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GEOMETRY OF 2-STEP NILPOTENT GROUPS WITH A LEFT INVARIANT METRIC

BY PATRICK EBERLEIN

ABSTRACT. – We consider properties of closed geodesics in a compact nilmanifold $\Gamma\backslash N$, where N is a simply connected 2-step nilpotent Lie group with a left invariant metric and Γ is a discrete cocompact subgroup of N. Among other results we show 1) There is an obstruction (resonance) to the density in $T_1(\Gamma\backslash N)$ of the set of vectors P that are periodic with respect to the geodesic flow. In particular P is not always dense in $T_1(\Gamma\backslash N)$, but P is dense in $T_1(\Gamma\backslash N)$ for any Γ if N is of Heisenberg type. 2) Every free homotopy class of closed curves in $\Gamma\backslash N$ contains a closed geodesic of largest period. Define the maximal length spectrum of $\Gamma\backslash N$ to be the collection with multiplicities of these largest periods. If $\Gamma\backslash N$, $\Gamma^*\backslash N^*$ are compact 2–step nilmanifolds with the same marked maximal length spectrum, then we show that $\Gamma\backslash N$, $\Gamma^*\backslash N^*$ are equivalent up to isometry and Γ-almost inner automorphism in the sense of Gordon and Wilson.

Introduction

Nilpotent Lie groups play an important role in many areas of mathematics, and 2-step nilpotent groups have a special significance. They are the nonabelian Lie groups that come as close as possible to being abelian, but they admit interesting phenomena that do not arise in abelian groups. In this paper we study the differential geometry of simply connected, 2-step, nilpotent Lie groups N with a left invariant Riemannian metric $\langle\, ,\rangle$. We are especially interested in those geometric properties of $\{N, \langle\, ,\rangle\}$ that do not depend on the choice of $\langle\, ,\rangle$. One would expect to find some properties that are similar to those in flat Euclidean space, which in this context one may regard as a simply connected, abelian Lie group of translations with a canonical left invariant metric. Such properties do exist, but other geometric properties of $\{N, \langle\, ,\rangle\}$ are foreign to Euclidean geometry. For example, J. Wolf in [Wo1] proved that any nonabelian nilpotent Lie group with a left invariant metric must admit both positive and negative sectional curvatures, and J. Milnor in [M] extended this result to Ricci curvatures. More generally, Milnor showed in [M] that the geometry of any Lie group G with a left invariant metric reflects strongly the algebraic structure of the Lie algebra $\mathcal{G}$. Many of the results of this paper illustrate that principle.

The geometry of simply connected nilpotent Lie groups N with a left invariant metric is also relevant to the study of simply connected homogeneous spaces $\tilde{M}$ whose sectional

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curvatures are bounded above by a negative constant. In this context the groups N arise as groups of isometries of $\tilde{M}$ that fix a point x in the boundary sphere $\tilde{M}(\infty)$ and act simply transitively on each horosphere at x. Each horosphere H at x with the induced metric from $\tilde{M}$ is isometric to N with an appropriate left invariant metric $\langle\, ,\rangle$ which depends on H. If $\tilde{M}$ is symmetric, then N has 2-steps and $\{N, \langle\, ,\rangle\}$ is a manifold of Heisenberg type [*see* below and also in (1.6)]. For a discussion of homogeneous manifolds of negative sectional curvature see [Hei].

The literature does not seem to contain much discussion of the geometry of nilpotent Lie groups with a left invariant metric. For this reason we include in the first two sections some basic geometric facts that are probably known to individual researchers but have not been written down. In the last three sections of the paper we consider the behavior of geodesics in a simply connected, 2-step nilpotent Lie group N with a left invariant metric. In particular, we study aspects of the behavior of closed geodesics in a compact quotient manifold $\Gamma\backslash N$, where Γ is a discrete cocompact subgroup of N that acts on N by left transformations. We present the main results below but avoid a more detailed discussion of the organization of the paper.

To study the geometry of 2-step nilpotent groups with a left invariant metric we adopt the approach of A. Kaplan used in [K1], and we now describe its main features. Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\, ,\rangle$. Let N be the unique, simply connected, 2-step nilpotent Lie group whose Lie algebra is $\mathcal{N}$, and equip N with the left invariant metric determined by the inner product $\langle\, ,\rangle$ on $\mathcal{N} = T_e N$. Let $\mathcal{Z}$ denote the center of $\mathcal{N}$, and let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$. Each element Z of $\mathcal{Z}$ defines a skew symmetric linear map $j(Z) : \mathcal{V} \to \mathcal{V}$ given by $j(Z)X = (\mathrm{ad}\, X)^*(Z)$ for all $X \in \mathcal{V}$, where $(\mathrm{ad}\, X)^*$ is the adjoint of ad X relative to the inner product $\langle\, ,\rangle$. Equivalently and more usefully $j(Z)$ is defined by the equation $\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$ for all $X, Y \in \mathcal{V}$.

Conversely, for each pair of positive integers m, n and each linear map $j : \mathbb{R}^n \to \mathrm{so}(m)$ we obtain a metric, 2-step nilpotent Lie algebra $\mathcal{N} = \mathbb{R}^n \oplus \mathbb{R}^m$ (orthogonal direct sum), where $\mathcal{Z} = \mathbb{R}^n$ is the center of $\mathcal{N}$ and the Lie bracket on $\mathcal{V} = \mathbb{R}^m$ is defined by the equation above. *See* (1.5).

All of the basic geometry of $\{N, \langle\, ,\rangle\}$ can be described by the maps $\{j(Z) : Z \in \mathcal{Z}\}$ as we show in section 2. This was first made clear by A. Kaplan in [K1], who used the maps $j(Z)$ to study the geometry of groups of **Heisenberg type**, those groups $\{N, \langle\, ,\rangle\}$ for which $j(Z)^2 = -|Z|^2\, \mathrm{Id}$ for every Z $\in \mathcal{Z}$. Many of his proofs are valid without change in the general 2-step nilpotent case, and others require only small modifications. The spaces $\{N, \langle\, ,\rangle\}$ of Heisenberg type should be regarded as the model spaces in the class of simply connected, 2-step nilpotent Lie groups with a left invariant metric. In this class they play a role that is similar to the role played by the Riemannian symmetric spaces in the class of all Riemannian manifolds. Groups of Heisenberg type have especially large isometry groups, and the geodesic symmetries at each point preserve the Riemannian volume form [K2]. Moreover, every unit speed geodesic in a group N of Heisenberg type lies in at least one 3-dimensional totally geodesic submanifold of N. This property also characterizes groups of Heisenberg type in the class of simply connected, 2-step nilpotent Lie groups with a left invariant metric [E1]. If $\tilde{M}$ is a symmetric space of strictly negative sectional curvature, then 2-step nilpotent groups N of Heisenberg type arise as groups of isometries of $\tilde{M}$ that act simply transitively on horospheres of $\tilde{M}$. In particular, if $G = \mathrm{KAN}$ is an Iwasawa decomposition of $G = I_0(\tilde{M})$, then the group N with a natural left invariant metric is a group of Heisenberg type [Ko]. *See* also the examples following (1.3).

We now describe the main results of the paper. Let N denote a simply connected, 2-step, nilpotent group with a left invariant metric, and let Γ denote a lattice in N; that is, Γ is a discrete subgroup of N such that the quotient manifold $\Gamma\backslash N$ is compact, where Γ acts on $\mathcal{N}$ by left translations. Let $\{g^t\}$ denote the geodesic flow in the unit tangent bundles SN or $S(\Gamma\backslash N)$.

1. First integrals for $\{g^t\}$. – In Corollary (3.3) we show that $\{g^t\}$ admits a smooth $\mathcal{Z}$-valued first integral f; that is, there exists a smooth function $f : SN \to \mathcal{Z}$ such that $f(g^t v) = f(v)$ for all $v \in SN$ and for all $t \in \mathbb{R}$. More precisely, let $\xi \in T_n N$ be any vector and write $\xi = dL_n(X_0 + Z_0)$, where L_n is the left translation by n and X_0, Z_0 are uniquely determined vectors in $T_e N = \mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ such that $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. Then $f(\xi) = Z_0$ defines a first integral for $\{g^t\}$ on TN and hence also on SN by restriction. The first integral f is clearly invariant under $\{dL_n : n \in N\}$, and hence it descends to a $\mathcal{Z}$-valued first integral for $\{g^t\}$ on $S(\Gamma\backslash N)$ for any discrete subgroup Γ of N. In particular the geodesic flow in $S(\Gamma\backslash N)$ does not have a dense orbit since the first integral f is nonconstant.

The first integral f is reminiscent of the canonical $\mathbb{R}^n$-valued first integral of the geodesic flow in flat Euclidean space $\mathbb{R}^n$. In fact, the subspace $\mathcal{Z}$ of $\mathcal{N} = T_e N$ defines an integrable left invariant distribution $\mathcal{Z}$ in N whose maximal integral manifolds are flat, totally geodesic imbedded submanifolds, the orbits of the center $Z = \exp(\mathcal{Z})$ of N, where Z acts by left translations.

2. Density of periodic vectors in $S(\Gamma\backslash N)$. – Let Γ be a lattice in N. A unit vector $v \in S(\Gamma\backslash N)$ is **periodic** relative to the geodesic flow $\{g^t\}$ on $S(\Gamma\backslash N)$ if $g^\omega v = v$ for some $\omega > 0$; that is, v is tangent to a closed geodesic of $\Gamma\backslash N$. For any flow $\{\phi^t\}$ on a space X it is a basic problem to determine if the periodic vectors for the flow are dense in X. We show that this is not always the case for $\{g^t\}$ on $S(\Gamma\backslash N)$.

The density of periodic vectors for $\{g^t\}$ on $S(\Gamma\backslash N)$ turns out to be related to a property of the skew symmetric linear maps $\{j(Z) : Z \in \mathcal{Z}\}$ which we call resonance. Given $Z \in \mathcal{Z}$ a map $j(Z) : \mathcal{V} \to \mathcal{V}$ is said to be in **resonance** if the ratio of any two nonzero eigenvalues of $j(Z)$ is a rational number. Note that this ratio is always a real number since the eigenvalues of $j(Z)$ are purely imaginary. If N is of Heisenberg type, then every map $j(Z)$, $Z \in \mathcal{Z}$, is in resonance since the condition $j(Z)^2 = -|Z|^2 \,\mathrm{Id}$ implies that $j(Z)$ has eigenvalues $\pm i|Z|$. In (5.6) and (5.7) we prove the following two results. The first of these has recently been generalized in [Ma].

1. Let N be a simply connected, 2-step nilpotent Lie group of Heisenberg type, and let Γ be any lattice in N. Then the periodic vectors for the geodesic flow $\{g^t\}$ in $S(\Gamma\backslash N)$ are dense in $S(\Gamma\backslash N)$.

2. Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric and a 1-dimensional center. Then the following properties are equivalent.

a) The linear map $j(Z) : \mathcal{V} \to \mathcal{V}$ is in resonance for every $Z \in \mathcal{Z}$.

b) For some lattice Γ in $\mathcal{N}$ the periodic vectors for $\{g^t\}$ in $S(\Gamma\backslash N)$ are dense in $S(\Gamma\backslash N)$.

c) For every lattice Γ in N the periodic vectors for $\{g^t\}$ in $S(\Gamma\backslash N)$ are dense in $S(\Gamma\backslash N)$.

We do not know if the hypothesis that N have a 1-dimensional center can be removed. Note that checking the hypothesis *a*) in 2) reduces to checking it in a single case for any nonzero vector Z of $\mathcal{Z}$.

In (5.8) we construct a lattice Γ in a 5-dimensional simply connected, 2-step nilpotent group N with 1-dimensional center such that none of the maps $j(Z)$, $Z \in \mathcal{Z}$, are in resonance. It follows from 2) above that the periodic vectors in $S(\Gamma\backslash N)$ are not dense in $S(\Gamma\backslash N)$, and the same is true for any lattice in N.

3. THE ASSOCIATED FLAT TORI T_B AND T_F. – If N is a simply connected nilpotent Lie group with Lie algebra $\mathcal{N}$, then the exponential map $\exp : \mathcal{N} \to N$ is a diffeomorphism. Let $\log : N \to \mathcal{N}$ denote the inverse of exp. Let Γ be a lattice in a simply connected, 2-step, nilpotent Lie group N with Lie algebra $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$, and let $\pi_{\mathcal{V}} : \mathcal{N} \to \mathcal{V}$ denote the projection map. It is elementary to show that $\pi_{\mathcal{V}} \log \Gamma$ and $\log \Gamma \cap \mathcal{Z}$ are vector lattices in $\mathcal{V}$ and $\mathcal{Z}$ respectively. Define flat tori $T_B = \mathcal{V}/(\pi_{\mathcal{V}} \log \Gamma)$ and $T_F = \mathcal{Z}/(\log \Gamma \cap \mathcal{Z})$. In (5.5) we show that there exists a Riemannian submersion of $\Gamma\backslash N$ onto T_B whose fibers are imbedded, flat, totally geodesic tori isometric to T_F. These fibers are also the orbits in $\Gamma\backslash N$ of $I_0(\Gamma\backslash N)$, which acts freely on $\Gamma\backslash N$. The (closed geodesic) length spectra of T_B and T_F are closely related to the length spectrum of $\Gamma\backslash N$ and in fact determine the length spectrum of $\Gamma\backslash N$ if N is of Heisenberg type (5.17). However, the isometry classes of T_B and T_F do not determine the isomorphism class of the fundamental group Γ of $\Gamma\backslash N$ (5.23).

We note that R. Palais and T. Stewart in [PS] showed that the compact 2-step nilmanifolds are precisely the total spaces of principal torus bundles over a torus.

4. LENGTH SPECTRUM OF $\Gamma\backslash N$. – Let Γ be a lattice in N, and let $\mathcal{C}$ denote a free homotopy class of closed curves in $\Gamma\backslash N$. Let $l(\mathcal{C})$ denote the collection of lengths of closed geodesics of $\Gamma\backslash N$ that belong to $\mathcal{C}$. The **length spectrum** of $\Gamma\backslash N$ is the collection of all ordered pairs (L, m), where L is the length of a closed geodesic in $\Gamma\backslash N$ and m is the multiplicity of L, *i. e.* the number of free homotopy classes $\mathcal{C}$ for which $L \in l(\mathcal{C})$. Compact nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are said to have the same **marked length spectrum** if there exists an isomorphism ϕ of Γ onto Γ^* such that $l(\phi_* \mathcal{C}) = l(\mathcal{C})$ for all free homotopy classes $\mathcal{C}$ in $\Gamma\backslash N$, where ϕ_* denotes the bijection induced by ϕ between free homotopy classes of closed curves in $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$.

The set $l(\mathcal{C})$ contains in general more than one number for each free homotopy class $\mathcal{C}$ [(4.8) and (4.11)]. However, $l(\mathcal{C})$ always contains a largest number $l^*(\mathcal{C})$, which is explicitly computable (4.5). The **maximal length spectrum of** $\Gamma\backslash N$ is the collection of all ordered pairs (L, m), where $L = l^*(\mathcal{C})\}$ for some free homotopy class $\mathcal{C}$ of closed curves in $\Gamma\backslash N$ and m is the number of free homotopy classes $\mathcal{C}$ for which $L = l^*(\mathcal{C})$. Compact nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are said to have the same **marked maximal length spectrum** if there exists an isomorphism ϕ of Γ onto Γ^* such that $l^*(\phi_* \mathcal{C}) = l^*(\mathcal{C})$ for all free homotopy classes $\mathcal{C}$ in $\Gamma\backslash N$.

The closed geodesics with maximal length $l^*(\mathcal{C})$ in a free homotopy class $\mathcal{C}$ of $\Gamma\backslash N$ have both geometric and dynamical significance. If γ is a closed geodesic in $\mathcal{C}$ with length $l^*(\mathcal{C})$, then γ is the projection of a geodesic in N of the form $t \to n \cdot \exp(t\,\xi)$ for some $\xi \in \mathcal{N}$. The geodesics of $\mathcal{N}$ are rarely left translates of 1-parameter subgroups of N (*see* (3.9)). To explain the dynamical significance of $l^*(\mathcal{C})$ we define for each number ω in $l(\mathcal{C})$ a set $SN_\omega(\mathcal{C})$ consisting of those unit vectors in $S(\Gamma\backslash N)$ that are tangent to a closed geodesic of length ω that belongs to $\mathcal{C}$. Clearly each set $SN_\omega(\mathcal{C})$ is invariant under the geodesic flow $\{g^t\}$. If $\omega^* = l^*(\mathcal{C})$ and if ω is any number in $l(\mathcal{C})$ distinct from ω^*, then we show in (4.18) that the dimension of $SN_{\omega^*}(\mathcal{C})$ is strictly smaller than the dimension of $SN_\omega(\mathcal{C})$. Moreover, in (4.17) we show that $SN_{\omega^*}(\mathcal{C})$ is a smooth submanifold of $S(\Gamma\backslash N)$, but we do not know if this is true for $SN_\omega(\mathcal{C})$ with $\omega \neq \omega^*$.

In (5.20) we show that there are essentially only two ways that compact, 2-step nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ can have the same marked maximal length spectrum (Note that if $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum then they have the same marked maximal length spectrum.)

1. There exists an isomorphism ψ of N onto N^* such that ψ is also an isometry and $\psi(\Gamma) = \Gamma^*$.

2. $N = N^*$ and $\Gamma^* = \psi(\Gamma)$, wherre ψ is a Γ-almost inner automorphism of N; that is, for each element γ of the lattice Γ there is an element a of N, possibly depending on γ, such that $\psi(\gamma) = a \cdot \gamma \cdot a^{-1}$.

The importance of Γ-almost inner automorphisms of N was discovered by C. Gordon and E. Wilson, who proved in [GW1] that if ψ is a Γ-almost inner automorphism of N for some lattice Γ of N, then $\Gamma\backslash N$ and $\psi(\Gamma)\backslash N$ have the same spectrum of the Laplacian on functions but are not in general isometric. Later work (*cf.* [G1] and [DG1]) showed that if ψ is a Γ-almost inner automorphism of N, then $\Gamma\backslash N$ and $\psi(\Gamma)\backslash N$ have the same marked length spectrum and the same Laplacian spectrum on functions and differential forms. It follows from these facts and (5.20) that if $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are compact, 2-step nilmanifolds with the same marked maximal length spectrum, then they also have the same marked length spectum and the same Laplacian spectrum on functions and differential forms. Moreover, (5.20) shows that if $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are compact, 2-step nilmanifolds with the same marked maximal length spectrum, then the associated sets of tori $\{T_B, T_F\}$ and $\{T_{B^*}, T_{F^*}\}$ are pairwise isometric (Corollary 5.22).

The relationship between the (unmarked) maximal length spectrum and the marked maximal length spectum is somewhat mysterious. In (5.23) we construct two examples that illustrate the problem:

1. There exist homeomorphic compact, 2-step nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ that have the same maximal length spectra but do not have the same marked maximal length spectra for any choice of isomorphism ϕ of Γ onto Γ^*. In this case the associated tori T_B and T_{B^*} have the same length spectrum but are not isometric.

2. There exist compact, 2-step nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ with the same maximal length spectra such that Γ is not isomorphic to Γ^*. Hence their marked maximal length spectra are *a priori* different. This also shows that the maximal length spectrum does not

determine the isomorphism class of the fundamental group Γ of $\Gamma \backslash N$. However, in this example the associated sets of tori $\{T_B, T_F\}$ and $\{T_{B^*}, T_{F^*}\}$ are pairwise isometric.

In a compact, 2-step nilmanifold $\Gamma \backslash N$ the relationship between the length spectrum and the spectrum of the Laplacian acting on functions or differential forms is unclear, even in the case that N is of Heisenberg type. This relationship deserves further study. It is also unclear to what extent the results (if not the methods) of this paper generalize to simply connected nilpotent Lie groups with a left invariant metric and an arbitrary number of steps.

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1. Definitions and examples

Let $\mathcal{N}$ denote a finite dimensional Lie algebra over the real numbers. For each integer $i \geqq 1$ we define $\mathcal{N}^i = [\mathcal{N}, \mathcal{N}^{i-1}]$, where $\mathcal{N}^0 = \mathcal{N}$. The Lie algebra $\mathcal{N}$ is **nilpotent** if $\mathcal{N}^i = \{0\}$ for some positive integer i. A nilpotent Lie algebra $\mathcal{N}$ has a nontrivial center that contains $\mathcal{N}^{i-1}$ if $\mathcal{N}^i = \{0\}$. A nilpotent Lie algebra $\mathcal{N}$ is **k-step** if $\mathcal{N}^k = \{0\}$ but $\mathcal{N}^{k-1} \neq 0$.

Let N denote the unique simply connected nilpotent Lie group corresponding to a given nilpotent Lie algebra $\mathcal{N}$. Let $\exp : \mathcal{N} \to N$ denote the Lie group exponential map. It is known that exp is a diffeomorphism [R], p. 6. We let $\log : N \to \mathcal{N}$ denote the inverse of exp.

2-step nilpotent groups and algebras.

We are primarily interested in the case that N and $\mathcal{N}$ are 2-step nilpotent. In this case the Campbell-Baker-Hausdorff formula (*cf.* [Hel], p. 96) yields the following simple expression for the multiplication law in N.

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y]\right) \tag{1.1}$$

for arbitrary elements X, Y of $\mathcal{N}$.

From the expression above we obtain

(1.2) Let ϕ, ψ be arbitrary elements of N, and write $\phi = \exp(X)$, $\psi = \exp(Y)$ for suitable elements X, Y of $\mathcal{N}$. Then

a) $\phi\psi\phi^{-1} = \exp(Y + [X, Y])$

b) $[\phi, \psi] = \phi\psi\phi^{-1}\psi^{-1} = \exp([X, Y])$

c) ϕ commutes with ψ if and only if $[X, Y] = 0$

d) $\log(\phi \cdot \psi) = \log\phi + \log\psi + \frac{1}{2}[\log\phi, \log\psi]$.

The following description of the differential of the Lie group exponential map $\exp : \mathcal{N} \to N$ will be useful.

(1.3) LEMMA. – *Let $\mathcal{N}$ denote a 2-step nilpotent Lie algebra, and let N denote the simply connected 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$. Let* $\exp : \mathcal{N} \to N$ *denote the exponential map. Then for any elements ξ, A of $\mathcal{N}$ we have* $d\exp_\xi : T_\xi \mathcal{N} \to T_{\exp(\xi)} N$ *is given by*

$$d\exp_\xi(A_\xi) = dL_{\exp(\xi)}\left(A + \frac{1}{2}[A, \xi]\right)$$

where A_ξ denotes the initial velocity of the curve $t \to \xi + tA$ and $L_{\exp(\xi)}$ denotes left translation by $\exp(\xi)$.

Proof. – By definition $d\exp_\xi(A_\xi)$ is the initial velocity of the curve $t \to \exp(\xi + tA)$. By (1.1) we see that $dL_{\exp(\xi)}\left(A + \frac{1}{2}[A, \xi]\right)$ is the initial velocity of the curve $t \to \exp(\xi) \cdot \exp\left(t\left\{A + \frac{1}{2}[A, \xi]\right\}\right) = \exp(\xi + tA)$. $\square$

Examples.

It is easy to construct 2-step nilpotent Lie algebras. Let $\mathcal{V}$, $\mathcal{Z}$ be any finite dimensional real vector spaces with bases $\{V_1, \dots, V_n\}$ for $\mathcal{V}$ and $\{Z_1, \dots, Z_m\}$ for $\mathcal{Z}$. Let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ and define a bracket operation in $\mathcal{N}$ by

$$[V_i, V_j] = \sum_{\alpha=1}^{m} C_{ij}^{\alpha} \cdot Z_\alpha$$

where the constants $\{C_{ij}^\alpha\}$ are chosen so that $C_{ij}^\alpha = -C_{ji}^\alpha$ for $1 \leqq i, j \leqq n$, $1 \leqq \alpha \leqq m$, but not all of the constants are zero. Define $[Z_\alpha, \zeta] = 0$ for all $\zeta \in \mathcal{N}$, $1 \leqq \alpha \leqq m$. The Jacobi identity is automatically satisfied since $[\mathcal{N}, \mathcal{N}] \subseteqq \mathcal{Z}$, and $\mathcal{Z}$ lies in the center of $\mathcal{N}$.

We construct more explicit examples (*cf.* [K2], p. 39).

Example 1. – Heisenberg algebras

Let $n \geqq 1$ be any integer and let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be any basis of $\mathbb{R}^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be a 1-dimensional vector space spanned by an element Z. Define $[X_i, Y_i] = -[Y_i, X_i] = Z$ for $1 \leqq i \leqq n$ with all other brackets zero. The Lie algebra $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ is the $(2n+1)$-dimensional **Heisenberg algebra.**

Remark. – Let $\mathbb{C}H^n$ denote the complex hyperbolic space of real dimension $2n$. The normalized sectional curvatures $K(\Pi)$ satisfy $-4 \leqq K(\Pi) \leqq -1$. Let $G = I_0(\mathbb{C}H^n)$ and let $G = KAN$ be an Iwasawa decomposition of G. The group N is the $(2n+1)$-dimensional Heisenberg group, the simply connected, $(2n+1)$ -dimensional nilpotent Lie group whose Lie algebra is the $(2n+1)$-dimensional Heisenberg algebra. Geometrically, AN acts transitively on $\mathbb{C}H^n$ and fixes a unique point x in the boundary sphere $\mathbb{C}H^n(\infty)$. The group N acts simply transitively on each horosphere at x (*see* section 6 of [Ka] and [E2]).

