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BIFURCATION DIAGRAMS AND FOMENKO'S SURGERY ON LIOUVILLE TORI OF THE KOLOSOFF POTENTIAL $U = \rho + (1/\rho) - k \cos \varphi$

BY LJUBOMIR GAVRILOV, MOHAMMED OUAZZANI-JAMIL AND REGIS CABOZ

ABSTRACT. — By making use of the rich algebraic structure of the problem and Fomenko's theory of surgery on (bifurcations of) Liouville tori, we give a complete description of the topology and bifurcations of the invariant level sets of the Kolossoff system corresponding to the integrable potential $U = \rho + (1/\rho) - k \cos \varphi$.

I. Introduction

Consider the motion of a particle of unit mass on the plane (x, y) in a potential field

$$U = a\rho + \frac{b}{\rho} + c \cos \varphi + d \sin \varphi, \quad a, b, c, d \in \mathbf{R}$$

where $x = \rho \cos \varphi$, $y = \rho \sin \varphi$. Without loss of generality one may suppose (after a rotation and $\mathbf{R}$ -linear change of ρ and U) that

$$U(x, y) = \pm \rho \pm \frac{1}{\rho} - k \cos \varphi, k \in \mathbf{R}$$

The corresponding Hamiltonian function is:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + U(x, y)$$

and the energy level sets $\{H = h\} \subset \mathbf{R}^4$ are compact if $U = \rho + (1/\rho) - k \cos \varphi$. The Hamiltonian system

$$(1) \quad \begin{cases} x' = \frac{dH}{dp_x}, & p'_x = -\frac{dH}{dx} \\ y' = \frac{dH}{dp_y}, & p'_y = -\frac{dH}{dy} \end{cases} \quad (') = \frac{d}{dt}$$

where

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \rho + \frac{1}{\rho} - k \cos \varphi$$

is integrable and the second integral of motion reads:

$$F = -(k^2 + y^2)p_x^2 + 2y(x-k)p_xp_y - p_y^2(x-k)^2 - \frac{2k(x-k)(kx-1)}{\sqrt{x^2+y^2}}$$

The integrability of the system (1) was discovered by Kolossoff [8] who used it to linearize the celebrated Kovalevskaya top.

In the present paper we give a complete description of the topology of the level sets

$$\mathbf{A}_{\mathbf{R}} = \{ (x, y, p_x, p_y) \in \mathbf{R}^4 : H = h, F = f \} \subset \mathbf{R}^4.$$

For doing that we find first the bifurcation diagram $\mathbf{B}$ of the problem (1), *i. e.* the set of critical values of the energy-momentum mapping

$$(x, y, p_x, p_y) \rightarrow (F, H).$$

It turns out (like in Hénon-Heiles system [5], Gorjatchev-Tchaplygin [4] and Kovalevskaya top [9], [10]) that $\mathbf{B}$ is exactly the discriminant locus of a certain polynomial whose coefficients are functions in f, h, k . The latter is closely related to the algebraic structure of the complexified system (1). This structure is studied in section 2 where we prove that the complexified generic level set $\{H = h, F = f\}$ is an affine part of an Abelian variety (Theorem 1). Contrary to the most of the known examples [1], the Hamiltonian flows corresponding to H and F do not linearize on this Abelian variety. Thus the system (1) is not algebraically completely integrable in the sense of Adler and van Moerbeke [1]. For non-critical values of F and H the level set $\mathbf{A}_{\mathbf{R}}$ is, according to Liouville theorem, a finite union of two-dimensional tori. Their number is related to the number of ovals of an associated genus two Riemann surface and could be calculated by making use of the results of chapter 2 (*see* Theorem 2 of section 3). At last, in section 4, we describe the structure of singular level sets $\mathbf{A}_{\mathbf{R}}$. According to Fomenko's theory of surgery on (bifurcations of) Liouville tori they turn out to be homeomorphic to a finite list of two-dimensional complexes. To "guess" exactly which bifurcation takes place we use once again the reach algebraic structure of the problem. Namely, each bifurcation of Liouville tori is related to a bifurcation of ovals on a Riemann surface (the last being easily studied). Thus we find all generic bifurcations of Liouville tori as f and h pass through the bifurcation diagram $\mathbf{B}$ (Theorem 3 and Theorem 4 of section 4).

II. Algebraic structure

Denote by $\mathbf{A}_{\mathbf{C}}$ the complex affine algebraic variety:

$$\mathbf{A}_{\mathbf{C}} = \{ (x, y, p_x, p_y, z) \in \mathbf{C}^5 : H = h, F = f, x^2 + y^2 = z^2, z \neq 0 \} \subset \mathbf{C}^5,$$

where

$$H(x, y, p_x, p_y, z) = \frac{1}{2}(p_x^2 + p_y^2) + z + \frac{1}{z} - k \frac{x}{z},$$

$$F(x, y, p_x, p_y, z) = -(k^2 + y^2)p_x^2 + 2y(x - k)p_x p_y - p_y^2(x - k)^2 - \frac{2k(x - k)(kx - 1)}{z}$$

The variety $\mathbf{A}_{\mathbf{C}}$ is invariant under the (complex) flow of the (complexified) system (1). Consider also the polynomial

$$(2) \quad \varphi(u) = -2(u^3 - hu^2 + (1 - k^2)u - f/2)$$

and the corresponding hyperelliptic curve

$$(3) \quad \mathbf{K}: \{ w^2 = (u^2 - k^2)\varphi(u) \}.$$

Remark. — $\mathbf{K}$ is precisely the curve used by Kovalevskaya [11] to integrate the Kovalevskaya top.

