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<http://www.numdam.org/item?id=ASENS_1992_4_25_5_567_0>
ZEROS OF DIRICHLET L-FUNCTIONS

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Introduction

Let $\chi$ be a Dirichlet character and $L(s, \chi)$ the associated Dirichlet L-function. We are interested in the zeroes of $L(s, \chi)$ in the critical strip $0 < \Re(s) < 1$. In the past, most attention has focussed on this question near $s=1$. We shall be particularly interested in the situation near $s=1/2$.

It follows from classical results of Landau, Page and others (see Davenport [D] for example) that the number of real characters $\chi$ of conductor $\leq x$ for which $L(s, \chi)$ has a real zero in the region $1-(1/\log x) \leq \sigma \leq 1$ is $O(\log \log x)$. On the other hand, the situation near $s=1/2$ is more delicate and not as well understood.

(1) Research partially supported by a grant from NSERC.
Several authors have studied the frequency with which $L(1/2, \chi) \neq 0$. In [B] it is shown that there are at least $cq/(\log q)^{1000}$ characters $\chi \pmod{q}$ with $L(1/2, \chi) \neq 0$. (Here, and elsewhere, $c$ is a positive constant, though not necessarily the same constant at different occurrences.) In another direction, we can allow both $\chi$ and $q$ to vary while we fix the order of $\chi$. A result of Jutila [J] implies that there are at least $cx/(\log x)$ real characters $\chi$, of conductor at most $x$, for which $L(1/2, \chi) \neq 0$.

In both of these works, the method is to study the moments

$$\sum |L(1/2, \chi)|^k.$$

For example, by the Cauchy-Schwarz inequality,

$$\# \left\{ \chi \pmod{q} : L\left(\frac{1}{2}, \chi\right) \neq 0 \right\} \geq \left( \frac{\sum |L(1/2, \chi)|^2}{\sum |L(1/2, \chi)|^2} \right)^{\frac{1}{2}}.$$

From this, we see that it would suffice to have a lower bound for $\sum |L(1/2, \chi)|$ and an upper bound for $\sum |L(1/2, \chi)|^2$.

There are general conjectures which predict, in particular, the asymptotic growth of the above moments. However, even assuming these conjectures, it does not seem possible to use the Cauchy-Schwarz inequality to deduce that $L(1/2, \chi) \neq 0$ for a positive proportion of the characters $\chi$ to a given modulus, or that $L(1/2, \chi) \neq 0$ for a positive proportion of real characters $\chi$. This result may be viewed as a (partial) $q$-analogue of theorems of Levinson-Selberg type.

On the other hand, no example is known of a character $\chi$ for which $L(1/2, \chi) = 0$. However, Siegel [S] has shown the fundamental result that any point on the line $\sigma = 1/2$ is a limit point of zeroes of the $L(s, \chi)$ as $\chi$ ranges over all Dirichlet characters.

In this paper, we take a different approach from [B] and [J]. We consider characters to a prime modulus $q$. Our first main result is the following.

**Theorem.** – Let $q$ be a sufficiently large prime. Then, for a positive proportion of the characters $\chi \pmod{q}$, we have $L(1/2, \chi) \neq 0$.

Our proof shows that the proportion is $\geq .04$. (Using the explicit formula, Ram Murty [RM] has shown that this proportion can be improved to $\geq .5$ if we assume the Riemann Hypothesis.) Our method actually produces a more general result (Theorem 11.1) which applies to any point $1/2 \leq \sigma < 1$.

Our second main result (Theorem 12.1) gives a non-vanishing theorem which is uniform on a line segment.

**Theorem.** – Let $q$ be a sufficiently large prime. For a positive proportion of the characters $\chi \pmod{q}$, there are no real zeroes of $L(s, \chi)$ in the region $(1/2) + (c/\log q) \leq \sigma < 1$. Here, $c > 0$ is an absolute constant.

In proving our results, our new idea is to count the desired characters directly, without the intermediary of moments of $L$-functions. Let $\chi$ be a non-trivial character. Using
weights \( \{ \lambda(n) \} \) first defined by Barban and Vehov [BV], we consider a mollifier polynomial

\[
M(s, \chi) = \sum_{n \in \mathbb{Z}} \lambda(n) \chi(n) n^{-s}
\]

where \( Z = q^{1/2} \). The \( \lambda(n) \) (which are closely related to Selberg's sieve) will be chosen with the property that if we set

\[
a(n) = \sum_{d | n} \lambda(d),
\]

then \( a(1) = 1 \) and \( a(n) = 0 \) for \( 1 < n < Y \) for some \( 1 \leq Y < Z \). It turns out that to prove our non-vanishing result at a fixed point, the particular choice of \( Y \) is not so crucial and we could take \( Y = 1 \) if we wished. In the proof of the non-vanishing result on an interval, however, we need to take \( Y \) to be a power of \( q \). We choose \( Y = q^{1/4} \). Then, we consider the integral

\[
\frac{1}{2\pi i} \int_{(2)} L(s + w, \chi) M(s + w, \chi) X^w \Gamma(w) dw
\]

where we choose \( X = q \). On the one hand it is equal to

\[
S(s, \chi) = \sum_{n} \frac{a(n) \chi(n)}{n^s} e^{-n/X}
\]

and on the other, it is

\[
L(s, \chi) M(s, \chi) + \frac{1}{2\pi i} \int_{(-\eta)} L(s + w, \chi) M(s + w, \chi) X^w \Gamma(w) dw
\]

where \( \eta > 0 \) is chosen appropriately. Now if \( \chi \) is a primitive character, we can apply the functional equation to transform the integral into

\[
\frac{1}{2\pi i} \int_{(-\eta)} L(1-s-w, \overline{\chi}) M(s + w, \chi) \gamma(s + w, \chi) X^w \Gamma(w) dw
\]

where \( \gamma(s, \chi) \) is an appropriate quotient of \( \Gamma \)-functions. Now if we have \( \eta > \sigma \), we can expand \( L(1-s-w, \overline{\chi}) \) as a Dirichlet series. Splitting it into a Dirichlet polynomial of length \( Z \) and a tail, we get two integrals \( I(s, \chi) \) and \( J(s, \chi) \). Thus our basic equation is

\[
S(s, \chi) = L(s, \chi) M(s, \chi) + I(s, \chi) + J(s, \chi).
\]

If \( L(s_0, \chi) = 0 \) then \( S(s_0, \chi) \) is equal to \( I(s_0, \chi) + J(s_0, \chi) \). We show that this cannot happen too often by comparing mean-square estimates of \( S(s_0, \chi) \), \( I(s_0, \chi) \) and \( J(s_0, \chi) \). Thus, we obtain a lower bound for the number of \( \chi \pmod{q} \) with \( L(s_0, \chi) \neq 0 \). We then extend this to a lower bound for the number of \( \chi \pmod{q} \) for which \( L(s, \chi) \neq 0 \) in a circle of radius \( (\log q)^{-1} \) about \( s_0 \). Equivalently, we obtain an
upper bound for the number of \( \chi \) (mod \( q \)) for which \( L(s, \chi) \) does vanish in this circle. This bound decreases exponentially with \( (\Re s_0 - (1/2)) \). Choosing the point \( s_0 = (1/2) + j (\log q)^{-1} \) and summing over \( j \) produces our non-vanishing result on an interval.

The estimates for \( S \) and \( J \) are given in § 3 and § 4. The mean square of \( I \) is determined in § 10, after preparations in § 5-§ 9. The main results are proved in § 11 and § 12. For an exposition of some of the results and techniques of this paper, the reader may consult [KM].

It is a pleasure to thank J. Friedlander, M. Jutila, and R. Murty for encouraging and helpful discussions. We would also like to thank the referee for a careful reading of the manuscript.

**Notation.** — \( \sum_{\chi \text{ (mod } q)} \) denotes a sum over characters mod \( q \). We denote by \( d(n) \) the number of positive divisors of \( n \) and for \( r \in \mathbb{R} \), \( \sigma_r(n) \) denotes the sum \( \sum_{d|n} d^r \).

1. **The Barban-Vehov Weights.** — Let \( 1 \leq z_1 \leq z_2 \). Following Barban and Vehov [BV], we introduce the functions

\[
\Lambda_i(n) = \begin{cases} 
\mu(n) \log \left( \frac{z_i}{n} \right) & \text{if } n \leq z_i \\
0 & \text{if } n > z_i,
\end{cases}
\]

for \( i = 1, 2 \). We also define

\[
\lambda(n) = \frac{\Lambda_2(n) - \Lambda_1(n)}{\log \left( \frac{z_2}{z_1} \right)} = \begin{cases} 
\mu(n) & 1 \leq n \leq z_1 \\
\mu(n) \frac{\log \left( \frac{z_2}{n} \right)}{\log \left( \frac{z_2}{z_1} \right)} & z_1 \leq n \leq z_2 \\
0 & n > z_2.
\end{cases}
\]  

(1.1)

Let us define

\[
a(n) = \sum_{d|n} \lambda(d).
\]

Graham [Gr] has found asymptotic estimates for the mean square of the \( a(n) \). We recall his main result.

**Proposition (1.1).** — We have

\[
\sum_{n \leq N} |a(n)|^2 = \begin{cases} 
\frac{N \log (N/z_1)}{\log^2 (z_2/z_1)} + O \left( \frac{N}{\log^2 (z_2/z_1)} \right) & \text{if } z_1 < N < z_2 \\
\frac{N}{\log (z_2/z_1)} + O \left( \frac{N}{\log^2 (z_2/z_1)} \right) & \text{if } z_2 \leq N.
\end{cases}
\]
Applying the Cauchy-Schwarz inequality and Proposition (1.1), we deduce the following.

**Proposition (1.2).** Let \( r \leq N \) and \((b, r) = 1\). We have

\[
\sum_{n \leq N} |a(n)| \leq \frac{N}{\varphi(r)^{1/2} (\log z_2/z_1)^{1/2}}.
\]

We next obtain an estimate for a shifted convolution.

**Proposition (1.3).** Let \( 1 \leq k \in \mathbb{Z}, t \in \mathbb{R} \) and \( k \leq M < N \). Then we have

\[
\sum_{\substack{a(n) < n \leq N \atop a(n-k) < n-k \leq N}} a(n) a(n-k) \left( \frac{n}{n-k} \right)^{k} \leq \frac{N + z_2^2}{(\log z_2/z_1)^2} |P(t)| + \frac{N |t|^4 (\log z_2)^4}{(\log z_2/z_1)^2}
\]

where \( P(t) \) is a polynomial in \( t \) (depending on \( k \)) with complex bounded coefficients and of degree \( \leq 4 \).

The proof will require two preliminary results. We begin by recalling a result from Graham [Gr, Lemma 2].

**Lemma (1.4).** For any integer \( r \), and any \( c > 0 \),

\[
\sum_{\substack{n \leq Q \atop (n, r) = 1}} \mu(n) \log \left( \frac{Q}{n} \right) = \frac{r}{\varphi(r)} + O(c \log^{-c} (2Q)).
\]

**Lemma (1.5).** We have for \( 1 \leq d_1, d_2 \leq z_2 \) and \( r_1, r_2 \geq 1 \) that

\[
\sum_{\substack{1 \leq j_1, j_2 \leq z_1/d_1, 1 \leq j_2 \leq z_2/d_2 \atop (j_1, j_2) = (j_1, r_1) = (j_2, r_2) = 1}} \frac{\Lambda_1(d_1j_1) \Lambda_2(d_2j_2)}{j_1j_2} \leq \left( \frac{d_1 r_1}{\varphi(d_1 r_1) + \sigma_{-1/2}(d_1 r_1)} \right) \left( \frac{d_2 r_2}{\varphi(d_2 r_2) + \sigma_{-1/2}(d_2 r_2)} \right).
\]

The same estimate holds even if we drop the condition that \((j_1, j_2) = 1\).

**Proof.** The sum in question is

\[
\sum_{\substack{1 \leq j_1, j_2 \leq z_1/d_1 \atop (j_1, j_2) = (j_1, r_1) = (j_2, r_2) = 1}} \frac{\Lambda_1(d_1j_1) \Lambda_2(d_2j_2)}{j_1j_2} \sum_{e \mid (j_1, j_2)} \mu(e) = \sum_{e \mid z_1/d_1} \mu(e) \sum_{e \mid z_2/d_2} \frac{\Lambda_1(d_1j_1) \Lambda_2(d_2j_2)}{j_1j_2},
\]

the inner sum ranging over \( j_1, j_2 \) satisfying

\[
1 \leq j_1 \leq z_1/d_1, \quad 1 \leq j_2 \leq z_2/d_2, \quad j_1j_2 \equiv 0 \pmod{e}, \quad (j_1, r_1) = (j_2, r_2) = 1.
\]
Let us set \( r = r_1 r_2 \) and \( d = d_1 d_2 \). Then the sum is seen to be

\[
\sum_{e \leq z_1/d_1} \frac{\mu(e)}{e^2} \sum_{l_1 \leq z_1/d_1, \, l_2 \leq z_2/d_2} \frac{\Lambda_1(d_1 e l_1) \Lambda_2(d_2 e l_2)}{l_1 l_2}
\]

\[
= \mu(d_1) \mu(d_2) \sum_{e \leq z_1/d_1} \frac{\mu(e)}{e^2} \left\{ \sum_{l_1 \leq z_1/d_1} \frac{\mu(l_1) \log \left( z_1/d_1 e l_1 \right)}{l_1} \right\}
\]

\[
\times \left\{ \sum_{l_2 \leq z_2/d_2} \frac{\mu(l_2) \log \left( z_2/d_2 e l_2 \right)}{l_2} \right\}
\]

\[
= \mu(d_1) \mu(d_2) \sum_{e \leq z_1/d_1} \frac{\mu(e)}{e^2} \prod_{k=1}^{2} \left\{ \frac{d_k e r_k}{\varphi(d_k e r_k)} + O\left( \sigma_{-1/2} (d_k e r_k) \log^{-c} \left( \frac{2 z_k}{d_k e} \right) \right) \right\}
\]

using Lemma (1.4).