Example 2. – Quaternionic Heisenberg algebras.

Let $n \geqq 1$ be any integer. For each integer i with $1 \leqq i \leqq n$ let $\mathbb{H}^i$ denote a 4-dimensional real vector space with basis $\{X_i, Y_i, V_i, W_i\}$. Let $\mathcal{V} = \oplus \mathbb{H}^i$. Let $\mathcal{Z}$ be a 3-dimensional real vector space with basis $\{Z_1, Z_2, Z_3\}$, and let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$. We define a bracket operation in $\mathcal{N}$ as follows:

$[Z_j, \xi] = 0$ for $1 \leqq j \leqq 3$ and all $\xi \in \mathcal{N}$

$$[X_i, Y_i] = Z_1, \quad [X_i, V_i] = Z_2, \quad [X_i, W_i] = Z_3, \qquad \text{for } 1 \leqq i \leqq n$$

$$[Y_i, X_i] = -Z_1, \quad [Y_i, V_i] = Z_3, \quad [Y_i, W_i] = -Z_2, \qquad \text{for } 1 \leqq i \leqq n$$

$$[V_i, X_i] = -Z_2, \quad [V_i, Y_i] = -Z_3, \quad [V_i, W_i] = Z_1, \qquad \text{for } 1 \leqq i \leqq n$$

$$[W_i, X_i] = -Z_3, \quad [W_i, Y_i] = Z_2, \quad [W_i, V_i] = -Z_1, \qquad \text{for } 1 \leqq i \leqq n$$

all other brackets are zero

The resulting Lie algebra $\mathcal{N}$ is the **quaternionic Heisenberg algebra** of dimension $4n + 3$.

Remark. – Let N denote the simply connected $(4n+3)$-dimensional nilpotent Lie group whose Lie algebra is the $(4n+3)$-dimensional quaternionic Heisenberg Lie algebra. Let $\mathbb{H}H^n$ denote the quaternionic hyperbolic space of real dimension $4n$. The remark of the previous example now applies to N if one replaces $\mathbb{C}H^n$ by $\mathbb{H}H^n$.

(1.4) DEFINITION. – *A 2-step nilpotent Lie algebra $\mathcal{N}$ is* **nonsingular** *if ad $X : \mathcal{N} \to \mathcal{Z}$ is surjective for all $X \in \mathcal{N} - \mathcal{Z}$.*

Here ad $X(Y) = [X, Y]$ for all $X, Y \in \mathcal{N}$. The Heisenberg and quaternionic Heisenberg algebras are nonsingular for any positive integer n. The nonsingular 2-step nilpotent Lie algebras $\mathcal{N}$ form an important class of 2-step nilpotent Lie algebras, and in general one can say much more than in the general case about the geometry of the corresponding group N equipped with a left invariant metric.

Metric examples. – We now assume that our 2-step nilpotent Lie algebra $\mathcal{N}$ is equipped with a positive definite inner product $\langle\,,\,\rangle$. Let $\mathcal{Z}$ denote the center of $\mathcal{N}$, and let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$ relative to $\langle\,,\,\rangle$. For each element Z in $\mathcal{Z}$ we define a skew symmetric linear transformation $j(Z) : \mathcal{V} \to \mathcal{V}$ by

$$j(Z)X = (\text{ad } X)^* Z \quad \text{for all } X \in \mathcal{V}$$

where $(\text{ad } X)^*$ denotes the adjoint of ad X. Equivalently one has the following more useful characterization:

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle \quad \text{for all } X, Y \in \mathcal{V}, \text{ all } Z \in \mathcal{Z} \tag{1.5}$$

The transformations $\{j(Z) : Z \in \mathcal{Z}\}$ capture all of the geometry of N equipped with the left invariant metric determined by $\langle\,,\,\rangle$. The notation $j(Z)$ was apparently first introduced by A. Kaplan in [K1] to study 2-step nilpotent groups N of Heisenberg type.

Given a pair of positive integers m, n, each linear map $j : \mathbb{R}^n \to \mathrm{so}(m)$ determines a metric, 2-step nilpotent Lie algebra $\mathcal{N}$. Define $\mathcal{N}$ to be the orthogonal direct sum $\mathcal{N} = \mathbb{R}^n \oplus \mathbb{R}^m$, where each factor has the standard metric. Then equip $\mathcal{N}$ with the Lie bracket determined by (1.5), where $\mathbb{R}^n = \mathcal{Z}$ and $\mathbb{R}^m = \mathcal{V}$.

(1.6) DEFINITION. – *A 2-step metric nilpotent Lie algebra* $\{\mathcal{N}, \langle\, ,\,\rangle\}$ *is of* **Heisenberg type** *if*

$$j(Z)^2 = -|Z|^2\,\mathrm{Id} \quad \text{on } \mathcal{V}$$

for every choice of $Z \in \mathcal{Z}$. A simply connected 2-step nilpotent Lie group $\{N, \langle\, ,\,\rangle\}$ with a left invariant metric is of Heisenberg type if its Lie algebra $\{\mathcal{N}, \langle\, ,\,\rangle\}$ is of Heisenberg type.

If $\{\mathcal{N}, \langle\, ,\,\rangle\}$ is of Heisenberg type, then from the definitions we immediately obtain the following facts.

$$(1.7)\quad \begin{cases} a) & \langle j(Z)X, j(Z^*)X\rangle = \langle Z, Z^*\rangle\,|X|^2 \quad \text{for all } Z, Z^* \in \mathcal{Z},\ \text{all } X \in \mathcal{V} \\ b) & \langle j(Z)X, j(Z)Y\rangle = |Z|^2\langle X, Y\rangle \quad \text{for all } Z \in \mathcal{Z},\ \text{all } X, Y \in \mathcal{V} \\ c) & |j(Z)X| = |X|\,|Z| \quad \text{for all } Z \in \mathcal{Z},\ \text{all } X \in \mathcal{V} \\ d) & j(Z)\circ j(Z^*) + j(Z^*)\circ j(Z) = -2\langle Z, Z^*\rangle\,\mathrm{Id} \quad \text{for all } Z, Z^* \in \mathcal{Z}. \end{cases}$$

The 2-step nilpotent groups $\{N, \langle\, ,\,\rangle\}$ of Heisenberg type may be regarded as the model spaces for the class of 2-step nilpotent groups $\{N, \langle\, ,\,\rangle\}$ with a left invariant metric. The groups of Heisenberg type have especially large groups of isometries and have a special status analogous to that of the Riemannian symmetric spaces in the class of Riemannian manifolds. *See* [K1,2] as well as the discussion below in Example 4 of (2.11).

The Heisenberg and quaternionic Heisenberg algebras equipped with a natural inner product become 2-step nilpotent Lie algebras of Heisenberg type. In general, for any positive integer m there exist infinitely many nonisometric Lie algebras $\{\mathcal{N}, \langle\, ,\,\rangle\}$ of Heisenberg type whose centers $\mathcal{Z}$ have dimension m. *See* [K2], p. 36.

We now define natural inner products on the Heisenberg and quaternionic Heisenberg Lie algebras and describe the maps $\{j(Z) : Z \in \mathcal{Z}\}$ in each case.

Example 1. – Let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ be the Heisenberg algebra of dimension $2n+1$, where as above $\mathcal{V} = \mathrm{span}\{X_1, Y_1, \dots, X_n, Y_n\}$ and $\mathcal{Z} = \mathrm{span}\{Z\}$. Identify $\mathcal{V}$ with $\mathbb{C}^n$ as follows: if $z_j = \alpha_j + \sqrt{-1}\,\beta_j \in \mathbb{C}$, where $\alpha_j, \beta_j \in \mathbb{R}$, then identify $(z_1, \dots, z_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n}\{\alpha_j X_j + \beta_j Y_j\} \in \mathcal{V}$. Give $\mathcal{N}$ the inner product such that the vectors $\{X_i, Y_j, Z : 1 \leqq i, j \leqq n\}$ form an orthonormal basis. With these identifications it follows that for any real number a,

$$j(aZ)(z_1, \dots, z_n) = (a\sqrt{-1})(z_1, \dots, z_n) = (a\sqrt{-1}\,z_1, \dots, a\sqrt{-1}\,z_n).$$

Clearly $j(aZ)^2 = -a^2\,\mathrm{Id} = -|aZ|^2\,\mathrm{Id}$.

Example 2. – Let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ be the quaternionic Heisenberg algebra of dimension $4n+3$, and let $\{X_i, Y_i, V_i, W_i : 1 \leqq i \leqq n\}$ and $\{Z_1, Z_2, Z_3\}$ be the bases of $\mathcal{V}$, $\mathcal{Z}$ defined above. Give $\mathcal{N}$ the inner product such that these basis vectors form an orthonormal basis for $\mathcal{N}$.

Generalizing the previous example, we show that the action on $\mathcal{V}$ of a map $j(Z)$, $Z \in \mathcal{Z}$, corresponds to left quaternion multiplication on $\mathbb{H}^n \approx \mathcal{V}$ by a purely imaginary quaternion ([K2], p. 39).

Recall that $\mathbb{H}$ is $\mathbb{R}^4$ with a basis $\{1, i, j, k\}$ and noncommutative multiplication given by $ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$ and $i^2 = j^2 = k^2 = -1$. Identify $\mathcal{V}$ with $\mathbb{H}^n$ as follows: if $h_r = \alpha_r + \beta_r i + \gamma_r j + \delta_r k$, $1 \leqq r \leqq n$, then identify $(h_1, \ldots, h_n)$ with $\sum_{r=1}^{n} \{\alpha_r X_r + \beta_r Y_r + \gamma_r V_r + \delta_r W_r\}$. If $Z = \sum_{s=1}^{3} c_s Z_s$ is any element of $\mathcal{Z}$, then it is routine to show that $j(Z)(h_1, \ldots, h_n) = \xi \cdot (h_1, \ldots, h_n) = (\xi h_1, \ldots, \xi h_n)$, where $\xi = c_1 i + c_2 j + c_3 k \in \mathbb{H}$. It follows that $j(Z)^2 = -(c_1^2 + c_2^2 + c_3^2)\,\mathrm{Id} = -|Z|^2\,\mathrm{Id}$ for all $Z \in \mathcal{Z}$.

We conclude this section with a useful characterization of nonsingular 2-step nilpotent Lie algebras.

(1.8) LEMMA. – *Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra. Then $\mathcal{N}$ is nonsingular if and only if for any positive definite inner product $\langle\, , \rangle$ on $\mathcal{N}$ the maps $\{j(Z) : Z \in \mathcal{Z}\}$ are nonsingular on $\mathcal{V} = \mathcal{Z}^{\perp}$ for every nonzero $Z \in \mathcal{Z}$.*

Proof. – If $Z \in \mathcal{Z}$ and $X \in \mathcal{V}$ are any elements, then it follows from the definitions that $j(Z) X = 0$ if and only if Z is orthogonal to ad $X(\mathcal{N}) = [X, \mathcal{N}]$. $\square$

2. Geometry of 2-step nilpotent groups with a left invariant metric

Let $\{\mathcal{N}, \langle\, , \rangle\}$ denote a 2-step nilpotent Lie algebra with a positive definite inner product. Let $\{N, \langle\, , \rangle\}$ denote the simply connected 2-step nilpotent group N with Lie algebra $\mathcal{N}$ and left invariant metric $\langle\, , \rangle$; that is, the left translations L_n, $n \in N$, are isometries of $\{N, \langle\, , \rangle\}$. In this section we derive some basic formulas for the curvature and Ricci tensors of N. We also give examples of complete, totally geodesic submanifolds of N.

In this section and in the sequel we shall sometimes regard the elements X of $\mathcal{N}$ as left invariant vector fields on N determined by their values at the identity e of N.

Covariant derivative and curvature ([CE], p. 64).

If X, Y are elements of $\mathcal{N}$ regarded as left invariant vector fields on N, then the function $n \to \langle X(n), Y(n) \rangle$ is a constant function on N. The formula for the covariant derivative $\nabla_X Y$ of smooth vector fields on a Riemannian manifold normally contains 6 terms (*cf.* [Hel], p. 48), but in this case 3 of them vanish since $\langle X, Y \rangle$ is constant. One obtains

$$\nabla_X Y = \frac{1}{2}\{[X, Y] - (\mathrm{ad}\, X)^*(Y) - (\mathrm{ad}\, Y)^*(X)\} \tag{2.1}$$

where $(\mathrm{ad}\, X)^*$, $(\mathrm{ad}\, Y)^*$ denote the adjoints of ad X, ad Y. We may regard ∇ as a bilinear mapping from $\mathcal{N} \times \mathcal{N}$ into $\mathcal{N}$ since $\nabla_X Y$ is a left invariant vector field if X, Y are left invariant vector fields.

From (2.1) one obtains routinely

$$\begin{cases} a) \quad \nabla_X Y = \dfrac{1}{2}[X, Y] & \text{for all } X, Y \in \mathcal{V} \\ b) \quad \nabla_X Z = \nabla_Z X = -\dfrac{1}{2} j(Z) X & \text{for all } X \in \mathcal{V},\ Z \in \mathcal{Z} \\ c) \quad \nabla_Z Z^* = 0 & \text{for all } Z, Z^* \in \mathcal{Z}. \end{cases} \tag{2.2}$$

Curvature tensor.

If ξ_1, ξ_2, ξ_3 are vector fields on $\mathcal{N}$ then recall that the curvature tensor is given by

$$R(\xi_1, \xi_2)\xi_3 = -\nabla_{[\xi_1, \xi_2]}\xi_3 + \nabla_{\xi_1}(\nabla_{\xi_2}\xi_3) - \nabla_{\xi_2}(\nabla_{\xi_1}\xi_3).$$

If ξ_1, ξ_2, ξ_3 are left invariant vector fields, then $R(\xi_1, \xi_2)\xi_3$ is also left invariant, and we may regard R as a multilinear map from $\mathcal{N} \times \mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$. From (2.2) we obtain

$$(2.3)\quad \begin{cases} a) \quad R(X, Y)X^* = \dfrac{1}{2} j([X, Y])X^* - \dfrac{1}{4} j([Y, X^*])X + \dfrac{1}{4} j([X, X^*])Y \\ \qquad\qquad \text{for all } X, Y, X^* \in \mathcal{V}. \text{ In particular} \\ \quad\;\; R(X, Y)X = \dfrac{3}{4} j([X, Y])X \\ b) \quad R(X, Z)Y = -\dfrac{1}{4}[X, j(Z)Y] \quad \text{for all } X, Y \in \mathcal{V}, \text{ all } Z \in \mathcal{Z} \\ \quad\;\; R(X, Y)Z = -\dfrac{1}{4}[X, j(Z)Y] + \dfrac{1}{4}[Y, j(Z)X] \\ c) \quad R(X, Z)Z^* = -\dfrac{1}{4}\{j(Z) \circ j(Z^*)X\} \\ \quad\;\; R(Z, Z^*)X = -\dfrac{1}{4}\{j(Z^*) \circ j(Z)X\} + \dfrac{1}{4}\{j(Z) \circ j(Z^*)X\} \\ \qquad\qquad \text{for all } X \in \mathcal{V}, \text{ all } Z, Z^* \in \mathcal{Z} \\ d) \quad R(Z_1, Z_2)Z_3 = 0 \quad \text{for all } Z_1, Z_2, Z_3 \in \mathcal{Z} \end{cases}$$

The entire curvature tensor can be computed from these formulas and the Bianchi identities.

Sectional Curvature.

Let $\Pi \subseteqq T_n N$ be a 2-dimensional subspace, and let X, Y be orthonormal elements of $\mathcal{N}$ such that $\operatorname{span}\{X(n), Y(n)\} = \Pi$. The sectional curvature $K(\Pi)$ equals $K(X, Y) = \langle R(X, Y)Y, X\rangle$. From (2.3) we immediately obtain

$$(2.4)\quad \begin{cases} a) \quad \text{If } X, Y \text{ are orthonormal elements of } \mathcal{V}, \text{ then} \\ \qquad\qquad K(X, Y) = -\dfrac{3}{4}|[X, Y]|^2 \\ b) \quad \text{If } X \in \mathcal{V} \text{ and } Z \in \mathcal{Z} \text{ are orthonormal, then} \\ \qquad\qquad K(X, Z) = -\dfrac{1}{4}|j(Z)X|^2 \\ c) \quad \text{If } Z, Z^* \text{ are orthonormal elements of } \mathcal{Z}, \text{ then } K(Z, Z^*) = 0. \end{cases}$$

Ricci tensor.

For arbitrary elements X, Y of $\mathcal{N}$ we recall that the Ricci tensor of N is given by $\operatorname{Ric}(X, Y) = \operatorname{trace}(\xi \to R(\xi, X)Y, \xi \in \mathcal{N})$. Symmetries of the curvature tensor imply that Ric is a symmetric, bilinear form on $\mathcal{N} \times \mathcal{N}$, and hence there exists a symmetric linear transformation $T : \mathcal{N} \to \mathcal{N}$ such that $\mathrm{Ric}(\xi_1, \xi_2) = \langle T\xi_1, \xi_2 \rangle$ for all $\xi_1, \xi_2 \in \mathcal{N}$.

(2.5) PROPOSITION. – *a*) $\mathrm{Ric}(X, Z) = 0$ *for all* $X \in \mathcal{V}$, $Z \in \mathcal{Z}$.

In particular T leaves $\mathcal{V}$ and $\mathcal{Z}$ invariant

b) *If* $\{Z_1, \ldots, Z_m\}$ *is an orthonormal basis of* $\mathcal{Z}$, *then*

$$T|_{\mathcal{V}} = \frac{1}{2} \sum_{k=1}^{m} j(Z_k)^2.$$

In particular, $T|_{\mathcal{V}}$ *is negative definite, and*

$$\mathrm{Ric}(X, X) < 0 \quad \text{for all nonzero } X \in \mathcal{V}.$$

c) $\mathrm{Ric}(Z, Z^*) = -\frac{1}{4}$ *trace* $\{j(Z) \circ j(Z^*)\}$ *for all* $Z, Z^* \in \mathcal{Z}$. *In particular* $T|_{\mathcal{Z}}$ *is positive semidefinite. The kernel of T in* $\mathcal{N} = \{Z \in \mathcal{Z} : j(Z) \equiv 0\} = \{Z \in \mathcal{Z} : Z$ *is orthogonal to* $[\mathcal{N}, \mathcal{N}]\}$.

(2.6) COROLLARY. – (*cf.* [K1], p. 134.) – *Let* $\{N, \langle\, ,\, \rangle\}$ *be a simply connected 2-step nilpotent Lie group with a left invariant metric. Then the left invariant distributions* $\mathcal{V}$ *and* $\mathcal{Z}$ *in* N *are left invariant by every isometry of* N.

Proof of the Corollary. – At any point n of N the distribution $\mathcal{V}$ is the subspace of $T_n N$ spanned by the eigenvectors of the Ricci transformation T corresponding to negative eigenvalues of T. The distribution $\mathcal{Z}$ is similarly described by the nonnegative eigenvalues of T. $\square$

Proof of the Proposition. – Assertions *a*) and *c*) follow routinely from (2.3). We prove *b*). Let X, Y be arbitrary elements of $\mathcal{V}$, and let $\{V_1, \ldots, V_n\}$ and $\{Z_1, \ldots, Z_m\}$ be orthonormal bases of $\mathcal{V}$ and $\mathcal{Z}$ respectively. From (2.3), the skew symmetry of $j(Z_k)$ and the fact that $[V_i, X] = \sum_{k=1}^{m} \langle j(Z_k) V_i, X \rangle Z_k$ we obtain

$$(\star) \quad \sum_{i=1}^{n} \langle R(V_i, X)Y, V_i \rangle = -\frac{3}{4} \sum_{i=1}^{n} \langle j([V_i, X]) V_i, Y \rangle = \frac{3}{4} \sum_{k=1}^{m} \langle j(Z_k)^2 X, Y) \rangle.$$

By (2.3)

$$\sum_{k=1}^{m} \langle R(Z_k, X)Y, Z_k \rangle = -\frac{1}{4} \sum_{k=1}^{m} \langle j(Z_k)^2 X, Y) \rangle$$

and *b*) now follows from $(\star)$. $\square$

Euclidean de Rham factor of N.

The next result explains the geometric significance of the nullity of the Ricci tensor.

(2.7) PROPOSITION. – *Let* $\{N, \langle\, ,\, \rangle\}$ *be a simply connected 2-step nilpotent Lie group with a left invariant metric. If* $\mathcal{N}$ *denotes the Lie algebra of* N *let* $\mathcal{E} = \{Z \in \mathcal{Z} : j(Z) \equiv 0\}$, *and let* $\mathcal{N}^*$ *denote the orthogonal complement of* $\mathcal{E}$ *in* $\mathcal{N}$ *relative to* $\langle\, ,\, \rangle$. *Then*

1) *$\mathcal{E}$ and $\mathcal{N}^*$ are commuting ideals in $\mathcal{N}$, and N is the direct product of the subgroups $N^* = \exp(\mathcal{N}^*)$ and $E = \exp(\mathcal{E})$.*

2) *N is isometric to the Riemannian product of the totally geodesic submanifolds N^*, E, and E is the Euclidean de Rham factor of N.*

Proof. – Assertion 1) follows immediately from the definition of $\mathcal{E}$. We prove 2). Let $\mathcal{E}$, $\mathcal{N}^*$ also denote the left invariant distributions in N determined by the subspaces $\mathcal{E}$, $\mathcal{N}^*$ of $\mathcal{N} = T_e N$. The subgroups $N^* = \exp(\mathcal{N}^*)$ and $E = \exp(\mathcal{E})$ are maximal integral manifolds of $\mathcal{N}^*$, $\mathcal{E}$.

The distributions $\mathcal{N}^*$, $\mathcal{E}$ are not integrable but parallel (i.e. invariant under parallel translation along arbitrary curves). It suffices to verify this for $\mathcal{E}$ since the orthogonal complement of a parallel distribution is also parallel. From the definition of $\mathcal{E}$ and (2.2) we see that $\nabla_{\xi^*} \xi = 0$ if $\xi \in \mathcal{E}$ and $\xi^* \in \mathcal{N}$, and hence $\mathcal{E}$ is parallel. It now follows from the de Rham theorem (see for example Theorem 6.1 of [KN], p. 187) that N is isometric to the Riemannian product $N^* \times E$. The totally geodesic submanifold E is flat by (2.3) and the fact that $\mathcal{E} \subseteq \mathcal{Z}$. If Z is a nonzero element of $\mathcal{N}^* \cap \mathcal{Z}$, then $j(Z) \not\equiv 0$ by the definition of $\mathcal{E}$ and $\mathcal{N}^*$. The subgroup N^* is totally geodesic [cf. (2.9) below], and the Ricci tensor of N^* is nondegenerate by (2.5c). In particular N^* has no Euclidean de Rham factor, and we conclude that E is the Euclidean de Rham factor of N. $\square$

Isometry group of N.

(2.8) PROPOSITION. – *Let $\{N, \langle , \rangle\}$ be a simply connected, nilpotent Lie group with a left invariant metric, and let $I(N)$ denote the isometry group of N. Let $A(N) = I(N) \cap \mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ denotes the automorphism group of N. Let N also denote the subgroup of $I(N)$ consisting of left translations by elements of N. Then N is a normal subgroup of $I(N)$; $N \cap A(N) = \{e\}$ and $I(N) = N \cdot A(N) = A(N) \cdot N$.*

Proof. – See Theorem 4.2 of [Wo2] and Theorem 2 of [Wi]. A simple direct proof of this result for 2-step nilpotent groups of Heisenberg type can be found in [K1], and this proof is valid without change in the general 2-step nilpotent case in view of Corollary (2.6) above. $\square$

Totally geodesic submanifolds and subgroups.

Let $\{N, \langle , \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric. A connected subgroup N^* of N with Lie algebra $\mathcal{N}^* \subseteq \mathcal{N}$ is a totally geodesic submanifold of N if and only if it is totally geodesic at the identity; left translations by elements of N^* are isometries of N that leave N^* invariant. Hence

(2.9) A connected subgroup N^* of N is a totally geodesic submanifold of N if and only if $\nabla_{\xi_1} \xi_2 \in \mathcal{N}^*$ whenever $\xi_1, \xi_2 \in \mathcal{N}^*$, where $\mathcal{N}^* \subseteq \mathcal{N}$ is the Lie algebra of N^*.

(2.10) DEFINITION. – *A Lie algebra $\mathcal{N}^* \subseteq \mathcal{N}$ is totally geodesic if $\nabla_{\xi_1} \xi_2 \in \mathcal{N}^*$ whenever $\xi_1, \xi_2 \in \mathcal{N}^*$.*

(2.11) EXAMPLES OF TOTALLY GEODESIC SUBGROUPS. – A complete, connected totally geodesic submanifold of N need not be a connected, totally geodesic subgroup of N, but a 2-step group $\{N, \langle , \rangle\}$ admits many totally geodesic subgroups. Many of these are flat, which is not too surprising since $\{N, \langle\,,\,\rangle\}$ should be similar in some sense to flat Euclidean space. We list some basic examples.

