \delta) \to \mathbf{K},$$

which correspond bijectively to the λ_i*'s and satisfy*

$$|\mathrm{grad}\, \varphi_i| \leqq M,$$

and these graphs are contained in $\mathcal{C}_\varepsilon (x_0, 0) \cap X$.

Proof. – Let $x_0 = 0$, $\lambda = \lambda_i$ and denote $\{\eta \in \mathbf{K}^{n-1}; |\eta| < \varepsilon\}$ by U_ε. By (*c*) of Definition 5.1

$$|D \ln (\lambda)| \leqq C \tag{5.1}$$

(in the real case we assume λ to be positive). Hence

$$e^{-C|\eta|} \leqq |\lambda (\eta)/\lambda (0)| \leqq e^{C|\eta|}. \tag{5.2}$$

Let $\Psi : U_\varepsilon \to \mathbf{K}^{n-1}$ be defined by $\Psi (\eta) = \lambda (\eta) \eta$. Then

$$\begin{aligned}|D\Psi (\eta) - \lambda (0)\, \mathbf{Id}| &\leqq |D\lambda (\eta)|\, |\eta| + |(\lambda (\eta) - \lambda (0))\, \mathbf{Id}| \\ &\leqq C\, |\lambda (\eta)|\, |\eta| + |D\lambda (\eta')|\, |\eta| \leqq 2\, C |\lambda (0)| e^{C|\eta|}\, |\eta|,\end{aligned}$$

and thus, for $|\eta| \leqq \varepsilon_0 (C, \varepsilon, n)$, $D\Psi (\eta)$ is an isomorphism and

$$|(D\Psi (\eta))^{-1}| \leqq C' |\lambda (0)|^{-1}.$$

Claim. – There are constants $\varepsilon_2 > \varepsilon_1 > 0$ and $\delta > 0$, $\varepsilon_2 \leqq \varepsilon_0$, such that Ψ is injective on U_{ε_2} and $\Psi (U_{\varepsilon_2}) \supseteq B (0, |\lambda (0)| \delta) \supseteq \Psi (U_{\varepsilon_1})$. In particular, Ψ^{-1} is defined on $B (0, |\lambda (0)| \delta)$.

Since Ψ preserves the lines through the origin, it is enough to prove the claim only for $n = 2$. The argument, we present, is different in the complex and real case.

Case $\mathbf{K} = \mathbf{R}$. – Then Ψ is injective on U_{ε_0}. If $|\eta| < (4\, C)^{-1}$, then by (5.1) and (5.2)

$$|\Psi'(\eta)| \geqq |\lambda(\eta)| - |\lambda'(\eta)\eta| \geqq |\lambda(0)|e^{-C|\eta|} - C|\eta|\,|\lambda(0)|e^{C|\eta|} \geqq e^{-1/4}|\lambda(0)|/4.$$

Put $\varepsilon_2 = \min\{(4C)^{-1}, \varepsilon_0\}$, $\delta = e^{-1/4}\varepsilon_2/4$. Then $\Psi(U_{\varepsilon_2}) \supseteq B(0, |\lambda(0)|\delta)$. On the other hand, for $|\eta| < (4C)^{-1}$,

$$|\Psi'(\eta)| \leqq |\lambda(\eta)| + |\lambda'(\eta)\eta| \leqq 4e^{1/4}|\lambda(0)|,$$

so $B(0, |\lambda(0)|\delta) \supseteq \Psi(U_{\varepsilon_1})$ for $\varepsilon_1 = e^{-1/4}\varepsilon_2/4$. This shows the claim in the real case:

Case $\mathbf{K} = \mathbf{C}$. – By (5.2) there is ε' such that

$$|\lambda(\eta) - \lambda(0)| \leqq |\lambda(0)|/3 \quad \text{if} \quad |\eta| \leqq \varepsilon'.$$

By Rouché's theorem, for $|y| \leqq |\lambda(0)|\varepsilon'/3$ the equation $y = \lambda(\eta)\eta$ has exactly one solution on $U_{\varepsilon'}$. Therefore, Ψ^{-1} is well defined on $B(0, |\lambda(0)|\varepsilon'/3)$ and $\Psi^{-1}(B(0, |\lambda(0)|\varepsilon'/3)) \supseteq U_{\varepsilon'/4}$. Then $\varepsilon_2 = \varepsilon'/4$, $\delta = |\lambda(0)|\varepsilon_2/3$, and $\varepsilon_1 = \varepsilon_2/4$ satisfy the claim.

In both cases we define $\varphi : B(0, |\lambda(0)|\delta) \to \mathbf{K}$ corresponding to λ by $\varphi(y) = \lambda(\Psi^{-1}(y))$. If $\eta = \Psi^{-1}(y)$, then

$$\begin{aligned}|D\varphi(y)| &= |D\lambda(\eta)\,D(\Psi^{-1})(y)| \leqq C|\lambda(\eta)|\,|(D\Psi(\eta))^{-1}| \\ &\leqq C|\lambda(0)|\,e^{C|\eta|}\,4e^{1/4}\,|\lambda(0)|^{-1} \leqq M,\end{aligned}$$

as required. This ends the proof. □

Fix $d \in \{1, \ldots, n-1\}$. We generalize the notion of regular projection to the case of projections onto the subspaces of $\mathbf{K}^n$ of arbitrary codimension d. Let $G(n, d)$ denote the Grassmannian of d-dimensional linear subspaces of $\mathbf{K}^n$. The scalar product on $\mathbf{K}^n$ induces an isomorphism $G(n, d) \simeq G(n, n-d)$. By $p : E \to G(n, d)$ we denote the tautological bundle over $G(n, d)$ and by $p^\perp : E^\perp \to G(n, d)$ the bundle induced by the tautological bundle over $G(n, n-d)$. Assume also that we have fixed a metric $d(V, V')$ on $G(n, d)$. For $V \in G(n, d)$ we denote by $\pi_V : \mathbf{K}^n \to V^\perp$ the orthogonal projection (along V) onto the orthogonal complement $V^\perp$ of V. We shall identify $G(n, d)$ with the zero section of E.

Definition 5.3. – Let $X \subseteq \mathbf{K}^n$. For $(C, \varepsilon) \in \mathbf{R}_+^2$ and $V \in G(n, d)$, we say that $\pi = \pi_V : \mathbf{K}^n \to V^\perp$ is (C, ε)*-regular at* $x_0 \in \mathbf{K}^n$ (with respect to X) if:

(a) $\pi_{|X}$ is finite (set-theoretically);

(b) the intersection of X with the open cone

$$\mathcal{C}_\varepsilon(x_0, V) = \{x_0 + v;\ v \in V' \setminus 0,\ d(V', V) < \varepsilon\}$$

is a $\mathbf{K}$-analytic smooth submanifold of pure dimension $n-d$ (or an empty set);

(c) for every $x \in X \cap \mathcal{C}_\varepsilon(x_0, V)$ the angle between $T_x X$ and V is bounded from zero by C by which we mean that: for every $V' \ni x - x_0$ such that $d(V', V) < \varepsilon$ and every $v \in T_x X$ we have $|\pi_{V'}(v)| \geqq |v|/C$ (in particular $T_x X \cap V' = 0$).

Remarks. – (i) For $d = 1$ Definitions 5.1 and 5.3 coincide (the constants C, ε are different but related).

(ii) Regularity of projection is preserved under small deformations of V that is if π_V is (C, ε)-regular at x_0, then for V' close to V, $\pi_{V'}$ is (C', ε')-regular at x_0 for some C', ε' close to C, ε.

DEFINITION 5.4. – Let q be a germ (at $0 \in \mathbf{K}$) of a $\mathbf{K}$-analytic curve in $\mathbf{K}^n$. We say that π_V is *(C, ε)-regular at q* if it is so at every $q(t)$ for $t \neq 0$ and small. We say that π_V is *regular* at a point $x \in \mathbf{K}^n$ or at a germ q if it is (C, ε)-regular for some positive C, ε. Finally, we call a subset $\mathcal{P} \subseteq \mathrm{G}(n, d)$ *a set of regular projections for X* if there exist $C, \varepsilon > 0$ such that for every germ q of a $\mathbf{K}$-analytic curve (and so for every point x) one can find $V \in \mathcal{P}$ such that π_V is (C, ε)-regular at q.

THEOREM 5.5 (Regular Projections Theorem for subanalytic sets). – *Let $X \subset \mathbf{R}^n$ be compact and subanalytic and let* $\dim X \leqq n - d$. *Then for generic* $V_1, V_2, \ldots, V_{n+1}$ (i. e. *from an open dense subset of* $G(n, d)^{n+1}$) *the set* $\{V_1, V_2, \ldots, V_{n+1}\}$ *is a set of regular projections for X.*

Proof. – In the next section we show a similar theorem for "complex subanalytic sets". To stress the similarities between the proofs we divide them into several corresponding steps.

Step 1. Condition (a). – We show the existence of projections satisfying the statement for the condition *(a)* of Definition 5.3. By the Koopman-Brown Lemma (*see* [KB] or [Ł], §22), if $\dim X < n$, then the restriction of a generic linear projection $\mathbf{R}^n \to \mathbf{R}^{n-1}$ to X is a finite map. In general the following lemma holds.

LEMMA 5.6. – *Let X be a compact subanalytic subset of* $\mathbf{R}^n$ *and let* $\dim X \leqq n - d$. *Then, the subset of all such* $V \in G(n, d)$ *that* $\pi_{V|X}$ *is not finite is subanalytic and nowhere dense in* $G(n, d)$.

