ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 26, nº 3 (1993), p. 281-308
<http://www.numdam.org/item?id=ASENS_1993_4_26_3_281_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 26, 1993, p. 281 à 308.

A CLASSIFICATION OF MONGE-AMPÈRE EQUATIONS

BY V. V. LYCHAGIN, V. N. RUBTSOV AND I. V. CHEKALOV

ABSTRACT. — A classical problem of a local classification of non-linear equation arising in S. Lie works is studied for the most natural class of Monge-Ampère equations (M.A.E.) on a smooth manifold M^n.

We solve this problem for a generic classical ($n=2$) case and give full proofs of S. Lie classical classification theorems.

For multidimensional generalizations of M.A.E. we reduce the problem to a problem in invariant theory which we solve for $n=3$ and give a partial description for $n \geqq 4$. C^∞-classification is obtained for $n=3$.

Our approach is based on the deep relations between M.A.E. on M^n and contact geometry of $J^1 M^n$ — the first jets space of M^n. This approach provides a possibility to apply symplectic and contact geometry methods in classical invariant theory and for calculation Spencer cohomologies as well.

In 1874 Sophus Lie raised the following problem: find the classes of local equivalence of non-linear 2nd order differential equations with respect to the group of contact diffeomorphisms. He formulated theorems on reducing the Monge-Ampère type differential equations to quasilinear and linear form [11]. As far as we know, a complete proof of this theorem had never been published.

The class of Monge-Ampère equations is a natural setting for a classification activity since the problem of local classification of arbitrary non-linear 2nd order equations contains as a subproblem the classification with respect to fractionally-linear transformations of all submanifolds in the space of quadrics.

In 1979 Morimoto [17] announced a number of statements on classification of Monge-Ampère equations of a special form based on the theory of G-structures.

Our approach to the classification problem is based on a relation between the differential forms on the manifold $J^1 M$ of 1-jets of smooth functions and Monge-Ampère equations [10]. We rely much upon [10].

Note also that the classification problem is in close connection with certain problems of the classical invariant theory which are, in our opinion, of an independent interest. One of them is the problem of description of orbits of the natural action of the symplectic group in the space of exterior forms. Notice that the analogy with differential equations enables us to understand better certain problems of the invariant theory for this group. In particular, this concerns the description of Sp-orbits of

3-forms on the 6-dimensional real symplectic space and also certain orbits of 4-forms on the 8-dimensional space.

The text consists of 6 sections (S).

In Section 0 we introduce basic concepts and constructions used below. The main source of definitions and ideas is paper [14].

In S. 1 we consider the classical two-dimensional problem of S. Lie on local classification of Monge-Ampère equations. We start with the algebraic model of our problem and study a symplectic equivalence of 2-forms in $2n$-dimensional space. The normal forms are listed.

The non-degenerate two-dimensional Monge-Ampère equation determines an additional geometric structure on 4-dimensional symplectic manifold. The elliptic equations define an almost-complex structure and hyperbolic – an almost-product structure. The Newlander-Nirenberg theorem (in the elliptic case) and Frobenius theorem in a hyperbolic one give necessary and sufficient conditions of equivalence of our Monge-Ampère to a constant coefficients equation.

At the end of the section we prove two classic S. Lie theorems on the reduction of non-linear Monge-Ampère to a quazilinear one and on the normal forms of Monge-Ampère admitting an intermediate integral.

Sections 2-5 are devoted to the algebraic problems arising in a high-dimensional ($n \geqq 3$) classification.

In S. 2 we solve the problem of symplectic classification of effective 3-forms on 6-dimensional real space. We list normal forms and indicate that even in the 3-dimensional case the Monge-Ampère equations corresponding to generic orbits are not linearizable even at a point.

Theorem 2.6 generalizes the corresponding results of J.-I. Igusa [8] and V. Popov [19] in the case of an algebraically non-closed field. Moreover we directly built an invariant of the classification problem.

In the next section (S. 3) we give a short outline of the description procedure for normal forms in the dimension greater than 3 (Theorem 3.4). In theorem 3.5 the normal forms of effective 4-forms on 8-dimensional symplectic space are listed (under some natural conditions).

We establish a relation between the set of all transvections admitted by a given form and the symplectic classification of effective forms. After that the stabilizers of effective n-forms are described. We also calculate the stationary subalgebras of the most important types of effective forms.