Example 1. – Let $\xi \in \mathcal{N}$ be arbitrary. The 1-parameter subgroup $\exp(t\,\xi)$ is a geodesic of N if and only if $(\operatorname{ad}\xi)^*\ \xi = 0$ by (2.1). This condition holds if and only if ξ is orthogonal to $[\xi, \mathcal{N}]$. In particular if $\xi \in \mathcal{V}$ or $\xi \in \mathcal{Z}$, then $t \to \exp(t\,\xi)$ is a geodesic of N that starts at the identity. These are the only possibilities for a 1-parameter subgroup to be a geodesic if $\mathcal{N}$ is nonsingular.

Example 2. – Let $\mathcal{N}^*$ be an abelian subspace of $\mathcal{V}$; that is, $[X, Y] = 0$ for all $X, Y \in \mathcal{N}^*$. Then $\mathcal{N}^*$ is a totally geodesic subalgebra in view of (2.1) since $\nabla_X Y = \dfrac{1}{2}[X, Y]$ for all $X, Y \in \mathcal{V}$. Moreover, $K(X, Y) = 0$ for all $X, Y \in \mathcal{N}^*$ by (2.4). Hence $N^* = \exp(\mathcal{N}^*)$ is a complete, flat, totally geodesic subgroup of N that contains the identity.

This example generalizes a part of the first example. Abelian subspaces $\mathcal{N}^*$ of $\mathcal{V}$ of dimension at least 2 arise whenever $\dim\mathcal{V} \geqq 2 + \dim\mathcal{Z}$. In such a case the map $\operatorname{ad}X$: $\mathcal{V} \to \mathcal{Z}$ has a kernel of dimension at least 2 for every nonzero $X \in \mathcal{V}$, and we define $\mathcal{N}^* = \operatorname{span}\{X, X_1\}$, where X_1 is any element of $\ker(\operatorname{ad}X)$ that is not collinear with X. More generally, if $\dim\mathcal{V} \geqq 1 + r + r\dim\mathcal{Z}$, for some integer $r \geqq 2$, then every nonzero element X of $\mathcal{V}$ lies in an abelian subspace $\mathcal{N}^*$ of $\mathcal{V}$ of dimension $r+1$. One may construct $\mathcal{N}^*$ as follows. Define $\mathcal{V}_0 = \ker(\operatorname{ad}X)$, a subspace of $\mathcal{V}$ of dimension $\geqq v - z \geqq 2$, where $v = \dim(\mathcal{V})$ and $z = \dim(\mathcal{Z})$. Let X_1 be any nonzero element of $\mathcal{V}_0$ that is linearly independent from X and define $\mathcal{V}_1 = \mathcal{V}_0 \cap \ker(\operatorname{ad}X_1)$. Continuing in this fashion we let X_j be any nonzero element of $\mathcal{V}_{j-1}$ that is linearly independent from $\{X, X_1, \ldots, X_{j-1}\}$ and define $\mathcal{V}_j = \mathcal{V}_{j-1} \cap \ker(\operatorname{ad}X_j)$. If $1 \leqq j \leqq r$, then this construction is possible since $\dim\mathcal{V}_{j-1} \geqq v - jz \geqq v - rz \geqq r + 1 \geqq j + 1$. Hence X_r exists, and by construction the subspace $\mathcal{N}^* = \operatorname{span}\{X, X_1, \ldots, X_r\}$ is abelian and has dimension $r + 1$.

Example 3. – If Z denotes the center of N, then it follows easily from (2.2) and (2.3) that the orbits of Z in N under left multiplication are complete, flat, totally geodesic submanifolds of N.

If N is of Heisenberg type, then one can find many additional totally geodesic submanifolds of N.

Example 4. – Let $\{N, \langle\,,\,\rangle\}$ be a 2-step nilpotent Lie group of Heisenberg type. Then every unit speed geodesic γ of N is contained in a complete, 3-dimensional totally geodesic subgroup N^* of N. After rescaling the metric of N^* by a positive constant depending on the geodesic γ, the group $\{N, \langle\,,\,\rangle\}$ is isometric to the 3-dimensional Heisenberg group corresponding to the 3-dimensional Heisenberg algebra $\{\mathcal{N}, \langle\,,\,\rangle\}$ constructed above in section 1.

We verify the assertions of the example above. It suffices to consider the case that the geodesic γ satisfies $\gamma(0) = e$, the identity of N. Let $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. We first consider the case that X_0 and Z_0 are both nonzero. Let $\mathcal{N}^* = \operatorname{span}\{X_0, Z_0, j(Z_0)X_0\}$ and let $N^* = \exp(\mathcal{N}^*)$. We assert that $\mathcal{N}^*$ is a 3-dimensional, totally geodesic subalgebra of $\mathcal{N}$, and it will then follow that N^* is a complete, 3-dimensional, totally geodesic subgroup of N that contains γ.

From (1.7) we obtain the following

LEMMA. – *Let N be a 2-step nilpotent group of Heisenberg type. Then*

$$[X, j(Z)X] = |X|^2 Z \quad \text{for all elements } X \in \mathcal{V},\ Z \in \mathcal{Z}.$$

It follows from the lemma that the subspace $\mathcal{N}^*$ defined above is a subalgebra of $\mathcal{N}$. From (2.2) we see that $\mathcal{N}^*$ contains $\nabla_{X_0} Z_0 = \nabla_{Z_0} X_0$, $\nabla_{X_0} j(Z_0) X_0 = -\nabla_{j(Z_0) X_0} X_0$ and $\nabla_{Z_0} j(Z_0) X_0 = \nabla_{j(Z_0) X_0} Z_0$. It follows that $\nabla_{\xi_1} \xi_2 \in \mathcal{N}^*$ for all $\xi_1, \xi_2 \in \mathcal{N}^*$, and hence $\mathcal{N}^*$ is a totally geodesic subalgebra of $\mathcal{N}$, provided that both X_0 and Z_0 are nonzero. If $\gamma'(0) = X_0 + Z_0$, where either X_0 or Z_0 is zero,then $\gamma'(0)$ lies in infinitely many totally geodesic subalgebras of the type $\mathcal{N}^*$ above. The group $N^* = \exp(\mathcal{N}^*)$ is of Heisenberg type since $\mathcal{N}^* \cap \mathcal{V}$ is invariant under $j(Z_0)$. It is easy to see that there is a unique 3-dimensional, 2-step Lie algebra of Heisenberg type up to isometry and multiplication of the metric by a positive constant. This completes the discussion of Example 4.

Remark. – The property of example 4 characterizes groups of Heisenberg type in the class of all 2-step, simply connected nilpotent Lie groups N with a left invariant metric $\langle\, , \rangle$. More precisely we have

(2.12) THEOREM. – *Let $\{N, \langle\, , \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric. Suppose that every unit speed geodesic through the identity e of N is tangent to at least one 3-dimensional totally geodesic submanifold of N that intersects the center Z in a submanifold of positive dimension at e. Then N is of Heisenberg type if one replaces $\langle\, , \rangle$ by $c^2 \langle\, , \rangle$ for a suitable positive constant c.*

We omit the proof, which will appear in [E1].

3. Geodesics

To describe the geodesics of $\{N, \langle\, , \rangle\}$ it suffices to describe those geodesics that begin at the identity of N. Let $\gamma(t)$ be a curve with $\gamma(0) = e$, and let $\gamma'(0) = X_0 + Z_0 \in \mathcal{N}$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. In exponential coordinates we write

$$\gamma(t) = \exp(X(t)) + Z(t), \qquad \text{where} \quad X(t) \in \mathcal{V},\ Z(t) \in \mathcal{Z} \quad \text{for all } t$$

and

$$X'(0) = X_0, \qquad Z'(0) = Z_0$$

(3.1) PROPOSITION. – *The curve $\gamma(t)$ is a geodesic if and only if the following equations are satisfied:*

a) $X''(t) = j(Z_0) X'(t)$ for all $t \in \mathbb{R}$

b) $Z'(t) + \dfrac{1}{2}[X'(t), X(t)] \equiv Z_0$ for all $t \in \mathbb{R}$

Proof. – These equations were derived by A. Kaplan in [K1] to study 2-step nilpotent groups N of Heisenberg type, but the proof is valid without change in the general 2-step nilpotent case. These equations can be completely integrated if N is of Heisenberg type (cf. (3.8) below, [K1, 2] and [Ko]). In the general 2-step case the equations can also be integrated but the answer involves the eigenvalues of $j(Z_0)$ as we show in (3.5).

The next result will show that the geodesic flow in $\{N, \langle , \rangle\}$ has $\dim \mathcal{Z}$ linearly independent first integrals, a fact reminiscent of the geodesic flow in flat Euclidean spaces.

(3.2) PROPOSITION. – *Let $\{N, \langle , \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$. Write $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V} \subseteq \mathcal{N}$, $Z_0 \in \mathcal{Z} \subseteq \mathcal{N}$ and $\mathcal{N} = T_e N$. Then*

$$\gamma'(t) = dL_{\gamma(t)}(e^{tj(Z_0)} X_0 + Z_0) \quad \text{for all } t \in \mathbb{R}$$

where $e^{tj(Z_0)} = \sum_{n=0}^{\infty} (t^n j(Z_0)^n)/n!$

Proof. – Write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$, and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$. Using the result and the notation of (1.3) and the second equation of (3.1) we obtain

$$\begin{aligned}\gamma'(t) &= d\exp_{X(t)+Z(t)}(X'(t) + Z'(t))_{X(t)+Z(t)} \\ &= dL_{\gamma(t)}\left(X' + Z' + \frac{1}{2}[(X' + Z'), (X + Z)]\right) \\ &= dL_{\gamma(t)}\left(X' + Z' + \frac{1}{2}[X', X]\right) = dL_{\gamma(t)}(X' + Z_0).\end{aligned}$$

By integrating the first equation of (3.1) we obtain $X'(t) = e^{tj(Z_0)} X_0$, which completes the proof. □

(3.3) COROLLARY. – *Let $\{g^t\}$ denote the geodesic flow in TN. Let $n \in N$ and $X_0, Z_0 \in \mathcal{N} = T_e N$ be given, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. Then*

$$g^t(dL_n(X_0 + Z_0)) = dL_{\gamma(t)}(e^{tj(Z_0)} X_0 + Z_0),$$

where $\gamma(t)$ is the unique geodesic with $\gamma'(0) = dL_n(X_0 + Z_0)$.

Proof. – Straightforward. □

(3.4) COROLLARY. – *Define $f: TN \to \mathcal{Z}$ by $f(dL_n X) = \Pi_{\mathcal{Z}} X$, where $\Pi_{\mathcal{Z}}: \mathcal{N} \to \mathcal{Z}$ is the projection map and $n \in N$, $X \in \mathcal{N} = T_e N$ are arbitrary. Then $f \circ g^t = f$ for all $t \in \mathbb{R}$. If $\Gamma \subseteq N$ is any discrete subgroup acting on N by left translations, then f induces a function $F: T(\Gamma\backslash N) \to \mathcal{Z}$ such that $F \circ g^t = F$ for all $t \in \mathbb{R}$.*

Proof. – These assertions follow routinely from (3.3). □

By choosing a basis for $\mathcal{Z}$ we can define $m = \dim \mathcal{Z}$ linearly independent first integrals from the $\mathcal{Z}$-valued first integrals f, F above. Alternatively, [Ba] if $\Gamma \subseteq N$ is any discrete subgroup, then each nonzero element Z of $\mathcal{Z}$ defines a Noether first integral (*cf.* [A], pp. 88-91) $h: S(\Gamma\backslash N) \to \mathbb{R}$ as follows: Extend Z to a biinvariant vector field on N and define $H: SN \to \mathbb{R}$ by $H(v) = \langle v, Z(p(v))\rangle$, where $p: SN \to N$ is the projection map. The flow transformations of Z are isometries of N, and hence the restriction of Z to any geodesic γ_v of N is a Jacobi vector field of constant length. It follows that for each $v \in SN$ the function $H(g^t v) = \langle \gamma_v'(t), Z(\gamma_v t)\rangle$ is a bounded affine linear function on $\mathbb{R}$ and must therefore be constant. The function H is invariant under dL_n for all $n \in N$, and hence H induces a first integral $h: S(\Gamma\backslash N) \to \mathbb{R}$ for any discrete subgroup Γ of N.

Integration of the geodesic equations.

We give a solution to the equations (3.1), but the equations obtained are expressed in terms of eigenvalues of the transformation $j(Z_0)$ as well as the initial data X_0, Z_0. These equations simplify if $\{N, \langle\, ,\, \rangle\}$ is nonsingular, especially if $\{N, \langle\, ,\, \rangle\}$ is of Heisenberg type, but do not simplify further in the general case.

Again, it suffices to consider the case that γ is a geodesic of N with $\gamma(0) = e$. We write $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. Let $J : \mathcal{V} \to \mathcal{V}$ denote the skew symmetric transformation $j(Z_0)$, and write $\mathcal{V}$ as a direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ where $\mathcal{V}_1$ is the kernel of J and $\mathcal{V}_2$ is the orthogonal complement of $\mathcal{V}_1$ in $\mathcal{V}$. Note that J leaves $\mathcal{V}_2$ invariant and is nonsingular on $\mathcal{V}_2$.

Let $\{i\,\theta_1, -i\,\theta_1, \ldots, i\,\theta_N, -i\,\theta_N\}$ be the distinct nonzero eigenvalues of J, where $\theta_i > 0$ for all i, and write $\mathcal{V}_2$ as an orthogonal direct sum $\bigoplus_{j=1}^{N} W_j$, where J leaves invariant each W_j and $J^2 = -\theta_j^2\,\mathrm{Id}$ on W_j. We write

$$X_0 = X_1 + X_2, \qquad \text{where} \quad X_j \in \mathcal{V}_1 \quad \text{for } j = 1,\, 2.$$

$$X_2 = \sum_{j=1}^{N} \xi_j, \qquad \text{where} \quad \xi_j \in W_j \quad \text{for } 1 \leqq j \leqq N$$

(3.5) PROPOSITION. – *Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric. Let $\gamma(t)$ be a geodesic with $\gamma(0) = e$. Write $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. Write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$ and $X'(0) = X_0$, $Z'(0) = Z_0$. Then with respect to the notation above we have*

1) $X(t) = t\,X_1 + (e^{tJ} - \mathrm{Id}\,(J^{-1} X_2)$

2) $Z(t) = t\,Z_1 + Z_2(t)$, where

a) $Z_1(t) = Z_0 + \dfrac{1}{2}[X_1, (e^{tJ} + \mathrm{Id})(J^{-1} X_2)] + \dfrac{1}{2}\displaystyle\sum_{j=1}^{N} [J^{-1}\xi_j, \xi_j]$

b) $Z_2(t)$ is a function of uniformly bounded absolute value given by

$$\begin{aligned} Z_2(t) = {} & [X_1, (\mathrm{Id} - e^{tJ})(J^{-2} X_2)] + \frac{1}{2}[e^{tJ} J^{-1} X_2, J^{-1} X_2] \\ & - \frac{1}{2} \sum_{i \neq j = 1}^{N} \{1/(\theta_j^2 - \theta_i^2)\} \{[e^{tJ} J\,\xi_i, e^{tJ} J^{-1}\xi_j] - [e^{tJ}\xi_i, e^{tJ}\xi_j]\} \\ & + \frac{1}{2} \sum_{i \neq j = 1}^{N} \{1/(\theta_j^2 - \theta_i^2)\} \{[J\,\xi_i, J^{-1}\xi_j] - [\xi_i, \xi_j]\} \end{aligned}$$

Proof. – We verify that the expressions for $X(t)$ and $Z(t)$ given above satisfy the equations in (3.1) together with the initial conditions $X(0) = Z(0) = 0$; $X'(0) = X_0$ and $Z'(0) = Z_0$. First we note

a) J commutes with e^{tJ} for all $t \in \mathbb{R}$ and $\dfrac{d}{dt}(e^{tJ}) = J\,e^{tJ} = e^{tJ}\,J$.

b) $J^2 = -\theta_j^2 \,\mathrm{Id}$ and $J = -\theta_j^2 \, J^{-1}$ on W_j for each $1 \leqq j \leqq N$.

It is now straightforward to verify that $X(t)$ satisfies the first equation of (3.1) and the initial conditions $X(0) = 0$ and $X'(0) = X_0$. Moreover it is routine to show

c) $[X'(t), X(t)] = [X_1, (e^{tJ} - \mathrm{Id})(J^{-1} X_2)] + t\,[e^{tJ} X_2, X_1]$
$+[e^{tJ} X_2, (e^{tJ} - \mathrm{Id})(J^{-1} X_2)]$

Next we observe that the derivative of $[e^{tJ} \xi_j, e^{tJ} J^{-1} \xi_j]$ is zero by *a)* and *b)* for all $t \in \mathbb{R}$ and $1 \leqq j \leqq N$. Hence we obtain

d) $[e^{tJ} \xi_j, e^{tJ} J^{-1} \xi_j] \equiv [\xi_j, J^{-1} \xi_j]$ for all $t \in \mathbb{R}$ and $1 \leqq j \leqq N$

From *d)* and the fact that $X_2 = \sum_{j=1}^{N} \xi_j$ we obtain

$$e)\ [e^{tJ} X_2, e^{tJ} J^{-1} X_2] = \sum_{i \neq j=1}^{N} [e^{tJ} \xi_i, e^{tJ} J^{-1} \xi_j] + \sum_{j=1}^{N} [\xi_j, J^{-1} \xi_j]$$

From *a)* and *b)* we obtain

$$Z_2'(t) = -[X_1, e^{tJ} J^{-1} X_2] + \frac{1}{2} [e^{tJ} X_2, J^{-1} X_2] - \frac{1}{2} \sum_{i \neq j=1}^{N} [e^{tJ} \xi_i, e^{tJ} J^{-1} \xi_j]$$

Combining this with e) we obtain

$$f)\ Z_2'(t) = -[X_1, e^{tJ} J^{-1} X_2] - \frac{1}{2} [e^{tJ} X_2, (e^{tJ} - \mathrm{Id})(J^{-1} X_2)] - \frac{1}{2} \sum_{j=1}^{N} [J^{-1} \xi_j, \xi_j]$$

Next we compute $Z'(t) = Z_1(t) + t\, Z_1'(t) + Z_2'(t)$ and from f) we obtain

$$g)\ Z'(t) = Z_0 - \frac{1}{2} [e^{tJ} X_2, e^{tJ} J^{-1} X_2]$$
$$-\frac{1}{2} [X_1, e^{tJ} - ld)(J^{-1} X_2)] + \frac{1}{2} [e^{tJ} X_2, J^{-1} X_2] - \frac{1}{2} t\,[e^{tJ} X_2, X_1]$$

Finally from *c)* and *g)* we see that $Z'(t) + \frac{1}{2} [X'(t), X(t)] \equiv Z_0$, which is the second equation of (3.1). It is evident that $Z(0) = 0$ and from g) we see that $Z'(0) = Z_0$. $\square$

(3.6) *Remark.* – Using assertion *b)* above it follows that

$$e^{tJ} = \cos(t\,\theta_j)\,\mathrm{Id} + \{\sin(t\,\theta_j)/\theta_j\}\, J \quad \text{on } W_j, \qquad 1 \leqq j \leqq N.$$

From this it follows that $Z_2(t)$ is uniformly bounded in absolute value for all $t \in \mathbb{R}$.

(3.7) *Remark.* – Using (3.6) and the fact that $X_2 = \sum_{j=1}^{N} \xi_j$ we can rewrite the equation for $X(t)$ in (3.5) in the following form:

$$X(t) = t\, X_1 + \sum_{j=1}^{N} X_j^*(t),$$

where

$$X_j^*(t) = (e^{tJ} - \mathrm{Id})(J^{-1}\xi_j) = \{\cos(t\theta_j) - 1\} J^{-1}\xi_j + \{\sin(t\theta_j)/\theta_j\}\xi_j.$$

Each curve $X_j^*(t)$ is a circle in W_j with center $-J^{-1}\xi_j$, radius $|J^{-1}\xi_j|$ and period $\dfrac{2\pi}{\theta_j}$.

(3.8) *Remark.* – If the Lie algebra $\mathcal{N}$ is nonsingular [cf. (1.4)] and if $Z_0 \neq 0$, then $J = j(Z_0)$ is invertible by (1.8), and it follows that $X_1 = 0$ and $X_2 = X_0$ in the notation of (3.5). The equations of (3.5) simplify in this case. If in addition N is of Heisenberg type, then $i\,|Z_0|$ and $-i\,|Z_0|$ are the only eigenvalues of J, and with the help of (3.6) and (3.7) the equations in (3.5) become

1) $X(t) = (e^{tJ} - Id)(J^{-1}X_0) = \{\cos(t\,|Z_0|) - 1\} J^{-1}X_0 + \{\sin(t\,|Z_0|)/|Z_0|\} X_0$

2) $Z(t) = t\,Z_1(t) + Z_2(t)$, where

a) $Z_1(t) \equiv Z_0 + \dfrac{1}{2}[J^{-1}X_0, X_0] = \left(1 + \dfrac{|X_0|^2}{2\,|Z_0|^2}\right) Z_0$ by (1.7) and the lemma in example 4 of (2.11).

b) $Z_2(t) = \dfrac{1}{2}[e^{tJ}J^{-1}X_0, J^{-1}X_0] = \dfrac{1}{2}\{\sin(t\,|Z_0|)/|Z_0|\}[X_0, J^{-1}X_0] = \left\{\dfrac{-\sin(t\,|Z_0|)\,|X_0|^2}{2\,|Z_0|^3}\right\} Z_0$ by (1.7) and the lemma in example 4 of (2.11).

Note that $X(t)$ is a circle with center $J^{-1}X_0$, radius $|J^{-1}X_0| = \dfrac{|X_0|}{|Z_0|}$ and period $\dfrac{2\pi}{|Z_0|}$. Moreover $Z(t)$ is a multiple of Z_0 for all $t \in \mathbb{R}$.

(3.9) *Remark.* – If $X_0 = 0$ or $Z_0 = 0$, then the geodesic equations in (3.1) or (3.5) become respectively $\gamma(t) = \exp(t\,Z_0)$ or $\gamma(t) = \exp(t\,X_0)$. More generally, it follows from the equations in (3.1) that if $\gamma(t)$ is the unique geodesic with $\gamma'(0) = X_0 + Z_0$, then $\gamma(t) = \exp(t\,(X_0 + Z_0))$ if and only if $j(Z_0)X_0 = 0$ if and only if Z_0 is orthogonal to $[X_0, \mathcal{N}]$. This fact can also be deduced from (2.1).

We conclude this section with two results about the behavior of geodesics in N that are tangent to the left invariant distributions $\mathcal{V}$ and $\mathcal{Z}$ in N.

(3.10) PROPOSITION. – *Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric. Let $\gamma(t)$ be a unit speed geodesic of the form $\gamma(t) = \alpha \cdot \exp(t\,Z^*)$, where $\alpha \in N$ is arbitrary and Z^* is any unit vector in $\mathcal{Z}$. Then $j(Z^*) \equiv 0$ if and only if no two points of γ are conjugate. If $j(Z^*) \not\equiv 0$, then $\gamma(0)$ is conjugate to $\gamma(b)$ for some $b > 0$.*

Remark. – If N has no Euclidean factor, then $j(Z^*) \not\equiv 0$ if $Z^* \neq 0$ by (2.7).

(3.11) PROPOSITION. – *Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric. Let V^* be a nonzero element of $\mathcal{V}$, and let $d(\, ,\,)$ denote the left invariant metric of N. Then $d(e, \exp(V^* + Z^*)) \geqq |V^*|$ for all $Z^* \in \mathcal{Z}$ with equality if and only if $Z^* = 0$. Hence if $\gamma(t)$ is a unit speed geodesic of the form $\gamma(t) = \alpha \cdot \exp(t\,V^*)$, where $\alpha \in N$ is arbitrary and V^* is any unit vector in $\mathcal{V}$, then γ minimizes the distance between any two of its points.*

Proof of Proposition 3.10. – Since left translations are isometries it suffices to consider the case that $\alpha = e$ and $\gamma(t) = \exp(t\,Z^*)$, where Z^* is a unit vector in $\mathcal{Z}$. If $j(Z^*) \equiv 0$, then the sectional curvature formulas in (2.4) imply that the sectional curvature of any 2-plane containing $\gamma'(t)$ is zero for any $t \in \mathbb{R}$. Standard arguments then show that no two points of γ are conjugate.

