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THE BASED RING OF THE LOWEST TWO-SIDED CELL OF AN AFFINE WEYL GROUP, II

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ABSTRACT. – We show that the lowest based ring of an affine Weyl group W is very interesting to understand some simple representations of the corresponding Hecke algebra H_{q_0} ($q_0 \in \mathbb{C}^*$) even when q_0 is a root of 1.

Let H_{q_0} be the Hecke algebra (over $\mathbb{C}$) attached by Iwahori and Matsumoto [IM] to an affine Weyl group W and to a parameter $q_0^2 \in \mathbb{C}^*$.

When q_0 is not a root of 1 or $q_0^2=1$, the simple H_{q_0}-modules have been classified (*see* [KL 2]). However we know little about the simple H_{q_0}-modules when q_0 is a root of 1. In this paper we give some discussion to the representations of H_{q_0} with q_0 a root of 1. Namely, let J be the asymptotic Hecke algebra defined in [L 3, III]. There exists a natural injection $\phi_{q_0}: H_{q_0} \to J$. Let K (J) [resp. K (H_{q_0})] be the Grothendieck group of J-modules (resp. H_{q_0}-modules) of finite dimension over $\mathbb{C}$, then ϕ_{q_0} induces a surjective homomorphism $(\phi_{q_0})_*: K(J) \to K(H_{q_0})$, when q_0 is not a root of 1 or $q_0^2=1$, $(\phi_{q_0})_*$ is an isomorphism (*loc. cit.*). For each two-sided cell c of W, we can define the direct summand $K(J_c)$ [resp. $K(H_{q_0})_c$] of K (J) [resp. $K(H_{q_0})$]. Thus $(\phi_{q_0})_*$ induces a homomorphism $(\phi_{q_0})_{*,c}: K(J_c) \to K(H_{q_0})_c$. The map $(\phi_{q_0})_{*,c}$ remains surjective and is an isomorphism if q_0 is not a root of 1 or $q_0^2=1$. In this paper we mainly discuss the map $(\phi_{q_0})_{*,c_0}$, where c_0 is the lowest two-sided cell of W.

1. Introduction

1.1. Let G be a simply connected, almost simple complex algebraic group and T a maximal torus. Let $P \subseteq X = \mathrm{Hom}(T, \mathbb{C}^*)$ be the root lattice. The Weyl group $W_0 = N_G(T)/T$ of G acts on X in a natural way and this action is stable on P. Thus we can form the affine Weyl group $W_a = W_0 \ltimes P$, which is a normal subgroup of the extended affine Weyl group $W = W_0 \ltimes X$. There exists a finite abelian subgroup Ω of W such that $W = \Omega \ltimes W_a$. Let $S = \{r_0, r_1, \ldots, r_n\}$ be the set of simple reflections of W_a with $r_0 \notin W_0$. Then we have a standard length function l on W_a which can be extended

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to W by defining $l(\omega w)=l(w)$ for any $\omega\in\Omega$, $w\in W_a$. We keep the same notation for the extension of l.

1.2. For any $u=\omega_1 u_1$, $w=\omega_2 w_1$, ω_1, $\omega_2\in\Omega$, u_1, $w_1\in W_a$, we define $P_{u,w}$ to be P_{u_1,w_1}, as in [KL 1] if $\omega_1=\omega_2$ and $P_{u,w}$ to be zero if $\omega_1\neq\omega_2$. We say that $u\underset{LR}{\leqq} w$ or $u\underset{L}{\leqq} w$ if $u_1\underset{LR}{\leqq} w_1$, or $u_1\underset{L}{\leqq} w_1$ in the sense of [KL 1], we say that $u\underset{R}{\leqq} w$ if $u^{-1}\underset{L}{\leqq} w^{-1}$. These relations generate equivalence relations $\underset{LR}{\sim}$, $\underset{L}{\sim}$, $\underset{R}{\sim}$ in W, respectively, and the corresponding equivalence classes are called two-sided cells, left cells, right cells of W, respectively. The relation $\underset{LR}{\leqq}$ (resp. $\underset{L}{\leqq}$, $\underset{R}{\leqq}$) in W then induces a partial order $\underset{LR}{\leqq}$ (resp. $\underset{L}{\leqq}$, $\underset{R}{\leqq}$) in the set of two-sided (resp. left, right) cells of W. We extend the Bruhat order $\leqq$ in W_a to W by defining $u\leqq w$ if and only if $\omega_1=\omega_2$ and $u_1\leqq w_1$.

Let q be an indeterminate and let $A=\mathbb{C}[q, q^{-1}]$. Let H be the Hecke algebra of W over A, that is a free A-module with basis T_w $(w\in W)$ and multiplication defined by

$$(T_r-q^2)(T_r+1)=0 \quad \text{if } r\in S \qquad \text{and} \qquad T_w T_{w'}=T_{ww'} \quad \text{if } l(ww')=l(w)+l(w').$$

For each $w\in W$, let

$$C_w=q^{-l(w)}\sum_{u\leqq w} P_{u,w}(q^2)\,T_u\in H.$$

And we write

$$C_w C_u=\sum_z h_{w,u,z}\,C_z\in H, \qquad h_{w,u,z}\in A.$$

For each $z\in W$, there is a well defined integer $a(z)\geqq 0$ such that

$$q^{a(z)}h_{w,u,z}\in\mathbb{C}[q] \quad \text{for all } w, u\in W$$
$$q^{a(z)-1}h_{w,u,z}\notin\mathbb{C}[q] \quad \text{for some } w, u\in W$$

(*see* [L 3, I, 7.3]). We have $a(z)\leqq l(w_0)$, where w_0 is the longest element of W_0. It is known that

$$c_0=\{\,w\in W \mid a(w)=l(w_0)\,\}$$

is a two-sided cell of W (*see* [S, I]) which is the lowest one for the partial order $\underset{LR}{\leqq}$.

1.3. Let $\gamma_{w,u,z}$ be the constant term of $q^{a(z)}h_{w,u,z}\in\mathbb{C}[q]$. We have $\gamma_{w,u,z}\in\mathbb{N}$. Moreover (*see* [L 3, II])

$$(a) \qquad \gamma_{w,u,z}\neq 0 \;\Rightarrow\; w\underset{L}{\sim}u^{-1},\quad u\underset{L}{\sim}z,\quad w\underset{R}{\sim}z.$$

Let J be the $\mathbb{C}$-vector space with basis $(t_w)_{w\in W}$. This is an associative $\mathbb{C}$-algebra with multiplication

$$t_w t_u=\sum_z \gamma_{w,u,z}\,t_z.$$

It has a unit element $1=\sum_{d\in\mathscr{D}} t_d$, where $\mathscr{D}=\{d\in W_a \mid a(d)=l(d)-2\deg P_{e,d}\}$ (e is the unit of W) (*see* [L 3, II]).

