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RIGIDITY OF HYPERSURFACES IN COMPLEX PROJECTIVE SPACE

By Gary R. JENSEN and Emilio MUSSO

Abstract. – Using the method of moving frames, we prove that if two nondegenerate hypersurfaces in projective space have the same quotient of cubic to quadratic forms, then they are projectively congruent.

Keywords: Projective geometry, quadratic and cubic forms, moving frames, complex conformal structure, Cartan connection.

1. Introduction

In a series of papers dating from 1916, G. Fubini studied the deformation of hypersurfaces in projective space P^{n+1}. For his notion of deformation he generalized to projective space Gauss's notion of applicability of surfaces in Euclidean space. Gauss had considered the problem of when there can be a correspondence preserving distances between two surfaces. A formulation of this which generalizes to any homogeneous space is the following (*cf.* [8] and [11]). Let G be the Euclidean group of motions acting on Euclidean space E^3. Two immersed surfaces, $f, \hat{f}: X \to E^3$ are applicable if there exists a smooth map $a: X \to G$ such that for every $p \in X$, the Taylor expansions about p of $a(p) \circ \hat{f}$ and f agree through first order terms. This is equivalent to the condition that the induced metrics agree: $\langle df, df \rangle = ds^2 = \langle d\hat{f}, d\hat{f} \rangle$.

In the case of projective space $P^{n+1} = G/G_0$, where G is the full group of projective transformations of P^{n+1}, Fubini's generalized notion of applicability must go to the second order. Two immersed hypersurfaces, $f, \hat{f}: X \to P^{n+1}$ are applicable if there exists a smooth (holomorphic in the complex case) map $a: X \to G$ such that for every $p \in P^{n+1}$, the Taylor expansions about p of $a(p) \circ \hat{f}$ and f agree through second order terms.

In his analysis [4] of the projective deformation problem in P^3, Fubini introduced a quadratic form φ and a cubic form ψ on X, defined by the immersion. These forms are symmetric, and in the complex case they are holomorphic. He showed that if the

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two immersions are applicable, then

$$\hat{\psi}/\hat{\varphi} = \psi/\varphi \tag{1.1}$$

(*cf.* § 3). Conversely, he showed that (1.1) implies applicability in the case of surfaces in P^3.

In [5] Fubini defined the quadratic and cubic forms for hypersurfaces in P^{n+1}. He characterized hypersurfaces for which φ is identically zero (hyperplanes) and those for which ψ is identically zero (quadrics and developables). He gives an unsatisfactory proof of the claim that two nondegenerate (*see* § 2) hypersurfaces are applicable if and only if (1.1) holds. He promises a more satisfactory proof in a forthcoming paper, which we assume to be [6]. The proof in [6] remains unsatisfactory. Finally, in the book [2] (p. 605-629) a readable proof appears that uses a normalization of the forms which is valid only in the real case.

In 1920 E. Cartan [1] applied his method of exterior differential systems to a study of the projective deformation problem. He rightly pointed out that Fubini's resolution of the problem in terms of the forms φ and ψ failed to answer the basic question of whether there actually exist any nontrivial (*i.e.*, the map $a: X \to G$ is noncontrast) projective deformations. (See Fubini's response in the note [7]). He showed that for generic, nondegenerate surfaces in P^3, there are no nontrivial deformations. He also showed that there do exist special families of surfaces which allow nontrivial deformations. He showed that, when $n > 2$, no nontrivial deformations exist for hypersurfaces in P^{n+1} for which the quadratic form has rank $\geqq 2$ at every point. Using his method of moving frames, Cartan proved Fubini's Theorem in the case of $n = 2$: two nondegenerate surfaces in P^3 are applicable if and only if (1.1) holds.

In this paper we use Cartan's method of moving frames to give a simple, elementary proof of the remaining doubtful case in Fubini's Theorem: If (1.1) holds for two nondegenerate hypersurfaces in $\mathbf{C}P^{n+1}$, then the hypersurfaces are projectively congruent. Although the paper is written exclusively for the complex case, the same proof works in the real case without change, except that one must assume that certain zero divisors are sufficiently thin that the complement of their union is a connected, dense, open subset of X.

Our proof is constructive in the sense that it gives an algebraic procedure, involving only the diagonalization of a symmetric bilinear form and the solution of linear equations, by which one can find the projective group element which brings the one hypersurface into congruence with the other. In more detail, we show in section 2 how to construct a local fourth order frame field e along f. In Proposition 3.2 we show, by a process involving only the solution of linear equations, how to construct a local fourth order frame field $\hat{e}$ along $\hat{f}$, from an arbitrary fourth order frame field along $\hat{f}$, such that (3.4) of Proposition 3.2 is satisfied. By Proposition 3.3 we then have $a\hat{e} = e$ for some constant $a \in G$. That is, $a = e(p)\hat{e}(p)^{-1}$ for *any* $p \in X$, and this a is the element of G sending $\hat{f}(X)$ onto $f(X)$.

We became aware of this problem while reading the paper [9], in Appendix B of which Griffiths and Harris formulate (not quite correctly) Fubini's Theorem, and indicate the part remaining in doubt. To a close approximation, their proposed idea of a proof provided us with the conceptual framework in which to find our proof. In fact, the original version of our proof was more conceptual than what appears here, but it was also four times as long. It involves interpreting the quadratic form φ as a complex conformal structure on X in the sense of LeBrun [12]. Such a structure possesses an analogue of the bundle of orthonormal frames in Riemannian geometry, called the Möbius bundle, on which there is a unique normal, conformal connection (*cf* [13]). More details of this conceptualization are given at the end of section 4.

Our direct proof eschews the need to develop this complex conformal structure theory. We use four frame reductions, in the sense of the method of moving frames (*cf.* [1], [8] or [10]). These reductions were first considered by Musso in [15] for the real case. We are able to show directly that the condition (1.1) implies that for each local fourth order frame field along f there exists an essentially unique local fourth order frame field along $\hat{f}$ such that the pull-backs of the Maurer-Cartan form of G coincide for the two frame fields. The projective congruence then follows from the uniqueness part of the Cartan-Darboux Theorem (*cf.* [14]).

Throughout this paper we will use the Einstein summation convention on repeated indices (even when both are up or both are down). We will also adhere to the following index ranges:

$$0 \leqq \mathrm{I}, \mathrm{J} \leqq n+1$$

$$1 \leqq i, j, k, l, p, q, s \leqq n$$

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2. Hypersurfaces in complex projective space

Projective frames. – Let $G = SL(n+2, \mathbf{C})$ act on $\mathbf{C}P^{n+1}$ in the usual way: given a point $[z] \in \mathbf{C}P^{n+1}$ represented by the non-zero vector $z \in \mathbf{C}^{n+2}$, and given $A \in G$, then $A[z] = [Az]$. This action is transitive and almost effective and gives all the projective transformations of $\mathbf{C}P^{n+1}$. Its Lie algebra $\mathcal{G} = sl(n+2\mathbf{C})$ is identified with the Lie algebra of all left invariant holomorphic vector fields on G.

The holomorphic Maurer-Cartan 1-form of G is denoted $\omega = A^{-1} dA$. The entries of ω are denoted $\omega^{\mathrm{I}}_{\mathrm{J}}$, for $\mathrm{I}, \mathrm{J} = 0, \ldots n+1$. They give a holomorphic coframing on G and satisfy the structure equations

$$d\omega^{\mathrm{I}}_{\mathrm{J}} = -\omega^{\mathrm{I}}_{\mathrm{K}} \wedge \omega^{\mathrm{K}}_{\mathrm{J}}. \tag{2.1}$$

Let $\varepsilon_0, \ldots, \varepsilon_{n+1}$ denote the standard basis of $\mathbf{C}^{n+2}$. Designate the origin of $\mathbf{C}P^{n+1}$ by $[\varepsilon_0]$. The (1, 0) tangent space of $\mathbf{C}P^{n+1}$ at $[\varepsilon_0]$ is naturally identified with the span of $\{\varepsilon_1, \ldots, \varepsilon_{n+1}\}$. For any $A \in G$ let $A_{\mathrm{I}} = A\varepsilon_{\mathrm{I}}$ denote the I-th column of A. Then the (1, 0)-tangent space of $\mathbf{C}P^{n+1}$ at $A[\varepsilon_0] = [A_0]$ has a basis $\{A_1, \ldots, A_{n+1}\}$.