Conversely, if $j(Z^*) \not\equiv 0$, then by (2.5) $\mathrm{Ric}(Z^*) = \mathrm{Ric}(\gamma'(0) = c > 0$. It follows that $\mathrm{Ric}(\gamma'(t)) \equiv c > 0$ since $\gamma'(t) = dL_{\exp(tZ^*)}\gamma'(0)$, which proves that $\gamma(0)$ and $\gamma(b)$ are conjugate for some number $b > 0$ (*see* Theorem 1.26 of [CE]). This completes the proof of Proposition (3.10). $\square$

Proof of Proposition 3.11. – Let $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$ be arbitrary elements, where $V^* \neq 0$. Let $\gamma: [0, 1] \to N$ be a shortest geodesic from $e = \gamma(0)$ to $\exp(V^* + Z^*) = \gamma(1)$, and let $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. We write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ and $X(0) = Z(0) = 0$. By (1.3) or (3.2) and the geodesic equations in Proposition 3.1 we see that

$$\gamma'(t) = dL_{\gamma(t)}\left(X' + Z' + \frac{1}{2}[X', X]\right) = dL_{\gamma(t)}(X' + Z_0).$$

Hence

$$\begin{aligned} d(e, \exp(V^* + Z^*)) &= \int_0^1 |\gamma'(t)|\, dt = \int_0^1 (|X'(t)|^2 + |Z_0|^2)^{1/2}\, dt \\ &\geqq \int_0^1 |X'(t)|\, dt \geqq d(0, V^*) \\ &= |V^*| \text{ since } X(0) = 0 \text{ and } X(1) = V^*. \end{aligned}$$

It is now routine to complete the proof. $\square$

4. Isometry invariant geodesics

Let $\{N, \langle\, ,\, \rangle\}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let ϕ be an arbitrary nonidentity element of N. We say that ϕ translates a unit speed geodesic $\gamma(t)$ in N by an amount ω if $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$.

The number ω is called a **period** of ϕ. If ϕ belongs to a discrete group $\Gamma \subseteq N$, then the periods of ϕ are precisely the lengths of the closed geodesics in $\Gamma \backslash N$ that belong to the free homotopy class of closed curves in $\Gamma \backslash N$ determined by ϕ. Elements of Γ that are conjugate in Γ have the same periods and determine the same free homotopy classes in $\Gamma \backslash N$.

In this section we show that every nonidentity element ϕ in N translates some geodesic of N. Moreover ϕ has both a minimal and a maximal period, which coincide if $\mathcal{N}$ is nonsingular and ϕ does not lie in the center of N. For each period ω of ϕ let $N_\omega(\phi)$ denote the union of all unit speed geodesics of N that are translated an amount ω by ϕ. Let $SN_\omega(\phi)$ denote the set of unit vectors in N that are tangent to a unit speed geodesic of N that is translated an amount ω by ϕ. Each set $N_\omega(\phi)$ is invariant under $Z(\phi) = \{\psi \in N : \phi\psi = \psi\phi\}$, and $N_\omega(\phi)$ is a single $Z(\phi)$ orbit if and only if ω is the maximal period ω^* of ϕ. Moreover, for any period ω of ϕ the dimension of the set $SN_\omega(\phi)$ is at least equal to the dimension of $SN_{\omega^*}(\phi)$ with equality if and only if $\omega = \omega^*$.

By the dimension of a set we mean the largest integer k such that the set contains an imbedded open k-disk.

Remark. – Some of the results of this section were obtained earlier by C. Gordon [G3]. In particular she essentially obtained the first three equivalences of Proposition 4.3 and showed (in the notation of Proposition 4.5) that $\phi = \exp(V^* + Z^*)$ translates the 1-parameter group $t \to \exp\left(\frac{t}{\omega^*}(V^* + Z^{**})\right)$.

(4.1) PROPOSITION. – *Let $\phi \in N$ be a nonidentity element, and let $d_\phi : N \to \mathbb{R}$ be the displacement function defined by $d_\phi(n) = d(n, \phi n)$. Then d_ϕ assumes a minimum value $\omega > 0$ on N, and ϕ translates some unit speed geodesic γ of N by an amount ω. The number ω is the smallest period of ϕ.*

Proof. – Choose $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$ so that $\phi = \exp(V^* + Z^*)$. If $Z_{V^*} = \{\exp[V^*, \xi] : \xi \in \mathcal{N}\}$, then Z_{V^*} is a closed subgroup of Z that equals the identity if and only if $V^* = 0$. Therefore the set $\phi \cdot Z_{V^*} = Z_{V^*} \cdot \phi$ is closed in N, and we may choose an element $\psi^* \in \phi \cdot Z_{V^*}$ such that $d(e, \psi^*) \leqq d(e, \psi)$ for all $\psi \in \phi \cdot Z_{V^*}$. We assert that $\omega = d(e, \psi^*)$ is the minimum value of d_ϕ and that d_ϕ assumes its minimum value at $\exp(\xi^*)$, where $\xi^* \in \mathcal{N}$ is any element such that $\psi^* = \phi \cdot \exp([V^*, \xi^*]) = \exp(V^* + Z^* + [V^*, \xi^*])$. Moreover, ϕ translates any minimizing geodesic from ξ^* to $\phi\xi^*$.

Let $\xi^* \in \mathcal{N}$ be chosen as above. If $\xi \in \mathcal{N}$ is arbitrary, then $\psi = \exp(-\xi) \cdot \phi \cdot \exp(\xi) = \exp(V^* + Z^* + [V^*, \xi]) = \phi \cdot \exp([V^*, \xi]) \in \phi \cdot Z_{V^*}$. Hence $d_\phi(\exp \xi^*) = d(e, \exp(-\xi^*) \cdot \phi \cdot \exp(\xi^*)) = d(e, \psi^*) \leqq d(e, \psi) = d_\phi(\exp \xi)$ by the choice of ψ^*. This proves that d_ϕ assumes its minimum value $\omega = d(e, \psi^*)$ at $\exp \xi^*$. Standard arguments now show that ω is the smallest period of ϕ, and ϕ translates any minimizing geodesic from $\exp(\xi^*)$ to $\phi \cdot \exp(\xi^*)$. $\square$

Next we define a number ω^* which later will turn out to be the maximal period of $\phi = \exp(V^* + Z^*)$.

(4.2) PROPOSITION. – *Let $\phi \in N$ be an arbitrary element and write $\phi = \exp(V^* + Z^*)$ for suitable elements $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Let Z^{**} be the component of Z^* orthogonal to $[V^*, \mathcal{N}]$, and let $\omega^* = \{|V^*|^2 + |Z^{**}|^2\}^{1/2} = |V^* + Z^{**}|$. Let $\xi \in \mathcal{N}$ be chosen so that $Z^{**} = Z^* + [V^*, \xi]$, and let $\gamma(t) = \exp(\xi) \cdot \exp\left(\frac{t}{\omega^*}(V^* + Z^{**})\right)$. Then $\gamma(t)$ is a unit speed geodesic such that $\phi \cdot \gamma(t) = \gamma(t + \omega^*)$ for all $t \in \mathbb{R}$.*

Proof. – If $a = \exp(\xi)$ we define $\phi^* = a^{-1} \cdot \phi \cdot a = \exp(V^* + Z^* + [V^*, \xi]) = \exp(V^* + Z^{**})$. The condition that $\phi \cdot \gamma(t) = \gamma(t + \omega^*)$ for all $t \in \mathbb{R}$ is equivalent to the condition that $\phi^* \cdot \gamma^*(t) = \gamma^*(t + \omega^*)$ for all $t \in \mathbb{R}$, where $\gamma^*(t) = a^{-1} \cdot \gamma(t) = \exp\left(\frac{t}{\omega^*}(V^* + Z^{**})\right)$. This latter condition is routine to verify.

Note that $\gamma^{*\prime}(0) = \dfrac{V^* + Z^{**}}{\omega^*}$ is a unit vector by the definition of ω^*. The condition that Z^{**} be orthogonal to $[V^*, \mathcal{N}]$ is equivalent to the condition that $j(Z^{**}) V^* = 0$. It follows immediately that $\gamma^*(t)$ satisfies the geodesic equations in (3.1), and hence $\gamma(t) = a \cdot \gamma^*(t)$ is a unit speed geodesic. *See* also (3.9). $\square$

Now we describe some general criteria for an element ϕ to translate a geodesic γ.

(4.3) PROPOSITION. – *Let $\phi \in N$ be an arbitrary element and write $\phi = \exp(V^* + Z^*)$ for suitable elements $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = a$ and $\gamma(\omega) = \phi \cdot a$, and let $\gamma'(0) = dL_a(X_0 + Z_0)$ for suitable elements $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. Let J denote $j(Z_0)$. Let $a^{-1} \cdot \gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$ and $X(0) = Z(0) = 0$. Then the following assertions are equivalent.*

1. $X(t) + V^* = X(t + \omega)$

$$Z(t) + Z^* + \frac{1}{2}[V^*, X(t)] = Z(t + \omega) \quad \text{for all } t \in \mathbb{R}$$

2. $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$
3. $\gamma'(0)$ is orthogonal to the orbit $Z_{V^*} \cdot a$, where $Z_{V^*} = \exp([V^*, \mathcal{N}]) \subseteq Z$
4. $\gamma'(\omega)$ is orthogonal to the orbit $Z_{V^*} \cdot \phi \cdot a$
5. $e^{\omega J}$ fixes X_0

Proof. – By straightforward arguments similar to those used in the proof of (4.2) it suffices to consider the case that $a = e$, the identity in N. Assertions 1) and 2) are equivalent by the multiplication law (1.1). To prove the other equivalences we proceed in the cyclic order 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 2).

We prove 2) $\Rightarrow$ 3). Write $\mathcal{V}$ as an orthogonal direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1$ is the kernel of $J = j(Z_0)$. Write $X_0 = X_1 + X_2$, where $X_1 \in \mathcal{V}_1$ and $X_2 \in \mathcal{V}_2$.

LEMMA. – *$V^* = \omega X_1$ and $e^{\omega J}$ fixes X_2.*

Suppose for the moment that the lemma has been proved. The lemma implies that $J(V^*) = 0$, which is equivalent to Z_0 being orthogonal to $[V^*, \mathcal{N}] = T_e Z_{V^*}$. Since $\gamma'(0) = X_0 + Z_0$ we conclude that $\gamma'(0)$ is orthogonal to Z_{V^*} at $e = \gamma(0)$, which proves the assertion 2) $\Rightarrow$ 3).

Proof of the lemma. – By Proposition 3.5 we have

(i) $X(n\omega) = n\omega X_1 + (e^{n\omega J} - \mathrm{Id})(J^{-1} X_2)$ for every positive integer n.

By induction we obtain from the equations above in 1) of the proposition

(ii) $X(n\omega) = n V^*$ for every positive integer n.

If we write $V^* = V_1^* + V_2^*$, where $V_1^* \in \mathcal{V}_1$ and $V_2^* \in \mathcal{V}_2$, then from (i) and (ii) we obtain

(iii) $n V_1^* = n\omega X_1$ for every positive integer n

(iv) $n V_2^* = (e^{n\omega J} - \mathrm{Id})(J^{-1} X_2)$ for every positive integer n.

The right hand side of (iv) is uniformly bounded in norm for all n since $e^{n\omega J}$ is an orthogonal transformation. This implies that $V_2^* = 0$ and hence $V^* = V_1^* = \omega X_1$ by (iii). From (iv) in the case $n = 1$ we see that $e^{\omega J}$ fixes $J^{-1} X_2$ and hence $e^{\omega J}$ fixes X_2 since $e^{\omega J}$ commutes with J. $\square$

We prove 3) $\Rightarrow$ 4). Since $\gamma'(0) = X_0 + Z_0$ the hypothesis 3) is equivalent to Z_0 being orthogonal to $[V^*, \mathcal{N}] = T_e Z_{V^*}$. By Proposition 3.2 we see that $\gamma'(\omega) = dL_{\gamma(\omega)}(e^{\omega J} X_0 + Z_0)$, where $J = j(Z_0)$. Now $\phi = \gamma(\omega)$ and $T_\phi(Z_{V^*} \cdot \phi) = dL_\phi T_e(Z_{V^*})$. Since Z_0 is orthogonal to $T_e(Z_{V^*})$ we conclude that $\gamma'(\omega)$ is orthogonal to $T_\phi(Z_{V^*} \cdot \phi)$. $\square$

We prove 4) $\Rightarrow$ 5). Since $\gamma(\omega) = \phi = \exp(V^* + Z^*)$ it follows that $Z^* = Z(\omega)$ and $V^* = X(\omega)$. Write $\mathcal{V}$ as an orthogonal direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1$ is the kernel of $J = j(Z_0)$, and write $X_0 = X_1 + X_2$, where $X_1 \in \mathcal{V}_1$ and $X_2 \in \mathcal{V}_2$. By (3.5) $X(\omega) = \omega X_1 + (e^{\omega J} - \mathrm{Id})(J^{-1} X_2)$ and hence we obtain

$$(\star) \qquad V^* - \omega X_1 = (e^{\omega J} - \mathrm{Id})(J^{-1} X_2)$$

From the description of $\gamma'(\omega)$ in the proof of 3) $\Rightarrow$ 4) we see that Z_0 is orthogonal to $[V^*, \mathcal{N}]$ if 4) holds. Hence $JV^* = 0$. We conclude that $e^{\omega J}$ fixes $J^{-1} X_2$ and hence also X_2 since the left hand side of (*) lies in $\mathcal{V}_1$ while the right hand side lies in $\mathcal{V}_2$. Finally $e^{\omega J} X_0 = e^{\omega J} X_1 + e^{\omega J} X_2 = X_1 + X_2 = X_0$ since $JX_1 = 0$ and $e^{\omega J}$ fixes X_2. $\square$

We prove 5) $\Rightarrow$ 2). Assertion 2) is equivalent to the assertion $dL_\phi \gamma'(0) = \gamma'(\omega)$; in this case the geodesics $\phi \cdot \gamma(t)$ and $\gamma(t + \omega)$ would be equal since they would have the same velocity at $t = 0$. By Proposition 3.2 and 5) we see that $\gamma'(\omega) = dL_{\gamma(\omega)}(e^{\omega J} X_0 + Z_0) = dL_\phi(X_0 + Z_0) = dL_\phi \gamma'(0)$. This completes the proof of Proposition 4.3. $\square$

The next result follows directly from (4.3).

(4.4) Corollary. – *Let $\phi \in N$ be an arbitrary element and write $\phi = \exp(V^* + Z^*)$ for unique elements $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Let $a \in N$ be given and write $a = \exp(\xi)$ for a unique $\xi \in \mathcal{N}$. Then the following are equivalent:*

1) There exists a unit speed geodesic γ of N with $\gamma(0) = a$ such that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$.

2) There exists a unit speed geodesic γ^* of N with $\gamma^*(0) = e$, $\gamma^{*\prime}(0)$ orthogonal to $[V^*, \mathcal{N}]$ and $\gamma^*(\omega) = \exp([V^*, \xi]) \cdot \phi$ for some $\omega > 0$.

Proof. – Suppose first that 1) holds. If $\phi^* = a^{-1} \cdot \phi \cdot a$ and $\gamma^*(t) = a^{-1} \cdot \gamma(t)$, then γ^* is a unit speed geodesic such that $\gamma^*(0) = e$ and $\phi^* \cdot \gamma^*(t) = \gamma^*(t + \omega)$ for all $t \in \mathbb{R}$. From Proposition 4.3 it follows that $\gamma^{*\prime}(0)$ is orthogonal to $T_e Z_{V^*} = [V^*, \mathcal{N}]$. Moreover, $\gamma^*(\omega) = \phi^* \cdot \gamma^*(0) = \phi^* = \exp(V^* + Z^* + [V^*, \xi]) = \exp([V^*, \xi]) \cdot \phi$ by the definition of ϕ^*.

Conversely suppose that 2) holds. Let $\phi^* = \exp([V^*, \xi]) \cdot \phi = a^{-1} \cdot \phi \cdot a$. Then $\phi^* \cdot \gamma^*(t) = \gamma^*(t + \omega)$ for all $t \in \mathbb{R}$ by the equivalences of 2) and 3) in Proposition 4.3, and it follows that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$, where $\gamma(t) = a \cdot \gamma^*(t)$. $\square$

We now prove that the constant ω^* defined in Proposition 4.2 is the largest possible period for $\phi = \exp(V^* + Z^*)$.

(4.5) Proposition. – *Let ϕ be an arbitrary element of N and write $\phi = \exp(V^* + Z^*)$ for unique elements $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Let Z^{**} be the component of Z^* orthogonal to $[V^*, \mathcal{N}]$. Let $\gamma(t)$ be a unit speed geodesic such that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$. Let $a = \gamma(0)$ and $\omega^* = \{|V^*|^2 + |Z^{**}|^2\}^{1/2}$. Then*

1) $|V^*| \leqq \omega \leqq \omega^*$

2) $\omega = \omega^*$ if and only if the following conditions hold

(i) $\gamma(t) = a \cdot \exp\left(\dfrac{t}{\omega^*}(V^* + Z^{**})\right)$ for all $t \in \mathbb{R}$

(ii) $Z^{**} = Z^* + [V^*, \xi]$, where $a = \exp(\xi)$.

3) $\omega = |V^*|$ if and only if $Z^* = 0$ if and only if $\gamma(t) = a \cdot \exp\left(\dfrac{tV^*}{|V^*|}\right)$ for all $t \in \mathbb{R}$.

(4.6) COROLLARY. – *Let ϕ be a nonidentity element of N that does not lie in the center of N. Assume that Z^* is contained in $[V^*, \mathcal{N}]$. Then*

1) ϕ has a unique period $|V^*|$.

2) Let $\gamma(t)$ be a unit speed geodesic in N with $\gamma(0) = a = \exp(\xi)$ for $\xi \in \mathcal{N}$. Then $\phi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$ if and only if $[\xi, V^*] = Z^*$ and $\gamma(t) = a \cdot \exp\left(\frac{tV^*}{\omega}\right)$ for all $t \in \mathbb{R}$.

In particular the corollary applies to all elements ϕ of N if $\mathcal{N}$ is nonsingular.

Proof of the corollary. – The hypothesis of the corollary says that $Z^{**} = 0$ in the notation of Proposition 4.5. The first assertion of the corollary now follows from 1) of Proposition 4.5. Next let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = a = \exp(\xi)$ such that $\phi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, where $\omega = |V^*|$. Since $Z^{**} = 0$ it follows that $[\xi, V^*] = Z^*$ and $\gamma(t) = a \cdot \exp\left(\frac{tV^*}{\omega}\right)$ for all $t \in \mathbb{R}$ by 2) and 3) of Proposition 4.5. Conversely, if $[\xi, V^*] = Z^*$ and $\gamma(t) = a \cdot \exp\left(\frac{tV^*}{\omega}\right) = \exp(\xi) \cdot \exp\left(\frac{tV^*}{\omega}\right)$, then it follows immediately that $\gamma(t)$ is a unit speed geodesic of N with $\gamma(0) = a$ such that $\phi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$. $\square$

Proof of Proposition 4.5. – We first note that 2(ii) follows from 2(i). If $\omega = \omega^*$ and 2(i) holds, then

$$\begin{aligned} \exp\left(V^* + Z^* + \xi + \frac{1}{2}[V^*, \xi]\right) &= \exp(V^* + Z^*) \cdot \exp(\xi) = \phi \cdot a = \gamma(\omega^*) \\ &= a \cdot \exp(V^* + Z^{**}) \\ &= \exp\left(V^* + Z^{**} + \xi + \frac{1}{2}[\xi, V^*]\right). \end{aligned}$$

This proves 2(ii). Conversely, if 2(i) and 2(ii) hold, then $\phi \cdot \gamma(t) = \gamma(t + \omega^*)$ for all $t \in \mathbb{R}$ by Proposition 4.2, and it follows that $\omega = \omega^*$.

It remains to prove assertions 1) and 3) and the part of 2) which says that if $\omega = \omega^*$ then 2(i) holds. By arguing as in the proof of (4.2) it suffices to consider the case that $\gamma(0) = a = e$, the identity of N. We let $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. By assertion 3) of Proposition 4.3 it follows that Z_0 is orthogonal to $[V^*, \mathcal{N}] = T_e Z_{V^*}$, where $Z_{V^*} = \exp([V^*, \mathcal{N}])$.

We now proceed as in the proof of Proposition 4.3, and we write $\mathcal{V}$ as an orthogonal direct sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1$ is the kernel of $J = j(Z_0)$. Write $X_0 = X_1 + X_2$, where $X_1 \in \mathcal{V}_1$ and $X_2 \in \mathcal{V}_2$. Let $\mathcal{V}_2$ be written as an orthogonal direct sum $\mathcal{V}_2 = \bigoplus_{j=1}^{N} W_j$ as in the discussion preceding the statement of Proposition 3.5. Write $X_2 = \sum_{j=1}^{N} \xi_j$, where $\xi_j \in W_j$ for $1 \leqq j \leqq N$.

(4.7) LEMMA. $Z^* = \omega Z_0 + [V^*, J^{-1} X_2] + \frac{\omega}{2} \sum_{j=1}^{N} [J^{-1} \xi_j, \xi_j]$

Proof. – By hypothesis $\gamma(\omega) = \phi \cdot \gamma(0) = \phi = \exp(V^* + Z^*)$. If we write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$, then we obtain

(i) $X(\omega) = V^*$ and $Z(\omega) = Z^*$

By the lemma in the proof of 2) $\Rightarrow$ 3) of Proposition 4.3 we have

(ii) $V^* = \omega X_1$ and $e^{\omega J}$ fixes X_2, where $J = j(Z_0)$. Hence $e^{\omega J}$ fixes each ξ_j, $1 \leqq j \leqq N$.

We write $Z(t) = t Z_1(t) + Z_2(t)$, where $Z_1(t)$ and $Z_2(t)$ are given explicitly in the statement of Proposition 3.5. Using (ii) and the formula for $Z_2(t)$ we find that $Z_2(\omega) = 0$. Hence from (i), (ii) and Proposition 3.5 we obtain

$$Z^* = Z(\omega) = \omega \cdot Z_1(\omega) = \omega \left\{ Z_0 + \frac{1}{2}[X_1, (e^{\omega J} + \mathrm{Id})(J^{-1} X_2)] + \frac{1}{2}\sum_{j=1}^{N}[J^{-1}\xi_j, \xi_j] \right\}$$

$$= \omega Z_0 + [V^*, J^{-1} X_2] + \frac{\omega}{2}\sum_{j=1}^{N}[J^{-1}\xi_j, \xi_j]. \quad \square$$

We now prove statement 1) of Proposition 4.5. By the definition of Z^{**} we may write $Z^{**} = Z^* + [V^*, \xi]$ for some $\xi \in \mathcal{N}$. We observed at the beginning of the proof that Z_0 is orthogonal to $[V^*, \mathcal{N}]$. Hence from the lemma above we obtain

$$\text{(i)} \qquad \langle Z^{**}, Z_0 \rangle = \langle Z^*, Z_0 \rangle = \omega |Z_0|^2 + \frac{\omega}{2}|X_2|^2$$

Now let $\mathcal{Z}_0$ be any 2-dimensional subspace of $\mathcal{Z}$ that contains both Z_0 and Z^{**}. Introduce coordinates in $\mathcal{Z}_0$ so that $Z^{**} = (\alpha, 0)$ and $Z_0 = (x, y)$. Equation i) then becomes

$$\text{(ii)} \qquad \alpha x = \omega(x^2 + y^2) + \frac{\omega}{2}|X_2|^2$$

Since $1 = |Z_0|^2 + |X_1|^2 + |X_2|^2$ and $V^* = \omega X_1$ equation ii) implies

$$\text{(iii)} \qquad \left(x - \frac{\alpha}{\omega}\right)^2 + y^2 = \frac{|V^*|^2 + \alpha^2 - \omega^2}{\omega^2}$$

Since the left hand side of iii) is nonnegative it follows that

$$\text{(iv)} \qquad \omega^2 \leqq \alpha^2 + |V^*|^2 = |Z^{**}|^2 + |V^*|^2 = (\omega^*)^2.$$

Moreover $1 \geqq |X_1| = \dfrac{|V^*|}{\omega}$, which impies $\omega \geqq |V^*|$. We have proved statement 1).

We prove statement 2) of Proposition 4.5. From the discussion above and (iii) we see that $\omega = \omega^*$ if and only if $x = \dfrac{\alpha}{\omega}$ and $y = 0$, which is equivalent to the condition $Z_0 = \dfrac{Z^{**}}{\omega}$. Furthermore, if $\omega = \omega^*$ then by (ii) of (4.7) we obtain

$$1 = |X_1|^2 + |X_2|^2 + |Z_0|^2 = \frac{|V^*|^2}{\omega^2} + \frac{\alpha^2}{\omega^2} + |X_2|^2$$

$$= \frac{(\omega^*)^2}{\omega^2} + |X_2|^2 = 1 + |X_2|^2,$$

which implies that $X_2 = 0$. Hence $X_0 = X_1 + X_2 = X_1 = \dfrac{V^*}{\omega}$, and it follows that $\gamma'(0) = X_0 + Z_0 = \dfrac{V^* + Z^{**}}{\omega^*}$.

The curve $t \to \exp\left(\dfrac{t(V^* + Z^{**})}{\omega^*}\right)$ is a geodesic in N by the equations in (3.1) or (3.9) since Z^{**} is orthogonal to $[V^*, \mathcal{N}]$ or equivalently $j(Z^{**})(V^*) = 0$. Hence $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) = \dfrac{tV^*}{\omega^*} \in \mathcal{V}$ and $Z(t) = \dfrac{tZ^{**}}{\omega^*} \in \mathcal{Z}$. We have shown that 2(i) holds if $\omega = \omega^*$.

