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PATCHING LOCAL UNIFORMIZATIONS

BY O. E. VILLAMAYOR U.

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

ABSTRACT. – In [Vi] we introduced a canonical algorithm for resolution of singularities in characteristic zero (*see* section 8 of this paper for explicit examples). This proved the constructiveness of the theorem of resolution.

The novelty in this paper is a synthetic definition of groves and a notion of funtions on groves. Indeed we are able to give an almost selfcontained account of main aspects of resolution of singularities, a detailed presentation of the results in [Vi] and also to include a proof of the lifting of group actions on a scheme to its constructive resolution.

At the same time this paper is strongly focused on the algorithm itself as to motivate explicit computations.

An embedded resolution of singularities of a reduced subscheme of a regular scheme consists of a resolution of singularities of the subscheme together with a condition of normal crossings of the final outcome with the exceptional hypersurfaces introduced by this procedure. This process, in terms, can be expressed as a concatenation of intermediate "resolutions" called resolutions of groves.

In section 8 we exemplify both, resolutions and lifting of group actions and discuss a natural outcome of our algorithmic (or constructive) resolution of singularities: the patching of local uniformizations.

Introduction

The theorem of resolution of singularities, as we know it, is an inductive theorem. So the first point is to clarify what we mean by this induction.

To fix ideas we think of a hypersurface H in a smooth scheme W and a point $x \in H$. If one adds to this data a *system of coordinates* at x, then one can somehow attach to H (in a non-unique way) an *equation*. Now among the many ways of choosing coordinates we will select some, through a notion of *convenient* system of coordinates.

Once H is locally expressed by an equation in convenient coordinates and, as in the case of the Weierstrasse presentation lemma, here we will also have a notion of coefficients (free from a preassigned variable).

Induction means that some form of improvement of the original singularity (of H at x) can be formulated as a new problem now involving *only these coefficients*.

The first difficulty can be stated as follows:

D_1. Induction is done in terms of convenient coordinates, but these are not unique.

This immediately confronts us with the need of a unified treatment for all the different inductive formulations arising from the different choises of coordinates.

Suppose that we can give a very settled notion of induction (locally at x) so that D_1 is overcomed, but now we replace x by an equally bad neighboring point $y (\in H)$ and we are confronted with a global problem:

D_2. Do these local inductive formulations patch?

Briefly speaking, we will distinguish within the problem of resolution, both local and global aspects. On the local side is the Tschirnhausen presentation which consists on attaching to a singularity of a hypersurface:

(i) a (*local*) system of coordinates.

(ii) a nice expression of the equation in terms of these coordinates.

The strength of a Tschirnhausen presentation will rely on:

(A) after applying a permissible monoidal transformation one can the outcome by finitely many local (affine) charts so that at each such chart there is a notion of transform of the system of coordinates (i) to a new system, say (i'). And there is a nice expression [say (ii')] of the equation of the strict transform on the coordinates (i') (stability of the convenient coordinates).

(B) The problem of improving the singularity by permissible monoidal transformations translates to a problem involving coefficients that arise from the expression (ii), and the problem on the coefficients requires a similar treatment but now in a smooth scheme of lower dimension (inductive approach to resolution).

In any case, from (A) and (B) we see that the need to consider both *permissible transformations* and (open) *restrictions* is justified by our "attachment to coordinates" and our will to understand the resolution itself as an "algorithm" on the equation (on the coordinates).

A basic ingredient for the development of these ideas are the notions of *trees* and *groves*. Roughly speaking a tree on a singularity is a concatenation of both operations (permissible monoidal transformations and restrictions). And a grove over a singularity consists of all possible trees on it.

Permissible transformations and restrictions also show up in Zariski's line of thinking which is: proceed by applying *convenient* monoidal transformations followed by affine open restrictions locally at an exceptional point. Where convenient means that regularity is achieved by this procedure locally at the last point. He formalized this idea by what the called a reduction of singularities along a valuation ring or "local uniformizations".

We took Zariski's proposal in the following setup (*see* [Vi], 2.2). A **constructive resolution of singularities** is defined by an *upper-semicontinuous function on the singular locus* so that the maximal value is achieved along a *convenient* permissible center. Indeed, there is an *improvement* of these functions after each such permissible monoidal transformation. Moreover, repeating this procedure again and again one will solve the singularity.

The important feature in the construction of our upper-semicontinuous function was the fact that the value of the function at a point is defined by an "algorithm" at the local ring of the point so it has a *good restriction property*, that is, the function will also provide the convenient center through any given point. Therefore a constructive resolution defines on the one hand an "algorithm" of resolution along any valuation ring (local uniformization) and at the same time all these local uniformizations patch to a global resolution.

This is the difference between the theorem of constructive (or "algorithmic") resolution of singularities proved in [Vi] and the existencial proof of resolution as given in [H1]. The bridge between existential and constructive resolution was the notion of *birth* which stated exactly where and when the inductive approach given by a Tschirnhausen presentation is to be done [Vi].

In the way of the new presentation, we remark the fact that the theory of local idealistic presentation [H2] shows that the problem of resolution of singularities reduces to a good understating of the problem on hypersurfaces. And that this important reduction can be done "locally" at any singular point. But for schemes of finite type over a field, local is to be understood in the sense of the etale topology.

Both etale maps and open restriction are in particular smooth maps. Other smooth maps arise in Hironaka's notion of groves. All this leads naturally to consider a common notion of restriction relative to general smooth maps.

It was suggested by Giraud to let "restriction" mean the pullback by a *smooth maps*, to define trees as concatenations of monoidal transformations and "restrictions", and that constructive resolution should "restricts well" in this sense.

With this as starting point we define here *functions on groves* as function on the singular locus with a good "restriction property" (4.1) and we show that the function defining the constructive resolution in our previous work is indeed a function on a grove.

As an outcome from this fact question as:

1. lifting the action of a group on a scheme to the constructive resolution (7.6.3), or

2. formally isomorphic points undergo the "same" resolution (7.6.1)

have a simple proof with our approach. Actually 1 and 2 are consequence both of [H2] together with [Vi], but in this selfcontained account we want to present these results after a conveniently developed "language of resolution" or "language of groves" with rudiments already in [H2]. Indeed, the core of this presentation is to develop this language and show that is suitable for a synthetic approach to resolution, it clarifies why constructive resolution simplifies the web of induction in Hironaka's theorem, and it leads naturally to the concept of gluing or patching local uniformizations.

The development of this language allows a set theoretical approach to resolutions from which the answer to both D_1 and D_2 will grow out.

Briefly speaking to the problem of reduction of singularities we will associate a *grove* which now means a sheaf of sets, say G, where the elements of the sets are trees. The answer to D_2 will rely on the fact that these sheaves naturally glue. D_1 finds an suitable answer with the concept of immersion of groves.

The simplicity of this language of groves is based on the strong geometric meaning of the trees of the grove, so if x and y are two points of the hypersupersurface H and suppose that there is a local isomorphism between (H, x) and (H, y), then this isomorphism defines a natural one to one correspondence between the corresponding stalks of germs G_x and G_y.

Recall that the value of the function defining the constructive resolution at a point x of H, is the outcome of an algorithm at the local ring $O_{H, x}$. An important result is (5.5.1) which states that the outcome of this algorithm is expressible in terms of G_x. Moreover, if (H, x) and (H, y) are linked by a local isomorphism, the expressions of these outcomes in G_x and G_y (respectively) are linked by the correspondence between these stalks mentioned before. Now both 1 and 2 will follow from this fact.

As opposed to the constructive resolution of locally embedded schemes given in [Vi], we present here the algorithm of a scheme embedded in a fixed regular scheme.

As starting in section 2.7 of [Vi], the construction is also given here in terms of the functions $n(x)$ and $w\text{-}ord(x)$. Enough ideas are developed as to give a selfcontained account of the algorithm as of the notions of trees, groves, birth etc.

Example 1 is the constructive resolution of the Whitney umbrella and Example 2 illustrates the lifting of group actions by these resolutions.

I profited of earlier conversations with J. Giraud who gave me this perspective on the problem and with whom these topics were discussed. I also acknowledge suggestion from my colleagues A. Campillo and Z. Hajto.

0. Notation and conventions

The theorem of constructive resolution of singularities in the algebraic context grows from applications of analytic methods in algebraic geometry. From a technical point of view there is little difference between constructive resolutions in the R or C-analytic case or the algebraic case as one can check in our development, this is an important advantage over the original theorem of resolution.

As in [Vi], in this presentation we also work in the algebraic case, k will denote a field of characteristic zero, schemes will be noetherian, of finite type over k and all morphisms compatible with this k-structure.

If W is a smooth scheme (over k) we say that a collection of strictly proper subschemes $\{F_\lambda\}_{\lambda \in J}$ have *normal crossings* if, at any point $x \in W$:

1. only finitely many subschemes F_λ contain x, say $\{F_{\lambda(1)}, \ldots, F_{\lambda(s)}\}$, $s \geqq 0$,

2. $\bigcup_{\lambda \in J} F_\lambda = F_{\lambda(1)} \cup \ldots \cup F_{\lambda(s)}$ at some neighborhood of x,

3. there is a regular system of parameters $\{x_1, \ldots, x_n\}$ and s *non empty* subset $J(i) (\subset \{1, \ldots, n\})$ $i=1, \ldots, s$, so that (locally at x) $F_{\lambda(i_0)}$ is defined by the ideal $\langle x_j / j \in J(i_0) \rangle (\subset \mathcal{O}_{W, x})$.

We say that i: $X \rightarrow W$ is a closed immersion if it identifies X with a *closed* subscheme of W.

1. Trees, operations on trees

1.1. DEFINITION. — (W, E) is said to be a *pair with index set* $\wedge$ if:

(*a*) W is smooth of finite type over a field k.

(*b*) $E = \{H_\lambda\} \lambda \in \wedge$, where each H_λ is either empty or a closed and smooth hypersurface of W, and E is a collection with normal crossings (*i.e.* $\bigcup H_\lambda$ is a divisor with normal crossings [Ha]. V. 3.8.1).

1.2. All the constructions in this work are based on two elementary transformations.

If C is a smooth proper and closed subscheme of W, set π: $W_1 \to W$ the monoidal transformation with center C.

1.2.1. DEFINITION. — (A) Let (W, E) and $\wedge$ be as before, the map π: $W_1 \to W$ is said to be *permissible for the pair* (W, E) if C has normal crossings with E (*i.e.* $\{C\} \bigcup E$ has normal crossings).

In such case we define a pair (W_1, E_1) with index set $\wedge_1 = \wedge \bigcup \{\delta\}$ and $E_1 = \{H'_\lambda\}_{\lambda \in \wedge_1}$ so that H'_λ is the strict transform of H_λ if $\lambda \in \wedge$ and H_δ is $\Pi^{-1}(C)$ (the exceptional locus of the monoidal transformation). (W_1, E_1) is called the *transform* of (W, E) by the permissible transformation.

(B) Given a pair (W, E) and $\wedge$ as before and now a smooth map f: $W_1 \to W$, then W_1 is smooth over the field k, and we set $E_1 = \{f^{-1}(H_\lambda)\}_{\lambda \in \wedge}$ indexed by the same set $\wedge$.

(W_1, E_1) is clearly a pair indexed by $\wedge$ called the *transform* of the pair (W, E) by the *permissible* map f.

1.3. Now that our two basic permissible transformations [of type (A) and (B)] have been defined together with a notion of transformation of pairs, we define compositions of permissible transformations and iterate transformations of pairs.

Consider a transformation of type (A) over (W, E) and suppose furthermore that there are global coordinates on W, then W_1 must be covered by open charts in order to define coordinates at each open set. So blowing ups followed by open restriction is a very natural thing to do if one is interested in coordinates, which is the case here. Note that the restriction to an open set is in particular a transformation of type (B). And after such restriction is done, then the setup is that of the beginning and one can start again (blow up and then restrict). A tree will be a concatenation of such procedures, the reason why we take smooth maps instead of (simply) open restrictions will be justified with the further development.

1.3.1. NOTATION. — If (W_1, E_1) is the transform of (W, E) by a permissible transformation g: $W_1 \to W$ we simply denote this by

$$(W, E) \overset{g}{\leftarrow} (W_1, E_1)$$

in general we will not write down the corresponding index sets.

1.3.2. Definition. — Given two permissible transformations say:

$$(W_1, E_1) \overset{g}{\leftarrow} (W_2, E_2) \qquad \text{and} \qquad (W_3, E_3) \overset{h}{\leftarrow} (W_4, E_4)$$

then a *concatenation* of g with h is possible if $(W_2, E_2)=(W_3, E_3)$ and there is natural identification of the index sets of E_2 and E_3 (if $\lambda \in \wedge$ then $H_\lambda (\in E_2)$ and $H'_\lambda (\in E_3)$ coincide as subsets of $W_2 = W_3$).

1.3.3. Remark. — If the concatenation of g with h is possible and both are permissible transformations of type B (1.2.1), then:

(i) $h.g$: $W_4 \rightarrow W_1$ is smooth of finite type.

(ii) the transform of (W_1, E_1) by $h.g$ is (W_4, E_4).

For (i) *see* [Ha], III, 10.1. c). (ii) is clear from the definition.

1.4 Definition. — A *tree* is a concatenation of blowing ups and "restrictions" (smooth maps), we define a notion of length on trees to count how many blowing ups are involved.

A tree of *length zero on* (W, E) is a permissible transformation of type B say $(W, E) \overset{g}{\leftarrow} (W_1, E_1)$ and (W_1, E_1) is called the *transform* of (W, E) by g.

A *tree* of *length one on* (W, E) is a concatenation of a transformation of type (A) say $(W, E, C) \leftarrow (W_1, E_1)$ with a transformation of type (B) over (W_1, E_1), say: $(W_1, E_1) \leftarrow (W^1, E^1)$ (1.2.1.). We denote this concatenation by $(W, E, C) \leftarrow (W^1, E^1)$ or symply $(W, E) \leftarrow (W^1, E^1)$. (W^1, E^1) is called the *transform* of (W, E).

A *concatenation* of two trees (as defined above), say $(W_1, E_1) \leftarrow (W_2, E_2)$ and $(W_3, E_3) \leftarrow (W_4, E_4)$ is possible if $(W_2, E_2)=(W_3, E_3)$ and there is an identification of the index sets (as in 1.3.2.).

1.4.1. Remark. — 1. A concatenation of two trees of length zero is a tree of length zero in the sence of Remark 1.3.3.

2. A concatenation of a tree of length one and a tree of length zero (in that order) is naturally a tree of length one since there is concatenation of two trees of length zero involved.

In particular any concatenation of trees of length zero or one can be naturally reformulated as tree of length zero followed by a concatenation of trees of length one, unless all trees are of length zero in which case the concatenation itself is a tree of length zero (as in 1).

1.5. Definition. — For $s>0$ we define a tree of length s to be a concatenation of s trees of length one: T_i, $i=1, \ldots, s$ usually denoted by:

1.5.1. $T: (W_1, E_1, C_1) \leftarrow (W_2, E_2, C_2) \leftarrow \ldots \leftarrow (W_s, E_s, C_s) \leftarrow (W_{s+1}, E_{s+1})$

where T_i: $(W_i, E_i, C_i) \leftarrow (W_{i+1}, E_{i+1})$. T is said to be a *tree on* (W_1, E_1) and (W_{s+1}, E_{s+1}) is called the *transform* of (W_1, E_1) by the tree T, sometimes denoted by (W_T, E_T). $\wedge_T$ denotes the index set of the pair (W_T, E_T).

1.6. Now we define some operations on trees which turn out being the key for a good definition of groves. We first state two lemmas.

1.6.1. Lemma. – *Let* $(W_i, E_i) \to (W, E)$ $i=1,2$ *be two transformations of type* B (1.2.1.) *and consider the fiber product of these two smooth maps if finite type together will the natural projections say* p_i: $U \to W_i$ $i=1, 2$. *Then:*

1. p_1 *and* p_2 *are smooth maps of finite type,*

2. *the transform of* (W_1, E_1) *by* p_1 *coincides with that of* (W_2, E_2) *by* p_2.

Proof: 1. is [Ha], III, 10.1 (*d*), and 2. follows from the definition of fiber products.

1.6.2. Lemma. – *Let* $(W, E, C) \overset{\pi}{\leftarrow} (W_1, E_1)$ *be a transform of type* (A), *and* $(W, E) \leftarrow (\bar{W}, \bar{E})$ *of type* (B) *notation as in* 1.3.1) *with fiber product:*

$$\begin{array}{ccc} W_1 & \overset{p_1}{\longleftarrow} & \bar{W}_1 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_2} \\ W & \overset{f}{\longleftarrow} & \bar{W} \end{array}$$

Then: (*a*) p_2 *is permissible of type* (A) *or of type* (B) *for* $(\bar{W}, \bar{E})$ *according to* $f^{-1}(C)$ *being non-empty or empty,*

(*b*) p_1 *is of type* (B) *for* (W_1, E_1),

(*c*) *the transform of* $(\bar{W}, \bar{E})$ *by* p_2 *is the same as that of* (W_1, E_1) *by* p_1.

Proof. – Smooth maps are flat so p_2 is the blowing up at the sheaf of ideals defining $f^{-1}(C)$. Since $(\bar{W}, \bar{E})$ is the transform of (W, E), if $f^{-1}(C)$ is not empty then p_2 is permissible of type (A) over $(\bar{W}, \bar{E})$. If $f^{-1}(C)$ is empty then p_2 is the identity map, in particular of type (B).

(*b*) is the stability of smoothness by a base changes ([Ha], III, 10.1. *b*), and (*c*) follows from the definition of transformations of pairs.

1.6.3. Corollary. – *Let* T *be as in* 1.5.1. *and* $(W_1, E_1) \leftarrow \bar{W}_1, \bar{E}_1)$ *a tree of length zero* (*i.e. a transformation of type* (B) (1.2.1). *Then there is a natural lifting of T to a tree* $\bar{T}$: $(\bar{W}_1, \bar{E}_1, \bar{C}_1) \leftarrow \ldots \leftarrow (\bar{W}_{s+1}, \bar{E}_{s+1})$ *of length at most s, and smooth maps* r_k: $\bar{W}_k \to W_k$ $k=i, \ldots, s+1$. *Moreover each* $(\bar{W}_k, \bar{E}_k)$ *is the transform of* (W_k, E_k) *via* r_k *i.e.* r_k: $(\bar{W}_k, \bar{E}_k) \to (W_k, E_k)$ *are trees of length zero* (1.4). *And all square diagrams are commutative.*

Proof: Recall that a tree of length one is a concatenation of a transformation of type (A) with one of type (B) (1.4). To prove the assertion in the case of length one apply first 1.6.2 and then 1.6.1. So the general case follows inductively.

1.7. Definition. – (*a*) If T is a tree of length s on (W_1, E_1) as in 1.5.1., for any $k \in \{1, \ldots, s+1\}$ set the *k-truncation* of T to be

$$[T]_k: \quad (W_1, E_1, C_1) \leftarrow \ldots \leftarrow (W_{k-1}, E_{k-1}, C_{k-1}) \leftarrow (W_k, E_k)$$

(*b*) Fix T, *s* and *k* be as in (*a*), and let f: $(\bar{W}_k, \bar{E}_k) \to (W_k, E_k)$ be a tree of length zero. According to 1.6.3 there is a lifting of the tree

$$S: \quad (W_k, E_k, C_k) \leftarrow \ldots \leftarrow (W_s, C_s, E_s) \leftarrow (W_{s+1}, E_{s+1})$$

to

$$\bar{S}: \quad (\bar{W}_k, \bar{E}_k, r^{-1}{}_k(C_k)) \leftarrow \ldots \leftarrow (\bar{W}_{s+1}, \bar{E}_{s+1})$$

and trees of length zero r_i: $(\bar{W}_i, \bar{E}_i) \to (W_i, E_i)$ $i=k, \ldots, s+1$ so that all squares commute.

In particular there is a tree

$$\bar{T}: \quad (W_1, E_1, C_1) \leftarrow \ldots \leftarrow (W_{k-1}, E_{k-1}, C_{k-1}) \leftarrow (\bar{W}_k, \bar{E}_k, r_k^{-1}(C_k)) \leftarrow \ldots \leftarrow (\bar{W}_{s+1}, \bar{E}_{s+1})$$

called a *restriction* of T.

If $k>1$ $\bar{T}$ is a tree *on* (W_1, E_1), if $k=1$ $\bar{T}$ is a tree *on* $(\bar{W}_1, \bar{E}_1)$ (in the sence of 1.5).

When f: $(\bar{W}_k, \bar{E}_k) \to (W_k, E_k)$ is an open immersion, then $(\bar{W}_{s+1}, \bar{E}_{s+1}) \to (W_{s+1}, E_{s+1})$ arises from an open immersion $\bar{W}_{s+1} \subsetneq W_{s+1}$ and in this case T is said to be an *open restriction* of T.

In any case, for a restriction $\bar{T}$ of T there is a tree of length zero $(W_T, E_T) \stackrel{r_T}{\leftarrow} (\bar{W}_T, \bar{E}_T)$ linking the final transforms (1.6.3).

1.8. If T is a tree on (W, E) and (W_T, E_T) is the transform of (W, E) by T (1.5), then (W_T, E_T) is also a pair and one can define trees on (W_T, E_T). To end this section we complete 1.4 defining concatenations of trees of any length.

1.8.1. Definition. − If T_1 is a tree on (W, E) and T_2 is a tree on (W^1, E^1), then a concatenation of T_1 and T_2 (in that order) is possible if $(W_{T_1}, E_{T_1}) = (W^1, E^1)$ and there is a good identification of the index sets as in 1.3.2.

2. Groves

2.1. We start now with a fix pair (W, E) and for *any* tree of length zero $(W_1, E_1) \to (W, E)$ (1.4) we will select some trees on (W_1, E_1).

2.1.1. Definition. − A groves G *over* (W, E) consists of:

1. (i) a closed subset $F \subset W$.

(ii) all trees f: $(W_1, E_1) \to (W, E)$ of length zero, and for any such f the closed subset $f^{-1}(F) \subset W_1$.