The main terms contribute an amount

\[
\frac{\mu(d_1) \mu(d_2) dr}{\varphi(d_1 r_1) \varphi(d_2 r_2)} \sum_{e \leq z_1/d_1} \frac{\mu(e)}{e^2} \leq \frac{dr}{\varphi(d_1 r_1) \varphi(d_2 r_2)}.
\]

The product of the \( O \)-terms contributes an amount

\[
\ll \sum_{e \leq z_1/d_1} \frac{1}{e^2} \sigma_{-1/2} (d_1 r_1) \sigma_{1/2} (d_2 r_2) \sigma_{-1/2} (e) \leq \sigma_{-1/2} (d_1 r_1) \sigma_{1/2} (d_2 r_2).
\]

The cross-terms contribute an amount

\[
\ll \sum_{e \leq z_1/d_1} \frac{1}{e^2} \left\{ \frac{d_1 e r_1}{\varphi(d_1 e r_1)} \cdot \sigma_{-1/2} (d_2 e r_2) \log^{-c} \left( \frac{2 z_2}{d_2 e} \right) \right\}
\]

\[
+ \frac{d_2 e r_2}{\varphi(d_2 e r_2)} \cdot \sigma_{-1/2} (d_1 e r_1) \log^{-c} \left( \frac{2 z_1}{d_1 e} \right) \}
\]

\[
\ll \left\{ \frac{d_1 r_1}{\varphi(d_1 r_1)} \sigma_{-1/2} (d_2 r_2) + \frac{d_2 r_2}{\varphi(d_2 r_2)} \sigma_{-1/2} (d_1 r_1) \right\}
\]

since the series \( \sum \sigma_{-1/2} (e)/e \varphi(e) \) converges. This proves the first statement. The second statement is easy to verify since there is now no condition relating \( j_1 \) and \( j_2 \). We argue as above setting \( e = 1 \).

Now we are ready to prove the estimate of the shifted convolution.
Proof of Proposition 1.3. — Again, we consider the sum

\[(1.2) \sum_{M<n\leq N} \left( \sum_{d \mid n} \Lambda_1(d) \right) \left( \sum_{e \mid n-k} \Lambda_2(e) \right) \left( \frac{n}{n-k} \right)^u \]

and we find that it is equal to

\[(1.3) \sum_{d, e} \Lambda_1(d) \Lambda_2(e) \sum_{M<n\leq N} \left( \frac{n}{n-k} \right)^u. \]

We see that the inner sum is zero unless \((d, e) \mid k\). Consider the identity

\[
\left( \frac{n}{n-k} \right)^u = (1 + k)^u \left( 1 - \frac{k}{1 + k} \left( 1 - \frac{1}{n-k} \right) \right)^u.
\]

We have an expansion

\[
\left( \frac{n}{n-k} \right)^u = (1 + k)^u \sum_{j=0}^{4} P_j(t) (n-k)^{-j} + O(\|t\|^4)
\]

where \(P_j(t)\) is a polynomial in \(t\) of degree \(\leq 3\) with complex coefficients which are absolutely bounded and depend on \(k\). Using this, we see that

\[(1.4) \sum_{M<n\leq N} \left( \frac{n}{n-k} \right)^u = (1 + k)^u \sum_{j=0}^{4} P_j(t) \sum_{M<n\leq N} (n-k)^{-j} + O(\|t\|^4) \sum_{M<n\leq N} 1.\]

Inserting this into (1.3), we get a main term of

\[(1.5) (1 + k)^u \sum_{j=0}^{4} P_j(t) \sum_{d, e} \Lambda_1(d) \Lambda_2(e) \sum_{M<n\leq N} (n-k)^{-j}.\]

If \(j=0\), the innermost sum is

\[
\frac{N-M}{[d, e]} + O(1)
\]

and if \(j=1\), it is

\[
\log \left( \frac{N-k}{M-k} \right) \frac{1}{[d, e]} + O(1).
\]
For \( j \geq 2 \) it is \( O(1) \). Thus, (1.5) is

\[
(1.6) \quad (1+k)^{\mu} \left( P_0(t) (N-M) + P_1(t) \log \left( \frac{N-k}{M-k} \right) \right) \sum_{d,e} \frac{\Lambda_1(d) \Lambda_2(e)}{(d,e) \mid k} \sum_{d,e} |\Lambda_1(d) \Lambda_2(e)| + O(|t|^3 \sum_{d,e} |\Lambda_1(d) \Lambda_2(e)|)
\]

The \( O \)-term is easily seen to be

\[
\ll z_1 z_2 (|t| + 1)^3.
\]

To evaluate the main term, we see that the sum over \( d, e \) is

\[
(1.7) \quad \sum_{d,e} \frac{\Lambda_1(d) \Lambda_2(e)}{(d,e) \mid k} \sum_{m \mid d} \sum_{e \mid m} \varphi(m).
\]

This is seen to be equal to

\[
\sum_{m \mid k} \frac{\varphi(m)}{m^2} \sum_{d_0, e_0} \frac{\Lambda_1(m d_0) \Lambda_2(m e_0)}{d_0 e_0}.
\]

Here, the inner sum ranges over pairs \( d_0, e_0 \) satisfying

\[
1 \leq d_0 \leq \frac{z_1}{m}, \quad 1 \leq e_0 \leq \frac{z_2}{m},
\]

\((d_0, m) = (e_0, m) = 1.\)

Also note that in the outer sum \( m \) must be squarefree for otherwise \( \Lambda_1(m d_0) = \Lambda_2(m e_0) = 0 \). Thus, invoking Lemma (1.5), we find that the main term in (1.7) is

\[
\ll \sum_{m \mid k} \frac{\mu^2(m) \varphi(m)}{m^2} \left( \frac{m}{\varphi(m)} + \sigma_{-1/2}(m) \right)^2 \ll \frac{k}{\varphi(k)}.
\]

Hence the main term in (1.6) is

\[
\ll \frac{k}{\varphi(k)} \left( |P_0(t)| N + |P_1(t)| \log N \right).
\]

Summarizing, the main term of (1.4) contributes to (1.3) an amount

\[
\ll \frac{k}{\varphi(k)} \left( |P_0(t)| N + |P_1(t)| \log N \right) + z_1 z_2 (|t| + 1)^3.
\]
The error term in (1.4) contributes to (1.3) an amount
\[ N |t|^4 \sum_{d, e \mid (d, e) \mid k} \left| \frac{\Lambda_1(d) \Lambda_2(e)}{[d, e]} \right| + |t|^4 z_1 z_2. \]

The first term above is estimated by
\[ \sum_{d, e \mid (d, e) \mid k} \left| \frac{\Lambda_1(d) \Lambda_2(e)}{[d, e]} \right| \leq \sum_{m \mid k} \phi(m) \mu(m)^2 \left( \sum_{d_0, m} \frac{1}{d_0 m} \log \frac{z_1}{m d_0} \right) \left( \sum_{e_0 m} \frac{1}{e_0 m} \log \frac{z_2}{me_0} \right) \leq k \frac{(\log z_1)^2 (\log z_2)^2}{\varphi(k)}. \]

Summarizing, the error term in (1.4) contributes to (1.3) an amount
\[ \leq \frac{k}{\varphi(k)} |t|^4 N (\log z_1)^2 (\log z_2)^2 + z_1 z_2 |t|^4. \]

The Proposition follows.

2. THE MOLLIFIER POLYNOMIAL. - We shall now introduce the following parameters. Let us set
\[ Y = (\log q) \]
\[ Z = q^{1/2} \]

Corresponding to the choices \( z_1 = Y \) and \( z_2 = Z \), we have from § 1 the weights
\[ \lambda(n) = \frac{\Lambda_2(n) - \Lambda_1(n)}{\log (Z/Y)}. \]

We define the Dirichlet polynomial
\[ M(s, \chi) = \sum_{n \leq Z} \frac{\lambda(n) \chi(n)}{n^s} \]
where \( \chi \) is a Dirichlet character. Then, we have
\[ L(s, \chi) M(s, \chi) = \sum_{n=1}^{\infty} \frac{a(n) \chi(n)}{n^s} \]
where
\[ a(n) = \sum_{d \mid n} \lambda(d) \]
satisfies
\[ a(1) = 1 \]
\[ a(n) = 0 \quad \text{for} \quad 1 < n \leq Y. \]

We record the following estimate.

**Lemma (2.1).** — For \(|\sigma| < 1/2\), and \(\sigma\) bounded away from \(1/2\), we have
\[
\sum_{\chi \pmod{q}} |M(s, \chi)|^2 \ll (q + Z) \sum_{n \leq Z} \frac{|\lambda(n)|^2}{n^{2\sigma}} \left( \frac{q^{1/2 - \sigma}}{1 - 2\sigma} \cdot \frac{1}{(\log q)^2} + Y^{1 - 2\sigma} \right).
\]

**Proof.** — We use the large sieve inequality [D] to get
\[
\sum_{\chi \pmod{r}} |M(s, \chi)|^2 \ll (Z + q) \sum_{n \leq Z} \frac{|\lambda(n)|^2}{n^{2\sigma}} \ll (q + Z) \left\{ \sum_{n \leq Y} \frac{1}{n^{2\sigma}} + \sum_{Y < n \leq Z} \left( \frac{\log Z / n}{\log Z / Y} \right)^2 \cdot \frac{1}{n^{2\sigma}} \right\} \ll (q + Z) \left\{ \frac{Y^{1 - 2\sigma}}{1 - 2\sigma} + \frac{Z^{1 - 2\sigma}}{1 - 2\sigma} \cdot \frac{1}{(\log Z / Y)^2} \right\}.
\]

The result follows from our choices of \(Y\) and \(Z\).

3. The Basic Equation. — Let us define
\[
S(s, \chi) = S(s, \chi, q) = \sum_{n=1}^{\infty} \frac{a(n) \chi(n)}{n^s} e^{-n/q}.
\]

Let \(s \in \mathbb{C}\) with \(1 > \sigma = \text{Re}(s) \geq 1/2\). Using the well-known identity
\[
\frac{1}{2\pi i} \int_{(2)} X^w \Gamma(w) dw = e^{-1/X},
\]
we find that for a character \(\chi\),
\[
S(s, \chi) = \frac{1}{2\pi i} \int_{(2)} L(s + w, \chi) M(s + w, \chi) q^w \Gamma(w) dw.
\]

Moving the line of integration to the left, we find that
\[
(3.1) \quad S(s, \chi) = L(s, \chi) M(s, \chi) + \frac{1}{2\pi i} \int_{(-\eta)} L(s + w, \chi) M(s + w, \chi) q^w \Gamma(w) dw
\]

where \(\sigma < \eta < 1\).
We can decompose the integral along the line \(-\eta\) into two parts as follows. Suppose that \(\chi\) is non-trivial. We apply the functional equation

\[
L(s, \chi) = \gamma(s, \chi) L(1-s, \overline{\chi})
\]

where

\[
\gamma(s, \chi) = \frac{\tau(\chi)}{\rho q^{1/2}} \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{2\pi}{q} \right)^{s-(1/2)} \sin \left( \frac{\pi}{2} (a + s) \right) \Gamma(1-s).
\]

[Here \(\tau(\chi)\) is the Gauss sum, \(a=0, 1\) and \(\chi(-1) = (-1)^a\).] Then we truncate the Dirichlet series expansion of \(L(1-s-w, \overline{\chi})\) at \(Z\). Let us set

\[
I(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n \leq Z} \frac{\overline{\chi}(n)}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) \, dw
\]

and

\[
J(s, \chi) = \frac{1}{2\pi i} \int_{(-\eta)} \gamma(s+w, \chi) \left\{ \sum_{n \leq Z} \frac{\overline{\chi}(n)}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) \, dw
\]

Thus, we get

\[
S(s, \chi) = L(s, \chi) M(s, \chi) + I(s, \chi) + J(s, \chi).
\]

If \(L(s, \chi) = 0\), then \(S(s, \chi)\) and \(I(s, \chi) + J(s, \chi)\) are equal. We will therefore try to show that, in general, they are not equal and for this purpose we study their mean values. We begin with \(J(s, \chi)\) which is the easiest of the three to estimate.

**Proposition (3.1).** For \(|\text{Im } s| < 1\), and \(0 \leq \sigma \leq 1\), we have

\[
\sum_{1 \neq \chi (\text{mod } q)} |J(s, \chi)| \ll \epsilon \frac{q^{(3/2) - \sigma}}{\log q}.
\]

**Proof.** From Stirling’s formula, we know that

\[
\gamma(s, \chi) \ll (q (|s| + 1))^{(1/2) - \sigma}.
\]

Using this and the definition, we find that

\[
\sum_{1 \neq \chi (\text{mod } q)} |J(s, \chi)| \ll q^{(1/2) - \sigma + \epsilon} q^{-\eta} \sum_{\chi (\text{mod } q)} \int_{(-\eta)} \left( \left| w \right| + 1 \right)^{(1/2) - \sigma + \epsilon} \left| \sum_{n \leq Z} \overline{\chi}(n) n^{1-s-w} \right| \left| M(s+w, \chi) \right| \left| \Gamma(w) \right| \, dw
\]
which by a double application of the Cauchy-Schwarz inequality is

\[ \ll q^{(1/2)-\sigma} \sum_{\chi \pmod{q}} \left( \int \left( \left| \frac{\chi(n)}{n^{1-s-w}} \sum_{n \equiv \gamma} \right|^2 \left| \Gamma(w) \right| \left| dw \right| \right)^{1/2} \times \left( \int \left| M(s+w, \chi) \right|^2 \left| \Gamma(w) \right| \left| dw \right| \right)^{1/2} \]

\[ \ll q^{(1/2)-\sigma} \left( \sum_{\chi \pmod{q}} \left( \left| \frac{\chi(n)}{n^{1-s-w}} \sum_{n \equiv \gamma} \right|^2 \left| \Gamma(w) \right| \left| dw \right| \right)^{1/2} \times \left( \sum_{\chi \pmod{q}} \left( \left| M(s+w, \chi) \right|^2 \left| \Gamma(w) \right| \left| dw \right| \right)^{1/2} \right) \]

Using the large sieve inequality and Lemma (2.1), we find that

\[ \sum_{1 \not\equiv \chi \pmod{q}} |J(s, \chi)| \ll q^{(1/2)-\sigma} \left\{ \sum_{n \equiv \gamma} \left( q^2 \right)^{(2\sigma+2\eta-1)} \right\}^{1/2} \times \left\{ \frac{q+Z}{1-2(\sigma-\eta)} \left( \frac{q^{(1/2)-\sigma+\eta} (\log q)^2}{1-2(\sigma-\eta)} + Y^{1-2(\sigma-\eta)} \right) \right\}^{1/2} \ll q^{(1/2)-\sigma} Z^{\sigma-\eta} \left\{ \frac{q}{|2(\sigma-\eta)-1|} \right\}^{1/2} \times \left\{ \frac{q+Z}{1-2(\sigma-\eta)-1} \right\}^{1/2} \frac{q^{(1/2)-\sigma+n}}{\log q} \right\}^{1/2} \]

Now, let us choose \( \eta \) so that it satisfies

\[ \frac{1}{4} > |\eta - \sigma| > \frac{1}{8} \text{ (say)} \]

if \( \sigma < 3/4 \).