The isotropy subgroup at $[\varepsilon_0]$ is the closed complex subgroup

$$G_0=\left\{b(r, B, v)=\begin{pmatrix} r & {}^t v \\ 0 & B \end{pmatrix}: B\in GL(n+1, \mathbf{C}),\ v\in\mathbf{C}^{n+1},\ r\in\mathbf{C},\ r\det B=1\right\}.$$

The projection map

$$\pi:\quad G\to \mathbf{C}P^{n+1}$$

is then a holomorphic principal G_0-bundle.

A projective frame field in $\mathbf{C}P^{n+1}$ is a holomorphic map

$$e:\quad U\to G,$$

from an open subset $U\subset \mathbf{C}P^{n+1}$, such that $\pi\circ e=[e_0]$; that is, a local section of $\pi: G\to \mathbf{C}P^{n+1}$. The columns e_A of e are $\mathbf{C}^{n+2}$-valued functions satisfying

$$de_A=e_B\, e^*\,\omega^B_A.$$

Let

$$f:\quad X\to \mathbf{C}P^{n+1}$$

be an immersed, connected, complex holomorphic hypersurface. A local projective frame field along f is a holomorphic map

$$e:\quad U\to G,$$

where U is an open subset of X, such that

$$f=[e_0]=\pi\circ e.$$

For any such frame field we put

$$\theta=e^*\,\omega=(\theta^I_J).$$

Then

$$de_0=\theta^0_0\, e_0+\theta^i_0\, e_i+\theta^{n+1}_0\, e_{n+1}. \tag{2.2}$$

Given a projective frame field $e: U\to G$, any other on U is given by

$$\tilde{e}=eb, \tag{2.3}$$

where $b: U\to G_0$ is a holomorphic map. If $\tilde{\theta}=\tilde{e}^*\,\omega$, then

$$\tilde{\theta}=b^{-1}\,\theta\, b+b^{-1}\, db. \tag{2.4}$$

The totality of projective frames on X is the holomorphic principal G_0-bundle $\pi_0: \mathscr{F}_0(f)\to X$, where

$$\mathscr{F}_0(f)=\{(x, A)\in X\times G \mid f(x)=\pi(A)\}.$$

First order frames. — The projective frame field $e: \mathrm{U} \to \mathrm{G}$ is of *first order* if

$$\theta_0^{n+1} = 0. \tag{2.5}$$

It is easily seen that first order frame fields exist locally. This follows from the fact that the linear isotropy representation of G_0 acts transitively on complex hyperplanes, which means geometrically that e can be chosen so that $e_1, \ldots, e_n$ span the holomorphic tangent space of X at each point. By (2.2), such a frame field is of first order.

If e is a first order frame field, then any other on U is given by (2.3), where $b: \mathrm{U} \to \mathrm{G}_1$ is a holomorphic map, and

$$\mathrm{G}_1 = \left\{ b = b(r, s, \mathrm{B}, x, y, t) = \begin{pmatrix} r & {}^t y & t \\ 0 & \mathrm{B} & x \\ 0 & 0 & s \end{pmatrix} \in \mathrm{G}_0 \right\}$$

where

$$r, s, t \in \mathbf{C}; \qquad x, y \in \mathbf{C}^n; \qquad \mathrm{B} \in \mathrm{GL}(n:\mathbf{C}); \qquad rs \det \mathrm{B} = 1.$$

The totality of first order frames is the holomorphic principal G_1-bundle π_1: $\mathscr{F}_1(f) \to \mathrm{X}$, where

$$\mathscr{F}_1(f) = \{ (p, e(p)) \in \mathrm{X} \times \mathrm{G} \},$$

where e is any local first order frame field along f. Then $\mathscr{F}_1(f) \subset \mathscr{F}_0(f)$ and is an integral manifold of the exterior differential system on $\mathscr{F}_0(f)$ given by

$$\omega_0^{n+1} = 0$$

$$d\omega_0^{n+1} \equiv -\omega_i^{n+1} \wedge \omega_0^i \mod (\omega_0^{n+1}).$$

Fubini's quadratic form. — If e is a first order frame field along f, then differentiating (2.5), and applying the structure equations and Cartan's Lemma, we have that

$$\theta_i^{n+1} = h_{ij} \theta_0^j, \tag{2.6}$$

where $h_{ij} = h_{ji}$ are holomorphic functions on $\mathrm{U} \subset \mathrm{X}$. *Fubini's quadratic form* is the holomorphic symmetric bilinear form defined on U by

$$\varphi = h_{ij} \theta_0^i \theta_0^j. \tag{2.7}$$

It depends on the choice of e.

If $\tilde{e} = eb$ is any other first order frame field on U, where $b = b(r, s, \mathrm{B}, x, y, t)$: $\mathrm{U} \to \mathrm{G}_1$, then

$$\tilde{\theta}_0^i = r (\mathrm{B}^{-1})_j^i \theta_0^j \tag{2.8}$$

$$\tilde{h}_{ij} = \frac{1}{rs} \mathrm{B}_i^k h_{kl} \mathrm{B}_j^l. \tag{2.9}$$

From these transformation formulas it follows that if $\tilde{\varphi}$ denotes Fubini's quadratic form with respect to $\tilde{e}$, then

$$\tilde{\varphi} = \frac{r}{s}\varphi. \tag{2.10}$$

It is easily seen that $\varphi = 0$ on X if and only if $f(\mathrm{X})$ is a hyperplane (Fubini). If we define $\det\varphi = \det(h_{ij})$, then $\det\varphi = 0$ on X if and only if $f(\mathrm{X})$ is the dual of a lower dimensional variety (Griffiths-Harris). If $\det\varphi \neq 0$ at every point of X, then φ defines a complex conformal structure on X in the sense of LeBrun.

We shall say that f is non-degenerate if $\det\varphi \neq 0$ at every point of X.

Fubini's cubic form. — If e is a first order frame field along f, then differentiating (2.6), and applying the structure equations (2.1) and Cartan's Lemma, we have that

$$-dh_{ij} + h_{ik}\theta^k_j + h_{jk}\theta^k_i - h_{ij}(\theta^0_0 + \theta^{n+1}_{n+1}) = \mathrm{F}_{ijk}\theta^k_0, \tag{2.11}$$

where the F_{ijk} are holomorphic functions on U symmetric in i, j and k.

Fubini's cubic form is the holomorphic symmetric cubic form ψ defined on U by

$$\psi = \mathrm{F}_{ijk}\theta^i_0\theta^j_0\theta^k_0. \tag{2.12}$$

It depends on the choice of first order frame field e.

