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BIFURCATION DIAGRAMS AND FOMENKO’S SURGERY ON LIOUVILLE TORI OF THE KOLOSOFF POTENTIAL $U = \rho + (1/\rho) - k \cos \varphi$

BY LJUBOMIR GAVRILOV, MOHAMMED OUAZZANI-JAMIL AND REGIS CABOZ

ABSTRACT. — By making use of the rich algebraic structure of the problem and Fomenko’s theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential $U = \rho + (1/\rho) - k \cos \varphi$.

I. Introduction

Consider the motion of a particle of unit mass on the plane $(x, y)$ in a potential field

$$U = \rho \frac{b}{\rho} + c \cos \varphi + d \sin \varphi, \quad a, b, c, d \in \mathbb{R}$$

where $x = \rho \cos \varphi, y = \rho \sin \varphi$. Without loss of generality one may suppose (after a rotation and $\mathbb{R}$-linear change of $\rho$ and $U$) that

$$U(x, y) = \pm \rho \pm \frac{1}{\rho} - k \cos \varphi, k \in \mathbb{R}$$

The corresponding Hamiltonian function is:

$$H = \frac{1}{2} (p_x^2 + p_y^2) + U(x, y)$$

and the energy level sets $\{ H = h \} \subset \mathbb{R}^4$ are compact if $U = \rho + (1/\rho) - k \cos \varphi$. The Hamiltonian system

$$\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x}, \\
\dot{p}_x &= -\frac{\partial H}{\partial x} \\
\dot{y} &= \frac{\partial H}{\partial p_y}, \\
\dot{p}_y &= -\frac{\partial H}{\partial y}
\end{align*}$$

\text{(1)}$$

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where

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + \rho + \frac{1}{\rho} - k \cos \varphi \]

is integrable and the second integral of motion reads:

\[ F = -(k^2 + y^2) p_x^2 + 2 y (x - k) p_x p_y - p_y^2 (x - k)^2 - \frac{2 k (x - k) (k x - 1)}{\sqrt{x^2 + y^2}} \]

The integrability of the system (1) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets

\[ \mathcal{A}_R = \{ (x, y, p_x, p_y) \in \mathbb{R}^4 : H = h, F = f \} \subset \mathbb{R}^4. \]

For doing that we find first the bifurcation diagram \( \mathcal{B} \) of the problem (1), \( i.e. \) the set of critical values of the energy-momentum mapping

\[ (x, y, p_x, p_y) \to (F, H). \]

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that \( \mathcal{B} \) is exactly the discriminant locus of a certain polynomial whose coefficients are functions in \( f, h, k \). The latter is closely related to the algebraic structure of the complexified system (1). This structure is studied in section 2 where we prove that the complexified generic level set \( \{ H = h, F = f \} \) is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to \( H \) and \( F \) do not linearize on this Abelian variety. Thus the system (1) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1].

For non-critical values of \( F \) and \( H \) the level set \( \mathcal{A}_R \) is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (see Theorem 2 of section 3). At last, in section 4, we describe the structure of singular level sets \( \mathcal{A}_R \). According to Fomenko's theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To "guess" exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as \( f \) and \( h \) pass through the bifurcation diagram \( \mathcal{B} \) (Theorem 3 and Theorem 4 of section 4).
II. Algebraic structure

Denote by $\mathbb{A}_c$ the complex affine algebraic variety:

$$\mathbb{A}_c = \{ (x, y, p_x, p_y, z) \in \mathbb{C}^5 : H = h, F = f, x^2 + y^2 = z^2, z \neq 0 \} \subset \mathbb{C}^5,$$

where

$$H(x, y, p_x, p_y, z) = \frac{1}{2} (p_x^2 + p_y^2) + z + \frac{1}{z} - k \frac{x}{z},$$

and

$$F(x, y, p_x, p_y, z) = -(k^2 + y^2) p_x^2 + 2 y (x - k) p_x p_y - p_y^2 (x - k)^2 - \frac{2k(x - k)(kx - 1)}{z}.$$

The variety $\mathbb{A}_c$ is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$\varphi(u) = -2 (u^3 - hu^2 + (1 - k^2)u - f)/2$$

and the corresponding hyperelliptic curve

$$K: \{ w^2 = (u^2 - k^2) \varphi(u) \}.$$

Remark. - $K$ is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

Theorem 1. - If the polynomial $(u^2 - k^2) \varphi(u)$ has no double roots then the affine algebraic variety $\mathbb{A}_c$ is a smooth complex manifold which is biholomorphically equivalent to the complex manifold $\mathbb{A}_c \setminus \mathcal{D}$, where $\mathbb{A}_c$ is a complex algebraic torus (Abelian variety) and $\mathcal{D}$ is a divisor. $\mathbb{A}_c$ is a two-sheeted unramified covering of the Jacobi variety $\text{Jac}(K)$ of the genus two algebraic curve $K$. The trajectories of the Hamiltonian flow generated by $H$ on $\mathbb{A}_c$ are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by $H + sF$, $s \neq 0$ on $\mathbb{A}_c$ are not straight lines.