We would then have

\[ (3.3) \sum_{1 \not\equiv \chi \pmod{q}} |J(s, \chi)| \ll \frac{q^{(3/2)-\sigma}}{\log q} \]

which proves the result.

4. The mean and mean square of \( S(s, \chi) \).

Proposition (4.1). — For any \( \varepsilon > 0 \), we have

\[ \sum_{\chi \pmod{q}} S(s, \chi) = \varphi(q) + O_{\varepsilon}(q^{1-\sigma+\varepsilon}) \]

Moreover, the same estimate holds if we sum only over non-trivial characters.
Proof. — By definition, we have that

\[
\sum_{\chi \pmod{q}} S(s, \chi) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n/q} \sum_{\chi \pmod{q}} \chi(n)
\]

\[
= \varphi(q) \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n/q}.
\]

Using the bound \(|a(n)| \leq d(n) \ll n^\varepsilon\), we find that the sum is

\[
e^{-1/q} + O\left( \frac{1}{q^\sigma} \right) \sum_{\sigma} \frac{1}{\sigma^q} \exp(-\tau).
\]

The \(O\)-term is

\[
\ll \varepsilon q^{-\sigma + \varepsilon}.
\]

It thus follows that

\[
\sum_{\chi \pmod{q}} S(s, \chi) \leq \varphi(q) + O\left(q^{1-\sigma + \varepsilon}\right).
\]

Finally,

\[
S(s, 1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n/q} \ll q^{1-\sigma + \varepsilon}
\]

as before. This proves the result.

**Proposition (4.2).** — We have

\[
\sum_{\chi \pmod{q}} \left| S\left(\frac{1}{2} + it, \chi\right) \right|^2 = \frac{5}{2} \varphi(q) + O\left((1 + |t|)^4 q (\log q)^{-1/2}\right) + O\left(|t|^4 q (\log q)^{7/2}\right).
\]

For \(1/2 < \sigma \leq 1\), we have

\[
\sum_{\chi \pmod{q}} \left| S(\sigma + it, \chi) \right|^2 = \varphi(q) - \frac{4 \varphi(q) q^{1/2} - \sigma}{(1 - 2 \sigma)^2 (\log q)^2} + \frac{4 \varphi(q) q^{1-2 \sigma} (\log q)^2 (1 - 2 \sigma)^2}{(\log q)(1 - 2 \sigma)} + \frac{2 \varphi(q) q^{1-2 \sigma}}{(\log q)(1 - 2 \sigma)} + O\left(\frac{\varphi(q) q^{1-2 \sigma}}{(\log q)^2 (1 - 2 \sigma)}\right) + O\left(\frac{\varphi(q) q^{1-2 \sigma} (\log q)^{2 - \sigma}}{1 - \sigma}\right)
\]

\[
\{ (1 + |t|)^4 + (|t| \log q)^4 \}
\]

where for \(\sigma = 1\), we interpret \((1 - \sigma)^{-1}\) to be \(\log q\).
Proof. — We see that the sum is equal to

\[
\sum_{n_1, n_2 = 1}^{\infty} \frac{a(n_1) a(n_2)}{(n_1 n_2)^\sigma} \left( \frac{n_2}{n_1} \right)^i \exp \left( -\frac{(n_1 + n_2)}{q} \right) \sum_{\chi \mod q} \chi(n_1) \bar{\chi}(n_2)
\]

which is seen to be

\[
\varphi(q) \sum_{n_1, n_2 = 1}^{\infty} \frac{a(n_1) a(n_2)}{(n_1 n_2)^\sigma} \left( \frac{n_2}{n_1} \right)^i \exp \left( -\frac{(n_1 + n_2)}{q} \right),
\]

where the inner sum ranges over pairs \((n_1, n_2)\) satisfying

\[
n_1 \equiv n_2 \mod q, \quad (n_1, q) = (n_2, q) = 1.
\]

We split the double sum into three pieces \(\Sigma_1 + \Sigma_2 + \Sigma_3\). In \(\Sigma_1\) we have \(n_1 < n_2\), in \(\Sigma_2\) we have \(n_1 > n_2\), and in \(\Sigma_3\) we have \(n_1 = n_2\). The estimation of \(\Sigma_1\) and \(\Sigma_2\) is the same, so we only consider \(\Sigma_1\). We have

\[
\Sigma_1 = \sum_{n_1 = 1}^{\infty} \frac{a(n_1)}{n_1^\sigma} \sum_{n_2 = 1}^{\infty} \frac{a(n_2)}{n_2^\sigma} \left( \frac{n_2}{n_1} \right)^i \exp \left( -\frac{n_1}{q} \right) \exp \left( -\frac{n_2}{q} \right) \left( \frac{n_2}{n_1} \right)^i.
\]

We begin by considering the sum over \(n_2\). We must necessarily have \(n_2 > q\) for if \(n_2 \leq q\), then \(n_1 \leq q\) also and so the congruence \(n_2 \equiv n_1 \mod q\) would force \(n_1 = n_2\). We split \(\Sigma_1\) into three subsums \(\Sigma_{11}, \Sigma_{12}\) and \(\Sigma_{13}\) where

- in \(\Sigma_{11}\) we have \(n_2 \geq q \log q\)
- in \(\Sigma_{12}\) we have \(q \leq n_1 < q \log q\) and \(n_1 < n_2 < q \log q\)
- in \(\Sigma_{13}\) we have \(n_1 < q\) and \(q < n_2 < q \log q\).

In \(\Sigma_{11}\), we see, by partial summation, that the sum over \(n_2\) is

\[
\ll q^{-1} \int_{q \log q}^{\alpha} \left\{ \sum_{n \equiv n_1 \mod q} \frac{|a(n)|}{n^\sigma} \right\} u^{-\sigma} e^{-u/q} du.
\]

We have from Proposition (1.2) that

\[
\sum_{n \equiv n_1 \mod q} \frac{|a(n)|}{n^\sigma} \ll \frac{u}{\varphi(q)^{1/2} \log^2 q^{1/2}}.
\]

Thus, we find that the integral is

\[
\ll \frac{1}{q^{3/2} \log^2 q^{1/2}} \int_{q \log q}^{\alpha} u^{1-\sigma} e^{-u/q} du.
\]
and this is
\[ q^{1/2 - \sigma} (\log q)^{-1/2} \int_{\log q}^{\infty} \frac{v^{-\sigma} e^{-v}}{v} dv. \]
\[ \ll q^{1/2 - \sigma} (\log q)^{-1/2 - \sigma}. \]

Inserting this into the \( n_1 \) sum, using Proposition (1.1), the Cauchy-Schwarz inequality and partial summation, we have
\[ \sum_{n_1} \ll \frac{q^{1 - \sigma}}{(1 - \sigma)(\log q)^{1/2}} \frac{(\log q)^{1/2 - \sigma}}{q^{1/2 + \sigma}} \ll \frac{q^{1/2 - 2\alpha} (\log q)^{-\sigma}}{1 - \sigma}. \]

Now we consider the contribution of \( \Sigma_{12} \). This is
\[ \sum_{q \leq n_1 < q \log q} \frac{a(n_1) e^{-n_1/q}}{n_1^\alpha} \sum_{n_1 < n_2 < q \log q, n_2 \equiv n_1 (\text{mod } q)} \frac{a(n_2) e^{-n_2/q}}{n_2^\alpha} \left( \frac{n_2}{n_1} \right)^u. \]

We split the \( n_1 \) sum into \( O(\log \log q) \) sums of the form
\[ \sum_{U < n_1 \leq 2U} \frac{a(n_1) e^{-n_1/q}}{n_1^\alpha} \sum_{n_1 < n_2 < q \log q, n_2 \equiv n_1 (\text{mod } q)} \frac{a(n_2) e^{-n_2/q}}{n_2^\alpha} \left( \frac{n_2}{n_1} \right)^u. \]

Let us write \( n_2 = n_1 + jq \). The above double sum may therefore be written as
\[ \sum_{j < \log q} e^{-j} \sum_{U < n_1 \leq 2U} e^{-2n_1/q} \frac{a(n_1) a(n_1 + jq)}{n_1^\alpha (n_1 + jq)^\alpha} \left( \frac{n_1 + jq}{n_1} \right)^u. \]

If we drop the condition \( (n_1, q) = 1 \), then we introduce an additional sum
\[ \sum_{j < \log q} e^{-j} \sum_{U < n_1 \leq 2U} e^{-2n_1/q} \frac{a(qk) a((k + j)q)}{(kq)^\alpha ((k + j)q)^\alpha} \left( \frac{k + j}{k} \right)^u. \]

Observe that as \( q \) is prime, and \( \lambda(n) = 0 \) for \( n > Z = q^{1/2} \), we have
\[ a(qk) = \sum_{d \mid kq} \lambda(d) = \sum_{d \mid k} \lambda(d) = a(k). \]

Therefore, we have the estimate
\[ |a(qk)| \leq d(k) \ll \varepsilon k^{2}. \]
A similar estimate holds for \( a((k+j)q) \). Using this in (4.4), we see that it is

\[
\ll q^{-2\sigma} \sum_{j < \log q} e^{-j} \sum_{U < qk \leq 2U} \frac{e^{-2k}}{k^{-\varepsilon}(k+j)^{\sigma-\varepsilon}}
\]

and this is

\[
\ll q^{-2\sigma}.
\]

The sum in (4.3) may thus be replaced by

(4.5) \[
\sum_{j < \log q} e^{-j} \sum_{U < n_1 \leq 2U} e^{-2n_1/q} \frac{a(n_1) a(n_1 + jq)}{n_1^\alpha (n_1 + jq)^\alpha} \left( \frac{n_1 + jq}{n_1} \right)^u
\]

Let us set

\[
G(u) = \sum_{U < n_1 \leq u} a(n_1) a(n_1 + jq) \left( \frac{n_1 + jq}{n_1} \right)^u.
\]

By Proposition (1.3), we see that for \( U < u \),

\[
G(u) \ll \frac{j}{\varphi(j)} \left( \frac{(u+j+1)q}{(\log q)^2} P(t) \right) + (u+jq)^4 (\log q)^4.
\]

The sum over \( n_1 \) in (4.5) can be estimated using partial summation. We find that it is equal to

\[
\frac{G(u) e^{-2u/q}}{u^\sigma (u+jq)^\sigma} \int_0^{2U} G(u) d\left( \frac{e^{-2u/q}}{u^\sigma (u+jq)^\sigma} \right).
\]

Using the estimate for \( G(u) \) quoted above, we see that for \( \sigma \neq 1 \), this is

\[
e^{-2U/q} \frac{j}{\varphi(j)} \frac{(U+jq)^{1-\sigma}}{U^{1-\sigma}} \frac{U}{q(1-\sigma)} (\log q)^{-2} (P(t) + (|t| \log q)^4)
\]

If \( \sigma = 1 \), then we can suppress the term \((1-\sigma)^{-1}\). Note that though the coefficients of \( P(t) \) depend on \( j \) and \( q \), they are absolutely bounded. Thus,

\[
|P(t)| \ll (1 + |t|)^4.
\]

Incorporating these estimates into the sum over \( j \), we find that (4.5) is for \( \sigma \neq 1 \)

\[
\ll \sum_{j < \log q} (U+jq)^{1-\sigma} \frac{U^{1-\sigma}}{q(1-\sigma)} e^{-2U/q} (\log q)^{-2} \frac{j}{\varphi(j)} e^{-j} (P(t) + (|t| \log q)^4)
\]
which is
\[
\ll \frac{U_{1}^{-\sigma} e^{-2U/q}}{q (1 - \sigma)(\log q)^2} \left( |P(t)| + (|t| \log q)^2 \right) \sum_{j < \log q} e^{-j} \sum_{j < \log q} \left( U + jq \right)^{1 - \sigma}.
\]
\[
\ll q^{1 - \sigma} (\log q)^{-1 - \sigma} \frac{U_{1}^{-\sigma}}{q (1 - \sigma)} e^{-2U/q} (|P(t)| + (|t| \log q)^4).
\]

Now summing this over \( U \), we find it is
\[
\ll q^{1 - 2\sigma} (\log q)^{\sigma - 1} (1 - \sigma)^{-1} (|P(t)| + (|t| \log q)^4).
\]

For \( \sigma = 1 \), we can suppress the term \((1 - \sigma)^{-1}\).