THEOREM 1. — *If the polynomial $(u^2 - k^2)\varphi(u)$ has no double roots then the affine algebraic variety $\mathbf{A}_{\mathbf{C}}$ is a smooth complex manifold which is biholomorphically equivalent to the complex manifold $\tilde{\mathbf{A}}_{\mathbf{C}} \setminus \mathcal{D}$, where $\tilde{\mathbf{A}}_{\mathbf{C}}$ is a complex algebraic torus (Abelian variety) and $\mathcal{D}$ is a divisor. $\tilde{\mathbf{A}}_{\mathbf{C}}$ is a two-sheeted unramified covering of the Jacobi variety $\text{Jac}(\mathbf{K})$ of the genus algebraic two curve $\mathbf{K}$. The trajectories of the Hamiltonian flow generated by H on $\mathbf{A}_{\mathbf{C}}$ are straight lines on which, however, the motion is non-linear. The trajectories of the Hamiltonian flows generated by $H + sF$, $s \neq 0$ on $\mathbf{A}_{\mathbf{C}}$ are not straight lines.*

Theorem 1 will be proved later in this section. We recall that the Hamilton-Jacobi equation corresponding to (1) separates in the following (λ, μ) coordinates (see [8] for details):

$$(4) \quad \left\{ \begin{array}{l} x = \frac{\lambda\mu}{k} + k \\ y = \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} \end{array} \right.$$

The canonical variables $(p_\lambda, p_\mu, \lambda, \mu)$ on $\mathbf{T}^*\mathbf{R}^2$ are given by

$$(5) \quad \begin{cases} p_x = \frac{(\lambda^2 - k^2)\mu p_\lambda - (\mu^2 - k^2)\lambda p_\mu}{k(\lambda^2 - \mu^2)} \\ p_y = \frac{\sqrt{(\lambda^2 - k^2)(\mu^2 - k^2)}(\lambda p_\lambda - \mu p_\mu)}{k(\lambda^2 - \mu^2)} \end{cases}$$

In these new variables the integrals of motion take the form

$$\begin{aligned} H &= \frac{(\lambda^2 - k^2)p_\lambda^2 - (\mu^2 - k^2)p_\mu^2 + 2(1 - k^2)(\lambda - \mu) + 2(\lambda^3 - \mu^3)}{2(\lambda^2 - \mu^2)}, \\ F &= \frac{-\mu^2(\lambda^2 - k^2)p_\lambda^2 + \lambda^2(\mu^2 - k^2)p_\mu^2 - 2\lambda\mu(\lambda\mu + k^2 + k^2 - 1)(\lambda - \mu)}{(\lambda^2 - \mu^2)} \end{aligned}$$

and hence on each level set $\mathbf{A}_c$ holds

$$(6) \quad p_\lambda = \sqrt{\frac{\varphi(\lambda)}{\lambda^2 - k^2}}, \quad p_\mu = \sqrt{\frac{\varphi(\mu)}{\mu^2 - k^2}}.$$

For a further use we note also the relation

$$(7) \quad F = p_\mu^2(\mu^2 - k^2) + 2\mu^3 - 2\mu^2 H + 2\mu(1 - k^2).$$

Denote by d/dt_s the time derivative along the Hamiltonian flow of the function $H_s = H + sF$. By making use of the equations

$$\frac{d\lambda}{dt_s} = \frac{\partial H}{\partial p_\lambda}, \quad \frac{d\mu}{dt_s} = \frac{\partial H_s}{\partial p_\mu}$$

and (6) one obtains

$$(8) \quad \begin{cases} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = -2s dt_s \\ \frac{\lambda^2 d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{\mu^2 d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = dt_s \end{cases}$$

The system (8) can be also written in the following equivalent form

$$(9) \quad \begin{cases} \frac{d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = -2s dt_s \\ \frac{\lambda d\lambda}{\sqrt{\varphi(\lambda)(\lambda^2 - k^2)}} + \frac{\mu d\mu}{\sqrt{\varphi(\mu)(\mu^2 - k^2)}} = \frac{1 - 2s\lambda\mu}{\lambda + \mu} dt_s \end{cases}$$

The flow of Kolosoff system (1) corresponds to $s=0$, and obviously $t_s|_{s=0} = t$. The system (9) implies, roughly speaking, that our initial system linearizes on a Jacobian

variety after using a “new time”

$$(10) \quad d\tau = \frac{dt}{\lambda + \mu}.$$

The time τ will play an important role and it is exactly the “Kovalevskaya time” (see [8] for details).

Define now the Abel-Jacobi map

$$\zeta : S^2 K \rightarrow \text{Jac}(K) : (P_1, P_2) \rightarrow \left(\int_{P_\infty}^{P_1} \omega_1 + \int_{P_\infty}^{P_2} \omega_1, \int_{P_\infty}^{P_1} \omega_2 + \int_{P_\infty}^{P_2} \omega_2 \right)$$

where

$$\omega_1 = \frac{du}{\sqrt{\varphi(u)(u^2 - k^2)}}, \quad \omega_2 = \frac{u du}{\sqrt{\varphi(u)(u^2 - k^2)}}$$

$P_1, P_2 \in K \setminus P_\infty$ is the “infinite” point on K and $S^2 K$ is the second symmetric product of K .

Solving the Jacobi inversion problem (9), we obtain the explicit solutions of our initial problem (1) [2]. Thus $x, y, p_x, p_y, z = \sqrt{x^2 + y^2}$ can be expressed in terms of genus two theta functions living on the Jacobi variety $\text{Jac}(K)$. These functions however are not single-valued as it can be seen from (4). Indeed to each point on the symmetric product $S^2 K$ of the curve K (which is birational to $\text{Jac}(K)$ according to Jacobi theorem) correspond two values of (x, y, p_x, p_y) . On the other hand these functions do not have branch points on $\text{Jac}(K)$ and hence they are root functions (Wurzelfunktionen [14]) on $\text{Jac}(K)$.

Consider the Abelian variety $\tilde{A}_C = \mathbb{C}^2 / \mathbb{Z} \{e_1, e_2, e_3, 2e_4\}$ where

$$\text{Jac}(K) = \mathbb{C}^2 / \mathbb{Z} \{e_1, e_2, e_3, e_4\}.$$

If the basis (e_1, e_2, e_3, e_4) of the period lattice is chosen in a proper way then the function x, y, p_x, p_y, z become single-valued on $\tilde{A}_C$. Let us fix such a basis. The natural projection

$$(11) \quad \pi : \tilde{A}_C \rightarrow \text{Jac}(K)$$

corresponds to the involution

$$(12) \quad (x, y, p_x, p_y, z) \rightarrow (x, -y, p_x, -p_y, z)$$

on $\tilde{A}_C$. Consider the mapping

$$i : \mathbb{C}^5 \rightarrow \mathbb{CP}^7 : (x, y, z, p_x, p_y) \rightarrow [f_0, f_1, \dots, f_7]$$

where

$$(13) \quad \left\{ \begin{array}{l} f_0 = 1 \\ f_1 = x \\ f_2 = y \\ f_3 = z \\ f_4 = xp_y - yp_x \\ f_5 = f_4^2 \\ f_6 = f_3(f_4 - kp_y) \\ f_7 = (p_x^2 - p_y^2)y - 2p_x p_y x - 2f_2 f_3. \end{array} \right.$$