2. For any f: $(W_1, E_1) \to (W, E)$ as in 1, a class $G(W_1, E_1)$ of trees on (W_1, E_1) and for any $T \in G(W_1, E_1)$ a closed set $F_T (\subset (W_T)$ (1.5) subject to the following conditions:

2. (i) if T belongs to the class, any truncation of T belongs to the class (1.7.*a*)

2. (ii) if T belongs the class, any restriction $\bar{T}$ (of T) belongs to the class and:

$$r_T^{-1}(F_T) = F_{\bar{T}} \quad \text{[notation as in 1.7(}b\text{)]}$$

2. (iii) if T belongs to the class $G(W_1, E_1)$ and T_1 is *any* tree of length one over $((W_1)_T, (E_1)_T)$, say T_1: $((W_1)_T, (E_1)_T, C) \leftarrow (W^1, E^1)$. Then the concatenation of T with T_1 belongs to the class *if and only if* $C \subset F_T$.

If T belongs to a class $G(W_1, E_1)$ as before, we say that T belongs to G or simply $T \in G$.

2.2. *The Hilbert Samuel grove* (general references [O], [H-I-O]):

2.2.1. To fix ideas we start with a pair (W, E) and a fixed hypersurface $H (\subset W)$, set

$$F(s, H) = \{x \in H/\text{multicity of H at } x = \text{mult}(H, x) \text{ is } \geqq s\}$$

For any $s \in \mathbb{Z}$ the set $F(s, H)$ is a closed. Now let b be the highest possible multiplicity at points of H ($F(b, H) \neq \varnothing$ and $F(b+1, H) = \varnothing$).

Let $(W_1, E_1) \rightarrow (W, E)$ be a transformation of type (A) (1.2.1) with center C included in $F(b, H)$, let H_1 denote the strict transform of H and b_1 the highest possible multiplicity at points of H_1. As we shall *see* latter in 3.3.1: $b_1 \leqq b$, and in particular $F(b, H_1)$ is either empty (if the inequality holds) or a non-empty closed subset of W_1.

Now let $(W_1, E_1) \rightarrow (W, E)$ be a transformation of type (B) (1.2.1) defined by the smooth map f: $W_1 \rightarrow W$, and let H_1 now denote the pullback of H by f and b_1, as before, the highest multiplicity at points of H_1. Note that in this case, for any $x \in H_1$, $\text{mult}(H_1, x) = \text{mult}(H, f(x))$. In particular $b_1 \leqq b$ and again $F(b, H_1)$ is either empty (if inequality holds) or a non empty closed subset of W_1. In any case:

$$f^{-1}(F(b, H)) = F(b, H_1)$$

So if we fix b as before we have a natural criterion to define a grove over (W, E) which is of course linked to H and b. These are, par excellence, the groves to be considered.

In general we will deal with a subscheme of W which is not a hypersurface in such case the notion of multiplicity will be replaced by a sequence of numbers obtained in terms, from the Hilbert-Samuel function. Now we define things properly.

2.2.2. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, for a map p: $\mathbb{N}_0 \rightarrow \mathbb{Z}$ set $p^{(1)}$: $\mathbb{N}_0 \rightarrow \mathbb{Z}$ where $p^{(1)}(k) = \Sigma\, p(j)$ $0 \leqq j \leqq k$, and in general for $r \in \mathbb{N}$ set $p^{(r)} = (p^{(r-1)})^{(1)}$.

Now we will consider datas of the form (W, X, p) where W is a smooth scheme of finite type over k, $X (\subset W)$ is a subscheme and p: $\mathbb{N}_0 \rightarrow \mathbb{Z}$ is a function.

If $W' \rightarrow W$ is a transformation either of type (A) or (B) (1.2.1) we define the transform of the data (W, X, p) to a data (W', X', p') as follows:

(*a*) if π: $W' \rightarrow W$ is the birational map corresponding to a transformation of type (A) (1.2.1) then set:

(i) (formally) the *transform* of p as p itself.

(ii) the *transform* of X: X', as the strict transform of X at W'.

(*b*) if f: $W' \rightarrow W$ is a smooth map corresponding to a transformation of type (B) then define:

(i) the *transform* of p as $p^{(r)}$ where r is the relative dimension of the smooth map f ([Ha] III, 10) (assume that the relative dimension is constant along W').

(ii) the *transform* of X to be $f^{-1}(X)(\subset W')$.

Now we fix a subscheme X of W. At any point $x \in X$ there is a function $H^{(1)}_{X, x}: \mathbb{N}_0 \to \mathbb{Z}$, and ultimately a map $H^{(1)}_X: X \to \mathbb{Z}^{\mathbb{N}_0}$ defined by $H^{(1)}_X(x) = H^{(1+d(x))}_{X, x}$, where $d(x)$ denotes the transcendance degree of the residue field of $O_{X, x}$ over k.

One of Bennett's results asserts that $H^{(1)}_X$ is upper semicontinuous along X (given on $\mathbb{Z}^{\mathbb{N}_0}$ the lexicographic order).

At any closed point x, $H^1_{X, x}$ expresses the rank at x of the coherent sheaves of principal parts of X over k ([O], Th. 2.6. (Bennett-Giraud)). In particular, since X is noetherian, there is a closed point $y_0 \in X$ such that H^1_{X, y_0} is maximal. Set (and fix) p as H^1_{X, y_0} and

$$F = \{ x \in X / H^{(1+d(x))}_{X, x} = p \text{ for } d(x) = \operatorname{tran}[k(x): k] \}.$$

F is a closed set and a closed point x belong to F if and only if $H^{(1)}_{X, x} = p$.

In this way, for $X(\subset W)$ we have a well defined data (W, X, p) and a closed subset $F \subset W$.

Another result of Bennett (the stability theorem) ([O] Th 3.1) states that for any transformation of type (A) say $(W_1, E_1) \to (W, E, C)$, if $C \subset F$ then at any closed point $x \in X'$ (the transform of X): $H^{(1)}_{X', x} \leqq H^{(1)}_{X, \pi(x)}$.

In particular if we set at X'; $F' = \{ x \in X' / H^{(1+d(x))}_{X', x} = p \}$ then F' is closed (and may be empty). In fact if F' is not empty, the setup for (W', X', p) and F' is that of (W, X, p) and F.

On the other hand if $h: W' \to W$ is a smooth map of relative dimension r then $H^{(1)}_{X', x} = H^{(r+1)}_{X, h(x)}$ for any $x \in W'$. So

$$F' = \{ x \in X' / H^{(1+d(x))}_{X', x} = p^{(r)} \} = h^{-1}(F)$$

and (W', X', $p^{(r)}$) (the transform of (W, X, p)) is again in the same setup as (W, X, p).

Now for any h, (W', X', $p^{(r)}$) as before, set $(W_1, X_1, p_1) = (W', X', p^{(r)})$ and define $G(W_1, E_1)$ to be the set of all trees T: $(W_1, E_1, C_1) \leftarrow \ldots \leftarrow (W_r, E_r)$, (W_i, X_i, p_i) is the transform of $(W_{i-1}, X_{i-1}, p_{i-1})$ and $C_i \subset F_i = \{ x \in X_i / H^1_{X_i, x} = p_i \}$.

In this way, for X and its maximal function p, we define a grove called the *Hilbert-Samuel grove* ([H2] 3).

2.3 Definition. – If G is a grove over a pair (W, E), the closed subset $F \subset W$ of 2.1.1.1 is called the *singular locus* of G at W and denoted as Sing G. ($F = \operatorname{Sing} G \subset W$). Moreover for any tree T of the grove, the closed set $F_T(\subset W_T)$ 2.1.1.2 is called the *singular locus* of the grove G at (W_T, E_T), and we denote F_T by $\operatorname{Sing} G_T(\subset W_T)$.

2.3.1. Remark. – Let G be a grove over (W, E) and T a tree of the grove. If $\bar{T}$ is a restriction of T then:

(*a*) $\bar{T}$ is also a tree of G. And if $r_T: W_{\bar{T}} \to W_T$ is the smooth map as in 1.7(*b*), then

(*b*) $r_T^{-1}(\operatorname{Sing} G_T) = \operatorname{Sin} G_{\bar{T}}(\subset W_{\bar{T}})$.

2.4. Main example of groves: *idealistic situations*.

Now we fix a smooth scheme W as before and define some special groves over pairs (W, E) which turn out being basic in the sence that all groves to be considered are constructed in terms of these.

2.4.1. DEFINITION. — A *couple* $(\mathscr{L}, b)$ on W consists of a coherent sheaf of ideals $\mathscr{L} \subset \mathcal{O}_W$ so that $\mathscr{L}_x \neq 0$ at any point $x \in W$, and a positive integer b. Now set

$$\operatorname{Sing}(\mathscr{L}, b) = \{ x \in W / \upsilon_x(\mathscr{L}_x) \geqq b \}$$

$[\upsilon_x(\mathscr{L}_x)$ stands for the order of the ideal $\mathscr{L}_x$ at the local regular ring $\mathcal{O}_{W, x}]$.

2.4.2. REMARK. — Since W is assumed to be a scheme over k, there is a coherent sheaf of $\mathcal{O}_W$-modules: Ω^1_W called the sheaf of k-differentials. For any $x \in W$, $\Omega_{W, x}$ is the module of k-differentials of the local regular ring $\mathcal{O}_{W, x}$.

If now x is a closed point and n is the dimension of the local ring $\mathcal{O}_{W, x}$, then $\Omega_{W, x}$ is a free module over $\mathcal{O}_{W, x}$ and for *any choise* of a regular system of parameters $\{y_1, \ldots, y_n\}$, the differentials $\{dy_1, \ldots, dy_n\}$ form a base for $\Omega_{W, x}$.

The theory of Fitting ideals will state the existence of a coherent sheaf of ideals $\Delta(\mathscr{L})(\subset \mathcal{O}_W)$ such that at any closed point $x \in W$ and for any selection of a regular system of parameters $\{y_1, \ldots, y_n\}$ at $\mathcal{O}_{W, x}$:

$$\Delta(\mathscr{L})_x = \left\langle f; \frac{\partial f}{\partial y_j} \middle/ f \in \mathscr{L}_x, j = 1, \ldots, n \right\rangle$$

We can think of Δ as an operator acting on the coherent sheaves of ideals. Clearly $\mathscr{L} \subset \Delta(\mathscr{L})(\subset \mathcal{O}_W)$ and $\operatorname{Sing}(\mathscr{L}, b)(\subset W)$ is the closed subscheme $\vee(\Delta^{b-1}(\mathscr{L}))$ defined by the sheaf of ideals $\Delta^{b-1}(\mathscr{L})$ (here Δ^r stands for the composition of the operator Δ with itself r-times). In particular $\operatorname{Sing}(\mathscr{L}, b) = \varnothing$ if and only if $\Delta^{b-1}(\mathscr{L}) = \mathcal{O}_W$.

2.4.3. DEFINITION. — 1. Let $\pi: W_1 \to W$ be the monoidal transformation on a smooth and closed center $C \subset \operatorname{Sing}(\mathscr{L}, b)$. The sheaf of ideals $\mathscr{L}\mathcal{O}_{W_1}$ admits an expression of the form

$$\mathscr{L}\mathcal{O}_{W_1} = \mathscr{I}^b \,.\, \mathscr{L}'$$

where $\mathscr{I}$ and $\mathscr{L}'$ are coherent ideals and $\mathscr{I}$ is the sheaf of ideals defining $\pi^{-1}(C)$. We say π *permissible for* $(\mathscr{L}, b)$ and define the *transform* of the couple $(\mathscr{L}, b)$ to be the couple $(\mathscr{L}', b)$ at W_1.

2. If $f: W_1 \to W$ is any smooth map we say that f is *permissible* for $(\mathscr{L}, b)$ and define the *transform* of $(\mathscr{L}, b)$ at W_1 to be the couple $(\mathscr{L}', b)$ where $\mathscr{L}' = \mathscr{L}\mathcal{O}_{W_1}$.

A local analysis at a closed point $x \in W_1$ shows that $\upsilon_x(\mathscr{L}'_x) = \upsilon_{f(x)})$ and $(\Delta^{b-1}(\mathscr{L}'))_x = \Delta^{b-1}(\mathscr{L}) \,.\, \mathcal{O}_{W_1, x}$.

3. Given a pair (W, E), a tree of length one on (W, E), say

$$(W, E, C) \leftarrow (U, E') \quad (1.4)$$

is said to be *permissible* for $(\mathscr{L}, b)$ if $C \subset \operatorname{Sing}(\mathscr{L}, b)$. In such case the *transform* of $(\mathscr{L}, b)$ at (U, E′) is defined as a combination of 1 and 2.

A tree of length zero on (W, E) is always *permissible* for $(\mathscr{L}, b)$ and the *transform* is defined as in 2.

4. A tree of length r on

$$(W_1, E_1)\,T\colon (W_1, E_1, C_1) \leftarrow (W_2, E_2, C_2) \leftarrow \ldots \leftarrow (W_{r+1}, E_{r+1})$$

is *permissible* for $(\mathscr{L}, b)$ if $C_i \subset \operatorname{Sing}(\mathscr{L}_i, b)$ where $(\mathscr{L}_i, b)$ denotes the transform of $(\mathscr{L}_{i-1}, b)$. In such case $(\mathscr{L}_{r+1}, b)$ is called the *transform* of $(\mathscr{L}, b)$ by T sometimes denoted by $(\mathscr{L}_T, b)$.

2.4.4. Proposition. — *A tree* T *on* (W, E) *is permissible for the couple* $(\mathscr{L}, b)$ *if and only if it is permissible for* $(\mathscr{L}^n, bn)$ (*for any* $n>0$).

Proof. — (Q) Let $\pi_1\colon W_1 \to W$ be as in 2.4.3.1 one can check that:

(i) $\operatorname{Sing}(\mathscr{L}, b) = \operatorname{Sing}(\mathscr{L}^n, nb)$ (at W).

(ii) if $(\mathscr{L}', b)$ is the transform of $(\mathscr{L}, b)$ and $((\mathscr{L}^n)', bn)$ that of $(\mathscr{L}^n, nb)$, then $(\mathscr{L}^n)' = (\mathscr{L}')^n (\subset \mathcal{O}_{W'})$.

(*b*) if $f\colon W_1 \to W$ is as in 2.4.3.2 then

(i) $f^{-1}(\operatorname{Sing}(\mathscr{L}, b)) = \operatorname{Sing}(\mathscr{L}', b)$

(ii) $(\mathscr{L}')^n = (\mathscr{L}\mathcal{O}_{W'})^n$

The proof of the proposition is a combination of parts (*a*) and (*b*).

2.4.5. Proposition. — Fix a couple $(\mathscr{L}, b)$ on W and a pair (W, E). The set of all trees on $(W_1 . E_1)$ permissible for $(\mathscr{L}_1 . b)$, where $f\colon (W_1, E_1) \to (W, E)$ is any tree of length zero and $(\mathscr{L}_1, b)$ the transform of $(\mathscr{L}, b)$ via f, defines a grove G over (W, E) by setting,

$$F_T = \operatorname{Sing}(\mathscr{L}_T, b)$$

for any such tree T.

Proof: If $\bar{T}$ is a restriction of T, then there is a smooth map $r_T\colon \bar{W}_{\bar{T}} \to W_T$ and a tree of length zero $(W_T, E_T) \leftarrow (\bar{W}_T, E_{\bar{T}})$. We want to know that

$$r_{\bar{T}}^{-1}(\operatorname{Sing}(\mathscr{L}_T, b)) = \operatorname{Sing}(\mathscr{L}_{\bar{T}}, b)$$

in order to fulfill the conditions of 2.1.1. But this is 2.4.4 (*b*) (i).

2.4.6. Definition. — A grove G over a pair (W, E) is an *idealistic situation* if there exists a couple $(\mathscr{L}, b)$ $(\mathscr{L} \subset \mathcal{O}_W)$ such that G is the grove of permissible trees over $(\mathscr{L}, b)$ (2.4.5.).

2.4.6.1. *Main Example.* — Let the setup be that of 2.2.1 where we started with a pair (W, E) and a fixed hypersurface $H (\subset W)$, and b denoted the highest possible order of H at a point. Recall that $F(b, H) = \{x \in H/\text{multipicity of H at } x = \operatorname{mult}(H, x) \geqq b\}$.

Now let $\mathscr{L}(H)$ denote the locally principal sheaf of ideals defining H, and check that $F(b, H)=\operatorname{Sing}(\mathscr{L}(H), b)$. Moreover, one can also check that the the Hilbert-Samuel grove attached to H and s in 2.2.1 is the grove defined by the couple $(\mathscr{L}(H), b)$ over the pair (W, E). So this Hilbert-Samuel grove is an idealistic situation.

One of the strongest results of resolution of singularities, which we shall mention latter in 7.5, is that of "local idealistic presentations" which states that for $X(\subset W)$ of any codimension and for p as in 2.2.2 (the heighest possible Hilbert-Samuel function at points of X) then the Hilbert-Samuel grove defined by (W, X, p), *is locally* an idealistic situation.

2.4.7. Remark. — $(\mathscr{L}, b)$ is not uniquely determined by this condition as shown in 2.4.4. Now consider all possible couples $(\mathscr{L}_\lambda, b_\lambda)$ on a pair (W, E). One can check that the relation: $(\mathscr{L}_\lambda, b_\lambda) \sim (\mathscr{L}_\beta, b_\beta)$ *if they define the same grove over* (W, E), is an equivalence relation. An equivalence class in this sence is caled an *idealistic exponent* (H2], 1. Def. 3).

2.4.8. Definition. — If G_i $i=1,2$ are groves over (W, E), set $G_1 \cap G_2$ (formally) as the trees that belong to both groves.

2.4.9. Proposition. — *Given G_i $i=1,2$ and (W, E) as before, if each G_i is an idealistic situation defined by a couple $(\mathscr{L}_i, b_i)$, then $G_1 \cap G_2$ is naturally the idealistic situation defined by the couple*

$$(\langle \mathscr{L}_1^{b_2}, \mathscr{L}_2^{b_1} \rangle, b_1 \,.\, b_2)$$

Proof. — Applying 2.4.4. we know that the grove defined by $(\mathscr{L}_1, b_1)$ is the same as that defind by $(\mathscr{L}_1^{b_2}, b_1 \,.\, b_2)$ and the same holds for $(\mathscr{L}_2, b_2)$ and $(\mathscr{L}_2^{b_1}, b_1 \,.\, b_2)$. The statement follows easily from these remarks.

2.4.10. Remark. — Let $Z \subsetneq W$ be a closed immersion of smooth schemes and let $\mathscr{A} \subset \mathscr{O}_W$ be the sheaf of ideals defining Z. Clearly Sing $(\mathscr{A}, 1)=Z$ and one can check that:

(i) for any map π: $W_1 \to W$ as in 2.4.3.1 (with center $C \subset Z$), if $Z_1 (\subset W_1)$ denotes the strict transform of Z and $(\mathscr{A}', 1)$ is the transform of $(\mathscr{A}, 1)$, then $Z_1 = \operatorname{Sing}(\mathscr{A}', 1)$

(ii) for any smooth map f: $W_1 \to W$ if $(\mathscr{A}', 1)$ is the transform of $(\mathscr{A}, 1)$ (2.4.3.2), then $\operatorname{Sing}(\mathscr{A}', 1)=f^{-1}(Z)$.

2.4.11. Definition. — Let Z and $\mathscr{A}$ be as in 2.4.10, Z is said to have *maximal contact* with a grove G over a pair (W, E) if the idealistic situation G_1 defined by the couple $(\mathscr{A}, 1)$ is such that $G_1 \cap G$ (2.4.8) is all G. Or equivalently if any tree of G is permissible for $(\mathscr{A}, 1)$ (2.4.3.4).

2.4.12. Remark. — With the setup as in 2.4.11: $\operatorname{Sing} G \subset Z = \operatorname{Sing}(\mathscr{A}, 1)$, and for any tree $T \in G$: $\operatorname{Sing} G_T \subset Z_T = \operatorname{Sing}(\mathscr{A}_T, 1)$ (2.4.10).

In fact any closed point $x \in \operatorname{Sing} G$ is a centre for a permissible transformation, so $x \in Z = \operatorname{Sing}(\mathscr{A}, 1)$. The same argument proves that $\operatorname{Sing} G_T \subset Z_T$.

2.5. TRANSFORMATIONS AND RESTRICTIONS OF GROVES. — Let G be a grove over a pair (W, E) and fix a tree T of G of length $k-1$.

$$T: \quad (W, E_1, C_1) \ldots \leftarrow (W_{k-1}, E_{k-1}, C_{k-1}) \leftarrow (W_k, E_k).$$

2.5.1. DEFINITION. — Given $T \in G$ as above and a tree of length zero T_0: $(W_k, E_k) \leftarrow (W'_k, E'_k)$ the concatenation of T with T_0 is a particular case of a restriction of T (1.7 (*b*)) which we call a *terminal restriction* of T.

2.5.2. PROPOSITION. — *Fix* T *and the notation as above, then the set of trees* T' of the grove G with the property that the k-truncation of T': $[T']_k$ (1.7 (*a*)) is either T or a terminal restriction of T, define naturally trees on the pair (W_T, E_T) or on a pair (U, E) where $(U, E) \to (W_T, E_T)$ is a tree of length zero. This defines a grove G_T over the pair (W_T, E_T), where $F_{T'} \subset W_{T'}$ (2.1.1) is given as before by $\operatorname{Sing} G_{T'}$ (2.3), in other words:

$$\operatorname{Sing}(G_T)_{T'} = \operatorname{Sing}(G)_{T'}$$

where T' is seen as a tree of G_T in the left term and as a tree of G in the right term.

Proof. — Follows straightforwards from the definition of groves (2.1.1).

2.5.3. DEFINITION. — (*a*) Given a grove G over (W, E) and a tree T of G, denote by G_T the grove on (W_T, E_T) constructed in 2.5.2 and call if the *transform of the grove* G *by the tree* T.

(*b*) if T is a tree of length zero T: $(W, E) \leftarrow (W_1, E_1)$ the transform of G by T is called the *restriction* of G to (W_1, E_1).

2.5.4. EXAMPLE. — Let G be an idealistic situation defined by a comple $(\mathscr{L}, b)$ over a pair (W, E), and T a tree of G (2.4.6).

One can check that G_T as a grove over (W_T, E_T) is the idealistic situation defined by the couple $(\mathscr{L}_T, b)$ (2.4.3.4).

2.6. IMMERSION OF GROVES. — Let Z be a closed and smooth subscheme of a regular scheme W, and let (W, E) be a pair indexed by $\wedge$.

If the elements of E together with Z have normal crossings, then the hypersurfaces of Z_1 (any irreducible component of Z) of the form $H_\lambda \cap Z_1$, also have *normal crossings at* Z_1. So if we assume in addition that $H_\lambda \not\supset Z_1$ for any $\lambda \in \wedge$ and any Z_1 as before, then the conditions of 1.1 are fulfilled by (Z, E_Z) where:

$$E_Z = \{ H_\lambda \cap Z / \lambda \in \wedge \}$$

is also indexed by $\wedge$.