For each two-sided cell c of W, let J_c be the subspace of J spanned by t_w, $w\in c$, then $J=\bigoplus_c J_c$, where the sum is over the set of all two-sided cells of W. By (a) we see that J_c is a two-sided ideal of J and in fact is an associative $\mathbb{C}$-algebra with unit $\sum_{d\in\mathscr{D}\cap c} t_d$.

1.4. For each $q_0\in\mathbb{C}^*$, we denote $H_{q_0}=H\otimes_A\mathbb{C}$, where $\mathbb{C}$ is an A-algebra with q acting as scalar multiplication by q_0. We shall denote $T_w\otimes 1$, $C_w\otimes 1$ in H_{q_0} again by T_w, C_w. We also use the notation $h_{w,u,z}$ for the specialization at $q_0\in\mathbb{C}^*$ of $h_{w,u,z}$.

The A-linear map $\phi: H\to J\otimes_{\mathbb{C}} A$ defined by

$$\phi(C_w)=\sum_{\substack{d\in\mathscr{D}\\ z\in W\\ a(z)=a(d)}} h_{w,d,z}\,t_z$$

is a homomorphism of A-algebra with 1 (*see* [L 3, II]). Let $\phi_{q_0}: H_{q_0}\to J$ be the induced homomorphism for any $q_0\in\mathbb{C}^*$.

Any (left) J-module E gives rise, via $\phi_{q_0}: H_{q_0}\to J$, to a (left) H_{q_0}-module E_{q_0}. We denote by K(J) [resp. $K(H_{q_0})$] the Grothendieck group of (left) J-modules (resp. H_{q_0}-modules) of finite dimension over $\mathbb{C}$. The correspondence $E\to E_{q_0}$ defines a homomorphism $(\phi_{q_0})_*: K(J)\to K(H_{q_0})$.

We similarly define $K(J_c)$ for any two-sided cell c of W. Then we have $K(J)=\bigoplus_c K(J_c)$, where the sum is over the set of all two-sided cells of W. Now we define $K(H_{q_0})_c$. For any simple H_{q_0}-module M, we attach to M a two-sided cell c_M of W by the following two conditions:

$$C_w M\neq 0 \text{ for some } w\in c_M$$

$$C_w M=0 \text{ for any } w \text{ in a two-sided cell } c \text{ with } c\underset{LR}{\leqq} c_M,\ c\neq c_M.$$

Then c_M is well defined since there are only a finite number of two-sided cells in W. Let $K(H_{q_0})_c$ be the subgroup of $K(H_{q_0})$ spanned by simple H_{q_0}-modules M with $c_M=c$. Obviously we have $K(H_{q_0})=\bigoplus_c K(H_{q_0})_c$. Thus for a two-sided cell c of W, $(\phi_{q_0})_*$ induces a homomorphism

$$(\phi_{q_0})_{*,c}:\quad K(J_c)\to K(H_{q_0})_c.$$

The following result is due to Lusztig (*see* [L 3, III, 1.9 and 3.4]).

PROPOSITION **1.5.** — *The map* $(\phi_{q_0})_{*,c}$ *is surjective for any* $q_0\in\mathbb{C}^*$, *moreover,* $(\phi_{q_0})_{*,c}$ *is an isomorphism when* q_0 *is not a root of* 1 *or* $q_0^2=1$.

Now we state a conjecture.

CONJECTURE **1.6.** — The map $(\phi_{q_0})_{*,c}$ is injective if $(\phi_{q_0})_{*,c'}$ is injective for some two-sided cell c' of W with $c' \underset{LR}{\leq} c$.

By proposition 1.6 one knows that $(\phi_{q_0})_{*,c}$ is injective is equivalent to that $(\phi_{q_0})_{*,c}$ is bijective.

We mainly discuss $(\phi_{q_0})_{*,c_0}$, where c_0 is the lowest two-sided cell of W. We prove that if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$, then $(\phi_{q_0})_{*,c_0}$ is injective (*see* Theorem 3.4) and show that $(\phi_{q_0})_{*,c_0}$ is likely not injective if $\sum_{w \in W_0} q_0^{2l(w)} = 0$ (*see* Theorem 3.6).

1.7. Let H'_{q_0} be the subalgebra of H_{q_0} spanned by T_w, $w \in W_0$. And let J′ be the subspace of J spanned by t_w, $w \in W_0$. J′ is a $\mathbb{C}$-algebra with unit $\sum_{d \in \mathcal{D} \cap W_0} t_d$. Let $\phi'_{q_0} : H'_{q_0} \to J'$ be defined by

$$\phi'_{q_0}(C_w) = \sum_{\substack{d \in \mathcal{D} \cap W_0 \\ z \in W_0 \\ a(d) = a(z)}} h_{w,d,z}(q_0)\, t_z, \qquad w \in W_0,$$

then ϕ'_{q_0} is a $\mathbb{C}$-algebra homomorphism preserving 1.

As in 1.4 we define $K(H'_{q_0})$, $K(J')$, $K(H'_{q_0})_{c'}$, $K(J'_{c'})$, $(\phi'_{q_0})_*$, $(\phi'_{q_0})_{*,c'}$, etc., where c' is a two-sided cell of W_0. We also have

PROPOSITION **1.8.** — $(\phi'_{q_0})_{*,c'}$ *is surjective for any* $q_0 \in \mathbb{C}^*$. *Moreover* $(\phi_{q_0})_{*,c'}$ *is an isomorphism when* q_0 *is not a root of* 1 *or* $q_0^2 = 1$.

CONJECTURE **1.9.** — $(\phi_{q_0})_{*,c'}$ is injective if $(\phi'_{q_0})_{*,c''}$ is injective for some two-sided cell c'' of W_0 with $c'' \underset{LR}{\leq} c'$.

When c' is the lowest two-sided cell of W_0, it is easy to see that $(\phi'_{q_0})_{*,c'}$ is injective if and only if $\sum_{w \in W_0} q_0^{2l(w)} \neq 0$.

2. The two-side cell c_0 and the ring J_{c_0}

In this section we recall and prove some results on c_0 and J_{c_0}.

2.1. We denote by w_0 the longest element in W_0. Let

$$\mathfrak{S} = \{ w \in W \mid l(ww_0) = l(w) + l(w_0) \quad \text{and} \quad ww_0 r \notin c_0 \quad \text{for any } r \in S \cap W_0 \}.$$

Then $\mathcal{D}_0 = \mathcal{D} \cap c_0 = \{ ww_0 w^{-1} \mid w \in \mathfrak{S} \}$ and $|\mathfrak{S}| = |W_0|$ (*see* [S, II]).