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following $(\lambda, \mu)$ coordinates (see [8] for details):

$$\begin{align*}
\begin{cases}
x = \frac{\lambda \mu}{k} + k \\
y = \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)}
\end{cases}
\end{align*}$$
The canonical variables \((p_\lambda, p_\mu, \lambda, \mu)\) on \(T^* \mathbb{R}^2\) are given by

\[
\begin{align*}
    p_\lambda &= \frac{(\lambda^2 - k^2) \mu p_\lambda - (\mu^2 - k^2) \lambda p_\mu}{k (\lambda^2 - \mu^2)} \\
    p_\mu &= \frac{\sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} (\lambda p_\lambda - \mu p_\mu)}{k (\lambda^2 - \mu^2)}
\end{align*}
\]

In these new variables the integrals of motion take the form

\[
H = \frac{(\lambda^2 - k^2) p_\lambda^2 - (\mu^2 - k^2) p_\mu^2 + 2 (1 - k^2) (\lambda - \mu) + 2 (\lambda^3 - \mu^3)}{2 (\lambda^2 - \mu^2)},
\]

\[
F = \frac{-\mu^2 (\lambda^2 - k^2) p_\lambda^2 + \lambda^2 (\mu^2 - k^2) p_\mu^2 - 2 \lambda \mu (\lambda \mu + k^2 + k^2 - 1) (\lambda - \mu)}{(\lambda^2 - \mu^2)}
\]

and hence on each level set \(A_c\) holds

\[
p_\lambda = \frac{\varphi(\lambda)}{\lambda^2 - k^2}, \quad p_\mu = \frac{\varphi(\mu)}{\mu^2 - k^2}.
\]

For a further use we note also the relation

\[
F = p_\mu^2 (\mu^2 - k^2) + 2 \mu^3 - 2 \mu^2 H + 2 \mu (1 - k^2).
\]

Denote by \(d/dt_s\) the time derivative along the Hamiltonian flow of the function \(H_s = H + s F\). By making use of the equations

\[
\frac{d\lambda}{dt_s} = \frac{\partial H}{\partial p_\lambda}, \quad \frac{d\mu}{dt_s} = \frac{\partial H}{\partial p_\mu}
\]

and (6) one obtains

\[
\begin{align*}
    \frac{d\lambda}{dt_s} &= \frac{\varphi(\lambda)}{\lambda^2 - k^2} + \frac{\varphi(\mu)}{\mu^2 - k^2} = -2 s dt_s \\
    \frac{\lambda^2 d\lambda}{\varphi(\lambda) (\lambda^2 - k^2)} + \frac{\mu^2 d\mu}{\varphi(\mu) (\mu^2 - k^2)} = dt_s
\end{align*}
\]

The system (8) can be also written in the following equivalent form

\[
\begin{align*}
    \frac{d\lambda}{dt_s} &= \frac{\varphi(\lambda)}{\lambda^2 - k^2} + \frac{\varphi(\mu)}{\mu^2 - k^2} = -2 s dt_s \\
    \frac{\lambda d\lambda}{\varphi(\lambda) (\lambda^2 - k^2)} + \frac{\mu d\mu}{\varphi(\mu) (\mu^2 - k^2)} &= \frac{1 - 2 s \lambda \mu}{\lambda + \mu} dt_s
\end{align*}
\]

The flow of Kolossoff system (1) corresponds to \(s = 0\), and obviously \(t_s|_{s=0} = t\). The system (9) implies, roughly speaking, that our initial system linearizes on a Jacobian
variety after using a "new time"

\[ d\tau = \frac{dt}{\lambda + \mu}. \]  

The time \( \tau \) will play an important role and it is exactly the "Kovalevskaya time" (see [8] for details).

Define now the Abel-Jacobi map

\[ \zeta : S^2 K \to \text{Jac}(K) : (P_1, P_2) \mapsto \left( \int_{P_\infty}^{P_1} \omega_1, \int_{P_\infty}^{P_2} \omega_1, \int_{P_\infty}^{P_1} \omega_2, \int_{P_\infty}^{P_2} \omega_2 \right) \]

where

\[ \omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2 - k^2)}}, \quad \omega_2 = \frac{u \, du}{\sqrt{\varphi(u)(u^2 - k^2)}} \]

\( P_1, P_2, \in K \) \( P_\infty \) is the "infinite" point on \( K \) and \( S^2 K \) is the second symmetric product of \( K \).

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus \( x, y, p_x, p_y, z = \sqrt{x^2 + y^2} \) can be expressed in terms of genus two theta functions living on the Jacobi variety \( \text{Jac}(K) \). These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symmetric product \( S^2 K \) of the curve \( K \) (which is birational to \( \text{Jac}(K) \) according to Jacobi theorem) correspond two values of \( (x, y, p_x, p_y) \). On the other hand these functions do not have branch points on \( \text{Jac}(K) \) and hence they are root functions (Wurzelfunktionen [14]) on \( \text{Jac}(K) \).

Consider the Abelian variety \( \tilde{A}_C = \mathbb{C}^2 / \mathbb{Z} \{ e_1, e_2, e_3, 2e_4 \} \) where

\[ \text{Jac}(K) = \mathbb{C}^2 / \mathbb{Z} \{ e_1, e_2, e_3, e_4 \}. \]

If the basis \( (e_1, e_2, e_3, e_4) \) of the period lattice is chosen in a proper way then the function \( x, y, p_x, p_y, z \) become single-valued on \( \tilde{A}_C \). Let us fix such a basis. The natural projection

\( \pi : \tilde{A}_C \to \text{Jac}(K) \)

corresponds to the involution

\[ (x, y, p_x, p_y, z) \mapsto (x, -y, p_x, -p_y, z) \]

on \( \tilde{A}_C \). Consider the mapping

\[ i : \mathbb{C}^5 \to \mathbb{C}P^7 : (x, y, z, p_x, p_y) \to [f_0, f_1, \ldots, f_7] \]
where

\[
\begin{align*}
    f_0 &= 1 \\
    f_1 &= x \\
    f_2 &= y \\
    f_3 &= z \\
    f_4 &= xp_y - yp_x \\
    f_5 &= f_4^2 \\
    f_6 &= f_3(f_4 - kp_x) \\
    f_7 &= (p_x^2 - p_y^2)y - 2p_xp_yx - 2f_2f_3.
\end{align*}
\]

(13)

**Lemma 1.** — The functions \( f_i, \ i = 0, 1, \ldots, 7 \) considered as single-valued meromorphic functions on \( \bar{\mathbb{A}}_c \) provide a smooth embedding of \( \bar{\mathbb{A}}_c \) into \( \mathbb{C}P^7 \).