Proof. – Let $p' : \mathbf{R}^n \times G(n, d) \to E^\perp$ be defined by $p'(x, V) = (\pi_V(x), V^\perp)$. By the existence of stratification of a proper subanalytic map, the subset $Y \subseteq E^\perp$ of the points over which $p'_{|X \times G(n, d)}$ is not finite is subanalytic and relatively compact. The set of those $V \in G(n, d)$ that $\pi_V|_X$ is not finite equals just $p^\perp(Y)$, so is also subanalytic. By the Koopman-Brown Lemma, if $d = 1$, it is also nowhere dense. Next, by induction on d, we show that it is always nowhere dense.

Step 2. Restriction to $\mathbf{B}^n$. – The statement is local in $G(n, d)^{n+1}$. Take $(V_1, V_2, \ldots, V_{n+1}) \in G(n, d)^{n+1}$ and assume that $V_1 \cap V_2 \cap \ldots \cap V_{n+1} = \varnothing$. Then, for $x \in \mathbf{R}^n$ with $|x|$ sufficiently big and for any $(V_1', V_2', \ldots, V_{n+1}')$ sufficiently close to $(V_1, V_2, \ldots, V_{n+1})$, one of $\mathcal{C}_\varepsilon(x, V_i') \cap X$ is empty. Hence, corresponding $\pi_{V_i'}$ is regular at x (with arbitrary C and ε depending only on $\{V_1, V_2, \ldots, V_{n+1}\}$). Therefore, to avoid the problem of non-compactness of $\mathbf{R}^n$, we can consider only the points from a big ball $\mathbf{B}^n \subset \mathbf{R}^n$.

Step 3. Restriction to generic curve. – We show the statement of theorem (that is regularity of projections) only for the case of generic curve. That means that we establish the existence of a nowhere dense compact subanalytic subset S of $\mathbf{B}^n$ such that the statement holds for all (germs of) curves $q(t)$ such that $q(t) \notin S$ for $t \neq 0$. Note that this is the case we need in this paper. The method of proof can be carried over to the general case (*see* Step 9).

Step 4. Main construction. – The proof is based on the following geometric construction. Let $\rho : E \to \mathbf{R}^n$ be the canonical map and let $F : \mathbf{R}^n \times E \to \mathbf{R}^n$ be defined by

$$F(x, v) = x + \rho(v).$$

Let $\Pi : \mathbf{R}^n \times E \to \mathbf{R}^n \times G(n, d)$ denote the projection induced by p. Then $\tilde{X} = F^{-1}(X) \cap \Pi^{-1}(\mathbf{B}^n \times G(n, d))$ is a compact subanalytic subset of $\mathbf{B}^n \times E$ and

$$(5.3) \qquad \mathcal{C}_\varepsilon(x_0, V) = \Pi^{-1}(\{(x_0, V'); d(V, V') < \varepsilon\}) \setminus (\{x_0\} \times G(n, d)).$$

In particular, if $\Pi_{|\tilde{X}}$ is a finite analytic covering over $\{(x_0, V'); d(V, V') < \varepsilon'\}$, then π_V satisfies conditions *(b)* and *(c)* of Definition 5.3 (with some constants) at x_0. The following lemma, which follows easily from the assumption on dimension of X, shows that for generic V and fixed x_0 it is always the case.

LEMMA 5.7. – *Let $\tilde{X} \subset \mathbf{R}^n \times E$ be as above. Then, for each $x_0 \in \mathbf{B}^n$ the set of those $V \in G(n, d)$ for which $\tilde{X} \cap \Pi^{-1}(x_0, V)$ is finite is subanalytic and dense in $G(n, d)$.*

Our task is to assure the existence of regular projections uniformly on $\mathbf{B}^n$. By Lemma 5.7 there is a compact nowhere dense subanalytic subset $Y \subset \mathbf{B}^n \times G(n, d)$ such that $\Pi|_{\tilde{X}}$ is an analytic covering over $\mathbf{R}^n \times G(n, d) \setminus Y$. Enlarging Y, if necessary, we may assume that the induced map

$$\tilde{X}^0 = \tilde{X} \setminus (\mathbf{R}^n \times G(n, d) \cup \Pi^{-1}(Y)) \to \mathbf{R}^n \times G(n, d) \setminus Y$$

is also an analytic covering. These are our preparations to handle with condition (b).

To work with condition *(c)*, we consider the regular part (of dimension $n-d$) $\operatorname{Reg}(X)$ of X. Then $\tilde{X}^0 \subseteq F^{-1}(\operatorname{Reg}(X))$. The function $\gamma' : \operatorname{Reg}(X)^0 \times G(n, d) \to [0, 1]$ given by

$$\gamma'(x, V) = \min_{v \in T_x X \setminus 0} |\pi_V(v)|/|v|$$

induces a continuous subanalytic function γ on $\tilde{X}^0$ by setting

$$\gamma(x, v) = \gamma'(F(x, v), p(v)).$$

Let $\varphi : Y \to \mathbf{R}^n$ be the induced projection. Consider its fibres $Y_x = \varphi^{-1}(x)$ as subsets of $G(n, d)$. We leave the proof of the following observation to the reader.

LEMMA 5.8. – *If $V \notin Y_{x_0}$, then π_V satisfies condition (b) of Definition 5.3 with any ε such that $\varepsilon < \operatorname{dist}(V, Y_{x_0})$. Moreover, if $\gamma(x_0, v)$ is bounded from below on*

$$\tilde{X}^0 \cap \Pi^{-1}\{(x_0, V'); \operatorname{dist}(V, V') < \varepsilon\}$$

by $\delta > 0$, then the condition (c) is satisfied with $C = \delta^{-1}$.

Step 5. Application of the local flattening theorem. – Let $Z' \subset \mathbf{R}^n \times E \times [0, 1]$ be the closure of the graph of γ and let Z be its projection in $\mathbf{R}^n \times G(n, d) \times [0, 1]$. Then Z is compact and subanalytic and the generic fibres of the projection $\psi : Z \to \mathbf{R}^n$ are nowhere dense in $G(n, d) \times [0, 1]$. Also the generic fibres of $\varphi : Y \to \mathbf{R}^n$ are nowhere dense in $G(n, d)$. Let $\{s_\alpha : W_\alpha \to \mathbf{R}^n\}$ and $K_\alpha \subseteq W_\alpha$ satisfy the statement of the statement of Corollary 4.3 [and Remark (iii) after Proposition 4.2] for φ and ψ, and $K = \mathbf{B}^n$. Then, all the fibres of:

$$\varphi_\alpha : \quad Y_\alpha = \overline{Y \times_{\mathbf{R}^n} (W_\alpha \setminus E_\alpha)} \to W_\alpha$$

$$\psi_\alpha : \quad Z_\alpha = \overline{Z \times_{\mathbf{R}^n} (W_\alpha \setminus E_\alpha)} \to W_\alpha$$

are nowhere dense in $G(n, d)$, $G(n, d) \times [0, 1]$ respectively.

Step 6. Estimate of the number of projections. – For given $V \in G(n, d)$ we put $(W_\alpha)_V = \{\tilde{x} \in K_\alpha;\ (\tilde{x}, V) \in Y_\alpha\}$. By the lemma below, for all α and generic $V_1, V_2, \ldots, V_{n+1}$

$$\bigcap_{i=1}^{n+1} (W_\alpha)_{V_i} = \emptyset. \tag{5.4}$$

LEMMA 5.9. – *Let G, W be compact subanalytic sets and let Y be a subanalytic subset of $W \times G$. We denote the projections of Y in G and W by π_G and π_W respectively and for $g \in G$ we put $W_g = \pi_W(\pi_G^{-1}(g))$. Let $\dim W \leqq n$ and assume that all fibres of π_W are nowhere dense in G. Then for each $k = 1, 2, \ldots, n+1$ and $g_1, \ldots, g_k$ generic (that is from an open dense subanalytic subset of G^k)*

$$\dim \bigcap_{i=1}^{k} W_{g_i} \leqq n - k, \ \textit{in particular},\ \bigcap_{i=1}^{n+1} W_{g_i} = \emptyset.$$

Proof. – Induction on k. Let $m = \dim G$. Then $\dim Y < n + m$ and $\dim \pi_G^{-1}(g) < n$ for generic n. This follows the statement for $k = 1$. Assume that the statement holds for $k - 1$. Then, by the assumption on the fibers of π_W

$$\dim \pi_W^{-1}\left(\bigcap_{i=1}^{k-1} W_{g_i}\right) \leqq m + n - k,$$

for generic $g_1, \ldots, g_{k-1} \in G$. Hence, for generic g

$$\dim\left(\pi_W^{-1}\left(\bigcap_{i=1}^{k-1} W_{g_i}\right) \cap \pi_G^{-1}(g)\right) \leqq n - k,$$

and since $\pi_W(\pi_W^{-1}(A) \cap \pi_G^{-1}(g)) = A \cap W_g$ for any $A \subseteq W$, we get the result. □

Step 7. Condition (b). – Now we show the theorem for condition *(b)* of Definition 5.3. Take $V_1, V_2, \ldots, V_{n+1}$ satisfying (5.4). There exists $\varepsilon' > 0$ such that for each α and each $\tilde{x} \in K_\alpha$ we may find $V(\tilde{x}) \in \{V_1, V_2, \ldots, V_{n+1}\}$ such that $(\tilde{x}, V) \notin Y_\alpha$ if only $|V - V(\tilde{x})| < \varepsilon'$.