By the discussion at the beginning of the proof of Proposition 4.5 assertion 2(ii) now follows, and conversely if 2(i) and 2(ii) hold then $\omega = \omega^*$. This completes the proof of 2).

We prove statement 3). If $Z^* = 0$, then $\omega = |V^*|$ by 1) of Proposition 4.5. Observe that $1 \geqq |X_1| = \dfrac{|V^*|}{\omega}$ with equality if and only if $\gamma'(0) = X_1 = \dfrac{V^*}{\omega}$. The curve $t \to \exp\left(\dfrac{tV^*}{|V^*|}\right)$ is a geodesic of N by (3.9), and hence if $|V^*| = \omega$ it follows that $\gamma(t) = \exp\left(\dfrac{tV^*}{|V^*|}\right) = \exp\left(\dfrac{tV^*}{\omega}\right)$. Moreover, $Z^* = 0$ since $\exp(V^* + Z^*) = \phi = \gamma(\omega) = \exp(V^*)$. Conversely, if $\gamma(t) = \exp\left(\dfrac{tV^*}{|V^*|}\right)$, then $\exp(V^* + Z^*) = \phi = \gamma(\omega) = \exp\left(\dfrac{\omega V^*}{|V^*|}\right)$. This implies that $\omega = |V^*|$ and $Z^* = 0$ and concludes the proof of Proposition 4.5. $\square$

(4.8) AN EXAMPLE OF NONUNIQUE PERIODS. – If N and $\mathcal{N}$ are nonsingular, and if $\phi = \exp(V^* + Z^*)$ is an element that does not lie in the center of N (*i.e.* $V^* \neq 0$), then Corollary 4.6 shows that ϕ has a unique period $\omega = |V^*|$. If N fails to be nonsingular, then the noncentral elements of N may have more than one period as the following example shows.

Example. – Let $\mathcal{N}$ be a 5-dimensional real vector space with basis $\{X, Y_1, Y_2, Z_1, Z_2\}$ and bracket relations as follows:

a) Z_i lies in the center $\mathcal{Z}$ of $\mathcal{N}$ for $i = 1, 2$.

b) $[X, Y_1] = Z_1$, $[X, Y_2] = Z_2$.

c) $[Y_1, X] = -Z_1$, $[Y_1, Y_2] = 0$.

d) $[Y_2, X] = -Z_2$, $[Y_2, Y_1] = 0$.

A routine argument shows that $\mathcal{Z} = \operatorname{span}\{Z_1, Z_2\}$. The Lie algebra $\mathcal{N}$ is 2-step nilpotent but fails to be nonsingular since $\dim[Y_i, \mathcal{N}] = 1$ for $i = 1, 2$. Note that $[X, \mathcal{N}] = \operatorname{span}\{Z_1, Z_2\} = \mathcal{Z}$ so the Lie algebra $\mathcal{N}$ has no Euclidean factor [cf. (2.7)].

Let N denote the simply connected, 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$, and equip N with the left invariant metric for which the set $\{X, Y_1, Y_2, Z_1, Z_2\}$ forms an orthonormal basis of $\mathcal{N} = T_e N$.

Let α, β be any nonzero constants and define $\beta^* = 1 + \frac{1}{2}\beta^2$. Define $\omega = 2\pi(1+\alpha^2+\beta^2)^{1/2}$ and $\omega^* = 2\pi(\alpha^2+\beta^{*^2})^{1/2}$. Let γ and γ^* be the unit speed reparametrizations of the geodesics beginning at e with initial velocities $\beta X + \alpha Y_2 + Z_1$ and $\alpha Y_2 + \beta^* Z_1$ respectively. Let $\phi = \exp(2\pi\alpha Y_2 + 2\pi\beta^* Z_1)$.

ASSERTION. – *$\phi = \gamma(\omega) = \gamma^*(\omega^*)$ and ϕ translates the geodesics γ, γ^* by ω, ω^* respectively.*

To prove the assertion we begin by defining

$$\sigma^*(t) = \exp(t\{\alpha Y_2 + \beta^* Z_1\}) \qquad \text{and} \qquad \sigma(t) = \exp(X(t) + Z(t)),$$

where

$$X(t) = \beta \sin t\, X + \beta(1-\cos t)\, Y_1 + \alpha t Y_2$$

and

$$Z(t) = \left\{\beta^* t - \frac{1}{2}\beta^2(\sin t)\right\} Z_1 + \alpha\beta\left\{1 - \cos t - \frac{1}{2} t \sin t\right\} Z_2$$

The curve σ^* satisfies $\sigma^*(0) = e$, $\sigma^{*\prime}(0) = \alpha Y_2 + \beta^* Z_1$, and σ^* is a geodesic by (3.9) since $j(Z_1) Y_2 = 0$. The curve $\sigma(t)$ satisfies $\sigma(0) = e$, $\sigma'(0) = \beta X + \alpha Y_2 + Z_1$. A routine computation shows that $\sigma(t)$ satisfies the geodesic equations of (3.1) since $j(Z_1) X = Y_1$, $j(Z_1) Y_1 = -X$ and $Z_0 = Z'(0) = Z_1$. Hence the unit speed geodesics γ, γ^* are reparametrizations of σ, σ^*. One computes that $\phi = \sigma(2\pi) = \gamma(\omega)$ and $\phi = \sigma^*(2\pi) = \gamma^*(\omega^*)$.

Finally we conclude that ϕ translates γ, γ^* by ω, ω^* using 3) of Proposition 4.3; note that $\gamma(0) = \gamma^*(0) = e$ and $\gamma'(0)$, $\gamma^{*\prime}(0)$ are both orthogonal to $[Y_2, \mathcal{N}] = \operatorname{span}\{Z_2\}$.

Remark. – The 1-parameter subgroup $\gamma^*(t)$ is the natural "direct" route between $e = \gamma^*(0)$ and $\phi = \gamma^*(\omega^*)$, but $\gamma^* \mid [0, \omega^*]$ is longer than the apparently circuitous route $\gamma \mid [0, \omega]$. In fact ω^* is the largest period of ϕ as Proposition 4.5 shows.

Periods of central elements of N.

(4.9) PROPOSITION. – *Let ϕ be a nonidentity element of Z, the center of N. Let $a \in N$ be arbitrary, and let γ be any unit speed geodesic of N such that $\gamma(0) = a$ and $\gamma(\omega) = \phi \cdot a$ for some number $\omega > 0$. Then $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$.*

Proof. – In the notation of Proposition 4.3 we write $\phi = \exp(V^* + Z^*)$, where $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Note that $V^* = 0$ since $\phi \in Z$. The assertion of (4.9) now follows from 3) of Proposition 4.3 since $Z_{V^*} = \exp([V^*, \mathcal{N}]) = e$ and the submanifold $Z_{V^*} \cdot a$ is the point $\{a\}$. □

(4.10) COROLLARY. – *There exists a neighborhood O of e in Z such that there exists a unique geodesic in N from e to any point of O. Moreover, if ϕ is any element of O, then ϕ has a unique period.*

Proof. – Let U be a metric ball centered at the origin in $\mathcal{N} = T_e N$ such that the Riemannian exponential map of N at e is a diffeomorphism of U onto its image W in N. Let U^* be a metric ball centered at the origin in $\mathcal{N}$ such that $\exp(U^*) \subseteq W$ and exp is a diffeomorphism of U^* onto its image, where $\exp: \mathcal{N} \to N$ is the Lie group exponential map. We show that $O = \exp(U^*) \cap Z$ satisfies the assertions of the corollary. Let ϕ be any element of O and write $\phi = \exp(Z^*)$ for a unique element Z^* of $U^* \cap \mathcal{Z}$. The curve $\gamma^*(t) = \exp(tZ^*)$, $0 \leqq t \leqq 1$, is a geodesic of N with length $\omega^* = |Z^*|$ by (3.9), and γ^* is contained in W by the definition of O. By the choice of W it follows that γ^* is the unique geodesic in W and the unique shortest geodesic in N from e to ϕ. However, the quantity $\omega^* = |Z^*|$ is the largest possible period for ϕ by Proposition 4.5. By (4.9) ϕ translates any geodesic γ in N from e to ϕ, and hence the length of γ, which is a period of ϕ, must be at most ω^*. It follows that γ^* is the unique geodesic in N from e to ϕ, and ϕ has a unique period ω^*. □

The result above shows that if $|Z^*|$ is sufficiently small for $Z^* \in \mathcal{Z}$, then $\phi = \exp(Z^*)$ has a unique period $|Z^*|$. This is no longer true if $|Z^*|$ is sufficiently large.

(4.11) Proposition. – *Let $Z^* \in \mathcal{Z}$ be an element such that $j(Z^*) \not\equiv 0$, and let $\phi = \exp(Z^*)$. Then there exists a positive integer N_0 such that ϕ^n has at least two periods for all $n \geqq N_0$.*

Remark. – We point out again that if N has no Euclidean factor, then $j(Z^*) \not\equiv 0$ if $Z^* \neq 0$ by (2.7).

Proof. – Let Z^*, ϕ be as above and define $\gamma(t) = \exp\left(\frac{tZ^*}{|Z^*|}\right)$. The curve $\gamma(t)$ is a unit speed geodesic of N by (3.9). By Proposition 3.10 there exists a number $b > 0$ such that $\gamma(0) = e$ and $\gamma(b)$ are conjugate along γ. Choose a positive integer N_0 such that $N_0|Z^*| > b$. If $n \geqq N_0$ is any integer, then $\gamma_|[0, n|Z^*|]$ is not a shortest geodesic in N between its endpoints by the choice of N_0. Hence there exists a minimizing unit speed geodesic $\sigma : [0, t_0] \to N$ with $\sigma(0) = \gamma(0)$ and $\sigma(t_0) = \gamma(n|Z^*|) = \phi^n$, where $t_0 = d(e, \phi^n) < n|Z^*|$. By Proposition 4.9 ϕ translates both γ and σ with periods $n|Z^*|$ and t_0 respectively. □

Resonance

We may regard $\mathbb{R}^n$ as a simply connected, additive abelian group, and the standard metric of $\mathbb{R}^n$ is invariant under left (=right) translations by elements of $\mathbb{R}^n$. If $\gamma(t)$ is any geodesic of $\mathbb{R}^n$ with the standard metric, then there exists an element ϕ in $\mathbb{R}^n$ such that $\phi + \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$; in fact, one can find such an element ϕ for any given $\omega > 0$. If $\{N, \langle\ ,\ \rangle\}$ is a simply connected 2-step nilpotent Lie group with a left invariant metric, then there is an obstruction to the geometric property just described. It seems appropriate to call this obstruction **resonance**; *see* (4.14) below.

(4.12.) Definition. – *If Z^* is a nonzero element of $\mathcal{Z}$, then we say that $j(Z^*)$ is* **in resonance** *if the ratio of any two nonzero eigenvalues of $j(Z^*)$ is a rational real number.*

Remark. – The eigenvalues of $j(Z^*)$ are purely imaginary, and the nonzero ones occur in conjugate pairs since $j(Z^*)$ is skew symmetric.

The next observation is an easy exercise.

(4.13) LEMMA. – *Let Z^* be any nonzero element of $\mathcal{Z}$. Then $j(Z^*)$ is in resonance if and only if $e^{\omega j(Z^*)}$ is the identity on $\mathcal{V}$ for some number $\omega > 0$.*

The role that resonance plays is most easily described in the case that N and $\mathcal{N}$ are nonsingular. We do not know to what extent the next result is true for an arbitrary 2-step nilpotent group N.

(4.14) PROPOSITION. – *Let N be a nonsingular, simply connected, 2-step nilpotent Lie group with a left invariant metric $\langle\ ,\ \rangle$. Then the following properties are equivalent.*

1. *$j(Z^*)$ is in resonance for all nonzero elements $Z^* \in \mathcal{Z}$.*
2. *Every unit speed geodesic of N is translated by some element ϕ of N.*
3. *Every unit speed geodesic of N that is not tangent to the left invariant distribution $\mathcal{V}$ in N is translated by some element ϕ belonging to the center $\mathbf{Z}$ of N.*
4. *Every unit speed geodesic γ of N that is not tangent to the left invariant distribution $\mathcal{V}$ in N meets the orbit $\mathbf{Z} \cdot \gamma(0)$ infinitely often at $\gamma(t)$ for $t > 0$.*

Remark. – If N is of Heisenberg type, then $j(Z^*)$ is in resonance for all nonzero $Z^* \in \mathcal{Z}$ since $j(Z^*)$ has eigenvalues $i|Z^*|$ and $-i|Z^*|$.

Proof of the proposition. – We prove the equivalence of these assertions in the cyclic order $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$.

$1) \Rightarrow 2)$. As usual it suffices to consider a unit speed geodesic $\gamma(t)$ with $\gamma(0) = e$. Let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = e$ and write $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. If $X_0 = 0$ or $Z_0 = 0$, then $\gamma(t) = \exp(t Z_0)$ or $\exp(t X_0)$ respectively by (3.9), and we may choose $\phi = \exp(\omega Z_0)$ or $\phi = \exp(\omega X_0)$ respectively for any $\omega > 0$. Hence we need only consider the case that X_0 and Z_0 are both nonzero. Let J denote $j(Z_0)$. By hypothesis J is in resonance, and by Lemma 4.13 there exists a positive number ω such that $e^{\omega J}$ is the identity on $\mathcal{V}$. Write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$. Since $\mathcal{N}$ is nonsingular it follows by the geodesic equations in Proposition 3.5 that $X(t) = (e^{tJ} - \mathrm{Id})(J^{-1} X_0)$. Hence $X(\omega) = 0$ and it follows that $\gamma(\omega) = \phi$ lies in Z. By Proposition 4.9 $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$.

$2) \Rightarrow 3)$. As in the proof above it suffices to consider a unit speed geodesic $\gamma(t)$ such that $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{V}$, $Z_0 \in \mathcal{Z}$ and X_0 and Z_0 are both nonzero. By hypothesis there exists $\phi = \exp(V^* + Z^*)$ and $\omega > 0$ such that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$; as usual $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. By assertion 3) of Proposition 4.3 we note that Z_0 is orthogonal to $[V^*, \mathcal{N}]$. If V^* were nonzero, then $[V^*, \mathcal{N}]$ would equal $\mathcal{Z}$ and Z_0 would be zero by the nonsingularity of $\mathcal{N}$. Therefore $V^* = 0$ and ϕ lies in the center of N, which completes the proof of the assertion.

$3) \Rightarrow 4)$. Let $\gamma(t)$ be a unit speed geodesic not tangent to $\mathcal{V}$. By hypothesis there exists $Z^* \neq 0$ in $\mathcal{Z}$ and $\omega > 0$ such that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$, where $\phi = \exp(Z^*) \in \mathbf{Z}$. Hence if n is any positive integer, then $\gamma(n\omega) = \phi^n \cdot \gamma(0) \in Z \cdot \gamma(0)$.

$4) \Rightarrow 1)$. Let Z^* be any nonzero element of $\mathcal{Z}$. Let $J = j(Z^*)$ and let $\{\pm i\theta_1, \cdots, \pm i\theta_N\}$ be the distinct eigenvalues of J, where $\theta_i > 0$ for $1 \leqq i \leqq N$.

Write $\mathcal{V}$ as an orthogonal direct sum $\bigoplus_{k=1}^{N} W_k$, where each subspace W_k is invariant under J and $J^2 = -\theta_k^2 \operatorname{Id}$ on W_k, $1 \leqq k \leqq N$. Let X_k^* be any nonzero element of $\mathcal{V}_k$ and define $X^* = \sum_{j=1}^{N} X_k^*$. Let $\gamma(t)$ be the geodesic of N such that $\gamma(0) = e$ and $\gamma'(0) = X^* + Z^*$.

If we write $\gamma(t) = \exp(X(t) + Z(t))$, where $X(t) \in \mathcal{V}$ and $Z(t) \in \mathcal{Z}$ for all $t \in \mathbb{R}$, then $X(t) = (e^{tJ} - \operatorname{Id})(J^{-1} X^*)$ by (3.5). By hypothesis $\gamma(\omega) \in Z$ for some positive number ω, and it follows that $X(\omega) = 0$. We conclude that $e^{\omega J}$ fixes $J^{-1} X^*$, X^* and X_k^* for each k, $1 \leqq k \leqq N$. By (3.6) it follows that $\omega \theta_k = 2\pi N_k$ for some positive integer N_k, $1 \leqq k \leqq N$, which proves that $j(Z^*)$ is in resonance. $\square$

The sets $N_\omega(\phi)$ and $SN_\omega(\phi)$.

(4.15) DEFINITION. – *Given an element $\phi \neq 1$ in N we define*

$$N_\omega(\phi) = \{\, n \in N : \textit{there exists a unit speed geodesic } \gamma \textit{ of } N \textit{ such that } \gamma(0) = n \textit{ and } \phi \cdot \gamma(t) = \gamma(t+\omega) \textit{ for all } t \in \mathbb{R} \,\}.$$

$$SN_\omega(\phi) = \{\, \xi \in SN : \phi \cdot \gamma_\xi(t) = \gamma_\xi(t+\omega) \textit{ for all } t \in \mathbb{R} \},$$

where SN denotes the unit tangent bundle of N and γ_ξ denotes the unique unit speed geodesic of N with initial velocity ξ.

Note that if $\phi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, then $\gamma(t) \in N_\omega(\phi)$ for all $t \in \mathbb{R}$. Hence $N_\omega(\phi)$ may also be described as the union of all unit speed geodesics in N that are translated an amount ω by ϕ.

If $p : SN \to N$ denotes the natural projection map of a unit vector onto its basepoint, then clearly $p(SN_\omega(\phi)) = N_\omega(\phi)$. It is evident from the definition that $SN_\omega(\phi)$ is invariant under the geodesic flow in SN.

Let ϕ be a nonidentity element of a discrete subgroup Γ of N and let $\pi : N \to \Gamma\backslash N$ denote the Riemannian covering projection onto the quotient manifold obtained by letting Γ act on N by left multiplication. Then $\pi(N_\omega(\phi))$ is the union of all smoothly closed geodesics of period ω that belong to the free homotopy class of closed curves in $\Gamma\backslash N$ that is determined by ϕ.

We now investigate the structure and dimension of the sets $N_\omega(\phi)$ and $SN_\omega(\phi)$, where ϕ is a nonidentity element of N and ω is a period of ϕ. If $\omega = \omega^*$, the maximal period of ϕ, then we show that $Z(\phi) = \{\, \psi \in N : \psi\phi = \phi\psi \,\}$, the centralizer of ϕ, acts simply transitively by left multiplication on $N_{\omega^*}(\phi)$. Hence $N_{\omega^*}(\phi)$ is a smooth submanifold of N whose dimension is the same as $Z(\phi)$. We also show that $dZ(\phi)$ acts simply transitively on $SN_{\omega^*}(\phi)$ and $p : SN_{\omega^*}(\phi) \to N_{\omega^*}(\phi)$ is a diffeomorphism. If ω is a period of ϕ strictly smaller than the maximal period ω^*, then $SN_\omega(\phi)$ always has bigger dimension than $SN_{\omega^*}(\phi)$. However, it is not clear if $N_\omega(\phi)$ or $SN_\omega(\phi)$ is a smooth submanifold of N or SN respectively when $\omega < \omega^*$.

(4.16) LEMMA. – *Let ϕ be any nonidentity element of N, and let $\omega > 0$ be any period of ϕ. Then $N_\omega(\phi)$ is invariant under left multiplication L_ψ by any element ψ of the centralizer $Z(\phi)$. $SN_\omega(\phi)$ is invariant under dL_ψ for any $\psi \in Z(\phi)$.*

Proof. – Straightforward. □

(4.17) PROPOSITION. – *Let ϕ be a nonidentity element of N and write $\phi = \exp(V^* + Z^*)$, where $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$. Let Z^{**} denote the component of Z^* orthogonal to $[V^*, \mathcal{N}]$, and let $\omega^* = (|V^*|^2 + |Z^{**}|^2)^{1/2}$. Then*

1. *For each $n \in N_{\omega^*}(\phi)$ there exists a unique geodesic $\gamma(t)$ such that $\gamma(0) = n$ and $\phi \cdot \gamma(t) = \gamma(t + \omega^*)$ for all $t \in \mathbb{R}$. In fact, $\gamma(t) = n \cdot \exp\left(\frac{t}{\omega^*}(V^* + Z^{**})\right)$.*

2. *$Z(\phi)$ acts simply transitively on $N_{\omega^*}(\phi)$; that is, $N_{\omega^*}(\phi) = Z(\phi) \cdot n$ for any $n \in N_{\omega^*}(\phi)$ and the right translation $R_n : Z(\phi) \to N_{\omega^*}(\phi)$ is a diffeomorphism.*

3. *$dL_{Z(\phi)} = \{ dL_\psi : \psi \in Z(\phi) \}$ acts simply transitively on $SN_{\omega^*}(\phi)$.*

Proof. – Assertion 1) is part of assertion 2) of Proposition 4.5, and assertion 1) implies that the projection $p : SN_{\omega^*}(\phi) \to N_{\omega^*}(\phi)$ is a bijection. Hence 3) follows from 1) and 2). We prove 2). Let $n \in N_{\omega^*}(\phi)$ be given and write $n = \exp(\xi)$ for a unique element $\xi \in \mathcal{N}$. From assertion 2) of Proposition 4.5 it follows that

$$(\star) \qquad Z^{**} = Z^* + [V^*, \xi]$$

Next let $n = \exp(\xi)$ and $n^* = \exp(\xi^*)$ be any two elements of $N_{\omega^*}(\phi)$. By $(^*)$ above it follows that $[V^*, \xi - \xi^*] = 0$. If $\mathcal{Z}(\phi) = \{ \xi' \in \mathcal{N} : [V^*, \xi'] = 0\}$, then $\mathcal{Z}(\phi)$ is the Lie algebra of $Z(\phi)$ by (1.2). Hence $\beta = \exp\left(\xi - \xi^* - \frac{1}{2}[\xi, \xi^*]\right)$ lies in $Z(\phi)$. We compute $\beta \cdot n^* = \exp\left(\xi - \xi^* - \frac{1}{2}[\xi, \xi^*]\right) \cdot \exp(\xi^*) = \exp(\xi) = n$. This proves that $Z(\phi)$ acts transitively on $N_{\omega^*}(\phi)$ since the elements n, n^* were arbitrary. Clearly $R_n : Z(\phi) \to N_\omega(\phi)$ is a diffeomorphism. □

Thickness of $SN_\omega(\phi)$.

A subset A of N or SN will be said to have dimension k with respect to the induced topology from N or SN if A contains a homeomorphic image of an open k-ball in $\mathbb{R}^k$ but not of an open $(k+1)$-ball in $\mathbb{R}^{k+1}$.

(4.18) PROPOSITION. – *Let ϕ* be a nonidentity element of N with maximal period $\omega^* = (|V^*|^2 + |Z^{**}|^2)^{1/2}$as defined in (4.17). Let ω be a period of ϕ that is strictly smaller than ω^*. Then $\dim(SN_\omega(\phi)) > \dim SN_{\omega^*}(\phi))$.

Proof. – Write $\phi = \exp(V^* + Z^*)$, where $V^* \in \mathcal{V}$ and $Z^* \in \mathcal{Z}$, and let Z^{**} be the component of Z^* orthogonal to $[V^*, \mathcal{N}]$. Let ω be a period of ϕ and let $\gamma(t)$ be a unit speed geodesic of N such that $\phi \cdot \gamma(t) = \gamma(t + \omega)$ for all $t \in \mathbb{R}$. By (4.16) and the discussion following (4.15) we see that $N_\omega(\phi)$ contains the union of the $Z(\phi)$ orbits of $\gamma(t)$ for all $t \in \mathbb{R}$. The set $SN_{\omega^*}(\phi)$ is a smooth manifold diffeomorphic to $Z(\phi)$ by (4.17). The assertion of (4.18) will now be a consequence of the following

LEMMA. – *Let $\xi \in SN_\omega(\phi)$ be given arbitrarily, where $\omega < \omega^*$ is a period of ϕ, and let γ_ξ denote the geodesic with initial velocity ξ. Identify $\mathcal{N}$ with $T_e N$ and write $\xi = dL_n(X_0 + Z_0)$, where $n = \gamma_\xi(0)$, $X_0 \in \mathcal{V} \subseteqq \mathcal{N}$ and $Z_0 \in \mathcal{Z} \subseteqq \mathcal{N}$. Then $j(Z_0) X_0 \neq 0$ and exactly one of the following occurs:*

1. *$\gamma'_\xi(t)$ is not tangent to the orbit $Z(\phi)\cdot\gamma_\xi(t)$ for some $t\in\mathbb{R}$.*

2. *$\gamma_\xi(t)\in Z(\phi)\cdot n$ for all $t\in\mathbb{R}$. In this case there exists a nonsingular curve $\xi:[0,\omega]\to S_nN$ (unit vectors at n) such that*

a) $\xi(0)=\xi(\omega)=\xi$.

b) $\gamma_{\xi(s)}(0)=n$ and $\gamma_{\xi(s)}(\omega)=\phi\cdot n$ for all $s\in[0,\omega]$.

c) $\phi\cdot\gamma_{\xi(s)}(t)=\gamma_{\xi(s)}(t+\omega)$ for all $s,t\in\mathbb{R}$; that is, $\xi(s)\in SN_\omega(\phi)$ for all $s\in[0,\omega]$.