In S. 4 we make an algebraic digression and study the finiteness conditions on type of effective forms stabilizers. First we classify reductive subalgebras $\mathscr{J}$ in EndV with non-trivial first Cartan prolongation. If the representation $\mathscr{J} \rightarrow \mathrm{EndV}$ is irreducible the results are known (Theorem 4.2.1). Theorem 4.3.1 solves this problem for reducible representations. Then we study stabilizers. Theorem 4.4.1 states the general result on finiteness of the stationary subalgebra of a regular element. We also give several reformulations and corollaries of this theorem for stabilizers of effective forms under a symplectic action. At the end of the section we explicitly calculate Cartan prolongations of the stabilizers of several important types of effective forms.

In S.5 we study involutiveness of the stabilizers of effective forms. The importance of these questions to the classification problem is explained in 6 where we identify the symbol of a Monge-Ampère equation corresponding to the homology equation of the classification problem with the stabilizor of the corresponding form. The involutiveness of the symbol is one of the conditions of the criterion for formal integrability.

S.6 is the central one from the classification problem viewpoint. Theorem 6.4.1 gives conditions for reducibility of an equation with analytic coefficients by an analytic symplectic diffeomorphism to an equation with constant coefficients in $\mathbb{R}^n$.

The finiteness of stabilizer condition enables us to strengthen this theorem and generalize it to C^∞-setting (Theorem 6.6.1).

At the end of the section Theorem 6.6.1 is applied to the classification of effective forms and the corresponding Monge-Ampère equations on the 3-dimensional manifolds. The main results were published in [5, 15, 16].

0. Formulation of the problem

0.1. Let (V, Ω) be a symplectic space over $\mathbb{R}$ with the structure form $\Omega \in \Lambda^2(V^*)$ and $\dim V = 2n$. Denote by $\Gamma : V \to V^*$ the isomorphism determined by the structure form Ω *i.e.* $\Gamma(X) = i_X(\Omega)$ and by $\Gamma_s : \Lambda^s(V) \to \Lambda^s(V^*)$ its exterior powers, $\Gamma_s = \Lambda^s(\Gamma)$. For every $\omega \in \Lambda^s(V^*)$ denote by $v_\omega \in \Lambda^s(V)$ the s-vector corresponding to ω, *i. e.* $\Gamma_s(v_\omega) = \omega$.

In the algebra of exterior forms $\Lambda^*(V^*) = \bigoplus_{s \geqq 0} \Lambda^s(V^*)$ introduce two operators $T : \Lambda^s(V^*) \to \Lambda^{s+2}(V^*)$, the operator of exterior multiplication by the structure form Ω $T(\omega) = \omega \wedge \Omega$, and $\perp : \Lambda^s(V^*) \to \Lambda^{s-2}(V^*)$, the operator of inner multiplication by the canonical bivector v_Ω, $\perp(\omega) = i_{v_\Omega}(\omega)$.

An exterior form $\omega \in \Lambda^k(V^*)$, $k \leqq n$ will be called effective if $\perp \omega = 0$ or, equivalently, $\omega \in \Lambda^k(V^*)$ is effective if and only if $T_{s+1}\,\omega = 0$ for $s = n-k$, $T_s = (1/s!)\,T^s$ [14].

0.2. Let M be a smooth manifold, $J^1(M)$ the manifold of 1-jets of smooth functions on M, $U \in \Lambda^1(J^1 M)$ the universal 1-form on $J^1 M$ which determines the contact structure [12]. At each point $x \in J^1 M$ the restriction of dU_x onto $C(x) = \operatorname{Ker} U_x$ determines a symplectic structure and therefore the operators $T : \Lambda^s(C^*(x)) \to \Lambda^{s+2}(C^*(x))$ and $\perp : \Lambda^s(C^*(x)) \to \Lambda^{s-2}(C^*(x))$. The tangent space $T_x(J^1 M)$ splits into the direct sum $T_x(J^1 M) = C(x) \oplus \mathbb{R} X_1$ where X_1 is the contact vector field with the generating function 1 [12]. Therefore, if $\Lambda^s(C^*)$ the space of differential s-forms on $J^1 M$ degenerated along X_1, $(\Lambda^s(C^*))_x$ is naturally identified with $\Lambda^s(C^*(x))$ and, besides, we have

$$\Lambda^s(J^1 M) = \Lambda^s(C^*) \oplus \Lambda^{s-1}(C^*).$$

We will say that $\omega \in \Lambda^s(C^*)$ is an effective form on $J^1 M$ if $\perp \omega = 0$. Denote by Λ^s_ε the set of all the effective s-forms.