Let $S = \bigcup s_\alpha(K_\alpha \cap E_\alpha)$ and let $q(t)$ be (a germ of) an analytic curve in $\mathbf{B}^n$ such that $q(t) \notin S$ for $t \neq 0$. There is an α and a lifting $q_\alpha(t)$ of q by s_α such that $q_\alpha(0) \in K_\alpha$. By construction, for $t \neq 0$ the fibres $\varphi_\alpha^{-1}(q_\alpha(t))$ and $\varphi^{-1}(q(t))$ are equal (as subsets of $G(n, d)$). Take $V = V(q_\alpha(0))$. Then, by Lemma 5.8, π_V satisfies condition *(b)* at each $q(t)\,(t \neq 0$ and small) with $\varepsilon = \varepsilon'/2$.

Step 8. Condition (c). – Thanks to the following lemma a similar argument works for condition *(c)* of Definition 5.3.

LEMMA 5.10. – *For each α and $\tilde{x} \in W_\alpha$ the intersection $\psi_\alpha^{-1}(\tilde{x}) \cap (G(n, d) \times \{0\})$ is nowhere dense in $G(n, d) \times \{0\}$.*

Proof. – First we express γ' in local coordinates.

Let $(x_0, V_0) \in (\mathbf{R}^n \times G(n, d)) \setminus Y$. Assume, for simplicity, that x_0 is the origin in $\mathbf{R}^n$ and $V_0 = \mathbf{R}^d \times 0$. We parametrize $G(n, d)$ near V_0 by $\mathrm{Hom}(\mathbf{R}^d, \mathbf{R}^{n-d})$, that is we identify $V \in G(n, d)$ with the graph of $l \in \mathrm{Hom}(\mathbf{R}^d, \mathbf{R}^{n-d})$ and in particular V_0

with zero homomorphism l_0. As a trivialization of E near V_0 we take that given by the standard projection on $\mathbf{R}^d$.

By assumption, $\tilde{X}^0$ over a small neighbourhood U of $(x_0, V_0) = (0, l_0)$ is the union of graphs of finitely many nowhere vanishing analytic maps $\lambda^i : U \to \mathbf{R}^d$. Fix one of them $\lambda(l) = \lambda^i(0, l)$. We claim that for some universal constant C' and l close to l_0

$$(5.5) \qquad (\gamma(0, l, \lambda(l)))^{-1} \leqq C'(1 + |D\lambda(l)|/|\lambda(l)|).$$

Indeed, we may assume $\lambda(l) = (\lambda_1(l), 0, \ldots, 0)$. Since the image of U by $F(0, l, \lambda(l)) = (\lambda(l), l(\lambda(l)) \in \mathbf{R}^d \times \mathbf{R}^{n-d}$ is contained in X^0, the tangent space $T_x X^0$ to X^0 at $x = F(0, l, \lambda(l))$ is the image of

$$(5.6) \qquad l' \to (D\lambda(l)\, l' + l(D\lambda(l)\, l')) + l'\lambda(l)),$$

where l' varies over all $l' \in \operatorname{Hom}(\mathbf{R}^d, \mathbf{R}^{n-d})$. Note that the first summand is contained in the graph of l, that is in $V \in G(n, d)$ corresponding to l, and the second in $\mathbf{R}^{n-d}$. If $l' = \mathbf{e}_1^* \otimes w \in \mathbf{e}_1^* \otimes \mathbf{R}^{n-d} \subseteq \operatorname{Hom}(\mathbf{R}^d, \mathbf{R}^{n-d})$, then

$$l'(\lambda(l)) = \lambda_1(l)\, w.$$

Consequently $T_x X^0$ is the image of (5.6) with l' varying over $\mathbf{e}_1^* \otimes \mathbf{R}^{n-d}$. Now an easy computation gives (5.5).

To show the lemma we assume that it is false. Then, there exist an α and $(\tilde{x}_0, V_0) \in W_\alpha \times G(n, d) \setminus Y_\alpha$ such that $\psi_\alpha^{-1}(\tilde{x}_0)$ is contained [near $(V_0, 0)$] in $G(n, d) \times \{0\}$.

Let $\tilde{X}_\alpha^0 \subseteq W_\alpha \times E$ be the inverse image of $\tilde{X}^0$ by $(s_\alpha, \mathrm{id}) : W_\alpha \times E \to \mathbf{R}^n \times E$ and let $\tilde{\gamma}$ be a function on $\tilde{X}_\alpha^0$ induced by γ. By construction, the projection $\tilde{X}_\alpha^0 \to G(n, d) \times W_\alpha$ is a finite analytic covering outside $(E_\alpha \times G(n, d)) \cup Y_\alpha$. Let U_G be a small contractible neighbourhood of V_0 in $G(n, d)$ such that $(\{\tilde{x}_0\} \times U_G) \cap Y_\alpha = \emptyset$ and let U be a small neighbourhood of $\tilde{x}_0$. Then, by the curve selection lemma, there exists an analytic curve $\tilde{q} : [0, \varepsilon) \to W_\alpha$, $\tilde{q}(0, \varepsilon) \subset W_\alpha \setminus E_\alpha$ and $\tilde{q}(0) = \tilde{x}_0$, such that for some section $\tilde{\lambda}$ of $U \times E|_{U_G}$ for all $V \in U_G$

$$\tilde{\gamma}(\tilde{q}(t), V, \tilde{\lambda}(\tilde{q}(t), V)) \to 0 \qquad \text{if} \quad t \to 0.$$

Let $D_V \tilde{\lambda}$ denote the derivative of $\tilde{\lambda}$ in direction V. Then, by (5.5), for all $V \in U_G$

$$(|D_V \tilde{\lambda}(\tilde{q}(t), V)|/|\tilde{\lambda}(\tilde{q}(t), V)| \to \infty \qquad \text{if} \quad t \to 0,$$

which contradicts Pawłucki's version of Puiseux's theorem [Pa], that says $\tilde{\lambda}(\tilde{q}(t^s), V)$ is analytic near generic V and for some positive integer s.

This ends the proof of lemma. □

Step 9. – We show how to extend the proof to such curves q in $\mathbf{B}^n$ whose liftings q_α lie entirely in E_α. Take one such E_α. By construction, it is normal crossings, so taking a component we may assume that it is nonsingular and of dimension $n-1$. Let $F_\alpha : E_\alpha \times E \to \mathbf{R}^n$ be given by $F_\alpha(x, v) = F(s_\alpha(x), v)$. Let $\Pi_\alpha : E_\alpha \times E \to E_\alpha \times G(n, d)$ denote the projection induced by Π. Then $\tilde{X}_\alpha = F_\alpha^{-1}(X) \cap \Pi_\alpha^{-1}(\mathbf{K}_\alpha \times G(n, d))$ is a compact subanalytic subset of $E_\alpha \times E$. By Lemma 5.7 $\Pi_\alpha|_{\tilde{X}}$ is a finite analytic covering outside a nowhere dense subanalytic subset of $K_\alpha \times G(n, d)$. Thus, using the same method as above we can show the statement for generic curves in E_α. The general case follows by descending induction on $\dim E_\alpha$. □

6. Subanalytic sets in complex domain

To get the estimate (3.1) from Proposition 3.1 in the semi-analytic case we used in [P1] Regular Projections Theorem for complex analytic sets. Similarly, in the subanalytic case we use a regular projections theorem for subanalytic sets in complex domain (Theorem 6.5 below).

Since we need only the case of projections onto linear subspaces of (complex) codimension 1, we work rather in the language of Definition 5.1 than in more general context of Definition 5.3. In particular, we parameterize linear projections by $\mathbf{C}^{n-1}$ and associate to each $\xi \in \mathbf{C}^{n-1}$ the linear projection $\pi_\xi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ parallel to $(\xi, 1)$.

We consider only analytic spaces which are countable at infinity.

DEFINITION 6.1. – Let X be an subanalytic subset of complex analytic space Z. We say that $\dim_{\mathbf{C}} X \leqq k$ if there exist a complex analytic space Z', $\dim_{\mathbf{C}} Z' \leqq k$, a subanalytic subset $K \subseteq Z'$ and a complex analytic morphism $\varphi : Z' \to Z$ such that $\varphi|_{\overline{K}}$ is proper and $X = \varphi(K)$. We say $\dim_{\mathbf{C}} X = k$ if $\dim_{\mathbf{C}} X \leqq k$ and not $\dim_{\mathbf{C}} X \leqq k - 1$.

We call a subset $H \subseteq Z$ a *complex analytic leaf* if there exists an open subset $U \subseteq Z$ such that H is a connected, complex analytic and nonsingular subset of U. If $\dim_{\mathbf{C}} X \leqq k$, then X is contained in a countable union of complex anaytic leaves of dimension not bigger than k.

The properties of complex dimension of subanalytic sets are similar to those of real dimension.

LEMMA 6.2. – (1) *Let $\Phi : Z \to Z'$ be a morphism of complex analytic spaces and let $X \subseteq Z$ be subanalytic and* $\dim_{\mathbf{C}} X \leqq k$. *If $\Phi|_{\overline{X}}$ is proper, then $\Phi(X)$ is subanalytic and* $\dim_{\mathbf{C}} \Phi(X) \leqq k$.

(2) *Let $\Phi : Z \to Z'$ be as above and let $X' \subseteq Z'$ be subanalytic* $\dim_{\mathbf{C}} X' \leqq k$. *Assume that the fibres of Φ over X' have dimension not bigger than r. Then* $\dim_{\mathbf{C}} \Phi^{-1}(X') \leqq k+r$.