**Proof of theorem 1 assuming the above lemma.** — As the functions \( f_0, f_1, \ldots, f_7 \) provide an embedding of \( \bar{\mathbb{A}}_c \) into \( \mathbb{C}P^7 \) (Lemma 1) then the closure \( \overline{i(\mathbb{A}_c)} \) of \( i(\mathbb{A}_c) \) in \( \mathbb{C}P^7 \) is biholomorphically equivalent to \( \bar{\mathbb{A}}_c \). Consider the divisors \( \mathcal{D}_\infty \) and \( \mathcal{D'}_{2\infty} \) defined by

\[
(\lambda_\mu)_\infty = 2(\zeta(P_\infty) + \zeta(K)) = 2\mathcal{D}_\infty
\]

and

\[
(\lambda_k)_0 = (\lambda + \mu)_0 = \mathcal{D'}_{2\infty}
\]

Obviously \( \mathcal{D'}_{2\infty} \sim 2\mathcal{D}_\infty \). It is easily seen that \( \mathbb{A}_c \) is biholomorphically equivalent to \( \overline{i(\mathbb{A}_c)} \setminus \{ \mathcal{D}_\infty \cup \mathcal{D'}_{2\infty} \} \). Indeed \( i \) is a biholomorphic mapping between some neighbourhood \( \mathcal{V}_{\mathbb{A}_c} \) of \( \mathbb{A}_c \) in \( \mathbb{C}^3 \{ z \neq 0 \} \) and \( i(\mathbb{A}_c) \subset \mathbb{C}P^7 \). To check that it suffice to note that if \((x, y, p_x, p_y, z) \in \mathbb{A}_c \) then

\[
\begin{align*}
    \det \left( \frac{\partial (f_1, f_2, f_3, f_4, f_6)}{\partial (x, y, p_x, p_y, z)} \right) &= kyz \\
    \det \left( \frac{\partial (f_1, f_2, f_3, f_5, f_7)}{\partial (x, y, p_x, p_y, z)} \right) &= -4p_y(p_x^2y + p_x^3 - p_y^3 - p_yx^2)
\end{align*}
\]

and hence \( \text{rank}(i) = 5 \) (otherwise the equality \( y = p_y = 0 \) implies \( \text{disc}((k^2 - u^2)\varphi(u)) = 0 \)). As \( i(\mathbb{A}_c) = \bar{\mathbb{A}}_c \setminus \mathcal{D}_\infty \) is a smooth complex manifold, it is concluded that \( \mathbb{A}_c \) is also a smooth complex manifold. \( \triangle \)

**Proof of Lemma 1.** — For an arbitrary divisor \( \mathcal{D} \subset \bar{\mathbb{A}}_c \) we denote

\[
\mathcal{L}(\mathcal{D}) = \{ \text{f meromorphic on } \bar{\mathbb{A}}_c, (f) \geq -\mathcal{D} \}
\]

As \( \zeta(K) \) defines \((1, 1)\) polarization on \( \text{Jac}(K) \) then \( \mathcal{D}_\infty = \pi^{-1} : \zeta(K) \) defines \((1, 2)\) polarization on \( \mathbb{A}_c \). Thus \( 2\mathcal{D}_\infty \) defines \((2, 4)\) polarization on \( \bar{\mathbb{A}}_c \) and \( \text{dim } \mathcal{L}(2\mathcal{D}_\infty) = 2 \times 4 = 8 \), [7]. To prove lemma 1, it is enough to check that the functions \( f_0, f_1, \ldots, f_7 \) provide a basis of \( \mathcal{L}(2\mathcal{D}_\infty) \). First of all let us note that \( f_i \) blow up only along \( \mathcal{D}_\infty \). Indeed in \( \lambda, \mu \)
coordinates we have

\[ f_1 = 1 \]
\[ f_1 = \frac{\lambda \mu}{k} + k \]
\[ f_2 = \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} \]
\[ f_3 = \lambda + \mu \]
\[ f_4 = \frac{1}{(\lambda - \mu)} \left\{ \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{- \varphi(\mu)} \right\} \]
\[ f_5 = f_2^2 \]
\[ f_6 = \frac{1}{(\lambda - \mu)} \left\{ \mu \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \lambda \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{- \varphi(\mu)} \right\} \]
\[ f_7 = \frac{1}{k(\lambda - \mu)} \left\{ 2(\lambda \mu - k^2) \sqrt{\varphi(\lambda)} \sqrt{- \varphi(\mu)} - \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} (\varphi(\lambda) + \varphi(\mu)) \right\} - 2f_2f_3. \]

To prove that \( f_7 \in \mathcal{L}(2 \mathcal{D}_\omega) \) we shall find, following [1], the asymptotic expansions of \( x, y, z \) as functions of the time \( \tau \) (10) in a neighbourhood of a generic point \( \tau_0 \in \mathcal{D}_\omega \). Formulae (4) imply that \( \lambda + \mu = \sqrt{x^2 + y^2} \) and hence the changing of time in the system (1) is equivalent to multiplying each equation by \( z \). According to (9) and (4) the variables \( x, y, z \) are meromorphic in \( \tau \) and the corresponding Laurent series are:

\[
\begin{align*}
  x &= \sum_{j=0}^{\infty} x_j \tau^{j-2}, & p_x &= \sum_{j=0}^{\infty} p_{x_j} \tau^{j-1} \\
  y &= \sum_{j=0}^{\infty} y_j \tau^{j-2}, & p_y &= \sum_{j=0}^{\infty} p_{y_j} \tau^{j-1} \\
  z &= \sum_{j=0}^{\infty} z_j \tau^{j-2}
\end{align*}
\]