0.3. For every differential n-form $\omega \in \Lambda^n(J^1 M)$ determine a non-linear differential operator Δ_ω acting via the formula $\Delta_\omega(h) = j_1(h)^*(\omega)$, $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$ where $j_1(h)$ is the section determined by a function $h \in C^\infty(M)$. Two differential forms ω_1, $\omega_2 \in \Lambda^n(J^1 M)$ determine the same operator if and only if $\omega_1 - \omega_2 \in I_C^n$ where I_C^n is the n-th homogeneous component of the ideal $I_C \subset \Lambda^*(J^1 M)$ formed by the elements of the form $\omega_1 \wedge U + \omega_2 \wedge dU$. Since $\Lambda_\varepsilon^s(J^1 M) = \Lambda^s(J^1 M)/I_C^s$ then $\Delta\omega$ is determined by the effective part of the projection of ω onto $\Lambda^s(C^*)$. In what follows we will assume that $\Delta\omega$ is given by an effective form ω. We will call the operators $\Delta_\omega : C^\infty(M) \to \Lambda^n(M)$ the Monge-Ampère operators.

0.4. Determine the action of the group $Ct(J^1 M)$ of contact transformations of $J^1 M$ onto the Monge-Ampère operators setting $\alpha(\Delta_\omega)\Delta_{\alpha^*(\omega)}$ for $\alpha \in Ct(J^1 M)$. Similarly define the action of the Lie algebra $ct(J^1 M)$ of contact vector fields on $J^1 M$ setting $X_f(\Delta_\omega) = L_{X_f}(\omega)$; here L_X is the Lie derivative along X and X_f is a contact vector field with a generating function $f \in C^\infty(J^1 M)$ [12].

0.5. We will be interested in the problem of local classification of Monge-Ampère operators (equations) with respect to the group $Ct(m)$ of the germs of contact diffeomorphisms preserving a point m.

Hereafter we will assume that in a neighbourhood of m there exists an infinitesimal contact symmetry X_f, where $f(m) \neq 0$, of Δ_ω. Then there exists a local contact diffeomorphism sending X_f to X_1 so that we may assume that $L_{X_1}(\omega) = 0$. This means that ω can be considered as a form on $T^* M$ and the classification problem for operators (equations) given by such forms as a classification problem of differential k-forms on $T^* M$ with respect to the group of symplectic diffeomorphisms.

1. Classification of Monge-Ampère equations on 2-dimensional manifolds. S. Lie's theorems

1.1. Let (V, Ω) be a symplectic space over a field $k = \mathbb{R}$ or $\mathbb{C}$, $\dim V = 2n$.

For any $\omega \in \Lambda^2(V^*)$ define its Pfaffian $Pf(\omega) \in k$ from the formula $\omega^n = Pf(\omega)\Omega^n$.

The coefficients $P_j(\omega)$ of the characteristic polynomial $P_\omega(\lambda) = Pf(\omega - \lambda\Omega)$ are invariants of the natural $Sp(V)$-action on $\Lambda^2(V^*)$ and $P_\omega(\lambda) = \sum_{0 \leqq j \leqq n} P_j \lambda^j$, *i.e.* these invariants completely determine generic orbits. Making use of the symplectic structure we may associate with any 2-form ω a linear operator $A_\omega : V \to V$, where $\omega(X, Y) = \Omega(A_\omega X, Y)$ for any $X, Y \in V$. Since ω is antisymmetric, then A_ω is symmetric with respect to the structure form, *i.e.*

$$\Omega(A_\omega X, Y) = \Omega(X, A_\omega Y) \qquad \text{for} \quad X, Y \in V.$$

Remark. – X and $A_\omega X$ are skew orthogonal, *i.e.* $\Omega(X, A_\omega X) = \omega(X, X) = 0$ for any $X \in V$. There is a relation among A_ω, its characteristic polynomial $P_{A_\omega}(\lambda) = \det \| A_\omega - \lambda E \|$ and P_ω; namely, $[Pf(\omega)]^2 = \det A_\omega$, in particular $[P_\omega(\lambda)]^2 = P_{A_\omega}(\lambda)$.

1.2. Let us establish a normal form of a 2-form $\omega \in \Lambda^2(V^*)$ with respect to the Sp (V)-action similarly to the case of symmetric forms for $k=\mathbb{C}$. Let λ_1, λ_2 be two roots of $P_\omega(\lambda)=0$ and $K(\lambda_1)$, $K(\lambda_2)$ the subspaces of V formed by the vectors annihilated by a power of $A_\omega - \lambda_1$ or $A_\omega - \lambda_2$, respectively.