(3) *Let $X \subseteq \mathbf{C}^n$ be subanalytic and* $\dim_{\mathbf{C}} X \leqq k$. *Then,* $\dim_{\mathbf{R}} X \cap \mathbf{R}^n \leqq k$.

(4) (Koopman-Brown Lemma [KB] for subanalytic sets in complex domain) *Let $X \subseteq \mathbf{C}^n$ be relatively compact subanalytic and contained in a countable union of complex analytic leaves of dimension smaller than n. Then, for generic $\eta \in \mathbf{C}^{n-1}$ (that is from the complement of a subanalytic set contained in a countable union of complex analytic leaves of dimension smaller than $n-1$) the projection $\pi_\eta|_X : X \to \mathbf{C}^{n-1}$ is finite (set-theoretically).*

Proof. – (1), (2) and (3) are clear.

To show (4) we note, that by the same argument as in the proof of Lemma 5.6, that the set T of those $\eta \in \mathbf{C}^{n-1}$ where π_η is not finite is subanalytic in $\mathbf{C}^{n-1}$ [even in $\mathbf{C}P(n-1)$]. We prove that T is contained in a countable union of complex analytic leaves of dimension smaller than $n-1$. It suffices to show this for an analytic leaf. We follow the idea of proof of Koopman and Brown [KB].

Let $H \subset U \subseteq \mathbf{C}^n$ be an analytic leaf such that there is a complex analytic function $f : U \to \mathbf{C}$, not identically equal to zero on all components of U, and $H \subseteq f^{-1}(0)$. We may assume that $H = f^{-1}(0)$. Let

$$Z = \{(x, \eta) \in H \times \mathbf{C}^{n-1};\ \pi_{\eta|H} \text{ is not finite at } x\}.$$

Z is an analytic subset of $H \times \mathbf{C}^{n-1}$ and T equals the image of its projection in $\mathbf{C}^{n-1}$. Therefore, T is a countable union of complex analytic leaves. Suppose that one of them is of dimension $n-1$. Then, there is an open subset $V \subseteq \mathbf{C}^{n-1}$ and a complex analytic map $\varphi : V \to \mathbf{C}^n$ such that $\varphi(V) \subseteq H$ and $\pi_{\eta|H}$ is not finite at $\varphi(\eta)$. In particular, for $\eta \in V$ and λ from a neighbourhood of 0 in $\mathbf{C}$,

$$f(\varphi(\eta) + \lambda(\eta, 1)) \equiv 0,$$

which is not possible since the Jacobian of $\varphi(\eta) + \lambda(\eta, 1)$ has a nonzero term λ^{n-1}. Thus we get the contradiction. This ends the proof. □

The following lemma is a complex analog of the fibre cutting lemma ((7.3.5) of [H1] or Lemma 3.6 of [BM1]).

LEMMA 6.3. – *Let $\varphi : Z \to Z'$ be a morphism of complex analytic spaces and K a subanalytic subset of Z such that $\varphi|_{\overline{K}}$ is proper. Then*

$$\dim_{\mathbf{C}} \varphi(K) \leqq \dim_{\mathbf{C}} Z - l,$$

where $l = \min_{x \in K} \{\dim_{\mathbf{C}} \varphi^{-1}(\varphi(x))_x\}$.

Proof. – It is easy to see that it suffices to consider only the following case:

(i) Z' is an analytic subset of $\mathbf{C}^n$ and then we may assume that Z' is $\mathbf{C}^n$ itself;

(ii) K is compact;

(iii) Z is irreducible.

(iv) replacing Z by an analytic subset of a neighbourhood of K, if necessary, we assume that $\min_{x \in Z} \{\dim_{\mathbf{C}} \varphi^{-1}(\varphi(x))_x\} = l$;

Let $d = \dim Z - l$ and let $\pi : \mathbf{C}^n \to \mathbf{C}^d$ be a generic linear projection. Then, the generic fibres of $\Psi = \pi \circ \varphi : Z \to \mathbf{C}^d$ are of dimension l. Let $\{\sigma_\alpha : U_\alpha \to \mathbf{C}^d, K_\alpha\}$ satisfy the statement of Theorem 4.1 for Ψ, K and $\Psi(K)$. Consider the induced diagram

$$\begin{array}{ccc} Z_\alpha & \overset{\tilde{\sigma}_\alpha}{\to} & Z \\ {\scriptstyle \varphi_\alpha} \downarrow & & \downarrow {\scriptstyle \varphi} \\ U_\alpha \times \mathbf{C}^{n-d} & \to & \mathbf{C}^n \\ {\scriptstyle \pi_\alpha} \downarrow & & \downarrow {\scriptstyle \pi} \\ U_\alpha & \overset{\sigma_\alpha}{\to} & \mathbf{C}^d \end{array}$$

where $\Psi_\alpha = \pi_\alpha \circ \varphi_\alpha : Z_\alpha \to U_\alpha$ is the strict transform of Ψ. Let $\tilde{K}_\alpha = \tilde{\sigma}_\alpha^{-1}(K) \cap \Psi_\alpha^{-1}(K_\alpha)$. Then

$$\bigcup_\alpha \tilde{\sigma}_\alpha(K_\alpha) = K.$$

All fibres of Ψ_α and so of φ_α are of pure dimension l. Hence, the images of φ_α (locally on Z_α) are analytic sets of dimension d. Then, X is the union of the images of $\varphi_\alpha(\tilde{K}_\alpha)$ by the induced maps $\varphi_\alpha(Z_\alpha) \to \mathbf{C}^n$. This ends the proof. □

DEFINITION 6.4. – Let $X \subset \mathbf{C}^n$. For $\xi \in \mathbf{C}^{n-1}$ we say that $\pi = \pi_\xi$ is *weakly regular at* $x_0 \in \mathbf{C}^n$ *(with respect to X)* if there exist positive constants C, ε such that:

(a) $\pi_{|X}$ is finite (set theoretically);

(b) the intersection of X with the open cone

$$\mathcal{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1);\ |\eta - \xi| < \varepsilon,\ \lambda \in \mathbf{C} \setminus 0\}$$

is an empty set or is contained in a finite not necessarily disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1);\ |\eta - \xi| < \varepsilon\},$$

where λ_i are $\mathbf{C}$-analytic nowhere vanishing functions defined for $|\eta - \xi| < \varepsilon$;

(c) the functions λ_i from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$|\operatorname{grad} \lambda_i(\eta)| \leqq C|\lambda_i(\eta)|.$$

Similarly to Definition 5.4 we define the notions of: *weak* (C, ε)*-regularity* at a germ of a $\mathbf{C}$-analytic curve, *weak regularity* and a *set of weakly regular projections.*

THEOREM 6.5 (Regular Projections Theorem for subanalytic sets in complex domain). – *Let* $X \subset \mathbf{C}^n$ *be compact subanalytic and let* $\dim_{\mathbf{C}} X < n$. *Then for generic* $\xi_1, \xi_2, \ldots, \xi_{n+1}$ [*that is from an open dense* $\Omega \subseteq (\mathbf{C}^{n-1})^{n+1}$)] *the set* $\{\xi_1, \xi_2, \ldots, \xi_{n+1}\}$ *is a set of weakly regular projectons for X. Moreover,* Ω *can be chosen in such a way that* $\Omega \cap (\mathbf{R}^{n-1})^{n+1}$ *is open and dense in* $(\mathbf{R}^{n-1})^{n+1}$.

Proof. – Let $X = \rho(K)$, where $\rho : Z \to \mathbf{C}^n$ is a morphism of complex analytic spaces, $K \subseteq Z$ is compact and subanalytic, and $\dim_{\mathbf{C}} Z \leqq n - 1$.

In the proof we follow quite closely the pattern of the proof of the real analytic case (Theorem 5.5). We sketch the proof of similar parts stressing the particularities of the complex case.

Step 1. – The existence of projections satisfying condition (a) of Definition 6.4 follows from our version of Koopman-Brown Lemma [Lemma 6.2 (4)].

Step 2. – We consider only germs of $\mathbf{C}$-analytic curves contained in a big closed ball $(\mathbf{B}^{2n})$ and projections from a big ball $\mathbf{B}^{2n-2} \subset \mathbf{C}^{n-1}$.

Step 3. – We show the statement for generic curves (that is not contained entirely in some subanalytic subset of $\mathbf{C}^n$ of complex dimension smaller than n). (*See* also Step 9.)

Step 4. – The proof is based on the following construction.

Let $F : \mathbf{C}^n \times \mathbf{C}^{n-1} \times \mathbf{C} \to \mathbf{C}^n$ be defined by

$$F(x, \eta, \lambda) = x + \lambda(\eta, 1)$$

and let $\tilde{Z}$ be the fibre product

$$\begin{array}{ccc} Z & \leftarrow & \tilde{Z} \\ \downarrow \rho & & \downarrow \tilde{\rho} \\ \mathbf{C}^n & \overset{F}{\leftarrow} & \mathbf{C}^n \times \mathbf{C}^{n-1} \times \mathbf{C}. \end{array}$$

Let $\tilde{K} \subseteq \tilde{Z}$ be the set of points corresponding to K, $\mathbf{B}^{2n}$, $\mathbf{B}^{2n-2}$ and let $\tilde{X} = \tilde{\rho}(\tilde{K})$. Then $\tilde{X} \subseteq F^{-1}(X)$ is compact and $\dim_{\mathbf{C}} \tilde{X} < 2n$. Let $\Pi : \mathbf{C}^n \times \mathbf{C}^{n-1} \times \mathbf{C} \to \mathbf{C}^n \times \mathbf{C}^{n-1}$ and $\operatorname{pr} : \mathbf{C}^n \times \mathbf{C}^{n-1} \to \mathbf{C}^n$ be the standard projections. Then

$$\mathcal{C}_\varepsilon(x_0, \xi) = \Pi^{-1}(\{x_0, \eta);\ |\eta - \xi| < \varepsilon\}) \setminus (\{x_0\} \times \mathbf{C}^{n-1})),$$

and we shall study the set of points where $\Pi|_{\tilde{X}}$ is not a finite analytic covering.