Now we discuss the contribution of \( \Sigma_{13} \). By the Cauchy-Schwarz inequality, we see that
\[
|\Sigma_{13}| \ll \left( \sum_{n_1 < q} \frac{a(n_1)}{n_1^{2\sigma}} \exp \left( -2 \frac{n_1}{q} \right) \right)^{1/2} \left( \sum_{n_1 \leq q} \frac{\sum_{n_2 \leq q \log q} a(n_2) e^{-\frac{n_2}{q}}}{n_2^{1/2} n_2^{1/2}} \right)^{1/2}.
\]

The first factor above is \( O(1) \) as can be seen from our discussion of \( \Sigma_3 \) below. As for the second factor, we see that it is equal to
\[
\sum_{q < n_2, n_2 \leq n \log q \mod q} \frac{a(n_2) e^{-\frac{n_2}{q}}}{n_2^{1/2}} \frac{a(n'_2) e^{-\frac{n'_2}{q}}}{(n'_2)^{1/2}} \left( \frac{n_2}{n'_2} \right)^{it}.
\]

Again, we split this sum into three sums according as \( n_2 < n'_2, n_2 = n'_2, \) and \( n_2 > n'_2 \). The third is the same as the first. Also, we note that the first sum is just \( \Sigma_{12} \) which we have estimated above as being (for \( \sigma \neq 1 \))
\[
\ll q^{1 - 2\sigma} (\log q)^{\sigma - 1} (1 - \sigma)^{-1} (|P(t)| + (|t| \log q)^4).
\]

If \( \sigma = 1 \), then as before, we may suppress the \((1 - \sigma)^{-1}\) term. As for the second, we see that it is equal to
\[
\sum_{q \leq n_2 < q \log q} \frac{a(n_2)^2 e^{-2n_2/q}}{n_2^{2\sigma}}.
\]

Using Proposition (1.1) and partial summation, this is
\[
\ll q^{1 - 2\sigma} \log q.
\]

Inserting this into the above, we deduce that
\[
\Sigma_{13} \ll q^{(1/2) - \sigma} (\log q)^{-(\sigma + 1)/2} (|P(t)| + (|t| \log q)^4)^{1/2}.
\]
Finally, we discuss the estimation of $\Sigma_3$, namely the terms with $n_1 = n_2$. Thus,

$$
\Sigma_3 = \sum_{n=1}^{\infty} \frac{a(n)^2}{n^{2\sigma}} \exp\left(-2\frac{n}{q}\right) = \sum_{n \leq Y} + \sum_{Y < n \leq q} + \sum_{n > q}.
$$

Since $a(n) = 0$ for $1 < n \leq Y$, we have

$$
\sum_{n \leq Y} = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise}. \end{cases}
$$

Also, by partial summation and Proposition (1.1), we find that

$$
\sum_{n > q} \ll \frac{q^{1-2\sigma}}{\log(z_2/z_1)}.
$$

Thus, we see from (4.6)-(4.8) that

$$
\Sigma_3 = 1 + \sum_{Y < n \leq q} \frac{a(n)^2}{n^{2\sigma}} \exp\left(-2\frac{n}{q}\right).
$$

Let us denote the sum on the right by $S$. We find that

$$
S = \sum_{Y < n \leq q} \frac{a(n)^2}{n^{2\sigma}} \left(1 + O\left(\frac{n}{q}\right)\right).
$$

Now, the $O$-term is

$$
\ll \frac{1}{q} \sum_{Y < n \leq q} \frac{a(n)^2}{n^{2\sigma - 1}} \ll \frac{1}{q} \log(z_2/z_1) \frac{q^{2-2\sigma}}{(1-\sigma)} \ll \frac{q^{1-2\sigma}}{(1-\sigma)\log q}.
$$

The main term is equal to

$$
\sum_{Y < n \leq q} \frac{a(n)^2}{n^{2\sigma}}.
$$

Finally, using Proposition (1.1),

$$
\sum_{n < q} \frac{a(n)^2}{n^{2\sigma}} = \sum_{1 \leq n \leq Y} + \sum_{Y < n \leq Z} + \sum_{Z < n \leq q} \frac{a(n)^2}{n^{2\sigma}}.
$$
The first sum is equal to 1 since \( a(n) = 0 \) for \( 1 < n \leq Y \). Using Proposition (1.1) and partial summation, we see that the second sum is

\[
\sum_{Y < n \leq Z} \frac{(\log n/Y)}{(\log Z/Y)^2} \cdot \frac{1}{n^{2\sigma}} + O\left(\frac{1}{\log Z/Y}\right).
\]

If \( \sigma = 1/2 \) this is

\[
\frac{1}{2} + O\left(\frac{1}{\log q}\right)
\]

and if \( \sigma > 1/2 \), this is

\[
\frac{2 Z^{1-2\sigma}}{(1-2\sigma)(\log q)} \left(1 - \frac{2}{(1-2\sigma)(\log q)} + O\left(\frac{1}{\log q}\right)\right) + \frac{4 Y^{1-2\sigma}}{(1-2\sigma)^2(\log q)^2}.
\]

Similarly, the third sum is

\[
\sum_{Z < n < q} \frac{1}{\log Z/Y} \cdot \frac{1}{n^{2\sigma}} + O\left(\frac{1}{\log q}\right)
\]

which is

\[
= \begin{cases} 
1 + O\left(\frac{1}{\log q}\right) & \text{if } \sigma = \frac{1}{2} \\
\frac{1}{1-2\sigma} \cdot \frac{2}{(\log q)} (q^{1-2\sigma} - Z^{1-2\sigma}) \left(1 + O\left(\frac{1}{\log q}\right)\right) & \text{if } \sigma > \frac{1}{2}.
\end{cases}
\]

Putting these together we deduce that

\[
\sum_{n < q} \frac{a(n)^2}{n} = \frac{5}{2} \left(1 + O\left(\frac{1}{\log q}\right)\right)
\]

and for \( \sigma > 1/2 \)

\[
\sum_{n < q} \frac{a(n)^2}{n^{2\sigma}} = 1 - \frac{4 Z^{1-2\sigma}}{(\log q)^2 (1-2\sigma)^2} + \frac{4 Y^{1-2\sigma}}{(\log q)^2 (1-2\sigma)^2} + \frac{2 q^{1-2\sigma}}{(\log q)(1-2\sigma)} + O\left(\frac{Z^{1-2\sigma}}{(\log q)^2 (1-2\sigma)}\right).
\]

This completes the proof of the proposition.

In the next sections, we shall study the mean square of the integral \( I(s, \chi) \).
5. THE INTEGRAL $R_a(s, \chi)$. — The purpose of the next few sections is to obtain an asymptotic formula for the mean square of $I(s, \chi)$. Recall that for $\chi \neq 1$, we have

$$I(s, \chi) = \frac{1}{2\pi i} \int_{(\eta)} \gamma(s+w, \chi) \left\{ \sum_{n \in \mathbb{Z}} \frac{\chi(n)}{n^{1-s-w}} \right\} M(s+w, \chi) q^w \Gamma(w) \, dw.$$ 

Thus

$$I(s, \chi) = \sum_{m,n \in \mathbb{Z}} \frac{\chi(n) \chi(m) \lambda(m)}{m^n n^{1-s}} \left( \frac{2\pi}{q} \right)^{s-(1/2)} \left( \frac{2}{\pi} \right)^{1/2} \tau(\chi) \frac{R_a(s, 2\pi n m)}{m}$$

where

$$R_a(s, y) = \frac{1}{2\pi i} \int_{(\delta)} y^w \sin \left( \frac{\pi}{2} (s+w+a) \right) \Gamma(1-s-w) \Gamma(w) \, dw.$$

Here $a=0, 1, \chi(-1)=(-1)^a$ and $-1<\delta<0$ is arbitrary, $y>0$ and $0<\Re(s)<1$. Notice that

$$R_a(s, y) = R_a(s, y).$$

The integrand has simple poles at $w=-k$ and $w=1-s+k$ where $0 \leq k \in \mathbb{Z}$. Since $1/2 \leq \Re(s)<1$, these are distinct points. We have the expansion

$$R_a(s, y) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} y^{-k} \sin \left( \frac{\pi}{2} (s-k+a) \right) \Gamma(1-s+k) \quad \text{for} \quad y \geq 1.$$

Indeed, this is just the sum of the residues at the points $w=-k$, $0 \leq k \in \mathbb{Z}$. The condition $y \geq 1$ ensures that it converges. Indeed, we have the following asymptotic expansion.

**Lemma (5.1).** — For $s=\sigma + it$ with $1/2 \leq \sigma < 1$, and $|t|<\sigma/10$, $y \geq 1$, and $0 \leq K \in \mathbb{Z}$, we have

$$R_a(s, y) = \sum_{k=1}^{K} \frac{(-1)^k}{k!} y^{-k} \sin \left( \frac{\pi}{2} (s-k+a) \right) \Gamma(1-s+k)$$

$$+ O \left( y^\delta \frac{1}{K^{\alpha/5}} \left| \Gamma((1-\delta-K-s) \Gamma(1+\delta+K)) \right| \right)$$

for any $\delta \in (-K-1, -K)$.

**Proof.** — We need only estimate the integral defining $R_a$ along a line $-K-1 < \delta < -K$. We write $w=K-\eta$, $0<\Re(\eta)<1$. Write $s=\sigma + it$ and $\eta = \beta + i\gamma$. Then,

$$\left| \Gamma(1-s-w) \Gamma(w) \right| = \prod_{j=1}^{K} \left| 1 - \frac{s}{j+\eta} \right| \left| \Gamma(1+\eta-s) \Gamma(-\eta) \right|.$$
Now,
\[
\left| 1 - \frac{s}{j+\eta} \right| \leq 1 - \frac{\sigma}{j+\eta} + \left| \frac{t}{j+\eta} \right|
\]
and
\[
\left| 1 - \frac{\sigma}{j+\eta} \right|^2 = \left( 1 - \frac{\sigma(j+\beta)}{(j+\beta)^2 + \gamma^2} \right)^2 + \frac{\sigma^2 \gamma^2}{|j+\eta|^4}.
\]
If \( j+\beta > 2\gamma \), we see that
\[
\left| 1 - \frac{\sigma}{j+\eta} \right|^2 \leq \left( 1 - \frac{4\sigma}{5(j+\beta)} \right)^2 + \frac{\sigma^2}{4|j+\beta|^2}.
\]
Therefore,
\[
\left| 1 - \frac{s}{j+\eta} \right| \leq 1 - \frac{4\sigma}{5(j+\beta)} + \frac{(1/2)\sigma + |t|}{j+\beta}
\]
which simplifies to
\[
\left| 1 - \frac{s}{j+\eta} \right| \leq 1 - \frac{3\sigma/10 - |t|}{j+\beta}.
\]
Let us set
\[
u = u(\eta) = \max \left( [2\gamma - \beta], 0 \right) + 1
\]
where \([x]\) denotes the greatest integer \(\leq x\). We deduce that
\[
\prod_{j=u}^{K} \left| 1 - \frac{s}{j+\eta} \right| \leq \prod_{j=u}^{K} \left( 1 - \frac{3\sigma/10 - |t|}{j+\beta} \right) \ll \left( \frac{u}{K} \right)^{(3/10)\sigma - |t|}.
\]
Moreover
\[
\prod_{j \leq u} \left| 1 - \frac{s}{j+\eta} \right| \leq \left( 1 + \frac{3\sigma}{5\gamma} \right)^u \ll 1.
\]
Note that the sine term in the integrand is bounded as a function of \( k \).
There is a similar expression and estimate when \( y \leq 1 \).
**Lemma (5.2).** For \( s = \sigma + it \) with \( 1/2 \leq \sigma < 1, \quad |t| < \sigma/10, \quad 0 < y \leq 1, \quad 0 \leq K \in \mathbb{Z} \) and any \( \delta \in (1 - \sigma + K, 2 - \sigma + K) \), we have

\[
R_a(s, y) = -\sin \left( \frac{\pi}{2} (s + a) \right) \Gamma(1 - s) - \sum_{k=1}^{K} \frac{(-1)^k}{k!} y^{1-s+k} \sin \left( \frac{\pi}{2} (a + k + 1) \right) \Gamma(1 - s + k) + O \left( y^{\delta} \Gamma^{-\sigma/5} \right).
\]

In both cases, we see that for \( s \) as above (that is, \( s = \sigma + it \), and \( 1/2 \leq \sigma < 1 \) and \( |t| < \sigma/10 \)),

\[
R_a(s, y) \ll 1.
\]

Finally, we define

\[
(5.4) \quad \omega_k = \omega_k(s) = \frac{(-1)^k}{k!} y^{1-k} \sin \left( \frac{\pi}{2} (a + k + 1) \right) \Gamma(1 - s + k).
\]

The argument of Lemma (5.1) shows that for \( s = \sigma + it \), with \( 1/2 \leq \sigma < 1 \), we have

\[
(5.5) \quad \omega_k(s) \ll k^{-\sigma+1/2}.
\]

**6. An expression for the mean square of \( I(s, \chi) \).** From (5.1) and (5.3), we see that for a fixed \( a = 0 \) or \( 1 \), and an \( s \), we have

\[
\sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 = \frac{2}{\pi} \left( \frac{2 \pi}{q} \right)^{2a-1} \sum_{m_1, m_2, n_1, n_2 \in \mathbb{Z}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^2 m_2^2 n_1^{-s-n_2^{-s}}} \times R_a \left( s, \frac{2 \pi n_1}{m_1} \right) R_a \left( s, \frac{2 \pi n_2}{m_2} \right) \sum_{1 \neq \chi \pmod{q}} \chi(m_1 n_2) \overline{\chi(n_1 m_2)}.
\]
Notice that we can drop the condition \((m, q) = \ldots = (n_2, q) = 1\) since \(1 \leq m_1, \ldots, n_2 \leq \mathbb{Z} < q\). Observe that for \((n, q) = 1\), and \(q\) odd, we have

\[
\chi(m) \overline{\chi(n)} = \frac{1}{2} \sum_{\chi \not\equiv \chi^*} \chi(m) \overline{\chi(n)} + \frac{1}{2} (-1)^n \sum_{\chi \not\equiv \chi^*} \chi(-m) \overline{\chi(n)}
\]

\[
= \frac{1}{2} \sum_{\epsilon = \pm 1} \epsilon^n \sum_{\chi \not\equiv \chi^*} \chi(\epsilon m) \overline{\chi(n)}
\]

\[
= \begin{cases} 
\frac{1}{2} \varphi(q) - 1 & \text{if } m \equiv n \text{ and } a = 0 \\
-1 & \text{if } m \not\equiv n \text{ and } a = 0 \\
\frac{1}{2} \varphi(q) & \text{if } m \equiv n \text{ and } a = 1 \\
-\frac{1}{2} \varphi(q) & \text{if } m \equiv -n \text{ and } a = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Applying this to the innermost sum, we see that this is

\[
\frac{1}{\pi} (2\pi)^{-2a-1} \varphi(q) q^{1-2a} \sum_{\epsilon} \sum_{m_1, m_2, n_1, n_2 < Z} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} R_a \left( s, \frac{2\pi n_1}{m_1} \right) R_a \left( \frac{2\pi n_2}{m_2} \right)
\]

minus

\[
\delta(a) \frac{1}{\pi} (2\pi)^{-2a-1} q^{1-2a} \sum_{m_1, m_2, n_1, n_2 < Z} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} R_0 \left( s, \frac{2\pi n_1}{m_1} \right) R_0 \left( \frac{2\pi n_2}{m_2} \right)
\]