LEMMA 1.2.1. — *Let* $\lambda_1 \neq \lambda_2$. *Then* $K(\lambda_1)$ *and* $K(\lambda_2)$ *are skew orthogonal.*

Proof. — Let $C(\lambda)$ be the subspace of eigenvectors of A_ω corresponding to the eigenvalue λ, $C(\lambda) \subset K(\lambda)$. First let us show that $C(\lambda_1)$ and $C(\lambda_2)$ are skeworthogonal; if $X \in C(\lambda_1)$, $Y \in C(\lambda_2)$ then (1.1) implies that $(\lambda_1 - \lambda_2)\Omega(X, Y)=0$ hence $\Omega(X, Y)=0$.

Hence, the lemma is valid for a semi-simple operator. As for an arbitrary symmetric operator A_ω its semi-simple part is symmetric too.

LEMMA 1.2.2. — *Let* $\omega \in \Lambda^2(V^*)$ *be such that* A_ω *is nilpotent. Then there exists a basis* $f^i_1, \dots f^i_{n_i}, g^i_1, \dots, g^i_{n_i}$, $i=1, 2, \dots$ *in* V, *where* $f^i_1, g^i_1 \dots$ *is a basis in* $\operatorname{Ker} A_\omega$ *such that*

$$A_\omega f^i_k = f^i_{k-1}, \qquad A_\omega g^i_k = g^i_{k-1}, \qquad \Omega(f^i_s, g^j_l) = \delta_{ij} \cdot \delta_{s+l,\, n_i+1}.$$

Proof. — Let f_{n_1} be a vector of maximal height with respect to A_ω *i.e.* $A_\omega^{n_1} f_{n_1}=0$, $A_\omega^{n_1-1} f_{n_1} \neq 0$. Set $f_m = A_\omega^{n_1-m} g_{n_1}$ and select a vector g_{n_1} so that $\Omega(f_1, g_{n_1})=1$. Then g_{n_1} is also a vector of height n_1 since $\Omega(f_{n_1}, A_\omega^{n_1-1} g_{n_1}) = \Omega(f_1, g_{n_1}) \neq 0$. Let $g_k = A_\omega^{n_1-m} g_{n_1}$. Let us show that we can modify $g_2, \dots, g_{n_1}$ somewhat so as to preserve $A_\omega g_m = g_{m-1}$ but with $\Omega(f_{n_1}, g_s)=0$ for $3 \geqq 2$. For this replace g_2 by $\tilde{g}_2 = g_2 - a_2 g_1$, where a_2 is selected so as $\Omega(f_{n_1}, \tilde{g}_2) = \Omega(f_{n_1}, g_2) - a_2 \Omega(f_{n_1}, g_1)$, *i.e.* $a_2 = \Omega(f_{n_1}, g_2)$.

Suppose that $\tilde{g}_2, \dots, \tilde{g}_{k_1}$ with this property are already constructed and set $\tilde{g}_m = g_m - a_2 g_{m-1} - \dots - a_m g_1$ provided $\tilde{g}_{m-1} = g_{m-1} - a_2 g_{m-2} - \dots - a_{m-1} g_1$. The coefficient a_m is uniquely determined by the equation $\Omega(f_{n_1}, \tilde{g}_m)=0$ provided $a_2, \dots, a_{m-1}$ are determined; in other words

$$a_m = \Omega(f_{n_1}, g_m) - a_2 \Omega(f_{n_1}, g_{m-1}) - \dots - a_{m-1} \Omega(f_{n_1}, g_2)$$

Assuming that $g_{n_1}, \dots, g_1$ is a set of vectors such that $\Omega(f_{n_1}, g_s) = \delta_{s,\,1}$ for arbitrary s, l, we get

$$\Omega(f_l, g_s) = \delta_{l+s,\, n_1+1}$$

Denote by F or G the subspace spanned by $f_1, \dots, f_{n_1}$ or $g_1, \dots, g_{n_1}$, respectively. Thanks to Remark 1.1 F and G are isotropic subspaces. In fact, the vectors X and $A_\omega X$ are skeworthogonal. Let us show that moreover $\Omega(X, A_\omega^k X)=0$ for all $\Omega m \geqq 0$. If $m=2l$ then $\Omega(X, A_\omega^{2l} X) = \Omega(A_\omega^l X, A_\omega^l X)=0$ and if $m=2l+1$ then $\Omega(X, A_\omega^{2l+1} X) = \Omega(Y, A_\omega Y)=0$, where $Y = A_\omega^l X$. Therefore V represents in the form $V = V' \oplus (F \oplus G)$ where V' is invariant with respect to A_ω and $\Omega \mid V'$ is nondegenerate. The induction in 1/2 dim V finishes the proof.