2.6.1. DEFINITION. — If W, Z, E and $\wedge$ are as before we say that $(Z, E_Z) \subsetneq (W, E)$ is an *immersion of pairs*.

2.6.1.1. *Remark*. — Given now a subset $\wedge' \subset \wedge$, one can naturally define $E' (\subset E)$ and $E'_Z (\subset E_Z)$ by choosing $\lambda \in \wedge'$. In the conditions of 2.6 the immersion $(Z, E_Z) \subsetneq (W, E)$ induces for any subset $\wedge'$ an immersion of pairs $(Z, E'_Z) \subsetneq (W, E')$.

2.6.2. LEMMA. — *Let* $(Z, E_Z) \subsetneq (W, E)$ *be an immersion of pairs indexed by* $\wedge$, *and* $f: W' \to W$ *a smooth map. Then the fiber product over* W:

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & W' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \underset{i}{\hookrightarrow} & W \end{array}$$

is such that:

(*a*) *i' is a closed immersion*

(*b*) *f' is a smooth map*

(*c*) *i' defines a closed immersion between the transform of* (Z, E_Z) *by f' and the transform of* (W, E) *by f.*

Proof. — Both (*a*) and (*b*) are well known, as for (*c*) recall that we do not change the index set when transforming pairs by smooth maps (1.2.1.B).

2.6.3. LEMMA. — *Let* $(Z, E_Z) \subsetneq (W, E)$ *be an immersion of pairs and* C *a smooth closed and proper subscheme of* Z (*and therefore of* W). *Then:*

(*a*) C *has normal crossings with* E_Z (*at* Z) *if and only if it has normal crossings with* E (*at* W).

(*b*) *Consider the commutative diagram*

$$\begin{array}{ccc} Z_1 & \xrightarrow{i'} & W_1 \\ {\scriptstyle \bar\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \underset{i}{\hookrightarrow} & W \end{array}$$

where the vertical maps are the monoidal transformations on C. *And i' is the inclusion of* Z_1 *in* W_1 *as the strict transform of* Z.

Then the transform (Z_1, E_{Z_1}) *of* (Z, E_Z) (*via* $\bar\pi$) *and the transform* (W_1, E_1) (*via* π) *of* (W, E) *are linked by a closed immersion of pairs*

$$i': \quad (Z_1, E_{Z_1}) \subsetneq (W_1, E_1).$$

Proof. — In the conditions of this Lemma, (*a*) follows from the definition of normal crossings. As for (*b*), the index set of E_{Z_1}, is given by $\wedge \cup \{\beta\}$ (1.2.1.A) and $\bar{H}_\beta \in E_{Z_1}$ is $\bar\pi^{-1}(C)$.

The same holds for E_1, where $H_\beta \in E_1$ is $\pi^{-1}(C)$. But now $\bar{H}_\beta = Z_1 \cap H_\beta$, and the other intersection conditions of normal crossings are preserved after the monoidal transformations.

2.6.4. THEOREM. — *Let* $(Z, E_Z) \xrightarrow{i} (W, E)$ *be an immersion of pairs* (2.6.1) *and* G *a grove over* (Z, E_Z). *Then there is a natural definition of a grove* $i(G)$ *over* (W, E).

Proof. — Set Sing(G) $\subset$ Z (2.3) and identify Sing(G) with the closed set i (Sing(G)) in W. 2.6.2 asserts that any tree T of length zero on (W, E) restricts to a tree of length zero on (Z, E_Z), and 2.6.3 that if T: (W, E, C) $\leftarrow$ (W_1, E_1) is a tree of length one on (W, E), then is restricts to a tree of length one on (Z, E_Z) with the only condition that C $\subset$ i(Sing(G)).

In both cases the setup for G_T, (Z_T, $(E_Z)_T$) and (W_T, E_T) is that of the beginning. The statement follows then by induction on the length of the trees T.

2.6.5. REMARK: If (Z, E_Z) $\subsetneq$ (W, E) and G are as before. Any tree T $\in i$(G) restricts to a tree T on G, defines a closed immersion of pairs i_T: (Z_T, E_T) $\subsetneq$ (W_T, E_T) (2.6.1) and $i_T(G_T) = i(G)_T$ (2.5.3). In particular $i_T(\mathrm{Sing}\, G_T)) = (\mathrm{Sing}\,(i(G))_T)$ (2.3).

2.6.6. THEOREM (idealistic Tchirnhausen). — *Let* G *be an idealistic situation over a pair* (W, E) *and* $\mathscr{A}$ ($\subset \mathcal{O}_W$) *a sheaf of ideals defining a smooth subscheme* Z (*of* W). *Assume that:*

(*a*) Z(=Sing($\mathscr{A}$, 1)) *has maximal contact with* G (2.4.11)

(*b*) x *is a closed point of* Sing($\subset$ W) *at which the inclusion* $(\mathrm{Sing}\, G)_x \subset Z_x$ (2.4.12) *is strict* (*at an open neighborhood of* x), *and* $x \notin H_\lambda$ *for any* H_λ *of* E.

Then, there is an open neighborhood W' *of* x, *so that* $E_{|W'} = \varnothing$ (*the restriction of hypersurfaces of* E *at* W' *is empty*), *and an idealistic situation* $\bar{G}$ *at* $Z' = Z \cap W'$ *so that* $i(\bar{G}) = G$ *via the immersion* i: (Z', $\varnothing$) $\rightarrow$ W', $\varnothing$).

Proof: *See* [H2], (8. Th.5, p 111). The assumption (*b*) of strict inclusion will force the idealistic exponent of $\bar{G}$ on Z to be defined by a ($\mathscr{L}$, b) (on Z) where $\mathscr{L}_y \neq 0\ \forall y \in Z$, a condition required for couples (2.4.1).

2.7. THE INDEX SET OF A GROVE. — We have defined at 1.1.1 the notion of an index set $\wedge$ to indicate subschemes $H_\lambda \subset W$ ($\lambda \in \wedge$) for a given pair (W, E).

We do not exclude the existence of $\lambda \in \wedge$ so that $H_\lambda = \varnothing$. On the other hand definition 1.2.1(B) forces us to consider this case.

With the notation as in 1.5.1 we fix a tree T permissible for (W_1, E_1) (with index set $\wedge$); $\wedge_T$ was defined as the union of $\wedge$ with the set $\{\beta(1), \ldots, \beta(s)\}$ corresponding to the trees of length one in the concatenation.

Define $\wedge\{T\} = \{\beta(1), \ldots, \beta(s)\}$, so $\wedge\{T\}$ ($\subset \wedge_T$) consist of the indices introduced by T. And now define at any point $x \in W_T$:

$$\wedge_T(x) = \{\lambda \in \wedge_T / x \in H_\lambda\}; \quad \wedge\{T\}(x) = \{\lambda \in \wedge\{T\} / x \in H_\lambda\}$$

and

$$E_T(x) = \{H_\lambda / \lambda \in \wedge_T(x)\} \, (= H_\lambda \in E_T / x \in H_\lambda\})$$

If $\bar{T}$ is a restriction of T then $\wedge_{\bar{T}}(x) = \wedge_T(r_T(x))$ and $\wedge\{\bar{T}\}(x) = \wedge\{T\}(r_T(x))$ for r_T: $W_{\bar{T}} \rightarrow W_T$ as 1.7(*b*) and any $x \in W_T$. Indeed, this follows from the existence of the trees r_i: ($\bar{W}_i$, $\bar{E}_i$) $\rightarrow$ (W_i, E_i) given also as in 1.7 together with the invariance of the index set for transformation of type (B) (1.2.1(B)).

Another nice property is that the set $\wedge \{T\}$ and its subsets $\wedge \{T\}(x)$ for $x \in W_T$ are naturally ordered ("$\beta(i) \leqq \beta(j)$" if $i \leqq j$) and this order is preserved by the equality $\wedge \{\bar{T}\}(x) = \wedge \{T\}(r_T(x))$ mentioned before.

Finally for any tree T and any hypersurface $H_\lambda \in E_T$, $\lambda \in \wedge_T(x)$, there is valuation ring $\mathcal{O}_{W_T, H_\lambda}$ (localization of $\mathcal{O}_{W_T, x}$). Now if f: $(W_2, E_2) \to (W_1, E_1)$ is a tree of length zero, for any $x \in W_2$: $\Lambda_2(x) = \Lambda_1(f(x))$; and any $H_\lambda \in E_1(f(x))$ induces a unique arrow reversing injection of the quotient fields of $\mathcal{O}_{W_1, f(x)}$ in that of $\mathcal{O}_{W_2, x}$ and an inclusion $\mathcal{O}_{W_1, H_\lambda} \subsetneq \mathcal{O}_{W_2, f^{-1}(H_\lambda)}$ (of valuation rings).

These observation suggest the existence of a *universal set of indices* for a pair (W, E), say: $\wedge_U$ so that for any T and $x \in W_T$: $\wedge_T(x) \subset \wedge_U$. In fact such $\wedge_U$ can be taken as the set of valuation rings of a universal extension field (in this sence) of $k(W)$ (the quotient field of W if it is irreducible) (*see* [H2] 9, 7.1).

3. The inductive theorems

In what follows $\mathscr{L}$ denotes a coherent sheaf of ideals at $\mathcal{O}_W$ so that $\mathscr{L}_x \neq 0$ at any $x \in W$. We fix the notation as that in 2.4.2.

3.1. Lemma. — *Let x be a closed point at* W. *If $\mathscr{L}_x$ is a proper ideal at $\mathcal{O}_{W, x}$ and* $b \in \mathbb{Z} > 0$, *then the following conditions are equivalent*

(*a*) $\upsilon_x(\mathscr{L}_x) = b$ (υ_x *denotes the* order *of an ideal at the local regular ring* $\mathcal{O}_{W, x}$).

(*b*) $\upsilon_x(\Delta \mathscr{L}_x) = b - 1$.

Proof. — ([H.2] 8. Lemma 5.4).

3.2. Corollary. — *The following are equivalent*

(*a*) *the maximal order achieved by $\mathscr{L}$ at points of* W *is* b,

(*b*) $\Delta^{b-1}(\mathscr{L})$ *is a proper sheaf of ideals and* $\Delta^b(\mathscr{L}) = \mathcal{O}_W$,

(*c*) *the maximal order achieved by* $\Delta^{b-1}(\mathscr{L})$ *at points of* W *is one.*

Now we can formulate a major theorem for inductive resolutions. The hypothesis on the caracteristic (zero) of the field k was necessary both 3.1 and 3.2.

3.3.Theorem(Giraud-Hironaka). *Let* (W, E) *and* $\mathscr{L} \subset \mathcal{O}_W$ *be as before. Assume that*

3.3.0. $$b = \operatorname{Max}\{\upsilon_x(\mathscr{L}_x)/x \in W\}$$

Let $\mathscr{A}(\subset \mathcal{O}_W)$ *be a sheaf of ideals which is regular of height one* (i.e. *locally principal and defines a smooth hypersurface*), *and* $\mathscr{A} \subset \Delta^{b-1}(\mathscr{L})$. *Then:*

1. $\operatorname{Sing}(\mathscr{A}, 1) \supset \operatorname{Sing}(\mathscr{L}\, b)$,

2. *if* T *is a tree on* (W, E) *permissible for* ($\mathscr{L}$, b) (2.4.3) *then* T *is also permissible for* ($\mathscr{A}$, 1),

3. (*stability of* 1) *let* T *be as in* 2, *set* ($\mathscr{L}_T$, b) *and* ($\mathscr{A}_T$, 1) *the transforms of* ($\mathscr{L}$, b) *and* ($\mathscr{A}$, 1) *by* T (2.4.3), *then* $\mathscr{A}_T \subset \Delta^{b-1}(\mathscr{L}_T)$.

Proof. – It is clear from 3.1 that (set theoretically) $\operatorname{Sing}(\mathscr{L}, b) = V(\Delta^{b-1}(\mathscr{L}))$ [the zeros of $\Delta^{b-1}(\mathscr{L})$], now 1 is clear since $\operatorname{Sing}(\mathscr{A}, 1) = V(\mathscr{A})$ and $\mathscr{A} \subset \Delta^{b-1}(\mathscr{L})$.

As for 2 and 3 both results are a major simplification in the theory, introduced by Giraud, with an approach to maximal contact via analytic methods. For a proof *see* [H.2] (theorem 5, p. 114).

3.3.1. COROLLARY. – *Either* $\operatorname{Sing}(\mathscr{L}_T, b) = \varnothing$ *or* $\mathscr{L}_T$ *is a sheaf of ideals of maximal order equal to b.*

Proof. – In fact if $\operatorname{Sing}(\mathscr{L}_T, b)$ is not empty, then 3.3, 1 asserts that $\mathscr{A}_T \subset \Delta^{b-1}(\mathscr{L}_T)$, now one can check that $\mathscr{A}_T$ a sheaf of ideals defining a smooth hypersurface of W_T ($\mathscr{A}_T$ is a sheaf of regular ideals of height one as shown in 2.4.10).

Now we come to our main inductive theorem. Recall 2.4.11 for the notion of maximal contact of a smooth subscheme with a grove.

3.4. THEOREM (of maximal contact). – *Let* $\mathscr{L} \subset \mathscr{O}_W$ *and b be as in* 3.3.0. *At each closed point* $x \in \operatorname{Sing}(\mathscr{L} b)$ *there is a* neighborhood W′ *of x so that the restriction* $(\mathscr{L}', b)$ $(\mathscr{L}' = \mathscr{L}/W')$ *satisfies:*

(*a*) *at* W′ *there is a sheaf of ideals* $\mathscr{A} \subset \Delta^{b-1}(\mathscr{L}')$ *as in* 3.3.

$Z = \operatorname{Sing}(\mathscr{A}, 1) (\subset W')$ *is a smooth hypersurface. Now*

(*b*) *If the inclusion* $\operatorname{Sing}(\mathscr{L}', b) \subset Z$ (3.3.1) *is strict locally at x, then there* exists *a couple* $(\mathscr{L}'', b')$, $\mathscr{L}'' \subset \mathscr{O}_Z$, *defining an idealistic situation* G *over the pair* $(Z, \varnothing)$ *such that the grove* $i(G)$ *induced by the immersion of pairs* $(Z, \varnothing) \subsetneq (W', \varnothing)$ (2.6.4) *is the grove defined by* $(\mathscr{L}', b)$ *over* $(W', \varnothing)$ (2.4.6).

Proof: Is a combination of 3.3 and 2.6.6.

3.4.1. EXAMPLE (*analytic case*). – $f = Y^b + a_2 Y^{b-2} + \ldots + a_b \in \mathbb{C}\{Y, X_1, \ldots, X_n\}$ with $a_i \in \mathbb{C}\{X\}$ of order $\geqq i$. $\mathscr{L} = \langle f \rangle$, $\mathscr{A} = \langle Y \rangle$ (both in $\mathbb{C}\{Y, X\}$); $\mathscr{L}'' = \Sigma \langle a_i^{b!/i} \rangle$ (sum of ideals in $\mathbb{C}\{X\}$) and $b'' = b!$.

3.5. REMARK. – $Z = \operatorname{Sing}(\mathscr{A}, 1)$ has maximal contact with the idealistic situation defined by $(\mathscr{L}', b)$ on $(W', \varnothing)$ (2.4.11).

3.6. COROLLARY. – *In the conditions of* 3.3.0 *the following are equivalent:*

(i) $\dim_x(\operatorname{Sing}(\mathscr{L}, b)) = \dim_x(W)$-1 ($\dim_x$ = *dimension locally at x*).

(ii) $\operatorname{Sing}(\mathscr{A}, 1) = \operatorname{Sing}(\mathscr{L}, b)$ *locally at x*

(iii) $\Delta^{b-1}(\mathscr{L}) = \mathscr{A}$ *at* $\mathscr{O}_{W, x}$.

Proof. – Recall from 2.4.2 that $\operatorname{Sing}(\mathscr{L}, b) = \vee(\Delta^{b-1}(\mathscr{L})) \subset W$. We assume here that $\mathscr{A} \subset \Delta^{b-1}(\mathscr{L})$ and $\mathscr{A}$ is a regular ideal.

3.7. DEFINITION. – With the setup as before define $R(1)(\mathscr{L}, b)$ or simply:

$$R(1) = \{ x \in \operatorname{Sing}(\mathscr{L}, b) / \dim_x(\operatorname{Sing}(\mathscr{L}, b)) = \dim_x(W) - 1 \}.$$

3.8. REMARK. – R(1) is a smooth scheme and a *union of connected components of* $\operatorname{Sing}(\mathscr{L}, b)$ as follows from 1) of 3.3 and 3.6.

3.9. PROPOSITION. — *With the setup as in* 3.3, *so that* 3.3.0 *holds:*

(*a*) *Let* $f: W' \to W$ *be a smooth map,* $(W', E') \to (W, E)$ *the induced tree of length zero and* $(\mathscr{L}', b)$ *the transform of* $(\mathscr{L}, b)$ (2.4.3.2). *Then* $R(1)(\mathscr{L}', b) = f^{-1}(R(1)(\mathscr{L}, b))$

(*b*) *If* $\pi : W_1 \to W$ *is a monoidal transformation as in* 2.4.3.1) [permissible for $(\mathscr{L}, b)$], *and if* $(\mathscr{L}_1, b)$ *is the transform of* $(\mathscr{L}, b)$ *at* W_1. *Then* $R(1)(\mathscr{L}_1, b)$ *is the* strict transform *of* $R(1)(\mathscr{L} b)$.

Proof. — (*a*) is the stability of the codimension by pullbacks of smooth maps and (*b*) is 3.6 (iii) and 3.3, 3.

4. Functions on groves

4.1. DEFINITION. — Let G be a grove over a pair (W, E) (2.1.1) and $(I, \leqq)$ a totally ordered set. A *function f on the grove* G *with values at* I will mean a function

$$f(G_T) \text{ or } f_T : \operatorname{Sing} G_T (\subset W_T) \to I$$

for any tree T of the grove (2.3), subject to the following conditions (*a*) and (*b*):

(*a*) f_T is locally finite (that locally it takes only finite different values).

Let $\tilde{T}$ be a restriction of T, $(W_T, E_T) \overset{r_T}{\to} (W_{\tilde{T}}, E_{\tilde{T}})$ the tree of length zero at the final transforms as defined in 1.7 (*b*) and $r_T^{-1}(\operatorname{Sing} G_T) = \operatorname{Sing} G_{\tilde{T}}$ as in 2.3.1 (*b*)

(*b*) $f_{\tilde{T}(x) = f_T}(r_T(x))$, $\forall x \in \operatorname{Sing} G_{\tilde{T}}$.

4.2. EXAMPLE. — Let G be a grove over a pair (W, E). For any $T \in G$ set:
codim : $\operatorname{Sing}(G_T) \to \mathbb{Z}$, $\operatorname{codim}(x) = \operatorname{codim}_x(\operatorname{Sing} G_T)$ (codim_x : codimension locally at x).

To check that this is function apply 2.3.1 (*b*) and the stability of codimension by pullbacks of smooth maps.

4.3. DEFINITION. — With the notation as in 4.1, for any $T \in G$ set:

(i) $\operatorname{Max} f_T$ (or $\operatorname{Max} f(G_T)$) = maximal value achieved by f_T alongs points of Sing G_T.

(ii) $\underline{\operatorname{Max}} f_T$ (or $\underline{\operatorname{Max}} f G_T)$) $= \{ x \in \operatorname{Sing} G_T / f_T(x) = \operatorname{Max} f_T \} \subset \operatorname{Sing}(G_T)$.

4.4. DEFINITION. — A function f from G to I is said to be *upper semicontinuous* if in addition it satifies:

(*c*) for any tree T of G the function $f_T : \operatorname{Sing} G_T \to I$ is uppersemicontinuous (*i.e.* for any $\alpha \in I$ the set $\{ x \in \operatorname{Sing} G_T / f_T(x) \geqq \alpha \}$ is closed).

4.5. PROPOSITION. — *Let* G *be a grove over* (W, E), T *a tree of* G *and f a function from* G *to* I (4.1), *f defines naturally a function f′ from the grove* G_T *over* (W_T, E_T) (2.5.2) *to* I.

Proof. — Recall from 2.5.2 that G_T as a grove over (W_T, E_T) was defined in such a way that for a tree T′ on G_T : $\operatorname{Sing}(G_T)_{T'} = \operatorname{Sing}(G)_{T'}$. In particular f defines for any such T′ a function

$$f_{T'}: \quad \operatorname{Sing}(G_T)_{T'} \rightarrow I.$$

Setting $f'_{T'} = f_{T'}$ one can check that (*a*) (*b*) and eventually (*c*) of (4.1) and (4.4) are fulfilled.

4.6. PROPOSITION. – *Let* $(Z, E_Z) \rightarrow (W, E)$ *be a closed immersion of pairs with index set* $\wedge$ (2.6.1). *Let* G *be a grove over* (Z, E_Z) *and* $i(G)$ *the induced grove* (W, E) (2.6.4). *If f is a function from* G *to* I *then f defines naturally a function from* $i(G)$ *to* I.

Proof: Any tree $T \in i(G)$ induces a *closed immersion* $Z_T \overset{i_T}{\hookrightarrow} W_T$ so that $i_T(\operatorname{Sing} G_T) = \operatorname{Sing}(i(G))_T)$ (2.6.5). The proof runs nows as in 4.5.

4.7. DEFINITION. – A function f from a grove G over a pair (W, E) to an ordered set (I, $\leqq$) (4.1) is said to be a *strong function* if both (A) and (B) hold.

(A) f is upper semicontinuous (4.4).

Let $T \in G$ and $T_1 = (W_T, E_T, C) \leftarrow (W_1, E_1)$ a tree of length one over (W_T, E_T), with the condition that $C \subset \underline{\operatorname{Max}} f_T (\subset \operatorname{Sing} G_T)$. Then the concatenation T′ of T with T_1 belongs to G (2.1.1.2) (iii). Now we state the next condition:

(B) Given T, T_1 and T′ as before:

$$f_T(\pi(x)) \geqq f_{T'}(x) \text{ (at I) for any } x \in \operatorname{Sing} G_{T'}.$$

4.8. REMARK. – Let f be a function from a grove G to I (do not assume that f is upper semicontinuous or strong). If T′ is the concatenation of $T (\in G)$ with a tree T_1 of length one, say:

$$T_1 : (W_T, E_T, C) \overset{\pi}{\leftarrow} (W_1, E_1)$$

so that $C \subset \operatorname{Sing} G_{T'}$. Then $T' \in G$ and for any $x \in \operatorname{Sing} G_{T'} - \pi_1^{-1}(C)$:

(i) $\pi(x) \in \operatorname{Sing} G_T$ (in particular $\pi(\operatorname{Sing} G_{T'}) \subset \operatorname{Sing} G_T$), and (ii) $f_{T'}(x) = f_T(\pi(x))$.