LEMMA 1. — *The functions f_i , $i=0, 1, \dots, 7$ considered as single-valued meromorphic functions on $\tilde{\mathbf{A}}_{\mathbf{C}}$ provide a smooth embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$.*

Proof of theorem 1 assuming the above lemma. — As the functions $f_0, f_1, \dots, f_7$ provide an embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$ (Lemma 1) then the closure $\overline{i(\mathbf{A}_{\mathbf{C}})}$ of $i(\mathbf{A}_{\mathbf{C}})$ in $\mathbf{CP}^7$ is biholomorphically equivalent to $\tilde{\mathbf{A}}_{\mathbf{C}}$. Consider the divisors $\mathcal{D}_{\infty}$ and $\mathcal{D}'_{2\infty}$ defined by

$$(\lambda\mu)_{\infty} = 2(\zeta(P_{\infty}) + \zeta(K)) = 2\mathcal{D}_{\infty}$$

and

$$(z)_0 = (\lambda + \mu)_0 = \mathcal{D}'_{2\infty}$$

Obviously $\mathcal{D}'_{2\infty} \sim 2\mathcal{D}_{\infty}$. It is easily seen that $\mathbf{A}_{\mathbf{C}}$ is biholomorphically equivalent to $\overline{i(\mathbf{A}_{\mathbf{C}})} \setminus \{\mathcal{D}_{\infty} \cup \mathcal{D}'_{2\infty}\}$. Indeed i is a biholomorphic mapping between some neighbourhood $V_{\mathbf{A}_{\mathbf{C}}}$ of $\mathbf{A}_{\mathbf{C}}$ in $\mathbf{C}^5 \setminus \{z \neq 0\}$ and $i(V_{\mathbf{A}_{\mathbf{C}}}) \subset \mathbf{CP}^7$. To check that it suffices to note that if $(x, y, p_x, p_y, z) \in \mathbf{A}_{\mathbf{C}}$ then

$$\det \left(\frac{\partial(f_1, f_2, f_3, f_4, f_6)}{\partial(x, y, p_x, p_y, z)} \right) = kyz$$

$$\det \left(\frac{\partial(f_1, f_2, f_3, f_5, f_7)}{\partial(x, y, p_x, p_y, z)} \right) = -4p_y(p_x x^2 y + p_x y^3 - p_y x^3 - p_y x y^2)$$

and hence $\text{rank}(i) = 5$ (otherwise the equality $y = p_y = 0$ implies $\text{disc}((k^2 - u^2)\varphi(u)) = 0$). As $i(\mathbf{A}_{\mathbf{C}}) = \tilde{\mathbf{A}}_{\mathbf{C}} \setminus \mathcal{D}_{\infty}$ is a smooth complex manifold, it is concluded that $\mathbf{A}_{\mathbf{C}}$ is also a smooth complex manifold. $\triangle$

Proof of Lemma 1. — For an arbitrary divisor $\mathcal{D} \subset \tilde{\mathbf{A}}_{\mathbf{C}}$ we denote

$$\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } \tilde{\mathbf{A}}_{\mathbf{C}}, (f) \geq -\mathcal{D}\}$$

As $\zeta(K)$ defines (1, 1) polarization on $\text{Jac}(K)$ then $\mathcal{D}_{\infty} = \pi^{-1} \circ \zeta(K)$ defines (1, 2) polarization on $\mathbf{A}_{\mathbf{C}}$. Thus $2\mathcal{D}_{\infty}$ defines (2, 4) polarization on $\tilde{\mathbf{A}}_{\mathbf{C}}$ and $\dim \mathcal{L}(2\mathcal{D}_{\infty}) = 2 \times 4 = 8$, [7]. To prove lemma 1, it is enough to check that the functions $f_0, f_1, \dots, f_7$ provide a basis of $\mathcal{L}(2\mathcal{D}_{\infty})$. First of all let us note that f_i blow up only along $\mathcal{D}_{\infty}$. Indeed in λ, μ

coordinates we have

$$\begin{aligned}
 f_1 &= 1 \\
 f_1 &= \frac{\lambda\mu}{k} + k \\
 f_2 &= \frac{1}{k} \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} \\
 f_3 &= \lambda + \mu \\
 f_4 &= \frac{1}{(\lambda - \mu)} \{ \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\varphi(\mu)} \} \\
 f_5 &= f_4^2 \\
 f_6 &= \frac{1}{(\lambda - \mu)} \{ \mu \sqrt{(k^2 - \mu^2)} \sqrt{\varphi(\lambda)} - \lambda \sqrt{\varphi(\lambda)} - \sqrt{(\lambda^2 - k^2)} \sqrt{-\varphi(\mu)} \} \\
 f_7 &= \frac{1}{k(\lambda - \mu)} \{ 2(\lambda\mu - k^2) \sqrt{\varphi(\lambda)} \sqrt{-\varphi(\mu)} - \sqrt{(\lambda^2 - k^2)(k^2 - \mu^2)} (\varphi(\lambda) + \varphi(\mu)) \} - 2f_2 f_3.
 \end{aligned}$$

To prove that $f_i \in \mathcal{L}(2\mathcal{D}_\infty)$ we shall find, following [1], the asymptotic expansions of x, y, z as functions of the time τ (10) in a neighbourhood of a generic point $\tau^0 \in \mathcal{D}_\infty$. Formulae (4) imply that $\lambda + \mu = \sqrt{x^2 + y^2}$ and hence the changing of time in the system (1) is equivalent to multiplying each equation by z . According to (9) and (4) the variables x, y, z are meromorphic in τ and the corresponding Laurent series are:

$$(14) \quad \left\{ \begin{aligned} x &= \sum_{j=0}^{\infty} x_j \tau^{j-2}, & p_x &= \sum_{j=0}^{\infty} p_{xj} \tau^{j-1} \\ y &= \sum_{j=0}^{\infty} y_j \tau^{j-2}, & p_y &= \sum_{j=0}^{\infty} p_{yj} \tau^{j-1} \\ z &= \sum_{j=0}^{\infty} z_j \tau^{j-2} \end{aligned} \right.$$