These lemmas imply the following

THEOREM 1.2.3. – *The exterior 2-forms over* $\mathbb{R}$ *or* $\mathbb{C}$ *are* $\mathrm{Sp}(V)$*-equivalent if and only if* A_{ω_1} *and* A_{ω_2} *are equivalent with respect to* $\mathrm{GL}(V)$.

1.3. This theorem directly implies the list of normal forms of $\mathrm{Sp}(V)$-action on $\Lambda^2(V^*)$.

THEOREM 1.3.1. – *For any* $\omega \in \Lambda^2(V^*)$ *over* $\mathbb{R}$ *there exists a decomposition of* V *into the skew orthogonal with respect to* Ω *direct sum of subspaces* $V = \bigoplus_{j, \lambda} V_j(\lambda, \bar{\lambda})$ *where* λ *runs the roots of* $P_\omega(\lambda)$ *and* $1 \leqq j \leqq 1/2 \dim K(\lambda)$ *so that* $\omega = \sum \omega_{\lambda, j}$ *and* $\omega_{\lambda, j} \in \Lambda^2(V_j)$ *in the symplectic basis* $(e_1, \ldots, e_r, f_1 \ldots, f_r)$ of $V_\lambda(\lambda, \bar{\lambda})$ *is of the following form*:

(1) λ *is real*, $\dim V_\lambda(\lambda, \bar{\lambda})$ *is even, then*

$$\omega = \lambda \sum_{1 \leqq i \leqq r} e_i^* \wedge f_i^* + \sum_{1 \leqq i \leqq r-1} e_i^* \wedge f_{i+1}^*$$

(2) $\lambda = \sigma + i\tau$, $\dim V_j$ *is a multiple of* 4 *then*

$$\omega_{\lambda, j} = \sigma \sum_{1 \leqq i \leqq r} e_i^* \wedge f_i^* + \tau\Big(\sum_{2i \leqq r} e_{2i}^* \wedge f_{2i-1}^* - \sum_{2i+2 \leqq r} e_{2i+1}^* \wedge f_{2i+2}^*\Big) + \sum_{2 \leqq i \leqq r} e_i^* \wedge f_{i-2}^*$$

COROLLARY 1. – *If all the roots of the characteristic equation* $P_\omega(\lambda) = 0$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are real and different then* $\omega = \sum_{1 \leqq i \leqq r} \lambda_i e_i^* \wedge f_i^*$ *in a symplectic basis.*

COROLLARY 2. – *Any effective form* $\omega \in \Lambda_\varepsilon^2(V_4^*)$ *on a 4-dimensional symplectic space* V_4 *can be transformed by a symplectic transformation to one of the following forms*

(1) $\omega = \lambda(f_1^* \wedge e_2^* - f_2^* \wedge e_1^*)$

(2) $\omega = \lambda(f_1^* \wedge e_1^* - f_2^* \wedge e_2^*)$

(3) $\omega = f_1^* \wedge e_2^*$

COROLLARY 3. – *Any effective form* $\omega \in \Lambda_\varepsilon^2(V_6^*)$ *on a 6-dimensional symplectic space can be transformed by a symplectic transformation to one of the following forms*

(1) $\omega = \lambda_1 e_1^* \wedge f_1^* + \lambda_2 e_2^* \wedge f_2^* + \lambda_3 e_3^* \wedge f_3^*$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$

(2) $\omega = 2\lambda e_1^* \wedge f_1^* - \lambda(e_2^* \wedge f_2^* + e_3^* \wedge f_3^*) + \nu(e_3^* \wedge f_2^* - e_2^* \wedge f_3^*)$

(3) $\omega = 2\lambda e_1^* \wedge f_1^* - \lambda(e_2^* \wedge f_2^* + e_3^* \wedge f_3^*) + e_2^* \wedge f_3^*$

(4) $\omega = e_2^* \wedge f_1^* + e_3^* \wedge f_2^*$

1.4. Let us return to the 2-dimensional case. For the effective forms $\omega \in \Lambda_\varepsilon^2(V_4^*)$ we have $P_\omega(\lambda) = \mathrm{Pf}(\omega) + \lambda^2$ and its $\mathrm{Sp}(4)$-orbit is completely determined by the values of the Pfaffian.