In fact in this case $\pi(x) \notin C$ and there are open restrictions of T and T_1 (1.7 (*b*)) say $\tilde{T}$, $\tilde{T}_1$, so that $\pi(x) \in \operatorname{Sing} G_{\tilde{T}}$ and $\tilde{T}_1$ becomes a tree of length zero. It suffices to take the open and terminal restriction $(W_T, E_T) \leftarrow (U, E_U)$ where $U = W_T - C$ (1.7 (*b*), 2.5.1).

4.9. Let f be a strong function from a grove G over (W, E) to an ordered set (I, $\leqq$) (4.7). Recall that f is upper semicontinuous so that for any tree $T \in G$ the set $\underline{\operatorname{Max}} f_T (\subset \operatorname{Sing} G_T)$ is closed (4.3 (ii)).

4.9.1. DEFINITION. – Let f be a strong function as before let G_f denotes all those trees T of G such that if T is the concatenation of trees $T_i : (W_i, E_i, C_i) \leftarrow (W_{i+1}, E_{i+1})$ of length one $i = 1, 2, \ldots, s$ (1.4.1), then:

$$C_k \subset \underline{\operatorname{Max}} f_{[T]_k} (\subset \operatorname{Sing} G_{[T]_k}) \qquad \text{For} \quad k = 1, \ldots, s.$$

4.10 Proposition. — *Let* G *be a grove over* (W, E) *and* f *an upper semicontinuous function* (4.4) [*a strong function* (4.7)]. *Then for any tree* $T \in G$, *the induced function* f' *from* G_T *to* I (see 4.5) *is an upper semicontinuous functions* (*is a strong function*).

Proof. — Follows from two facts:

1. any tree $T' \in G_T$ has an interpretation as a tree $T' \in G$ and moreover $\operatorname{Sing}(G_T)_{T'} = \operatorname{Sing} G_{T'}$ (2.5.2) and

2. $f'_{T'} : \operatorname{Sing}(G_T)_{T'} \to I$ is defined as $f_{T'} : \operatorname{Sing} G_{T'} \to I$ (4.5).

4.11. Proposition. — *Let* $(Z, E_Z) \to (W, E)$ *be a closed immersion of pairs and* G *and* $i(G)$ *as in* 2.6.4.

If f *is an upper semicontinuous function* (*a strong function*) *from* G *to* I *then* f *induces an upper semicontinuous function* (*a strong function*) *from* $i(G)$ *to* I.

Moreover if f *is a strong function and* f' *is the induced strong function on* $i(G)$ *then any tree* $T \in i(G)_{f'}$ *restricts to a tree* $T \in G_f$.

Proof. — Follows as that of 4.10 replacing results of 2.5.2 by 2.6.5 and 4.5 by 4.6.

4.12. *Remark.* — Let G be a grove over (W, E) and f a strong function from G to $(I, \leqq)$. If $T \in G_f$ (4.9.1) is a tree of length $s \geqq 1$, say

$$T: \quad (W_1, E_1, C_1) \leftarrow \dots \leftarrow (W_{S+1}, E_{S+1})$$

then $\operatorname{Max} f_{[T]_k} \geqq \operatorname{Max} f_{[T]_{k+1}}$.
In fact $T \in G_f$ if and only if $C_k \subset \underline{\operatorname{Max}} f_{[T]_k}$ (4.3) so the assertion follows from 4.7 (B).

4.13 Definition. — With the conditions and notation as in 4.12 let the *birth* or *f-birth* of T be the smallest index k, such that $\operatorname{Max} f_{[T]_k} = \operatorname{Max} f_T$

4.14. Proposition. — *If* f *is a strong function on a grove* G *over* (W, E) *and the maximal value achieved by* f *at* $\operatorname{Sing} G (\subset W)$ *is:* $\operatorname{Max} f_{id} = \alpha$, *set* $G_{f,\alpha}$ *as the trees* T *of* G_f *such that* $\operatorname{Max} f_{[T]_k} = \alpha$ *for* $k = 1, \dots, s$ (*notation as in* 4.12). *Then* $G_{f,\alpha}$ *is a grove over* (W, E) *setting for any* $T \in G_{f,\alpha}$: $\operatorname{Sing}(G_{f,\alpha})_T = \{ x \in \operatorname{Sing} G_T / f_T(x) = \alpha \}$.

In particular, if $\operatorname{Max} f_T < \alpha$ *then* $\operatorname{Sing}(G_{f,\alpha})_T = \emptyset$.

Proof. — Since f is a function on the grove, one can check that all conditions of 2.1.1 are fulfilled for trees $T \in G_{f,\alpha}$ setting $F_T = \underline{\operatorname{Max}} f_T$ with the additional condition that $\operatorname{Max} f_T = \alpha$.

5. Function on idealistic situations

5.1. This section is devoted to the functions introduced in [VI] and their behavior on the idealistic situations. We emphasize here two facts: (i) that *all the functions derive from the function* ord( ) introduced in [H2] and (ii) *the value of these functions at a point are expressible in terms of the grove*. The strength of both condition will show up in 6.

In this section $\mathbb{A}^1$ denotes the affine line over k ($\mathbb{A}^1 = \mathrm{Spec}(k[T])$).

5.1.1. Proposition. — *Let* G *be the idealistic situation defined by* $(\mathscr{L}, b)$ *on* (W, E) (2.4), *and* $x \in \mathrm{Sing}\, G$ *a closed point*

(*a*) *the rational number* $\upsilon_x(\mathscr{L}_x)/b$ *is expressible in terms of the grove* (*so if* $(\mathscr{L}, b) \sim (\mathscr{L}', b')$ (2.4.7) *then* $\upsilon_x(\mathscr{L}_x)/b = \upsilon_x(\mathscr{L}'_x)/b'$).

(*b*) *the trees of* G *involved in this expression are defined as a sequence of* monoidal transformations *over the smooth scheme* $U \times \mathbb{A}^1_k$, *the sequence being independent of the neighborhood* U *of* x *and permissible for the restricted idealistic situation.*

Proof. — We sketch the main ideals and leave [H2] 2. Prop 8 (p. 68) for details.

First set $W_0 = W \times \mathbb{A}^1_k$ and $f : W_0 \to W$ the projection. Define: $L_0 = f^{-1}(x) \simeq \mathbb{A}^1_k$ and $x_0 = O \in L_0$.

Suppose that for some index $k : x_k$, L_k and W_k have been defined, then set: $\pi_k : W_{k+1} \to W_k$ the quadratic transformation at x_k, L_{k+1} the strict transform of L_k, H_{k+1} the exceptional locus of π_k, and $x_{k+1} = L_{k+1} \cap H_{k+1}$.

f induces a tree of length zero: $f : (W_0, E_0) \to (W, E)$ with transforms $(\mathscr{L}, b)$ to $(\mathscr{L}_0, b)$ (2.4.3.2). Now set: $S_N : (W_0, E_0, x_0) \leftarrow \ldots (W_{N-1}, E_{N-1}, x_{N-1}) \leftarrow (W_N, E_N)$.

S_N is a tree of the grove for any N. Moreover if $\upsilon_x(\mathscr{L}_x) = m$ ($\geqq b$ since $x \in \mathrm{Sing}\, G$) and $(\mathscr{L}_k, b)$ (*a* couple on (W_k, E_k)) is the transform of $(\mathscr{L}_{k-1}, b)$, one can check that:

$$\upsilon_{x_k}(\mathscr{L}_k) = m + k(m-b)$$

In particular $\upsilon_{x_{N-1}}(\mathscr{L}_{N-1}) = m + (N-1)(m-b)$ and therefore the order of $\mathscr{L}_N$ at the generic point of H_N is $m + (N-1)(m-b) - b$ (2.4.3.1) *i.e.*:

$$\upsilon_{H_N}(\mathscr{L}_N) = (m-b) + (N-1)(m-b) = N(m-b)$$

Denote by $S_1(N, \beta)$ the concatenation of S_N with β monoidal transformations with center (always) at the *hypersurface* H_N. The monoidal transformation of W_N with center H_N is the identity map but still the *transform of the pair* $(\mathscr{L}_N, b)$ undergoes a change.

Now check that $S_1(N, \beta)$ is a tree of the grove if and only if

$$\beta \leqq \left[\frac{N.(m-b)}{b} \right]$$

([ ] denotes integral part). The equation in brackets is a *linear equation on* N *with rational coefficients*. Since the integral part of this linear equation is determined by the grove *for any* N, the equation itself is determined by the grove, in particular the slope which is:

$$(m-b)/b = (\upsilon_x(\mathscr{L}_x)/b) - 1.$$

5.2. Definition. — With the setup as before we denote:

$$\mathrm{ord}(G)(x) = \upsilon_x(\mathscr{L}_x)/b \; (\in \mathbb{Q} \geqq 1) \; \forall x \in \mathrm{Sing}(G)$$

5.3. COROLLARY. – *Let* $(Z_i, \bar{E}_i) \overset{j_i}{\to} (W, E)$ $i=1, 2$ *be two immersion of pairs. And let* G_i *be an idealistic situation over* $(Z_i, \bar{E}_i)$ *such that* $j_1(G)=j_2(G_2)=G$ *(they induce the same grove over* (W, E)) (2.6.4).

If $\dim_x(Z_1)=\dim_x(Z_2)$ *at a closed point* $x \in \operatorname{Sing} G$ $(=j_i(\operatorname{Sing} G_i)$ (2.6.4)), *then*

$$\operatorname{ord}(G_1)(x)=\operatorname{ord}(G_2)(x) \qquad (\in \mathbb{Q})$$

where $\operatorname{ord}(G_i)$ *is defined at* x *as* $\operatorname{ord}(G_i)(j_i^{-1}(x))$ (4.6).

Proof. – The value $\operatorname{ord}(G_i)(x)$ was given in terms of trees $S(N, \beta)$ which were trees over $Z_i \times \mathbb{A}^1$. The immersions j_i extend to immersions $(Z_i \times \mathbb{A}^1, E_i) \to (W \times \mathbb{A}^1, E^1)$ (2.6.2).

One can check that a tree $S(N, \beta)$ is a tree of G_i if and only if it is a tree of G. In particular $S(N, \beta) \in G_1$ if an only if $S(N, \beta) \in G_2$. In fact the equality of the dimensions at x asserts that a *hypersurface* of the transform of $Z_1 \times \mathbb{A}^1$ by S_N corresponds via the other immersion to *hypersurface* at the transform of $Z_2 \times \mathbb{A}^1$, and of course the converse also holds. Therefore $\operatorname{ord}(G_1)((j_1)^{-1}(x))=\operatorname{ord}(G_2)((j_2)^{-1}(x))$.

5.4. Let (W, E) be a pair, $(\mathscr{L}, b)$ a couple on W (2.4.1), C a smooth center included in $\operatorname{Sing}(\mathscr{L} b)$ and $\pi_1 : W_1 \to W$ monoidal transformation with center C. With notation as in 2.4.3.1:

$$\mathscr{L}\mathcal{O}_{W_1}=\mathscr{I}^b . \mathscr{L}'$$

Now let C_1 be an irreducible component of C with generic point $x_1 \in W$. $C_1 \subset C \subset \operatorname{Sing}(\mathscr{L}, b)$ so $b^1=\upsilon_{x_1}(\mathscr{L}_{x_1}) \geqq b$.

At any closed point $y \in \operatorname{Sing}(\mathscr{L}', b) \subset W_1$ such that $\pi_1(y) \in C_1$ there is an expression of the form

$$\mathscr{L}'_y=\mathscr{I}^{b^1-b} . \bar{\mathscr{L}}_y \quad (\text{at } \mathcal{O}_{W_1, y}).$$

Let $\wedge$ be the index set of E [of the pair (W, E)] and $\wedge_1$ that of (W_1, E_1), $\wedge_1=\wedge \cup \{\delta\}$ and $H_\delta=\pi^{-1}(C) \in E_1$ (1.2.1 (A)). Now we express formaly (at $\mathcal{O}_{W_1}$)

$$\mathscr{L}'=(\mathscr{I}_\delta)^{\beta(\delta)} . \bar{\mathscr{L}}'$$

where $\beta(\delta)$ is constantly equal to b^1-b along $\pi^{-1}(C_1)$ with is a connected component of H_δ. In this way $\beta(\delta)$ is defined as a locally constant function on $H_\delta=\pi^{-1}(C)$.

Let T be a tree of length k permissible for $(\mathscr{L}, b)$ (2.4.3.4). Say

$$T: \quad (W_1, E_1, C_1) \leftarrow (W_2, E_2, C_2) \leftarrow \ldots (W_k, E_k, C_k) \overset{\pi}{\leftarrow} (W_{k+1}, E_{k+1})$$

we have defined a pair $(\mathscr{L}_i, b)$ at W_i [the transform of $(\mathscr{L}_1, b)$] and suppose inductively that at W_k a formal expression

$$\mathscr{L}_k=\Pi(\mathscr{I}_\lambda)^{\beta(\lambda)} . \bar{\mathscr{L}}_k$$

is given, such that $\beta(\lambda)$ is a locally constant functions along $H_\lambda \in E_k$. Then we set at $\mathcal{O}_{W_{k+1}}$ an expression for $\mathcal{L}_{k+1}$ of the form

$$\mathcal{L}_{k+1} = (\Pi (\mathcal{I}_\lambda)^{\beta(\lambda)}).(\mathcal{I}_\gamma)^{\beta(\gamma)} \bar{\mathcal{L}}_{k+1}$$

where (i) $\gamma \in \wedge_{k+1}$ is the index corresponding to $\pi^{-1}(C_k)$.

(ii) $\mathcal{I}_\lambda$ the sheaf of ideals defining H'_λ (the strict transform of H_λ) for $\lambda \in \wedge_k \subset \wedge_{k+1}$ (1.2.1 (A)).

(iii) $\mathcal{I}_\gamma$ the sheaf of ideals defining $\pi^{-1}(C_k)$.

Finally if $y \in H_\gamma = \pi^{-1}(C_k)$ and $\pi(y) \in \bar{C}$ an irreducible component of C_k with generic point $x_1 \in \operatorname{Sing}(\mathcal{L}_k, b)$, set:

$$\beta(\gamma)(y) = \sum_{\bar{C} \subset H_\lambda} \beta(\gamma)(\pi(y)) + \upsilon_{x_1}(\bar{\mathcal{L}}_{k+1}) - b$$

and $\beta(\gamma)(y) = 0$ if $y \notin H_\gamma$. It is clear that $\beta(\gamma)$ is a locally constant function.

5.5. Definition. — Let T be a tree of length k of the idealistic situation defined by the couple $(\mathcal{L}, b)$ at (W, E) (2.4). Consider for the transform $(\mathcal{L}_{k+1}, b)$ of $(\mathcal{L}_1, b)$ at W_{k+1}, the expression

$$\mathcal{L}_{k+1} = (\Pi (\mathcal{I}_\lambda)^{\beta(\lambda)}).\bar{\mathcal{L}}_{k+1}$$

as in 5.4. And now define at $\operatorname{Sing}(\mathcal{L}_{k+1}, b)$:

1. for each $\lambda \in \wedge_{k+1}$

$$\alpha(\lambda) : \operatorname{Sing}(\mathcal{L}_{k+1}, b) \to \mathbb{Q}, \qquad \alpha(\lambda)(x) = \frac{\beta(\lambda)(x)}{b} \qquad (x \in \operatorname{Sing}(\mathcal{L}_{k+1}, b))$$

2. w-ord : $\operatorname{Sing}(\mathcal{L}_{k+1}, b) \to \mathbb{Q}$,

$$w\text{-ord}(x) = \frac{\upsilon_x(\bar{\mathcal{L}}_{k+1, x})}{b} \qquad (\in \mathbb{Q} \geqq 0)$$

3. ord: $\operatorname{Sing}(\mathcal{L}_{k+1}, b) \to \mathbb{Q}$,

$$\operatorname{ord}(x) = \frac{\upsilon_x(\mathcal{L}_{k+1, x})}{b} \qquad (\in \mathbb{Q} \geqq 1).$$

5.6. Remark. — $\operatorname{ord}(x) = \Sigma \alpha(\lambda)(x) + \omega\text{-ord}(x)$.

5.7. Theorem. — *Let* G *be the idealistic situation over* (W, E) *defined by the couple* $(\mathcal{L}, b)$ (2.4), *and* $\wedge_U$ *a universal set of indices for* (W, E) (2.7). *Then:*

(i) ω-ord *and* ord *are functions on the grove* G *in the sence of* 4.1, *and so is* $\alpha(\lambda)\ \forall \lambda \in \wedge_U$.

(ii) *These functions depend on the grove* G *and not at the particular pair* $(\mathcal{L}, b)$ *defining* G.

Proof. — We begin with the proof of (ii). Let $(\mathscr{L}', b')$ be another couple at W defining the same grove as $(\mathscr{L}, b)$ over (W, E). Any tree T of length k permissible for one couple is permissible for the other, and $\mathrm{Sing}(\mathscr{L}_T, b)=\mathrm{Sing}(\mathscr{L}'_T, b')=\mathrm{Sing}\, G_T$ (at W_T) (2.5.4 and 2.4.5). Applying 5.1.1 (*a*) both for $(\mathscr{L}_T, b)$ and $(\mathscr{L}'_T, b')$:

$$\frac{\upsilon_x(\mathscr{L}_{T,x})}{b}=\frac{\upsilon_x \mathscr{L}'_{T,x})}{b'}$$

for any $x \in \mathrm{Sing}\, G_T$.

The value of the functions $\alpha(\lambda)$ at $x \in \mathrm{Sin}\, G_T (\subset W_T = W_{k+1})$ depends on the functions defined at $\mathrm{Sing}\, G_k \subset W_k$, and on

$$\frac{\upsilon_{x_1}(\overline{\mathscr{L}}_k)}{b}$$

where x_1 is a generic point of an irreducible component of C_k (5.4). Applying again 5.1.1 (*a*):

$$\frac{\upsilon_{x_1}(\mathscr{L}_k)}{b}=\frac{\upsilon_{x_1}(\mathscr{L}'_k)}{b'}.$$

So an inductive argument together with 5.6 proves (ii).

As for the proof of part (i) let T be a restriction of T. There is a tree of length zero $(W_T, E_T) \overset{r_T}{\leftarrow} W_{\tilde{T}}, E_{\tilde{T}})$ linking the final transforms (1.7 (*b*)). We want to show that conditions 4.1 (*a*) and (*b*) hold for these functions. In the case of ord( ) 4.1 (*a*) is clear and 4.1 (*b*) is proved in 2.4.3,2. But then the conditions also hold for the other functions since they are constructed inductively in terms of the function ord ( ) (5.4).

5.8. Remark (On the good condition: Max ord(G) = 1). — 1. Now that we know that ord is a function on groves we express condition 3.3.0 on acouple $(\mathscr{L}, b)$ (at (W, E)) as Max ord(G) = 1 (4.3(i)) where G denotes the idealistic situation defined by $(\mathscr{L}, b)$ over (W, E).

2. Let $(\mathscr{L}_i, b_i)$ and $G_i\, i=1, 2$ be as in 2.4.9. If condition 3.3.0 holds at $(\mathscr{L}_1, b_1)$, then it also holds at $(\langle(\mathscr{L}_1^{b_2}, \mathscr{L}_2^{b_1}\rangle, b_1 . b_2)$. So $\mathrm{Max\, ord}(G_1)=1$ implies that $\mathrm{Max\, ord}(G_1 \cap G_2)=1$.

5.9. Proposition. — *Let* G *be an idealistic situation over* (W, E) (2.4.6), *then:*

(i) w-ord *is a strong function on* G *with values at* $\mathbb{Q}$ (4.7).

(ii) For a tree $T \in G_{w-\mathrm{ord}}$ (4.9.1) set $w_0 = \mathrm{Max}(w-\mathrm{ord}(G_T))$ (4.3(i)). The grove $(G_T)_{w-\mathrm{ord},\, w_0}$ (4.14) is an idealistic situation over (W_T, E_T). And

$$\mathrm{Sing}((G_T)_{w-\mathrm{ord},\ w_0})=\underline{\mathrm{Max}}\ w\text{-ord}(G_T)=\{x \in \mathrm{Sing}\, G_T / f_T(x)=w_0\}.$$

(iii) if $w_0>0$, then: $\mathrm{Max\ ord}((G_T)_{w-\mathrm{ord},\ w_0})=1$ (5.8).

Proof. — The proof is given in [Vi] (1.17.6 and 1.17.7). There a couple $w(\mathrm{J}, b)$ is constructed so it defines $\mathrm{G}_{w-\mathrm{ord},\, w_0}$ as in idealistic situation (if (J, b) defines G). A tree T is said to be w-permissible in [Vi] (1.17.4) if $\mathrm{T}\in\mathrm{G}_{w-\mathrm{ord}}$. The construction and behavior of $w(\mathrm{J}, b)$ grow from [H1] Remark 3, p. 325.

With our setup if G is defined over (W, E) by $(\mathscr{L}, b)$ and $\mathrm{T}\in\mathrm{G}$, then $\mathscr{L}_{\mathrm{T}}=\Pi(\mathscr{I}_{\lambda})^{\beta(\lambda)}.\bar{\mathscr{L}}_{\mathrm{T}}$. Condition (A) of 4.7 is clear since it reduces to the study of the function $\operatorname{ord}(\bar{\mathscr{L}}_{\mathrm{T}})$. Condition (B) of 4.7 was proved in 3.3 and 3.3.1.