(here τ stays for $\tau - \tau_0$). After substituting the above series in the Kolossoff system (1) one obtains a recurrent system of linear equations for the coefficients x_j, y_j, z_j . The general solution (14) depends effectively upon three free parameters α, γ, δ :

$$(15) \quad \left\{ \begin{aligned} x &= \frac{\alpha}{\tau^2} + \frac{(k\beta^3 - 4\gamma\alpha)}{4\beta} + \delta\tau + \dots \\ y &= \frac{\beta}{\tau^2} - \frac{(k\alpha\beta + 4\gamma)}{4} - \frac{\alpha\delta}{\beta}\tau + \dots \\ z &= \frac{-2}{\tau^2} + \frac{2\gamma}{\beta} + \dots \end{aligned} \right.$$

where $\alpha^2 + \beta^2 = 4$ (for details about the general procedure of finding the series (15) we refer the reader to [1] or [6, 15]). After substituting (15) in (14), we obtain

$$(16) \quad \left\{ \begin{array}{l} f_0 = 1 \\ f_1 = \frac{\alpha}{\tau^2} + \dots \\ f_2 = \frac{\beta}{\tau^2} + \dots \\ f_3 = -\frac{2}{\tau^2} + \dots \\ f_4 = \frac{k\beta}{\tau} + \dots \\ f_5 = \frac{k^2\beta^2}{\tau^2} + \dots \\ f_6 = \frac{12\delta}{\beta\tau^2} + \dots \\ f_7 = -2\frac{(k\alpha\beta + 6\gamma)}{\tau^2} + \dots \end{array} \right.$$

The complex constants α (or β such that $\alpha^2 + \beta^2 = 4$), γ, δ parametrize the pole divisor $\mathcal{D}_\infty$. Indeed substituting (15) in $\{H=h, F=f, z^2=x^2+y^2\}$ we obtain the genus three curve

$$(17) \quad \left\{ \begin{array}{l} \gamma = \frac{2h\beta - k\alpha\beta}{16}, \\ \delta^2 = \frac{\beta}{72}(k^3\alpha\beta^3 + 8k^2\gamma\beta^2 - 2k(1+k^2)\alpha\beta - 32k^2\gamma - 2f\beta), \\ \alpha^2 + \beta^2 = 4 \end{array} \right.$$

$\mathcal{D}_\infty$ is a double unramified covering of the genus two curve

$$(18) \quad \delta^2 = \frac{(\alpha^2 - 4)}{144}(k^3\alpha^3 + 2hk^2\alpha^2 + 4k(1-k^2)\alpha + 4f)$$

and obviously this curve (18) coincides with (3) after making the substitution

$$\alpha \rightarrow \frac{2u}{k}, \quad \delta \rightarrow \frac{w}{3k}.$$

Equations (16) and (18) imply that $f_0, f_1, \dots, f_7$ are linearly independent on $\tilde{A}_C$ which completes the proof of lemma 1. $\triangle$

III. Topology of Regular Level Sets

In this section we shall describe the topological type of A_R for all generic constants $f, h, k \in \mathbb{R}$. The system (1) will be considered as a real system of differential equations.

According to Theorem 1 A_R is a smooth real manifold if the polynomial $(k^2 - u^2)\varphi(u)$ has no double roots. Define the bifurcation set

$$(19) \quad B = \{(f, h, k) \in \mathbb{R}^3 : \text{disc}((u^2 - k^2)\varphi(u)) = 0\}.$$

It is clear that the topological type of A_R may change only as (f, h, k) passes through B . Thus in each connected component of the set $\mathbb{R}^3 \setminus B$ the level set A_R has the same topological type. Note that the bifurcation set $B \subset \mathbb{R}^3 \setminus \{f, h, k\}$ is invariant under the involution

$$(f, h, k) \rightarrow (f, h, -k)$$

and the topological type of the level set A_R is one and the same at the points (f, h, k) and $(f, h, -k)$. Thus it is enough to consider $k \geq 0$.

THEOREM 2. — *The set $\{\mathbb{R}^3 \setminus B\} \cap \{k \geq 0\}$ consists of 12 connected components. The sections of these components with the plane $\{k = \text{const.}\}$ are shown on figure 1. If $(f, h, k) \in \mathbb{R}^3 \setminus B$ the level set A_R is (diffeomorphic to) a torus, to a disjoint union of two tori, or it is the empty set as it is shown in table I.*

Remark. — The notation 2T in table I means a disjoint union of 2 two-dimensional tori.

Proof of Theorem 2. — The complex conjugation

$$(20) \quad (x, y, z, p_x, p_y) \rightarrow (\bar{x}, \bar{y}, \bar{z}, \bar{p}_x, \bar{p}_y)$$

acts as an antiholomorphic involution on A_C . The set of its fixed points is the real part $\Re(A_C)$ of A_C and $A_R = \Re(A_C) \cap \{z > 0\}$. Consider also the natural antiholomorphic involution τ of the Kovalevskaya curve (3) given in (w, u) coordinates by:

$$\tau: (w, u) \rightarrow (\bar{w}, \bar{u}).$$

It induces an antiholomorphic involution on the symmetric product $S^2 K$ and hence on $\text{Jac}(K)$ and $\tilde{A}_C$. Formulae (4), (5), (6) imply that this involution coincides with the complex conjugation (20) on A_C . The upshot is that in order to describe A_R it is enough to study the projection

$$\pi: A_C \rightarrow \text{Jac}(K)$$

and the pair (K, τ) .

Remark. — The pair (K, τ) where K is a Riemann surface and τ is an antiholomorphic involution on K is called Klein surface. For the theory of Klein surfaces we refer the reader to [12].

Fig. 1.1

Fig. 1.2

Fig. 1.3

Fig. 1.4

Fig. 1.5

Fig. 1.6

Fig. 1. — The set $\mathbf{B} \cap \{k = \text{const.}\}$ for $k \geq 0$.

TABLE I

Topological type of $\mathbf{A}_{\mathbf{R}}$ and real roots of the polynomial $(u^2 - k^2)\varphi(u)$
for $(f, h, k) \in \mathbf{R}^3 \setminus \mathbf{B}$ (see fig. 1).