Any other first order frame field on U is given by $\tilde{e} = eb$ where

$$b = b(r, s, \mathrm{B}, x, y, t) : \mathrm{U} \to \mathrm{G}_1$$

is a holomorphic map. The components of the cubic form then transform by

$$\begin{aligned} \tilde{\mathrm{F}}_{ijk} = {} & \frac{1}{r^2 s}\mathrm{B}^p_i\mathrm{B}^q_j\mathrm{B}^s_k\mathrm{F}_{pqs} + \frac{1}{r^2 s}[y_i\mathrm{B}^s_j h_{sq}\mathrm{B}^q_k + y_j\mathrm{B}^s_i h_{sq}\mathrm{B}^q_k + y_k\mathrm{B}^s_i h_{sq}\mathrm{B}^q_j] \\ & - \frac{1}{r^2 s^2}[\mathrm{B}^q_i h_{qp}x^p\mathrm{B}^s_j h_{sr}\mathrm{B}^r_k + \mathrm{B}^q_j h_{qp}x^p\mathrm{B}^s_i h_{sr}\mathrm{B}^r_k + \mathrm{B}^q_k h_{qp}x^p\mathrm{B}^s_i h_{sr}\mathrm{B}^r_j]. \end{aligned} \tag{2.13}$$

It follows then that

$$\begin{aligned} \tilde{\psi} &= \frac{r}{s}\left[\psi + (y_i(\mathrm{B}^{-1})^i_j\theta^j_0 - \frac{1}{s}h_{ij}x^i\theta^j_0)\varphi\right] \\ &= \frac{r}{s}(\psi + \alpha\varphi), \end{aligned} \tag{2.14}$$

where α is the holomorphic 1-form defined on U by

$$\alpha = y_i(\mathrm{B}^{-1})^i_j\theta^j_0 - \frac{1}{s}h_{ij}x^i\theta^j_0. \tag{2.15}$$

It was shown by Fubini that $\psi=0$ on X if and only if $f(X)$ is projectively congruent to an open subset of the quadric

$$2z^0 z^{n+1}+\sum_0^n (z^i)^2=0.$$

Second order frames. — Under our assumption of nondegeneracy of f, and since we are working over the complex numbers, it follows from the transformation formula (2.9) that there always exist local first order frame fields e with respect to which

$$h_{ij}=\delta_{ij}. \tag{2.16}$$

A first order frame field e is of *second order* if it satisfies (2.16) at every point of U.

By our comment above, a second order frame field exists on a neighborhood of any point of X. Computing the isotropy of the action of G_1 defined in (2.9), we find that if e is a second order frame field on U, then any other is given by $\tilde{e}=eb$, where $b: U \to G_2$ is a holomorphic map and

$$G_2=\{b=b(r, s, B, x, y, t)\in G_1 : {}^t BB=rs I, rs \det B=1\}.$$

Notice that then $(rs)^{n+2}=1$.

The totality of second order frames is the holomorphic principal G_2-bundle $\pi_2 : \mathscr{F}_2(f) \to X$, where

$$\mathscr{F}_2(f)=\{(p, e(p))\in X\times G\},$$

where e is any second order frame field defined about p. Then $\mathscr{F}_2(f)\subset\mathscr{F}_1(f)$ and is an integral manifold of the exterior differential system on $\mathscr{F}_0(f)$ given by

$$\begin{aligned}
\omega_0^{n+1}&=0\\
\omega_i^{n+1}-\omega_0^i&=0\\
d\omega_0^{n+1}&\equiv 0 \bmod (\omega_0^{n+1}, \omega_i^{n+1}-\omega_0^i)\\
d(\omega_i^{n+1}-\omega_0^i)&\equiv[\omega_k^i+\omega_i^k-\delta_k^i(\omega_0^0+\omega_{n+1}^{n+1})]\wedge\omega_0^k \bmod (\omega_0^{n+1}, \omega_i^{n+1}-\omega_0^i).
\end{aligned}$$

If e is a second order frame field, then (2.16) implies that (2.11) becomes

$$\theta_j^i+\theta_i^j-\delta_{ij}(\theta_0^0+\theta_{n+1}^{n+1})=F_{ijk}\theta_0^k. \tag{2.17}$$

Any other second order frame field on U is given by $\tilde{e}=eb$, where

$$b=b(r, s, B, x, y, t): U \to G_2.$$

Fubini's cubic forms with respect to e and $\tilde{e}$ are related by (2.14) where now (2.15) becomes

$$\alpha=\frac{1}{s}\left(\frac{1}{r}y_i B_i^j-x^j\right)\theta_0^j. \tag{2.18}$$

The action of G_2 defined by (2.13) seems too complicated to analyse directly. In (2.17) we set $i=j$ and sum on i. Recalling that $\theta_1^1=0$, we find that

$$\theta_0^0+\theta_{n+1}^{n+1}=-\frac{1}{n+2}F_{iij}\theta_0^j. \tag{2.19}$$

Writing (2.13) for second order frame fields, we find that these contracted components transform by

$$\tilde{F}_{iij}=\frac{1}{r^2 s}F_{iik}B_j^k+\frac{n+2}{r}\left(y_j-\frac{1}{s}B_j^k x^k\right) \tag{2.20}$$

Third order frames. — A second order frame field e is of *third order* if, with respect to it,

$$\theta_0^0+\theta_{n+1}^{n+1}=0. \tag{2.21}$$

From (2.19) and (2.20) it follows that third order frame fields exist on a neighborhood of any point of X. In addition, if e is a third order frame field on U, then any other is given by $\tilde{e}=eb$, where $b=b(r, s, B, x, y, t): U \to G_3$ is a holomorphic map into the complex subgroup of G_2 given by

$$G_3=\left\{b(r, s, B, x, y, t)\in G_2 : {}^t y=\frac{1}{s}{}^t x B\right\}.$$

The totality of third order frames is the holomorphic principal G_3-bundle π_3: $\mathscr{F}_3(f) \to X$, where

$$\mathscr{F}_3(f)=\{(p, e(p))\in X \times G\}$$

such that e is any local third order frame field about p. Then $\mathscr{F}_3(f) \subset \mathscr{F}_0(f)$ and is an integral manifold of the exterior differential system on $\mathscr{F}_0(f)$ given by

$$\begin{aligned}
\omega_0^{n+1}&=0\\
\omega_i^{n+1}-\omega_0^i&=0\\
\omega_0^0+\omega_{n+1}^{n+1}&=0\\
d\omega_0^{n+1}&\equiv 0 \bmod \{I\}\\
d(\omega_i^{n+1}-\omega_0^i)&\equiv(\omega_k^i+\omega_i^k)\wedge\omega_0^k \bmod \{I\}\\
d(\omega_0^0+\omega_{n+1}^{n+1})&\equiv(\omega_{n+1}^i-\omega_i^0)\wedge\omega_0^i \bmod \{I\},
\end{aligned}$$

where $\{I\}$ denotes the algebraic ideal generated by the 1-forms on the left side of the first three equations. The independence condition of this system is

$$\Omega \wedge \omega_3,$$

where $\Omega=\omega_0^1 \wedge \ldots \wedge \omega_0^n$ and ω_3 is any left-invariant volume element of G_3.

If e is a third order frame field, then any other on U is given by $\tilde{e}=eb$, where $b=b(r, s, \mathrm{B}, x, y, t):\mathrm{U}\to\mathrm{G}_3$ is any holomorphic map. From (2.18) it follows that $\alpha=0$, and that

$$\tilde{\psi}=\frac{r}{s}\psi. \tag{2.22}$$

PROPOSITION 2.1. — *Suppose that $n=\dim\mathrm{X}>2$. Let e be a third order frame field defined on a neighborhood of a point $p\in\mathrm{X}$. If $\psi(p)\neq 0$, then $\varphi(p)$ and $\psi(p)$ are relatively prime in the symmetric algebra* $\mathrm{S}(\mathrm{T}_p^{1,0}\mathrm{X})$.

Proof. — Since $\det\varphi\neq 0$ and $n>2$, it follows that φ is irreducible. If $\psi=\varphi\alpha$, for some (1, 0)-form $\alpha=a_i\theta_0^i(p)$, then by (2.12),

$$\mathrm{F}_{ijk}=\frac{1}{3}(a_i\delta_{jk}+a_j\delta_{ik}+a_k\delta_{ij}).$$

Thus, by (2.19) and (2.20), $0=\mathrm{F}_{iij}=(n+2)a_j$, for every j, which means that $\alpha=0$, which is impossible when $\psi(p)\neq 0$.

