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Annales scientifiques de l’É.N.S. 4e série, tome 25, n° 6 (1992), p. 679-685

<http://www.numdam.org/item?id=ASENS_1992_4_25_6_679_0>
FIXED POINTS OF POLYNOMIAL MAPS
I. ROTATION SUBSETS OF THE CIRCLES

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ABSTRACT. — We give a combinatorial analysis of rational rotation subsets of the circle. These are invariant subsets that have well-defined rational rotation numbers under the standard self-covering maps of $S^1$. This analysis has applications to the classification of dynamical systems generated by polynomials in one complex variable.

0. Introduction

Late in the 1800's, Poincaré showed that every homeomorphism of the circle has a well defined rotation number which measures asymptotically, the average distance each point is moved by the map. Since its inception, this concept has played a fundamental role in the theory of dynamical systems in one and two dimensions.

This article focuses on dynamical systems generated by the standard $d$-fold self-coverings of the circle $S^1$. We give a combinatorial classification of rational rotation subsets of $S^1$. By definition, these are invariant subsets that have well defined rational rotation numbers. For $d=2$, these sets are always periodic cycles, and they arise in a variety of different contexts ([B], [Bu], [GH], [GLT], [V1], [V2]). Other points of view that are not, to my knowledge, in the literature, have been taught to me by Charles Tresser.

There is an important application of rotation sets to the problem of classifying dynamical systems generated by polynomials in a single complex variable. A repelling fixed point of a degree $d$ polynomial admits a set of external arguments $\Theta = \{ \theta_0, \ldots, \theta_{n-1} \}$ which constitute a degree $d$ rotation subset of the circle [DH]. This application will be explored at length in a joint project with John Milnor, that makes up Part II of this work.

(1) Research supported in part by the Alfred P. Sloan Foundation, the National Science Foundation, and by a PSC-CUNY Research Award.
Acknowledgement

The author wishes to thank John Milnor for numerous and significant contributions to this article.

1. Notation and Definition

Parametrize the unit circle $S^1$ by the interval $[0, 1)$. Let $d \geq 2$ and consider the $d$-fold covering map

$$f_d : \theta \mapsto d\theta \mod 1$$

Let $m$ and $n$ be non-negative integers satisfying $0 \leq m \leq n$. We will adopt the convention throughout that an indexed subset $\Theta = \{ \theta_0, \ldots, \theta_{n-1} \}$ of $S^1$ satisfies

$$0 \leq \theta_0 < \ldots < \theta_{n-1} < 1.$$

**Definition.** A finite subset $\Theta = \{ \theta_0, \ldots, \theta_{n-1} \}$ of $S^1$ is a degree $d$ $m/n$-rotation set if $f_d^i(\theta) = \theta + i\frac{m}{n}$ for $i = 0, \ldots, n-1$.

In general the numbers $m$ and $n$ need not be relatively prime, so that $m/n = kp/kq$ for some $k \geq 1$ with $p$ and $q$ relatively prime. In this case, we say that the rotation number of the set $\Theta$ is $p/q$. It follows that the set $\Theta$ is a union of $k$ cyclic orbits which are regularly interspersed, each of which has the order type any orbit of the rotation $\theta \mapsto \theta + p/q \mod 1$. Hence, each of these $k$ cyclic subsets of $\Theta$ will be called a degree $d$ $p/q$-rotation cycle.

**Remark.** Most finite sets invariant under $f_d$ are not rotation sets. Consider the 4-cycle generated by the angle $1/5$ whose base 2 expansion is .00110011...

To begin our analysis, we isolate the special case of rotation number zero. Here a rotation set is any non-vacuous set of fixed points of the map $f_d$. There fixed points are precisely the angles $j/(d-1)$ with $0 \leq j < d-1$.

![Fig. 1. Three Quadratic Rotation Sets.](image-url)
Henceforth, we will assume $0 < p < q$.

**Lemma 1.** — For $q \geq 2$, the $q$-cycles under $f^r_d$ are in one-to-one correspondence with orbits of period $q$ under the one-sided d-shift.

**Sketch Proof.** — Label the $d$ arcs obtained by removing the points $\{i/d\}$ from $S^1$ counterclockwise from 0 with the digits 0, 1, ... , $d-1$. Let $\theta$ be a period $q$ periodic point for $f^r_d$. If $\theta \in S^1$ is not a fixed point of $f^r_d$, let $\gamma(\theta) \in \{0, \ldots, d-1\}$ denote the label of the arc containing $\theta$. Define the word

$$a = \gamma(\theta) \gamma(f^0_d \theta) \ldots \gamma(f^{q-1}_d \theta).$$

The base $d$ expansion of $\theta$, is then given by $\theta = .aaaaa...$

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### 2. Existence and Uniqueness of Rotation Sets

As we will see below, rotation sets with all possible rotation numbers exist in all degrees $d \geq 2$; furthermore, quadratic rotation sets are completely classified by their rotation numbers. This is not true in higher degrees, as is indicated by examples in Figure 2. Two of the rotation sets in Figure 2 can be distinguished from the remaining three by the number of elements they contain, however a finer invariant is needed to distinguish all five examples. For each degree $d$ rotation set, we will record the deployment of the elements with respect to the fixed points of the map $f^r_d$.