Proof of the proposition. – Before proving the lemma we use it to prove (4.18). Let $\xi\in SN_\omega(\phi)$ be an arbitrary element. Let $M_0=dL_{Z(\phi)}(\xi)=\{dL_\psi(\xi):\psi\in Z(\phi)\}$. Then M_0 is a smooth submanifold of the unit tangent bundle SN that is diffeomorphic to $Z(\phi)$ under the map $T:Z(\phi)\to M_0$ given by $T(\psi)=dL_\psi(\xi)$. Note that the projection $p:SN\to N$ is a diffeomorphism of M_0 onto the orbit $Z(\phi)\cdot(p(\xi))$ since $p\circ T=R_{p(\xi)}$ on $Z(\phi)$, where $R_{p(\xi)}$ denotes right translation by $p(\xi)$. The submanifold M_0 is contained in $SN_\omega(\phi)$ by (4.16).

Suppose that ξ satisfies case 1) of the lemma, and assume without loss of generality that $\xi=\gamma'_\xi(0)$ is not tangent to $Z(\phi)\cdot(p(\xi))$. If we define $\xi(t)=\gamma'_\xi(t)$ for all $t\in\mathbb{R}$, then the curve $\xi(t)$ lies in $SN_\omega(\phi)$ and is not tangent to M_0 at $t=0$ by hypothesis. By (4.16) $SN_\omega(\phi)$ contains the union of the submanifolds $M_t=dL_{Z(\phi)}(\xi(t))$ for all $t\in\mathbb{R}$, and hence $SN_\omega(\phi)$ contains an imbedded disk around ξ of dimension $1+\dim(M_0)=1+\dim(Z(\phi))$. By (4.17) $SN_{\omega^*}(\phi)$ is diffeomorphic to $Z(\phi)$, and we conclude that $\dim(SN_\omega(\phi))\geqq 1+\dim(SN_{\omega^*}(\phi))$ in this case.

Now assume that ξ satisfies case 2) of the lemma. Let $\xi:[0,\omega]\to SN_\omega(\phi)$ be the curve such that $\xi(0)=\xi(\omega)=\xi$ that is described in the statement of case 2). The curve $\xi(t)$ is not tangent to M_0 at $t=0$ since the vectors tangent to $\xi(t)$ lie in the kernel of dp, $p:SN\to N$, while p is nonsingular on M_0. Using the same argument as above we conclude that $\dim(SN_\omega(\phi))\geqq 1+\dim(SN_{\omega^*}(\phi))$ in this case also. This completes the proof of the proposition. $\square$

Proof of the lemma. – We prove first that $j(Z_0)X_0\neq 0$ if $\omega<\omega^*$. Note that $|X_0|^2+|Z_0|^2=|\xi|^2=1$. We assume that $j(Z_0)X_0=0$ and obtain a contradiction. By (3.9) it would follow that $\gamma_\xi(t)=n\cdot\exp(t\{X_0+Z_0\})$ for all $t\in\mathbb{R}$. Write $n=\exp(\xi_0)$ for a unique element $\xi_0\in\mathcal{N}$. We assert

$$(\star)\qquad \omega Z_0=Z^*+[V^*,\xi_0]$$

To prove this we observe that on the one hand $\gamma_\xi(\omega)=n\cdot\exp(\omega\{X_0+Z_0\})=\exp(\xi_0)\cdot\exp(\omega X_0+\omega Z_0)=\exp\left(\omega X_0+\omega Z_0+\xi_0+\frac{\omega}{2}[\xi_0,X_0]\right)$. On the other hand $\gamma_\xi(\omega)=\phi\cdot n=\exp(V^*+Z^*)\cdot\exp(\xi_0)=\exp\left(V^*+Z^*+\xi_0+\frac{1}{2}[V^*,\xi_0]\right)$. Comparing the $\mathcal{V}$-components of the two expressions of $\log(\gamma_\xi(\omega))$ we see that $\omega X_0=V^*$. Using this fact and comparing the $\mathcal{Z}$-components of the two expressions of $\log(\gamma_\xi(\omega))$ we obtain $(\star)$.

By (3) of Proposition 4.3 we see that Z_0 is orthogonal to $[V^*,\mathcal{N}]$ since by hypothesis $\phi\cdot\gamma_\xi(t)=\gamma_\xi(t+\omega)$ for all $t\in\mathbb{R}$. Hence from $(\star)$. we obtain

$$(\star\star)\qquad \omega Z_0=Z^{**}$$

where Z^{**} is the component of Z^* orthogonal to $[V^*, \mathcal{N}]$. Finally from ($**$) and the fact that $\omega X_0 = V^*$ we obtain $(\omega^*)^2 = |V^*|^2 + |Z^{**}|^2 = \omega^2 |X_0|^2 + \omega^2 |Z_0|^2 = \omega^2$, which contradicts our assumption that $\omega < \omega^*$. We have proved that $j(Z_0) X_0 \neq 0$ if $\omega < \omega^*$.

Now let $\xi \in SN_\omega(\phi)$ be given and assume that case 1) of the lemma does not hold. Then $\gamma'_\xi(t)$ is tangent to the orbit $Z(\phi) \cdot \gamma_\xi(t)$ for all $t \in \mathbb{R}$. Since the tangent spaces to orbits of $Z(\phi)$ form an integrable distribution in N it follows that $\gamma_\xi(\mathbb{R})$ lies in a single integral manifold of this distribution, namely the orbit $Z(\phi) \cdot n$, where $n = \gamma_\xi(0)$.

Let J denote $j(Z_0)$. We define $\xi(t) = dL_n(e^{tJ} X_0 + Z_0) \in S_n N$, the unit vectors of N at n. Clearly $\xi(0) = \xi$ and $\xi(t)$ is nonsingular since $J(X_0) \neq 0$. The fact that ϕ translates γ_ξ by an amount ω implies that $e^{\omega J}$ fixes X_0 by 5) of Proposition 4.3. Hence $\xi(\omega) = \xi$, which proves $2a$).

We prove $2b$) and $2c$). If $\beta(t) = \gamma_\xi(t) \cdot n^{-1}$, then since $\gamma_\xi(\mathbb{R}) \subseteqq Z(\phi) \cdot n$ we obtain

(1) $\beta(t) \in Z(\phi)$ for all $t \in \mathbb{R}$

By (3.3) we have $g^t \xi = \gamma'_\xi(t) = dL_{\gamma_\xi(t)}(e^{tJ} X_0 + Z_0) = dL_{\{\gamma_\xi(t) \cdot n^{-1}\}} \xi(t) = dL_{\beta(t)} \xi(t)$. If we fix a number t, then we conclude

(2) $\gamma_{\xi(t)}(s) = \beta(t)^{-1} \cdot \gamma_\xi(t+s)$ for all $s \in \mathbb{R}$

since both sides of the equation (2) are geodesics in the parameter s that have the same initial velocity $\xi(t)$. Assertions $2b$) and $2c$) of the lemma now follow immediately from (1) and (2). $\square$

5. Lattices and closed geodesics

Let N be a simply connected, nilpotent Lie group with a left invariant metric, and let $\Gamma \subseteqq N$ be a discrete subgroup of N. The group Γ is said to be a **lattice** in N if the quotient manifold $\Gamma \backslash N$ obtained by letting Γ act on N by left translations is compact. In this context this is not an oversimplification of the usual definition of lattice. Noncompact lattices do not exist in N; see for example Theorem 2.1 of [R].

In this section we discuss three problems related to the smoothly closed geodesics in $\Gamma \backslash N$, where Γ is a lattice in N and $\Gamma \backslash N$ is equipped with the metric that makes the projection $\pi : N \to \Gamma \backslash N$ a Riemannian covering projection. We give partial answers to the first two problems and a complete answer to a variation of the third, where one considers the marked length spectrum instead of the length spectrum.

PROBLEM 1. – Are the vectors tangent to smoothly closed unit speed geodesics of $\Gamma \backslash N$ dense in the unit tangent bundle $S(\Gamma \backslash N)$?

PROBLEM 2 . – Can one describe the length spectrum of the smoothly closed geodesics in $\Gamma \backslash N$ in terms of $\log \Gamma \subseteqq \mathcal{N}$, where $\log : N \to \mathcal{N}$ is the inverse of $\exp : \mathcal{N} \to N$?

PROBLEM 3 . – Given a lattice Γ in N can one describe the automorphisms ψ of N such that $\Gamma \backslash N$ and $\psi(\Gamma) \backslash N$ have the same length spectrum of smoothly closed geodesics?

Existence of lattices.

Not every simply connected, nilpotent Lie group N admits a lattice $\Gamma \subseteq N$; *see* section 2.14 of [R] for an example. A proof of the next result can be found in Theorem 2.12 of [R].

(5.1) THEOREM. – *Let N be a simply connected, nilpotent Lie group, and let $\mathcal{N}$ be its Lie algebra. Then N admits a lattice Γ if and only if $\mathcal{N}$ admits a basis $\{ X_1, \cdots, X_n \}$ such that*

$$[X_i, X_j] = \sum_{\alpha=1}^{n} C_{ij}^{\alpha} X_\alpha \text{ for all } i, j, \text{ where the constants } \{ C_{ij}^{\alpha} \} \text{ are all rational.}$$

Extension properties of homomorphisms.

(5.2) PROPOSITION. – *Let N, N^* be simply connected, nilpotent groups, and let Γ, Γ^* be lattices in N, N^*. Then any homomorphism $\phi : \Gamma \to \Gamma^*$ has a unique extension to a continuous homomorphism $\phi : N \to N^*$.*

Proof. – See Theorem 2.11 of [R], p 33. $\square$

Logarithm of a lattice.

(5.3) PROPOSITION. – *Let N be a simply connected, 2-step nilpotent Lie group with a left invariant metric, and Γ be a lattice in N. Let* $\log : N \to \mathcal{N}$ *denote the inverse of* $\exp : \mathcal{N} \to N$, *and let $\pi_{\mathcal{V}} : \mathcal{N} \to \mathcal{V}$ denote the projection onto $\mathcal{V} = \mathcal{Z}^{\perp}$. Then*

1. $\Gamma \cap Z$ *is a lattice in Z and* $\log \Gamma \cap \mathcal{Z}$ *is a lattice in $\mathcal{Z}$.*
2. *Let $\xi \in \log \Gamma$ and ξ^*, ξ_1^*, $\xi_2^* \in \log \Gamma \cap \mathcal{Z}$. Then*
   a) $\xi_1^* + \xi_2^* \in \log \Gamma \cap \mathcal{Z}$.
   b) $\xi + \xi^* \in \log \Gamma$.
   c) $k\,\xi \in \log \Gamma$ *for any integer k.*
3. $\pi_{\mathcal{V}}(\log \Gamma)$ *is a lattice in $\mathcal{V}$.*
4. $\Gamma \cap Z = Z(\Gamma) = \{ \phi \in \Gamma : \phi \cdot \psi = \psi \cdot \phi \text{ for all } \psi \in \Gamma \}$.

Proof. – A lattice in a vector space W is a discrete additive subgroup Γ such that W/Γ is compact in the quotient topology. Equivalently, a lattice Γ is the set of integer linear combinations of some basis of W. The first assertion of 1) holds by [R, Proposition 2.17], and the second assertion of 1) follows since $\exp : \mathcal{Z} \to Z$ is an isomorphism. The assertions in 2) follow easily from (1.2*d*). We prove 3). By (1.2*d*) the map $\tau_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \log : N \to \mathcal{V}$ is a continuous surjective homomorphism with kernel Z, and hence $\tau_{\mathcal{V}}(\Gamma)$ is a discrete subgroup of the additive group $\mathcal{V}$. There exists a compact subset A of N such that $\Gamma \cdot A = N$, and hence $\mathcal{V}/\tau_{\mathcal{V}}(\Gamma)$ is compact since $\mathcal{V} = \tau_{\mathcal{V}}(\Gamma) + \tau_{\mathcal{V}}(A)$. To prove 4) it suffices to show that $Z(\Gamma) \subseteq Z \cap \Gamma$. This follows from (5.2) since an inner automorphism by an element of $Z(\Gamma)$ is the identity on Γ and hence on all of N. $\square$

The associated tori T_B and T_F.

Let $\{ N, \langle\ ,\ \rangle \}$ be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let Γ be a lattice in N. To the compact nilmanifold $\Gamma \backslash N$ we associate two flat tori $T_F = \mathcal{Z}/(\log \Gamma \cap \mathcal{Z})$ and $T_B = \mathcal{V}/\pi_{\mathcal{V}} \log \Gamma$. It is clear from (5.3) that T_F, T_B are tori. We shall see that the length spectrum of $\Gamma \backslash N$ is closely related to the length spectra of T_F and T_B. Moreover, we shall show that $\Gamma\backslash N$ is a Riemannian submersion over T_B with fibers isometric to T_F. For this reason we regard T_F, T_B as the fiber torus, base torus respectively.

Remark. – Palais and Stewart in [PS] showed that the total spaces of principal torus bundles over a torus are precisely the compact 2-step nilmanifolds $\Gamma\backslash N$.

We need a preliminary result to prove the submersion assertion above. This result holds also in the case that N is nilpotent with an arbitrary number of steps.

(5.4) PROPOSITION. – *Let N be a simply connected, nilpotent Lie group with a left invariant metric, and let Γ be a lattice in N. Let T^m denote the m-torus $Z/(\Gamma \cap Z)$, where m is the dimension of Z. Let $k : Z \to Z/(\Gamma \cap Z)$ and $p : N \to \Gamma\backslash N$ denote the projection maps. Let $F : T^m \to I_0(\Gamma\backslash N)$ be the map defined by $F(k(z))(p(n)) = p(z \cdot n)$ for every $z \in Z$, $n \in N$. Then*

1. *F is an isomorphism of T^m onto $I_0(\Gamma\backslash N)$.*
2. *$I_0(\Gamma\backslash N)$ acts freely on $\Gamma\backslash N$, and the orbits of $I_0(\Gamma\backslash N)$ are flat, totally geodesic imbedded m-tori isometric to T_F.*

Proof. – We recall that $\Gamma \cap Z$ is a lattice in Z by (5.3) so that $T^m = Z/(\Gamma \cap Z)$ is indeed an m-torus. It is easy to show that the map $F(k(z)) : \Gamma\backslash N \to \Gamma\backslash N$ is a well defined map on $\Gamma\backslash N$ for each element $z \in Z$, and $F(k(z))$ has an inverse $F(k(z^{-1}))$. Each map $F(k(z))$ is an isometry of $\Gamma\backslash N$ since the maps $p : N \to \Gamma\backslash N$ and $L_z : N \to N$ are local isometries. Hence $F(T^m)$ is a connected subgroup of $I(\Gamma\backslash N)$, and it is routine to show that $F : T^m \to I_0(\Gamma\backslash N)$ is an injective homomorphism. To prove 1) it remains only to show that F is surjective.

LEMMA. – *Let N be a connected, nilpotent Lie group with a left invariant metric. Any inner automorphism of N that is also an isometry of N must be the identity.*

Proof of the lemma. – We actually only need this result in the special case that N is simply connected. Let ϕ an isometry of N that is also an inner automorphism of N determined by an element g of N. Then $\mathrm{Ad}(g) : \mathcal{N} \to \mathcal{N}$ is a linear isometry since $\mathrm{Ad}(g)$ is the differential map $(d\phi)_e : T_e N \to T_e N$. Since N is nilpotent the exponential map $\exp : \mathcal{N} \to N$ is surjective (*cf.* [Hel], p. 229) and we may choose $X \in \mathcal{N}$ such that $\exp(X) = g$. Hence $d\phi = \mathrm{Ad}(g) = e^{\mathrm{ad}X}$ is unipotent since $\mathrm{ad}X$ is nilpotent. A unipotent linear isometry must be the identity. $\square$

We now complete the proof of 1) by showing that $F : T^m \to I_0(\Gamma\backslash N)$ is surjective. Let an element α of $I_0(\Gamma\backslash N)$ be given, and let $\varphi : [0, 1] \to I_0(\Gamma\backslash N)$ be a continuous curve such that $\varphi(0) = \mathrm{Id}$ and $\varphi(1) = \alpha$. Let $\varphi^* : [0, 1] \to I_0(N)$ be a continuous lift of φ starting at Id; that is, $\varphi^*(0) = \mathrm{Id}$ and $p \circ \varphi^*(t) = \varphi(t) \circ p$ for all t, where $p : N \to \Gamma\backslash N$ is the projection. By (2.8) we can find elements $g(t)$ in N and $A(t)$ in $\mathrm{Aut}(N) \cap I(N)$ such that $\varphi^*(t) = L_{g(t)} \circ A(t)$ for all t. Since Γ is discrete it is easy to see $\varphi^*(t)$ commutes with L_γ for all t in [0, 1] and all γ in Γ. It follows that $A(t)$ acting on Γ is the inner automorphism by $g(t)^{-1}$ for every t. By (5.2) and the lemma above $A(t) = \mathrm{Id}$, $g(t) \in Z$ and $\varphi^*(t) = L_{g(t)}$ for all t. Hence $p \circ L_{g(1)} = p \circ \varphi^*(1) = \varphi(1) \circ p = \alpha \circ p$, which is equivalent to the assertion that $F(k(g(1))) = \alpha$.

We prove 2). From 1) it is easy to see that $I_0(\Gamma\backslash N)$ acts freely on $\Gamma\backslash N$, and hence the orbits of $I_0(\Gamma\backslash N)$ are imbedded m-tori. It follows immediately from (2.1) that $\nabla_Z Z^* = 0$ for all elements Z, Z^* of $\mathcal{Z}$, regardless of the number of steps of $\mathcal{N}$. Hence the orbits of Z in N are flat, totally geodesic submanifolds of N. It follows that the orbits of $I_0(\Gamma\backslash N)$ are flat, totally geodesic submanifolds of $\Gamma\backslash N$ since by 1) they are projections of orbits of Z in N under the local isometry $p : N \to \Gamma\backslash N$.

It remains only to show that the orbits of $I_0(\Gamma\backslash N)$ are isometric to $T_F = \mathcal{Z}/(\log \Gamma \cap \mathcal{Z})$. For each point $n \in N$ we define a map $I_n : \mathcal{Z} \to \Gamma\backslash N$ by $I_n = p \circ L_n \circ \exp$. Note that I_n is a local isometry since $p : N \to \Gamma\backslash N$ is a local isometry while $L_n : N \to N$ and $\exp : \mathcal{Z} \to Z$ are isometries. It is routine to show that $I_n(\xi) = I_n(\xi^*)$ for elements ξ, $\xi^* \in \mathcal{Z}$ if and only if $\xi^* = \xi + \log\gamma$ for some element $\gamma \in \Gamma \cap Z$. Hence I_n induces an isometry I_n^* from $T_F = \mathcal{Z}/(\log \Gamma \cap \mathcal{Z})$ into $\Gamma\backslash N$. The image of I_n^* is the orbit of $p(n)$ under $I_0(\Gamma\backslash N)$ by 1), and hence I_n^* is an isometry onto this orbit. $\square$

(5.5) PROPOSITION. – *Let Γ be a lattice in a simply connected, 2-step nilpotent Lie group N with a left invariant metric. Let $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ be the Lie algebra of N*, and let $\pi_{\mathcal{V}} : \mathcal{N} \to \mathcal{V}$ *denote the projection. Let $T_B = \mathcal{V}/\pi_{\mathcal{V}}(\log \Gamma)$. Then*

1. *T_B is a flat torus of dimenion n, where $n = \dim \mathcal{V}$.*

2. *There exists a Riemannian submersion $\pi : \Gamma\backslash N \to T_B$ whose fibers are the orbits of $I_0(\Gamma\backslash N)$. Hence $\Gamma\backslash N$ is a principal torus bundle over T_B whose fibers are flat, totally geodesic imbedded tori isometric to T_F.*

Proof. – The first assertion is obvious since $\pi_{\mathcal{V}}(\log \Gamma)$ is a vector lattice in $\mathcal{V}$ by Proposition (5.3). We prove 2). By *d*) of (1.2) the map $\tau_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \log : N \to \mathcal{V}$ is a group homomorphism whose kernel is Z, where $\mathcal{V}$ is regarded as an additive abelian group. Define $\pi : \Gamma\backslash N \to T_B$ by

$$(*) \qquad \pi \circ p = Q \circ \tau_{\mathcal{V}}$$

where $Q : \mathcal{V} \to T_B$ is the projection homomorphism. The projection π is well defined and since $Q \circ \tau_{\mathcal{V}} : N \to T_B$ is a homomorphism with kernel$= Z \cdot \Gamma$ it follows from $(\star)$ and 1) of (5.4) that the fibers of π are the orbits of $I_0(\Gamma\backslash N)$.

It remains only to prove that $\pi : \Gamma\backslash N \to T_B$ is a Riemannian submersion. If $\xi \in T_{p(n)}(\Gamma\backslash N)$ is orthogonal to $I_0(\Gamma\backslash N)(p(n))$, the fiber of π through $p(n)$, then it is easy to see from 1) of (5.4) that $\xi = (dp \circ dL_n)(X)$ for some $X \in \mathcal{V}$. Hence by $(\star)$ $d\pi(\xi) = (dL_{\pi(p(n))} \circ dQ \circ d\tau_{\mathcal{V}})(X)$ since $\pi \circ p = Q \circ \tau_{\mathcal{V}}$ is a homomorphism. The maps p, L_n, $L_{\pi(p(n))}$ and Q are local isometries, and $d\tau_{\mathcal{V}}$ is the identity on $\mathcal{V}$. Hence $|d\pi(\xi)| = |X| = |\xi|$. $\square$

We now consider the behavior of closed geodesics in $\Gamma\backslash N$, where Γ is a lattice in a simply connected, 2-step nilpotent Lie group with a left invariant metric.

Density of smoothly closed geodesics.

In this section we show that the vectors tangent to smoothy closed unit speed geodesics are not always dense in the unit tangent bundle $S(\Gamma\backslash N)$, where Γ is a lattice in a simply connected, 2-step nilpotent Lie group N with a left invariant metric. In particular, if N has

1-dimensional center we find necessary and sufficient conditions for this density property to hold. However, if N is of Heisenberg type, then this density property always holds.

(5.6) PROPOSITION. – *Let N be a simply connected, 2-step nilpotent Lie group of Heisenberg type, and let Γ be any lattice in N. Then the vectors tangent to smoothly closed unit speed geodesics in $\Gamma\backslash N$ are dense in $S(\Gamma\backslash N)$.*

Remark. – Mast has recently shown in [Ma] that the density result above holds under the weaker condition that N be nonsingular and in resonance. The proof in [Ma] is a generalization of the one given here.

Proof. – Let N be of Heisenberg type, and let Γ be a lattice in N. Let U be any open subset of $\mathcal{N} = T_e N$. We will show that there exist a nonzero element $\xi \in U$ and a nonidentity element φ of $\Gamma \cap Z$ such that $\varphi \cdot \gamma_\xi(t) = \gamma_\xi(t+\omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$, where γ_ξ denotes the geodesic with $\gamma_\xi(0) = e$ and $\gamma_\xi{}'(0) = \xi$. Since $\varphi \in Z$ it then follows that $\varphi \cdot (L_n \circ \gamma_\xi(t)) = (L_n \circ \gamma_\xi(t+\omega))$ for all $t \in \mathbb{R}$ and all $n \in N$. This will complete the proof of (5.6).

Let ξ be any nonzero vector of $\mathcal{N}$-$\mathcal{V}$, and write $\xi = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. By (3.8) we have

(1) $\gamma_\xi(\omega) = \exp(F(\xi)) \in Z$, where $\omega = \dfrac{2\pi}{|Z_0|}$ and $F(\xi) = \dfrac{2\pi}{|Z_0|}\left(1 + \dfrac{|X_0|^2}{2|Z_0|^2}\right) Z_0$

Further inspection of (3.8) shows

(2) ω is the first positive number for which $\gamma_\xi(t)$ lies in Z.

(3) $\gamma_\xi(n\omega) = \exp(n F(\xi))$ for every positive integer n.

It is straightforward to verify that $F : \mathcal{N}\text{-}\mathcal{V} \to \mathcal{Z}$ has maximal rank $m = \dim \mathcal{Z}$ at all points ξ of $\mathcal{N}$-$\mathcal{V}$; in fact, for every $\xi \in \mathcal{N}\text{-}\mathcal{V}$ the differential map dF is nonsingular on the m-dimensional subspace $\mathcal{Z}_\xi = \{Z_\xi \in T_\xi \mathcal{N} : Z \in \mathcal{Z}\}$, where Z_ξ denotes the initial velocity of $Z(t) = \xi + tZ$.

Now let U be any open subset of $\mathcal{N}$. Since F has maximal rank on $U_0 = U \cap (\mathcal{N}\text{-}\mathcal{V})$ it follows that $F(U_0)$ contains an open subset W of $\mathcal{Z}$. Let n be a positive integer such that for any integer $m \geqq n$ the set $mW = \{m\alpha : \alpha \in W\}$ contains a nonzero element of the vector lattice $\log \Gamma \cap \mathcal{Z}$. Let $\xi = X_0 + Z_0$ be an element of U_0 such that $nF(\xi) \in \log \Gamma \cap \mathcal{Z}$. By (3) we see that $\varphi = \gamma_\xi(n\omega)$ is a nonidentity element of $\Gamma \cap Z$, where $\omega = \dfrac{2\pi}{|Z_0|}$. Hence $\varphi \cdot \gamma_\xi(t) = \gamma_\xi(t + n\omega)$ for all $t \in \mathbb{R}$ by (4.9). $\square$

Groups N with 1-dimensional center.