Domain	Roots	Topological type
1	$-k < u_1 < u_2 < k < u_3$	T
1'	$u_1 < -k < u_2 < u_3 < k$	$\emptyset$
2	$-k < u_1 < k < u_2 < u_3$	T
2'	$u_1 < u_2 < -k < u_3 < k$	$\emptyset$
3	$u_1 < -k < k < u_2 < u_3$	2 T
3'	$u_1 < u_2 < -k < k < u_3$	$\emptyset$
4	$u_1 < -k < k$	$\emptyset$
4'	$-k < k < u_1$	$\emptyset$
5	$-k < u_1 < k$	$\emptyset$
6	$-k < u_1 < u_2 < u_3 < k$	$\emptyset$
7	$-k < k < u_1 < u_2 < u_3$	$\emptyset$
7'	$u_1 < u_2 < u_3 < -k < k$	$\emptyset$

DEFINITION. — A connected component of the set of fixed points of τ on K is called an oval.

To determine the ovals of K it suffices to study the real roots of the polynomial $(u^2 - k^2)\varphi(u)$ for different values of f , h and k . These roots are shown on table I. Using the formulae (4), (5), (6) and the condition $(x, y, z, p_x, p_y) \in \mathbf{R}^5$ we obtain that $\mathbf{A}_{\mathbf{R}} \neq \emptyset$ only if (f, h, k) belongs to domain 1, 2 or 3. There we find exactly two “admissible”

TABLE II

Domain	1	2	3
Projection of the “admissible” ovals on z -plane	$\begin{cases} \Delta_1 = [u_1, u_2] \\ \Delta_2 = [k, u_3] \end{cases}$	$\begin{cases} \Delta_1 = [u_1, k] \\ \Delta_2 = [u_2, u_3] \end{cases}$	$\begin{cases} \Delta_1 = [-k, k] \\ \Delta_2 = [u_2, u_3] \end{cases}$

ovals whose projections on the z -plane are given by the intervals Δ_1 and Δ_2 (see table II). The product of the “admissible” ovals in $S^2 K$ [and hence in $\text{Jac}(K)$] gives a Liouville torus. Thus we proved that $\pi(\mathbf{A}_{\mathbf{R}})$ consists of a torus T . There are two possibilities for $\mathbf{A}_{\mathbf{R}} = \pi^{-1}(T)$ (recall that $\mathbf{A}_{\mathbf{C}}$ is a double covering of $\text{Jac}(K) \setminus \mathcal{D}$ and the projection is given by the map (11)):

- $\mathbf{A}_{\mathbf{R}}$ is a disjoint union of two copies of T ;
- $\mathbf{A}_{\mathbf{R}}$ is homeomorphic to a torus two times “longer” than T .

To determine which case arises it suffices to note that when λ (respectively μ) makes one turn around the interval Δ_1 (respectively Δ_2) in a complex domain then the function y does not change in the first case, whereas in the second case it changes the sign [we recall that the projection π corresponds to the involution (20)]. Thus we find that in domain 1 and 2 $\mathbf{A}_{\mathbf{R}}$ is a torus and in domain 3 it is a disjoint union of two tori. $\triangle$

At last we shall find the topological type of the regular energy-level surface $\{H=h\}$

LEMMA 2. — *The bifurcation set Σ of the family of surfaces*

$$Q_{h,k} = \left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} - k \cos \varphi = h \right\}$$

is given by the union of two lines $\Sigma = \{h=2+k\} \cup \{h=2-k\} \subset \mathbf{R}^2\{h, k\}$. The set $\mathbf{R}^2 \setminus \Sigma$ consists of 4 components shown on figure 2. The topological type of $Q_{h,k}$ in each of these domains is given in table 3.

Fig. 2. — The set Σ .

Remarks. — We note that the three dimensional constant-energy surfaces most often met in mathematical physics and theoretical mechanics are: S^3 (the sphere), $\mathbf{RP}^3$ (the projective space), T^3 (the torus) and $S^2 \times S^1$ (the direct product), see [13] for details.

TABLE III

Topological type of the energy level set
 $Q_{h,k} = \{H=h\}$ for $(h,k) \in \mathbf{R}^2 \setminus \Sigma$ (see fig. 2).

Domain	1	2	3	4
Topological type	$\emptyset$	S^3	$S^2 \times S^1$	S^3

Proof of Lemma 2. — The function H has exactly two critical points $p_x=p_y=0$, $y=0$, $x=\pm 1$, for $k \neq 0$ and a critical variety $\{p_x=p_y=0, x^2+y^2=1\}$ for $k=0$ with

corresponding critical values $h = 2 \pm k$ ($k \neq 0$) and $h = 2$ ($k = 0$). Let us compute the topological type of $Q_{h,k}$. If $k = 0$ then

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} + 2 \geq 2$$

and hence for $h < 2$ we have $Q_{h,k} = \emptyset$. This implies that in domain 1 $Q_{h,k} = \emptyset$. Suppose now that $k = 0$. On the surface $H = 2 + \varepsilon$ where ε is small and positive, $\rho - 1$ is small together with ε . As

$$\left\{ \frac{p_x^2 + p_y^2}{2} + \rho + \frac{1}{\rho} = 2 + \varepsilon \right\}$$

can be written as

$$\left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{(\rho - 1 + 1)} = \varepsilon \right\} \Leftrightarrow \left\{ \frac{p_x^2 + p_y^2}{2} + (\rho - 1)^2 - (\rho - 1)^3 + \dots = \varepsilon \right\} \sim S^2$$

Then $Q_{2+\varepsilon,0}$ is topologically equivalent to $S^2 \times S^1$ and hence in domain 3 the topological type of $Q_{h,k}$ is $S^2 \times S^1$. Consider at last $Q_{2,\varepsilon}$, for ε small and positive

$$Q_{2,\varepsilon} = \left\{ \frac{p_x^2 + p_y^2}{2} + \frac{(\rho - 1)^2}{\rho} = \varepsilon \cos \varphi \right\}.$$

The set $Q_{2,\varepsilon} \cap \{\varphi = \text{const.}\}$ is topologically equivalent to S^2 for $\varphi \in ((-\pi/2), (\pi/2))$ and to a point for $\varphi = \pm(\pi/2)$. Hence $Q_{2,\varepsilon}$ is topologically equivalent to S^3 . This implies that in domain 2 (and 4 by a symmetry) the topological type of $Q_{h,k}$ is S^3 . $\triangle$

IV. Topology of Singular Level Sets and Surgery on Liouville Tori

In this section we shall find the topological type of the level set $A_{\mathbf{R}}$ for generic values $(f, h, k) \in \mathbf{B}$ and thus we shall describe all generic bifurcations of Liouville tori (the non-generic ones are easily found by continuity). For doing that we shall use the Fomenko's classification theorem of bifurcations of (surgery on) Liouville tori [3].