![Fig. 2. - The Five Cubic Rotation Sets with Rotation Number 1/2.](image)

**Definition.** — Let

$$\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$$

be a finite subset of $S^1$. The degree $d$ deployment sequence of $\Theta$ is the nondecreasing sequence of non-negative integers $(s_2, \ldots, s_{d-1})$, where $s_j$ is the number of $\theta_i$'s in the interval $[0, i/(d-1))$.

The cubic rotation sets in Figure 2 have deployment sequences

$$(2, 2), (3, 4), (1, 2), (1, 4), (0, 2)$$

respectively. (Thus the proportion $s_1/s_2$ of angles in the upper half-circle is 1, 3/4, 1/2, 1/4, 0 respectively.)

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Remarks:
1. The last entry $s_{d-1}$ is just the cardinality of $\Theta$. In the case of a rotation set, it is always a product $kq$ with $1 \leq k \leq d-1$. (Compare Corollary 6.)
2. This invariant contains no information for $d=1$. (It is just the single number $(q)$.)
3. The degree $d$ deployment sequence of a rotation set locates the components of the set with respect to the fixed points of $f_d$, not with respect to the $f_d$-preimages of 0. Therefore it does not, a priori, determine the base $d$ expansions of the components.

Lemma 2 (Uniqueness). — A degree $d$ rotation set is completely determined by its rotation number $p/q$ together with its deployment sequence

$$0 \leq s_1 \leq s_2 \leq \ldots \leq s_{d-1} = kq.$$

The proof depends on the interplay between the fixed points of $f_d$ and its preimages of zero.

**Definition.** A point $\theta \in S^1$ is **advancing** if $f_d(\theta) > \theta$, **retreating** if $f_d(\theta) < \theta$. (Remember that all angles are reduced modulo 1 so as to lie in the half-open interval $[0, 1)$.

**Proof of Lemma 2.** For $j=1, \ldots, d-1$, let $U_j$ denote the arc $((j-1)/(d-1), j/(d-1))$. Each arc $U_j$ contains exactly one $f_d$-preimage of zero $j/d$ that divides it into a pair of subarcs

$$U_{j, \text{adv}} = \left(\frac{j-1}{d-1}, \frac{j}{d}\right).$$
These are labeled to reflect the fact that \((j-1)/(d-1)<0<f(\theta)<1\) on \(U_j,_{\text{adv}}\) and \(0<f(\theta)<0<j/(d-1)\) on \(U_j,_{\text{ret}}\).

Let \(\Theta=\{\theta_0, \ldots, \theta_{kq-1}\}\) be a degree \(dkp/kq\)-rotation set with deployment sequence \((s_1, s_2, \ldots, s_{d-1}=kq)\). Since \(\Theta\) is a \(kp/kq\)-rotation set, \(f_d\) advances \(\theta_0, \ldots, \theta_{kq-kp-1}\) and retreats the other \(\theta_i\)'s. If \(0 \leq i \leq kq-kp-1\), then

\[
\theta_i \in U_j,_{\text{adv}} = \left(\frac{j-1}{d-1}, \frac{j}{d}\right) \subset \left(\frac{j-1}{d-1}, \frac{j}{d}\right)
\]

and if \(q-p \leq i \leq q-1\),

\[
\theta_i \in U_j,_{\text{ret}} = \left(\frac{j}{d-1}, \frac{j+1}{d}\right) \subset \left(\frac{j}{d-1}, \frac{j+1}{d}\right)
\]

so that the location of the \(\theta_i\)'s vis-a-vis the \(f_d\) preimages of 0 is determined. Now, as in Lemma 1, the action of \(f_d\) yields the base \(d\) expansions of the \(\theta_i\)'s. □

We now turn to the question of existence. An examination of the proof of Lemma 2 gives an algorithm for constructing angles from the data consisting of a rotation number \(p/q\), and a candidate deployment sequence \((s_1, \ldots, s_{d-1}=kq)\). It is not difficult to check that the angles \(\theta_i\) resulting from this construction satisfy

\[0 \leq \theta_0 \leq \ldots \leq \theta_{kq-1} < 1.\]

However, these inequalities need not be strict, so the angles \(\theta_i\) will not be distinct in general. We give below, a necessary and sufficient condition for strict inequality, and hence for existence of a set of a set angles fitting the given combinatorial data.