(here \( \tau \) stays for \( \tau - \tau_0 \)). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients \( x_j, y_j, z_j \). The general solution (14) depends effectively upon three free parameters \( \alpha, \gamma, \delta \):

\[
\begin{align*}
  x &= \frac{\alpha}{\tau^2} + \frac{(k \beta^2 - 4 \gamma \alpha)}{4 \beta} + \delta \tau + \ldots \\
  y &= \frac{\beta}{\tau^2} - \frac{(k \alpha \beta + 4 \gamma)}{4} - \frac{\alpha \delta}{\tau} + \ldots \\
  z &= \frac{-2}{\tau^2} + \frac{2 \gamma}{\beta} + \ldots
\end{align*}
\]
where $\alpha^2 + \beta^2 = 4$ (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

$$
\begin{align*}
  f_0 &= 1 \\
  f_1 &= \frac{\alpha}{\tau^2} + \ldots \\
  f_2 &= \frac{\beta}{\tau^2} + \ldots \\
  f_3 &= -\frac{2}{\tau^2} + \ldots \\
  f_4 &= \frac{k \beta}{\tau} + \ldots \\
  f_5 &= \frac{k^2 \beta^2}{\tau^2} + \ldots \\
  f_6 &= \frac{12 \delta}{\beta \tau^2} + \ldots \\
  f_7 &= -2 \left( \frac{ka\beta + 6\gamma}{\tau^2} \right) + \ldots
\end{align*}
$$

(16)

The complex constants $\alpha$ (or $\beta$ such that $\alpha^2 + \beta^2 = 4$), $\gamma$, $\delta$ parametrize the pole divisor $D_\omega$. Indeed substituting (15) in \{ $H = h$, $F = f$, $z^2 = x^2 + y^2$ \} we obtain the genus three curve

$$
\begin{align*}
  \gamma &= \frac{2h\beta - k\alpha\beta}{16} \\
  \delta^2 &= \frac{\beta}{72} (k^3 \alpha^3 + 8k^2 \gamma\beta^2 - 2k(1+k^3)\alpha\beta - 32k^2 \gamma - 2f\beta), \\
  \alpha^2 + \beta^2 &= 4
\end{align*}
$$

(17)

$D_\omega$ is a double unramified covering of the genus two curve

$$
\delta^2 = \frac{(\alpha^2 - 4)}{144} (k^3 \alpha^3 + 2hk^2 \alpha^2 + 4k(1-k^2)\alpha + 4f)
$$

(18)

and obviously this curve (18) coincides with (3) after making the substitution

$$
\alpha \rightarrow \frac{2u}{k}, \quad \delta \rightarrow \frac{w}{3k}.
$$

Equations (16) and (18) imply that $f_0$, $f_1$, $\ldots$, $f_7$ are linearly independent on $\tilde{A}_c$ which completes the proof of lemma 1. \(\triangle\)
III. Topology of Regular Level Sets

In this section we shall describe the topological type of $\mathbb{A}_\mathbb{R}$ for all generic constants $f, h, k \in \mathbb{R}$. The system (1) will be considered as a real system of differential equations.

According to Theorem 1 $\mathbb{A}_\mathbb{R}$ is a smooth real manifold if the polynomial $(k^2 - u^2) \varphi (u)$ has no double roots. Define the bifurcation set

$$B = \{(f, h, k) \in \mathbb{R}^3: \text{disc} \left( (u^2 - k^2) \varphi (u) \right) = 0 \}. \tag{19}$$

It is clear that the topological type of $\mathbb{A}_\mathbb{R}$ may change only as $(f, h, k)$ passes through $B$. Thus in each connected component of the set $\mathbb{R}^3 \setminus B$ the level set $\mathbb{A}_\mathbb{R}$ has the same topological type. Note that the bifurcation set $B \subset \mathbb{R}^3 \setminus \{(f, h, k) \}$ is invariant under the involution

$$(f, h, k) \to (f, h, -k)$$

and the topological type of the level set $\mathbb{A}_\mathbb{R}$ is one and the same at the points $(f, h, k)$ and $(f, h, -k)$. Thus it is enough to consider $k \geq 0$.

**Theorem 2.** — The set $\{ \mathbb{R}^3 \setminus B \} \cap \{ k \geq 0 \}$ consists of 12 connected components. The sections of these components with the plane $\{ k = \text{const.} \}$ are shown on figure 1. If $(f, h, k) \in \mathbb{R}^3 \setminus B$ the level set $\mathbb{A}_\mathbb{R}$ is (diffeomorphic to) a torus, to a disjoint union of two tori, or it is the empty set as it is shown in table I.