Let $X^+=\{w\in W \mid l(rx)>l(x)$ for any $r\in S'\}$, where $S'=S\cap W_0$. Let $x_i\in X^+$ ($i\in\{1, 2, \ldots, n\}=I_0$) be the i-th basic dominant weight, then x_i has the properties: $l(x_i r_i)<l(x_i)$, $x_i r_j=r_j x_i$, $l(x_i r_j)=l(x_i)+1$ if $i\neq j\in I_0$. We have

$$c_0=\{w' w_0 x w^{-1} \mid w, w'\in\mathfrak{S}, x\in X^+\} \qquad (\textit{see } [\text{S, II}]).$$

Moreover $l(w' w_0 x w^{-1})=l(w')+l(w_0)+l(x)+l(w^{-1})$.

LEMMA 2.2. — *Let* $u\in c_0$, *then* $C_u=h C_{w_0} h'$ *for some* $h, h'\in H_{q_0}$, *i.e., the two-sided ideal* $\bigoplus_{u\in c_0} \mathbb{C} C_u$ of H_{q_0} *is generated by the element* C_{w_0}.

Proof. — Write $u=w' w_0 w$ for some $w', w\in W$ such that $l(u)=l(w')+l(w_0)+l(w)$. We use induction on $l(u)$ to prove that C_u is in the two-sided ideal N of H_{q_0} generated by C_{w_0}.

When $l(u)=l(w_0)$, then $C_u=C_\omega C_{w_0} C_{\omega'}$ for some $\omega, \omega'\in\Omega$. Now assume that $l(w')>0$. Let $s\in S$ be such that $sw'\leqq w'$, then

$$C_s . C_{su}=C_u+\sum_{\substack{z\in c_0\\ l(z)<l(u)}} a_z C_z, \qquad a_z\in\mathbb{N} \quad (\textit{see } [\text{KL 1}]).$$

By induction hypothesis we know that $C_u\in N$. Similarly we can prove that $C_u\in N$ if $l(w)>0$. The lemma is proved.

COROLLARY 2.3. — *For a simple* H_{q_0}*-module* M, *we have* $c_M=c_0$ *if and only if* $C_{w_0} M\neq 0$.

For $w\in W$, set $L(w)=\{r\in S \mid rw\leqq w\}$ and $R(w)=\{r\in S \mid wr\leqq w\}$.

LEMMA 2.4. — (i) *Let* w' *be the longest element in the Weyl group generated by* $L(w)$ (*or* $R(w)$), *then* $w=w' w''$ (*or* $w=w'' w'$) *for some* $w''\in W$ *and* $l(w)=(l(w')+l(w'')$.

(ii) *Let* w' *be the longest element in the Weyl group* W' *generated by* $S-L(w)$ [resp. $S-R(w)$], *then* $l(w' w)=l(w')+l(w)$ [resp. $l(ww')=l(w)+l(w')$].

Proof. — (i) follows from $T_{w'} C_w=q^{l(w')} C_w$ or $C_w T_{w'}=q^{l(w')} C_w$.

(ii) follows from the fact that w is the shortest element in $W' w$ or $w W'$.

Let Γ_0 be the left cell in c_0 containing w_0, then

$$\begin{aligned}\Gamma_0&=\{w w_0 x \mid x\in X^+, w\in\mathfrak{S}\}.\\ &=\{w\in W \mid R(w)=S'\}\end{aligned}$$

LEMMA 2.5. — *Any element* $u\in\Gamma_0$ *has the form* wxw_J, *where* $w\in W_0$, $x=\prod_{i=1}^{n} x_i^{a_i}\in X^+$. w_J *is the longest element in* $W_J=\langle r_j \mid a_j=0, j\in I_0\rangle$, moreover $l(u)=l(w)+l(x)+l(w_J)$.

Proof. — Choose $x=\prod_{i=1}^{n} x_i^{a_i}\in X^+$ such that $u\in\Gamma_0\cap W_0 x W_0$.

Then the shortest element in $W_0 x W_0$ is $x w_J w_0$ and the shortest element in $\Gamma_0 \cap W_0 x W_0$ is $x w_J$ by lemma 2.4 (i), where w_J is the longest element in $W_J = \langle r_j \mid a_j = 0, j \in I_0 \rangle$. The lemma is proved.

LEMMA 2.6. — (i) *Let* $J \subseteqq K \subseteqq I_0$, *then in* H_{q_0} *we have* $C_{w_J} C_{w_K} = C_{w_K} C_{w_J} = \eta_J C_{w_K}$, *where* w_J, w_K *are the longest element in* $W_J = \langle r_j | j \in J \rangle$, $W_K = \langle r_k | k \in K \rangle$, *respectively*, $\eta_J = q_0^{-l(w_J)} \sum_{w \in W_J} q_0^{2l(w)}$.

(ii) $C_{w w_J} = h C_{w_J}$, $C_{w_J w'} = C_{w_J} h'$ for some h, $h' \in H_{q_0}$ if $l(w w_J) = l(w) + l(w_J)$, $l(w_J w') = l(w_J) + l(w')$.

Proof. — First we prove (ii). We use induction on $l(w)$. Assume that $l(w) > 0$. Choose $r \in S$ such that $rw \leqq w$, then

$$C_r C_{r w w_J} = C_{w w_J} + \sum_{\substack{z \in W \\ l(z) < l(w w_J)}} a_z C_z, \qquad a_z \in \mathbb{N} \quad (\textit{see } [\text{KL } 1]).$$

Moreover $a_z \neq 0$ implies that $z \underset{L}{\leqq} r w w_J$. So $R(z) \supseteqq \{ r_j | j \in J \}$ (*see* [KL 1]).

By Lemma 2.4 we see that $z = z' w_J$ for some $z' \in W$ and $l(z) = l(z') + l(w_J)$. By induction hypothesis we know that $C_{w w_J} = h C_{w_J}$ for some $h \in H_{q_0}$. Similarly we have $C_{w_J w'} = C_{w_J} h'$ for some $h' \in H_{q_0}$.

(i) follows from $C_J C_J = \eta_J C_J$ and (ii).