Here, \(\delta(a) = 1 - a\). If we designate the second quantity as \(|I(s, 1)|^2\), then setting \(a = 0, 1\) and adding, we deduce that

\[
(6.1) \quad \sum_{\chi (\mod q)} \left| I(s, \chi) \right|^2 = \frac{1}{\pi} (2\pi)^{2a-1} \varphi(q) q^{1-2a} (S^+(s, q) + S^-(s, q))
\]

where

\[
S^\pm (s, q) = \sum_{m_1, m_2, n_1, n_2 < Z} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \times \left[ R_0 \left( s, \frac{2\pi n_1}{m_1} \right) R_0 \left( \frac{2\pi n_2}{m_2} \right) \pm R_1 \left( s, \frac{2\pi n_1}{m_1} \right) R_1 \left( \frac{2\pi n_2}{m_2} \right) \right].
\]
Note that if \( s = 1/2 \), this can be rewritten as

\[
S^\pm \left( \frac{1}{2}, q \right) = \sum_{m_1, m_2, n_1, n_2 < Z \atop (m_1, n_2, q) = (m_2, n_1, q) = 1} \frac{\lambda(m_1) \lambda(m_2)}{(m_1 m_2 n_1 n_2)^{1/2}} \times \left[ R_0 \left( \frac{1}{2}, \frac{2\pi n_1}{m_1} \right) R_0 \left( \frac{1}{2}, \frac{2\pi n_2}{m_2} \right) \pm R_1 \left( \frac{1}{2}, \frac{2\pi n_1}{m_1} \right) R_1 \left( \frac{1}{2}, \frac{2\pi n_2}{m_2} \right) \right]
\]

and

\[
(6.2) \quad \sum_{\chi \mod q} \left| \frac{1}{2}, \frac{1}{2}, \chi \right|^{1/2} = \frac{1}{\pi} \varphi(q) \left( S^+ \left( \frac{1}{2}, q \right) + S^- \left( \frac{1}{2}, q \right) \right).
\]

Let us also define, for \( a = 0, 1 \)

\[
S^\pm (s, q; a) = \sum_{m_1, m_2, n_1, n_2 < Z \atop m_1 n_2 \equiv \pm n_1 m_2 \mod q} \frac{\lambda(m_1) \lambda(m_2)}{m_1^{\sigma} m_2^{1-\sigma} n_1^{1-s} n_2^{1-s}} R_a \left( s, \frac{2\pi n_1}{m_1} \right) R_a \left( \bar{s}, \frac{2\pi n_2}{m_2} \right)
\]

so that

\[
S^\pm (s, q) = S^\pm (s, q; 0) \pm S^\pm (s, q; 1).
\]

Our estimations are complicated by the unusual way in which the four indices of summation \( m_1, m_2, n_1, n_2 \) are interlaced. Our goal in the next sections will be to show that the main contribution comes from those terms where \( m_1 n_2 = n_1 m_2 \) and \( n_1 \leq (1/2 \pi) m_1, n_2 \leq (1/2 \pi) m_2 \).

7. ESTIMATE OF THE NON-DIAGONAL TERMS. — We wish to show that the terms in \( S^+ (s, q; a) \) contribute a negligible amount to the right hand side of (6.1). Since \( m_1, m_2, n_1, n_2 < Z \) and \( m_1 n_2 \equiv -n_1 m_2 \mod q \), this means that \( m_1 n_2 = q - m_2 n_1 \). (Notice that for the same reason, the indices in \( S^+ (s, q; a) \) satisfy \( m_1 n_2 = m_2 n_1 \).)

**Lemma (7.1).** For \( 1/2 \leq \alpha < 1 \), we have

\[
S^- (s, q; a) \leq \frac{1}{(1-\alpha)^2 (\log q)^2} + \frac{q^{\alpha-1}}{(1-\alpha) \log q}.
\]

**Proof.** We wish to estimate the sum

\[
\sum_{1 \leq m_1, m_2, n_1, n_2 < Z \atop m_1 n_2 = q - m_2 n_1} \left| \frac{\lambda(m_1)}{m_1^{\sigma} m_2^{1-\sigma} n_1^{1-\alpha} n_2^{1-\alpha}} \right| R_a \left( s, \frac{2\pi n_1}{m_1} \right) R_a \left( \bar{s}, \frac{2\pi n_2}{m_2} \right).
\]

Without loss, we may suppose that \( m_2 n_1 < (1/2) q \). A consequence of this is that \( m_1 n_2 > (1/2) q \) and so

\[
\frac{1}{2} Z < m_1, n_2 < Z.
\]
We may also suppose that \( m_1, m_2, n_1, n_2 \) are squarefree. Notice that we must have \((m_1, m_2) = 1\). We consider two cases.

**Case 1.** \(- m_2 < n_2\).

In this case, we must have \( n_2 \equiv m_1 q (\text{mod } m_2) \) where \( m_1 \) denotes the inverse of \( m_1 \) modulo \( m_2 \). Moreover,

\[
\left| R_s \left( s, \frac{2\pi n_2}{m_2} \right) \right| \leq \frac{m_2}{n_2}.
\]

Thus, we can rewrite our sum as

\[
\sum_{m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^\sigma} \right| \sum_{(1/2)Z < m_1 < Z} \left| \frac{\lambda(m_1)}{m_1^\sigma} \right| \sum_{n_2 \equiv \bar{m}_1 q (\text{mod } m_2)} \frac{1}{n_2^1-s} \left( \frac{q-m_1 n_2}{m_2} \right)^{s-1} \frac{m_2}{n_2}
\]

\[
\leq Z^{\sigma-2} \sum_{m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^{2\sigma-2}} \right| \sum_{(1/2)Z < m_1 < Z} \left| \frac{\lambda(m_1)}{m_1^{\sigma}} \right| \sum_{n_2 > m_2} \left( \frac{1}{q-m_1 n_2} \right)^{1-s}
\]

\[
\leq Z^{\sigma-2} \sum_{m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^{2\sigma-1}} \right| \sum_{(1/2)Z < m_1 < Z} \left| \frac{\lambda(m_1)}{m_1^{1+s}} \right| \left( \frac{q-m_1 Z}{2} \right)^{s}
\]

\[
\leq Z^{2\sigma-2} \sum_{m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^{2\sigma-1}} \right| \sum_{(1/2)Z < m_1 < Z} \left| \frac{\lambda(m_1)}{m_1} \right| \frac{1}{m_1}
\]

\[
\leq Z^{2\sigma-1} \frac{1}{\log Z/Y} \left( \frac{\log Z}{2 - 2\sigma} + \frac{Z^{2-2\sigma} - 1}{(2-2\sigma)^2} \right) \frac{1}{\log Z/Y}
\]

\[
\leq \left( \frac{q^{\sigma-1}}{\sigma-1} + \frac{1}{(\log q) (1-\sigma)^2} \right) \frac{1}{\log q}.
\]

**Case 2.** \(- m_2 \geq n_2\).

In this case, we write the congruence condition as \( m_1 \equiv \bar{m}_1 q (\text{mod } m_2) \). Since \((1/2)Z < m_1 < Z\), this implies that there are at most two possible values for \( m_1 \). Thus, we see that our sum is

\[
\sum_{(1/2)Z < m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^\sigma} \right| \sum_{n_2 \leq m_2} \frac{1}{n_2^{1-s}} \sum_{(1/2)Z < m_1 < Z} \left| \frac{\lambda(m_1)}{m_1^\sigma} \right| \left( \frac{m_2}{q-m_1 n_2} \right)^{1-s}
\]

\[
\leq \sum_{(1/2)Z < m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^{2\sigma-1}} \right| \sum_{n_2 \leq m_2} \frac{1}{n_2^{1-s}} \frac{1}{\log Z/Y} \frac{1}{Z^{\sigma}} \left( \frac{q-Z n_2}{1-s} \right)^{1-s}
\]

\[
\leq \frac{1}{Z^{2-\sigma} (\log Z/Y)} \sum_{(1/2)Z < m_2 < Z} \left| \frac{\lambda(m_2)}{m_2^{2\sigma-1}} \right| \int_{m_2}^{m_3} \frac{dt}{(Z-t)^{1-s}}
\]

\[
\leq \int (\log Z/Y) \left( \frac{Z^{2-\sigma} - 1}{(2-2\sigma)^2} \right) \frac{1}{\log Z/Y}
\]

\[
\leq \left( \frac{q^{\sigma-1}}{\sigma-1} + \frac{1}{(\log q) (1-\sigma)^2} \right) \frac{1}{\log q}.
\]
This proves the result.

Finally in this section, we shall show that \( |I(s, 1)|^2 \) is negligible.

**Lemma (7.2).** We have for \( 1/2 \leq \sigma < 1 \)

\[
|I(s, 1)|^2 \ll \frac{q^{1-\sigma}}{(1-\sigma)^2} \left( 1 + \frac{q^{1-\sigma}}{(1-\sigma)^2 (\log q)^2} \right).
\]

If \( \sigma = 1 \) we have

\[
|I(s, 1)|^2 \ll (\log Z)^2.
\]

**Proof.** By definition, we have

\[
|I(s, 1)|^2 = \frac{2}{\pi} (2\pi)^{2\sigma - 1} q^{1-2\sigma} \sum_{m_1, m_2, n_1, n_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{-s} n_2^{-s}} R_0(s, \frac{2\pi n_1}{m_1}) R_0(s, \frac{2\pi n_2}{m_2})
\]

The \( n_1 \) and \( n_2 \) sums are estimated as \( \ll Z^\sigma/\sigma \). To estimate the sum over \( m_1 \) and \( m_2 \) we observe that for \( \sigma \neq 1 \),

\[
\sum_m |\lambda(m)| m^{-\sigma} \ll \sum \frac{\log(Z/m)}{\log(Z/Y)} \cdot \frac{1}{m^\sigma} + O\left( \frac{Y^{1-\sigma}}{(1-\sigma)} \right) \ll \frac{1}{(\log Z/Y)(1-\sigma)} \left( \frac{Z^{1-\sigma}}{1-\sigma} + \log Z \right).
\]

Using this estimate, we see that

\[
|I(s, 1)|^2 \ll q^{1-2\sigma} \frac{Z^{2\sigma}}{\sigma^2} \left\{ \frac{1}{(\log Z/Y)(1-\sigma)} \left( \frac{Z^{1-\sigma}}{1-\sigma} + \log Z \right) \right\}^2.
\]

and this simplifies to the stated expression, given our choices of \( Y \) and \( Z \).

**8. The diagonal terms.** We are now reduced to the study of the sum

\[
(8.1) \quad \frac{1}{\pi} (2\pi)^{2\sigma - 1} \phi(q) q^{1-2\sigma}
\]

\[
\times \sum_{m_1, m_2, n_1, n_2 < Z, (m_1, n_2, q) = 1} \frac{\lambda(m_1)\lambda(m_2)}{m_1^s m_2^s n_1^{-s} n_2^{-s}} \left( R_0\left(s, \frac{2\pi n_1}{m_1}\right) \right)^2 + \left| R_1\left(s, \frac{2\pi n_1}{m_1}\right) \right|^2.
\]
Let us define

\[(8.2) \quad D_s(s) = \sum_{m_1, m_2, n_1, n_2 < \infty \atop m_1 n_2 = m_2 n_1} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} |R_s \left( \frac{2\pi n_1}{m_1} \right)|^2.\]

For future reference let us denote by \(s\) the set of quadruples \((m, m, n, n)\) included in the above sum. Then, the sum in (8.1) can be written as

\[\frac{1}{\pi} (2\pi)^{2\sigma-1} \frac{\varphi(q)}{q^{2\sigma-1}} (D_0(s) + D_1(s)).\]

We define a splitting

\[D_s(s) = M_s(s) + E_s(s)\]

where in \(M_s(s)\), we range over those quadruples \((m, m, n, n)\) in (8.2) which in addition satisfy \(n_1 \leq (1/2\pi) m_1\). (Note that as \(n_1/m_1 = n_2/m_2\), we will then also have \(n_2 \leq (1/2\pi) m_2\).)

We shall henceforth assume that \(1/2 \leq \sigma \leq 1\). The case \(\sigma = 1\) can also be handled, but it will not be necessary for us. Moreover, we suppose that \(|t|\) is sufficiently small in the strong sense that

\[|t| \ll \frac{1}{\log q}.\]

We begin our study \(M_s(s)\) by replacing \(R_s(s, (2\pi n_1/m_1))\) with the Taylor expansion of Lemma (5.2). We find that

\[(8.3) \quad M_s(s) = \sum_{m_1, m_2, n_1, n_2 < \infty \atop m_1 n_2 = m_2 n_1} \frac{\lambda(m_1) \lambda(m_2)}{m_1^s m_2^s n_1^{1-s} n_2^{1-s}} \sin \left( \frac{\pi}{2} (s+\alpha) \right) \Gamma(1-s) + \sum_{k=1}^{K} \left( \frac{2\pi n_1}{m_1} \right)^{1-s+k} \omega_k
+ O \left( \left( \frac{2\pi n_1}{m_1} \right)^{\delta} K^{-\sigma} \Gamma(2-\delta+K-\sigma) \Gamma(\delta-K) \right)^2\]

where we recall (from (5.4)) that

\[\omega_k = \frac{(-1)^k}{k!} \sin \left( \frac{\pi}{2} (a+k+1) \right) \Gamma(1-s+k)\]

and \(0 \leq K \in \mathbb{Z}\) and \(1-\sigma+K < \delta < 2-\sigma+K\). Expanding, we find that \(M_s(s)\) splits into a main term and an \(O\)-term. We shall now analyze the \(O\)-term with the help of the following lemma.