Step 5. – Let $\lambda : \tilde{Z} \to \mathbf{C}$ be the function induced by $\tilde{Z} \xrightarrow{\tilde{\rho}} \mathbf{C}^n \times \mathbf{C}^{n-1} \times \mathbf{C} \to \mathbf{C}$, where the last map is the standard projection onto the last factor. We may assume that

$Z_0 = \lambda^{-1}(0)$ is of complex dimension smaller than $2n-1$ (if not we substitute $\tilde{Z}$ by the closure of $\tilde{Z} \setminus Z_0$).

Let $Z_1 \subseteq \tilde{Z}$ be the set of all those points at which $\Pi \circ \rho : \tilde{Z} \to \mathbf{C}^n \times \mathbf{C}^{n-1}$ is not a finite analytic covering. For $i = 0, 1$ we denote $\rho_i = \rho | Z_i$ and $\tilde{K}_i = \tilde{K} \cap Z_i$.

Let the family $\{s_\alpha : W_\alpha \to \mathbf{C}^{n-1}\}$, $K_\alpha \subseteq W_\alpha$ satisfies the statement of Theorem 4.1 for compact $\mathbf{B}^{2n} \subset \mathbf{C}^n$ and simultaneously for:

$$\varphi = \mathrm{pr} \circ \Pi \circ \rho : \tilde{Z} \to \mathbf{C}^n \text{ and } \tilde{K} \subset \tilde{Z};$$
$$\varphi_i = \mathrm{pr} \circ \Pi \circ \rho_i : Z_i \to \mathbf{C}^n \text{ and } \tilde{K}_i \subset Z_i, \qquad i = 0,\, 1.$$

Consider the induced diagrams

$$\begin{array}{ccc} \tilde{Z}_\alpha & \overset{\tilde{s}_\alpha}{\to} & \tilde{Z} \\ {\scriptstyle\psi_\alpha}\downarrow & & \downarrow{\scriptstyle\psi} \\ W_\alpha \times \mathbf{C}^{n-1} & \to & \mathbf{C}^n \times \mathbf{C}^{n-1} \\ {\scriptstyle\mathrm{pr}_\alpha}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}} \\ W_\alpha & \overset{s_\alpha}{\to} & \mathbf{C}^n \end{array} \qquad \begin{array}{ccc} Z_{i\alpha} & \overset{s_{i_\alpha}}{\to} & Z_i \\ {\scriptstyle\psi_{i_\alpha}}\downarrow & & \downarrow{\scriptstyle\psi_i} \\ W_\alpha \times \mathbf{C}^{n-1} & \to & \mathbf{C}^n \times \mathbf{C}^{n-1} \\ {\scriptstyle\mathrm{pr}_\alpha}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}} \\ W_\alpha & \overset{s_\alpha}{\to} & \mathbf{C}^n \end{array}$$

for $i = 0,\, 1$, where $\varphi_\alpha = \mathrm{pr}_\alpha \circ \psi_\alpha$ and $\varphi_{i_\alpha} = \mathrm{pr}_\alpha \circ \psi_{i_\alpha}$ are the strict transforms of φ, φ_i respectively. Let $\tilde{K}_\alpha \subseteq \tilde{Z}_\alpha$ (resp. $\tilde{K}_{i\alpha} \subseteq Z_{i\alpha}$) be the set of points corresponding to K_α and $\tilde{K}$ (resp. $\tilde{K}_i$). We denote also by $\lambda_\alpha : \tilde{Z}_\alpha \to \mathbf{C}$ the analytic function induced by λ.

Let $T_\alpha \subset \tilde{Z}_\alpha$ be the analytic subset of points at which ψ_α is not finite. Then, the set $\tilde{Y}_\alpha = \psi_\alpha(\tilde{K}_\alpha \cap T)$ is compact subanalytic and by Lemma 6.3, $\dim_{\mathbf{C}} \tilde{Y}_\alpha \leqq 2n-2$. Moreover, since the fibres of φ_α are of complex dimension smaller than n, $\dim_{\mathbf{C}} (\tilde{Y}_\alpha \cap p_\alpha^{-1}(x)) < n-1$ for any $x \in W_\alpha$. Also $Y_{i\alpha} = \psi_{i_\alpha}(K_{i_\alpha})$, $i = 0,\, 1$, satisfy similar properties. Therefore we have

LEMMA 6.6. – *The set* $Y_\alpha = \tilde{Y}_\alpha \cup Y_{0_\alpha} \cup Y_{1_\alpha}$ *is compact and subanalytic and* $\dim_{\mathbf{C}} Y_\alpha \leqq 2n-2$. *Also* $\dim_{\mathbf{C}} (Y_\alpha \cap (\{\alpha\} \times \mathbf{C}^{n-1})) < n-1$ *for all* $x \in W_\alpha$.

The set Y_α plays a similar role as in the proof of Theorem 5.5. By construction it satisfies the following property.

LEMMA 6.7. – *For every* $(x, \eta) \in W_\alpha \setminus Y_\alpha$ *there is a neighbourhood* $U_{x,\eta}$ *of* (x, η) *in* W_α *and* $\tilde{U}_{x,\eta}$ *of* $\tilde{K}_\alpha \cap \psi_\alpha^{-1}(x, \eta)$ *such that* $\tilde{U}_{x,\eta} \to U_{x,\eta}$ *is a finite analytic covering possibly branched over the exceptional divisor* E_α *of* s_α. *Furthermore, the zero set of* λ_α *intersected with* $\tilde{U}_{x,\eta}$ *lies over* E_α.

Step 6. – For given $\xi \in \mathbf{B}^{n-1}$, we put $(W_\alpha)_\xi = \{x \in K_\alpha;\, (x, \xi) \in Y_\alpha\}$. By complex analog of Lemma 5.9, for generic $\xi_1, \ldots, \xi_{n+1}$

$$\bigcup_{i=1}^{n+1} (W_\alpha)_{\xi_i} = \varnothing.$$

Moreover, by (3) of Lemma 6.2, $\xi_1, \ldots, \xi_{n+1}$ can be taken generic from $(\mathbf{R}^{n-1})^{n+1}$.

Step 7. – From Lemma 6.7 and Step 6 we show in the same way as in the real case that generic $\{\xi_1, \ldots, \xi_{n+1}\}$ form the set of weakly regular projection for the condition (*b*) of Definition 6.4 and for such germs of curves q in $\mathbf{C}^n$ which, for some α, have a lifting q_α to W_α such that $q_\alpha(0) \in K_\alpha$ and q_α is not entirely contained in E_α.

Step 8. – In some sense in the complex case condition (c) follows from condition (b) (*see* [M], Proposition 4.1). Namely, take $U_{x,\eta}$, $\tilde{U}_{x,\eta}$ as in Lemma 6.7. Let $\lambda : U_{x,\eta} \to \mathbf{C}$ be a multivalued function induced by $\tilde{\lambda}_\alpha$ on $\tilde{U}_{x,\eta}$. Let $q(t)$ be a germ of analytic curve and $q(0) = x$. Then, by Puiseux's theorem, for some positive integer s, $\lambda(q(t^s), \eta)$ induces analytic functions $\lambda_i(t, \eta)$ which are nonzero for $t \neq 0$. Hence $D_\eta \lambda_i/\lambda_i$ are locally bounded. Therefore, by the curve selection lemma, $D_\eta \lambda/\lambda$ (defined outside E_α) is bounded near (x, η). This shows tthe statement for condition (c).

Step 9. – Similarly to the real analytic case the proof extends to the case of all germs of curves. □

7. Subanalytic functions

In this section we show Proposition 3.1. The main tool in the proof is the product formula for subanalytic function (Theorem 7.5 below) which can be considered as a subanalytic version of the Weierstrass preparation theorem.

Consider subanalytic functions defined on subsets of $\mathbf{R}^n$. Since there are several different notions of subanalytic function (*see* [DŁS], [K1], [P4]), we give a precise definition. We follow mainly the notation of [K1].

DEFINITION 7.1. – 1. Let $U \subset \mathbf{R}^n$. We call a function $f : U \to \mathbf{R}$ *subanalytic* [and write $f \in SUB(\mathbf{R}^n)$] if the closure of the graph Γ_f of f is a subanalytic subset of $\mathbf{R}^n \times \mathbf{R}$;

2. We say that such $f \in SSUB(\mathbf{R}^n)$ if $\overline{\Gamma_f}$ is subanalytic in $\mathbf{R}^n \times \mathbf{RP}(1)$ [where $\mathbf{RP}(1) = \mathbf{R} \cup \infty$];

3. Let $f, g \in SUB(\mathbf{R}^n)$. We say that $f = g$ *almost everywhere on* $V \subset \mathbf{R}^n$ if there is a nowhere dense subanalytic subset $X \subset \mathbf{R}^n$ such that: f and g are defined on $V \setminus X$ and $f_{|V \setminus X} \equiv g_{|V \setminus X}$.

Note that we do not require f to be continuous. Likewise the Weierstrass preparation theorem does not hold for any semi-analytic function but only for the analytic ones our product formula for subanalytic functions holds for locally blow-analytic functions.