COROLLARY 2.7. — *Let* x, w_J *be as in* 2.5, *then in* H_{q_0} *we have*

$$C_{w_0} C_{x w_J} = \eta_J \sum_{\substack{y \in X^+ \\ w_0 y \leqq w_0 x}} a_{x,y} C_{w_0 y} \in \mathbb{C}, \qquad a_{x,y} \in \mathbb{C} \quad \textit{and} \quad a_{x,x} = 1.$$

Proof. — By 2.1 and 2.6 (ii) we see that $C_{x w_J} = C_{w_J x} = C_{w_J} h$, where

$$h = \sum_{\substack{w \in W \\ l(w_J w) = l(w_J) + l(w) \\ w_J w \leqq w_J x}} a_w T_w, \qquad a_w \in \mathbb{C}, \qquad a_x = q_0^{-l(x)}.$$

By (2.6 (i) we know that

$$(a) \qquad C_{w_0} \cdot C_{x w_J} = C_{w_0} \cdot C_{w_J} h = \eta_J C_{w_0} h.$$

Note that $h_{w_0, x w_J, z} \neq 0$ implies that $z \underset{L}{\sim} x w_J$, $z \underset{R}{\sim} w_0$ (*see* [L 3, I]), we have $z \in \Gamma_0 \cap \Gamma_0^{-1} = \{ w_0 y | y \in X^+ \}$. So by (*a*) we get

$$C_{w_0} C_{x w_J} = \eta_J \sum_{y \in X^+} a_{x,y} C_{w_0 y}, \qquad a_{x,y} \in \mathbb{C}.$$

Since $a_x=q_0^{-l(x)}$ and $l(w)<l(x)$ if $a_w\neq 0$, $w\neq x$. We havec $a_{x,x}=1$ and $a_{x,y}=0$ if $l(y)>l(x)$ or $l(y)=l(x)$ but $x\neq y$. Let $w\in W$ be such that $a_w\neq 0$. Consider the expression

$$C_{w_0}\cdot T_w=\sum_{z^{-1}\in\Gamma_0} b_z C_z, \qquad b_z\in\mathbb{C}.$$

Since $w_J w\leqq w_J x$, we have $b_z\neq 0$ implies that $z\leqq w_0 x$. Thus by (*a*) we know that $a_{x,y}\neq 0$ implies that $w_0 y\leqq w_0 x$. The Corollary is proved.

2.8. For any $x\in X$, we choose x', $x''\in X^+$ such that $x=x'x''^{-1}$ and then define $\tilde{T}_x=q_0^{-l(x')}T_{x'}(q_0^{-l(x'')}T_{x''})^{-1}$. $\tilde{T}_x$ is independent of the choices x' and x''. We denote the conjugacy class of $x\in X$ in W by O_x and let $z_x=\sum_{x'\in O_x}\tilde{T}_{x'}$. z_x is in the center of H_{q_0}. For $x\in X^+$, denote $d(x',x)$ the dimension of the x'-weight space $V(x)_{x'}$ of $V(x)$, where $V(x)$ is the irreducible representation of G with highest weight x. We set $S_x=\sum_{x'\in X^+} d(x',x)z_{x'}$, $x\in X^+$.

LEMMA **2.9**. (*see* [X]). — *In* H_{q_0} *we have* $C_{w'w_0w^{-1}}S_x=S_x C_{w'w_0w^{-1}}=C_{w'w_0xw^{-1}}$ *for any* w', $w\in\mathfrak{S}$, $x\in X^+$.

LEMMA **2.10**. — *Let* $u\in\Gamma_0$, *then*

$$C_u=\sum_{\substack{y\in X^+\\ I\subseteq I_0}} h_{I,y}C_{x_I w_{I'}}S_y,$$

where $h_{I,y}\in H'_{q_0}=\bigoplus_{w\in W_0}\mathbb{C}T_w=\bigoplus_{w\in W_0}\mathbb{C}C_w$, $x_I=\prod_{i\in I}x_i$, $I'=I_0-I$.

Proof. — By 2.5 we see that $u=wxw_J$, where $w\in W_0$, $x=\prod_{i=1}^{n}x_i^{a_i}$, $J=\{j\in I_0\mid a_j=0\}$.

We use induction on $l(u)$, when $w=e$, by 2.9 we see that $C_u=C_{x_{J'}w_J}S_y$, where $J'=I_0-J$, $y=\prod_{j\in J'}x_j^{a_j-1}$, *i.e.* the lemma is true. Now assume that $l(w)>0$ and choose $r\in S'$ such that $rw\leqq w$, then

$$C_r.C_{rwxw_J}=C_{wxw_J}+\sum_{\substack{z\in\Gamma_0\\ l(z)<l(wxw_J)}} a_z C_z, \qquad a_z\in\mathbb{N}.$$

By induction hypothesis we know that there exists $h_{I,y}\in H'_{q_0}$ such that $C_u=\sum_{\substack{y\in X^+\\ I\subseteq I_0}} h_{I,y}C_{x_I w_{I'}}S_y$. The lemma is proved.

2.11. Let R_G be the ring of the rational representations ring of G tensor with $\mathbb{C}$. Then R_G is a $\mathbb{C}$-algebra with a $\mathbb{C}$-basis $V(x)$, $x\in X^+$. Let $M_{\mathfrak{S}\times\mathfrak{S}}(R_G)$ be the $\mathfrak{S}\times\mathfrak{S}$ matrix ring over R_G. Then we have

THEOREM 2.12 (*see* [X]). – *There is a natural isomorphism* $J_{c_0} \xrightarrow{\sim} M_{\mathfrak{S}\times\mathfrak{S}}(R_G)$ *such that* $t_{w'w_0xw^{-1}} \to (m_{w_1, w_2}) \in M_{\mathfrak{S}\times\mathfrak{S}}(R_G)$, w', w^{-1}, $w_1\ w_2 \in \mathfrak{S}$,

$$m_{w_1, w_2} = \begin{cases} V(x) & \text{if } \ w_1 = w', \quad w_2 = w \\ 0 & \text{otherwise.} \end{cases}$$

Hereafter we identify J_{c_0} with $M_{\mathfrak{S}\times\mathfrak{S}}(R_G)$.

3. The homomorphism $(\phi_{q_0})_{*, c_0}$

3.1. For any semisimple conjugacy class s in G, we have a simple representation ψ_s of $J_{c_0} \simeq M_{\mathfrak{S}\times\mathfrak{S}}(R_G)$:

$$\psi_s: \quad M_{\mathfrak{S}\times\mathfrak{S}}(R_G) \to M_{\mathfrak{S}\times\mathfrak{S}}(\mathbb{C})$$

$$(m_{w, w'}) \to (tr(s, m_{w, w'})), \ w, \ w' \in \mathfrak{S}.$$

Any simple representation of J_{c_0} is isomorphic to some ψ_s (*see* [X]). Let E_s be the simple J_{c_0}-module providing the representation ψ_s. E_s gives rise, via

$$\phi_{q_0, c_0}: \quad H_{q_0} \to J \to J_{c_0},$$

to an H_{q_0}-module E_{s, q_0}. Note that $\phi_{q_0, c_0}(S_x) = \sum_{w\in\mathfrak{S}} t_{ww_0xw^{-1}}$ for any $x \in X^+$, we see that S_x acts on E_{s, q_0} by scalar $tr(s, V(x))$.