**Lemma (8.1).** — For any \(\beta > 0\) we have

\[\sum_{m_1, m_2, n_1, n_2 < \infty \atop m_1 n_2 = m_2 n_1} \frac{|\lambda(m_1)| |\lambda(m_2)|}{(m_1 m_2)^\beta (n_1 n_2)^{1-\sigma}} \left( \frac{n_1}{m_1} \right)^{\beta} \ll \frac{1}{(2\sigma+\beta-1)} \frac{1}{(2\pi)^{2\sigma+\beta-1}(\log Z)^3}.\]
Proof. — We use the fact that

\[ m_1 n_2 = m_2 n_1 \]

\[ n_1 \leq \frac{1}{2\pi} m_1, \quad n_2 \leq \frac{1}{2\pi} m_2 \]

and

\[ |\lambda(m)| \leq 1 \quad \text{for all } m. \]

Note that \(2\sigma + \beta > 1\). Then, denoting \(m_1 n_2\) by \(j\), we see that the sum is bounded by

\[ \sum_{m_1, m_2} \frac{1}{(m_1 m_2)^{2\sigma + \beta - 1}} \sum_j j^{2\sigma + \beta - 2} \]

where the inner sum ranges over integers \(j\) satisfying

\[ 1 \leq j \leq \frac{1}{2\pi} m_1 m_2 \]

\[ j \equiv 0 \mod [m_1, m_2]. \]

Let us set

\[ i = (m_1, m_2). \]

Then \([m_1, m_2] = m_1 m_2 / i\) and (8.4) is

\[ \ll \sum \frac{[m_1, m_2]^{2\sigma + \beta - 2}}{(m_1 m_2)^{2\sigma + \beta - 1}} \left( \frac{1}{2\pi} \frac{m_1 m_2}{[m_1, m_2]} \right)^{2\sigma + \beta - 1} \frac{1}{2\sigma + \beta - 1} \]

\[ \ll \frac{1}{2\sigma + \beta - 1} \cdot \frac{1}{(2\pi)^{2\sigma + \beta - 1}} \cdot \sum \frac{i}{m_1 m_2}. \]

Moreover,

\[ \sum \frac{i}{m_1 m_2} \ll \sum \frac{1}{m_1} \sum_{i \mid m_1} \sum_{m \leq (Z/i)} \frac{1}{m} \]

\[ \ll (\log Z)^3. \]

This proves the lemma.
Now, the O-term in (8.3) is, for any \( 0 \leq K \in \mathbb{Z} \) and any \( 1 - \sigma + K < \delta < 2 - \sigma + K \),
\[
\ll \sum_{(m_1m_2)\mathfrak{d}((m_1)\mathfrak{d})} \left| \frac{\lambda(m_1)\lambda(m_2)}{m_1m_2} \right| \frac{2\pi n_1}{m_1^{1-\sigma}} K^{-\sigma/5} \Gamma(2 - \sigma - \delta + K) \Gamma(\delta - K).
\]
\[
\left( \left| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| \Gamma(1 - s) \right) + \sum_{k=1}^{K} \left| \frac{2\pi n_1}{m_1} \right|^{1-\sigma+k} |\omega_k| + K^{-\sigma/5} \Gamma(2 - \sigma - \delta + K) \Gamma(\delta - K).
\]

Now using (5.5), we find that the above is
\[
\ll K^{-\sigma/5} (\log Z)^3 \left| \Gamma(2 - \sigma - \delta + K) \Gamma(\delta - K) \right| (2\pi)^6
\]
\[
\times \left\{ \left| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| \Gamma(1 - s) \left| \frac{1}{2\sigma + \delta - 1} \frac{1}{(2\pi)^{2\sigma+\delta+1}} \right|
\]
\[
+ \sum_{k=1}^{K} |\omega_k| (2\pi)^{1-\sigma+k} \cdot \frac{1}{\sigma + \delta + k} \frac{1}{(2\pi)^{\sigma+\delta+k}}
\]
\[
+ \left| \Gamma(2 - \sigma - \delta + K) \Gamma(\delta - K) \right| (2\pi)^6 K^{-\sigma/5} (2\sigma + 2\delta - 1)^{-1} (2\pi)^{1-2\sigma-2\delta}\right\}.
\]

Choosing \( \delta = (3/2) - \sigma + K \), this is
\[
\ll K^{-1-\sigma/5} (\log Z)^3 \left\{ \left| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| + K^{1-\sigma/5} + K^{-\sigma/5} \right\}.
\]

Finally, choosing
\[
K = (\log q)^{20}
\]
shows that the O-term in (8.3) is
\[
(8.5) \quad \ll |\Gamma(1 - s)| \cdot (\log q)^{-1}.
\]

Now we analyze the main term of (8.3), namely
\[
(8.6) \quad \sum_{m_1, m_2, n_1, n_2 \in \mathfrak{d}} \frac{\lambda(m_1)\lambda(m_2)}{m_1m_2n_1^{1-s}n_2^{1-s}} \sin \left( \frac{\pi}{2} (s + \alpha) \right) \Gamma(1 - s) + \sum_{k=1}^{K} \left( \frac{2\pi n_1}{m_1} \right)^{1-\sigma+k} |\omega_k|^2.
\]

For this purpose we utilise a more refined version of Lemma (8.1).

**Lemma (8.2).** — We have for any \( w \) with \( \beta = \operatorname{Re} w > 0 \)
\[
\sum_{m_1, m_2, n_1, n_2 \in \mathfrak{d}} \frac{\lambda(m_1)\lambda(m_2)}{m_1m_2n_1^{1-\sigma}n_2^{1-\sigma}} \cdot \left( \frac{n_1}{m_1} \right)^w \ll (2\pi)^{1-2\sigma-\beta} \frac{1}{1 - 2\sigma - \beta} \cdot \frac{1}{\log q}.
\]
Proof. — We see that the sum is

\[ T = \sum_{m_1, m_2 \in \mathbb{Z}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^{\sigma} m_2^{1-\sigma}} \left( \frac{n_1}{m_1} \right)^w = \sum_{1 \leq m_1, m_2 < Z} \frac{\lambda(m_1) \lambda(m_2)}{(m_1 m_2)^{2\sigma + 1 + w}} \sum j^{2\sigma - 2 + w} \]

where the inner sum ranges over integers \( j \) satisfying

\[ j \equiv 0 \pmod{[m_1, m_2]} \]
\[ 1 \leq j \leq \frac{1}{2\pi} m_1 m_2. \]

Setting \( j = j_0 [m_1, m_2], \) and \( i = (m_1, m_2) \) as before, we see that the sum is

\[ T = \sum_{1 \leq j_0, i \leq 1/(2\pi) Z} \frac{\lambda(m_1) \lambda(m_2) [m_1, m_2]^{2\sigma - 1 + w}}{(m_1 m_2)^{2\sigma + 1 + w}} \sum_{1 \leq j_0 \leq i/2\pi} j^{2\sigma - 2 + w}. \]

Since \([m_1, m_2] = m_1 m_2/i,\) this may be rewritten as

\[ (8.7) \quad T = \sum_{1 \leq j_0 \leq (1/2\pi) Z} j^{2\sigma - 2 + w} \sum_{2 \times j_0 \leq i \leq Z} \frac{1}{i^{2\sigma - 2 + w}} \sum_{1 \leq m_1, m_2 \leq Z} \frac{\lambda(m_1) \lambda(m_2)}{(m_1 m_2)^{2\sigma + 1 + w}}. \]

The innermost sum can be written

\[ (8.8) \quad S = \sum \frac{\lambda(j_1) \lambda(j_2)}{i^2 j_1 j_2} \]

where the summation ranges over pairs \((j_1, j_2)\) satisfying

\[ 1 \leq j_1 \leq \frac{Z}{i}, \quad 1 \leq j_2 \leq \frac{Z}{i} \]
\[ (j_1, j_2) = 1 \]

We may suppose that \(j_1, j_2\) are squarefree (else \(\lambda(j_1) \lambda(j_2)\) will be zero). In particular, this implies that \((j_1, i) = 1\) and \((j_2, i) = 1\). Applying Lemma (1.5) to (8.8), we find that

\[ (8.9) \quad S \ll \left( \frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2 \cdot \frac{1}{\log^2(Z/Y)} \cdot \frac{1}{i^2}. \]
Substituting this estimate into (8.7), we find that

\[
\begin{align*}
\tau & \ll \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq j_0 \leq (1/2n)Z} f_0^{2\sigma - 2 + \beta} \sum_{2n \leq l \leq Z} \frac{1}{l^{2\sigma - 2 + \beta}} \left( \frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2 \\
& \ll \frac{1}{(\log q)^2} \sum_{1 \leq j_0 \leq (1/2n)Z} f_0^{2\sigma - 2 + \beta} \sum_{2n \leq l \leq Z} \frac{1}{l^{2\sigma + \beta}} \\
& \ll \frac{1}{(1 - 2\sigma - \beta) (\log q)^2} (2\pi)^{1 - 2\sigma - \beta} \sum_{1 \leq j_0 \leq (1/2n)Z} f_0 \frac{1}{\log q} \frac{(2\pi)^{1 - 2\sigma - \beta}}{1 - 2\sigma - \beta}.
\end{align*}
\]

Here, we have used the fact that

\[
\left( \frac{i}{\varphi(i)} + \sigma_{-1/2}(i^2) \right)^2
\]

is bounded on average. This proves the Lemma.

We now apply Lemma (8.2) to analyze (8.6). We find that it is equal to

\[
\begin{align*}
\sum_{m_1} \frac{\lambda(m_1) \lambda(m_2)}{m_1^{s_1} m_2^{s_2} n_1^{s_1 - 1 - \sigma} n_2^{s_2 - 1 - \sigma}} & \left\{ \sin \left( \frac{\pi}{2} (s + a) \right) \Gamma (1 - s) \right\}^2 \\
& + 2 \sum_{k=1}^K \text{Re} \left( \sin \left( \frac{\pi}{2} (s + a) \right) \Gamma (1 - s) \left( \frac{2\pi n_1}{m_1} \right)^{1 - s + k} \omega_k \right) \\
& + \sum_{k_1, k_2 = 1}^K \left( \frac{2\pi n_1}{m_1} \right)^{1 - \sigma + k_1} \omega_{k_1} \left( \frac{2\pi n_1}{m_1} \right)^{1 - \sigma + k_2} \omega_{k_2} \\
& = \left| \sin \left( \frac{\pi}{2} (s + a) \right) \Gamma (1 - s) \right|^2 \sum_{m_1} \frac{\lambda(m_1) \lambda(m_2)}{m_1^{s_1} m_2^{s_2} n_1^{s_1 - 1 - \sigma} n_2^{s_2 - 1 - \sigma}} \\
& + O \left( \sum_{k=1}^K \frac{1}{k^{1 - |s|}} (2\pi)^{1 - \sigma + k} \left| \sin \left( \frac{\pi}{2} (s + a) \right) \Gamma (1 - s) \right| \frac{(2\pi)^{1 - 2\sigma - (1 - \sigma + k)}}{\log q} \frac{1}{1 - 2\sigma - (1 - \sigma + k)} \right) \\
& + O \left( \sum_{k_1, k_2 = 1}^K \frac{1}{(k_1 k_2)^{1 - |s|}} (2\pi)^{2 - 2\sigma + k_1 + k_2} \frac{1}{\log q} \frac{(2\pi)^{1 - 2\sigma - (2 - 2\sigma + k_1 + k_2)}}{1 - 2\sigma - (2 - 2\sigma + k_1 + k_2)} \right).
\end{align*}
\]

By our assumption that $|t| \ll (\log q)^{-1}$ we may ignore $|t|$ in the estimations below. We observe that the sum over $k$ in the first error term is

\[
\sum_{k=1}^K \frac{1}{k^{s}} \frac{(2\pi)^{1 - 2\sigma}}{(k + \sigma)} \ll 1.
\]
Since $\sigma \geq (1/2)$, the double sum over $k_1, k_2$ is

$$\sum_{k_1, k_2=1}^{K} \frac{(2\pi)^{1-2\sigma}}{(k_1 k_2)^\sigma (k_1 + k_2 + 1)} \ll \left\{ \begin{array}{ll} (\log K) & \text{always} \\ (2\sigma - 1)^{-1} & \text{if } \sigma > (1/2) \end{array} \right.$$ 

We also note that if $\sigma$ is close to $1/2$, it is sometimes more convenient to use the first estimate. Recalling that $K = (\log q)^{20}$, we deduce that

$$(8.10) \quad M_a(s) = \sum_{(m_1, m_2, n_1, n_2) \in \mathcal{S}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^a m_2^a n_1^{1-s} n_2^{1-s}} \left| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| \Gamma(1-s)$$

$$+ O \left( \frac{1}{(2\sigma - 1) \log q} \right) + O \left( \frac{1}{\log q} \left| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| \Gamma(1-s) \right)$$

where $2\sigma - 1$ is to be interpreted as $(\log \log q)^{-1}$ when $\sigma = 1/2$. By an entirely analogous argument, it can be shown that

$$(8.11) \quad E_a(s) = O \left( \frac{1}{(2\sigma - 1) \log q} \right) + O \left( \frac{1}{\log q} \right| \sin \left( \frac{\pi}{2} (s + \alpha) \right) \right| \Gamma(1-s) \right)$$

with the same interpretation of $2\sigma - 1$ as above.