Let \(\Theta=\{\theta_0, \theta_1, \ldots, \theta_{kq-1}\}\subset S^1\) be disjoint from the fixed points of \(f_d\). The complement of \(\Theta\) in \(S^1\) consists of \(kq\) arcs \(A_0, A_1, \ldots, A_{kq-1}\) labeled so that the arcs \(A_i\) is bounded by \(\theta_i\) and \(\theta_{i+1 \mod kq}\). We define the weight \(\omega(A_i)\) of the arc \(A_i\) to be the number of \(f_d\) fixed points is contains. Note that the length, \(l(A_i)\) of \(A_i\) equals the difference \(\theta_{i+1}-\theta_i\) when \(i<kq-1\) and equals \(1+\theta_0-\theta_{kq-1}\) when \(i=kq-1\).

**Lemma 3.** Let \(\Theta=\{\theta_0, \theta_1, \ldots, \theta_{kq-1}\}\) be a degree \(d\) rotation set with rotation number \(p/q\) and complementary arcs \(A_0, A_1, \ldots, A_{kq-1}\). Then the following equation holds:

\[d l(A_i) = l(A_{i+kp \mod kq}) + \omega(A_i)\]

Furthermore, the map \(f_d\) carries \(A_i\) homeomorphically onto \(A_{i+kp \mod kq}\) if and only if the weight \(\omega(A_i)\) is zero.

**Proof.** The image of an arc \(A_i\) under \(f_d\) covers the (disjoint) arc \(A_{i+kp \mod kq}\) and then winds some number \(N\) times around the circle. It is easy to check that each of these circumnavigators of \(S^1\) in \(A_i\) contains a unique fixed point of \(f_d\). Therefore, \(N=\omega(A_i)\). □
We can solve these linear equations \( \ell(A_i) \) for the angles \( \omega(A_i) \) as functions of the critical weights \( \omega(A_i) \). If we sum these equations over a residue class modulo \( k \), we obtain the equation

\[
(d-1)(\ell(A_i) + \ell(A_{i+k}) + \ldots + \ell(A_{i+k(q-1)})) = \omega(A_i) + \omega(A_{i+k}) + \ldots + \omega(A_{i+k(q-1)})
\]

for each \( i \) between 0 and \( k-1 \). That is, the total angular width of these \( q \) sectors is directly proportional to the total weight. In particular, at least one of these \( q \) sectors must contain a fixed point of \( f^d \). (More directly, if the \( \omega(A_{i+k}) \) were all zero, then each of these sectors would map homeomorphically onto a sector with strictly greater length, which is impossible.)

**Lemma 4.** For each \( i \) between 0 and \( k-1 \), the \( q \)-fold sum

\[
\omega(A_i) + \omega(A_{i+k}) + \ldots + \omega(A_{i+k(q-1)})
\]

must be strictly positive. In other words, each of the arcs \( A_i \) either contains a fixed point, or is mapped homeomorphically by an iterate of \( f^d \) onto an \( A_j \) that does contain a fixed point. \( \Box \)

**Remark.** In the sequel to this article, we will show that the weight \( \omega(A_i) \) is equal to the number of critical points contained in an associated region of the dynamical plane of a polynomial map. (Compare Part II, § 2.)

An equivalent formulation of Lemma 4 in terms of deployment sequences is the following. Fix any \( p/q \neq 0 \).

**Lemma 5.** A sequence \( 0 \leq s_1 \leq s_2 \leq \ldots \leq s_{d-1} = kq \) is realized by a degree \( d \) rotation set if and only if every residue class modulo \( k \) is realized by at least one of the \( s_i \)'s.

**Corollary 6.** We have \( k \leq d-1 \). That is, a degree \( d \) rotation subset with rotation number \( p/q \) contains at most \( (d-1)q \) points. \( \Box \)

We summarize the results from this section as

**Theorem 7.** A degree \( d \) rotation subset of the circle is uniquely determined by its rotation number and its deployment sequence. Conversely, a lowest terms fraction \( p/q \) and candidate deployment sequence

\[
0 \leq s_1 \leq s_2 \leq \ldots \leq s_{d-1} = kq
\]

determine a rotation subset of \( S^1 \) only if every class modulo \( k \) is realized by at least one of the \( s_i \)'s. \( \Box \)

**Corollary 8.** Quadratic rotation cycles are in one to one correspondence with the set of rational numbers modulo one. \( \Box \)
3. Counting Rotation Cycles

Recall that the number of ways to deploy \( q \) indistinguishable balls in \( N \) labeled boxes is equal to the binomial coefficient \( \binom{N+q-1}{q} \).

**Proposition 9.** - The map \( f_d \) has \( \binom{d+q-2}{q} \) rotation cycles with rotation number \( p/q \).

**Proof.** - The conditions of Theorem 7 are satisfied for every candidate deployment sequence \((s_1, s_2, \ldots, s_{d-1} = q)\) corresponding to a rotation cycle. Consequently, the number of \( p/q \)-rotation cycles in degree \( d \) is precisely to the number of ways to deploy \( q \) indistinguishable balls in \( d-1 \) labeled boxes. \( \square \)

**REFERENCES**


(Manuscript received December 11, 1990; revised December 9, 1991).