If e is a third order frame field along f, then differentiating (2.21), applying the structure equations (2.1) and Cartan's Lemma, we have that

$$\theta_i^0-\theta_{n+1}^i=\mathrm{M}_{ij}\theta_0^j, \tag{2.23}$$

where $\mathrm{M}_{ij}=\mathrm{M}_{ji}$ are holomorphic functions on U. If $\tilde{e}=eb$ is any other third order frame field on U, then

$$\tilde{\mathrm{M}}_{ij}=\frac{1}{r^2}\mathrm{B}_i^k\mathrm{B}_j^l\mathrm{M}_{kl}+\frac{1}{r}\delta_{ij}\left(\frac{x^k x^k}{s}-2t\right)-\frac{1}{r^2 s}\mathrm{B}_i^k\mathrm{B}_j^l\mathrm{F}_{klm}x^m.$$

The action of G_3 defined by this equation can be analyzed, but it has singular orbits, so that a full reduction of this action would involve non-degeneracy assumptions which we must avoid. As in the third order reduction we consider the contraction, M_{ii}, which transforms by

$$\tilde{\mathrm{M}}_{ii}=\frac{r}{s}\mathrm{M}_{ii}+\frac{n}{r}\left(\frac{x^i x^i}{s}-2t\right). \tag{2.24}$$

Fourth order frames. — A fourth order frame field along f is a third order frame field e for which

$$\mathrm{M}_{ii}=0. \tag{2.25}$$

From (2.24) it is clear that fourth order frame fields exist on a neighborhood of any point, and that if e is a fourth order frame field on U, then any other is given by $\tilde{e}=eb$, where $b:\mathrm{U}\to\mathrm{G}_4$ is a holomorphic map and

$$\mathrm{G}_4=\left\{b=b(r, s, \mathrm{B}, x, y, t)\in\mathrm{G}_3 : t=\frac{1}{2s}{}^t xx\right\}. \tag{2.26}$$

To be explicit, G_4 is the set of all matrices

$$b(r, s, \mathrm{B}, x, y, t)=\begin{pmatrix} r & {}^t y & t \\ 0 & \mathrm{B} & x \\ 0 & 0 & s \end{pmatrix}, \tag{2.27}$$

where $r, s, t\in\mathbf{C}$, $\mathrm{B}\in \mathrm{GL}(n; \mathbf{C})$, and $x, y\in\mathbf{C}^n$ satisfy

$$ {}^t\mathrm{BB}=rs\,\mathrm{I}, \qquad rs\det\mathrm{B}=1, \qquad {}^t y=\frac{1}{s}{}^t x\,\mathrm{B}, \qquad t=\frac{1}{2s}{}^t xx. \tag{2.28}$$

The totality of fourth order frames is the holomorphic principal G_4-bundle

$$\pi_4:\ \mathscr{F}_4(f)\to \mathrm{X},$$

where

$$\mathscr{F}_4(f)=\{(p, e(p))\in \mathrm{X}\times\mathrm{G}\}$$

such that e is any local fourth order frame field about p.

PROPOSITION 2.2. — *If $e:\mathrm{U}\to\mathrm{G}$ is a fourth order frame field along f, then*

$$\theta_0^{n+1}=0 \tag{2.29}$$

$$\theta_i^{n+1}=\theta_0^i \tag{2.30}$$

$$\theta_0^0+\theta_{n+1}^{n+1}=0 \tag{2.31}$$

$$\theta_j^i+\theta_i^j=\mathrm{F}_{ijk}\,\theta_0^k \tag{2.32}$$

$$\theta_i^0-\theta_{n+1}^i=\mathrm{M}_{ij}\,\theta_0^j, \tag{2.33}$$

where F_{ijk} and M_{ij} are holomorphic functions on U *totally symmetric in their indices and satisfying*

$$\mathrm{F}_{iij}=0=\mathrm{M}_{ii}. \tag{2.34}$$

Furthermore, by the structure equations,

$$d\theta_0^0=-\theta_i^0\wedge\theta_0^i \tag{2.35}$$

$$d\theta_{n+1}^{n+1}=-\theta_0^i\wedge\theta_{n+1}^i \tag{2.36}$$

$$d\theta_0^i=-(\theta_j^i-\delta_j^i\theta_0^0)\wedge\theta_0^j \tag{2.37}$$

$$d\theta_j^i=-\theta_k^i\wedge\theta_j^k-\theta_0^i\wedge\theta_j^0-\theta_{n+1}^i\wedge\theta_0^j. \tag{2.38}$$

$$d\theta_{n+1}^i=-\theta_0^i\wedge\theta_{n+1}^0-\theta_j^i\wedge\theta_{n+1}^j-\theta_{n+1}^i\wedge\theta_{n+1}^{n+1}. \tag{2.39}$$

With respect to e, Fubini's quadratic and cubic forms are given, respectively, by

$$\varphi=\sum(\theta_0^i)^2 \quad \textit{and} \quad \psi=\mathrm{F}_{ijk}\,\theta_0^i\,\theta_0^j\,\theta_0^k. \tag{2.40}$$

The reduced equations. — Let e be a first order frame field along f. For any point $q\in\mathrm{X}$, $[e_0(q)]=f(q)$, and the differential at $e_0(q)$ of the projection mapping $\mathbf{C}^{n+2}-\{0\}\to\mathbf{C}\,\mathrm{P}^{n+1}$ sends $e_1(q), \ldots, e_n(q)$ onto a basis of the tangent space at $f(q)$. It follows that in a neighborhood of $f(q)$ our hypersurface $f(\mathrm{X})$ is represented by the graph of a holomorphic function

$$(2.41)\qquad x^{n+1}=\frac{1}{2}a_{ij}x^i x^j+\frac{1}{6}a_{ijk}x^i x^j x^k+\frac{1}{24}a_{ijkl}x^i x^j x^k x^l+\ldots,$$

where the coefficients are totally symmetric in their indices. As such a representation holds for any point q in the domain of e, it follows that the coefficients in (2.41) are functions of q. Thus we have a map $\mathrm{F}(q, x)$ defined on a neighborhood of $\mathrm{X}\times\{0\}\subset \mathrm{X}\times\mathbf{C}^n$ given by

$$(2.42)\qquad \mathrm{F}(q, x)=e_0(q)+x^i e_i(q)+x^{n+1}(q, x)\,e_{n+1}(q),$$

such that $[\mathrm{F}(q, x)]\in f(\mathrm{X})$.

We can relate the coefficients in (2.41) to the invariants $h_{ij}(q)$, $\mathrm{F}_{ijk}(q)$ and $\mathrm{M}_{ij}(q)$ by considering the equation $[\mathrm{F}(q, x)]=$ a constant point in $\mathbf{C}\mathrm{P}^{n+1}$. This is expressed in $\mathbf{C}^{n+2}$ by the differential equation

$$(2.43)\qquad d\mathrm{F}=\lambda\,\mathrm{F},$$

for some 1-form λ in $\mathrm{X}\times\mathbf{C}^n$. In order to take the exterior derivative of (2.42), it is convenient to introduce the notation

$$(2.44)\qquad dx^{n+1}=\frac{\partial x^{n+1}}{\partial x^i}dx^i+d_q x^{n+1},$$

where

$$(2.45)\qquad d_q x^{n+1}=\frac{1}{2}da_{ij}x^i x^j+\frac{1}{6}da_{ijk}x^i x^j x^k+\ldots$$

Substituting (2.42) into (2.43), we find that

$$\begin{aligned}(2.46)\quad \lambda(e_0+x^i e_i+x^{n+1}e_{n+1})&=d(e_0+x^i e_i+x^{n+1}e_{n+1})\\ &=\theta_0^0 e_0+\theta_0^i e_i+dx^i e_i+x^i(\theta_i^0 e_0+\theta_i^j e_j+\theta_i^{n+1}e_{n+1})\\ &\qquad +dx^{n+1}e_{n+1}+x^{n+1}(\theta_{n+1}^0 e_0+\theta_{n+1}^i e_i+\theta_{n+1}^{n+1}e_{n+1}).\end{aligned}$$