To summarize, we deduce from (6.1), Lemma (7.1), Lemma (7.2), (8.1), (8.9) and (8.10), (8.11) that

$$\sum_{1 \neq \chi (\mod q)} \left| I(s, \chi) \right|^2$$

$$= \frac{1}{\pi} (2\pi)^{2\sigma - 1} \frac{\varphi(q)}{q^{2\sigma - 1}} \left| \sin \left( \frac{\pi}{2} \right) \right|^2$$

$$+ \left| \sin \left( \frac{\pi}{2} (s + 1) \right) \right|^2 \left| \Gamma(1-s) \right|^2 \sum_{(m_1, m_2, n_1, n_2) \in \mathcal{S}} \frac{\lambda(m_1) \lambda(m_2)}{m_1^a m_2^a n_1^{1-s} n_2^{1-s}}$$

$$+ O \left( \frac{\varphi(q)}{q^{2\sigma - 1} \log q} \min \left( \frac{1}{(2\sigma - 1)}, \log \log q \right) \right) + O \left( \frac{\varphi(q) |\Gamma(1-s)|}{q^{2\sigma - 1} (\log q)} \right) + O \left( \frac{q^{1-\sigma}}{(1-\sigma) \log q} \right)$$

$$+ O \left( \frac{q^{1-\sigma}}{(1-\sigma)^2} \right) + O \left( \frac{\varphi(q)}{q^{2\sigma - 1} (1-\sigma)^4 (\log q)^2} \right).$$

9. **Analysis of the Main Term.** — We shall now analyze the sum in the main term, namely,

$$N(\sigma) \overset{\text{def}}{=} \sum_{m_1^a m_2^a n_1^{1-s} n_2^{1-s}} \frac{\lambda(m_1) \lambda(m_2)}{}$$
where the sum ranges over quadruples \((m_1, m_2, n_1, n_2)\) satisfying
\[
1 \leq m_1, m_2, n_1, n_2 < Z
\]
\[
m_1 n_2 = m_2 n_1
\]
\[
n_1 \leq \frac{1}{2\pi} m_1.
\]

We note that \(N(\sigma)\) is well defined since the relation \(m_1 n_2 = m_2 n_1\) makes the right hand side independent of the imaginary part of \(s\).

As before, we set \(j = m_1 n_2 = m_2 n_1\), \(i = (m_1, m_2)\). Note that given \(m_1, m_2\) and \(j, n_1\) and \(n_2\) are uniquely determined. We may thus rewrite \(N(\sigma)\) as
\[
N(\sigma) = \sum_{1 \leq m_1, m_2 < Z} \frac{\lambda(m_1)\lambda(m_2)}{(m_1 m_2)^{\sigma-1}} \sum_j j^{\sigma-1}
\]
where the inner sum ranges over integers \(j\) satisfying
\[
1 \leq j \leq \frac{1}{2\pi} m_1, m_2
\]
\[
j \equiv 0 \pmod{[m_1, m_2]}.
\]

We can rewrite \(N(\sigma)\) as in (8.7). Thus, setting \(j = j_0 \lfloor m_1, m_2 \rfloor\) and \(i = (m_1, m_2)\), we get
\[
N(\sigma) = \sum_{j \equiv 0 \pmod{[m_1, m_2]}} \frac{\lambda(j_1)\lambda(j_2)}{j_1 j_2}
\]
where the sum over \(i\) ranges over
\[
2\pi j_0 \leq i \leq Z
\]
and the inner sum ranges over pairs \((j_1, j_2)\) satisfying
\[
1 \leq j_1 \leq \frac{Z}{i}, \quad 1 \leq j_2 \leq \frac{Z}{i}
\]
\[
(j_1, j_2) = 1.
\]

Notice that we can also stipulate that
\[
i \leq \min(m_1, m_2) \leq Z
\]
and that
\[
(j_1, i) = (j_2, i) = 1.
\]
We write the innermost sum of (9.1) as

\[
\sum_{j_1,j_2} \frac{\lambda(j_1) \lambda(j_2)}{j_1 j_2} \sum_{e | j_1} \mu(e) \sum_{e | j_2} \frac{\mu(e)}{e^2} \sum_{l_1,l_2} \frac{\lambda(iel_1) \lambda(iel_2)}{l_1 l_2}
\]

where on the right, \( e \) ranges over

\[1 \leq e \leq \frac{Z}{i}\]

and \( l_1, l_2 \) range over

\[1 \leq l_1 \leq \frac{Z}{ie}, \quad 1 \leq l_2 \leq \frac{Z}{ie}\]

\((el_1, i) = (el_2, i) = 1.\)

Writing

\[\lambda(m) = \frac{\Lambda_1(m) - \Lambda_2(m)}{(\log Z/Y)},\]

we find that (9.4) breaks up into four subsums of the form

\[
\frac{1}{(\log Z/Y)^2} \sum_{e | i} \mu(e) \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}
\]

where \( g, h \in \{1, 2\}, z_1 = Y, z_2 = Z.\)

**Lemma (9.1).** Define

\[X_{g,h} = \sum_{e | i} \mu(e) \left\{ \sum_{l_1} \frac{\Lambda_g(iel_1)}{l_1} \right\} \left\{ \sum_{l_2} \frac{\Lambda_h(iel_2)}{l_2} \right\}\]

and let \( z = \min(z_g, z_h). \) Then,

\[X = 0 \quad \text{if} \quad i > z.\]

If \( i \leq z, \) then

\[
X = \sum_{e | i} \frac{\mu(e)}{e^2} \mu(i e)^2 \left( \frac{ie}{\varphi(ie)} \right)^3 + O_s \left( \frac{\sigma_{-1/2}(i) i}{\varphi(i) (2Z/i)^c} \right) + O_s \left( \frac{\sigma_{-1/2}(i)^2}{(\log (2Z/i))^{2c}} \right).
\]

**Proof.** The first assertion is obvious from the definition of \( \Lambda_g \) and \( \Lambda_h. \) Therefore, suppose that

\[1 \leq e \leq \frac{Z}{i}.
\]
We have
\[
X_{g, h} = \sum_{\ell_1 \leq z / \ell} \frac{\mu(\ell_1)}{e^2} \left\{ \sum_{l_1 \leq z / \ell} \frac{\mu(i\ell_1)}{l_1} \log \left( \frac{z_{g, \ell_1}}{i\ell_1} \right) \right\} \left\{ \sum_{l_2 \leq z / \ell} \frac{\mu(i\ell_2)}{l_2} \log \left( \frac{z_h}{i\ell_2} \right) \right\} \\
= \sum_{\ell_1 \leq z / \ell} \frac{\mu(\ell_1)}{e^2} \mu(i\ell_1) \left\{ \sum_{l_1 \leq z / \ell} \frac{\mu(l_1)}{l_1} \log \left( \frac{z_{g, \ell_1}}{i\ell_1} \right) \right\} \left\{ \sum_{l_2 \leq z / \ell} \frac{\mu(l_2)}{l_2} \log \left( \frac{z_h}{i\ell_2} \right) \right\}.
\]

Using Lemma (1.4), we have for any \(c > 0\),
\[
X_{g, h} = \sum_{\ell_1 \leq z / \ell} \frac{\mu(\ell_1)}{e^2} \mu(i\ell_1) \left\{ \frac{i\ell}{\varphi(i\ell)} + O_x \left( \sigma_{-1/2}(i\ell) \log^{-c} \left( \frac{2z}{i\ell} \right) \right) \right\}^2.
\]

There are two error terms \(\mathcal{E}_1, \mathcal{E}_2\) (say). The first is
\[
\mathcal{E}_1 \ll c \sum_{\ell_1 \leq z / \ell} \frac{i\ell}{\varphi(i\ell)} \cdot \sigma_{-1/2}(i\ell) \log^{-c} \left( \frac{2z}{i\ell} \right) = \Sigma_1 + \Sigma_2
\]
where in \(\Sigma_1\), \(e < \sqrt{z/\ell}\) and in \(\Sigma_2\), \(\sqrt{z/\ell} \leq e < z/\ell\). We have
\[
\Sigma_1 \ll c \sum_{\ell_1 \leq z / \ell} \frac{i}{\varphi(i)} \cdot \sigma_{-1/2}(i\ell) \cdot \frac{1}{(\log(2z/\ell))^c} \cdot \frac{\sigma_{-1/2}(e)}{e \varphi(e)}
\]
\[
\ll c \frac{\sigma_{-1/2}(i)}{\varphi(i)} \left( \frac{i}{\varphi(i)} \right) \cdot \frac{1}{e} \cdot \frac{1}{\log(2z/\ell)^c}.
\]

Also,
\[
\Sigma_2 \ll c \sum_{\ell_1 \leq z / \ell} \frac{i}{\varphi(i)} \cdot \sigma_{-1/2}(i\ell) \cdot \frac{\sigma_{-1/2}(e)}{\varphi(e) e}
\]
\[
\ll c \frac{\sigma_{-1/2}(i)}{\varphi(i)} \cdot \frac{1}{\sqrt{z/\ell}} \cdot \frac{1}{\sqrt{z}}
\]
\[
\ll c \frac{\sigma_{-1/2}(i)}{\varphi(i)} \cdot \frac{1}{(\log(2z/\ell))^c}
\]
for any \(c > 0\). The second error term is
\[
\mathcal{E}_2 \ll c \sum_{\ell_1 \leq z / \ell} \frac{1}{e^2} \sigma_{-1/2}(i\ell)^2 \log^{-2c} \left( \frac{2z}{i\ell} \right)
\]
\[
\ll c \sigma_{-1/2}(i)^2 \sum_{\ell_1 \leq z / \ell} \frac{1}{e^2} \sigma_{-1/2}(e)^2 \cdot \log^{-2c} \left( \frac{2z}{i\ell} \right)
\]
\[
\ll c \sigma_{-1/2}(i)^2 \left( \frac{2z}{i} \right)^{-2c}
\]
\[
\ll c \sigma_{-1/2}(i)^2 \left( \frac{2z}{i} \right)^{-2c}
\]
\[
\ll c \sigma_{-1/2}(i)^2 \left( \frac{2z}{i} \right)^{-2c}
\]
\[
\ll c \sigma_{-1/2}(i)^2 \left( \frac{2z}{i} \right)^{-2c}
\]
for any $c > 0$. The last estimate is obtained by proceeding as with $\epsilon_1$. This proves the lemma.

Now, notice that if $i \leq Y$, then

$$X_{1,1} - 2X_{1,2} + X_{2,2} = \sum_{Y_{i,i} < e^{2\pi i}} \mu(e) e^{2\pi i} \mu(ie)^2 \left( \frac{ie}{\varphi(ie)} \right)^2 + O\left( \frac{\sigma_{-1/2}(i)}{\varphi(i) (\log(2Y/i))^c} \right) + O\left( \frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c} \right).$$

The contribution of such terms to $N(\sigma)$ is

$$\ll \frac{1}{(\log Z/Y)^2} \sum_{1 \leq j_0 \leq Z/2 \pi} j_0^{2\sigma - 2} \sum_{2 \pi j_0 \leq i \leq Y} i^{-2\sigma} \left( \frac{i^3}{\varphi(i)^2 Y} + \frac{\sigma_{-1/2}(i)}{\varphi(i) (\log(2Y/i))^c} + \frac{\sigma_{-1/2}(i)^2}{(\log(2Y/i))^c} \right)$$

and this is

$$\ll \frac{\log Z}{(\log Z/Y)^2} \quad \text{if} \quad \sigma = 1/2$$

$$\ll \frac{1}{(\log Z/Y)^2 (2\sigma - 1)} \quad \text{if} \quad \sigma > 1/2.$$

On the other hand, if $i > Y$, then $X_{1,1} = X_{1,2} = 0$. Since

$$N(\sigma) = \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma - 2} \sum_{2 \pi j_0 \leq i \leq Z} i^{-2\sigma} (X_{1,1} - 2X_{1,2} + X_{2,2})$$

we deduce that

$$(9.7) \quad N(\sigma) = \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma - 2} \sum_{2 \pi j_0 \leq i \leq Z} i^{-2\sigma} \sum_{1 \leq e \leq Z/2} \frac{\mu(e)}{e^2} \mu(ie)^2 \left( \frac{ie}{\varphi(ie)} \right)^2$$

$$+ O\left( \frac{1}{\log^2 (Z/Y)} \left\{ \frac{1}{1 - 2\sigma} \text{ or } \log Z \right\} \right)$$

$$+ O\left( \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq j_0 \leq (1/2\pi) Z} j_0^{2\sigma - 2} \sum_{i > Y} i^{-2\sigma} \left\{ \frac{\sigma_{-1/2}(i)}{(\log(2Z/i))^c} + \frac{i}{\varphi(i) (\log(2Z/i))^c} \right\} \frac{\sigma_{-1/2}(i)^2}{(\log(2Z/i))^c} \right).$$

The first $O$-term is

$$\ll \frac{1}{|1 - 2\sigma|(\log q)^2} \quad \text{if} \quad \sigma > \frac{1}{2}$$

and

$$\ll \frac{1}{(\log q)^2} \quad \text{if} \quad \sigma = \frac{1}{2}.$$
Let us simplify the second \( O \)-term. If \( \sigma > 1/2 \), we interchange the \( i \) and the \( \gamma_0 \) sums and we find that this \( O \)-term is
\[
\ll \frac{1}{(\log q)^2} \sum_{1 \leq \gamma_0 \leq i} i^{-2\sigma} \left( \log \frac{2Z}{i} \right)^{-\epsilon} \sum_{1 \leq j_0 \leq i} j_0^{2\sigma - 2}.
\]
\[
\ll \frac{1}{(\log q)^\epsilon} \frac{1}{2\sigma - 1}.
\]

If \( \sigma = 1/2 \), then the \( O \)-term is
\[
\ll \frac{1}{(\log 2Z)^\epsilon}.
\]
(The value of \( \epsilon \) is not the same at each occurrence.)

Summarizing, we have proved that
\[
(9.8) \quad N(\sigma) = \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq \gamma_0 \leq (1/2)Z} j_0^{2\sigma - 2} \sum_{2 \gamma_0 \leq i \leq Z} i^{-2\sigma} \sum_{2 \gamma_0 \leq j \leq Z} \frac{\mu(e)}{e^2} \mu(ie)^2 \left( \frac{ie}{\varphi(ie)} \right)^2
\]
\[
\ll \begin{cases} 
O\left( \frac{1}{\log q} \right) & \text{if } \sigma = \frac{1}{2} \\
O\left( \frac{1}{(\log q)^2} \frac{1}{2\sigma - 1} \right) & \text{if } \sigma > \frac{1}{2}.
\end{cases}
\]

Note that in the above sum, we may suppose that \( e \) and \( i \) are squarefree.

10. THE MAIN TERM: CONTINUED. — Let us define the constant
\[
C_2 = \frac{2}{3^{\zeta(2)}} \prod_{p > 2} \left( 1 + \frac{2}{(p-2)(p+1)} \right) \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \approx .45.
\]

The main result of this section is the following.