Let $w_0=\max\{\upsilon_x(\bar{\mathscr{L}}_{\mathrm{T}})/b,\ x\in\operatorname{Sing}\mathrm{G}_{\mathrm{T}}=\operatorname{Sing}(\mathscr{L}_{\mathrm{T}}, b)\}$, and define:

$$w(\mathscr{L}_{\mathrm{T}}, b)=(\bar{\mathscr{L}}_{\mathrm{T}}, b\,.\,w_0)\qquad\text{if}\quad w_0\geqq 1,$$

and

$$w(\mathscr{L}_{\mathrm{T}}, b)=(\bar{\mathscr{L}}_{\mathrm{T}}, b\,.\,w_0)\cap(\Pi(\mathscr{I}_{\lambda})^{\beta(\lambda)}, b-b\,.\,w_0)\qquad\text{if}\quad 0\leqq w_0<1$$

in both cases $w(\mathscr{L}_{\mathrm{T}}, b)$ is a couple (*see* 2.4.9) defining $(\mathrm{G}_{\mathrm{T}})_{w-\mathrm{ord},\, w_0}$ as an idealistic situation over $(\mathrm{W}_{\mathrm{T}}, \mathrm{E}_{\mathrm{T}})$. The second part of (ii) is in 4.14.

If $w_0>0$ then one can check that condition 3.3.0 holds for $(\bar{\mathscr{L}}_{\mathrm{T}}, b\,.\,w_0)$ so (iii) follows from 5.8.

The condition $w_0=0$ is equivalent to $\mathscr{L}_{\mathrm{T}}=\Pi(\mathscr{I}_{\lambda})^{\beta(\lambda)}$ (to $\bar{\mathscr{L}}_{\mathrm{T}}=\mathscr{O}_{\mathrm{W}_{\mathrm{T}}}$). This is a very simple case as we shall see latter, so our goal we be to "force w_0 to drop" to the case $w_0=0$.

6. Functions on idealistic spaces

6.1. In this section we define groves by *glueing* other groves and we also want to *glue* functions.

In the last section we introduced and discussed some functions on groves insisting on the fact that the values of these function at each point were expressible in terms of the grove.

The moral of this will be that *function on groves will glue when the groves themselves glue.*

1. (*Etale topology and sheaves*) Recall from 1.6.1. that the fiber product of smooth maps $f_i:\mathrm{W}_i\to\mathrm{W}$ is smooth and defines a smooth map on W, express formally the fiber product of f_1 and f_2 as $\mathrm{W}_1\cap\mathrm{W}_2$. Then $(\mathrm{W}_1\cap\mathrm{W}_2)\cap\mathrm{W}_3=\mathrm{W}_1\cap(\mathrm{W}_2\cap\mathrm{W}_3)$. Formally, one can define a topology on W generated by these "sets".

Smooth maps of relative dimension zero over W form a subclass closed by intersections ([Ha], III, 10). And in our context these are the etale maps (since we assume all maps to be of finite type over a field k of characteristic zero).

This subclass defines the etale topology on W ([Ha], III, 10.1 *d*) and [G4]).

If G is a grove over (W, E) and $f:\mathrm{W}_1\to\mathrm{W}$ is etale, we attach to f a tree of length zero $(\mathrm{W}_1, \mathrm{E}_1)\to(\mathrm{W}, \mathrm{E})$ (1.4) and to $(\mathrm{W}_1, \mathrm{E}_1)$ a set $\mathrm{G}(\mathrm{W}_1, \mathrm{E}_1)$ (*see* 2.1.1). Furthermore our definition of restriction of trees (1.7 (*b*)) is suitable enough so that ultimately the grove G on (W, E) defines a presheaf (of sets) on W with etale topology.

If G is an idealistic situation defined by a couple $(\mathscr{L}, b)$ on (W, E), then *b*) 5.1.1 can also be phrased by saying that the value $\upsilon_x(\mathscr{L}_x)/b\,(=w-\operatorname{ord}(G)(x))$ is expressable in terms of the stalk of the presheaf at x.

2. (*Patching functions*). An etale neighborhood of a point $x \in W$ is a *pair* (W_1, y), $y \in W_1$ and an etale map $f: W_1 \to W$ such that $f(y)=x$. The intersection of two neighborhoods of x, say $f_i : (W_i, x_i) \to (W, x)\ i=1, 2$ is an etale neighborhood of (W, x) and also of (W_i, x_i) for $i=1, 2$. Call it $(\bar{W}, \bar{x})$.

Now let G be a grove on (W, E) (W as before) and let $(W_i, E_i) \to (W, E)$ be the transform of the pair by the smooth maps $f_i\ i=1, 2$ (trees of length zero 1.4), and let G_i denote the "restriction" of G to (W_i, E_i) (2.5.3 (*b*)). Since the diagram of the fiber product

$$\begin{array}{ccc} W_1 & \longleftarrow & \bar{W} \\ \downarrow & & \downarrow \\ W & \longleftarrow & W_2 \end{array}$$

commutes the restriction $\bar{G}$ (of G to $(\bar{W}, \bar{E})$) is also a restriction of the groves G_i.

We shall say that two functions g_i on $G_i\ i=1, 2$ with values on the same ordered set $(I, \leq)$ will *glue* or *patch* if they induce the same function at *any common restriction*, in particular at $\bar{G}$.

Let T be a tree on G, there is a notion of restriction of T to trees $T_i \in G_i$ and $\bar{T}_i \in \bar{G}$ (1.7 (*b*)), together with smooth maps (etale in case both f_i are etale)

$$\begin{array}{ccc} (W_1)_{T_1} & \xleftarrow{\ p_1\ } & \bar{W}_{\bar{T}} \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ W_T & \longleftarrow & (W_2)_{T_2} \end{array}$$

an in these conditions $\operatorname{Sing}\bar{G}_{\bar{T}} = p_i^{-1}(\operatorname{Sing}(G_i)_{T_i})$ (2.3.1). Now set $x \in \operatorname{Sing}(G)_T$, $x_i \in \operatorname{Sing}(G_i)_{T_i}$ and $\bar{x} \in \operatorname{Sing}\bar{G}_{\bar{T}}$ as before. Each g_i is defined at x_i, if we assume that both $g_i' s$ glue we want a natural definition of "$g(x)$".

Since they define the same map at $\operatorname{Sing}\bar{G}_{\bar{T}}\,(\subset \bar{W}_{\bar{T}})$, $(g_1)_{\bar{T}}(\bar{x}) = (g_2)_{\bar{T}}(\bar{x})$. On the other hand both are function of groves so $(g_i)_{T_i}(x_i) = (g_i)_{\bar{T}}(\bar{x})$, $i=1, 2$, therefore $(g_1)_{T_1}(x_1) = (g_2)_{T_2}(x_2)$. Define now $g(x) = g_{T_i}(x_i)$ (for $i=1$ or 2).

For a "covering" $f_i : W_i \to W\,(\bigcup \operatorname{Img} f_i = W)$ and a grove G on (W, E), set G_i the restriction of G via f_i (as before) and assume the existence of functions g_i from G_i to a common set I so that all the $g_i' s$ glue. The associativity of the formal intersections $W_i \cap W_j$ allow a consistent (and unique) definition of a function g from G to I so that the $g_i' s$ are "restrictions" of g, or equivalently that g is obtained by glueing or patching the $g_i' s$.

6.2. Definition. — A grove G over a pair (W, E) is said to be an *d-dimensional idealistic space* if there is:

(i) a system of "charts" $\{(U_\beta, f_\beta)\}_{\beta \in J}$; where $f_\beta : U_\beta \to W$ are etale maps defining a covering of W (in the sense of the etale topology).

(ii) for each $\beta \in J$ a closed immersion of pairs (2.6.1) $(Z_\beta, \bar{E}_\beta) \overset{i_\beta}{\hookrightarrow} (U_\beta, E_\beta)$ where (U_β, E_β) is the transform of (W, E) via f_β (1.2.1.B)). And any irreducible component of the smooth scheme Z_β is of dimension *d*.

(iii) for each $\beta \in J$ an idealistic situation $\bar{G}_\beta$ on $(Z_\beta, \bar{E}_\beta)$ (2.4.6) such that

$$i(\bar{G}_\beta) = G_{U_\beta} \ (2.6.4)$$

where G_{U_β} is the restriction of G defined by the tree of length zero $(U_\beta, E_\beta) \to (W, E)$ (2.5.3 (*b*)).

6.2.1. Remark. — With the setup be as in 6.2, let $\wedge$ denote the index set of (W, E) and fix a subset $\wedge'(\subset \wedge)$. If (W, E′) is defined as in 2.6.1.1, then G induces naturally a *d*-dimensional idealistic space over the pair (W, E′).

6.3. Lemma. — With notation as before, the immersion of pairs $(Z_\beta, \bar{E}_\beta)$ in (U_β, E_β) defines (for each $\beta \in J$) functions codim, $\alpha(\lambda)$, *w*-ord and ord at the grove G_{U_β} over (U_β, E_β) (4.2 and 5.5).

Proof. — Follows from 5.7, 4.6 and 4.2.

6.4. Theorem. — *The functions* codim, $\alpha(\lambda)$, *w*-ord *and* ord *defined at each grove* G_{U_β} (*over* (U_β, E_β)), *glue to functions* (*called* codim, $\alpha(\lambda)$, *w*-ord *and* ord) *at the grove* G *over* (W, E).

Proof. — Set formally $U_\beta \cap U_\delta (\beta, \delta \in J)$ as in 6.1. There is an etale map $f : U_\beta \cap U_\delta \to U_\beta$ and by 2.6.2 a commutative diagram.

$$\begin{array}{ccc} Z'_{\beta_i} & \xrightarrow{j'_i} & U_{\beta_i} \cap U_{\beta_2} \\ {\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ Z_{\beta_i} & \xrightarrow[j_i]{} & U_{\beta_i} \end{array}$$

where the vertical maps are etale and j_i, j'_i are closed immersions. Now 5.3 states exactly that both immersions j'_i $i = 1, 2$ define the same function ord at the restriction of the grove to $U_{\beta_1} \cap U_{\beta_2}$. So they glue to a function on G.

As pointed out in 5.7, the functions $\alpha(\lambda)$ and *w*-ord grow from the function ord so thay will glue as well.

A similar argument can be used to glue the functions codim, however at each U_{β_i} codim reflects the codimension of the singular locus *at* Z_i (not at U_{β_i}).

6.5. PROPOSITION. — *Let* G *be an d-dimensional idealistic space over* (W, E), *then:*

(i) w-ord *is a strong function on* G *with values at* $\mathbb{Q}$ (4.7).

(ii) *Let* T *belong to* $G_{w-\text{ord}}$ (4.9.1) *and set* $w_0 = \text{Max}$ w-ord G_T. *Then* $(G_T)_{w-\text{ord}, w_0}$ *is a d-dimensional idealistic grove over* (W_T, E_T) (4.14), *and*

$$\text{Sing}((G_T)_{w-\text{ord}, w_0} = \underline{\text{Max}}\ w-\text{ord}(G_T) = \{ x \in \text{Sing}\, G_T / f_T(x) = w_0 \}.$$

(iii) *if* $w_0 > 0$, *then:* Max ord $((G_T)_{w-\text{ord}, w_0}) = 1$ (5.8).

Proof. — Is consecuence of 5.9 and 4.11. We only need to check that a function which is obtained by glueing strong functions is again a strong function, which is immediate from the definition (4.7). The second part of (ii) follows from (i) and 4.14.

6.6. REMARK. — Since w-ord is a strong function we can apply 4.12. So let T be a tree of $G_{w-\text{ord}}$ of length s. Then for any $1 \leqq k \leqq s+1$

$$\text{Max}\ w\text{-ord}\, G_{[T]_k} \geqq \text{Max}\ w\text{-ord}\, G_{[T]_{k+1}}$$

and the weighted-order birth of T is now the smallest index k such that $\text{Max}(w\text{-ord})(G_{[T]_k}) = \text{Max}(w\text{-ord})(G_T)$ (4.13).

6.7. *Idealistic spaces of maximal order one.* With the notation as in 6.2, assume that for each index $\beta \in J$, the idealistic situation $\bar{G}_\beta$ over $(Z_\beta, \bar{E}_\beta)$ satisfies the good condition Max ord $(\bar{G}_\beta) = 1$ (5.8).

6.7.1. OBSERVATION. — With the assumptions and notation as above, then Max ord G = 1.(4.3(i)). In fact the function ord(G) is obtained by glueing the functions ord$(\bar{G}_\beta)$. Moreover in this case for any $\beta \in J$ the function ord( ) is constantly equal to one along Sing$(\bar{G}_\beta)$ since in general ord$(x) \in \mathbb{Q} \geqq 1$ (5.5, 3).

6.7.2. DEFINITION. — If G is an d-dimensional idealistic space on (W, E) and Max ord (G) = 1 then set.

$$R(1)\,(\text{or}\ R(1)(G)) = \{ x \in \text{Sing}\, G / \text{codim}(x) = 1 \}$$

(*see* 3.7 and 6.4). Recall that if $x \in \text{Sing}\, G_\beta$ then codim$(x) = 1$ means that codim (Sing $\bar{G}_\beta)_x = 1$ at Z_β (4.2 and 6.4).

6.7.3. REMARK. — R(1)(G) is closed since codim ( ) is upper-semicontinuos, moreover the subscheme R(1)(G)($\subset$Sing G) is a smooth scheme of dimension $d-1$ and a union of connected components of Sing G (*see* 3.8).

6.7.4. THEOREM. — *With the setup and conditions of* 6.7.2:

(i) *If* $f: W_1 \to W$ is a monoidal transformation permissible for (W, E) (1.2.1 (*a*)) *defining a tree of length one* T : (W, E) $\leftarrow$ (W_1, E_1) *of the grove* G *then* R(1)(G_T) *is the strict transform of* R(1)(G).

(ii) *If* $f: W_1 \to W$ *is a smooth map and* T : $(W_1, E_1) \to$ (W, E) *the corresponding tree of length zero. Then* R(1)$(G_T) = f^{-1}$(R(1)(G)).

(iii) *If* $R(1)(G)=\varnothing$ *and* $E=\varnothing$ *then* G *has a structure of a* $d-1$ *dimensional idealistic space.*

Proof. — This result is essential for the inductive arguments. Let us go back to the definition of idealistic spaces (6.2) and in order to simplify assume that there is only one chart, so assume that Z is a smooth subscheme of W, that every irreducible component of Z is of dimension d, that $i:(Z, \bar{E}) \to (W, E)$ is an immersion of pairs and that $\bar{G}$ is the idealistic situation defined by the couple $(\mathscr{L}, b)$ over $(Z, \bar{E})$.

To prove (i) apply first 2.6.3 and then (at Z) 3.9 (*b*). Analogously for the proof of (ii) apply first 2.6.2 and then (at Z) 3.9 (*a*).

Similarly for (iii), the condition $\text{Max ord}(G)=1$ is equivalent to condition 3.3.0 on $(\mathscr{L}, b)$ (5.8.1) and now all conditions for 3.4 (*b*) are fulfilled since $R(1)(G)=\varnothing$!!. So we can choose a smooth hypersurface Z_{d-1} at Z and a couple at Z_{d-1} as in 3.4 (*b*) (locally at any singular point). The proof of (iii) is now clear.

6.8. *The obstruction function* $n(x)$ ([Vi] 2.3.2). — In 2.7 we associated to a tree T over a pair (W, E) (with index set $\wedge$), a subset $\wedge\{T\} \subset \wedge_T$ (the index set of the transform by T). Recall from 2.7 also the definitions of $\wedge_T(x)$, $\wedge\{T\}(x)$ and $E_T(x)$ for $x \in W_T$.

6.8.1. DEFINITION. — Let G be a d-dimensional idealistic space over (W, E) and T a tree of length s at $G_{w\text{-ord}}$. Set $\wedge\{T\}=\{\beta(1), \ldots, (\beta(s)\}(\subset \wedge_T)$ and let $k_0 \in \{1, \ldots, s\}$ denote the weighted order birth of T (6.6).

(i) At the pair (W_T, E_T) (1.5) we define an expression of E_T as a *disjoint* union of subset

$$E_T = E_T^+ \cup E_T^-$$

where $E_T^+ = \{H_{\beta(i)}/\beta(i) \in \wedge\{T\} \text{ and } i \geqq k_0\}$ and of course E_T^- is the complement of E_T^+ in E_T.

(ii) Now set $w_0 = \text{Max } w\text{-ord } G_T = \text{Max } w\text{-ord } G_{[T]k_0}$ (6.6). At any $x \in \text{Sing}(G)_T$ define

$$n(x) = \#\{H_\lambda \in E_T^-/x \in H_\lambda\} \quad (\#\text{ number of elements of the set})$$

if $x \in \underline{\text{Max}}\, w\text{-ord}(G_T)$ (4.3) (*i.e.* if $w\text{-ord}(G_T)(x)=w_0$), and

$$n(x) = \#\{H_\lambda \in E_T/x \in H_\lambda\} \qquad \text{if} \quad w\text{-ord}(G_T)(x) < w_0.$$

(iii) Now define

$$t_d: \ \text{Sing}((G)_T) \to \mathbb{Q} \times \mathbb{Z}, \qquad t_d(x) = (w\text{-ord}(x), n(x)).$$

Where d stands for the dimension of the idealistic space.

6.8.2. REMARK. — Along $\underline{\text{Max}}\, w\text{-ord}(G_T)$ the first coordinate of t_d is constant, in fact for x at $\underline{\text{Max}}\, w\text{-ord}(G_T)$: $w\text{-ord}(x) = \text{Max}(w\text{-ord}(G_T))$ (4.3 (ii)).

6.8.3. LEMMA. — *If* T *is a tree of* $G_{w-\mathrm{ord}}$ *and* $w_0 = \mathrm{Max}\ w\text{-ord}(G_T)$ *then:*

(i) $t_d(x)$ *and* $n(x)$ *are locally finite and upper semicontinuous along* $\mathrm{Sing}(G_T)_{w-\mathrm{ord},\, w_0}$.

(ii) *Suppose that* T *is the concatenation of trees* $T_i : i=1, \ldots, s$ *(trees of length one) so that* $T_s : (W_s, E_s, C_s) \overset{\pi}{\leftarrow} (W_{s+1}, E_{s+1})$. *Then for any* $x \in \mathrm{Sing}(G_T)_{w-\mathrm{ord},\, w_0}$: w-ord $(\pi(x)) \geqq w\text{-ord}(x)$, *and moreover if the equality holds then* $n(\pi(x)) \geqq n(x)$.

Proof. — Along the points of $\mathrm{Sing}(G_T)_{w-\mathrm{ord},\, w_0}$ ($\underline{\mathrm{Max}}\ w\text{-ord}(G_T)$ (4.14)) $n(x)$ is counting the hypersurface of E_T^- passing through x. The first coordinate of t_d is constant so (i) is now clear.

This first part of (ii) follows from the fact that w-ord is a strong function on the d-dimensional grove G over (W, E) (4.7, 6.5). If the equality holds then Max w-ord $(G_T) = \mathrm{Max}\ w\text{-ord}(G_{[T]_s})$ and in this case k_0 (the weighted order birth of T) is also the weighted order birth of $[T]_s$.

According to 6.8.1 in such case E_T^- consists of the strict transforms of elements of $E_{[T]_s}^-$, the second statement of (ii) follows easily from this fact.

6.8.4. PROPOSITION. — *Let* G *be a d-dimensional idealistic space over* (W, E), T *a tree of* $G_{w-\mathrm{ord}}$ *and* $w_0 = \mathrm{Max}\ w\text{-ord}(G_T)$.

Then the map t_d *defines a function (in the sence of* 4.1) *on the grove* $(G_T)_{w-\mathrm{ord},\, w_0}$ *(over the pair* (W_T, E_T)*) with values at* $Q \times \mathbb{Z}$. *Moreover this function is a strong function* (4.7).

Proof. — The first coordinate of t_d is w-ord(G_T), which is constant $(= w_0)$ along $\mathrm{Sing}(G_T)_{w-\mathrm{ord},\, w_0} = \underline{\mathrm{Max}}\ (w\text{-ord}(G_T))$ (4.14). We study the behavior of the second coordinate: $n(x)$.

Let S be a tree of $(G_T)_{w-\mathrm{ord},\, w_0}$, the concatenation T.S (of T with S) is also a tree of $G_{w-\mathrm{ord}}$. Moreover, either S is such that Max w-ord $(G_{T.S}) < w_0$ in which case $\mathrm{Sing}(((G_T)_{w-\mathrm{ord},\, w_0})_S) = \varnothing$ or Max w-ord$(G_{T.S}) = w_0$ and in this case the setup for T.S is the exactly that of T, so for simplification set S.T = T.

If T is the concatenation of T_i: $(W_i, E_i, C_i) \leftarrow (W_{i+1}, E_{i+1})$ and $\bar{T}$ is a restriction of T, then $\bar{T}$ is a concatenation of trees $\bar{T}_i$: $(\bar{W}_i, \bar{E}_i, \bar{C}_i) \leftarrow (\bar{W}_{i+1}, \bar{E}_{i+1})$ and there are trees of length zero r_i: $(\bar{W}_i, \bar{E}_i) \to (W_i, E_i)$ so that all square diagrams commute (1.6.3). Since w-ord is a function on G (6.5) for any $x \in \mathrm{Sing}(G_{\bar{T}})_{w-\mathrm{ord},\, w_0}$ $[= r_{s+1}^{-1}(\mathrm{Sing}(G_T)_{w-\mathrm{ord},\, w_0})]$: w-ord$(G_T)(r_{s+1}(x)) = w\text{-ord}(G_{\bar{T}})(x) = w_0$ and the link defined by the tree of length zero r_{s+1}: $(\bar{W}_{s+1}, \bar{E}_{s+1}) \to (W_{s+1}, E_{s+1})$ also links the partitions as E_{s+1} $(= E_T)$ and $\bar{E}_{s+1}$ $(= E_{\bar{T}})$ defined at 6.8.1. So $n(x) = n(r_{s+1}(x))$ and the proof follows now from 6.8.3.

6.8.5. REMARK. — The function $n(x)$: $\mathrm{Sing}\, G_T \to \mathbb{Z}$ was defined at 6.8.1 only for trees T of $G_{w-\mathrm{ord}}$. It is not hard to *extend* the formulation for any tree T of G as a function on the grove G with values at $\mathbb{Z}$. For simplification denote $G_{[T]_i}$ as G_i.

Suppose that T is a concatenation of trees $T_i:(W_i, E_i, C_i) \overset{\pi_i}{\leftarrow} (W_{i+1}, C_{i+1})$ of length one, we begin by defining subsets $E_i^-(x)$ ($\subset E_i(x)$) for any $x \in \operatorname{Sing} G_i$, $i=1, \ldots, s+1$:

1. $E_1^-(x) = E_1(x)$ (2.7) for any $x \in \operatorname{Sing} G$. Assume for $k-1>0$ a definition of:

$k-1$. $E_{k-1}^-(x)$ ($\subset E_{k-1}(x)$ (2.7)) for any $x \in \operatorname{Sing} G_{k-1}$.

k. If w-ord $(x) = w$-ord $(\pi_k(x))$ let $E_k^-(x)$ ($\subset E_k(x)$) be defined as:

$$E_k^-(x) = \{ H_i' \in E_k(x)/H_i' \text{ is the strict transform of } H_i \in E_{k-1}^-(\pi_k(x)) \}$$

and if w-ord $(x) \neq w$-ord $(\pi_k(x))$: $E_k^-(x) = E_k(x)$.