DEFINITION 7.2. – Let U be a subset of $\mathbf{R}^n$. We call a subanalytic function $f : U \to \mathbf{R}$ *locally blow-analytic in* $\mathbf{R}^n$ *if* $f \in SSUB(\mathbf{R}^n)$ and there exist a locally finite collection of real analytic morphisms $\sigma_\alpha : U_\alpha \to \mathbf{R}^n$ and subanalytic compacts $K_\alpha \subseteq U_\alpha$ such that:

(1) each U_α is isomorphic to $\mathbf{R}^n$ and $\bigcup \sigma_\alpha(K_\alpha) = \overline{U}$;

(2) each σ_α is the composition of finitely many local blowings-up with smooth nowhere dense centres and $f \circ \sigma_\alpha$ extends to a normal crossings on U_α (or is identically equal to zero).

Remark. – By the resolution of singularities ([H2] or [BM1]) we get the same class of functions if we require in (2) only that $f \circ \sigma_\alpha$ extends to an analytic function on U_α. This is the reason why we call such f locally blow-analytic. The class of locally blow-analytic functions does not coincide with the class of blow-analytic functions of Kuo [Ku]. Clearly each locally subanalytic function $f : U \to \mathbf{R}$ is locally bounded in $\overline{U}$. By [BM2] it is also arc-analytic and continuous in U.

In some sense locally blow-analytic functions generate $SSUB(\mathbf{R}^n)$. The following proposition, which is a combination of rectilinearization and Puiseux's theorem, follows from [P4], Theorem 2.7.

PROPOSITION 7.3. – *Let $f : U \to \mathbf{R}$ be a continuous subanalytic function such that $f \in SSUB(\mathbf{R}^n)$. Then, there exist a locally finite collection of real analytic morphisms $\phi_\alpha : W_\alpha \to \mathbf{R}^n$ and subanalytic compacts $K_\alpha \subset W_\alpha$ such that:*

(1) *each $W_\alpha \simeq \mathbf{R}^n$ and $\bigcup \phi_\alpha(K_\alpha) = \overline{U}$;*

(2) *for each α there exist $r_i \in \mathbf{N}$, $i = 1, 2, \dots, n$, such that $\phi_\alpha = \sigma_\alpha \circ \psi_\alpha$, where $\sigma_\alpha : U_\alpha \to \mathbf{R}^n$ is the composition of finitely many local blowings-up with smooth nowhere dense centres, U_α is isomorphic to $\mathbf{R}^n$ and in some systems of coordinates on W_α and U_α, $\psi_\alpha : W_\alpha \to U_\alpha$ is given by*

$$\psi_\alpha(x) = (x_1^{r_1}, \dots, x_n^{r_n}); \tag{7.1}$$

(3) *$f \circ \phi_\alpha$ extends to either a normal crossings on $W_\alpha \simeq \mathbf{R}^n$ or the inverse of a normal crossing (or is identically equal to zero).*

In the following corollary of Proposition 7.3 we express each subanalytic function in terms of locally blow-analytic functions. In general both below equations (7.2) and (7.3) are valid almost everywhere on some compact subanalytic sets L_α by which we mean outside a subanalytic subset of L_α that is of dimension smaller than n.

COROLLARY 7.4. – *Let $f \in SSUB(\mathbf{R}^n)$ and let $U \subset \mathbf{R}^n$ be the domain of f. Then, there exists a locally finite family $\{L_\alpha\}$ of compact subanalytic subsets of $\mathbf{R}^n$ such that $\bigcup L_\alpha = \overline{U}$ and:*

(i) *For each α, one may find bounded locally blow-analytic functions f_i, integers r_i, q_i such that $r_i > 0$ and $q_i \geq 0$, and an analytic function g nowhere vanishing on the closure of the image $(f_1^{1/r_1}, \dots, f_n^{1/r_n})(L_\alpha)$ and such that almost everywhere on L_α either $f \equiv 0$ or*

$$f \text{ or } f^{-1} = g(f_1^{1/r_1}, \dots, f_n^{1/r_n})\, f_1^{q_1/r_1} \dots f_n^{q_n/r_n}; \tag{7.2}$$

(ii) *Almost everywhere on each L_α (and so almost everywhere on every compact subset of $\overline{U}$) f or f^{-1} satisfies a unitary equation*

$$f^N + \sum g_i f_i^{N-i} \equiv 0 \tag{7.3}$$

with locally blow-analytic coefficients g_i.

Proof. – (i) follows from Proposition 7.3 by taking $L_\alpha = \phi_\alpha(K_\alpha)$ and $f_i = x_i^{r_i} \circ \phi_\alpha^{-1}$.

(ii) follows from (i), since for each analytic h, $h(x_1^{1/r_1}, \dots, x_n^{1/r_n})$ satisfies (locally) a similar equation with coefficients analytic in $(x_1, \dots, x_n)$. □

Let $\pi : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the standard projection. For $x \in \mathbf{R}^n$ we write $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. The standard projection of $\mathbf{R}^n$ onto the last factor we denote by π_n.

The following theorem in crucial in our proof of Proposition 3.1. It can be understood as an analog of the Weierstrass preparation theorem. Likewise Corollary 7.4, Theorem 7.5 is valid almost everywhere but unlike the ordinary Weierstrass preparation theorem it holds for any choice of coordinates in $\mathbf{R}^n$.

THEOREM 7.5. – *Let $U \subseteq \mathbf{R}^n$ and let $f : U \to \mathbf{R}$ be locally blow-analytic. We assume also that U is relatively compact in $\mathbf{R}^n$. Then there exists a finite family of compact subanalytic subsets $L_\alpha \subseteq \overline{U}$ such that $\bigcup L_\alpha = \overline{U}$ and almost everywhere on each L_α*

$$f(x) = g(f_1(x), \dots, f_n(x)) F(x), \tag{7.4}$$

where: f_i and F are bounded subanalytic functions defined almost everywhere on $\pi(L_\alpha)$; they are almost everywhere the finite products of functions of one of the forms:

$$u(x'),\ (x_n - \varphi(x')) \text{ or their reciprocals } (u(x'))^{-1},\ (x_n - \varphi(x'))^{-1} \tag{7.5}$$

where u, φ are bounded subanalytic functions on (a dense subset of) $\pi(L_\alpha)$; and g is analytic and nowhere vanishing in a neighbourhood of $\overline{(f_1, \dots, f_n)(L_\alpha)}$.

Moreover, there exists a morphism of complex analytic spaces $\rho : Z \to \mathbf{C}^n$ and a compact subanalytic subset $K \subseteq Z$ independent on the choice of coordinates in $\mathbf{R}^n$ and such that:

(i) $\dim_{\mathbf{C}} Z = n - 1$;

(ii) *for each φ from above there is a bounded subanalytic functions ψ such that the graph of $\varphi \pm \sqrt{-1}\,\psi$ is contained in $\rho(K)$.*

Proof. The begining. – It suffices to consider only such f that there exists a composition of local blowings-up with smooth nowhere dense centres $\sigma : \hat{U} \to \mathbf{R}^n$ such that $f \circ \sigma$ is analytic and the domain of f equals $\sigma(L)$ for a compact $L \subseteq \hat{U}$. Since the problem is also local on L we will work in a neighbourhood of $p \in L$.

Let $\sigma = \sigma_1 \circ \dots \circ \sigma_k$, where $\sigma_i : U_i \to U_{i-1}$ ($U_k = \hat{U}$ and $U_0 = \mathbf{R}^n$) are local blowings-up and $p_i \in U_i$ the subsequent images of $p = p_k$. We can assume that near each p_i, σ_{i+1} is a blowing up of a finitely generated ideal $I_i = (g_{i1}, \dots, g_{ik_i})$. Let $\mathcal{G}_i = \{g_{ij}\}$, $i = 0, \dots, k-1$, and for technical reason (*see* Step 3 of the proof) we add to $\mathcal{G}_i$ also the differences of g_{ij}'s Put $\mathcal{G}_k = \{f \circ \sigma\}$ and $g_{k0} = f \circ \sigma$.

Consider induced complexifications

$$\tilde{\sigma} : \quad \tilde{U}_k \xrightarrow{\tilde{\sigma}_k} \tilde{U}_{k-1} \to \dots \to \tilde{U}_1 \xrightarrow{\tilde{\sigma}_1} \tilde{U}_0 = \mathbf{C}^n \tag{7.6}$$

and denote by Z_{ij} the zero sets of a complexifications of $g_{ij} \in \mathcal{G}_i$.

Let Z be a formal union of all Z_{ij}'s and let $\rho : Z \to \mathbf{C}^n$ be induced by (7.6). Let $\rho' = \pi \circ \rho : Z \to \mathbf{C}^{n-1}$ and $\rho_n = \pi_n \circ \rho : Z \to \mathbf{C}$.

Structure of Proof. – The proof is divided into several steps:

(1) Using the local flattening theorem (Proposition 4.2) we can "modify" $\mathbf{C}^{n-1}$ to get ρ' finite. Then $\rho(Z_{ij})$ are contained in the zero sets of some unitary polynomials P_{ij} with respect to x_n.

(2) We replace σ by compositions of some different mappings making the ideals (P_{ij}) (and consequently I_i) invertible. Each of these mappings is the composition of a very general change of the first $(n-1)$-coordinates and in the last step a very restrictive global

blowing-up involving all the coodinates. In this construction we shall use a rectilinearization procedure given by Bierstone and Milman in the proof of Theorem 4.4 of [BM1].