In section 3 we found the topological type of level set $A_{\mathbf{R}}$ far from the bifurcation diagram. Suppose now that the constants f, h, k are changed in such a way, that (f, h, k) passes through the bifurcation diagram $\mathbf{B}$. Then the topological type of $A_{\mathbf{R}}$ may change

and bifurcations of (surgery on) Liouville tori takes place. Consider the following three types of bifurcations (see *fig. 3*).

Fig. 3. — Bifurcations of two-dimensional invariant Liouville tori and the corresponding graphs.

1) A (two-dimensional) torus T^2 is contracted to the axial circle S^1 and then vanishes. Denote this surgery as $T \rightarrow S^1 \rightarrow \emptyset$.

2) A torus T splits into two tori by passing through the complex $S^1 \times \{S^1 \wedge S^1\}$ where $S^1 \wedge S^1$ is a union of two circles having exactly one common point. Denote this bifurcation as $T \rightarrow 2T$.

3) A torus T becomes twice “shorter” as it spirals twice round a torus. The last complex is homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \rightarrow S^1$, and the corresponding bifurcation will be denoted as $T \rightarrow T$.

Following Fomenko [3] we present each of the above bifurcations by a graph shown on figure 3. An ordinary point denotes a non-singular Liouville torus. A black circle stands for a circle and a “branching” point (see *fig. 3*) stands for $\{S^1 \wedge S^1\} \times S^1$. At last asterisk denotes a set homeomorphic to a non-trivial section of the bundle $S^1 \wedge S^1 \rightarrow S^1$.

For fixed constants h and k let us consider the energy level surface $Q_{h,k} = \{H=h\}$. As f varies the Liouville tori contained in the level set $\{F=f\}|_{Q_{h,k}}$ may change its topological type. Denote by $\Gamma(Q_{h,k}, F)$ the graph describing the corresponding sequence of bifurcations of Liouville tori. The main result of this section is the following

Fig. 4. — The set $\mathcal{D}$ and the graphs $\Gamma(Q_{h,k}, F)$.

THEOREM 3. — *If (h, k) belongs to one and the same connected component of the set*

$$\mathcal{D} = \{h \neq 2 \pm k\} \cap \left\{h \neq \pm \left(k + \frac{1}{2k}\right)\right\} \subset \mathbf{R}^2\{h, k\}$$

then the graph $\Gamma(Q_{h,k}, F)$ is the same and it is shown on figure 4.

Theorem 3 also implies a description of all generic bifurcation of Liouville tori of our initial system (1). Namely, consider a parametrized smooth curve

$$\gamma(s) : s \rightarrow (f(s), h(s), k(s)) \in \mathbf{R}^3\{f, h, k\}$$

intersecting the bifurcation diagram $\mathbf{B}$ at $s = s_0$.

DEFINITION. — A bifurcation of Liouville tori contained in the level set

$$\mathbf{A}_{\mathbf{R}} \equiv Q_{h(s), k(s)} \cap \{F = f(s)\}$$

as s passes through s_0 is called generic, provided that $\mathbf{B}$ is smooth in a neighbourhood of $(f(s_0), h(s_0), k(s_0))$ and $\gamma(s)$ intersects $\mathbf{B}$ transversally.

THEOREM 4. — *All generic bifurcations of Liouville tori of the system (1) are given in table IV.*

TABLE IV

Generic bifurcations of the level set A_R .					
$1 \rightarrow 2$	$1 \rightarrow 6$	$1 \rightarrow 4'$	$2 \rightarrow 3$	$2 \rightarrow 5$	$3 \rightarrow 4$
$T \rightarrow T$	$T \rightarrow \emptyset$	$T \rightarrow \emptyset$	$T \rightarrow 2T$	$T \rightarrow \emptyset$	$T \rightarrow \emptyset$
					$T \rightarrow \emptyset$

Before proving Theorem 3 and Theorem 4 we shall formulate Fomenko's theorem [3] (adapted to our case).

DEFINITION. — A smooth function F on a manifold Q is a Bott function, provided that its critical points form nondegenerate critical smooth submanifolds. A critical submanifold of a smooth function F on a manifold Q is called nondegenerated, provided that the Hessian matrix $d^2 F$ is nondegenerate in normal planes to the submanifold.

Now we may state the Fomenko's classification theorem of bifurcations of two-dimensional Liouville tori.

THEOREM (Fomenko [3]). — *Let F be a Bott integral on a non-singular constant energy surface Q^3 of an integrable two-degrees of freedom Hamiltonian system. Suppose that each critical manifold of F on Q^3 is a union of circles. Then each bifurcation of Liouville tori contained in the level set $\{F=f\}$, as f varies, is a composition of the three bifurcations $T \rightarrow S^1 \rightarrow \emptyset$, $T \rightarrow 2T$, and $T \rightarrow T$ described above.*

Remark. — The condition that each critical manifold of F is a union of circles does not seem to be very restrictive. To our knowledge all studied integrable mechanical systems fall into this case (it may be a conjecture).

In order to apply Fomenko's theorem we need to check that F is a Bott function when restricted on an energy level surface $Q_{h,k}$.