Now we define: (*a*) $n(x) = -1$ if $n(\pi_k(x)) = -1$ or if the function w-ord is not constant along C_k locally at $\pi_k(x)$.

(*b*) $n(x) = \# E_k^-(x)$ ($\geqq 0$) in any other case.

By construction $n(x)$ is locally finite and the same proof given in 6.8.4 shows that for a restriction $\bar{T}$ of T, the link defined by the tree of length zero r_{s+1}: $(\bar{W}_{s+1}, \bar{E}_{s+1}) \to (W_{s+1}, E_{s+1})$ also links $E_{s+1}^-(x))$ ($\subset E_T(x)$) and $\bar{E}_{s+1}^-(x)$ ($\subset E_{\bar{T}}(x)$) for any $x \in \operatorname{Sing} G_{\bar{T}}$. So $n(x)$ is a function on the grove G (4.1).

6.9. *Two properties of the function* t_d ([Vi] 2.7.3) Now we fix a tree $T \in G_{w-\text{ord}}$ where G is a d-dimensional idealistic space over (W, E) (6.2).

Since t_d is strong function from $(G_T)_{w-\text{ord}, w_0}$ to $\mathbb{Q} \times \mathbb{Z}$ (6.8.4), define $((G_T)_{w-\text{ord}, w_0})_{t_d, \alpha}$ as in 4.14 where $\alpha = (w_0, n_0)$ ($\in \mathbb{Q} \times \mathbb{Z}$) is $\operatorname{Max} t_d$ at $\operatorname{Sing}(G_T)_{w-\text{ord}, w_0}$. For simplicity set $(G_T)_{t_d, \alpha} = ((G_T)_{w-\text{ord}, w_0})_{t_d, \alpha}$.

Property 1. — (i) The grove $(G_T)_{t_d, \alpha}$ over (W_T, E_T) is an idealistic space of dimension $d' = \dim W_T$.

(ii) If Max w-ord $(G_T) > 0$, then Max ord $((G_T)_{t_d, \alpha}) = 1$.

Proof. — Let the notation be as in 6.2 so that G is defined by "local charts" $(Z_\beta, E_\beta) \to (U_\beta, E_\beta)$.

In order to simplify the notation we fix β and just consider $(Z, \bar{E}) \to (U, E)$. G_U was $i(G')$ where G' is an idealistic situation over $(Z, \bar{E})$ defined (say) by the couple $(\mathscr{L}, b)$.

Since $T \in G_{w-\text{ord}}$, then $T \in G$ and there is an immersion $(Z_T, \bar{E}_T) \to (U_T, E_T)$ and a transform $(\mathscr{L}_T, b)$ defining the idealistic situation at $(Z_T, \bar{E}_T)$ (2.6.4; 2.6.5).

Set $(J, b) = (\mathscr{L}_T, b)$. Recall from the proof of 5.9 that in these conditions a couple $w(J, b)$ was defined (at Z_T) so that $(G_T)_{w-\text{ord}, w_0}$ is the idealistic space locally defined by this couple and the immersion of Z_T in W_T (5.9 (ii)).

Again, since $T \in G_{w-\text{ord}}$, there is a partition $E_T = E_T^+ \cup E_T^-$ (6.8.1) so that the function n: $\operatorname{Sing}(G_T)_{w-\text{ord}, w_0} \to \mathbb{Z}$ is defined by $n(y) = \# \{ H_\lambda \in E_T^- / y \in H_\lambda \}$.

Now fix $y \in \underline{\operatorname{Max}} \, t_d$. At the local regular ring $O_{W_T, y}$ there is an element of order 1, say $x_\lambda \in O_{W_T, y}$ such that $H_\lambda = \operatorname{Sing}(\langle x_\lambda \rangle, 1)$ (locally at y). Set formally.

6.9.1.1. $$t_d(J, b) = w(J, b) \cap \bigcap_{H_\lambda \in E^-(y)} (x_\lambda, 1) \qquad ([Vi]\ 2.7.1).$$

Here $w(J, b)$ is a couple at Z_T and each $(x_\lambda, 1)$ a couple on W_T. So we first want to express 6.9.1.1 as a couple at Z_T. Applying 2.4.9 and 2.4.10 one can express $(x_\lambda, 1) \cap Z_T$ as a couple on W_T. Now 2.4.11 states that Z_T has maximal contact with this couple.

If we assume that $(x_\lambda, 1) \cap Z_T$ is strictly included in Z_T (locally at y) then we apply 2.6.6, if not disregard that intersection so ultimately $t_d(J, b)$ can be regarded as a couple at Z_T (and 6.9.1.1 as an intersection of couples at Z_T) at some neighborhood of y. Moreover one can check that this couple at Z_T is defining $(G_T)_{t_d, \alpha}$ over (W_T, E_T) via the immersion in W_T (locally at y).

If $w_0 (= \text{Max } w\text{-ord } G_T)) > 0$ then 5.9 (iii) says that Max ord $(G_T)_{w-\text{ord}, w_0}$ is 1. But this means that the condition of Remark 1 of 5.8 hold for the couple $w(J, b)$ (at Z_T), now the formula 6.9.1.1 and 5.8,2 assert that Max ord$((G_T)_{t_d, \alpha}) = 1$.

Property 2 ([Vi] cond. 1), 2), 3) and 4), p. 24). — Any tree of the grove $(G_T)_{t_d, \alpha}$ over (W_T, E_T^+) is also a tree of G_T over (W_T, E_T).

Proof. — Of course a tree of $(G_T)_{t_d, \alpha}$ over (W_T, E_T) is also a tree of G_T over (W_T, E_T). The fact that we can disregard the subset $E_T^- (\subset E_T)$ is because: 1. locally at $y \in \text{Sing}(t_d(J, b))$ any center for a permissible transformation of this pair is contained in all $H_\lambda \in E_T^-$ (*see* formula 6.9.1.1), and 2. $n(y) = \# E_T^-(y)$ is upper semicontinuous along $\text{Sing } G_{w-\text{ord}, w_0}$ and maximal along $\text{Sing}(t_d(J, b))$ $(= \underline{\text{Max}}\ t_d)$.

From 1 and 2 one can check that for any closed irreducible subscheme $C \subset \text{Sing}(t_d(J, b))$ and for any $H_\lambda \in E_T^-$, either $C \subset H_\lambda$ or $C \cap H_\lambda = \phi$.

6.10. Let T denote again a tree of length s of $G_{w-\text{ord}}$ where G is a d-dimensional idealistic space over (W, E) (6.2), k_0 denotes the w-ord-birth of T, $w_0 = \text{Max } w\text{-ord } G_T$ (as in 6.8.1) and $\alpha = (w_0, n_0)$ is $\text{Max } t_d$.

6.10.1. THEOREM. — *If* $w_0 > 0$ *and suppose that* T *is a concatenation of (only) monidal transformations (so* $\dim W_T = \dim W$), *then:*

(*a*) $R(1)((G_T)_{t_d, \alpha})$ *is a smooth and permissible center for (a tree of)* $G_T, (W_T, E_T)$.

(*b*) *After blowing up* $R(1)((G_T)_{t_d, \alpha})$ *we may assume that* $R(1)((G_T)_{t_d, \alpha})$ *is empty.*

(*c*) *If* $R(1)((G_T)_{t_d, \alpha})$ *is empty, then the grove induced by* $(G_T)_{t_d, \alpha}$ *over* (W_T, E_T^+) (6.2.1) *has a natural structure of a* $d-1$*-dimensional idealistic space.*

Proof. — (*a*) Recall from 6.8.1 that at E_T there is a decomposition $E_T = E_T^- \cup E_T^+$ where E_T^+ is defined in terms of k_0 and consist of the hypersurfaces introduced by blowing up C_i, $i = k_0, k_0 + 1, \ldots, s-1$. Recall also that the grove $(G_T)_{t_d, \alpha_0}$ was constructed from $(G_T)_{w-\text{ord}, w_0}$ and the set E_T^- as a d-dimensional idealistic space satisfying the nice condition of having maximal order 1 (6.9 Property 1).

Since $\text{Sing}(G_T)_{t_d, \alpha_0} \subset \text{Sing}(G_T)_{w-\text{ord}, w_0} (\subset W_T)$ and both are singular locus of idealistic spaces of the same dimension d, then

$$R(1)(G_T)_{t_d, \alpha_0} \subset R(1)(G_T)_{w-\text{ord}, w_0}$$

and moreover any connected component of the first is a connected component of the second as follows from 6.7.2 and 6.7.3 and since both are of the same dimension $d-1$.

We may also assume from 6.7.3 that after a convenient open restriction at a neighborhood of $R(1)(G_T)_{t_d, \alpha_0}$ (1.7.4):

6.10.1.1. $$R(1)(G_T)_{w-\text{ord}, w_0} = R(1)(G_T)_{t_d, \alpha_0},$$

and

6.10.1.2. $$\text{Sing}(G_T)_{w-\text{ord}, w_0} = R(1)(G_T)_{w-\text{ord}, w_0}$$

But $R(1)(G_T)_{w-\text{ord}, w_0}$ is the strict transform (via the intermediate maps) of the *smooth* scheme $R(1)(G_{k_0})_{w-\text{ord}, w_0}$ (6.7.4(i)), so one can assume that the formula 6.10.1.2 holds not only for T but for any truncation of T with index between k_0 and s. Therefore if $H_\lambda \in E_k^+$, $H_\lambda \cap R(1)(G_T)_{w-\text{ord}, w_0}$ is either empty or transversal at each point, in fact in this last case H_λ arises by blowing up centers included in the intermediate strict transforms of $R(1)(G_{k_0})_{w-\text{ord}, w_0}$ (6.10.1.2). This shows that $R(1)((G_T)_{t_d, \alpha})$ is a permissible center for $(G_T)_{t_d, \alpha}$ over (W_T, E_T^+). Now 2 of 6.9 settles (*a*).

(*b*) Follows now from 6.7.4(i).

(*c*) This is a local problem. For any $x \in \text{Sing}(G_T)_{t_d, \alpha_0}$ let $x_{k'}$ denote the image of x at $W_{k'}$ (via the intermediate maps) where $k' (= k'(x))$ is the smallest index j such that

6.10.1.3. $$w\text{-ord}(x_j) = w\text{-}ord(x) = w_0.$$

Claim. – *In general* $k' \leqq k_0$, *but we may always identify the point* $x_{k'}$ *with a point of* $\text{Sing}(G_{k_0})$ $(\subset W_{k_0})$.

The proof of the claim follows from the facts that w-ord is a strong function (6.5), that T is a tree of $G_{w-\text{ord}}$ and finally from the definition of strong function (4.7). In fact $w\text{-ord}(x_{k_0}) \geqq w\text{-ord}(x)$ (4.7 B) and Max $w\text{-ord}(G_{k_0}) = w_0$, therefore $w\text{-ord}(x_{k_0}) = w_0$. So $k_0 \geqq k'$, but the intermediate centers of blowing ups (for $k_0 > j \geqq k'$) are chosen in $\underline{\text{Max}}$ w-ord and Max $w\text{-ord}(G_j) > w_0$ for any such j. The claim is now clear.

So actually we could have defined k' as k_0, but the interest of the definition of k' given in 6.10.1.3 will show up in the punctual description of the algorithm.

So as it stands $x \in \text{Sing}(G_T)_{t_d, \alpha_0}$ and $x_{k'} \in \text{Sing}(G_{k_0})_{w-\text{ord}, w_0}$. Since this grove is an idealistic space of dimension d and of maximal order 1, we will argue as in 6.7.4(iii) *locally* at $x_{k'}$ to find a smooth scheme of dimension $d-1$: $Z_{d-1} (\subset Z_{[T]_{k_0}})$ so that $x_{k'} \in Z_{d-1}$ and Z_{d-1} has maximal contact with the grove induced by the restriction of $(G_{k_0})_{w-\text{ord}, w_0}$ over the couple $(W_{k_0}, \varnothing)$ in the sence of 6.2.1. In fact the selection of Z_{d-1} with maximal contact with $w(\mathscr{L}_{k_0}, b)$ was given by an ideal $\mathscr{A} \subset \mathcal{O}_{Z_{[T]_{k_0}}}$ (as in 3.3) so that $Z_{d-1} = \text{Sing}(\mathscr{A}, 1)$ (locally at $x_{k'}$). Since $(G_T)_{w-\text{ord}, w_0}$ is the transform of $(G_{k_0})_{w-\text{ord}, w_0}$ by the intermediate maps, Theorem 3.3 asserts that the transform of $(\mathscr{A}, 1)$, say $(\mathscr{A}', 1)$, is in the same setup for $(G_T)_{w-\text{ord}, w_0}$ (for $w(\mathscr{L}_T, b)$ at Z_T).

Set $Z'_{d-1} = \text{Sing}(\mathscr{A}', 1)$, Z'_{d-1} is the strict transform of Z_{d-1} (2.4.10). Now the formula 6.9.1.1 shows that Z'_{d-1} will also have maximal contact with $(G_T)_{t_0, \alpha_0}$ (locally at x). Moreover the hypothesis ($R(1)=\varnothing$) is such that condition of 2.6.6 will hold for $t_d(J, b)$ at $\mathcal{O}_{Z_T}$ and for Z'_{d-1} so as to define a structure of $d-1$ dimensional grove for $(G_T)_{t_0, \alpha_0}$ over the pair $(W_T, \varnothing)$, locally at x. We claim however that this structure can be defined over the pair (W_k, E_k^+). In fact after convenient restrictions we see that Z'_{d-1} has transversal intersection with elements of E_T^+ since all these hypersurfaces arise by monoidal transformation with centers strictly included in Z_{d-1} and it's strict transforms by condition A. This proves (c).

7. Constructive resolutions

7.1. Definition ([Vi] 2.2). — A function defining a constructive resolutions on a grove G with values at I is a function Ψ from G to a totally ordered set $(I, \leqq)$ in the sense of 4.1, such that Ψ defines a tree of the grove

$$T: \quad (W_1, E_1, C_1) \leftarrow (W_2, E_2, C_2) \leftarrow \ldots \leftarrow (W_r, E_r)$$

where: (i) the maps $\Psi_{[T]_i}: \operatorname{Sing} G_{[T]_i} \to I$ are upper semicontinuous,

(ii) each $\pi_i: W_{i+1} \to W_i$ is the *monoidal transformation* with center C_i where $C_i = \underline{\operatorname{Max}}\, \Psi_{[T]_i}$.

(iii) (a) $\Psi_{[T]_i}(x) \leqq \Psi_{[T]_{i-1}}(\pi_{i-1}(x))$, $\forall x \in \operatorname{Sing} G_{[T]_i}$, with equality iff $x \notin C_i = \underline{\operatorname{Max}}\, \Psi_{[T]_i}$, in particular:

(b) $\operatorname{Max} \Psi_{[T]_i} < \operatorname{Max} \Psi_{[T]_{i-1}}$ for any $i > 1$ (4.3),

(iv) $\operatorname{Sing} G_T = \varnothing$ (at W_T).

In general a tree T is said to be a *resolution of the grove* if T is a concatenation of monoidal transformations (no smooth maps involved) and $\operatorname{Sing} G_T = \varnothing$.

7.2. (*the monomial case*). Let G be a d-dimensional idealistic space over (W, E) (6.2). Recall that if T is a tree of $G_{w-\text{ord}}$ the functions w-ord $(G_{[T]_k}): \operatorname{Sing} G_{[T]_k} \to \mathbb{Q}$ are such that $\operatorname{Max} w\text{-ord}(G_{[T]_{k+1}}) \leqq \operatorname{Max} w\text{-ord}(G_{[T]_k})$ $k = 1, \ldots, r-1$ (6.6).

Assume now that $\operatorname{Max} w\text{-ord}\, G_T = 0$. Since the grove G over (W, E) is a d-dimensional idealistic space, it is defined locally at W by a closed immersion of groves (6.2).

In order to simplify the notation assume that it is defined by *one* closed immersion say $(Z, \bar{E}) \subsetneq (W, E)$, that $\bar{G}$ is an idealistic situation induced by the couple $(\mathcal{L}, b)$ at (Z, E) and that $G = i(\bar{G})$ (notation as in 6.2).

Since T is a tree of G it induces a tree on (Z, E) and a couple $(\mathcal{L}_T, b)$ at (Z_T, E_T) defining G_T (2.6.5).

The assumption that $\operatorname{Max} w\text{-ord}(G_T) = 0$ means exactly that:

7.2.1. $$\mathcal{L}_T = \Pi\, \mathcal{I}_\lambda^{\beta(\lambda)} \cdot O_{Z_T} \quad \text{(as a sheaf of ideals at } Z_T!\text{) (5.5).}$$

So $\mathcal{L}_T$ is *locally monomial* at Z_T. In these conditions let us exhibit a constructive resolution of the grove G_T over (W_T, E_T).

Following Hironaka (*see* [H1], Lemma D1 p. 312) we first look at the irreducible components of $\mathrm{Sing}(\mathscr{L}_T, b)$ of maximal dimension, then choose among them those where the order of the monomial is maximal (*see* $a(x)$ and $c(x)$ below).

The point is to follow Hironaka's procedure but with a function satisfying the condition of 7.1. So we want to choose *one* of the irreducible components suggested by Hironaka in a unique way (*see* β below).

Any index λ in 7.2.1 is an index of $\wedge\{T\}$ and this is a totally ordered finite set ($\subset \mathbb{Z}$). The same holds for the subsets $\wedge\{T\}(x)$ (2.7), so $\wedge\{T\}(x) \subset \wedge\{T\} \subset \mathbb{Z}$.

To any ordered set $(I, \geqq)$ we "attach" a formal element: ∞ ($\infty \geqq n$, $\forall n \in I$).

Now set: $a(x) = -b(x)\,(\in \mathbb{Z})$ where

$$b(x) = \min\{k/\exists\, i_1 < \ldots < i_k / \Sigma\, \alpha(i_j)(x) \geqq 1\} \quad (\in \mathbb{N})$$

If $b(x) = b$ set

$$c(x) = \max\{\alpha(i_1)(x) + \ldots + \alpha(i_b)(x) / i_1 < \ldots < i_b\} \quad (\in \mathbb{Q})$$

Set $[\wedge\{T\}(x)]^b (\subset [\wedge\{T\}]^b \subset \mathbb{Z}^b$ "included in" $\mathbb{Z}^{\mathbb{N}}$ via the inclusion

$$(a_1, \ldots, a_b) \to (a_1, \ldots, a_b, \infty, \infty, \infty, \infty, \ldots)).$$

And define:

$$\beta = (\bar{\beta}_1, \ldots, \bar{\beta}_b) = \max\{(\beta_1, \ldots, \beta_b)/\beta_1 > \ldots > \beta_b$$

and

$$\alpha(\beta_1)(x) + \ldots + \alpha(\beta_b)(x) = c(x)\}.$$

Where the order involved in the definition of β is the lexicographic order.

This is how we arrange these functions in [V_i] 2.3.1 so that the maximal value is reached at *one* of the irreducible components suggested before and clearly in the conditions of 7.1, by a function, say $\Psi_d(0)$, with values at the totally ordered set, say $I_d(0) = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}^{\mathbb{N}}$ ordered lexicografically.

7.3. THEOREM. — *For each dimension d there is an ordered set I_d so that any d-dimensional idealistic space* G *over a pair* (W, E) (6.2) *admits a constructive resolution defined by a function Ψ_d with values at I_d* (7.1).

Proof. — For $d=0$ we may choose $I_0 = \varnothing$ since any zero dimensional idealistic space is non singular (*i.e.* $\mathrm{Sing}\, G = \varnothing$). In fact if $\dim Z_\beta = 0$ (notation as in 6.2) then O_{Z_β} is a direct sum of fields, our assumption $\mathscr{L}_x \neq 0$ on 2.4.1 states that $\mathrm{Sing}\, G = \varnothing$.

Assume that the theorem holds for $d-1$. We will first construct a tree of G say:

$$T : (W, E, C_1) \leftarrow (W_2, E_2, C_2) \leftarrow \ldots \leftarrow (W_h, E_h)$$

which will ultimately satisfy (i), (ii), (iii) and (iv) of 7.1, in fact, we will first construct T *as a sequence of monoidal transformations*, and then point out how it arises (as in 7.1) from a function on the grove. The construction and characterization of this particular tree T is based on the conditions (A), (B), (C) and (D) given below.

In order to simplify the notation denote the transform of G by a k-truncation ($G_{[T]_k}$) as G_k (2.5.3, 1.7 (*a*)).

(A) T *is a tree of* $G_{w-\text{ord}}$. So

$$\text{Max } w-\text{ord } G_i \geqq \text{Max } w\text{-ord } G_{i+1} \quad (6.6)$$

(i) Now fix $r (\in \{1, \ldots, h\})$ and let k' denote the weighted order birth of G_r and $w_0 = \text{Max } w\text{-ord}(G_r)$ as in 6.6. Recall that the intermediate tree

$$S: \quad (W_{k'}, E_{k'}, C_{k'}) \leftarrow \ldots \leftarrow (W_r, E_r)$$

is permissible for $(G_{k'})_{w-\text{ord}, w_0}$ and notice that $(G_r)_{w-\text{ord}, w_0}$ is the transform of the first by the tree S (6.5 (ii)).

(B) *If* $w_0 = 0$, *then* Max w-ord $(G_{[T]_{k'}}) = 0$. *Assume in this case that the procedure defined by* S (*between* k' *and* r) *consit of the* $r - k'$ *steps of the procedure described at* 7.2 *making use of the upper semicontinuous function mentioned there.*

(C) *If* $w_0 > 0$, *we will assume that the tree* S *is a tree of* $((G_{k'})_{w-\text{ord}, w_0})_{t_d}$ (6.8.4, 4.9.1).

Now set: (ii) $\alpha_0 = (w_0, n_0) = \text{Max } (t_d)_{[T]_r} (\in \mathbb{Q} \times \mathbb{Z})$, and

(iii) k = the t_d-birth of $[T]_r$ (the smallest index i such that $\text{Max}(t_d)_{[T]_i} = \alpha_0 = (w_0, n_0)$). We denote the grove $((G_k)_{w-\text{ord}, w_0})_{t_d, \alpha_0}$ (6.8.4, 4.14) by $(G_k)_{t_d, \alpha_0}$.