An effective form $\omega \in \Lambda_\varepsilon^2)V_4^*)$ is called

elliptic if $\mathrm{Pf}(\omega) > 0$,

hyperbolic if $\mathrm{Pf}(\omega) < 0$,

parabolic if $\mathrm{Pf}(\omega) = 0$.

The Monge-Ampère equation $\mathscr{E}_\omega$ on a two-dimensional manifold M determined by an effective form $\omega \in \Lambda^2(J^1 M)$ will be called elliptic, hyperbolic or parabolic at $m \in J^1 M$, if so is the exterior form ω_m on $\operatorname{Ker} U_{1,m}$.

THEOREM 1.5. – *The Monge-Ampère equation determined by a form* $\omega \in \Lambda^2(T^* M)$, $\dim M = 2$, *in a neighbourhood of* $m \in T^* M$ *where* $\mathrm{Pf}(\omega_m) \neq 0$, *is symplectic equivalent to a Monge-Ampère equation with constant coefficients with respect to a symplectic coordinate system if and only if*

$$d\omega = 1/2\, d\ln/\mathrm{Pf}(\omega)| \wedge \omega \tag{1.5}$$

Proof. – Necessity. Let $\omega_0 \in \Lambda^2(T^* M)$ be an effective 2-form with constant coefficients with respect to a canonical coordinate system and $|\mathrm{Pf}(\omega_0)| = 1$. Let also $F : T^* M \to T^* M$ be a symplectic diffeomorphism such that $F^*(\omega_0) = \lambda \omega_0$ for a function $\lambda \in C^\infty(T^* M)$, $\lambda \neq 0$. Then $\mathrm{Pf}(F^*(\omega)) = F^*(\mathrm{Pf}(\omega)) = \lambda^2$ and therefore $\omega_0 = F^*(|\mathrm{Pf}(\omega)|^{-1/2} \omega)$ hence $d(|\mathrm{Pf}(\omega)|^{-1/2} \omega) = 0$ implying (1.5).

Sufficiency. First consider an elliptic case, $\mathrm{Pf}(\omega) > 0$ normalizing the form $\omega \to (\mathrm{Pf}(\omega))^{-1/2} \omega$ assume that $\omega \in \Lambda^2_\varepsilon(T^* M)$ is an effective form such that $d\omega = 0$. Consider the family of operators $A_x : T_x(T^* M) \to T_x(T^* M)$ dual to ω with respect to the symplectic structure: $A_x = A_{\omega_x}$. The operators A_x satisfy the characteristic equation $P_\omega(A_x) = 0$, therefore $A_x^2 + 1 = 0$ and the field $x \to A_x$ determines an almost complex structure on $T^* M$. Since ω is closed, this structure is integrable.

Prove this statement. Remind that an almost complex structure determined by the field of operators $A = (A_x)$ is integrable if and only if the vector field

$$Z = [X, Y] + A[X, AY] + A[AX, Y] - [AX, AY]$$

vanishes for arbitrary vector fields X, Y.

The relations $i_{[X,Y]} = [L_X, i_Y]$ and $L_X = i_X d + d i_X$ imply that on the closed forms the following relation holds:

$$i_{[X,Y]} = i_X d i_Y - i_Y d i_X + d i_X i_Y$$

To prove that $Z = 0$ it suffices to show that $i_Z \Omega = 0$ we have

$$i_{[X,Y]} \Omega = i_X d i_Y \Omega - i_Y d i_X \Omega - d\Omega(X, Y),$$

$$\begin{aligned} -i_{A[X,AY]} \Omega = i_{[X,AY]} \omega = i_X d i_{AY} \omega - i_{AY} d i_X \omega \\ - d\omega(X, AY) = -i_X d i_Y \Omega - i_{AY} d i_X \omega + d\Omega(X, Y). \end{aligned}$$

$$i_{A[AX,Y]} \Omega = i_{AX} d i_Y \omega + i_Y d i_X \Omega + d\Omega(X, Y).$$

$$i_{[AX,AY]} \Omega = i_{AX} d i_{AY} \Omega - i_{AY} d i_{AX} \Omega - d\Omega(AX, AY) = i_{AX} d i_Y \omega - i_{AY} d i_X \omega + d\Omega(X, Y).$$

implying the desired.

Therefore by Newlander-Nirenberg's theorem A determines a complex structure on $T^* M$.

On $T^* M$ consider the complex-valued closed form $\theta = \Omega - i\omega$. Then making use of (1.2) we get $\theta(A_x, v_1, v_2) = \theta(v_1, A_x v_2) = i\theta(v_1, v_2)$ for all $v_1, v_2 \in T_x(T^* M)$.