Equating the coefficients of e_0, e_i, e_{n+1} in (2.46), we have

$$(2.47)\qquad \lambda=\theta_0^0+x^i\theta_i^0+x^{n+1}\theta_{n+1}^0$$

$$(2.48)\qquad \lambda x^i=\theta_0^i+dx^i+x^j\theta_j^i+x^{n+1}\theta_{n+1}^i$$

$$(2.49)\qquad \lambda x^{n+1}=x^i h_{ij}\theta_0^j+dx^{n+1}+x^{n+1}\theta_{n+1}^{n+1}.$$

The first $1+n$ equations determine λ and dx^i. Using (2.44), we can substitute these into (2.49) to arrive at

$$(2.50)\quad (\theta_0^0+x^i\theta_i^0+x^{n+1}\theta_{n+1}^0)\left(x^{n+1}-x^i\frac{\partial x^{n+1}}{\partial x^i}\right)=\left(h_{ij}x^j-\frac{\partial x^{n+1}}{\partial x^i}\right)\theta_0^i$$
$$+d_q x^{n+1}-x^j\frac{\partial x^{n+1}}{\partial x^i}\theta_j^i-x^{n+1}\frac{\partial x^{n+1}}{\partial x^i}\theta_{n+1}^i+x^{n+1}\theta_{n+1}^{n+1},$$

which is a power series in $x^1, \ldots, x^n$ whose coefficients are 1-forms in X.

We can easily read off the coefficient of x^j on each side of (2.50). As these must hold identically in X, we have

$$(2.51)\qquad 0=(h_{ij}-a_{ij})\theta_0^i.$$

As $\theta_0^1, \ldots, \theta_0^n$ are linearly independent at each point, it follows that

$$(2.52)\qquad a_{ij}=h_{ij}.$$

Suppose now that e is a second order frame field, so that $h_{ij}=\delta_{ij}$. We calculate the quadratic and cubic terms on each side of (2.50). For the left hand side these are

$$(2.53)\qquad -\frac{1}{2}\delta_{ij}\theta_0^0 x^i x^j-\left[\frac{1}{3}a_{ijk}\theta_0^0+\frac{1}{2}\delta_{ij}\theta_k^0\right]x^i x^j x^k,$$

and for the right hand side they are

$$(2.54)\quad -\frac{1}{2}[a_{ijk}\theta_0^k-\delta_{ij}\theta_{n+1}^{n+1}+2\theta_j^i]x^i x^j$$
$$+\frac{1}{6}[-a_{ijkl}\theta_0^l+da_{ijk}+a_{ijk}\theta_{n+1}^{n+1} 3a_{ikl}\theta_j^l-3\delta_{jk}\theta_{n+1}^i]x^i x^j x^k.$$

Equating the quadratic terms of (2.53) to those of (2.54), taking care to symmetrize all coefficients in i and j, we have

$$(2.55)\qquad \delta_{ij}\theta_0^0=a_{ijk}\theta_0^k-\delta_{ij}\theta_{n+1}^{n+1}+\theta_j^i+\theta_i^j.$$

Thus, from (2.11), we conclude that

$$(2.56)\qquad a_{ijk}=-\mathrm{F}_{ijk}.$$

Suppose now that e is a third order frame field so that (2.21) and (2.23) hold, as well as

$$(2.57)\qquad \theta_j^i+\theta_i^j=\mathrm{F}_{ijk}\theta_0^k.$$

Taking the exterior derivative of (2.57), and using the structure equations, we find

$$\begin{aligned}[-dF_{ijk}-F_{ijk}\theta_0^0+F_{ijl}\theta_k^l]\wedge\theta_0^k+(\theta_j^0-\theta_{n+1}^j)\wedge\theta_0^i+(\theta_i^0-\theta_{n+1}^i)\wedge\theta_0^j\\ =\theta_l^i\wedge\theta_j^l+\theta_l^j\wedge\theta_i^l\\ =(-\theta_i^l+F_{ilk}\theta_0^k)\wedge\theta_j^l+(-\theta_j^l+F_{jlk}\theta_0^k)\wedge\theta_i^l\\ =-(F_{ilk}\theta_j^l+F_{jlk}\theta_i^l)\wedge\theta_0^k.\end{aligned}$$

Thus, using (2.23), we find that

$$\begin{aligned}[dF_{ijk}+F_{ijk}\theta_0^0-F_{ijl}\theta_k^l-F_{ilk}\theta_j^l-F_{jlk}\theta_i^l]\,\theta_0^k\\ =M_{jl}\theta_0^l\wedge\theta_0^i+M_{il}\theta_0^l\wedge\theta_0^j\\ =[M_{jl}\delta_{ik}+M_{il}\delta_{jk}+M_{kl}\delta_{ij}]\,\theta_0^l\wedge\theta_0^k,\end{aligned}$$

where the last term is zero since $M_{kl}=M_{lk}$, and has been added in order to symmetrize in i, j and k the left hand side of (2.58) below. By Cartan's Lemma we conclude that

$$dF_{ijk}+F_{ijk}\theta_0^0-F_{ijl}\theta_k^l-F_{ilk}\theta_j^l-F_{jlk}\theta_i^l-(M_{jl}\delta_{ik}+M_{il}\delta_{jk}+M_{kl}\delta_{ij})\,\theta_0^l=-A_{ijkl}\theta_0^l, \tag{2.58}$$

where

$$A_{ijkl}=A_{ijlk}$$

are functions in X. In addition, since the left hand side of (2.58) is symmetric in i, j and k, it follows that A_{ijkl} is totally symmetric in all four indices.

Substituting (2.56) into (2.53) and (2.54), taking care to symmetrize the coefficients of $x^i x^j x^k$, we can equate the cubic terms to arrive at

$$\begin{aligned}2F_{ijk}\theta_0^0-\delta_{ij}\theta_k^0-\delta_{ki}\theta_j^0-\delta_{jk}\theta_i^0=-a_{ijkl}\theta_0^l-dF_{ijk}\\ -F_{ijk}\theta_{n+1}^{n+1}+F_{ikl}\theta_j^l+F_{jil}\theta_k^l+F_{kjl}\theta_i^l-\delta_{jk}\theta_{n+1}^i-\delta_{ij}\theta_{n+1}^k-\delta_{ki}\theta_{n+1}^j.\end{aligned}$$

Substituting (2.58) into this, we conclude that

$$a_{ijkl}=A_{ijkl}. \tag{2.59}$$

In summary, given a third order frame field e along f, then for any $q\in X$ there is a local lift of f given by

$$e_0(q)+x^i e_i(q)+x^{n+1}e_{n+1},$$

where x^{n+1} is a holomorphic function of $x^1, \ldots, x^n$ whose power series expansion is given by

$$x^{n+1}=\frac{1}{2}\sum(x^i)^2-\frac{1}{6}F_{ijk}(q)\,x^i x^j x^k+\frac{1}{24}A_{ijkl}(q)\,x^i x^j x^k x^l+O(4),$$

where F_{ijk} are given by (2.11) and A_{ijkl} are given by (2.58).