PROPOSITION **3.2.** – *The set* $\Lambda = \{(\phi_{q_0})_{*, c_0}(E_s) \mid s$ *semisimple conjugacy class of* $G\} - \{0\}$ *is a base of* $K(H_{q_0})_{c_0}$.

Proof. – It is easy to see that $(\phi_{q_0})_{*, c_0}(E_s) = \sum_{M} a_M M$, where the sum is over the set of composition factors M of E_{s, q_0} with $c_M = c_0$ and a_M is the multiplicity of M in E_{s, q_0}.

Now let $F_i = (\phi_{q_0})_{*, c_0}(E_{s_i}) \in \Lambda$, $1 \leqq i \leqq k$, and suppose that $\sum_{i=1}^{k} m_i F_i = 0$, $m_i \in \mathbb{Z}$. Let $F_i = \sum_{M_{ij}} a_{M_{ij}} M_{ij}$, M_{ij} simple H_{q_0}-module with $c_{M_{ij}} = c_0$. Since S_x acts on E_{s_i, q_0} by scalar $tr(s_i, V(x))$. S_x acts on M_{ij} by scalar $tr(s_i, V(x))$ if $a_{M_{ij}} \neq 0$. $F_i \in \Lambda$ implies that $a_{M_{ij}} \neq 0$ for some M_{ij}. Therefore $m_i = 0$. By 1.6 we know that $(\phi_{q_0})_{*, c_0}$ is surjective, hence Λ is a base of $K(H_{q_0})_{c_0}$. The proposition is proved.

COROLLARY **3.3.** – E_{s, q_0} *has at most one composition factor to which the attached two-sided cell is* c_0. *Moreover the multiplicity* a_M *is* 1 *if* E_{s, q_0} *has such a composition factor* M.

THEOREM 3.4. — *If* $\sum_{w \in W_0} q_0^{2l(w)} = q_0^{l(w)} \eta_{I_0} \neq 0$, *then* $(\phi_{q_0})_{*, c_0}$ *is injective, so* $(\phi_{q_0})_{*, c_0}$ *is an isomorphism by* 1.6.

Proof. — We have

$$\phi_{q_0, c_0}(C_{w_0}) = \sum_{\substack{w \in \mathfrak{S} \\ x \in X^+}} h_{w_0, w w_0 w^{-1}, w_0 x w^{-1}} t_{w_0 x w^{-1}} \in J_{c_0}.$$

We identify J_{c_0} with $M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$, then $\phi_{q_0, c_0}(C_{w_0}) = (m_{w', w}) \in M_{\mathfrak{S} \times \mathfrak{S}}(R_G)$ and

$$m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0, w w_0 w^{-1}, w_0 x w^{-1}} V(x), & \text{if } w' = e \\ 0 & \text{if } w' \neq e \end{cases}.$$

Note that $C_{w_0} C_{w_0} = \eta_{I_0} C_{w_0}$, we see that $m_{e, e} = \eta_{I_0} \neq 0$, where e is the unit in W. So $C_{w_0} E_{s, q_0} \neq 0$ for any semisimple conjugacy class s of G since $\psi_s \phi_{q_0, c_0}(C_{w_0}) \neq 0$. Now let $0 = F_0 \subseteqq F_1 \subseteqq \ldots \subseteqq F_k = E_{s, q_0}$ be a composition series of E_{s, q_0} and let i be the integer such that $C_{w_0} F_i \neq 0$ and $C_{w_0} F_{i-1} = 0$. Then $C_{w_0} M \neq 0$ where $M = F_i / F_{i-1}$, otherwise, $C_{w_0} F_i \subseteqq F_{i-1}$, choose $v \in F_i$ such that $C_{w_0} v \neq 0$, we have $C_{w_0}^2 v = \eta_{I_0} C_{w_0} v \neq 0$. A contradiction, so $C_{w_0} M \neq 0$, *i. e.*, $c_M = c_0$. That is to say $(\phi_{q_0})_{*, c_0}(E_s) \neq 0$. The theorem follows from proposition 3.2.

3.5. In the subsequent part of this section we assume that $\eta_{I_0} = 0$, *i. e.*, $\sum_{w \in W_0} q_0^{2l(w)} = 0$.

Let $\Delta_{q_0} = \{ I \subseteqq I_0 \mid \eta_{I'} \neq 0$ but $\eta_{I' \cup \{i\}} = 0$ for any $i \in I \}$. Here we use the convention that I′ always denotes the complement of I in I_0 *i. e.*, $I' = I_0 - I$.

THEOREM 3.6. — *Let* s *be a semisimple conjugacy class of* G, *then* $(\phi_{q_0})_{*, c_0}(E_s) = 0$ *if and only if* $\alpha_I = 0$ *for all* $I \in \Delta_{q_0}$, *where*

$$\alpha_I = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x}\, tr(s, V(x)) \quad \textit{for any } I \subseteqq I_0.$$

We need two lemmas.

LEMMA 3.7. — *The following conditions are equivalent.*

(i) $C_{w_0} E_{s, q_0} = 0$.

(ii) $\psi_s \phi_{q_0, c_0}(C_{w_0}) = 0$.

(iii) $\sum_{x \in X^+} h_{w_0, w w_0 w^{-1}, w_0 x w^{-1}}\, tr(s, V(x)) = 0$ *for all* $w \in \mathfrak{S}$.

(iv) $\sum_{x \in X^+} h_{w_0, w w_0, w_0 x}\, tr(s, V(x)) = 0$ *for all* $w \in \mathfrak{S}$.

(v) $\alpha_I = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x}\, tr(s, V(x)) = 0$ *for all* $I \subseteqq I_0$.

(vi) $\alpha_I = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x}\, tr(s, V(x)) = 0$ *for all* $I \in \Delta_{q_0}$.

Proof. — (i) and (ii) are obviously equivalent.

Note that $h_{w_0, ww_0w^{-1}, z} \neq 0$ implies that $z = w_0 x w^{-1}$ for some $x \in X^+$ and that $\phi_{q_0, c_0}(C_{w_0}) = (m_{w', w})$,

$$m_{w', w} = \begin{cases} \sum_{x \in X^+} h_{w_0, ww_0w^{-1}, w_0xw^{-1}} V(x), & \text{if } w' = e \\ 0, & \text{otherwise} \end{cases}.$$

we see that (ii) $\Leftrightarrow$ (iii).

By theorem 2.9 in [X] we have $h_{w_0, ww_0, w_0x} = h_{w_0, ww_0w^{-1}, w_0xw^{-1}}$. So we have (iii) $\Leftrightarrow$ (iv).