Clearly $k' \leqq k \leqq r$ and by assumption the grove $(G_k)_{w-\text{ord}, w_0}$ (over (W_k, E_k)) is the transform of $(G_{k'})_{w-\text{ord}, w_0}$ (over $(W_{k'}, E_{k'})$) [both d-dimensional idealistic spaces of maximal weighted order one (6.5)].

Recall from 6.8.1 that at E_k there is a decomposition $E_k = E_k^- \cup E_k^+$ where E_k^+ is defined now in terms of k' and consist of the hypersurfaces introduced by blowing up C_i, $i = k', k'+1, \ldots, k-1$. Recall also that the grove $(G_k)_{t_d, \alpha_0}$ was constructed from $(G_k)_{w-\text{ord}, w_0}$ and the set E_k^- as a d-dimensional idealistic space satisfying the nice condition of having maximal order 1 (6.9 Property 1).

Now define $\Psi^*_{d-1} : \text{Sing}(G_k)_{t_d, \alpha_0}) \to I_{d-1} \cup \{\infty\}$; $\Psi^*_{d-1}(x) = \infty$ if $x \in R(1)(\text{Sing}(G_k)_{t_d, \alpha_0}$ [if $\text{codim}(x) = 1$ where codim is now a function on the grove $(G_k)_{t_d, \alpha_0}$ (6.4)] and $\Psi^*_{d-1}(x) = \Psi_{d-1}(x) \in I_{d-1}$ if $x \notin R(1)$, where Ψ_{d-1} is defined by inductive assumption according to Remark 2.

So if $R(1) \neq \varnothing$ the maximal value is reached there, Theorem 6.10.1 states that $R(1)(G_k)_{t_d, \alpha_0}$ is permissible as a center over the full pair (W_k, E_k) and moreover after blowing up such center we may assume that $R(1)\,(G_k)_{t_d, \alpha_0} = \varnothing$.

We state now the last condition on the tree T as follows.

(D) If $w_0 > 0$, *assume that the intermediate steps*

$$(W_k, E_k, C_k) \leftarrow \ldots \leftarrow (W_r, E_r)$$

were constructed by first blowing up $R(1)\,((G_k)_{t_d, \alpha_0})$ *and then in accordance to the resolution of the* $d-1$ *dimensional space* $(G_k)_{t_d, \alpha_0}$ over (W_k, E_k^+).

Remark 3. — Through this construction of T, if $w_0>0$ we can force n_0 to drop. Since n_0 is always a natural number, at some point we force w_0 to drop. So ultimately (and uniquely) we come to the case $w_0=0$ and therefore the construction of T comes to an end by the procedure of 7.2.

Remark 4. — Now we come to the definition of a function Ψ_d and of the set I_d. Recall that in this theorem we expect Ψ_d to be a function on the grove G over a pair (W, E), with values on a fixed ordered set I_d(4.1) depending only on the dimension (d). We assumed inductively the theorem for $d-1$ dimensional idealistic spaces and therefore the existence of Ψ_{d-1} and I_{d-1}.

In 6.8.4 t_d was defined as a function only on the groves of the form $(G_T)_{w-\mathrm{ord},\, w_0}$ (where T a tree of $G_{w-\mathrm{ord}}$ and $w_0=\mathrm{Max}\ w\text{-ord}\,(G_T)$) with values at $\mathbb{Q}\times\mathbb{Z}$,

$$t_d(x)=(w\text{-ord}\,(x),\ n(x));\qquad w\text{-ord}\,(x)\in\mathbb{Q},\qquad \text{and}\qquad n(x)\in\mathbb{Z}.$$

Now 6.4 states that w-ord ( ) is a function on the full grove G and 6.8.4 states the same for n ( ).

On the other hand in 7.2 we define a function of groves $\Psi_d(0)$, for d-dimensional idealistic groves, say G′ over (W′, E′), with values at $I_d(0)=\mathbb{Z}\times\mathbb{Q}\times\mathbb{Z}^{\mathbb{N}}$, but with the additional hypothesis that Max w-ord (G′)=0. Now if T′ is any tree of G and $x\in\mathrm{Sing}\,(G_{T'})$ is such that $w\text{-ord}\,(x)=0$, there is an open restriction of T′ locally at x at which the condition Max w-ord $(G_{T'})=0$ is fulfilled by the upper semi continuity of w-ord (6.5 and 4.7). Now define:

$$I_d=(\mathbb{Q}\times\mathbb{Z})\times I_d(0)\times I_{d-1}$$

And for any tree T′ of G and any $x\in\mathrm{Sing}\,(G_{T'})$:

(a) if $w\text{-ord}\,(x)=0$, $\Psi_d(x)=(0,\ \infty,\ \Psi_d(0)(x),\ \infty)\ (\in(\mathbb{Q}\times\mathbb{Z})\times I_d(0)\times I_{d-1})$

(b) if w-ord $(x)>0$ and $n(x)=-1$ (*see* 6.8.5), set

$$\Psi_d(x)=(w\text{-ord}\,(x),\ n(x)=-1,\ \infty,\ \infty)$$

(c) if $w\text{-ord}\,(x)>0$ and $n(x)\geqq 0$ (*see* 6.8.5), then

$$\Psi_d(x)=(w\text{-ord}\,(x),\ n(x),\ \infty,\ \Psi^*_{d-1}(x))$$

where $\Psi^*_{d-1}:\mathrm{Sing}\,(G_k)_{t_d,\,\alpha_0})\to I_{d-1}\cup\{\infty\}$ is defined as follows. If $n(x)\geqq 0$ then (b) of 6.8.5 holds, and this case (b) has been so carefully selected that after a convenient open restriction of T′ we may assume:

(c1) $T'\in G_{w-\mathrm{ord}}$ (and $x\in\mathrm{Max}\ w$-ord $(G_{T'})$ since w-ord is upper semicontinuous 6.5 (i); 4.7)

(c2) $x\in\mathrm{Max}\ t_d(G_{T'})$ (6.8.4 and 4.7), so that $t_d(x)=\mathrm{Max}\,(t_d)$.

Therefore 6.9 applies for T′ and $x\in\mathrm{Sing}\ (G_{T'})_{t_d,\,\alpha_0})$, where $\alpha_0=t_d(x)=\mathrm{Max}\,(t_d)$, and $(G_{T'})_{t_d,\,\alpha_0}$ is a grove over $(W_{T'},\ E_{T'})$ as in Property 2) of 6.9. Now define: $\Psi^*_{d-1}(x)=\Psi_{d-1}(x)$ if this last grove is the transform of a $d-1$ dimensional grove (locally at x) and

$\Psi^*_{d-1}(x)=\infty$ in any other case.

Remark 5. — Now we come to an end by showing that the tree T constructed in 7.3 arises from the function Ψ_d (of remark 4) in the sence of 7.1. So T will denote that tree and $[T]_i$ denote the truncations.

Because of the way T was constructed one can check that for any i and any $x\in \mathrm{Sing}(G_{[T]_i})$ only (*a*) and (*c*) of Remark 4 are possible when defining $\Psi_d(x)$. Now one can check both (ii) and (iii) of 7.1 from this and from the inductive assumption on Ψ^*_{d-1}. Of course 7.1 (iv) holds by construction.

So now that we do know that T was constructed by chooseing inductively $C_i=\underline{\mathrm{Max}}\,\Psi_{[T]_i}$, we will finally show *why* (i) of 7.1 also holds. Recall that Ψ_d can take only finite different values along $\mathrm{Sing}(G_{[T]_i})$ for any index i. Let $\{\alpha(1), \ldots, \alpha(p)\}\subset I_d$ be the subset of all those possible values for the different truncations.

It suffices to show that for any $\beta\in I_d$ and for any fixed index i:

$$F_\beta=\{x\in \mathrm{Sing}(G_{[T]_i})/\Psi_{[T]_i}(x)\geqq\beta\}$$

is a closed subset of $\mathrm{Sing}(G_{[T]_i})$.

For any $j\geqq i$ let $\pi^j_i: W_j\to W_i$ be the composition of the intermediate blowing downs, and finally set $P(\beta)=\{k/\mathrm{Max}\,\Psi_{[T]_k}(x)\geqq\beta\}$.

Now applying 7.1 (iii) (*a*) one can check that

$$F_\beta=\bigcup\quad \pi^k_i(\underline{\mathrm{Max}}\,\Psi_{[T]_k})$$

where the union is taken over $k\in P(\beta)$. F_β is closed since all π^k_i are proper maps and the sets $C_k=\underline{\mathrm{Max}}\,\Psi_{[T]_k}$ are closed.

7.4. *Local idealistic presentation.* — If X is a *hypersurface* embedded in a smooth scheme W set $\mathscr{L}=I(X)\subset\mathcal{O}_W$ the sheaf of ideals of the subscheme X and b the maximal possible order of $\mathscr{L}$ at points of W.

The pair $(\mathscr{L}, b)$ defines an idealistic situation and a constructive resolution of this grove (7.1) is a sequence of *monoidal transformation* over W so that the final strict transform of X (say X′ included in W_r), has maximal order smaller then b.

So if X is a reduced subscheme and $b>1$, this strict transform is not empty and "closer" to a desingularization of X since the maximal order (b) has dropped.

And now we can start again with a new couple $(\mathscr{L}', b')$ where b' is the new maximal order, and a pair (W_r, R_r) so that $\mathscr{L}'\subset\mathcal{O}_{W_r}$ and E_r consist of the hypersurfaces introduced so far.

Now recall from 2.2 the notion of Hilbert-Samuel grove. The resolution of $(\mathscr{L}, b)$ mentioned above is nothing but a resolution of a Hilbert-Samuel grove for the particular case of a hypersurface. The following theorem of *local idealistic presentation* states however that this situation is quite general. This perhaps the major simplification of the theorem of resolution, proved in [G1] for the analytic case and by J. M. Aroca in the algebraic case. It is in this last case when etale topology comes in.

7.5. Let W be a smooth scheme of dimension m at each irreducible component, and X a subscheme of W with maximal Hilbert-Samuel function p. Therefore (W, X, p) is in the setup of 2.2. Finally set E in *any way* so that (W, E) is a pair (1.1).

7.5.1. THEOREM (*of local idealistic presentation*). – With the setup and notations as above, the Hilbert-Samuel grove of the data (W, X, p) (2.2) as a grove over (W, E) is an m-dimensional idealistic space over (W, E) (6.2).

Let us point out the fact that since m is already the dimension of W the maps i_β of 6.2 (ii) can be taken as the identity maps. Now we can apply our canonical resolution of idealistic spaces.

7.6. APPLICATION TO RESOLUTION OF SINGULARITIES. – A result of Hironaka states that resolution of singularities of embedded schemes $X \subset W$ is achieved by the resolution of the Hilbert-Samuel groves ([H2], 3, Remark 2), on the other hand the theorem of local idealistic presentation says that the Hilbert-Samuel grove is an idealistic spaces over the pair (W, E) no matter what set E as in 1.1 is considered.

7.6.1. THEOREM. – *Let* $X_1 \subset W_1$, $X_2 \subset W_2$ *be two immersions of schemes of finite type over a field k, both* W_i *smooth over k. Set* x_i *a point of* X_i $(i=1, 2)$ *and assume that there is an isomorphism of k-algebras* $\theta: \hat{\mathcal{O}}_{X_1, x_1} \to \hat{\mathcal{O}}_{X_2, x_2}$ *and that* $\dim_{x_1} W_1 = \dim_{x_2} W_2 = m$. *Then there are etale neighborhoods* U_i *of* $x_i (\in X_i)$ *which undergo the same constructive resolution of singularities.*

Proof. – First of all a corollary of Artin's approximation theory asserts that there is a common etale neighborhood (X', x') for both (X_1, x_1) and (X_2, x_2) ([Ar], 2, Corollary 2.6).

Set $A_i = \mathcal{O}_{W_i, x_i}$ and $\bar{A}_i = \mathcal{O}_{X_i, x_i}$. So $\mathcal{O}_{X', x'}$ is the localization at x' of an etale extension of both $\bar{A}_1$ and $\bar{A}_2$.

A simple application of the jacobian criterion for etale homomorphisms ([R] V, § 2, Th. 5) is that the surjections $A_i \to \bar{A}_i \to 0$ can lift to two surjections $S_i \to \mathcal{O}_{X', x'} \to 0$ in such a way that homomorphisms $A_i \to S_i$ exist and are etale.

In particular each S_i is a regular ring $(i=1,2)$ and we obtain two embeddings of X' locally at x'.

$$X' \subset W'_i; \qquad W'_i = \operatorname{Spec}(S_i)$$

and moreover $\dim_{x'} W'_1 = \dim_{x'} W'_2$.

Now the proof of the theorem is a consequence of the following proposition.

7.6.2. PROPOSITION. – *Let* X_i $i=1, 2$ *be two schemes which are isomorphic via a k-morphism* $\theta: X_1 \to X_2$. *Assume that* $X_i \subset W_i$ $(i=1,2)$ *are immersions in smooth schemes and that* $\dim_x(W_1) = \dim_{\theta(x)}(W_2)$ *for any* $x \in X_1$. *Then the constructive resolution of singularities over* $X_1 (\subset W_1)$ *coincides with that of* $X_2 (\subset W_2)$ *via the isomorphism* θ.

Proof. – Let x_1 be a closed point in X_1 and $x_2 = \theta(x_1) \in X_2$. Of course the functions $H^1_{X_1, x_1}$ and $H^1_{X_2, x_2}$ are the same and after convenient restrictions we may assume that these are the biggest Hilbert-Samuel functions. Indeed the map H_{X_1} defined in 2.2 was upper semicontinuous.

All functions that show up in the constructive resolution of idealistic spaces grow from the functions ord ( ) (5.2).

In 5.1.1 the value ord(x) ($\in \mathbb{Q}$) was expressed in terms of the grove, moreover in terms of the stalk of the grove at x (6.1.1). In fact in that proof some trees S(N. β) were defined, where S(N, β) consisted of a sequence of monoidal transformations over $W \times \mathbb{A}^1$.

Now the embedding $X_i \subset W_i$ lifts to an embedding $X_i \times \mathbb{A}^1 \subset W_i \times \mathbb{A}^1$ and θ lifts to an isomorphism:

$$\theta \times \mathrm{id} : X_1 \times \mathbb{A}^1 \to X_2 \times \mathbb{A}^1$$

From these remarks one can check that a tree S(N, β) belongs to the Hilbert-Samuel grove at x if and only if it belongs to the Hilbert-Samuel grove at $\theta(x)$. So the value ord(x) for one grove and ord$(\theta(x))$ for the other are the same. In other words: θ defines an isomorphism between the Hilbert-Samuel groves (locally at x and $\theta(x)$), and the correspondence between the stalks preserve the "expressions" of the values of ord(x) and ord$(\theta(x))$.

Now as stated in the proof of 5.7, the functions $\alpha(\lambda)$ and w-ord grow from the functions ord ( ) and one can check that the isomorphism that θ defines between the Hilbert-Samuel stratas of X_i locally at x_i, commutes with the functions ψ_m defining the constructive resolutions of both idealistic spaces. Then θ maps isomorphically the sets $\underline{\mathrm{Max}}$ (4.3) of both functions and therefore lifts to an isomorphism at the strict transform so that the set up after blowing-up is that of the very beginning, where now there is also a natural identification of the exceptional hypersurfaces introduced.

The proof of the proposition follows from these remarks.

7.6.3. Corollary. – *Given* $X_1 \subset W_1$ *as in* 7.6.1 *with* X_1 *reduced, and* G *a subgroup of* $\mathrm{Aut}_k(W_1)$ *inducing an action on* X_1. *Then* G *lifts uniquely to an action on the constructive resolution of* X_1 $(\subset W_1)$.

8. The algorithm of resolution, examples

8.1. We follow the notation of 1.5.1 for a tree T of length s, over a pair (W_1, E_1), let d denote the dimension of W_1.

To begin with and to motivate ideas, assume that G_1 is an *idealistic situation* (2.4.6) defined by a couple $(\mathscr{L}_1, b)$ over (W_1, E_1). We assume that $T \in G_1$ [*i.e.* T is permissible for $(\mathscr{L}_1, b)$] and will denote by $(\mathscr{L}_k, b)$ the transform of $(\mathscr{L}_1, b)$ at (W_k, E_k) (2.4.3,4) for $1 \leqq k \leqq s+1$.

As in 5.4 we attach at each W_k an expression

8.1.1. $$\mathscr{L}_k = I(H_1)^{\beta(1)} \dots I(H_{k-1})^{\beta(k-1)} \bar{\mathscr{L}}_k.$$

Recall that for any

$$x \in \operatorname{Sing}(\mathscr{L}_k, b): \operatorname{ord}(G_k)(x) = \upsilon_x(\mathscr{L}_k)/b$$

and

$$w\text{-ord}(G_k)(x) = \upsilon_x(\bar{\mathscr{L}}_k)/b \, (\in \mathbb{Q} \text{ see } 5.5),$$

where G_k is the idealistic situation defined by $(\mathscr{L}_k, b)$ over (W_k, E_k).

Now we come to the main point: to exhibit *the* tree T of the constructive resolution of G_1 over (W_1, E_1) as constructed in 7.3.

Condition (A) of 7.3 states that Max w-ord $(G_k) \leqq$ Max w-ord (G_{k-1}) (Max f = maximal possible value of the function f, where f is a function with values at an ordered set (4.3)). A condition which is fulfilled if the centers of the blowing ups involved in the construction of the tree, say C_i ($\subset W_i$), verify:

$$C_i \subset \underline{\operatorname{Max}}\, w\text{-ord}(G_i) (\subset W_i) \qquad (\text{where } \underline{\operatorname{Max}}\, f = \{ x / f(x) = \operatorname{Max}(f) \})$$

A tree in this condition is said to be a tree of $G_{w-\mathrm{ord}}$ (6.5, 4.9.1).

To characterize the tree T we first fix an index r and let now k' denote the w-ord-birth of the truncation $[T]_r$ (4.13, 6.6) which means: k' is the smallest index j such that

$$\operatorname{Max}\, w\text{-ord}(G_j) = \operatorname{Max}\, w\text{-ord}(G_r)$$

1. If Max w-ord$(G_r)=0$, then Max w-ord$(G_{k'})=0$ or equivalently $\bar{\mathscr{L}}_{k'} = \mathscr{O}_{W_{k'}}$. The assumption in this case (Condition B of 7.3) is that the procedure over $(W_{k'}, E_{k'})$ defined by T is that described at 7.2 (the monomial case).

2. If Max w-ord $(G_r) = w_0 > 0$, then Max w-ord $(G_{k'})_{= w0}$. For any r' such that $k' \leqq r' \leqq r$ define on $E_{r'}$ a decomposition as a disjoint union $E_{r'} = E_{r'}^+ \cup E_{r'}^-$ where $E_{r'}^+$ consists of the hypersurfaces of $E_{r'}$ that arised by blowing up C_i for $i = k', k'+1, \ldots, r'-1$. And for any such r' define also t_d: $\operatorname{Sing}(G_{r'}) \to \mathbb{Q} \times \mathbb{Z}$ as in 6.8.1. We assume inductively that

$$\operatorname{Max} t_d(G_{k'}) \geqq \operatorname{Max} t_d(G_{k'+1}) \geqq \ldots \geqq \operatorname{Max} t_d(G_r) \, (= \alpha_0 = (w_0, n_0))$$

Now let k denote the t_d-birth of r (*see* (iii) of 7.3):

$$k = \min \{ j / k' \leqq j \leqq r \text{ and } \operatorname{Max} t_d(G_1) = (w_0, n_0) \}$$

Then $\underline{\operatorname{Max}}\, t_d(G_k)$ is a closed in W_k, algebraic in our context, and if R(1) denotes the subset of points of $\underline{\operatorname{Max}}(t_d(G_k))$ where this set has codimension 1 in W_k, then R(1) is a union of connected components of $\underline{\operatorname{Max}}(t_d(G_k))$. Therefore the expression

8.1.2. $$\underline{\operatorname{Max}}\, t_d(G_k) = R(1) \cup F (\subset W_k)$$

is a disjoint union of closed sets.

Canonical resolution boils down to the following (*see* 6.9 and 6.10):

(2 A) R (1) is smooth and a permissible center for G_k (W_k, E_k).

(2 B) After blowing up R (1) we may assume that the new R (1) (that defined analogously for the transform of G_k (W_k, E_k)) is empty. In particular if $F \neq \varnothing$, this is already a resolution of the grove.

(2 C) If R (1) is empty, there is a naturally defined $d-1$ dimensional *idealistic space* $(G_k)_{t_d, \alpha_0}$; (W_k, E_k^+) such that:

(2 C (i)) Sing $((G_k)_{t_d, \alpha_0}) = \underline{\text{Max}}\, t_d (G_k) = F$ (of codim >1 in W_k, at any point).

(2 C (ii)) The concatenation (say T′) of $[T]_k$ with any tree of $(G_k)_{t_d, \alpha_0}$, (W_k, E_k^+) is also a tree of $G_{w-\text{ord}}$ (in particular of G_1, (W_1, E_1)).

(2 C (iii)) For any T′ as before $\text{Max}\, t_d (G_{T'}) \leqq \alpha_0$ and the equality holds iff the singular locus of the transform: $((G_k)_{t_d, \alpha_0})_{T'}$ is not empty, in which case

$$\text{Sing}\,((G_k)_{t_d, \alpha_0})_{T'} = \underline{\text{Max}}\, (t_d (G_{T'}))\, (= \{ x \in \text{Sing}\,(G_{T'}) / t_d (x) = \alpha_0 \})$$

MORAL: In case 2) R (1) is the center par excellence, and after blowing up R (1), the lowering of $\text{Max}\,(t_d) = \alpha_0$ is equivalent to the resolution of a $d-1$ dimensional idealistic space: $(G_k)_{t_d, \alpha_0}$, (W_k, E_k^+).

Condition (D) of 7.3 states that the tree between the levels k and r, defined by the T, consists of the blowing up at R (1) in the first place (if not empty of course), and then in accordance to the resolution of the $d-1$ dimensional: $(G_k)_{t_d, \alpha_0}$, (W_k, E_k^+).

The Remark 3 of 7.3 states that the lowering of $\text{Max}\,(t_d)$ implies resolution.