Therefore by Darboux theorem $\theta = dz_1 \wedge dz_2$ in a local complex coordinate system. Set $z_1 = q_1 - iq_2$, $z_2 = p_1 + ip_2$ for some functions q_1, q_2, p_1, p_2 on $T^* M$ which form a local coordinate system. Separating the real and imaginary parts in the relation $\theta = dz_1 \wedge dz_2$we get

$$\Omega = dq_1 \wedge dp_1 + dp_2 \wedge dp_2, \qquad \omega = dp_1 \wedge dq_2 - dp_2 \wedge dq_1.$$

Therefore (q, p) is a canonical coordinate system in which $\mathscr{E}_\omega$ is the Laplace equation.

Now consider the hyperbolic case assuming as above that $\mathrm{Pf}(\omega) = -1$, $d\omega = 0$. Then $\omega^2 + \Omega^2 = 0$ and since ω is effective, then $(\omega + \Omega)^2 = 0$ and $(\omega - \Omega)^2 = 0$.

Therefore $\omega + \Omega$ and $\omega - \Omega$ are primitive 2-forms. Let E_+ and E_- be distributions determined by these forms. Let us show that $E_\pm$ are completely integrable. In fact, the annihilator of, say E_+, is formed by the 1-forms $\gamma \in \Lambda^1(T^* M)$ such that $\gamma \wedge (\omega + \Omega) = 0$ but then $d\gamma \wedge (\omega + \Omega) = 0$. Select the integrals f_1, f_2 and g_1, g_2 of E_+ and E_-, respectively, so that $\omega + \Omega = df_1 \wedge df_2$, $\omega - \Omega = dg_1 \wedge dg_2$ in a neighbourhood of $m \in T^* M$. Then $\Omega = 1/2(df_1 \wedge df_2 + dg_1 \wedge dg_2)$ $\omega = 1/2(df_1 \wedge df_2 - dg_1 \wedge dg_2)$ and therefore in the canonical coordinate system $q_1 = 1/2 f_1$, $q_2 = 1/2 g_1$, $p_1 = f_2$, $p_2 = g_2$, $\mathscr{E}_\omega$ is the wave equation.

THEOREM (S. Lie) 1.6. – *Let* M *be an analytic manifold,* $\dim M = 2$ *and* $\mathscr{E}_\omega$ *a Monge-Ampère equation with analytic coefficients where* $\mathrm{Pf}(\omega) \neq 0$, $\omega \in \Lambda^2(J^1 M)$. *Then* $\mathscr{E}_\omega$ *is locally equivalent to a quasilinear equation in a neighbourhood of* $m \in J^1 M$.

The quasilinearity of $\mathscr{E}_\omega$ is equivalent to the fact that the fibers of $\mathscr{F}_{1,0} : J^1(M) \to J^0(M)$ are integral manifolds. By Cauchy-Kovalevsky theorem there exists a 3-parameter analytic family of solutions h_s which defines a foliation in a neighbourhood of $m \in J^1 M$. By Weinstein's theorem [13] there exists a canonical transformation preserving m and sending this foliation into $\mathscr{F}_{1,0}^{-1}(x)$, $x \in J^1(M)$. Then $F^*(\omega)$ defines a quasilinear equation.

1.7. We say that $f, g \in C^\infty(T^* M)$ define an intermediate integral for Δ_ω if $\omega = \lambda_1 df \wedge dg + \lambda_2 \Omega$, $\lambda_i \in C^\infty(M)$. Since we are interested in normal forms of equations we may assume that $\omega = df \wedge dg + \lambda\Omega$, $\lambda \in C^\infty(T^* M)$.

THEOREM. – (S. Lie) *Suppose a Monge-Ampère equation* $\mathscr{E}_\omega$ $\omega \in \Lambda^2(T^* M)$ *admits an intermediate integral* (f, g) *for which the subspace* $\mathscr{J} \subset C^\infty(T^* M)$ *generated by* $f, g, 1$ *over* $\mathbb{R}$ *forms a Lie algebra with respect to the Poisson bracket. Then* $\mathscr{E}_\omega$ *reduces to the one of the following normal forms:*

(1) *hyperbolic type:* $\omega = 1/2(dp_1 \wedge dq_1 - dp_2 \wedge dq_2)$

(2) *parabolic type:* $\omega = dp_1 \wedge dq_2$.