By Lemma 2.4 (i) we see that $x_I w_{I'} = ww_0$ for some $w \in W$. Using the method in [S] one knows that $w \in \mathfrak{S}$. Thus we have (iv) $\Rightarrow$ (v). Now we show that (v) $\Rightarrow$ (iv). Let $w \in \mathfrak{S}$, then $ww_0 \in \Gamma_0$, hence by 2.10

$$C_{ww_0} = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} h_{I, y} C_{x_I w_{I'}} S_y, \qquad h_{I, y} \in H'_{q_0}.$$

Since $C_{w_0} h_{I, y} = a_{I, y} C_{w_0}$ for some $a_{I, y} \in \mathbb{C}$, we have

$$\sum_{x \in X^+} h_{w_0, ww_0, w_0x}\, tr(s, V(x)) = \sum_{\substack{y \in X^+ \\ I \subseteq I_0}} a_{I, y} \alpha_I\, tr(s, V(y)) = 0.$$

Finally we prove that (v) and (vi) are equivalent.

One direction is obvious. Now assume that (vi) holds. Let $J \subseteq I_0$. We use induction on $l(x_J)$ to prove that $\alpha_J = 0$. When $\eta_{J'} = 0$ or $J \in \Delta_{q_0}$ we have $\alpha_J = 0$ by 2.7 or by (vi). Suppose $\eta_{J'} \neq 0$ and $J \notin \Delta_{q_0}$. Choose $j \in J$ such that $\eta_{J' \cup \{j\}} \neq 0$. Let $K = J - \{j\}$, then $K' = J' \cup \{j\}$. We have

$$C_{w_0} C_{x_J w_{J'}} = \frac{1}{\eta_{K'}} C_{w_0} C_{w_{K'}} C_{x_J w_{J'}} \quad \text{(by 2.6)}$$

$$= \frac{\eta_{J'}}{\eta_{K'}} C_{w_0} (C_{w_{K'} x_K x_j} + \sum_{\substack{I \subseteq I_0 \\ y \in X^+}} h_{I, y} C_{x_I w_{I'}} S_y), \qquad h_{I, y} \in H'_{q_0} \quad \text{(by 2.6, 2.10)}.$$

Let $C_{w_0} h_{I, y} = a_{I, y} C_{w_0}$, $a_{I, y} \in \mathbb{C}$. By 2.7 we see that $a_{I, y} \eta_{I'} \neq 0$ implies that $l(x_I y) < l(x_J)$. Obviously $l(x_K) < l(x_J)$. Using induction hypothesis we get

$$\alpha_J = \frac{\eta_{J'}}{\eta_{K'}} (\alpha_K\, tr(s, V(x_j)) + \sum_{\substack{I \subseteq I_0 \\ y \in X^+}} a_{I, y} \alpha_I\, tr(s, V(y))) = 0.$$

The lemma is proved.

LEMMA **3.8.** — $(\phi_{q_0})_{*, c_0}(E_s) = 0$ *if and only if* $C_{w_0} E_{s, q_0} = 0$.

Proof. — The "if" part is obvious. The "only if" part need to do a little more.

Assume that $C_{w_0}E_{s, q_0} \neq 0$. By 3.7 we see that $\alpha_I \neq 0$ for some $I \subseteq I_0$. As in [LX] we define an automorphism $\alpha: W \to W$ by

$$\alpha(wx) = w_0 w x^{-1} w_0, \qquad w \in W_0, \quad x \in X.$$

One verifies that α leaves stable W_0, X, S, S'. In particular, α induces a bijection $\alpha: I_0 \to I_0$ and an automorphism $\sigma: H_{q_0} \to H_{q_0}$ by defining $C_u \to C_{\alpha(u)}$, $u \in W$. Let $J = \alpha(I)$, we have $\alpha(x_I) = x_J$, $\alpha(w_{I'}) = w_{J'}$. Consider

$$\psi_s \phi_{q_0, c_0}(C_{x_J^{-1} w_{J'}}) = (n_{w', w}) \in M_{\mathfrak{S} \times \mathfrak{S}}(\mathbb{C}).$$

By 2.4 and 2.12, we know that $n_{w', w} = 0$ if $w' \neq e$ and

$$n_{e, w} = \sum_{x \in X^+} h_{x_J^{-1} w_{J'}, w w_0 w^{-1}, w_0 x w^{-1}} \, tr(s, V(x)).$$

In particular,

$$n_{e, e} = \sum_{x \in X^+} h_{x_J^{-1} w_{J'}, w_0, w_0 x} \, tr(s, V(x)).$$

We claim that $n_{e, e} = \alpha_I$. In fact, let ι be the antiautomorphism of H_{q_0} defined by $C_u \to C_{u^{-1}}$, $u \in W$. Apply ι to the equality

$$C_{w_0} C_{x_I w_{I'}} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{w_0 x}.$$

We get

$$C_{x_I^{-1} w_{I'}} C_{w_0} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{x^{-1} w_0}.$$

Apply σ to the above identity we obtain

$$C_{x_J^{-1} w_{J'}} C_{w_0} = \sum_{x \in X^+} h_{w_0, x_I w_{I'}, w_0 x} C_{w_0 x}.$$

Therefore $h_{x_J^{-1} w_{J'}, w_0, w_0 x} = h_{w_0, w_I w_{I'}, w_0 x}$ and $n_{e, e} = \alpha_I \neq 0$. By this and $n_{w', w} = 0$ if $w' \neq e$ we see that α_I is an eigenvalue of $\psi_s \phi_{q_0, c_0}(C_{x_J^{-1} w_{J'}})$. Let $0 \neq v \in E_{s, q_0}$ be such that $C_{x_J^{-1} w_{J'}} v = \alpha_I v$. Let F be the H_{q_0}-submodule of E_{s, q_0} generated by v. Then F has a maximal H_{q_0}-submodule F_0 which doesn't contain v. F/F_0 is an irreducible H_{q_0}-module. Moreover $C_{x_J^{-1} w_{J'}}(F/F_0) \neq 0$ since $v \notin F_0$. We have proved that $(\phi_{q_0})_{*, c_0}(E_s) \neq 0$.

Theorem 3.6 follows from 3.7 and 3.8.

3.9. There are two special cases. One is that $\eta_{I_0} = 0$ but $\eta_I \neq 0$ for any proper subset I of I_0. In this case we have $\Delta_{q_0} = \{\{i\} \mid i \in I_0\}$. Let $i' = I - \{i\}$. By 2.7 we have $h_{w_0, x_i w_{i'}, w_0 x} = \eta_{i'} a_{i, x}$ for some $a_{i, x} \in \mathbb{C}$. Moreover, $a_{i, x} \neq 0$ implies that $w_0 x \leqq w_0 x_i$ and $a_{i, x_i} = 1$. By this we see that the equation system

$$\alpha_{\{i\}} = \eta_{i'} \sum_{\substack{x \in X^+ \\ w_0 x \leqq w_0 x_i}} a_{i, x} \, tr(s, V(x)) = 0, \qquad i \in I_0$$

uniquely determines $tr(s, V(x_i))$, $i \in I_0$. In other words, there exists a unique semisimple conjugacy class s of G such that $\alpha_{\{i\}}=0$ for all $i \in I_0$. By 3.6 we have got the following.