**Remark.** — The notation $2T$ in table I means a disjoint union of 2 two-dimensional tori.

**Proof of Theorem 2.** — The complex conjugation

$$\tilde{x} = (x, y, z, p_x, p_y) \to (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}_x, \tilde{p}_y) \tag{20}$$

acts as an antiholomorphic involution on $\mathbb{A}_\mathbb{C}$. The set of its fixed points is the real part $\Re (\mathbb{A}_\mathbb{C})$ of $\mathbb{A}_\mathbb{C}$ and $\mathbb{A}_\mathbb{R} = \Re (\mathbb{A}_\mathbb{C}) \cap \{ z > 0 \}$. Consider also the natural antiholomorphic involution $\tau$ of the Kovalevskaya curve (3) given in $(w, u)$ coordinates by:

$$\tau: (w, u) \to (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symmetric product $S^2 K$ and hence on $\text{Jac}(K)$ and $\tilde{\mathbb{A}}_\mathbb{C}$. Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on $\mathbb{A}_\mathbb{C}$. The upshot is that in order to describe $\mathbb{A}_\mathbb{R}$ it is enough to study the projection

$$\pi: \mathbb{A}_\mathbb{C} \to \text{Jac}(K)$$

and the pair $(K, \tau)$.

**Remark.** — The pair $(K, \tau)$ where $K$ is a Riemann surface and $\tau$ is an antiholomorphic involution on $K$ is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

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Fig. 1. The set $B_0 \{A:=\text{const.}\}$ for $k^0$.

Fig. 1.1

Fig. 1.2

Fig. 1.3

Fig. 1.4

Fig. 1.5

Fig. 1.6

Fig. 1. – The set $B \cap \{k=\text{const.}\}$ for $k \geq 0$. 

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**Table I**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Roots</th>
<th>Topological type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-k &lt; u_1 &lt; k &lt; u_3$</td>
<td>T</td>
</tr>
<tr>
<td>1'</td>
<td>$u_1 &lt; -k &lt; u_3 &lt; k$</td>
<td>Ø</td>
</tr>
<tr>
<td>2</td>
<td>$-k &lt; u_1 &lt; k &lt; u_3$</td>
<td>T</td>
</tr>
<tr>
<td>2'</td>
<td>$u_1 &lt; u_3 &lt; -k &lt; k &lt; u_2$</td>
<td>Ø</td>
</tr>
<tr>
<td>3</td>
<td>$u_1 &lt; -k &lt; k &lt; u_3$</td>
<td>2 T</td>
</tr>
<tr>
<td>3'</td>
<td>$u_1 &lt; u_3 &lt; -k &lt; k &lt; u_2$</td>
<td>Ø</td>
</tr>
<tr>
<td>4</td>
<td>$-k &lt; k &lt; u_1$</td>
<td>Ø</td>
</tr>
<tr>
<td>4'</td>
<td>$-k &lt; k &lt; u_2$</td>
<td>Ø</td>
</tr>
<tr>
<td>5</td>
<td>$-k &lt; u_4 &lt; k$</td>
<td>Ø</td>
</tr>
<tr>
<td>6</td>
<td>$-k &lt; u_1 &lt; k$</td>
<td>Ø</td>
</tr>
<tr>
<td>7</td>
<td>$-k &lt; k &lt; u_1 &lt; u_2 &lt; u_3$</td>
<td>Ø</td>
</tr>
<tr>
<td>7'</td>
<td>$u_4 &lt; u_2 &lt; u_3 &lt; -k &lt; k$</td>
<td>Ø</td>
</tr>
</tbody>
</table>

**Definition.** — A connected component of the set of fixed points of $\tau$ on $K$ is called an oval.

To determine the ovals of $K$ it suffices to study the real roots of the polynomial $(u^2 - k^2)^2 (p(u))$ for different values of $f$, $h$, and $k$. These roots are shown on table I. Using the formulae (4), (5), (6) and the condition $(x, y, z, p_x, p_y) \in \mathbb{R}^5$ we obtain that $A_\mathbb{R} \neq Ø$ only if $(f, h, k)$ belongs to domain 1, 2 or 3. There we find exactly two “admissible” ovals whose projections on the $z$-plane are given by the intervals $\Delta_1$ and $\Delta_2$ (see table II). The product of the “admissible” ovals in $S^2K$ [and hence in Jac(K)] gives a Liouville torus. Thus we proved that $\pi(A_{\mathbb{R}})$ consists of a torus $T$. There are two possibilities for $A_{\mathbb{R}} = \pi^{-1}(T)$ (recall that $A_{\mathbb{C}}$ is a double covering of Jac(K) \Ø and the projection is given by the map (11)):

- $A_{\mathbb{R}}$ is a disjoint union of two copies of $T$;
- $A_{\mathbb{R}}$ is homeomorphic to a torus two times “longer” than $T$.

To determine which case arises it suffices to note that when $\lambda$ (respectively $\mu$) makes one turn around the interval $\Delta_1$ (respectively $\Delta_2$) in a complex domain then the function $y$ does not change in the first case, whereas in the second case it changes the sign [we recall that the projection $\pi$ corresponds to the involution (20)]. Thus we find that in domain 1 and 2 $A_{\mathbb{R}}$ is a torus and in domain 3 it is a disjoint union of two tori. △

---

**Table II**

<table>
<thead>
<tr>
<th>Domain</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Projection of the “admissible” ovals on $z$-plane</td>
<td>$\Delta_1 = [u_1, u_2]$</td>
<td>$\Delta_1 = [u_1, k]$</td>
<td>$\Delta_1 = [-k, k]$</td>
</tr>
<tr>
<td></td>
<td>$\Delta_2 = [k, u_3]$</td>
<td>$\Delta_2 = [u_2, u_3]$</td>
<td>$\Delta_2 = [u_2, u_3]$</td>
</tr>
</tbody>
</table>

**ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE**
At last we shall find the topological type of the regular energy-level surface \( \{ H = h \} \)

**Lemma 2.** — *The bifurcation set* \( \Sigma \) *of the family of surfaces*

\[
Q_{h,k} = \left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} - k \cos \varphi = h \right\}
\]

is given by the union of two lines \( \Sigma = \{ h = 2 + k \} \cup \{ h = 2 - k \} \subset \mathbb{R}^2 \{ h, k \} \). The set \( \mathbb{R}^2 \setminus \Sigma \) consists of 4 components shown on figure 2. The topological type of \( Q_{h,k} \) in each of these domains is given in table 3.