Proof. – Consider $\mathscr{J}$, then 1 belongs to the center of $\mathscr{J}$ and therefore it suffices to define $\{f, g\}$. Let $\{f, g\} = C_1 + C_2 f + C_3 g$, $C_i \in \mathbb{R}$ then up to isomorphism the 3 cases are possible:

$$(a)\ \{f, g\} = 1; \qquad (b)\ \{f, g\} = 0; \qquad (c)\ \{f, g\} = f.$$

By a contact diffeomorphism f, g can be transformed into p_1, q_1 in case (a), into p_1, q_2 in case (b) and in case (c), replacing g by g/f, we have $\{f, g/f\}=1$ and therefore are in case (a).

2. Symplectic classification of exterior effective 3-forms on 6-dimensional space

In this section we describe the Sp(V)-orbits in the space of effective forms $\Lambda^n_\varepsilon(V^*_{2u})$ for $n=3$.

2.1. Let V be a 6-dimensional real symplectic space with the structure form Ω and $\omega \in \Lambda^3_\varepsilon(V^*)$ an effective form. Set $\omega_X = i_X \omega \in \Lambda^2(V^*)$, $E_X = \operatorname{Ker} \omega_X$ for all $X \in V$. Let us construct an invariant of Sp-action, the quadratic form q_ω on V associated with an effective ω. Notice that $\omega_X \wedge \Omega = 0$ since the effectiveness of ω, $\omega \wedge \Omega = 0$, implies that $\omega_X \wedge \Omega = \omega \wedge \theta_x$ where $\theta_X = i_X \Omega$ and therefore $\omega_X \wedge \Omega^2 = \omega \wedge \theta_x \wedge \Omega = \omega \wedge \Omega \wedge \theta_X = 0$. Therefore the characteristic polynomial of ω_X is of the form (up to the volume form Ω^3):

$$(\omega_X - \lambda\Omega)^3 = -\lambda^3 \Omega^3 - 3\lambda\, \omega_X^2 \wedge \Omega,$$

with $\omega_X^3 = 0$ since ω_X is degenerate. Therefore the roots of $P_{\omega_X}(\lambda)$ are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm\sqrt{-1/4 \perp^2 \omega_X^2}$. In fact, for a non-zero root λ we have $\lambda^2 \Omega^3 = -3\, T \omega_X^2$ and since $\perp_3 : \Lambda^6(V^*) \to \mathbb{R}$ is an isomorphism, then $\lambda^2 \perp_3 \Omega^3 = -3 \perp_3 T \omega_X^2$. But $\perp_3 T \omega_X^2 = \perp_2 \omega_X^2$, $\perp_3 \Omega^3 = 6$ and therefore $\lambda^2 = -1/4 \perp^2 \omega_X^2$.

Making use of these remarks we define an Sp(V)-invariant, the quadratic form $q_\omega \in S^2(V^*)$:

$$q_\omega(X) = -1/4 \perp^2 \omega_X^2 \tag{2.1}$$

If $\omega_2 = F^*(\omega_1)$, $F \in \mathrm{Sp}(V)$ then $q_{\omega_2} = F^*(q_{\omega_1})$. Notice also that if $q_\omega(X) \neq 0$ then $\omega_X^2 \neq 0$ and since $\omega_X^3 = 0$, then ω_X is a form of rank 4 and $\dim E_X = 2$.

PROPOSITION 2.2. — *Let* $X, Y \in V$ *be such that* $Y \in E_X$. *Then* $\omega = \omega_Y \wedge \theta_X - \omega_X \wedge \theta_Y$ *if* $\Omega(X, Y) = 1$ *and* $\omega_X \wedge \theta_X = \omega_Y \wedge \theta_Y$ *if* $\Omega(X, Y) = 0$.

Proof. — The effectiveness of ω implies

$$i_X(\omega \wedge \Omega) = \omega_X \wedge \Omega - \omega \wedge \theta_X = 0$$

yielding

$$\begin{aligned} i_X i_Y(\omega \wedge \Omega) &= i_Y(\omega_X \wedge \Omega) - i_Y(\omega \wedge \theta_X) = (i_Y \omega_X) \wedge \Omega \\ &\quad + \omega_X \wedge \theta_Y - \omega_Y \wedge \theta_X + \omega . \theta_X(Y) = \omega . \Omega(X, Y) - \omega_Y \wedge \theta_X + \omega_X \wedge \theta_Y. \end{aligned}$$

PROPOSITION 2.3. — *If* $q_\omega(x) \neq 0$ *and* $E_X \subset V$ *is an isotropic space then* $\omega = e_1^* \wedge e_2^* \wedge e_3^*$ *in a canonical basis of* V.