For a third order frame field e, we have $F_{iik}=0$ by (2.19) and (2.21), and thus, contracting (2.58) on i and j we have

$$A_{iikl}=F_{imk}F_{iml}+(n+2)M_{kl}. \tag{2.60}$$

The frame e is of fourth order if and only if $M_{kk}=0$, which by (2.60) holds if and only if

$$A_{iikk}=F_{imk}F_{imk}.$$

3. Proof of Fubini's theorem

Fubini's hypothesis. — Let X be a connected complex manifold of dimension $n>2$. Let

$$f, \hat{f}: X \to \mathbf{C}P^{n+1}$$

be nondegenerate holomorphic immersions such that, calculated with respect to third order frame fields,

$$\hat{\psi}/\hat{\varphi}=\psi/\varphi \tag{3.1}$$

at every point of X. By (2.10) and (2.22), the quotient ψ/φ is independent of the choice of third order frame field used to compute ψ and φ.

If ψ is identically zero on X, then so is $\hat{\psi}$, and then Fubini has shown that $f(X)$ and $\hat{f}(X)$ are congruent to a quadric. If ψ is not identically zero, then let D denote its zero divisor, which must be the zero divisor of $\hat{\psi}$ as well. Then $X-D$ is a connected, open dense subset of X. By continuity, if $f(X-D)$ is congruent to $\hat{f}(X-D)$, then the same is true of $f(X)$ and $\hat{f}(X)$. Thus we may assume that D is empty.

PROPOSITION 3.1. — *Let $e, \hat{e}: U \to G$ be third order frame fields along f and $\hat{f}$, respectively. Then* (3.1) *holds if and only if*

$$\hat{\psi}=t\psi, \qquad \hat{\varphi}=t\varphi, \tag{3.2}$$

for some holomorphic function $t: U \to \mathbf{C}-\{0\}$.

Proof. — At every point of U we have $\varphi\hat{\psi}=\hat{\varphi}\psi$, and thus φ must divide $\hat{\varphi}$ by Proposition 2.1. Hence $\hat{\varphi}=t\varphi$ for some nowhere zero holomorphic function t on U. Cancelling φ, we have (3.2).

The condition (3.1) can be formulated in terms of first order frame fields. If φ and ψ are calculated with respect to a first order frame field $e: U \to G$ along f, then by (2.10) and (2.14), with respect to another first order frame field $e'=eb$, where $b=b(r, s, \mathbf{B}, x, y, t)$, we have

$$\varphi'=\frac{r}{s}\varphi, \qquad \psi'=\frac{r}{s}\psi+\frac{r}{s}\alpha\varphi,$$

for some holomorphic 1-form α on U.

Suppose that $\hat{\varphi}$ and $\hat{\psi}$ are calculated with respect to a first order frame field $\hat{e}: U \to G$ along $\hat{f}$. Suppose that $\hat{e}' = \hat{e}\hat{b}$, where $\hat{b} = b(\hat{r}, \hat{s}, \hat{B}, \hat{x}, \hat{y}, \hat{t})$, so that

$$\hat{\varphi}' = \frac{\hat{r}}{\hat{s}}\hat{\varphi}, \qquad \hat{\psi}' = \frac{\hat{r}}{\hat{s}}\hat{\psi} + \frac{\hat{r}}{\hat{s}}\hat{\alpha}\hat{\varphi}.$$

In particular, these hold if e' and $\hat{e}'$ are third order frames fields, in which case, by (3.2),

$$\hat{\varphi}' = t'\varphi', \qquad \hat{\psi}' = t'\psi',$$

for some holomorphic function t' on U. Hence,

$$\hat{\varphi} = u\varphi, \qquad \hat{\psi} = u\psi + \alpha'\varphi, \tag{3.3}$$

where $u = \hat{s}t'r/\hat{r}s$ is a holomorphic function on U and α' is a holomorphic 1-form on U.

DEFINITION. — *The Fubini Hypothesis on holomorphic immersions* $f, \hat{f}: X \to \mathbf{C}P^{n+1}$ *is that* (3.3) *holds with respect to any local first order frame fields. By the above discussion, for non-degenerate immersions, this is equivalent to either* (3.1) *or* (3.2) *with respect to third order frame fields.*

Isomorphism of the bundles of fourth order frames

PROPOSITION 3.2. — *Let* $f, \hat{f}: X \to \mathbf{C}P^{n+1}$ *be non-degenerate holomorphic immersions for which* (3.1) *holds. Let* $e: U \to G$ *be any fourth order frame field along* f. *Then there exists a fourth order frame field* $\hat{e}: U \to \mathbf{C}P^{n+1}$ *along* $\hat{f}$ *such that*

$$\hat{\theta}^i_0 = \theta^i_0 \quad \textit{and} \quad \hat{\theta}^0_0 = \theta^0_0. \tag{3.4}$$

Moreover, if U *is connected, then* $\hat{e}$ *is unique up to multiplication by a constant* r *such that* $r^{n+2} = 1$.

Proof. — If $\tilde{e}: U \to G$ is any fourth order frame field along $\hat{f}$, then by (2.40) and (3.2),

$$\sum (\tilde{\theta}^i_0)^2 = t \sum (\theta^i_0)^2. \tag{3.5}$$

Thus $\tilde{\theta}^i_0 = B^i_j \theta^j_0$ for some holomorphic, matrix valued function $B = (B^i_j)$ satisfying ${}^tBB = tI$.

If we let $b = b(1, t, B, 0, 0, 0): U \to G_4$, then $\hat{e} = \tilde{e}b$ is another fourth order frame field along $\hat{f}$. With respect to $\hat{e}$, by (2.8),

$$\hat{\theta}^i_0 = (B^{-1})^i_j \tilde{\theta}^j_0 = \theta^i_0 \tag{3.6}$$

$$\hat{\varphi} = \sum (\hat{\theta}^i_0)^2 = \sum (\theta^i_0)^2 = \varphi. \tag{3.7}$$

By (3.2), $\hat{\psi} = \psi$.

Thus, we may assume that $\tilde{e}$ was chosen so that

$$\tilde{\varphi} = \varphi, \qquad \tilde{\psi} = \psi, \qquad \tilde{\theta}^i_0 = \theta^i_0. \tag{3.8}$$

Now any other fourth order frame field along $\hat{f}$, for which (3.8) continues to hold, must be given by $\hat{e}=\tilde{e}b$, where

$$b=b(r, s, \mathrm{B}, x, y, t):\mathrm{U}\to \mathrm{G}_4$$

satisfies, by (2.8), $\mathrm{B}=r\mathrm{I}$, where I is the $n\times n$ identity matrix, and by (2.10), $s=r$. Furthermore, $1=rs\det \mathrm{B}=r^{n+2}$. Thus

$$b=b\left(r, r, r\mathrm{I}, x, \frac{1}{r}{}^t x, \frac{1}{2r}{}^t xx\right), \tag{3.9}$$

where $x:\mathrm{U}\to \mathbf{C}^n$ is a holomorphic map, and r is a constant satisfying $r^{n+2}=1$. (To be precise, r is constant if U is connected; otherwise, r is locally constant).

Using (2.4), we can calculate that

$$\begin{aligned}\hat{\theta}_0^0&=\tilde{\theta}_0^0-\frac{1}{r}x^i\tilde{\theta}_0^i+\frac{dr}{r}\\ &=\tilde{\theta}_0^0-\frac{1}{r}x^i\theta_0^i,\end{aligned} \tag{3.10}$$

by (3.8) and the fact that $dr=0$ on U. Thus, we will have

$$\hat{\theta}_0^0=\theta_0^0 \tag{3.11}$$

provided that we take $r=1$ and we let the x^i be determined by

$$\tilde{\theta}_0^0-\theta_0^0=x^i\theta_0^i, \tag{3.12}$$

which determines the holomorphic functions x^i because $\theta_0^1, \ldots, \theta_0^n$ give a basis of holomorphic 1-forms on U and $\tilde{\theta}_0^0-\theta_0^0$ is a given holomorphic 1-form on U.