**Remarks.** — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are: \( S^3 \) (the sphere), \( \mathbb{RP}^3 \) (the projective space), \( T^3 \) (the torus) and \( S^2 \times S^1 \) (the direct product), see [13] for details.

**Table III**

*Topological type of the energy level set* 
\( Q_{h,k} = \{ H = h \} \) for \( (h, k) \in \mathbb{R}^2 \setminus \Sigma \) (see fig. 2).

<table>
<thead>
<tr>
<th>Domain</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Topological type</td>
<td>( \emptyset )</td>
<td>( S^3 )</td>
<td>( S^2 \times S^1 )</td>
<td>( S^3 )</td>
</tr>
</tbody>
</table>

**Proof of Lemma 2.** — The function \( H \) has exactly two critical points \( p_x = p_y = 0, y = 0, x = \pm 1 \), for \( k \neq 0 \) and a critical variety \( \{ p_x = p_y = 0, x^2 + y^2 = 1 \} \) for \( k = 0 \) with
corresponding critical values \( h = 2 \pm k \) \((k \neq 0)\) and \( h = 2 \) \((k = 0)\). Let us compute the topological type of \( Q_{h,k} \). If \( k = 0 \) then

\[
H = \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} + 2 \geq 2
\]

and hence for \( h < 2 \) we have \( Q_{h,k} = \emptyset \). This implies that in domain 1 \( Q_{h,k} = \emptyset \). Suppose now that \( k = 0 \). On the surface \( H = 2 + \varepsilon \) where \( \varepsilon \) is small and positive, \( \rho - 1 \) is small together with \( \varepsilon \). As

\[
\left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} = 2 + \varepsilon \right\}
\]

can be written as

\[
\left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho - 1 + 1} = \varepsilon \right\} \iff \left\{ \frac{p_x^2 + p_y^2}{2} + (\rho - 1)^2 - (\rho - 1)^3 + \ldots = \varepsilon \right\} \sim S^2
\]

Then \( Q_{2,+,0} \) is topologically equivalent to \( S^2 \times S^1 \) and hence in domain 3 the topological type of \( Q_{h,k} \) is \( S^2 \times S^1 \). Consider at last \( Q_{2,-,} \) for \( \varepsilon \) small and positive

\[
Q_{2,\varepsilon} = \left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} = \varepsilon \cos \varphi \right\}
\]

The set \( Q_{2,\varepsilon} \cap \{ \varphi = \text{const.} \} \) is topologically equivalent to \( S^2 \) for \( \varphi \in (-(\pi/2), (\pi/2)) \) and to a point for \( \varphi = \pm (\pi/2) \). Hence \( Q_{2,\varepsilon} \) is topologically equivalent to \( S^3 \). This implies that in domain 2 (and 4 by a symmetry) the topological type of \( Q_{h,k} \) is \( S^3 \). \( \triangle \)

**IV. Topology of Singular Level Sets and Surgery on Liouville Tori**

In this section we shall find the topological type of the level set \( A_\mathbf{R} \) for generic values \((f, h, k) \in B\) and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set \( A_\mathbf{R} \) far from the bifurcation diagram. Suppose now that the constants \( f, h, k \) are changed in such a way, that \((f, h, k)\) passes through the bifurcation diagram \( B \). Then the topological type of \( A_\mathbf{R} \) may change
and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see fig. 3).

1) A (two-dimensional) torus $T^2$ is contracted to the axial circle $S^1$ and then vanishes. Denote this surgery as $T \rightarrow S^1 \rightarrow \emptyset$.

2) A torus $T$ splits into two tori by passing through the complex $S^1 \times \{S^1 \wedge S^1\}$ where $S^1 \wedge S^1$ is a union of two circles having exactly one common point. Denote this bifurcation as $T \rightarrow 2T$.

3) A torus $T$ becomes twice "shorter" as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \rightarrow S^1$, and the corresponding bifurcation will be denoted as $T \rightarrow T$.

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a "branching" point (see fig. 3) stands for $\{S^1 \wedge S^1\} \times S^1$. At last an asterisk denotes a set homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \rightarrow S^1$.

For fixed constants $h$ and $k$ let us consider the energy level surface $Q_{h,k} = \{H=h\}$. As $f$ varies the Liouville tori contained in the level set $\{F=f\}|_{Q_{h,k}}$ may change its topological type. Denote by $\Gamma(Q_{h,k}, F)$ the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following.
Fig. 4. — The set $\mathcal{D}$ and the graphs $\Gamma(Q_{h,k}, F)$.

**Theorem 3.** — If $(h, k)$ belongs to one and the same connected component of the set

$$\mathcal{D} = \{ h \neq 2 \pm k \} \cap \left\{ h \neq \pm \left( k + \frac{1}{2k} \right) \right\} \subseteq \mathbb{R}^2 \{ h, k \}$$

then the graph $\Gamma(Q_{h,k}, F)$ is the same and it is shown on figure 4.

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s): s \rightarrow (f(s), h(s), k(s)) \subseteq \mathbb{R}^3 \{ f, h, k \}$$

intersecting the bifurcation diagram $B$ at $s = s_0$.

**Definition.** — A bifurcation of Liouville tori contained in the level set

$$A_{s_0} = Q_{h(s_0), k(s_0)} \cap \{ F = f(s) \}$$

as $s$ passes through $s_0$ is called generic, provided that $B$ is smooth in a neighbourhood of $(f(s_0), h(s_0), k(s_0))$ and $\gamma(s)$ intersects $B$ transversally.
THEOREM 4. — All generic bifurcations of Liouville tori of the system (1) are given in table IV.