LEMMA 3. — *The second integral F is a Bott function on the non-singular energy level surface $Q_{h,k} = \{H=h\}$ provided that $h \neq \pm(k + (1/2)k)$.*

Proof of Lemma 3. — Suppose that $Q_{h,k}$ is a non-singular compact manifold, i.e. $h \neq 2 \pm k$ (Lemma 2). If F has a critical value f on $Q_{h,k}$ then the corresponding level surface $A_R = \{H=h, F=f\}$ is degenerated and hence the polynomial $(u^2 - k^2)\varphi(u)$ has multiple zeros. The condition $h \neq \pm(k + (1/2)k)$ means that $(u^2 - k^2)\varphi(u)$ has no triple zeros on the boundary of the domains 1, 2 and 3 on figure 1, as the h -coordinates of the points A, B, C' are $2+k$, $k + (1/2)k$, $2-k$ for $k > 0$ and $2-k$, $-k(1/2)k$, $2+k$ for $k < 0$. So let us suppose that the level set A_R is degenerated and consider a degenerated connected component of it. Such a component is parametrized locally by (λ, μ) , formulae (5), (6) and (7), at least for $\lambda \neq \mu$. If in addition λ and μ are far from a double root of $(u^2 - k^2)\varphi(u)$ then the equations (8) imply that the Hamiltonian flows of H and $H + sF$ are linearly independent and hence dH and dF are linearly independent at such point.

Thus critical points of $F|_{Q_{h,k}}$ correspond only to (λ, μ) such that λ (or μ) is a double root of $(u^2 - k^2)\varphi(u)$. This is an one-dimensional analytical set and hence it is a disjoint union of circles. The last follows from the fact that the flow of H on $Q_{h,k}$ has no stationary points and the critical set of F on $Q_{h,k}$ is invariant under the action of this flow.

Fig. 5. — Correspondence between bifurcation of roots of the polynomial $(u^2 - k^2)(u^3 - hu^2 + (1 - k^2)u - f/2)$ and bifurcations of invariant Liouville tori.

At last let us prove that the hessian matrix of $F|_{Q_{h,k}}$ is non-degenerated of the normal planes to these circles. Let $\mu = \mu_0$ be a double root of $(u^2 - k^2)\varphi(u)$. According to (7) we have

$$F|_{Q_{h,k}} = (\mu^2 - k^2)p_\mu^2 + 2\mu^3 - 2\mu^2 h + 2\mu(1 - k^2)$$

and a critical circle of the level set $\{F|_{Q_{h,k}}\} = f$ is given by $\mu = \mu_0$, $p_\mu = 0$. The normal directions to this circle are given by derivations with respect to μ and p_μ . We have

$$\frac{\partial^2 F}{\partial \mu \partial p_\mu} = \begin{pmatrix} 2(\mu_0^2 - k^2) & 0 \\ 0 & -\varphi''(\mu_0) \end{pmatrix}$$

and as $\mu_0 \neq \pm k$ then $\text{rank}(d^2(F|_{Q_{h,k}})) \geq 2$. On the other hand the Hessian $d^2(F|_{Q_{h,k}})$ is degenerated on tangent lines to the critical circle and hence $\text{rank}(d^2(F|_{Q_{h,k}})) = 2$ which completes the proof of lemma 3. $\triangle$

Proof of Theorem 3. — Let us fix a regular energy level set $Q_{h,k}$ with a Bott integral F on it, and let us consider the corresponding line $h = \text{const.}$ on figure 1 (plane $k = \text{const.}$, $h = \text{const.}$ in the space $\mathbf{R}^3\{f, h, k\}$). As f vary the topological type of $A_{\mathbf{R}} = \{Q_{h,k}\} \cap \{F = f\}$ may change. Using Theorem 2 and the Fomenko's classification theorem we identify several possible bifurcations. For example passing from domain 3

(where $A_R \sim 2T$) to domain 2 where ($A_R \sim T$) on figure 1 we may have the following surgeries: $2T \rightarrow T$, or composition of $T \rightarrow T$ and $T \rightarrow \emptyset$. To make the difference between the two possibilities it suffices to look at the bifurcations of roots of the polynomial $(u^2 - k^2)\varphi(u)$, and more specifically the four ends of the "admissible" ovals Δ_1 and Δ_2 . The correspondence between bifurcation of roots and tori is shown on figure 5. As the bifurcations of real roots of the polynomial $(u^2 - k^2)\varphi(u)$ are easily described on table 1 then we obtain a description of the bifurcations of invariant Liouville tori of our initial system (1). By making use of figure 1 we note that if (h, k) is fixed and belongs to one and the same connected component of the set

$$\mathcal{D} = \{h \neq 2 \pm k\} \cap \left\{h \neq \pm \left(k + \frac{1}{2k}\right)\right\},$$

then changing f the same bifurcations of roots of the polynomial $(u^2 - k^2)\varphi(u)$ take place. This implies that if (h, k) belongs to one and the same connected component of the set $\mathcal{D}$ the corresponding Fomenko's graph $\Gamma(Q_{h,k}, F)$ is the same and it is shown on figure 4. This completes the proof of theorem 3. $\triangle$

DEFINITION. — The straight line $l \subset \mathbb{R}^3 \{f, h, k\}$ is generic provided that it intersects $\mathbf{B}$ transversally.

To prove Theorem 4 we note that instead of a generic smooth curve $l \subset \mathbb{R}^3 \{f, h, k\}$ it suffice to consider a generic straight line

$$\{c_1 h + c_2 f + c_3 = 0, k = \text{const.}\} \subset \mathbb{R}^3 \{f, h, k\}.$$

Then Theorem 4 follows from the following

LEMMA 4. — Let $\{c_1 h + c_2 f + c_3 = 0, k = \text{const.}\}$ be a generic straight line in $\mathbb{R}^3 \{f, h, k\}$. Then $\{c_1 H + c_2 F + c_3 = 0\} \subset \mathbb{R}^4 \{x, y, p_x, p_y\}$ is a smooth surface, and F is a Bott integral on it.

Indeed, instead of H we may take for a Hamiltonian of (1) the function $c_1 H + c_2 F$. The same arguments as in the proof of Theorem 3 imply the desirable result (table IV).

To the end of the paper we shall prove Lemma 4 (which generalizes Lemma 2 and Lemma 3).