Thus, the frame field $\hat{e}$ satisfies the conditions of the Proposition. If $\bar{e}=\hat{e}b$ is another fourth order frame field along $\hat{f}$ on U satisfying (3.8) and (3.11), then b must be given by (3.9), and by (3.10) we have

$$\begin{aligned}\theta_0^0=\bar{\theta}_0^0&=\hat{\theta}_0^0-\frac{1}{r}x^i\theta_0^i\\ &=\theta_0^0-\frac{1}{r}x^i\theta_0^i.\end{aligned}$$

Hence, $x=0$ on U, and $b=b(r, r, r, \mathrm{I}, 0, 0, 0)$.

PROPOSITION 3.3. – *Let $f, \hat{f}:\mathrm{X}\to \mathbf{C}\mathrm{P}^{n+1}$ be nondegenerate holomorphic immersions for which* (3.1) *holds. Let $e:\mathrm{U}\to \mathrm{G}$ be any fourth order frame field along f, defined on a connected domain $\mathrm{U}\subset \mathrm{X}$. Let $\hat{e}:\mathrm{U}\to \mathrm{G}$ be a fourth order frame field along $\hat{f}$ satisfying* (3.4). *Then there exists an element $a\in \mathrm{G}$ such that*

$$a\hat{e}(p)=e(p), \quad \textit{for every } p\in \mathrm{U}. \tag{3.13}$$

Proof. — The idea of the proof is to show that $\hat{\theta}^I_J = \theta^I_J$, for $0 \leqq I,\ J \leqq n+1$, where $\hat{\theta}^I_J = \hat{e}^* \omega^I_J$ and $\theta^I_J = e^* \omega^I_J$. We then apply the Cartan-Darboux Theorem (*cf.* [14], p. 167-168) to conclude (3.13). By Proposition 2.2 and 3.2 we have

$$(3.14) \qquad \begin{cases} \hat{\theta}^{n+1}_0 = \theta^{n+1}_0 = 0 \\ \hat{\theta}^i_0 = \theta^i_0 \\ \hat{\theta}^{n+1}_j = \theta^{n+1}_j \\ \hat{\theta}^0_0 = \theta^0_0 \\ \hat{\theta}^{n+1}_{n+1} = \theta^{n+1}_{n+1} \end{cases}$$

Moreover, by (2.32) and (3.8) we have

$$(3.15) \qquad \hat{\theta}^i_j + \hat{\theta}^j_i = \theta^i_j + \theta^j_i.$$

By (2.37) and the second equation in (3.14) we have

$$(3.16) \qquad \hat{\theta}^i_j - \theta^i_j = A^i_{jk}\, \theta^k_0,$$

where

$$A^i_{jk} = A^i_{kj}$$

are holomorphic functions on U. But then (3.15) implies that

$$A^i_{jk} = -A^j_{ik}$$

as well. Hence, $A^i_{jk} = 0$ for all i, j and k, and we have

$$(3.17) \qquad \hat{\theta}^i_j = \theta^i_j.$$

Differentiating the last two equations in (3.14), using (2.35) and (2.36), and applying Cartan's Lemma, we have

$$(3.18) \qquad \hat{\theta}^0_i - \theta^0_i = R_{ij}\, \theta^j_0$$

$$(3.19) \qquad \hat{\theta}^i_{n+1} - \theta^i_{n+1} = T^i_j\, \theta^j_0,$$

where $R_{ij} = R_{ji}$ and $T^i_j = T^j_i$ are holomorphic functions on U. From (2.38) and (3.17) we then have

$$(\delta^i_k R_{jl} + T^i_k \delta_{jl})\, \theta^k_0 \wedge \theta^l_0 = 0,$$

from which, by Cartan's Lemma, we conclude that

$$\delta^i_k R_{jl} - \delta^i_l R_{jk} + T^i_k \delta_{jl} - T^i_l \delta_{jk} = 0.$$

Contracting i and k we find

$$(3.20) \qquad (n-1) R_{jl} - T^j_l + T\, \delta_{jl} = 0,$$

where $T = T^i_i$, while contracting j and l gives

$$(n-1)\,T^i_k - R_{ik} + R\,\delta^i_k = 0, \tag{3.21}$$

where $R = R_{ii}$. Contracting either (3.20) or (3.21) gives

$$R + T = 0. \tag{3.22}$$

Substituting (3.22) into (3.20) and (3.21), we find that $R_{ij} = -T^i_j$, for all i and j, which when substituted again into (3.20) and (3.21) gives

$$R_{ij} = \frac{R}{n}\delta_{ij} = -T^i_j, \tag{3.23}$$

for all i and j. Substituting this into (3.18) and (3.19), and using (2.33), we find that

$$2\frac{R}{n}\delta_{ij} = \hat{M}_{ij} - M_{ij}.$$

Contracting, and using (2.34), we conclude that $R = 0$, and thus that $R_{ij} = 0 = T^i_j$ for all i and j by (3.23). Hence

$$\begin{cases} \hat{\theta}^0_i = \theta^0_i \\ \hat{\theta}^i_{n+1} = \theta^i_{n+1}. \end{cases} \tag{3.24}$$

It now follows from (2.39) that

$$(\hat{\theta}^0_{n+1} - \theta^0_{n+1}) \wedge \theta^i_0 = 0$$

for all i. Since $n > 1$, it follows that

$$\hat{\theta}^0_{n+1} = \theta^0_{n+1}. \tag{3.25}$$

Combining (3.14), (3.17), (3.24) and (3.25) we conclude that $\hat{\theta}^I_J = \theta^I_J$ for all I and J. This completes the proof.

FUBINI'S THEOREM. — *Let* X *be a connected complex manifold of dimension* $n > 2$. *Let* $f : X \to \mathbf{C}P^{n+1}$ *and* $\hat{f} : X \to \mathbf{C}P^{n+1}$ *be holomorphic immersions for which none of* $\det\varphi$, $\det\hat{\varphi}$, $\hat{\psi}$ *and* ψ *is identically zero on* X. *If Fubini's Hypothesis holds for* f *and* $\hat{f}$, *then* $\hat{f}(X)$ *is projectively congruent to* $f(X)$*; that is, there is an element* $a \in G$ *such that* $a\hat{f}(X) = f(X)$.

Proof. — Let X′ be the complement of the union of the zero divisors of $\det\varphi$, $\det\hat{\varphi}$, ψ and $\hat{\psi}$. Then X′ is a connected, open, dense subset of X. Fix a point $p_0 \in X'$. There exists a connected domain U of X′, containing p_0 and on which there exists a fourth order frame field e along f. Let $\hat{e} : U \to G$ be a fourth order frame field along $\hat{f}$ given by Proposition (3.2). By Proposition 3.3 there exists an element $a \in G$ such that $a\hat{e}(p) = e(p)$ for every $p \in U$.

Suppose that V is another connected open subset of X′ on which there exists a fourth order frame field e_V along f. Suppose also that there is a point $q \in U \cap V$. Multiplying e_V on the right by the constant matrix $e_V(q)^{-1} e(q) \in G_4$, we may assume that $e_V(q) = e(q)$. If $\hat{e}_V : V \to G$ is a fourth order frame field along $\hat{f}$ given by Proposition 3.2, then by Proposition 3.3, there is a constant matrix $c \in G$ such that $c\hat{e}_V = e_V$. But then

$$(3.26) \qquad c\hat{e}_V(q) = e_V(q) = e(q) = a\hat{e}(q).$$

On the other hand, restricted to $U \cap V$, both $\hat{e}$ and $\hat{e}_V$ are fourth order frame fields along $\hat{f}$ satisfying the conditions of Proposition 3.2. By Proposition 3.3, on the connected component of $U \cap V$ containing q there must be a constant r, with $r^{n+2} = 1$, such that

$$(3.27) \qquad r\hat{e}_V = \hat{e}.$$

In particular, (3.17) holds at q, which combined with (3.16) implies that $c = ar$. But by Proposition 3.2, we may replace $\hat{e}_V$ by $r\hat{e}_V$, in which case we then have $c = a$. That is, we have $a\hat{e}_V = e_V$ on all of V.