If N is a simply connected, 2-step nilpotent group with a 1-dimensional center, then we can give a complete answer to the density of closed geodesics problem.

(5.7) PROPOSITION. – *Let N be a simply connected, 2-step nilpotent group with a 1-dimensional center and a left invariant metric. Then the following properties are equivalent:*

1) *For any lattice Γ in N the vectors tangent to smoothly closed unit speed geodesics in $\Gamma\backslash N$ are dense in $S(\Gamma\backslash N)$.*

2) *For some lattice* Γ *in* N *the vectors tangent to smoothly closed unit speed geodesics in* $\Gamma\backslash N$ *are dense in* $S(\Gamma\backslash N)$.

3) *The linear transformations* $j(Z) : \mathcal{V} \to \mathcal{V}$ *are in resonance for all nonzero elements* Z *in* $\mathcal{Z}$ [*see* (4.12)].

Proof. – We prove these assertions in the cyclic order $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 1)$. The assertion $1) \Rightarrow 2)$ is obvious. We prove $2) \Rightarrow 3)$. Let Z^* be one of the two unit vectors in $\mathcal{Z}$, and let $\mathcal{V} = W_1 \oplus \ldots \oplus W_k$ be an orthogonal direct sum decomposition of $\mathcal{V}$ into distinct eigenspaces W_i of $j(Z^*)^2$ that are invariant under $j(Z^*)$. We may assume that $k \geqq 2$ for otherwise $j(Z^*)$ is clearly in resonance. Hence if $\mathcal{N}_1$ denotes the set of unit vectors in $\mathcal{N}$, then there exists a dense open subset $\mathcal{N}_1{}^*$ of $\mathcal{N}_1$ with the following properties: Let ξ^* be an element of $\mathcal{N}_1{}^*$ with $\mathcal{V}$, $\mathcal{Z}$ components X, Z. Then

(*a*) X, Z are both nonzero.

(*b*) For $1 \leqq i \leqq k$ the component X_i of X in W_i is nonzero.

Since $\mathcal{Z}$ is 1-dimensional it suffices to prove that $j(Z_0)$ is in resonance for a single nonzero vector Z_0 in $\mathcal{Z}$. Let $\xi \in \mathcal{N}_1{}^*$ be given. By hypothesis there exists a lattice Γ in N such that the vectors in $S(\Gamma\backslash N)$ tangent to smoothly closed geodesics of $\Gamma\backslash N$ are dense in $S(\Gamma\backslash N)$. Hence there exists a nonidentity element φ in Γ, a point n in N close to e and a vector $\xi_0 \in \mathcal{N}_1{}^*$ close to ξ such that $\varphi \cdot \gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$, where $\gamma(t)$ is the unit speed geodesic in N such that $\gamma'(0) = dL_n(\xi_0)$. If we write $\xi_0 = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$, then X_0, Z_0 are both nonzero by *a*), and $e^{\omega j(Z_0)}$ fixes X_0 by 5) of (4.3). If X_r is the component of X_0 in W_r, then $X_r \neq 0$ for $1 \leqq r \leqq k$ by *b*). Moreover, we note that

$$(\star) \qquad e^{\omega j(Z_0)} \text{ fixes } X_r \qquad \text{for} \quad 1 \leqq r \leqq k$$

since $j(Z_0)$ leaves each subspace W_r invariant. Since $j(Z_0)^2 = -\theta_r{}^2 \,\mathrm{Id}$ on W_r for some positive number θ_r it follows from $(\star)$ and (3.6) that $\theta_r = 2\pi N_r/\omega$ for some nonzero integer N_r, $1 \leqq r \leqq k$. Hence $j(Z_0)$ is in resonance since $\{\pm i\theta_1, \ldots, \pm i\theta_k\}$ are the eigenvalues of $j(Z_0)$.

We prove $3) \Rightarrow 1)$. We begin by examining the geodesic equations in (3.5) in the case that N has 1-dimensional center Z. Let ξ be a nonzero vector in $\mathcal{N}$-$\mathcal{V}$ and write $\xi_0 = X_0 + Z_0$, where $X_0 \in \mathcal{V}$ and $Z_0 \in \mathcal{Z}$. The group N is nonsingular since Z is 1-dimensional, and hence in the terminology of (3.5) we have

a) $X_1 = 0$

In the expression for $Z_1(t)$ in 2*a*) of (3.5) the third term is collinear with Z_0 since $\mathcal{Z}$ is 1-dimensional, and hence it equals $\dfrac{1}{2}\dfrac{|X_0|^2}{|Z_0|^2}Z_0$. From 2*a*) we now obtain

b) $Z_1(t) = \left(1 + \dfrac{1}{2}\dfrac{|X_0|^2}{|Z_0|^2}\right) Z_0$ for all $t \in \mathbb{R}$

Now let Z^* be any unit vector in $\mathcal{Z}$. By hypothesis $j(Z^*)$ is in resonance, and by (4.13) there exists a positive number ω^* such that $e^{\omega^* j(Z^*)} = \mathrm{Id}$. We proceed as in the proof of (5.6). For a nonzero vector $\xi = X_0 + Z_0$ in $\mathcal{N}$-$\mathcal{V}$ we define $\omega(\xi) = \dfrac{\omega^*}{|Z_0|}$. From the definition of ω^* we obtain *c)* $e^{\omega(\xi) j(Z_0)} = \mathrm{Id}$

In the notation of (3.5), from (*c*) we obtain

d) $e^{n\omega(\xi) j(Z_0)} = \mathrm{Id}$ and $Z_2(n\omega(\xi)) = 0$ for every positive integer n

If we define $F : \mathcal{N}\text{-}\mathcal{V} \to \mathcal{Z}$ by

$$F(\xi) = \frac{\omega^*}{|Z_0|}\left(1 + \frac{1}{2}\frac{|X_0|^2}{|Z_0|^2}\right) Z_0$$

then from *a*), *b*), *c*), *d*) and (3.5) we obtain

e) $\gamma_\xi(n\omega(\xi)) = \exp(nF(\xi))$ for every positive integer n, where γ_ξ is the geodesic in N with initial velocity ξ.

As in the proof of (5.6) it is routine to show that F has maximal rank at every point of $\mathcal{N}\text{-}\mathcal{V}$. We now proceed exactly as in the last paragraph of the proof of (5.6) to show that for any lattice Γ in N the vectors in $S(\Gamma\backslash N)$ tangent to smoothly closed geodesics of $\Gamma\backslash N$ are dense in $S(\Gamma\backslash N)$. □

(5.8) AN EXAMPLE OF NONRESONANT BEHAVIOR. – Examples of nonresonant behavior are easy to construct. We construct a 5-dimensional, simply connected, 2-step nilpotent Lie group N with 1-dimensional center such that the linear transformations $j(Z) : \mathcal{V} \to \mathcal{V}$ are not in resonance for any nonzero Z in $\mathcal{Z}$. Moreover, the group N admits lattices Γ, one of which we describe explicitly. It follows from Proposition 5.7 that the vectors tangent to smoothly closed unit speed geodesics in $\Gamma\backslash N$ are not dense in $S(\Gamma\backslash N)$ for any choice of lattice Γ in N.

Define $\mathcal{N}$ to be the 5-dimensional real vector space with basis $\{X_1, X_2, X_3, X_4, Z\}$, and make $\mathcal{N}$ into a 2-step nilpotent Lie algebra by defining a bracket operation characterized by the relations $[X_1, X_2] = -[X_2, X_1] = Z$, and $[X_3, X_4] = -[X_4, X_3] = \lambda Z$, where $\lambda \neq 0$ is irrational; all other brackets between basis elements are zero. It is easy to check that $\mathcal{N}$ is 2-step nilpotent with a 1-dimensional center spanned by Z. Now give $\mathcal{N}$ the inner product that makes $\{X_1, X_2, X_3, X_4, Z\}$ into an orthonormal basis and give N the corresponding left invariant metric, where N is the simply connected, 2-step nilpotent Lie group with Lie algebra $\mathcal{N}$. It is routine to calculate

$$j(Z)X_1 = X_2, \qquad j(Z)X_2 = -X_1$$
$$j(Z)X_3 = \lambda X_4, \qquad j(Z)X_4 = -\lambda X_3$$

Hence $j(Z)^2 = -\mathrm{Id}$ on $W_1 = \mathrm{span}\{X_1, X_2\}$ and $-\lambda^2\,\mathrm{Id}$ on $W_2 = \mathrm{span}\{X_3, X_4\}$ so the eigenvalues of $j(Z)$ are $\{\pm i, \pm\lambda i\}$. It follows that $j(Z)$ is not in resonance since λ is irrational. Therefore $j(Z^*)$ is not in resonance for any nonzero $Z^* \in \mathcal{Z}$.

Next we show that N admits a lattice Γ. Consider the basis $S = \{X_1, X_2, (1/\lambda)X_3, X_4, (1/2)Z\}$ of $\mathcal{N}$. Observe that

$$(\star) \qquad \frac{1}{2}[\xi_1, \xi_2] \in S \cup \{0\} \quad \text{for all } \xi_1, \xi_2 \in S$$

It follows by the rationality criterion (5.1) that N admits a lattice Γ. We construct a lattice Γ explicity as follows. Let S^* denote the set of all integer linear combinations of elements of S, and let $\Gamma = \exp(S^*)$. The property $(\star)$ implies that $\xi + \eta + \frac{1}{2}[\xi, \eta] \in S^*$

whenever $\xi, \eta \in S^*$. It follows by (1.1) that Γ is a discrete subgroup of N, and it is easy to see that Γ is a lattice in N since S^* is a lattice in $\mathcal{N}$. $\square$

Length spectrum and maximal length spectrum.

(5.9) DEFINITION. – *Let M be a compact Riemannian manifold. For each nontrivial free homotopy class $\mathcal{C}$ of closed curves in M (i.e. $\mathcal{C}$ does not contain the constant curves) we define $l(\mathcal{C})$ to be the collection of all lengths of smoothly closed geodesics that belong to $\mathcal{C}$.*

If M is nonsimply connected, then one may write M as a quotient manifold $\tilde{M}/\Gamma$, where $\tilde{M}$ is simply connected and Γ is a nonidentity discrete subgroup of isometries of $\tilde{M}$. A free homotopy class of closed curves in M corresponds to a conjugacy class of an element ϕ in Γ as we observed at the beginning of section 4. The collection $l(\mathcal{C})$ is then precisely the set of periods of ϕ; note that conjugate elements of Γ have the same periods.

If $M_i = \tilde{M}_i/\Gamma_i$ for $i = 1, 2$ are compact Riemannian manifolds, and if $T : \Gamma_1 \to \Gamma_2$ is a homomorphism of the fundamental group of M_1 into the fundamental group of M_2, then T maps conjugacy classes in Γ_1 into conjugacy classes in Γ_2 and hence induces a map T_* from the set of free homotopy classes of closed curves in M_1 into the set of free homotopy classes of closed curves in M_2.

(5.10) DEFINITION. – *The* **length spectrum** *of a compact Riemannian manifold M is the collection of all ordered pairs (L, m), where L is the length of a closed geodesic in M and m is the multiplicity of L,* i.e. *m is the number of free homotopy classes $\mathcal{C}$ of closed curves in M that contain a closed geodesic of length L.*

(5.11) DEFINITION. – *Two compact Riemannian manifolds M_1, M_2 are said to have the same* **marked length spectrum** *if there exists an isomorphism T from the fundamental group of M_1 onto the fundamental group of M_2 such that $l(T_*(\mathcal{C})) = l(\mathcal{C})$ for all nontrivial free homotopy classes of closed curves in M_1, where T_* denotes the induced map on free homotopy classes.*

Let $M = \Gamma\backslash N$, where N is a simply connected, 2-step nilpotent Lie group with a left invariant metric and Γ is a lattice in N. We have seen in Proposition 4.5 that for each free homotopy class $\mathcal{C}$ of closed curves in M there is a maximum value ω^* in the collection $l(\mathcal{C})$. We let $l^*(\mathcal{C})$ denote this maximum value ω^*.

(5.12) DEFINITION. – *Let $M = \Gamma\backslash N$, where N is a simply connected, 2-step nilpotent Lie group with a left invariant metric and Γ is a lattice in N. The* **maximal length spectrum** *of M is the collection of all ordered pairs (L, m), where $L = l^*(\mathcal{C})$ for some free homotopy class $\mathcal{C}$ of closed curves in M and m is the number of free homotopy classes $\mathcal{C}$ for which $L = l^*(\mathcal{C})$.*

(5.13) DEFINITION. – *Let $M_i = \Gamma_i\backslash N_i$ for $i = 1, 2$, where Γ_i is a lattice in a simply connected, 2-step nilpotent Lie group N_i with a left invariant metric. We say that M_1 and M_2 have the same* **marked maximal length spectrum** *if there exists an isomorphism $T : \Gamma_1 \to \Gamma_2$ such $l^*(T_*\mathcal{C}) = l^*(\mathcal{C})$ for every nontrivial free homotopy class of closed curves in M_1.*

(5.14) *Remarks.* – 1. Manifolds $M_i = \Gamma_i\backslash N_i$, $i = 1, 2$, with the same marked length spectrum have the same marked maximal length spectrum.

2. Manifolds $M_i = \Gamma_i \backslash N_i$, $i = 1, 2$, with isomorphic fundamental groups Γ_i are homeomorphic. Moreover, any abstract isomorphism of Γ_1 onto Γ_2 is the restriction of an isomorphism of N_1 onto N_2. See (5.2) and Corollary 2 of [R], p. 34.

Calculation of the length spectrum.

For the rest of this section we consider only compact manifolds $M = \Gamma \backslash N$, where Γ is a lattice in a simply connected, 2-step nilpotent Lie group with a left invariant metric. In principle one would like to describe the length spectrum of $\Gamma \backslash N$ in terms of $\log \Gamma \subseteq \mathcal{N}$. This can be done for flat tori $T^n = \mathbb{R}^n/\Gamma$, where Γ is a lattice of translations in $\mathbb{R}^n$ [BGM], but this is rarely possible in our context for $\Gamma \backslash N$, except when N is of Heisenberg type; see (5.16). C. Gordon has calculated the length spectrum in the case that N has a 1-dimensional center [G1]. In the case that N is nonsingular with a center of arbitrary dimension we have the following partial result that is an immediate consequence of Corollary (4.6) and (5.12).

(5.15) PROPOSITION. – *Let $M = \Gamma \backslash N$, where N is nonsingular. The length spectrum of $T_B = \mathcal{V}/\pi_{\mathcal{V}} \log \Gamma$ is precisely the length spectrum of those free homotopy classes $\mathcal{C}$ of closed curves in $\Gamma \backslash N$ that do not contain an element in the center of $\Gamma = \pi_1(\Gamma \backslash N)$. The length spectrum of $T_F = \mathcal{Z}/(\log \Gamma \cap \mathcal{Z})$ is precisely the* **maximal** *length spectrum of those free homotopy classes $\mathcal{C}$ of closed curves in $\Gamma \backslash N$ that contain an element in the center of $\Gamma = \pi_1(\Gamma \backslash N)$.*

If ϕ is an element of $\Gamma \cap Z$ and $|\log \phi|$ is sufficiently large, then ϕ will have more than one period by Proposition 4.11, or equivalently $l(\mathcal{C})$ will contain more than one number, where the free homotopy class $\mathcal{C}$ corresponds to the element ϕ in Γ. In general one may calculate only the largest of the numbers in $l(\mathcal{C})$, namely, $l^*(\mathcal{C}) = |\log \phi|$ (cf. (4.5)). However, if N is of Heisenberg type, then one may calculate $l(\mathcal{C})$ completely. We omit the proof of the next result, which follows from (3.8).

(5.16) PROPOSITION. – *Let $M = \Gamma \backslash N$, where Γ is a lattice in a simply connected, 2-step nilpotent Lie group of Heisenberg type. Let ϕ be a nonidentity element of $\Gamma \cap Z$, and let $Z^* = \log \phi \in \mathcal{Z}$. Then ϕ has the following periods:*

$$\{|Z^*|,\ (4\pi k)^{1/2}(|Z^*| - \pi k)^{1/2},\ \text{where } k \text{ is an integer such that } \left\{1 \leqq k < \frac{1}{2\pi}|Z^*|\right\}.$$

In particular ϕ has only finitely many periods and a unique period if $|Z^*| \leqq 2\pi$.

(5.17) COROLLARY. – *Let N, N^* be simply connected, 2-step nilpotent Lie groups of Heisenberg type, and let Γ, Γ^* be lattices in N, N^*. Let $\{T_B, T_F\}$ and $\{T_{B^*}, T_{F^*}\}$ be the pairs of flat tori associated to $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ respectively. Assume that $\{T_B, T_{B^*}\}$ have the same length spectrum, and $\{T_F, T_{F^*}\}$ have the same length spectrum. Then $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ have the same length spectrum.*

Proof. – By (5.16) the maximal length spectrum determines the length spectrum for those nontrivial free homotopy classes of closed curves in $\Gamma \backslash N$ (respectively $\Gamma^* \backslash N^*$) that contain an element from the center of $\Gamma = \pi_1(\Gamma \backslash N)$ (respectively $\Gamma^* = \pi_1(\Gamma^* \backslash N^*)$). The result now follows immediately from (5.15). □

Lattices with the same marked length spectrum.

Let N, N^* be simply connected, 2-step nilpotent Lie groups, and let ψ be an isomorphism of N onto N^*. Given a lattice Γ in N we consider $\Gamma^* = \psi(\Gamma)$, which is a lattice in N^*.

Suppose now that N and N^* have left invariant metrics such that $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum (respectively marked maximal length spectrum) relative to the isomorphism $\psi : \Gamma \to \Gamma^*$. What, if anything, can one say about the isomorphism ψ? In what follows we give a complete description of the isomorphisms ψ with this length spectrum preserving property.

We recall from (5.14) that if Γ, Γ^* are isomorphic lattices in simply connected, 2-step nilpotent Lie groups N, N^*, then the isomorphism between Γ and Γ^* is the restriction of an isomorphism ψ of N onto N^*.

In the case that $N = N^*$ we describe two examples of length spectrum preserving automorphisms ψ of N. We shall then show that these two examples are essentially the only examples.

Example 1. – Let N be a simply connected, 2-step nilpotent group with a left invariant metric. Let ψ be an element of $\mathrm{Aut}\,(N) \cap I\,(N)$, where $\mathrm{Aut}\,(N)$ denotes the group of automorphisms of N and $I\,(N)$ denotes the isometry group of N. Clearly if Γ is any lattice in N, then $\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$ are isometric and hence have the same marked length spectrum and marked maximal length spectrum.

Example 2. – Let Γ be a lattice in N, and let ψ be an automorphism of N that is $\Gamma-$ **almost inner**; that is, for every element $\gamma \in \Gamma$ there exists an element $a \in N$, possibly depending on γ, such that $\psi\,(\gamma) = a^{-1} \cdot \gamma \cdot a$. Then $\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$ also have the same marked length spectrum and marked maximal length spectrum. Moreover, by [DG1] they have the same spectrum of the Laplacian acting on either functions or p-forms, $1 \leqq p \leqq \dim\ N$.

(5.18) *Remarks*. – 1. An automorphism ψ of N (simply connected, nilpotent with an arbitrary number of steps) is **almost inner** if for every element $\gamma \in N$ (not necessarily lying in some lattice Γ) there exists an element $a \in N$, possibly depending on γ, such that $\psi\,(\gamma) = a^{-1} \cdot \gamma \cdot a$. Clearly an almost inner automorphism ψ of N is $\Gamma-$ almost inner for every lattice Γ of N. The collection of almost inner automorphisms of N is a closed, normal Lie subgroup of $\mathrm{Aut}\,(N)$ ([GW1], Theorem 2.3), and is abelian if N has 2 steps ([DG1], pp. 368-369). The importance of almost inner automorphisms was discovered by C. Gordon and E. Wilson. They proved in [GW1] that if Γ is a lattice in N, then $\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$ have the same spectrum of the Laplacian acting on functions if ψ is almost inner. Actually the proof is valid if ψ is only $\Gamma-$ almost inner. They also showed that in general the group of almost inner automorphisms of N has a larger dimension than the group of inner automorphisms of N. Later C. Gordon and D. de Turck showed in [DG1] that $\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$ have the same spectrum of the Laplacian acting on$p$-forms if Γ is any lattice in N and if ψ is a $\Gamma-$ almost inner automorphism of N. C. Gordon proved in [G1] that a$\Gamma-$ almost inner automorphism ψ of N preserves the marked length spectra of$\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$.

2. If ψ is an inner automorphism of N, then $\Gamma\backslash N$ and $\psi\,(\Gamma)\backslash N$ are isometric, but this is not necessarily the case if ψ is an almost inner automorphism. In Theorem 5.5 of [GW1] Gordon and Wilson completely described the isometry classes of lattices $\psi\,(\Gamma)$, where Γ is a fixed lattice in N and ψ ranges over the group of almost inner automorphisms of N.

3. Recently, there has been much research by R. Brooks, D. de Turck, C. Gordon, H. Gluck, D. Webb and others on isospectral but nonisometric deformations of metrics on $\Gamma\backslash N$, where Γ is a lattice in a simply connected, 2-step nilpotent group with a left invariant metric. *See* for example [BG], [D], [DGGW1-5], [DG1-3], [G1, 2] and [GW1, 2].

For the sake of completeness we give a short proof of the assertion in Example 2 regarding length spectra. See also Lemma 1.7 of [G1].

(5.19) PROPOSITION. – *Let N be any simply connected, 2-step nilpotent Lie group with a left invariant metric. Let Γ be any lattice of N, and let ψ be any $\Gamma-$ almost inner automorphism of N. Then $\Gamma\backslash N$ and $\psi(\Gamma)\backslash N$ have the same marked length spectrum and marked maximal length spectrum relative to the isomorphism $\psi : \Gamma \to \psi(\Gamma)$.*

Proof. – Let N, Γ and ψ be as above. If ϕ is any nonidentity element of Γ, then ϕ and $\psi(\phi)$ have the same periods since they are conjugate. More precisely, if $\phi\cdot\gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$, some $\omega > 0$ and some unit speed geodesic γ of N, then $\psi(\phi)\cdot\gamma^*(t) = \gamma^*(t+\omega)$ for all $t \in \mathbb{R}$, where $\gamma^*(t) = a^{-1}\cdot\gamma(t)$ and $\psi(\phi) = a^{-1}\cdot\phi\cdot a$ for some element $a \in \Gamma$. Hence if $\mathcal{C}$ and $\psi_*(\mathcal{C})$ denote the free homotopy classes of closed curves in $\Gamma\backslash N$ and $\psi(\Gamma)\backslash N$ determined by ϕ and $\psi(\phi)$ respectively, then $l(\mathcal{C}) = l(\psi_*(\mathcal{C}))$ and $l^*(\mathcal{C}) = l^*(\psi_*(\mathcal{C}))$. $\square$

Our main result of this section is that the two examples above determine essentially all of the examples of lattices with the same marked length or marked maximal length spectra. A priori the marked maximal length spectrum seems to carry less information than the marked length spectrum, but in fact the two spectra are equivalent.

(5.20) THEOREM. – *Let Γ, Γ^* be lattices in simply connected, 2-step nilpotent Lie groups N, N^* with left invariant metrics. Assume that $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked maximal length spectrum, and let $\psi : \Gamma \to \Gamma^*$ be an isomorphism that induces this marking. Then $\psi = (\psi_1 \circ \psi_2)|_\Gamma$, where ψ_2 is a $\Gamma-$ almost inner automorphism of N and ψ_1 is an automorphism of N onto N^* that is also an isometry. In particular $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum and the same spectrum of the Laplacian on functions and differential forms.*

(5.21) *Remarks.* – 1. The last statement of the theorem follows from the earlier statements, the two examples above and the first remark following those examples.

2. Recall that an automorphism ψ of N defines a Lie algebra automorphism $d\psi$ of $\mathcal{N}$ that is characterized by the equation $\psi(\exp\xi) = \exp(d\psi(\xi))$ for all $\xi \in \mathcal{N}$. Let Γ be any lattice of N. Using the multiplication law (1.1) for 2-step nilpotent groups it is easy to show that ψ is a $\Gamma-$ almost inner automorphism of N if and only if for any element $\xi \in \log\Gamma$ there is an element $\xi^* \in \mathcal{N}$ such that $d\psi(\xi) = \xi + [\xi^*, \xi]$.