PROPOSITION. — *There exists a unique semisimple conjugacy class s of* G *such that* $(\phi_{q0})_{*, c0}(E_s)=0$ *when* $\eta_{I_0}=0$ *but* $\eta_I \neq 0$ *for any proper subset* I *of* I_0.

When W is of type $\tilde{A}_n$. We can determine the semisimple conjugacy class s in the proposition explicitly. We have $a_{i,x}=0$ if $x \neq x_i$ since x_i is a minimal dominant weight for any $i \in I_0$. So $\alpha_{\{i\}}=\eta_i, tr(s, V(x_i))$. Let T be the diagonal subgroup of $G=SL_{n+1}(\mathbb{C})$. We may require that $x_i \in \mathrm{Hom}(T, \mathbb{C}^*)$ is defined by $x_i(t)=t_1 t_2 \ldots t_i$ where $t=\mathrm{diag}(t_1, t_2, \ldots, t_{n+1}) \in T$. Thus, we have

$$tr(s, V(x_i)) = \sum_{\substack{j_a \in I_0 \cup \{n+1\} \\ j_a \neq j_b \text{ if } a \neq b}} t_{j_1} t_{j_2} \ldots t_{j_i}.$$

where $t=\mathrm{diag}(t_1, t_2, \ldots, t_{n+1}) \in s \cap T$, s a semisimple conjugacy class of G. Hence, $tr(s, V(x_i))=0$, $1 \leqq i \leqq n$ is equivalent to that t_i $(1 \leqq i \leqq n+1)$ is the solution of the equation $\lambda^{n+1}+(-1)^{n+1}=0$. So if $\eta_{I_0}=0$ but $\eta_I \neq 0$ for any proper subset I of I_0, $(\phi_{q0})_{*, c0}(E_s)=0$ if and only if the eigenpolynomial of s is $\lambda^{n+1}+(-1)^{n+1}$.

Another special case is that $q_0+q_0^{-1}=0$. In this case $\Delta_{q_0}=\{I_0\}$. So $(\phi_{q0})_{*, c0}(E_s)=0$ if and only if $\alpha_{I_0}=0$. If we identify the set $\{$semisimple conjugacy classes of G$\}$ with $\mathbb{C}^n$ through the bijection

$$s \to (tr(s, V(x_1)), tr(s, V(x_2)), \ldots, tr(s, V(x_n))),$$

then $\alpha_{I_0}=0$ defines a hypersurface in $\mathbb{C}^n$. That is to say, the set $\{$semisimple conjugacy class s of $G \mid (\phi_{q0})_{*, c0}(E_s)=0\}$ is a variety of dimension $n-1$.

When W_0 is of rank 2, if $\eta_{I_0}=0$, then either $\eta_I \neq 0$ for any proper subset $I \subseteqq I_0$ or $q_0+q_0^{-1}=0$. The above discussion shows that $(\phi_{q0})_{*, c0}$ is an isomorphism if and only if $\eta_{I_0} \neq 0$.

3.10. In general it is difficult to compute $C_{w_0} C_{x_I w_{I'}}$ in H. Now we compute it for the simplest case: x_I is the highest short root.

When $x_I \in X^+$ is the highest short root, $x_I w_{I'}=r_0 w_0$, and $w_0 x \leqq w_0 x_I$, $x \in X^+$ implies that $x=e$ or x_I. So by 2.7, in H we have

$$C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_I w_{I'}} = \sigma_{I'}(C_{w_0 x_I} + a C_{w_0}),$$

where $\sigma_{I'} \in A=\mathbb{C}[q, q^{-1}]$ is determined by $C_{w_{I'}} C_{w_{I'}} = \sigma_{I'} C_{w_{I'}}$, $a \in A$. We need to determine the coefficient a. Comparing the coefficient of T_e in both sides we get

$$q^{-l(w_0)-1} \sigma_{I_0} = q^{-l(w_0 w_I)} \sigma_{I'} P_{e, w_0 x_I}(q^2) + a q^{-l(w_0)} \sigma_{I'}.$$

i.e.

(*a*) $$\sigma_{I_0} = q^{1-l(x_I)} \sigma_{I'} P_{w_0, w_0 w_I}(q^2) + a q \sigma_{I'}.$$

Using the formula 8.10 in [L 2] we get the following

PROPOSITION **3.11.** — *If* x_{I} *is the highest short weight, then*

$$P_{w_0, w_0 x_{\mathrm{I}}} = \begin{cases} \sum_{i=1}^{n} q^{e_i - 1} & \text{for type } \tilde{A}_n, \tilde{D}_n, \tilde{E}_n. \\ 1 & \text{for type } \tilde{C}_n, \tilde{G}_2. \\ \dfrac{q^{2(n-1)} - 1}{q^2 - 1} & \text{for type } \tilde{B}_n. \\ q^4 + 1 & \text{for type } \tilde{F}_4. \end{cases}$$

where $e_1, \ldots, e_n$ *are the exponents of* W_0.

By the proposition and 3.10 (*a*) we obtain the following

PROPOSITION **3.12.** — *In* H *we have*

$$C_{w_0} C_{r_0 w_0} = C_{w_0} C_{x_{\mathrm{I}} w_{\mathrm{I}'}} = \sigma_{\mathrm{I}'} C_{w_0 x_{\mathrm{I}}} + \frac{\sigma_{\mathrm{I}_0}}{[e_n + 1]} [e_n] C_{w_0},$$

where e_n *is the largest exponent of* W_0 *and* $[i] = (q^i - q^{-i})/(q - q^{-1})$ *for any* $i \in \mathbb{N}$.