Let $k = k_0$ be fixed, $(f_0, h_0, k_0) \in \mathbf{B}$ be a generic point (i.e. in a neighbourhood of it $\mathbf{B}$ is a smooth manifold), and let $q = (x^0, y^0, p_x^0, p_y^0)$ be a point on the level set $\{H = h_0, F = f_0\}$. We shall prove that if

$$(21) \quad c_1 \text{grad}(H)|_q + c_2 \text{grad}(F)|_q = 0$$

then the straight line $\{c_1 h + c_2 f + c_3 = 0\}$ is tangent to $\mathbf{B}$ (and hence it is not generic). As the equation of a straight line tangent to $\mathbf{B}$ at the point (f_0, h_0, k_0) is given by

$$\{u_0^3 - hu_0^2 + (1 - k^2)u_0 - f/2 = 0\} \subset \mathbb{R}^2 \{f, h\}$$

where u_0 is the double root of the polynomial $P(u) = (u^2 - k^2)\varphi(u)$ then it is enough to prove that $c_1/c_2 = 2u_0^2$. In $(\lambda, \mu, p_\lambda, p_\mu)$ coordinates defined by (4), (5) we have the identity

(7)

$$F = p_\mu^2 (\mu^2 - k^2) + 2\mu^3 - 2\mu^2 H + 2\mu (1 - k^2).$$

Then, at least far from the locus we have

$$(22) \quad \{ \lambda = \mu \} \cup \{ (\lambda^2 - k^2)(\mu^2 - k^2) = 0 \}$$

$$(23) \quad \left\{ \begin{array}{l} \frac{\partial F}{\partial \mu} = 2\mu p_\mu^2 - \varphi'(\mu) - 2\mu^2 \frac{\partial H}{\partial \mu} \\ \frac{\partial F}{\partial p_\mu} = 2(\mu^2 - k^2)p_\mu - 2\mu^2 \frac{\partial H}{\partial p_\mu} \\ \frac{\partial F}{\partial \lambda} = -2\mu^2 \frac{\partial H}{\partial \lambda} \\ \frac{\partial F}{\partial p_\lambda} = -2\mu^2 \frac{\partial H}{\partial p_\lambda} \end{array} \right.$$

As $\text{grad}(H)$ and $\text{grad}(F)$ are colinear according to (21), then

$$(24) \quad \begin{cases} 2(\mu^2 - k^2)p_\mu = 0 \\ 2\mu p_\mu^2 - \varphi'(\mu) = 0 \end{cases}$$

and hence $p_\mu = 0$, $\varphi'(\mu) = 0$. Now (6) implies that $\varphi(\mu) = \varphi'(\mu) = 0$ and hence μ is a double root of the polynomial $(\mu^2 - k^2)\varphi(\mu)$. Suppose now that $\lambda^0, \mu^0, p_\lambda^0, p_\mu^0$ belongs to the locus (22) and let $(\lambda, \mu, p_\lambda, p_\mu)$ tends to $(\lambda^0, \mu^0, p_\lambda^0, p_\mu^0)$. The vectors $\text{grad}(H)$ and $\text{grad}(F)$ tend to some vectors $\text{grad}(H)^0$ and $\text{grad}(F)^0$ and let us suppose that these vectors are colinear. Using (6), (23) and (24) we conclude that

$$(\mu^2 - k^2)\varphi(\mu) \rightarrow 0 \quad \text{and} \quad 2\mu \frac{\varphi(\mu)}{\mu^2 - k^2} - \varphi'(\mu) \rightarrow 0$$

and hence μ^0 is a double root of the polynomial $(\mu^2 - k^2)\varphi(\mu)$. The upshot is that if $c_1 \text{grad}(H) + c_2 \text{grad}(F) = 0$ then $c_1/c_2 = 2\mu_0^2$, where μ_0 is the double root of the polynomial $(\mu^2 - k^2)\varphi(\mu)$, and hence the straight line

$$\{ c_1 h + c_2 f + c_3 = 0, k = \text{const.} \}$$

is tangent to **B**. This completes the proof of Lemma 4. $\triangle$

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REFERENCES

- [1] M. ADLER and P. VAN MOERBEKE, *Algebraic Completely Integrable Systems: a systematic approach, Perspectives in Mathematics*, Academic Press (to appear in 1992).
- [2] F. M. EL-SABAA, *Solution of Equations of Problem of Motion of a Heavy Rigid Body About a Fixed Point in the Kowalevskaya's Case Using θ -Function* (*Celestial Mech.*, Vol. 29, 1983, pp. 249-253.).
- [3] A. T. FOMENKO, *Integrability and Nonintegrability in Geometry and Mechanics*, Kluwer Acad. Publishers, 1988.
- [4] L. GAVRILOV, *On the Geometry of Gorjatchev-Tchaplygin top* (*Compt. rend. Acad. bulg. Sci.*, Vol. 40, No 9, pp. 33-36, 1987).
- [5] L. GAVRILOV, *Bifurcations of Invariant Manifolds in the Generalized Hénon-Heiles System* (*Physica D*, Vol. 34, 1989, pp.223-239).
- [6] L. GAVRILOV, *Non-Integrability of the Equation of Heavy Gyrostat* (*Comp. Mathematica*, Vol. 82, 1992, pp. 275-291).
- [7] P. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
- [8] G. KOLOSOFF, *Zur Rotation eines Körpers im Kowalewski'schen Falle*, (*Mathematische Annalen*, Vol. 56, 1903, pp. 265-272).
- [9] M. P. KHARLAMOV, *Bifurcation of Common Levels of First Integrals of the Kovalevskaya Problem* (*PMM U.S.S.R.*, Vol. 47, No 6, 1983, pp. 737-743).
- [10] M. P. KHARLAMOV, *Topological Analysis of Classical Integrable Systems in the Dynamics of the Rigid Body* (*Soviet Math. Dokl.*, Vol. 28, No 3, 1983, pp. 802-805).
- [11] S. KOVALEVSKAYA, *Sur le problème de la rotation d'un corps solide autour d'un point fixe* (*Acta Math.*, Vol. 12, 1889, pp. 177-232).
- [12] S. M. NATANZON, *Klein Surfaces* (*Russian Math. Surveys*, Vol. 45, 6, 1990, pp. 53-108).
- [13] NGUEN T'EN ZUNG and A. T. FOMENKO, *Topological classification of integrable non-degenerate Hamiltonians on a constant energy three-dimensional sphere* (*Russian Math. Surveys*, Vol. 45, 6, 1990, pp. 109-135).
- [14] *Encyklopädie der Mathematischen Wissenschaften*, Band II, 2 Teil (Leipzig 1901), pp. 766-768.
- [15] H. YOSHIDA (*Celestial Mech.*, Vol. 31, 1983, p. 363).

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