If ψ is a $\Gamma-$ almost inner automorphism of N, where N is simply connected and nilpotent with an arbitrary number of steps, then we assert that $d\psi : \mathcal{N} \to \mathcal{N}$ is unipotent. It follows from the definition in (5.18) that if $\xi \in \log\Gamma$, then there exists an element $a = a(\xi)$ in N such that $d\psi(\xi) = \mathrm{Ad}(a)(\xi)$. If $X = \log a \in \mathcal{N}$, then $\mathrm{Ad}(a) = e^{\mathrm{ad}(X)}$ is a unipotent linear transformation of $\mathcal{N}$ since ad X is nilpotent. The set $\log\Gamma$ spans $\mathcal{N}$ linearly (*see* for example (5.3) or [R, p. 36]), and hence $(d\psi - \mathrm{Id})^n = 0$ for some integer n. If $\mathcal{N}$ has 2 steps, then it follows that $(d\psi - \mathrm{Id})^2 = 0$ by the description of $d\psi$ in the previous paragraph.

Before proving the theorem we discuss two corollaries.

(5.22) COROLLARY. – *Let* Γ, Γ^* *be lattices in simply connected, 2-step nilpotent Lie groups* N, N^* *with left invariant metrics. Assume that* $\Gamma\backslash N$ *and* $\Gamma^*\backslash N^*$ *have the same marked maximal length spectrum. If* $\{T_B, T_F\}$ *and* $\{T_{B^*}, T_{F^*}\}$ *are the associated pairs of flat tori for* $\Gamma\backslash N$ *and* $\Gamma^*\backslash N^*$ *respectively, then* T_B *is isometric to* T_{B^*} *and* T_F *is isometric to* T_{F^*}.

Proof. – Let $\psi : \Gamma \to \Gamma^*$ be an isomorphism that induces the marking of the maximal length spectra of Γ and Γ^*. By theorem (5.20) we need only prove the corollary in the cases 1) ψ is an isometry of N onto N^* that is also an isomorphism 2) $\Gamma^* = \psi(\Gamma)$, where ψ is a Γ– almost inner automorphism of N. The corollary is obviously true in the first case. In the second case assertion 2) of (5.21) shows that $\pi_{\mathcal{V}} \log \Gamma = \pi_{\mathcal{V}}\, d\psi(\log \Gamma) = \pi_{\mathcal{V}} \log \psi(\Gamma)$ and $\log \Gamma \cap \mathcal{Z} = d\psi(\log \Gamma) \cap \mathcal{Z} = \log \psi(\Gamma) \cap \mathcal{Z}$ since $d\psi(\log \Gamma) = \log \psi(\Gamma)$. Hence the corollary is also true in this case. Assertions 1 and 5 of the proof of Theorem (5.20) also give a direct proof of the corollary that avoids consideration of the two separate cases above. $\square$

(5.23) COROLLARY. – *There exist examples of simply connected, nonsingular, 2-step nilpotent Lie groups* N, N^* *with left invariant metrics and lattices* Γ, Γ^* *in* N, N^* *that have the following properties:*

1) $\Gamma\backslash N$ *and* $\Gamma^*\backslash N^*$ *are homeomorphic manifolds with the same maximal length spectra but there exists no isomorphism of* Γ *onto* Γ^* *that preserves the marked maximal length spectra.*

2) $\Gamma\backslash N$ *and* $\Gamma^*\backslash N^*$ *have the same maximal length spectra but have nonisomorphic fundamental groups. In particular they cannot have the same marked maximal length spectra. However, if* $\{T_B, T_F\}$ *and* $\{T_{B^*}, T_{F^*}\}$ *are the associated pairs of flat tori for* $\Gamma\backslash N$ *and* $\Gamma^*\backslash N^*$ *respectively, then* T_B *is isometric to* T_{B^*} *and* T_F *is isometric to* T_{F^*} *[cf. Corollary (5.22)].*

Remark. – Carolyn Gordon has informed me that in example *d*) on pages 75 and 79 of [G1] the manifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same length spectra but different marked length spectra.

Proof. – Let T and T^* be $2n$-dimensional flat tori that have the same length spectrum but are not isometric. (Milnor has given an example of 16-dimensional tori with this property; see [BGM, pp. 154-158]). Write $T = \mathbb{R}^{2n}/L$ and $T^* = \mathbb{R}^{2n}/L^*$ where L and L^* are vector lattices in $\mathbb{R}^{2n}$. Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and $\{X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*\}$ be generating sets for L and L^* respectively. Let $\mathcal{Z}$, $\mathcal{Z}^*$ be 1-dimensional vector spaces with basis elements Z, Z^*. Let $\mathcal{N} = \mathbb{R}^{2n} \oplus \mathcal{Z}$ and $\mathcal{N}^* = \mathbb{R}^{2n} \oplus \mathcal{Z}^*$. Define a bracket operation in $\mathcal{N}$ such that $[X_i, Y_i] = -[Y_i, X_i] = Z$ for $1 \leqq i \leqq n$ with all other brackets of basis vectors $\{X_i, Y_j, Z\}$ being zero. Define a bracket operation in $\mathcal{N}^*$ in similar fashion. The Lie algebras $\mathcal{N}$ and $\mathcal{N}^*$ are 2-step nilpotent. Define an inner product in $\mathcal{N}$ such that $\mathbb{R}^{2n}$ has its canonical inner product, Z has length 1 in $\mathcal{Z}$ and $\mathbb{R}^{2n}$ is orthogonal to $\mathcal{Z}$. Define the analogous inner product in $\mathcal{N}^*$. Let N, N^* denote the simply connected, 2-step nilpotent groups with Lie algebras $\mathcal{N}$, $\mathcal{N}^*$, and equip N, N^* with the left invariant metrics determined by the inner products on $\mathcal{N}$, $\mathcal{N}^*$. The Lie algebras $\mathcal{N}$, $\mathcal{N}^*$ are nonsingular with 1-dimensional center.

We construct lattices Γ, Γ^* in N, N^*. Let S denote the $\mathbb{Z}$-span of

$$\left\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \frac{1}{2} Z\right\} \text{ in } \mathcal{N},$$

and let S^* denote the $\mathbb{Z}$-span of

$$\left\{X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*, \frac{1}{2} Z^*\right\} \text{ in } \mathcal{N}^*.$$

Let $\Gamma = \exp(S) \subseteqq N$ and let $\Gamma^* = \exp(S^*) \subseteqq N^*$. Note that if ξ, η are arbitrary elements of S, then $\xi + \eta + \frac{1}{2}[\xi, \eta]$ lies in S. An analogous statement holds for S^*. From these observations and (1.1) it is easy to verify that Γ, Γ^* are lattice subgroups in N, N^*. Let $\phi : \mathcal{N} \to \mathcal{N}^*$ be the linear transformation such that $\phi(Z) = Z^*$, $\phi(X_i) = X_i^*$ and $\phi(Y_i) = Y_i^*$ for $1 \leqq i \leqq n$. The bracket relations in $\mathcal{N}$ and $\mathcal{N}^*$ show that ϕ is a Lie algebra isomorphism of $\mathcal{N}$ onto $\mathcal{N}^*$ that carries S onto S^*. Hence ϕ is the differential map of an isomorphism ψ of N onto N^* that carries Γ onto Γ^*. The nilmanifolds $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ are homeomorphic by 2) of (5.14). By hypothesis the flat tori $T_B = T$ and $T_{B^*} = T^*$ have the same length spectrum. By construction the 1-dimensional tori T_F and T_{F^*} both have length spectrum $\frac{1}{2}\mathbb{Z} = \left\{\frac{1}{2} n : n \in \mathbb{Z}\right\}$. Hence $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ have the same maximal length spectrum by (5.15). However, the manifolds $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ cannot have the same marked maximal length spectrum by (5.22) since the flat tori $T_B = T$ and $T_{B^*} = T^*$ by hypothesis are not isometric. This completes the discussion of 1).

We construct nilmanifolds $\Gamma \backslash N$ and $\Gamma^* \backslash N^*$ satisfying the conditions in 2) above. Let $\mathcal{N} = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ be the 2-step nilpotent Lie algebra in which $[X_i, Y_i] = -[Y_i, X_i] = Z$ for $1 \leqq i \leqq n$ and all other brackets of basis vectors $\{X_i, Y_j, Z\}$ are zero. Let $\mathcal{N}^* = \operatorname{span}\{X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*, Z^*\}$ be the 2-step nilpotent Lie algebra in which $[X_i^*, Y_i^*] = -[Y_i^*, X_i^*] = 2Z^*$ for $1 \leqq i \leqq n$ and all other brackets of basis vectors $\{X_i^*, Y_j^*, Z^*\}$ are zero. Give $\mathcal{N}$ and $\mathcal{N}^*$ the inner products for which $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ and $\{X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*, Z^*\}$ are orthonormal bases. Let N, N^* be the simply connected, 2-step nilpotent Lie groups with Lie algebras $\mathcal{N}$, $\mathcal{N}^*$ and equip N, N^* with the left invariant metrics determined by the metrics on $\mathcal{N}$, $\mathcal{N}^*$. Define S, S^* and Γ, Γ^* as in the example in 1). Then Γ, Γ^* are lattices in N, N^*, and it is easy to see that the corresponding associated rectangular tori $\{T_B, T_{B^*}\}$ and $\{T_F, T_{F^*}\}$ are isometric. On the other hand it is not difficult to see that Γ is not isomorphic to Γ^*. To verify this let $\xi = \exp\left(\frac{1}{2} Z\right)$ and $\xi^* = \exp\left(\frac{1}{2} Z^*\right)$ denote the generators for the centers of Γ and Γ^*. Let Γ_1 and Γ_1^* denote the quotient groups $\Gamma / \langle \xi^4 \rangle$ and $\Gamma^* / \langle \xi^{*4} \rangle$ respectively, where $\langle \xi^4 \rangle$ and $\langle \xi^{*4} \rangle$ denote the infinite cyclic subgroups of Γ, Γ^* generated by ξ^4 and ξ^{*4}. If there existed an isomorphism ϕ of Γ onto Γ^*, then ϕ would carry $\langle \xi^4 \rangle$ onto $\langle \xi^{*4} \rangle$ and hence would induce an isomorphism of Γ_1 onto Γ_1^*. This however is impossible since Γ_1^* is abelian while Γ_1 is nonabelian. To see that Γ_1 is nonabelian note that $[\exp(X_i), \exp(Y_i)] = \exp([X_i, Y_i]) = \xi^2$ for every i by $(1.2\,b)$. Hence the projections of $\exp(X_i)$ and $\exp(Y_i)$ into Γ_1 do not commute in Γ_1. On the other hand $[\exp(X_i^*), \exp(Y_i^*)] = \exp([X_i^*, Y_i^*]) = \xi^{*4}$ for every i, and it follows that Γ_1^* is abelian. This completes the discussion of 2). $\square$

Remark. – The example just constructed in 2) is a special case of a much more general phenomenon described in Theorem 2.4 of [GW2].

Proof of the theorem 5.20. – We first prove the uniqueness of the decomposition $\psi = (\psi_1 \circ \psi_2)|_\Gamma$.

LEMMA. – *Let $\psi_1 \circ \psi_2 = \psi_1^* \circ \psi_2^*$, where ψ_1, ψ_1^* are isomorphisms of N onto N^* that are also isometries, and ψ_2, ψ_2^* are Γ-almost inner automorphisms of N. Then $\psi_1 = \psi_1^*$ and $\psi_2 = \psi_2^*$.*

Proof. – If $\phi = \psi_2^* \circ \psi_2^{-1}$, then $d\phi : \mathcal{N} \to \mathcal{N}$ is unipotent by 2) of (5.21). However $\phi = \psi_1^{*-1} \circ \psi_1$ also, and hence $d\phi$ is orthogonal. Therefore ϕ is the identity, and the proof of the Lemma is complete. $\square$

To prove the existence of the decomposition $\psi = (\psi_1 \circ \psi_2)|_\Gamma$ in the statement of the theorem will require several further steps. We now extend ψ to be an isomorphism of N onto N^*, which is possible by (5.14). We decompose $d\psi : \mathcal{N} \to \mathcal{N}^*$ as follows: given $V \in \mathcal{V}$ and $Z \in \mathcal{Z}$ we write

$$(\star) \qquad d\psi(V + Z) = A(V) + B(V) + C(Z)$$

where $C : \mathcal{Z} \to \mathcal{Z}^*$ is the restriction of $d\psi$ to $\mathcal{Z}$ and $A : \mathcal{V} \to \mathcal{V}^*$, $B : \mathcal{V} \to \mathcal{Z}^*$ are the linear transformations obtained by projecting $d\psi(V)$ onto $\mathcal{V}^*$ and $\mathcal{Z}^*$ respectively for all $V \in \mathcal{V}$.

We outline briefly the rest of the proof. We show first that $C : \mathcal{Z} \to \mathcal{Z}^*$ and $A : \mathcal{V} \to \mathcal{V}^*$ are isometries and there exists an isomorphism and isometry ψ_1 of N onto N^* such that $d\psi_1(V + Z) = A(V) + C(Z)$ for all $V \in \mathcal{V}$ and $Z \in \mathcal{Z}$. We then define a linear isomorphism $T : \mathcal{N} \to \mathcal{N}$ by $T(V + Z) = V + Z + (C^{-1} \circ B)(V)$ for all $V \in \mathcal{V}$ and $Z \in \mathcal{Z}$, and we show that *a)* $T = d\psi_2$ for some Γ-almost inner automorphism ψ_2 of N and *b)* $d\psi = d\psi_1 \circ d\psi_2$. Finally we conclude that $\psi = \psi_1 \circ \psi_2$, completing the proof of the theorem.

(1) $C : \mathcal{Z} \to \mathcal{Z}^*$ is an isometry.

Proof. – By hypothesis $\Gamma\backslash N$ and $\psi(\Gamma)\backslash N$ have the same marked maximal length spectrum and by the discussion following (5.11) this means that ϕ and $\psi(\phi)$ have the same maximal period for every $\phi \in \Gamma$. If $\phi \in \Gamma \cap Z$, then $\psi(\phi) \in Z^*$, the center of N^*, and ϕ, $\psi(\phi)$ have maximal periods $|\log\phi|$ and $|\log\psi(\phi)|$ respectively by Proposition 4.5. Therefore $|d\psi(\xi)| = |\xi|$ for every $\xi \in \log\Gamma \cap \mathcal{Z}$ since $\log \circ \psi = d\psi \circ \log$. It follows that $|d\psi(Z)| = |Z|$ for every unit vector $Z \in \mathcal{Z}$ of the form $\dfrac{\xi}{|\xi|}$, $\xi \in \log\Gamma \cap \mathcal{Z}$. The set of such unit vectors is dense in the unit sphere of $\mathcal{Z}$ since $\log\Gamma \cap \mathcal{Z}$ is a lattice in $\mathcal{Z}$, and hence $|d\psi(Z)| = |Z|$ for every unit vector $Z \in \mathcal{Z}$. It follows that $C = d\psi|_{\mathcal{Z}}$ is an isometry of $\mathcal{Z}$ onto $\mathcal{Z}^*$. $\square$

We begin the proof that $A : \mathcal{V} \to \mathcal{V}^*$ is an isometry. From the definitions of A and C we obtain

(2) $[A(V), A(V^*)] = C([V, V^*])$ for all $V, V^* \in \mathcal{V}$.

The fact that $d\psi$ is an isomorphism of $\mathcal{N}$ onto $\mathcal{N}^*$ implies that A is an isomorphism of $\mathcal{V}$ onto $\mathcal{V}^*$. Hence from (1) and (2) we obtain

(3) Let $V^* + Z^*$ be an arbitrary element of $\mathcal{N}$. If Z^{**} is the component of Z^* orthogonal to $[V^*, \mathcal{N}]$, then $C(Z^{**})$ is the component of $C(Z^*)$ orthogonal to $[A(V^*), \mathcal{N}^*]$.

Next we assert

(4) Let V be any element of $\pi_{\mathcal{V}} \log \Gamma$. Then $|A(V)| \leqq |V|$ with equality if and only if $B(V) \in [A(V), \mathcal{N}^*]$.

Proof of (4). – Let R be the diameter of some fundamental domain for the vector lattice $\Gamma \cap Z$ in $\mathcal{Z}$. Let V be any element of $\pi_{\mathcal{V}} \log \Gamma$. By (5.3) we can find an element $\xi \in \log \Gamma$ such that $\xi = V + Z$, where $Z \in \mathcal{Z}$ and $|Z| \leqq R$. By (5.3) $n V \in \pi_{\mathcal{V}} \log \Gamma$ for each positive integer n, and hence we can find an element $\xi_n \in \log \Gamma$ such that $\xi_n = n V + Z_n$, where $Z_n \in \mathcal{Z}$ and $|Z_n| \leqq R$. Let $\phi_n = \exp(\xi_n)$. Then $\psi(\phi_n) = \exp(d\psi(\xi_n))$ and $d\psi(\xi_n) = n A(V) + n B(V) + C(Z_n)$. By Proposition 4.5 and (3) the maximal period ω_n of $\psi(\phi_n)$ is given by

$$(a) \qquad \begin{aligned} \omega_n^2 &= n^2 |A(V)|^2 + |n B(V)^{\perp} + C(Z_n)^{\perp}|^2 \\ &= n^2 |A(V)|^2 + n^2 |B(V)^{\perp}|^2 + |C(Z_n^{\perp})|^2 \\ &\quad + 2n \langle B(V)^{\perp}, C(Z_n^{\perp})\rangle, \end{aligned}$$

where $B(V)^{\perp}$, $C(Z_n)^{\perp}$ are the components of $B(V)$, $C(Z_n)$ orthogonal to $[A(V), \mathcal{N}^*]$ and $Z_n^{\perp}$ is the component of Z_n orthogonal to $[V, \mathcal{N}]$.

By hypothesis the maximal period of ϕ_n is also ω_n since ψ preserves the marked maximal length spectrum. Hence we have

$$(b) \qquad \omega_n^2 = |n V|^2 + |Z_n^{\perp}|^2 = n^2 |V|^2 + |Z_n^{\perp}|^2$$

Since $C : \mathcal{Z} \to \mathcal{Z}^*$ is a linear isometry, from (*a*) and (*b*) we obtain

$$(c) \qquad n^2 |V|^2 = n^2 |A(V)|^2 + n^2 |B(V)^{\perp}|^2 + 2n \langle B(V)^{\perp}, C(Z_n^{\perp})\rangle$$

from which we obtain

$$(d) \qquad |A(V)|^2 = |V|^2 - |B(V)^{\perp}|^2 - \frac{2}{n} \langle B(V)^{\perp}, C(Z_n^{\perp})\rangle$$

Now $|\langle B(V)^{\perp}, C(Z_n^{\perp})\rangle| \leqq R |B(V)^{\perp}|$ for every n by the choice of ξ_n and Z_n. Hence the third term on the right hand side of $(d) \to 0$ as $n \to \infty$. Since the other terms in (d) do not depend on n we obtain

$$(e) \qquad |A(V)|^2 = |V|^2 - |B(V)^{\perp}|^2.$$

The assertion of (4) is now clear. □

(5) $A : \mathcal{V} \to \mathcal{V}^*$ is an isometry and $B(V) \in [A(V), \mathcal{N}^*]$ for every $V \in \pi_{\mathcal{V}}(\log \Gamma)$.

Proof. – We show first that $|A(V)| \leqq |V|$ for all $V \in \mathcal{V}$. By (4) this assertion is true for all unit vectors in $\mathcal{V}$ of the form $\dfrac{\xi}{|\xi|}$ for $\xi \in \pi_{\mathcal{V}} \log \Gamma$. Such vectors are dense in the unit sphere of $\mathcal{V}$ since $\pi_{\mathcal{V}}$ log Γ is a lattice in $\mathcal{V}$. Hence $|A(V)| \leqq |V|$ for all $V \in \mathcal{V}$ by the linearity and continuity of A.

By the observation above and (4) it suffices to prove that $|A(V)| = |V|$ for all $V \in \mathcal{V}$. If $V \in \mathcal{V}$ is arbitrary, then it is routine to show that $d(\psi^{-1})(A(V)) = V + Z$ for some element $Z \in \mathcal{Z}$. Applying (4) and the observation above to $\psi^{-1} : N^* \to N$ we obtain $|V| \leqq |A(V)| \leqq |V|$. $\square$

(6) There exists an isomorphism $\psi_1 : N \to N^*$ which is also an isometry such that $d\psi_1(V + Z) = A(V) + C(Z)$ for all $V \in \mathcal{V}$ and all $Z \in \mathcal{Z}$.

Proof. – We define a vector space isomorphism $S : \mathcal{N} \to \mathcal{N}^*$ by $S(V + Z) = A(V) + C(Z)$ for all $V \in \mathcal{V}$ and all $Z \in \mathcal{Z}$. The isomorphism S is a linear isometry since A and C are linear isometries by (1) and (5). Moreover, S is a Lie algebra isomorphism by (2). Define a map $\psi_1 : N \to N^*$ by $\psi_1(\exp\xi) = \exp(S(\xi))$ for every $\xi \in \mathcal{N}$. It is easy to verify that ψ_1 is a Lie group isomorphism and $d\psi_1 = S$. Hence ψ_1 is also an isometry since $S : \mathcal{N} \to \mathcal{N}^*$ is an isometry. $\square$

We are now ready to complete the proof of Theorem 5.20. We define a linear isomorphism $T : \mathcal{N} \to \mathcal{N}$ by setting $T(V + Z) = V + Z + (C^{-1} \circ B)(V)$ for all $V \in \mathcal{V}$ and $Z \in \mathcal{Z}$. It is routine to verify that

(*a*) $(d\psi_1 \circ T)(V + Z) = d\psi(V + Z)$ for all $V \in \mathcal{V}$ and $Z \in \mathcal{Z}$.

Next, by (2) and (5) we obtain

(*b*) $(C^{-1} \circ B)(V) \in [V, \mathcal{N}^*]$ for all $V \in \pi_{\mathcal{V}}(\log\Gamma)$.

From (*b*) and the definition of T we obtain

(*c*) If ξ is any element of log Γ, then there exists an element $\xi^* \in \mathcal{N}$ such that $T(\xi) = \xi + [\xi, \xi^*]$.

The definition of $T : \mathcal{N} \to \mathcal{N}$ shows that T is the identity on $\mathcal{Z}$, and hence T is a Lie algebra automorphism of $\mathcal{N}$. If $\psi_2 : N \to N$ is the map defined by $\psi_2(\exp\xi) = \exp(T(\xi))$ for all $\xi \in \mathcal{N}$, then it is routine to verify that ψ_2 is an automorphism of N such that $d\psi_2 = T$. Hence form (*c*) and 2) of Remark 5.21 we conclude that

(*d*) $d\psi_2 = T$ and ψ_2 is a $\Gamma-$ almost inner automorphism of N.

Finally from (*a*) and (*d*) we conclude that $d\psi = d\psi_1 \circ d\psi_2 = d(\psi_1 \circ \psi_2)$, which proves that $\psi = \psi_1 \circ \psi_2$ and concludes the proof of Theorem 5.20. $\square$

Conjugacy of geodesic flows.

An important special case where two compact 2-step nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum occurs when there is a conjugacy between the geodesic flows in the unit tangent bundles $S(\Gamma\backslash N)$ and $S(\Gamma^*\backslash N^*)$; that is, there exists a homeomorphism $F : S(\Gamma\backslash N) \to S(\Gamma^*\backslash N^*)$ such that $F \circ g^t = g^{*t} \circ F$, where $\{g^t\}$ denotes the geodesic flow in $S(\Gamma\backslash N)$ and $\{g^{*t}\}$ denotes the geodesic flow in $S(\Gamma^*\backslash N^*)$.

(5.24) Proposition. – Let Γ, Γ^* denote lattices in simply connected, 2-step nilpotent Lie groups N, N^* with left invariant metrics, and let $F : S(\Gamma\backslash N) \to S(\Gamma^*\backslash N^*)$ be a conjugacy of the geodesic flows. Then $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ have the same marked length spectrum.

Proof. – The unit tangent bundle $S(\Gamma\backslash N)$ admits a section since any left invariant unit vector field X on N induces a unit vector field X^* on $\Gamma\backslash N$. Since dim $N \geqq 3$ it follows by Theorem 17.7 of [St, p. 92] that the projection $p : S(\Gamma\backslash N) \rightarrow \Gamma\backslash N$ induces an isomorphism $p_1 : \pi_1(S(\Gamma\backslash N)) \rightarrow \pi_1(\Gamma\backslash N)$. Similarly the projection $p^* : S(\Gamma^*\backslash N^*) \rightarrow \Gamma^*\backslash N^*$ induces an isomorphism $p_1^* : \pi_1(S(\Gamma^*\backslash N^*)) \rightarrow \pi_1(\Gamma^*\backslash N^*)$. Let $F_1 : \pi_1 S(\Gamma\backslash N) \rightarrow \pi_1(S(\Gamma^*\backslash N^*))$ denote the isomorphism induced by the homeomorphism F, and let $\varphi : \pi_1(\Gamma\backslash N) \rightarrow \pi_1(\Gamma^*\backslash N^*)$ denote the isomorphism such that $\varphi \circ p_1 = p_1^* \circ F_1$. It is now routine to show that $l(\mathcal{C}) = l(\varphi * (\mathcal{C}))$ in the notation of (5.9) and (5.11), where $\mathcal{C}$ is any free homotopy class of closed curves in $\Gamma\backslash N$. $\square$

Remark. – It is an interesting open problem whether two compact 2-step nilmanifolds $\Gamma\backslash N$ and $\Gamma^*\backslash N^*$ are isometric if they have conjugate geodesic flows.

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