3.13. When W is of type $\tilde{A}_n$, the highest short weight is $x_1 x_n$.

$$\eta_{\mathrm{I}_0} = [2]_{q_0} [3]_{q_0} \ldots [n+1]_{q_0},$$

where $[i]_{q_0}$ is the specialization at $q_0 \in \mathbb{C}^*$ of $[i]$. By 3.12, in H_{q_0} we have

$$C_{w_0} C_{r_0 w_0} = [2]_{q_0} [3]_{q_0} \ldots [n-1]_{q_0} (C_{w_0 x_1 x_n} + [n]_{q_0}^2 C_{w_0}).$$

Now suppose $[n]_{q_0} = 0$ but $[i]_{q_0} \neq 0$ for i, $1 \leqq i \leqq n-1$, then $\Delta_{q_0} = \{\{1, n\}, \{2\}, \{3\}, \ldots, \{n-1\}\}$. By 3.9 and 3.12 we see that $\alpha_{\mathrm{I}} = 0$, $\mathrm{I} \in \Delta_{q_0}$ is equivalent to $tr(s, V(x_1 x_n)) = 0$, $tr(s, V(x_i)) = 0$, $2 \leqq i \leqq n-1$. Note that $tr(s, V(x_1 x_n)) = tr(s, V(x_1))\, tr(s, V(x_n)) - 1$, by 3.9, we know that $\alpha_{\mathrm{I}} = 0$, $\mathrm{I} \in \Delta_{q_0}$ if and only if the eigenpolynomial of s has the form $\lambda^{n+1} - a\lambda^n + (-1)^n a^{-1} \lambda + (-1)^{n+1}$, $a \in \mathbb{C}^*$. In other words, if $[n]_{q_0} = 0$, $[i]_{q_0} \neq 0$, $1 \leqq i \leqq n-1$, then $(\phi_{q_0})_{*, c_0}(E_s) = 0$ if and only if the eigenpolynomial of s has the form $\lambda^{n+1} - a\lambda^n + (-1)^n a^{-1} \lambda + (-1)^{n+1}$, $a \in \mathbb{C}^*$.

4. Examples

4.1. Type $\tilde{A}_1$. In this case $G = SL_2(\mathbb{C})$, $S = \{r_0, r_1\}$, $x_1 = r_0 \omega$, $\Omega = \{e, \omega\}$, $\eta_{\mathrm{I}_0} = q_0 + q_0^{-1}$. $c_0 = \{w \in W \mid l(w) > 0\}$. Another two-sided cell c of W is Ω.

J_c has two irreducible modules F_0, F_1. Both have dimension 1 and t_ω acts on F_i by scalar $(-1)^i$, $i = 0, 1$. Via, $\phi_{q_0, c} : H_{q_0} \to J \to J_c$, F_i becomes H_{q_0}-module F_{i, q_0}. T_ω acts on F_{i, q_0} by scalar $(-1)^i$ and T_{r_i} acts on F_{i, q_0} by scalar -1. $(\phi_{q_0})_{*, c}$ is an isomorphism for any $q_0 \in \mathbb{C}^*$.

For c_0, we have $J_{c_0}=M_{2\times 2}(R_G)$ and

$$\phi_{q_0, c_0}(C_{r_1})=\begin{pmatrix}\eta_{I_0} & V(x_1)\\ 0 & 0\end{pmatrix}$$
$$\phi_{q_0, c_0}(C_{r_0})=\begin{pmatrix}0 & 0\\ V(x_1) & \eta_{I_0}\end{pmatrix}$$
$$\phi_{q_0, c_0}(C_{\omega})=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Suppose that $\eta_{I_0}\neq 0$. Let s be the semisimple conjugacy class of G containing $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\in G$, then E_{s, q_0} is irreducible if and only if $\eta_{I_0}\neq\pm(t+t^{-1})$. When $\eta_{I_0}=\pm(t+t^{-1})$, $E_{s, q_0}/F_{i, q_0}\simeq M_{s, q_0}$, where $i=0$ if $\eta_{I_0}=-(t+t^{-1})$ and $i=1$ if $\eta_{I_0}=t+t^{-1}$. T_ω acts on M_{s, q_0} by scalar $(-1)^{i-1}$ and T_{r_i} acts on M_{s, q_0} by scalar q_0^2. $(\phi_{q_0})_{*, c}(E_s)=E_{s, q_0}$ if $\eta_{I_0}\neq\pm(t+t^{-1})$, $(\phi_{q_0})_{*, c_0}(E_s)=M_{s, q_0}$ if $\eta_{I_0}=\pm(t+t^{-1})$. In particular, when $\eta_{I_0}\neq 0$, $(\phi_{q_0})_*$ is an isomorphism.

When $\eta_{I_0}=0$, one verifies that E_{s, q_0} is irreducible if $t+t^{-1}\neq 0$ and $E_{s, q_0}=F_{0, q_0}\oplus F_{1, q_0}$ if $t+t^{-1}=0$. In particular rank $\ker(\phi_{q_0})_*=1$.

4.2. Type $\tilde{A}_2$. In this case we have $G=SL_3(\mathbb{C})$, $S=\{r_0, r_1, r_2\}$, $\Omega=\{1, \omega, \omega^2\}$ and $\omega r_0=r_1\omega$, $\omega r_1=r_2\omega$, $\omega r_2=r_0\omega$, $x_1=r_0 r_2\omega$, $x_2=r_0 r_1\omega^2$. W has three two-sided cells: $c=\Omega$, c_0, $c'=W-c\cup c_0$. c' is the two-sided cell of W containing r_0, r_1, r_2.

It is obviously $(\phi_{q_0})_{*, c}$ is an isomorphism.

Now consider $J_{c'}$. Any element in c' has one of the following forms: $\omega^i r_1 x_1^a\omega^j$, $\omega^{i+1}x_1^a\omega^j$, $\omega^{i+2}r_2 x_2^a\omega^{j+1}$, $\omega^{i+1}x_2^a\omega^{j+1}$, $i, j=0, 1, 2$. We define a $\mathbb{C}$-linear map θ: $J_{c'}\to M_{3\times 3}(A)$, $A=\mathbb{C}[q, q^{-1}]$, by $\theta(t_w)=(\mathscr{M}_{ab})\in M_{3\times 3}(A)$, $w\in c'$. Assume that w is of one of the above forms, then $m_{ab}=0$ except $(a, b)=(i+1, j+1)$ and

$$m_{i+1, j+1}=\begin{cases} q^{2a} & \text{if } w=\omega^i r_1 x_1^a\omega^j\\ q^{2a-1} & \text{if } w=\omega^{i+1}x_1^a\omega^j\\ q^{-2a} & \text{if } w=\omega^{i+2}r_2 x_2^a\omega^{j+1}\\ q^{-2a+1} & \text{if } w=\omega^{i+1}x_2^a\omega^{j+1}.\end{cases}$$

By [L 1, 3.8] we know that θ is a $\mathbb{C}$-algebra isomorphism. We have

$$\theta\phi_{q_0,c'}(C_{r_1})=\begin{pmatrix}[2]_{q_0} & q^{-1} & q\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$\theta\phi_{q_0, c'}(C_\omega)=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

Specialize q to $a\in\mathbb{C}^*$, we get a simple representation ψ_a of $J_{c'}=M_{3\times 3}(A)$ and any simple representation of $J_{c'}$ is isomorphic to some ψ_a, $a\in\mathbb{C}^*$. Let E_a be a simple $J_{c'}$-module providing ψ_a.