(3) Using the previous steps we show inductively on i that each g_{ij} (so finally $f \circ \sigma$) satisfies the statement of the theorem.

Step 1. – Since ρ' is a complexification of a real analytic map (Z is a complexification of a real analytic space), by Proposition 4.2, there is a finite family of analytic mappings (compositions of local blowings-up with smooth nowhere dense centres) $\tau'_\alpha : V'_\alpha \to \mathbf{R}^{n-1}$ and subanalytic compacts $K'_\alpha \subset V'_\alpha$ such that:

(i) $V'_\alpha \simeq \mathbf{R}^{n-1}$ and $\bigcup_\alpha \tau'_\alpha (K'_\alpha)$ is a neighbourhood of $\pi(p_0)$ in $\mathbf{R}^{n-1}$;

(ii) there exist complexifications $\tilde{\tau}'_\alpha : \tilde{V}'_\alpha \to \mathbf{C}^{n-1}$ of τ'_α such that the strict transforms $Z'_\alpha \to \tilde{V}'_\alpha$ of ρ' by $\tilde{\tau}'_\alpha$ are finite maps at the points corresponding to $\{p_i\}_{i=0,\dots,k}$.

Fix $\tau' = \tau'_\alpha$, $V' = V'_\alpha$ and $K' = K'_\alpha$. Let $\tau : V = V' \times \mathbf{R} \to \mathbf{R}^n$ be given by $\tau(y, x_n) = (\tau'(y), x_n)$ and let $\tilde{\tau} : \tilde{V} = \tilde{V}' \times \mathbf{C} \to \mathbf{C}^n$ the corresponding complexification of τ.

We claim that there is a neighbourhood of the set of points in $Z \times_{\mathbf{C}^n} \tilde{V}$ corresponding to $\{p_i\}$ and K' whose image in $\tilde{V}$ is contained in a proper analytic hypersurface $\hat{Z}$ of $\tilde{V}$. In fact, by (ii)

$$Z \times_{\mathbf{C}^{n-1}} \tilde{V}' = Z' \cup Z'',$$

where Z' is the strict transform and Z'' lies over the union of exceptional divisors of $\tilde{\tau}'$. Now the claim follows from the fact that the following map is an isomorphism

$$Z \times_{\mathbf{C}^n} \tilde{V} \ni (z, (v', \rho_n(z)) \leftrightarrow (z, v') \in Z \times_{\mathbf{C}^{n-1}} \tilde{V}'.$$

By the above, $\hat{Z} \subset \tilde{V}$ can be chosen as the zero set of an analytic function $h(x) = u(x') P(x)$, where u depends only on $x' \in \tilde{V}'$ and P is an unitary polynomial in x_n. Denote by $h_{ij}(x) = u_{ij}(x') P_{ij}(x)$ the factors of h corresponding to Z_{ij}. Note that all these functions can be chosen as complexifications of real analytic ones.

Step 2. – We transform $h(x) = u(x') P(x)$ (and so each u_{ij}, P_{ij}) to a normal crossings. Since P is a unitary polynomial in x_n

$$P(x) = \prod_{i=1}^{r} (x_n - \varphi_i(x')) \prod_{j=1}^{s} ((x_n - \phi_j(x'))^2 + \psi_j^2(x')),$$

where $\varphi(x')$, $\phi(x')$, $\psi(x')$ are bounded and subanalytic. Using Proposition 7.3 we transform them to normal crossings. Thus, we can assume that

$$h(x) = u(x') \prod_{i=1}^{r} (x_n - a_i(x')) \prod_{j=1}^{s} ((x_n - b_j(x'))^2 + c_j^2(x')),$$

where u, a_i, b_j, c_j are simultaneously normal crossings. For technical reason we make also all their differences normal crossings. This is all what we can be achieved by transforming only the first $(n-1)$ coordinates. Next we blow up in all the coordinates.

LEMMA 7.6 (Bierstone & Milman [BM1], Lemma 4. 7, Sussman [S], (6.VI)). – *Let $\alpha, \beta, \gamma \in \mathbf{N}^n$ and let $a(x)$, $b(x)$, $c(x)$ be invertible elements of $\mathbf{R}\{x\}$. If*

$$a(x) x^\alpha - b(x) x^\beta = c(x) x^\gamma,$$

then either $\alpha \leqq \beta$ or $\beta \leqq \alpha$ (in the lexicographic order).

In fact, then, up to permutation of α, β, γ, $\alpha = \beta \leqq \gamma$.

If f, g are analytic in U, we write $f \sim g$ if f equals g times a factor invertible on U.

LEMMA 7.7. – *Let $V \subseteq \mathbf{R}^n$ be an open subset and let $h : V \to \mathbf{R}$ be an analytic function on V such that*

$$h(x) \sim x'^{\delta} \prod_{i=1}^{r} (x_n - a_i(x')) \prod_{j=1}^{s} ((x_n - b_j(x'))^2 + c_j^2(x')),$$

where a_i, b_j, c_j are analytic and all of them and their differences are normal crossings (or identically equal to zero).

Then, there exist a composition of global blowings-up with smooth nowhere dense centres $\sigma : W \to V$ and a finite covering of W by open sets W_λ such that:

(a) for each λ there exist an open embedding $W_\lambda \subseteq \mathbf{R}^n$ such that the restrictions of coordinate functions to W_λ are the products of the original coordinate functions $x_1 \circ \sigma, \ldots, x_n \circ \sigma$, $(x_n - a_i(x')) \circ \sigma$, $(x_n - b_j(x')) \circ \sigma$ and their reciprocals;

(b) For every λ, the restriction of $h \circ \sigma$ to W_λ is normal crossings in these coordinates.

Proof. – We follow closely the proof of Case 2 of Theorem 4.4 of [BM1].

After a coordinate transformation $y_k = x_k$, $k = 1, \ldots, n-1$ and $y_n = x_n - a_1(x')$ [or $y_n = x_n - b_1(x)'$ if $r = 0$] we can assume that $a_1 \equiv 0$ ($b_1 \equiv 0$ resp.).

By Lemma 7.6, the exponents α^i, β^j, γ^j of nonzero $a_i(x') \sim x'^{\alpha^i}$, $b_j(x') \sim x'^{\beta^j}$, $c_j(x') \sim x'^{\gamma^j}$ are totally ordered (in the lexicographic order). Let ξ be the smallest of them. We show the lemma by induction on $(r+s, |\xi|)$. For simplicity we assume that $r > 0$ (if $r = 0$ the inductive step is similar).

If some $c_{j_0}(0) \neq 0$, then c_{j_0} as a normal crossing nowhere vanishes and $(x_n - b_{j_0}(x'))^2 + c_{j_0}^2(x')$ in invertible. Therefore we may assume that all $c_j(0) = 0$.

We may also assume that all $a_i(0) = b_j(0) = 0$. In fact, assume that this is not the case and let d equals one of $a_i(0)$, $b_j(0)$. Let $I_d = \{i;\ a_i(0) = d\}$ and $J_d = \{j;\ b_j(0) = d\}$. On $V_d = U \setminus \{(x', x_n);\ x_n = a_i(x')$ for some $i \notin I_d$ or $x_n = b_j(x')$ for some $j \notin J_d\}$

$$h \sim (x')^{\delta} \prod_{i \in I_d} (x_n - a_i(x')) \prod_{j \in J_d} ((x_n - b_j(x'))^2 + c_j^2(x')).$$

Since $\#I_d + \#J_d < r + s$ we may apply the inductive assumption. It follows from the construction below that all the centres of blowings-up over V_d lies in fact over $Y_d = \bigcup_{i \in I_d} \{x;\ x_n = a_i(x')\} \cup \bigcup_{j \in J_d} \{x;\ x_n = b_j(x')\}$. Note that if two of $\{a_i, b_j\}$ have different values at the origin, then, since their difference has normal crossings, they differ everywhere. Therefore, for $d \neq d'$, $Y_d \cap Y_{d'} = \varnothing$ and the blowings-up of various V_d glue to one global blowing-up.

We may also assume that not all a_i and b_j are identically equal to zero (if they are, we are done). Choose (arbitrarily) $k = 1, \ldots, n-1$ such that $\xi_k \neq 0$. Let $\sigma : V' \to V$ be the blowing-up of $\{x_k = x_n = 0\}$. Then V' can be covered by two coordinate charts V'_n and V'_k such that the restrictions $\sigma_n = \sigma_{|V'_n}$ and $\sigma_k = \sigma_{|V'_k}$ are given in coordinates $y_1, \ldots, y_n$ on V'_n (V'_k resp.) by:

$$\sigma_n = \begin{cases} x_k = y_k \, y_n \\ x_i \ = y_i \end{cases} \quad \text{if} \quad i \neq k$$

$$\sigma_k = \begin{cases} x_n = y_k \, y_n \\ x_i \ = y_i \end{cases} \quad \text{if} \quad i \neq n.$$

In particular

$$h \circ \sigma_n (y) \sim y_n^m (y')^\xi \prod_{i=1}^r (1 - \overline{a}_i (y)) \prod_{j=1}^s ((1 - \overline{b}_j (y'))^2 + \overline{c}_j^2 (y')),$$

where $m = \xi_k + r + 2\,s$. Since $\overline{a}_i$, $\overline{b}_j$, $\overline{c}_j$ vanish identically on $V'_n \setminus V'_k$, $g \circ \sigma_k$ has normal crossing in a neighbourhood of $V'_n \setminus V'_k$.