<table>
<thead>
<tr>
<th>Generic bifurcations of the level set $A_h$</th>
<th>$1 \rightarrow 2$</th>
<th>$1 \rightarrow 6$</th>
<th>$1 \rightarrow 4'$</th>
<th>$2 \rightarrow 3$</th>
<th>$2 \rightarrow 5$</th>
<th>$3 \rightarrow 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T \rightarrow T$</td>
<td>$T \rightarrow \emptyset$</td>
<td>$T \rightarrow \emptyset$</td>
<td>$T \rightarrow 2T$</td>
<td>$T \rightarrow \emptyset$</td>
<td>$T \rightarrow \emptyset$</td>
<td>$T \rightarrow \emptyset$</td>
</tr>
</tbody>
</table>

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko's theorem [3] (adapted to our case).

DEFINITION. — A smooth function $F$ on a manifold $Q$ is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function $F$ on a manifold $Q$ is called nondegenerated, provided that the Hessian matrix $d^2F$ is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko's classification theorem of bifurcations of two-dimensional Liouville tori.

THEOREM (Fomenko [3]. — Let $F$ be a Bott integral on a non-singular constant energy surface $Q^3$ of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of $F$ on $Q^3$ is a union of circles. Then each bifurcation of Liouville tori contained in the level set $\{F=f\}$, as $f$ varies, is a composition of the three bifurcations $T \rightarrow S^1 \rightarrow \emptyset$, $T \rightarrow 2T$, and $T \rightarrow T$ described above.

Remark. — The condition that each critical manifold of $F$ is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko's theorem we need to check that $F$ is a Bott function when restricted on an energy level surface $Q_{h,k}$.

LEMMA 3. — The second integral $F$ is a Bott function on the non-singular energy level surface $Q_{h,k} = \{H=h\}$ provided that $h \neq \pm (k+(1/2k))$.

Proof of Lemma 3. — Suppose that $Q_{h,k}$ is a non-singular compact manifold, i.e. $h \neq 2 \pm k$ (Lemma 2). If $F$ has a critical value $f$ on $Q_{h,k}$ then the corresponding level surface $A_h = \{H=h,F=f\}$ is degenerated and hence the polynomial $(u^2-k^2) \phi(u)$ has multiple zeros. The condition $h \neq \pm (k+(1/2k))$ means that $(u^2-k^2) \phi(u)$ has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the $h$-coordinates of the points A, B, C' are $2+k$, $k+(1/2k)$, $2-k$ for $k>0$ and $2-k$, $-k(1/2k)$, $2+k$ for $k<0$. So let us suppose that the level set $A_h$ is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by $(\lambda, \mu)$, formulae (5), (6) and (7), at least for $\lambda \neq \mu$. If in addition $\lambda$ and $\mu$ are far from a double root of $(u^2-k^2) \phi(u)$ then the equations (8) imply that the Hamiltonian flows of $H$ and $H+sF$ are linearly independent and hence $dH$ and $dF$ are linearly independent at such point.
Thus critical points of $F|_{Q_{h,k}}$ correspond only to $(\lambda, \mu)$ such that $\lambda$ (or $\mu$) is a double root of $(u^2 - k^2) \varphi(u)$. This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of $H$ on $Q_{h,k}$ has no stationary points and the critical set of $F$ on $Q_{h,k}$ is invariant under the action of this flow.

Fig. 5. – Correspondence between bifurcation of roots of the polynomial $(u^2 - k^2)(u^3 - hu^2 + (1 - k^2)u - f/2)$ and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of $F|_{Q_{h,k}}$ is non-degenerated of the normal planes to these circles. Let $\mu = \mu_0$ be a double root of $(u^2 - k^2) \varphi(u)$. According to (7) we have

$$F|_{Q_{h,k}} = (\mu^2 - k^2) p_{\mu}^2 + 2 \mu^3 - 2 \mu^2 h + 2 \mu (1 - k^2)$$

and a critical circle of the level set $\{ F|_{Q_{h,k}} \} = f$ is given by $\mu = \mu_0$, $p_{\mu} = 0$. The normal directions to this circle are given by derivations with respect to $\mu$ and $p_{\mu}$. We have

$$\frac{\partial^2 F}{\partial \mu \partial p_{\mu}} = \begin{pmatrix} 2(\mu_0^2 - k^2) & 0 \\ 0 & -\varphi''(\mu_0) \end{pmatrix}$$

and as $\mu_0 \neq \pm k$ then $\text{rank}(d^2 (F|_{Q_{h,k}})) \geq 2$. On the other hand the Hessian $d^2 (F|_{Q_{h,k}})$ is degenerated on tangent lines to the critical circle and hence $\text{rank}(d^2 (F|_{Q_{h,k}})) = 2$ which completes the proof of lemma 3. △