Any point $p \in X'$ can be joined with p_0 by a finite chain of open sets of X′ such that any adjacent pair U_α and $U_{\alpha+1}$ has the same properties as the above pair U and V. By induction up the chain, then, it follows that for each α, there exist fourth order frame fields $\hat{e}_\alpha, e_\alpha : U_\alpha \to G$ along $\hat{f}$ and f, respectively, such that $a\hat{e}_\alpha = e_\alpha$ for the same constant matrix $a \in G$. It follows then that $a\hat{f}_{X'} = f_{X'}$. By continuity, $a\hat{f} = f$ on all of X.

4. Complex conformal structures

For a nondegenerate hypersurface in $\mathbf{C}P^{n+1}$, Fubini's quadratic form φ defines a complex conformal structure on X as defined by LeBrun in [12]. In this section we briefly outline the theory of such structures in order to show the role they play in our proof of Fubini's Theorem.

A complex conformal structure on X assigns to each local complex coordinate system $z^1, \ldots, z^n$ on $U \subset X$ a holomorphic symmetric bilinear form $\varphi = h_{ij}\, dz^i\, dz^j$, where $h_{ij} = h_{ji}$ are holomorphic functions on U and $\det(h_{ij}) \neq 0$ at every point of U. In addition, if $\tilde{z}^1, \ldots, \tilde{z}^n$ is another complex coordinate system on $\tilde{U}$, with $U \cap \tilde{U} \neq \emptyset$, then on $U \cap \tilde{U}$, $\tilde{\varphi} = r\varphi$, where r is a nowhere zero holomorphic function on $U \cap \tilde{U}$. We will denote such a structure on X by $[\varphi]$.

Let

$$S = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 0 & & & & 0 \\ \vdots & & I_n & & \vdots \\ 0 & & & & 0 \\ -1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Consider the representation of the complex orthogonal group

$$O(n+2;\ \mathbf{C}) = \{ \mathrm{A} \in \mathrm{GL}(n+2;\ \mathbf{C}) : {}^t\mathrm{ASA} = \mathrm{S} \},$$

and the closed complex subgroup

$$\mathrm{H}(n,\ \mathbf{C}) = \left\{ \begin{pmatrix} 1/r & {}^tz\mathrm{B} & r/2\,{}^tzz \\ 0 & \mathrm{B} & rz \\ 0 & 0 & r \end{pmatrix} : 0 \neq r \in \mathbf{C},\ \mathrm{B} \in O(n,\ \mathbf{C}),\ z \in \mathbf{C}^n \right\}.$$

We observe that $\mathrm{H}(n, \mathbf{C})$ is a semidirect product of $\mathbf{C}\,O(n, \mathbf{C})$ by $\mathbf{C}^n$.

PROPOSITION 4.1. — *There exists a natural holomorphic principal* $\mathrm{H}(n, \mathbf{C})$*-bundle* $\mathrm{P} \to \mathrm{X}$. *It is called the Möbius bundle of the complex conformal space* X, $[\varphi]$.

Naturality here means that if $\tilde{\mathrm{X}}$, $[\tilde{\varphi}]$ is another complex conformal space, and if $\mathrm{F} : \mathrm{X} \to \tilde{\mathrm{X}}$ is a biholomorphic map preserving the complex conformal structures, then F induces a holomorphic bundle isomorphism from $\mathrm{P} \to \mathrm{X}$ to $\tilde{\mathrm{P}} \to \tilde{\mathrm{X}}$.

A *Cartan connection* on $\mathrm{P} \to \mathrm{X}$ is a holomorphic 1-form on P taking values in $\mathcal{O}(n+2;\ \mathbf{C})$, the Lie algebra of $O(n+2;\ \mathbf{C})$. We refer the reader to [13] for the definitions of Cartan connection and of what it means for such a connection to be torsion free.

PROPOSITION 4.2. — *On the Möbius bundle* $\mathrm{P} \to \mathrm{X}$ *of a complex conformal space* X, $[\varphi]$ *there exists a unique torsion free Cartan connection* ψ *whose curvature* Ψ *satisfies the conditions*

a) Ψ *takes values in the Lie algebra of* $\mathrm{H}(n;\ \mathbf{C})$;

b) $\Psi^0_0 = 0$;

c) $\Psi^i_j = (1/2)\,\mathrm{F}^i_{jkl}\,\psi^k_0 \wedge \psi^l_0$, *where* $\mathrm{F}^i_{jil} = 0$.

This Cartan connection is called the normal conformal connection of X, $[\varphi]$.

As a consequence of the uniqueness in Proposition 4.2, if $\mathrm{F} : \mathrm{X} \to \tilde{\mathrm{X}}$ is a biholomorphic map preserving the complex conformal structures, then the induced bundle isomorphism $\mathrm{P} \to \tilde{\mathrm{P}}$ preserves the normal conformal connections.

Let $f : \mathrm{X} \to \mathbf{C}\mathrm{P}^{n+1}$ be a nondegenerate holomorphic hypersurface with induced complex conformal structure $[\varphi]$. Let G_4 be the complex Lie group defined in (2.26), and let $\mathcal{F}_4(f) \to \mathrm{X}$ denote the holomorphic principal G_4-bundle defined in section 2. Let $e : \mathcal{F}_4(f) \to \mathrm{G}$ denote the projection onto the second factor in $\mathrm{X} \times \mathrm{G}$, and let $\theta^{\mathrm{I}}_{\mathrm{J}} = e^* \omega^{\mathrm{I}}_{\mathrm{J}}$. Then equations (2.29) through (2.39) hold on $\mathcal{F}_4(f)$.

PROPOSITION 4.3. — *Let* $f : \mathrm{X} \to \mathbf{C}\mathrm{P}^{n+1}$ *be a nondegenerate holomorphic hypersurface with induced complex conformal structure* $[\varphi]$. *Then* G_4 *is isomorphic to* $\mathrm{H}(n;\ \mathbf{C})$ *and* $\mathcal{F}_4(f) \to \mathrm{X}$ *is isomorphic to the Möbius bundle of* X, $[\varphi]$. *Under this bundle isomorphism,* *the normal conformal Cartan connection is given on* $\mathcal{F}_4(f)$ *by*

$$(4.1)\qquad \begin{cases} \psi_0^0 = \theta_0^0, \\ \psi_0^i = \theta_0^i, \\ \psi_j^i = \theta_j^i - \dfrac{1}{2} F_{ijk}\,\theta_0^k, \\ \psi_i^0 = \theta_i^0 + M_{ij}\,\theta_0^j - S_{ij}\,\theta_0^j \end{cases}$$

where the S_{ij} *are holomorphic functions on* $\mathcal{F}_4(f)$ *defined by*

$$S_{ij} = -\frac{3n}{2(n-2)} M_{ij} - \frac{3}{4(n-2)} \left(F_{ikl} F_{jkl} - \frac{1}{2(n-1)} F_{klm} F_{klm} \delta_{ij} \right).$$

The idea of our proof of Fubini's Theorem can now be expressed as follows. If $f, \tilde{f}: X \to \mathbf{C}P^{n+1}$ are nondegenerate holomorphic hypersurfaces which satisfy the Fubini Hypothesis of section 3, then the complex conformal structures induced by f and $\tilde{f}$ are equivalent. Thus, by Proposition 4.1, the corresponding Möbius bundles are isomorphic, and by Proposition 4.2 this isomorphism preserves the normal conformal connections. Thus, by Proposition 4.3, the bundles $\mathcal{F}_4(f)$ and $\mathcal{F}_4(\tilde{f})$ are isomorphic, and under this isomorphism, equations (3.14) and (3.17) hold. The local version of Fubini's Theorem then follows by the proof of Proposition 3.3, and the proof of the theorem itself is the same as that given in section 3.

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