On V'_k

$$h \circ \sigma_k (y) \sim (y')^\delta (y'_k)^{r+2\,s} \prod_{i=1}^r (y_n - \tilde{a}_i (y')) \prod_{j=1}^s ((y_n - \tilde{b}_j (y'))^2 + \tilde{c}_j^2 (y')),$$

where, as is easy to check, all nonzero $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_j$ and their differences are normal crossings. Assume that they all vanish at the origin. Then, the exponents $\tilde{\alpha}^i$ of $\tilde{a}_i$ satisfy $\tilde{\alpha}^i_m = \alpha^i_m$ for $m \neq k$ and $\tilde{\alpha}^i_k = \alpha^i_k - 1$ (the same holds for β^j and γ^j). Therefore, the smallest exponent $\tilde{\xi}$ satisfies $|\tilde{\xi}| < |\xi|$ and the statement follows from the inductive assumption. □

Step 3. – Let $\mu_\lambda : W_\lambda \to V_\alpha \xrightarrow{\tau_\alpha} \mathbf{R}^n$ be the compositions of the maps constructed in the Steps 1 and 2 and let $\tilde{\mu}_\lambda : \tilde{W}_\lambda \to \mathbf{C}^n$ be their complexifications. We consider W_λ as a subset of $\mathbf{R}^n$. By the diagram

$$\begin{array}{ccccc} Z & \leftarrow & Z \times_{\mathbf{C}^n} \tilde{V}_\alpha & \leftarrow & Z \times_{\mathbf{C}^n} \tilde{W}_\lambda \\ \downarrow^{\rho} & & \downarrow & & \downarrow^{\rho_\lambda} \\ \mathbf{C}^n & \leftarrow & \tilde{V}_\alpha & \leftarrow & \tilde{W}_\lambda . \end{array}$$

(7.7) the image by ρ_λ of some neighbourhood of the points of $Z \times_{\mathbf{C}^n} W_\lambda$ corresponding to $\{p_i\}_{i=0,\dots,k}$ is contained in the union of coordinate hyperplanes. Now we factor μ_λ through the original blowing-up $\sigma : U_k \to \mathbf{R}^n$.

LEMMA 7.8. – *For each $i = 0, \dots, k$ there are $W_{\lambda i} \subseteq W_\lambda$ and subanalytic compacts $K_{\lambda i} \subseteq W_{\lambda i}$ such that*:

(1) *There is an analytic map $\mu_{\lambda i} : W_{\lambda i} \to U_i$ such that $\sigma_1 \circ \dots \circ \sigma_i \circ \mu_{\lambda i} \equiv \mu_{\lambda | W_{\lambda i}}$.*

(2) *$\bigcup_\lambda \mu_{\lambda i} (K_{\lambda i})$ is a neighbourhood of p_i.*

(3) *for each $g_{ij} \in \mathcal{G}_i$, $g_{ij} \circ \mu_{\lambda i}$ have normal crossings.*

Proof. Induction on i.

Case $i = 0$. – The family μ_λ satisfies (1) and (2). Shrinking W_λ and taking appropriate $K_{\lambda 0}$ we get (3) from (7.7), since the inverse image by $\tilde{\mu}_\lambda$ of a small neighbourhood of p_0 in Z_{0j} is contained in the union of coordinate hyperplanes.

Inductive step. – Assume the lemma holds for i. Then the induced generators of $\mu^*_{\lambda i} (I_i)$ and their differences are normal crossings. Thus, by Lemma 7.6, $\mu^*_{\lambda i} (I_i)$ is invertible and consequently $\mu_{\lambda i}$ factors through the blowing-up of I_i. Shrinking $W_{\lambda i}$ we can factor it through U_{i+1}. For such $\mu_{\lambda i+1} : W_{\lambda i+1} \to U_{i+1}$ we argue like in Case $i = 0$. This ends the proof of the Lemma.

Since all the coordinate functions on W_λ are of the form (7.5), Lemma 7.8 for $i = k$ shows the theorem. □

Proof of Proposition 3.1. Induction on k. – Since both sides of (3.1) are continuous on U it suffices to show (3.1) for x from a dense subset of U. Let $\mathbf{A} \subset SSUB(\mathbf{R}^k)$ be the subset of all such $f \in SSUB(\mathbf{R}^k)$ that:

(i) the domain U of f is bounded;

(ii) There is a stratification of a big closed ball $\mathbf{B} \subset \mathbf{R}^k$ satisfying the statement of Proposition 3.1 (*see* the remark after Proposition 3.1).

To show Proposition 3.1 it remains to prove that every $f \in SSUB(\mathbf{R}^k)$ belongs to $\mathbf{A}$.

We start by listing some properties of $\mathbf{A}$. For stressing the domain U of a function f we often write (f, U) instead of f.

LEMMA 7.9. – (1) *if* $(f, U) \in \mathbf{A}$ *and* $V \subseteq U$, *then* $(f, V) \in \mathbf{A}$;

(2) *if* $(f, U) \in \mathbf{A}$ *and* $(g, U) \in \mathbf{A}$, *then* $(fg, U) \in \mathbf{A}$;

(3) *if* $f \in \mathbf{A}$, *then* $1/f \in \mathbf{A}$;

(4) *if* $f \in \mathbf{A}$ *and r is a positive integer, then* $f^{1/r} \in \mathbf{A}$ *(for r even we assume f to be nonnegative);*

(5) *if* $f(x) = g(a_1(x), a_2(x), \ldots, a_s(x))$, *where* $a_i \in \mathbf{A}$ *are bounded,* $|g| > \varepsilon$ *and* Dg *is bounded, then* $f \in \mathbf{A}$;

(6) *if* $U = \bigcup_{i=1}^{k} U_i$ *and* $(f|_{U_i}, U_i) \in \mathbf{A}$ *for each i, then* $(f, U) \in \mathbf{A}$;

(7) *Let* $f(x) = x_k - \varphi(x') \in SSUB(\mathbf{R}^k)$ *with bounded domain, where* φ *is a bounded subanalytic function with bounded derivatives. Then* $f \in \mathbf{A}$.

Proof. – (1) is obvious. (2) follows from the Leibniz rule. Similarly we show (3) and (4). We prove (5). Let $a(x) = (a_1(x), a_2(x), \ldots, a_s(x))$. Then

$$|Df(x)v(x)| = |Dg(a(x))Da(x)v(x)| \leqq CL|a(x)| \leqq C'L|f(x)|.$$

(6) follows from the existence of a stratification of $\mathbf{R}^k$ which refines given finitely many stratifications of $\mathbf{R}^k$.

To show (7) we take a stratification $\mathcal{S}$ of $\mathbf{B}$ compatible with the graph Γ_φ of φ. Then each $\mathcal{S}$-compatible vector field v on $\mathbf{B}$ is tangent to Γ_φ. It means, that if we write v in the coordinates as $v = (v', v_k)$, then $v_k(x', \varphi(x')) = D\varphi(x')v'(x', \varphi(x'))$. Therefore, if v is Lipschitz with constant L, then

$$\begin{aligned}|Df(x)v(x)| &= |v_k(x) - D\varphi(x')v'(x)| \\ &\leqq |v_k(x) - v_k(x', \varphi(x'))| + |D\varphi(x')(v'(x) \\ &\quad - v'(x', \varphi(x')))| \leqq CL|x_k - \varphi(x')|.\end{aligned}$$

It is interesting to note that we are unable to prove directly that: $f \in \mathbf{A}$ and $g \in \mathbf{A}$ follow $f + g \in \mathbf{A}$.

Remark. – In [P1] (Lemmas 4.4 and 4.5) we showed Proposition 3.1 for semi-analytic functions. The proof was based on the following observation ([P1], Key Lemma) which we do not use in this paper: if f satisfies an equation $f^s + \sum_{i=1}^{s} a_i f^{s-i} \equiv 0$ and $a_i \in \mathbf{A}$, then $f \in \mathbf{A}$.

By (i) of Corollary 7.4 and Lemma 7.9 it suffices to show the statement for locally blow-analytic functions.

Assume that f is locally blow-analytic. We apply to f Theorem 7.5 (with $n = k$) and let L_α, $\rho : Z \to C^k$ and $K \subset Z$ be given by the statement. By Theorem 6.5 there exists a finite set of weakly regular projections $\mathcal{P} \in (\mathbf{R}^{k-1})^{k+1}$ for $\rho(K)$.

Fix $\pi \in \mathcal{P}$ and choose a system of coordinates in which $\pi : \mathbf{C}^k \to \mathbf{C}^{k-1}$ is the standard projection. Then $X = \rho(Z) \cap \pi^{-1}(\mathbf{R}^{k-1})$ is the closure of the union of the graphs of complex-valued functions φ_β. Fix $C > 0$ and let $V'_C \subseteq \mathbf{R}^{k-1}$ be the closure of the set of those x' that $|D\alpha_\beta(x')| \leqq \mathbf{C}$ for all β. Let $V_C = \pi^{-1}(V'_C) \cap L_\alpha$. We claim that $(f, V_C) \in \mathbf{A}$. Indeed, f is of the form (7.4), and by Lemma 7.9 it is enough to show that all function from (7.5) belong to $\mathbf{A}$. For $u(x')$ it follows from the inductive hypothesis and for $x_k - \varphi(x')$ by Lemma 7.9.

Since $\mathcal{P}$ is a set of weakly regular projections, if C sufficiently big, the V_C's constructed for all projections defined by $\mathcal{P}$, cover the domain of f. Then, (6) of Lemma 7.9 gives the result. □

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Adam Parusiński,
School of Mathematics and Statistics,
University of Sydney,
Sydney, NSW 2006, Australia.