**Proof of Theorem 3.** – Let us fix a regular energy level set $Q_{h,k}$ with a Bott integral $F$ on it, and let us consider the corresponding line $h = \text{const.}$ on figure 1 (plane $k = \text{const.}$, $h = \text{const.}$ in the space $\mathbb{R}^3 \{ f, h, k \}$). As $f$ vary the topological type of $A_k = \{ Q_{h,k} \} \cap \{ F = f \}$ may change. Using Theorem 2 and the Fomenko’s classification theorem we identify several possible bifurcations. For example passing from domain 3
(where \( A_n \sim 2T \)) to domain 2 where \((A^T) \) on figure 1 we may have the following surgeries: \( 2T \to T \), or composition of \( T \to T \) and \( T \to \emptyset \). To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial \((u^2 - k^2) \varphi(u)\), and more specifically the four ends of the "admissible" ovals \( \Delta_1 \) and \( \Delta_2 \). The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial \((u^2 - k^2) \varphi(u)\) are easily described on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if \((h, k)\) is fixed and belongs to one and the same connected component of the set

\[
\mathcal{D} = \{ h \neq 2 \pm k \} \cap \left\{ h \neq \pm \left( k \pm \frac{1}{2} k \right) \right\},
\]

then changing \( f \) the same bifurcations of roots of the polynomial \((u^2 - k^2) \varphi(u)\) take place. This implies that if \((h, k)\) belongs to one and the same connected component of the set \( \mathcal{D} \) the corresponding Fomenko’s graph \( \Gamma(Q_h, k, F) \) is the same and it is shown on figure 4. This completes the proof of theorem 3. △

**Definition.** — The straight line \( l \subset \mathbb{R}^3 \{ f, h, k \} \) is generic provided that it intersects \( B \) transversally.

To prove Theorem 4 we note that instead of a generic smooth curve \( l \subset \mathbb{R}^3 \{ f, h, k \} \) it suffice to consider a generic straight line

\[
\{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \} \subset \mathbb{R}^3 \{ f, h, k \}.
\]

Then Theorem 4 follows from the following

**Lemma 4.** — Let \( \{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \} \) be a generic straight line in \( \mathbb{R}^3 \{ f, h, k \} \). Then \( \{ c_1 H + c_2 F + c_3 = 0 \} \subset \mathbb{R}^4 \{ x, y, p_x, p_y \} \) is a smooth surface, and \( F \) is a Bott integral on it.

Indeed, instead of \( H \) we may take for a Hamiltonian of (1) the function \( c_1 H + c_2 F \). The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (which generalizes Lemma 2 and Lemma 3).

Let \( k = k_0 \) be fixed, \((f_0, h_0, k_0) \in B\) be a generic point (i.e. in a neighbourhood of it \( B \) is a smooth manifold), and let \( q = (x^0, y^0, p^0_x, p^0_y) \) be a point on the level set \( \{ H = h_0, F = f_0 \} \). We shall prove that if

\[
(21) \quad c_1 \frac{\partial H}{\partial q} + c_2 \frac{\partial F}{\partial q} = 0
\]

then the straight line \( \{ c_1 h + c_2 f + c_3 = 0 \} \) is tangent to \( B \) (and hence it is not generic). As the equation of a straight line tangent to \( B \) at the point \((f_0, h_0, k_0)\) is given by

\[
\{ u_0^3 - hu_0^2 + (1 - k^2) u_0 - f/2 = 0 \} \subset \mathbb{R}^2 \{ f, h \}
\]

where \( u_0 \) is the double root of the polynomial \( P(u) = (u^2 - k^2) \varphi(u) \) then it is enough to prove that \( c_1/c_2 = 2u_0^2 \). In \((\lambda, \mu, \rho, \rho')\) coordinates defined by (4), (5) we have the identity
\( F = p_\mu^2 (\mu^2 - k^2) + 2 \mu^3 - 2 \mu^2 H + 2 \mu (1 - k^2). \)

Then, at least far from the locus we have

\[
\begin{align*}
\lambda = \mu & \cup \{ (\lambda^2 - k^2) (\mu^2 - k^2) = 0 \}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial F}{\partial \mu} &= 2 \mu p_\mu^2 - \varphi'(\mu) - 2 \mu^2 \frac{\partial H}{\partial \mu} \\
\frac{\partial F}{\partial p_\mu} &= 2 (\mu^2 - k^2) p_\mu - 2 \mu^2 \frac{\partial H}{\partial p_\mu}
\end{align*}
\]

As \( \text{grad}(H) \) and \( \text{grad}(F) \) are colinear according to (21), then

\[
\begin{align*}
2 (\mu^2 - k^2) p_\mu &= 0 \\
2 \mu p_\mu^2 - \varphi'(\mu) &= 0
\end{align*}
\]

and hence \( p_\mu = 0, \varphi'(\mu) = 0. \) Now (6) implies that \( \varphi(\mu) = \varphi'(\mu) = 0 \) and hence \( \mu \) is a double root of the polynomial \( (\mu^2 - k^2) \varphi(\mu). \) Suppose now that \( (\lambda^0, \mu^0, p_\mu^0, p_\mu^0) \) belongs to the locus (22) and let \( (\lambda, \mu, p_\mu, p_\mu) \) tends to \( (\lambda^0, \mu^0, p_\mu^0, p_\mu^0). \) The vectors \( \text{grad}(H) \) and \( \text{grad}(F) \) tend to some vectors \( \text{grad}(H)^0 \) and \( \text{grad}(F)^0 \) and let us suppose that these vectors are colinear. Using (6), (23) and (24) we conclude that

\[
(\mu^2 - k^2) \varphi(\mu) \to 0 \quad \text{and} \quad 2 \mu \frac{\varphi(\mu)}{\mu^2 - k^2} - \varphi'(\mu) \to 0
\]

and hence \( \mu^0 \) is a double root of the polynomial \( (\mu^2 - k^2) \varphi(\mu). \) The upshot is that if \( c_1 \text{grad}(H) + c_2 \text{grad}(F) = 0 \) then \( c_1/c_2 = 2 \mu_0^2 \), where \( \mu_0 \) is the double root of the polynomial \( (\mu^2 - k^2) \varphi(\mu), \) and hence the straight line

\[
\{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \}
\]

is tangent to \( B. \) This completes the proof of Lemma 4. \( \triangle \)

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