8.2. Remark. — Although we started 8.1 considering an idealistic situation, in 2 (C) we are forced to deal with the more ample notion of idealistic spaces, the need of this ample notion already shows up in the theorem of local idealistic presentation (7.5). So of course the problem of patching is a central issue.

The notion of idealistic space of dimension d was introduced in 6.2, where G is a grove over a pair (W, E) and there is a "covering" of W and closed immersions. So the setup is not exactly that of 8.1, the datas (W_k, E_k) are to be replaced by closed immersions $(Z_k, E_k) \to (W_k, E_k)$. Now W_1 belongs to the covering, Z_1 is a smooth and closed subscheme of dimension d, and the expression 8.1.1 is now a product of scheaves of ideals at $\mathcal{O}_{Z_k}$.

We defined *functions on groves* in such a way to extend the role of the function codim, w-ord, t_d, etc. (4.1, 4.2) which allows an extension of 8.1 to the general case of d-dimensional idealistic spaces. The smooth centers chosen either in case 1) or in case 2), are now smooth centers at the (smooth) scheme Z_k, therefore smooth at W_k.

8.3 (*On the punctual description and local uniformizations*). Let G be a grove over a pair (W, E) which is an idealistic space of dimension d. Let T denote *the* tree of the constructive resolution of G, we express now the function defining the constructive resolution at any singular point of G_r, (W_r, E_r) (the transform by the r-truncation $[T]_r$) *i.e.* for $x \in \text{Sing}\, G_r$ we want to express $\Psi_d (x)\, (\in I_d)$. For any index $j \leqq r$ let x_j denote the image of $x=x_r$ ($\in W_r$) at W_j, via the intermediate blowing downs. Since the tree T is constructed from Ψ as in 7.1, either x_{j+1} is an exceptional point over x_j after blowing *up at* $(x_j\in)\,\underline{\mathrm{Max}}\,\Psi_{[T]_j}$ if $\Psi_{[T]_j}(x_j)=\mathrm{Max}\,\Psi_{[T]_j}$, or $\Psi_{[T]_j}(x_j)\neq\mathrm{Max}\,\Psi_{[T]_j}$ and then $x_j=x_{j+1}$. Disregarding those indices where the second case occurs, and renumbering increasingly those left, then the sequence of local blowing ups is defined again by the function Ψ by blowing up at the locally closed and smooth $(x_j\in)\,\underline{\mathrm{Max}}\,\Psi_{[T]_j}$. This is how constructive resolutions define "constructive resolutions along valuation rings" or *local uniformizations*. The point is that now these *local* algorithms patch.

With notation as in 8.1 and 8.2, fix $x(=x_r)\in \mathrm{Sing}(\mathscr{L}_r, b)\subset Z_r\subset W_r$:

1. w-ord$(G_r)(x)=0$. This means that at $\mathcal{O}_{Z_r}$ the expression of 8.1.1 is such that $(\mathscr{L}_r)_x=\mathcal{O}_{Z_r, x}$. This is within case (*a*) of Remark 4 of 7.3, and

$$\Psi_d(x)=(w\text{-ord}(G_r)(x)=0,\ \infty,\ \Psi_d(0)(x),\ \infty)\,(\in(\mathbb{Q}\times\mathbb{Z})\times I_d(0)\times I_{d-1})$$

so $\Psi_d(x)$ depends essentially on the values $\Psi_d(0)(x)$ which is defined in 7.2 (the monomial case) ([Vi], 2.8, p. 26).

2. w-ord$(G_r)(x)=w_0>0$. This means that $\upsilon_x(\mathscr{L}_r)=b\,.\,w_0>0$ (υ_x: the order in the usual sence at the local regular ring $\mathcal{O}_{Z_r, x}$). We are in case (*c*) of Remark 4 of 7.3, and

$$\Psi_d(x)=(w\text{-ord}(x),\ n(x),\ \infty,\ \Psi^*_{d-1}(x));\qquad \Psi^*_{d-1}:\mathrm{Sing}(G_k)_{t_d,\alpha_0})\to I_{d-1}\cup\{\infty\}$$

Since the first coordinate of the upper semi continuous map $\Psi_{[T]_j}$ is the map w-ord(G_j) in general w-ord$(x_j)\geqq w$-ord(x_{j+1}) (7.3 iii)). As in 6.10.1.3 let $x_{k'}$ denote the image of x at $W_{k'}$ (via the intermediate maps) where $k'(=k'(x))$ is the smallest index j such that:

$$w\text{-ord}(x_j)=w\text{-ord}(x)=w_0.$$

So $\upsilon_{x_{k'}}(\mathscr{L}_{k'})=b\,.\,w_0$ and furthermore for any $k'\leqq j\leqq r$:

(i) $\upsilon_{x_j}(\mathscr{L}_j)=b\,.\,w_0$ for any $k'\leqq j\leqq r$, and

(ii) $x_j\in\underline{\mathrm{Max}}\,\Psi_{[T]_k}(\subset\underline{\mathrm{Max}}\ w\text{-ord}(G_j))$, or $x_j=x_{j+1}$ and then try with $j=j+1$.

Now we make use the notation and conventions of 6.8.5 to study the values $n(x_j)$ for $k'\leqq j\leqq r$. It is clear that $E^-_{k'}(x_{k'})=E_{k'(xk')}$ (since w-ord$(xk'-1)>w$-ord$(x_{k'})$), and for any $j>k'$:

$$E^-_j(x_j)=\{H'_i\in E_j(x_j)/H'_i \text{ the s.t. of } H_i\in E^-_{j-1}(x_{j-1})\}.$$

Since $n(x_j)=\#E^-_j(x_j)$:

$$t_d(x_{k'})\geqq\ldots\geqq t_d(x_j)\geqq\ldots\geqq t_d(x_r)\,(=\alpha_0=(w_0, n_0)).$$

Set $k=\min\{j/k'\leqq j\leqq r \text{ and } t_d(x_j)=\alpha_0\}$. In 6.9.1.1 (in 5.9) a couple $t_d(\mathscr{L}_k, b)$ [a couple $w(\mathscr{L}_k, b)$] at $Z_k(\subset W_k)$ is defined so that $\mathrm{Sing}\,t_d(\mathscr{L}_k, b)=\underline{\mathrm{Max}}\,t_d$ [so that $\mathrm{Sing}\,w(\mathscr{L}_k, b)=\underline{\mathrm{Max}}\ w$-ord] locally at x_k. Recall that $\overline{8.3.1}$ $t_d(\mathscr{L}_k, b)=w(\mathscr{L}_k, b)\cap\subset(x_\lambda, 1)$ where $\mathrm{Sing}\,(x_\lambda, 1)=H_\lambda\in E^-_k(x_k)$.

(iii) Clearly $\mathrm{Sing}\,(t_d(\mathscr{L}_k, b))\subset\mathrm{Sing}\,w(\mathscr{L}_k, b)$.

We know from 5.9 that in this case $(w_0>0)\, w(\mathscr{L}_{k'}, b)$ fulfills the condition 3.3.0 locally at $x_{k'}$ at $Z_{k'}$ [*see* 5.9 (iii)]. So let $\mathscr{A}\,(\subset \mathcal{O}_{Z_{k'}})$ be in the conditions of 3.3 and set $H_{k'}=\operatorname{Sing}(\mathscr{A}, 1)$ which is a smooth hypersurface of $Z_{k'}$ containing $x_{k'}$.

From (i) and (ii) we know that $w(\mathscr{L}_{k'}, b)$ is the transform of $w(\mathscr{L}_k, b)$ by the intermediate transformations. So let $(\mathscr{A}', 1)$ be the transform of $(\mathscr{A}, 1)$ at Z_k, now Theorem 3.3 states that the setup for $\mathscr{A}'$ and $w(\mathscr{L}_k, b)$ is the same as that of $\mathscr{A}$ and $w(\mathscr{L}_{k'}, b)$. And now from 8.3.1 we see that $\mathscr{A}'$ is also in the setup of 3.3 for the couple $t_d(\mathscr{L}_k, b)$.

Now set $H_k=\operatorname{Sing}(\mathscr{A}', 1)$ which is a smooth hypersurface of Z_k. Theorem 3.3 together with 2.4.10 (i) state that H_k is the strict transform of $H_{k'}$ and that the centers of these blowing ups are included in the intermediate strict transforms of $H_{k'}$. On the other hand $E_k^+(x_k)\,(\subset E_k(x_k))$ consists exactly of the hypersurfaces that arise by the local blowing ups between $x_{k'}$ and x_k. So $(H_k;\, E_k^+) \rightsquigarrow (Z_k;\, E_k^+)$ is a closed immersion locally at x_k.

(2 A) if codim of Sing $t_d(\mathscr{L}_k, b)$ $(=\underline{\operatorname{Max}}\, t_d)$ at x_k is 1 in Z_k, then $x_k=x_{k+1}\ldots=x_r\,(=x)$ and $\Psi^*_{d-1}(x_r)\ (=\Psi^*_{d-1}(x_k))=\infty$ ([Vi] 2.8 p. 26).

(2 B) if codim of Sing t_d $(\mathscr{L}_k, b)$ $(=\underline{\operatorname{Max}}\, t_d)$ at x_k is <1 in Z_k, we first apply 3.4 for the immersion $(H_k;\, \varnothing) \subsetneq (Z_k;\, \varnothing)$, which, as pointed out before is extensive so the immersion $(H_k;\, E_k^+) \subsetneq (Z_k;\, E_k^+)$ locally at x_k. In this way we define a $d-1$ dimensional idealistic space $(G_k)_{t_d,\,(w_0,\,n_0)}$ over $(W_k,\, E_k^+)$. For any $k \leqq j \leqq r$:

(iv) $x_j \in \underline{\operatorname{Max}}\, \Psi_{[T]_j}\,(\subset \underline{\operatorname{Max}}\, t_d(G_j)=\operatorname{Sing}(G_j)_{t_d,\,(w_0,\,n_0)})$, or $x_j=x_{j+1}$ and try with $j=j+1$.

(v) the grove $(G_r)_{t_d,\,(w_0,\,n_0)}$ over $(W_r,\, E_r^+)$ is the transform of $(G_k)_{t_d,\,(w_0,\,n_0)}$ over $(W_k,\, E_k^+)$.

Now $x_r \in \operatorname{Sing}(G_r)_{t_d,\,(w_0,\,n_0)}$ and $x_k \in \operatorname{Sing}(G_k)_{t_d,\,(w_0,\,n_0)}$, so set $\Psi^*_{d-1}(x_r)$ as $\Psi_{d-1}(x_r)$ where we assume this last value known by inductive assumption ([Vi], 2.10, p. 30).

REMARK 8.4. — The function w-ord is an extension of the function ord. Our strategy consists in forcing the condition Max w-ord$(G_k)=0$, in this case

$$\mathscr{L}_k=I(H_1)^{\beta(1)}\ldots I(H_{k-1})^{\beta(k-1)}$$

what we call the monomial case, which is already a very simple case. However if Max-ord(G_1) is 1, then one can check that all exponents $\beta(i)=0$ and all functions w-ord(G_s) coincide ith the functions ord(G_s) and furthermore for any index s, either Sing(G_s) is empty or again Max ord$(G_s)=1$. But of course our procedure of resolution of idealistic spaces still applies here too.

Let $X \subsetneq W$ be a closed immersion where (W, E) is a pair (1.1) and we also assume that X is an irreducible and smooth subscheme of W. As in 2.4.10 set $\mathscr{A}\,(\subset \mathcal{O}_W)$ the sheaf of ideals of X and consider the idealistic situation G defined by $(\mathscr{A}, 1)$ over (W, E). This is a case in which Max ord G $=1$ and of course $X=\operatorname{Sing}(\mathscr{A}, 1)$. So along the tree T of the constructive resolution of G over (W, E) we will never come to the case of weighted order zero. But still we will force the pairs Max $t_d(G_{[T]_i})$ to drop again and again, for the different truncations $[T]_i$, where the first coordinate (Max ord $G_{[T]_i}$) is always equal to 1.

Suppose furthermore that X is not included in any hypersurface of E, then for some truncation $[T]_k$ we reach the case

$$\operatorname{Max} t_d (G_{[T]_k}) = (1,0).$$

If X_k denotes the strict transform of X at W_k, then X_k $(=\operatorname{Sing}(\mathscr{A}_{[T]_k}), 1)$ 2.4.10 is smooth and moreover one can check that X_k now has normal crossings with E_k.

This is in general the role of the obstruction function $n(x)$ (6.8) along a smooth subscheme of the singular locus: it tells you how far it is from a sufficient condition for being permissible (1.2.1 (*a*)).

REMARK 8.5. – Let G_1 be idealistic space over (W_1, E_1). If Max w-ord$(G_1)=1$ and E_1 is empty, then:

(*a*) Max $t_d=(1,0)$.

(*b*) $\underline{\operatorname{Max}}(t_d)=\operatorname{Sing}(G_1)$.

(*c*) If T denotes the tree of the constructive resolution of G_1, (W_1, E_1) and G_k, (W_k, E_k) the transform by the different truncations, then both (*a*) and (*b*) hold for G_k, (W_k, E_k).

So in this case the lowering of Max t_d already implies resolution.

In fact in the case of Max ord $G_1=1$ and $E_1=\varnothing$, t_d is constant along Sing G_k for any k:

$$\operatorname{Sing} G_k = \underline{\operatorname{Max}}\ w\text{-ord}\ G_k = \underline{\operatorname{Max}}\ t_d(G_k) = \operatorname{Sing}(G_k)_{t_d,\alpha} \qquad \text{for} \quad \alpha=(1,0).$$

since clearly $G_k=(G_k)_{t_d,\alpha}$ for all k. So after blowing up R(1) we are solving a $d-1$ dimensional idealistic space. This will be the case in our two examples where $E_1=\varnothing$, dim $W_1=d=3$ and Max ord $G_1=1$ so we will be concerned with a resolution of a 2-dimensional idealistic space.

If $d=1$ the resolution reduces of course to quadratic transformation.

8.6. – Let X be a singular subscheme of a smooth scheme W of dimension m and p the biggest Hilbert Samuel function along closed points at X.

The theorem of local idealistic presentation (7.5.1) together with the existence of constructive resolutions of idealistic spaces (7.3) states that there is a sequence of monoidal (permissible) transformation over X so that the biggest Hilbert-Samuel function drops. In doing so a number of hypersurfaces with normal crossings appear, but we can start again and force the new biggest Hilbert-Samuel function to drop according to 7.5.

Now if X is irreducible all these permissible transformation induce a sequence of monoidal (birational) transformations over X and ultimately we come to a resolution of singularities $X_r(\subset W_r)$ on the one hand (*see* [H 2], Remark 2, p. 72) and to a set E_r of hypersurfaces with normal crossings at W_r.

The theorem of embedded resolutions requires $\{X_r\} \cup E_r$ to have normal crossings.

Now the setup for the immersion $X_r \subsetneq W_r$ and E_r is that of 8.1.2, and this is how embedded resolution is achieved.

Example 1 (Constructive resolution of the Whitney umbrella):

$$W=\operatorname{Spec}(k[X_1, X_2, X_3]); \quad E=\varnothing; \quad \mathscr{L}=\langle f \rangle,$$
$$f=X_3^2-X_1 X_2^2; \quad b=2; \quad \mathscr{A}=\langle X_3 \rangle$$

($X_3=1/2\ \partial f/\partial X_3 \in \Delta^1(\mathscr{L})$); $Z=\operatorname{Spec}\ (k[X_1,\ X_2, X_3]/\langle X_3 \rangle)\ (Z, \varnothing) \subsetneq (W, \varnothing)$ is an immersion of pairs (2.6.4). And finally G_1 is the idealistic situation defined by the couple $(X_1 X_2^2, 2)$ over $(Z, \varnothing)$. First check that all the condition are given as in 3.3 and 3.4.

If F is a closed subscheme of W, then $I(F)$ ($\subset \mathcal{O}_W$) denotes the associated sheaf of ideals. We also denote by F any strict transform of this set.

At each level k we will define:

(*a*) the transform $(\mathscr{L}'_k, 2)$ of $(\mathscr{L}'_1, 2)=(X_1 X_2^2, 2)$.

(*b*) the expression $\mathscr{L}'_k=I(H_1)^{\beta(1)} \ldots I(H_{k-1})^{\beta(k-1)}\, \bar{\mathscr{L}}'_k$ (5.4).

(*c*) the center C_k ($=\underline{\operatorname{Max}}\, \Psi_{[T]_k}$ as in 7.2 (i))

$k=1$: Set $F_1=\{X_1=0\}$ $F_2=\{X_2=0\}$ ($\subset Z$), $(X_1 X_2^2, 2)=(I(F_1)\, I(F_2)^2, 2)=(\mathscr{L}'_1, 2)$ and $p=F_1 \cap F_2$.

Now $\operatorname{Sing}(\mathscr{L}'_1, 2)=F_2$, $t_2(G_1)(p)=(3/2,0)$ and $t_2(G_1)(x)=(1,0)\ \forall x \in F_2\text{-}\{p\}$.

So $C_1=\underline{\operatorname{Max}}\, \Psi=p$.

$k=2$: $\mathscr{L}'_1=I(F_1)\, I(F_2)^2\, I(H_1)$ and $\bar{\mathscr{L}}'_1=I(F_1)\, I(F_2)^2$. Now $F_1 \cap F_2=\varnothing$ and

$$\operatorname{Sing}(\mathscr{L}'_2, 2)=\{F_1 \cap H_1\} \cup F_2.$$

Set $q_1=F_1 \cap H_1$, $q_2=F_2 \cap H_1$, then

$$w\text{-ord}(G_2)(q_1)=1/2\,(<w\text{-ord}(\pi(q_1)=p)=3/2)\ w\text{-ord}(G_2)(q_2)=1\,(<3/2),$$

and since the weighted order drops: $n(q_1)=n\ (q_2)=1$ (look at the formulation of $n(x)$ at 6.8.5). Therefore

$$t_2(G_2)(q_1)=(1/2,1)$$
$$t_2(G_2)(q_2)=(1,1)$$

so $C_2=q_2$.

$k=3$: $\mathscr{L}'_3=I(F_1)\, I(F_2)^2\, I(H_1)\, I\, H_2)$, $\bar{\mathscr{L}}'_3=I(F_1)\, I(F_2)^2$.

$$\operatorname{Sing}(G_3)=F_2 \cup \{H_1 \cap H_2\} \cup \{H_1 \cap F_1\}.$$

Set $r_1=H_1 \cap H_2$, $r_2=H_1 \cap F_1$ ($=q_1$).

w-ord (G_3) $(r_1)=0$ (monoidal case).

$$t_2(G_3)(r_2)=(t_2(G_2)(q_1)=(1/2, 1)$$
$$t_2(G_3)(x)=(1, 0) \quad \text{for any } x \in F_2.$$

So now $C_3 = F_2$.

$k=4$: $\mathscr{L}'_4 = I(F_1)\, I(H_1)\, I(H_2)$, $\bar{\mathscr{L}}'_4 = I(F_1)$. Sing $(G_4) = \{H_1 \cap H_2\} \cup \{H_1 \cap F_1\}$.

Set $s_1 = H_1 \cap H_2\ (=r_1)$ $s_2 = H_1 \cap F_1\ (=r_2 = q_1)$. Now

$$t_2(G_4)(s_2) = (1/2, 1) \qquad \text{and} \qquad w\text{-ord}(G_4)(s_1) = 0.$$

Therefore $C_4 = s_2$.

$k=5$: $\mathscr{L}'_5 = I(F_1)\ I(H_1)\ I(H_2)$, $\bar{\mathscr{L}}'_5 = I(F_1)$. Sing$(\mathscr{L}'_5, 2) = H_1 \cap H_2$ so Max w-ord $G_5 = 0$, and the singularity is solved setting $C_5 = H_1 \cap H_2$.

Example 2 (lifting a group action to the resolution). – Let the setup be as in Example 1 replacing f by $g = X_3^2 - X_1^2 X_2^2$, now $\mathbb{Z}/2\mathbb{Z}$ is acting by interchanging x_1 and x_2. And $(\mathscr{L}'_1, b) = (X_1^2 X_2^2, 2)$.

The point now is to check is that all center C_k are invariant by the inductive lifting of the action at the level k.

$k=1$: $\mathscr{L}'_1 = I(F_1)^2 I(F_2)^2$.

Sing $(\mathscr{L}'_1, 2) = F_1 \cup F_2$. Set

$$p = F_1 \cap F_2 . t_2(G_1)(p) = (2,0), \qquad t_2(G_1)(x) = (1,0)$$

$\forall\, x \in \text{Sing } G_1 - \{p\}$. So $C_1 = p$.

$k=2$: $\mathscr{L}'_2 = I(F_1)^2\ I(F_2)^2\ I(H_1)^2$; $\bar{\mathscr{L}}'_2 = I(F_1)^2\ I(F_2)^2$, Sing $G_2 = F_1 \cup F_2 \cup H_1$.

Set $q_1 = F_1 \cap H_1$, $q_2 = F_2 \cap H_1$, then:

$$t_2(G_2)(q_1) = t_2(G_2)(q_2) = (1,1)$$

$$t_2(G_2)(x) = (1,0), \quad \forall\, x \in F_1 \cup F_2 - \{q_1, q_2\}; \qquad w\text{-ord}(G_2)(y) = 0, \quad \forall\, y \in H_1 - \{q_1, q_2\}.$$

So $C_2 = \{q_1, q_2\}$

$k=3$: $\mathscr{L}'_3 = I(F_1)^2 I(F_2)^2 I(H_1)^2 I(H_2)^2$

$$\bar{\mathscr{L}}'_3 = I(F_1)^2 I(F_2)^2, \qquad \text{Sing } G_3 = F_1 \cup F_2 \cup H_1 \cup H_2,$$

w-ord $(G_3) > 0$ only for $x \in F_1 \cup F_2$. Now $t_2(G_3)\ (x) = (1,0)$ for any x along $F_1 \cup F_2$.

So $C_3 = F_1 \cup F_2$.

$k=4$: $\mathscr{L}'_4 = I(H_1)^2\ I(H_2)^2$; $\bar{\mathscr{L}}'_4 = \mathcal{O}_{Z_4}$ we are in the monoidal case (Max w-ord $= 0$) and condition (B) of 7.3 says that Ψ is now determined as in 7.2.

Now check that $\underline{\text{Max}}\, \Psi = H_2\ (= C_4)$

$k=5$: $\mathscr{L}'_5 = I(H_1)^2$. Now $C_5 = H_1$ and the resolution is achieved.

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