**Proposition (10.1).** — We have
\[
\sum_{1 \not\equiv x \pmod{q}} |I(s, \chi)|^2 = c(s, q) \varphi(q) + \delta(\sigma).
\]

Here,
\[
c\left( \frac{1}{2} + it, q \right) = \frac{C_2}{\pi} \left| \Gamma\left( \frac{1}{2} - it \right) \right|^2 (\cosh \pi t)
\]
and for \( 1 > \sigma > 1/2 \),
\[
c\left( \sigma + it, q \right) = \frac{2C_2}{\pi} \frac{1}{2\sigma - 1} \left| \Gamma\left( 1 - \sigma - it \right) \right|^2 (\cosh \pi t) \frac{q^{1-2\sigma}}{\log q}.
\]
Also,

\[ \mathfrak{S} \left( \frac{1}{2} \right) \ll \frac{q (\log \log q)}{\log q} \]

and

\[ \mathfrak{S} (\sigma) \ll \frac{q^{2-2\sigma}}{\log q} \min \left( \frac{1}{(2 \sigma - 1)}, \log \log q \right) \]

if \(1/2 < \sigma \leq 3/4\) (say), while

\[ \mathfrak{S} (\sigma) \ll q^{2-2\sigma} (\log q)^2 \]

if \(3/4 \leq \sigma < 1 - (1/\log q)\).

**Proof.** – We saw in (6.1), § 7, and § 8 that

\[ (10.1) \sum_{1 \neq \chi (\text{mod } q)} |I(s, \chi)|^2 \]

\[ = \frac{1}{\pi} (2\pi)^{2\sigma - 1} |\Gamma(1-s)|^2 \text{N} (\sigma) \varphi (q) q^{1-2\sigma} \left( \left| \sin \left( \frac{\pi s}{2} \right) \right|^2 + \left| \sin \left( \frac{\pi s + 1}{2} \right) \right|^2 \right) \]

\[ + \mathcal{O} \left( \frac{\varphi (q)}{q^{2\sigma - 1} \log q} \min \left( \frac{1}{(2 \sigma - 1)}, \log \log q \right) \right) + \mathcal{O} \left( \frac{\varphi (q) \Gamma(1-s)}{q^{2\sigma - 1} (\log q)^2} \right) \]

\[ + \mathcal{O} \left( \frac{q^{1-\sigma}}{(1-\sigma) \log q} \right) + \mathcal{O} \left( \frac{q^{1-\sigma}}{(1-\sigma)^2} \right) + \mathcal{O} \left( \frac{\varphi (q)}{q^{2\sigma - 1} (1-\sigma)^{\frac{3}{2}} (\log q)^2} \right) \]

We shall now study the main term of \( N(\sigma) \). From (9.8), we see that it is

\[ (10.2) = \frac{1}{\log^2 (Z/Y)} \sum_{1 \leq j_0 \leq (1/2 \pi) Y} j_0^{2\sigma - 2} \sum_{i \geq Y} j_0^{\sigma - 2} \frac{i^{2-2\sigma} \mu (i)^2}{\varphi (i)^2} \sum_{\sigma \in Z/\varphi} \frac{\mu (e)}{\varphi (e)^2} \]

\[ = \frac{4}{(\log q)^2} \sum_{2 \leq i \leq 1/2 \pi, \varphi (i)^2} j_0^{\sigma - 2} \sum_{1 \leq j_0 \leq i/2 \pi} j_0^{\sigma - 2} \left\{ \sum_{e \leq Z/\varphi} \frac{\mu (e)}{\varphi (e)^2} \right\} \]

We note that

\[ \sum_{1 \leq j_0 \leq i/2 \pi} j_0^{\sigma - 2} = \left\{ \begin{array}{ll}
\frac{i}{2 \pi} \left( \frac{1}{2 \sigma - 1} \right) \frac{1}{2 \sigma - 1} + \mathcal{O} \left( i^{2\sigma - 2} \right) & \text{if } \sigma \neq 1/2 \\
\log \left( \frac{i}{2 \pi} \right) + \mathcal{O} (1) & \text{if } \sigma = 1/2.
\end{array} \right. \]
We easily check that the contribution of the $O$-terms is
\[
\ll \frac{1}{(\log q)}
\]
which is negligible.

If we replace in (10.2) the sum over $e$ with
\[
\sum_{e=1}^{\infty} \frac{\mu(e)}{\phi(e)^2} \left( \prod_{p \mid (e, i)} \frac{1}{(p-1)^2} \right)
\]
we introduce an error of
\[
\ll \frac{1}{(\log q)^2} \frac{1}{2\sigma - 1} \quad \text{if} \quad \sigma \neq \frac{1}{2}
\]
and of
\[
\ll \frac{1}{\log q} \quad \text{if} \quad \sigma = \frac{1}{2}.
\]
In any case, it is negligible.

Notice that
\[
\sum_{e=1}^{\infty} \frac{\mu(e)}{\phi(e)^2} = \left\{ \begin{array}{ll}
0 & \text{if } i \text{ is even} \\
\prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) & \text{if } i \text{ is odd}.
\end{array} \right.
\]

We see that for $\sigma > 1/2$,
\[
N(\sigma) = \frac{1}{(2\pi)^{2\sigma - 1}} \frac{1}{2\sigma - 1} \frac{4}{(\log q)^2} \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \sum_{i \text{ even}} \frac{\mu(i)^2 i}{\phi(i)^2} \prod_{p \mid (i/2)} \left( 1 - \frac{1}{(p-1)^2} \right)^{-1} + O\left( \frac{1}{\log q} \right) + O\left( \frac{1}{(\log q)^2} \frac{1}{2\sigma - 1} \right).
\]

On the other hand, for the case $\sigma = 1/2$ we have
\[
N\left(\frac{1}{2}\right) = \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \frac{4}{(\log q)^2} \sum_{i \text{ even}} \frac{i\mu^2(i)}{\phi(i)^2} \log \left( \frac{i}{2\pi} \right) \prod_{p \mid (i/2)} \left( 1 - \frac{1}{(p-1)^2} \right)^{-1} + O\left( \frac{1}{\log q} \right).
\]

Here, the sum over $i$ has range
\[
\max (Y, 2\pi) \leq i \leq Z, \quad i \text{ even}.
\]
Also, note that as $i$ may be assumed squarefree, $i/2$ is odd. We observe that

$$\frac{i\mu^2(i)}{\phi(i)\phi^2} \prod_{p \mid i(2)} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} = 4 \frac{i\mu^2(i)}{\phi(i)\phi^2} \prod_{p \mid i(2)} \frac{p^2}{(p-1)^2} \cdot \frac{(p-1)^2}{((p-1)^2-1)} = 2 \frac{i\mu^2(i)}{\phi(i)\phi^2},$$

where $\varphi_1(n)$ is defined by

$$\varphi(n) = \sum_{d \mid n} \varphi_1(n).$$

Thus, the sum over $i$ is

$$2 \sum_{i=1}^{x} \frac{i\mu^2(i)}{\phi_1(i)\phi^2} \log \left(\frac{i}{2\pi}\right) \text{ if } \sigma = \frac{1}{2}.$$

$$2 \sum_{i=1}^{x} \frac{i\mu^2(i)}{\phi_1(i)\phi^2} \text{ if } \sigma > \frac{1}{2}.$$

This sum can be estimated as follows. We first observe that for $\Re(s) > 1$

$$\sum_{i=1, \text{odd}}^{\infty} \frac{i\mu^2(i)}{\phi_1(i)\phi^2} = \prod_{p \geq 2} \left(1 + \frac{p}{(p-2)p^s}\right).$$

Now,

$$\left(1 + \frac{p}{(p-2)p^s}\right) = \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 + \frac{2}{(p-2)(p^s+1)}\right).$$

Thus,

$$\sum_{i=1, \text{odd}}^{\leq x} \frac{i\mu^2(i)}{\phi_1(i)} = C_1 x + O(x^{3/4})$$

where

$$C_1 = \frac{2}{3\zeta(2)} \prod_{p \geq 2} \left(1 + \frac{2}{(p-2)(p+1)}\right).$$

By partial summation, it follows that

$$2 \sum_{i=1, \text{odd}}^{Z/2} \frac{i\mu^2(i) \log i}{\phi_1(i)} = 2 \int_{1-}^{Z/2} \left(\frac{\log u}{u}\right) d \left(\sum_{n \leq u, \text{odd}}^{\phi_1(n)} \frac{i\mu^2(n)n}{\phi_1(n)}\right)$$

$$= C_1 (\log Z)^2 + O(\log Z).$$
Similarly, 

\[ 2 \sum_{i \leq Z/2, \varphi_1(i) \text{ odd}} \varphi_1(i) = 2 C_1 \log(Z/2) + O(1). \]

Substituting this information into (10.3), we find that 

\[ N(\frac{1}{2}) = C_2 + O\left(\frac{1}{\log q}\right) \]

where 

\[ C_2 = C_1 \cdot \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \approx 45. \]

Similarly, if \( \sigma > 1/2 \), 

\[ N(\sigma) = 4 C_2 \cdot \frac{2\pi}{2\sigma-1} \cdot \frac{1}{(\log q)^2} (\log q + O(1)) + O\left(\frac{1}{(\log q)^2 (2 \sigma - 1)}\right). \]

Inserting this into (10.1), and observing that 

\[ |\sin(x + iy)|^2 + |\cos(x + iy)|^2 = \cosh 2y \]

we deduce that 

\[ \sum_{1 \neq \chi \pmod{q}} |I(s, \chi)|^2 = c(s, q) \varphi(q) + \mathcal{E} \]

where 

\[ c(s, q, t) = \begin{cases} \frac{C_2}{\pi} \left| \Gamma\left(\frac{1}{2} - it\right)\right|^2 (\cosh\pi t) & \text{if } \sigma = \frac{1}{2} \\ \frac{4 C_2}{\pi} \frac{1}{2\sigma-1} \left| \Gamma(1 - \sigma - it)\right|^2 (\cosh\pi t) \frac{q^{1-2\sigma}}{\log q} & \text{if } \sigma > \frac{1}{2} \end{cases} \]

and 

\[ \mathcal{E} = O\left(\frac{q^{2-2\sigma}}{\log q} \min\left(\frac{1}{(2\sigma-1)}, \log \log q\right)\right) + O\left(\frac{q^{2-2\sigma}}{(1-\sigma)(\log q)}\right) + O\left(\frac{q^{2-2\sigma}}{(\log q)^2 (1-\sigma)^4}\right) \]

\[ + O\left(\frac{q^{1-\sigma}}{(1-\sigma)^2}\right) + \mathcal{E}_1(\sigma) \]

and 

\[ \mathcal{E}_1\left(\frac{1}{2}\right) \ll \frac{q}{\log q} \]
and for $1/2 < \sigma < 1$,

$$\varepsilon_1(\sigma) \ll \frac{q^{2-2\sigma}}{(1-\sigma)^2 (2\sigma - 1) (\log q)^2}.$$  

In particular, we see that if $\sigma = 1/2$

$$\varepsilon \ll \frac{q (\log \log q)}{\log q}.$$  

If $1/2 < \sigma \leq 3/4$ (say),

$$\varepsilon \ll \frac{q^{2-2\sigma}}{\log q} \min \left( \frac{1}{(2\sigma - 1)}, \log \log q \right)$$  

and if $3/4 \leq \sigma < 1 - (1/\log q)$

$$\varepsilon \ll q^{2-2\sigma} (\log q)^2.$$  

This proves the result.

11. Non-vanishing at a fixed point. — The main result of this section is the following.

**Theorem (11.1).** — Fix a $\sigma$ in the interval $1/2 \leq \sigma < 1$. Then, for all sufficiently large primes $q$,

$$L(\sigma, \chi) \neq 0$$

for a positive proportion of the characters $\chi \pmod{q}$.

**Remark.** — The proof will produce a lower bound for this proportion. Notice that how large $q$ must be taken will depend on $\sigma$.

**Proof.** — Let us fix $s_0 \in \mathbb{C}$ with $1/2 \leq \text{Re} s_0 < 1 - (1/\log q)$. We return now to (3.2). For $\chi \neq 1$, we have

$$S(s_0, \chi) = L(s_0, \chi) M(s_0, \chi) + I(s_0, \chi) + J(s_0, \chi).$$

Thus,

$$\sum_{\chi \neq 1} S(s_0, \chi) = \sum' (I(s_0, \chi) + J(s_0, \chi)) + \sum'' S(s_0, \chi)$$

where $\sum'$ ranges over $\chi \neq 1$ such that $L(s_0, \chi) = 0$ and $\sum''$ over the remaining non-trivial $\chi \pmod{q}$. By Proposition (4.1), we have

$$\sum_{\chi \neq 1} S(s_0, \chi) = \varphi(q) + O_{\varepsilon}(q^{1-\sigma + \varepsilon}).$$

Thus, we have

$$\sum'' S(s_0, \chi) = \varphi(q) - \sum' (I(s_0, \chi) + J(s_0, \chi)) + O_{\varepsilon}(q^{1-\sigma_0 + \varepsilon})$$
and consequently,
\[ \sum'' S(s_0, \chi) \geq \varphi(q) - |\sum I(s_0, \chi) + J(s_0, \chi)| + O\left(q^{1-\sigma + \varepsilon}\right). \]

Now, if we assume that \(|\text{Im} \, s_0| < 1\) (say), then
\[ |\sum (I(s_0, \chi) + J(s_0, \chi))| \leq \sum |I(s_0, \chi)| + \sum |J(s_0, \chi)| \leq \varphi(q)^{1/2} (\sum |I(s_0, \chi)|^2)^{1/2} + O\left(\frac{q^{3/2-\sigma}}{\log q}\right) \]

by Proposition (3.1). Now using the main result of § 10, namely
\[ \sum |I(s_0, \chi)|^2 = c(s_0, q) \cdot \varphi(q) + \delta'(\sigma) \]
we have
\[ |\sum (I(s_0, \chi) + J(s_0, \chi))| \leq \sqrt{c(s_0, q)} \varphi(q) + O\left(\sqrt{\varphi(q) \delta'(\sigma)}\right) + O\left(q^{3/2-\sigma} (\log q)^{-1}\right). \]

Thus,
\[ |\sum'' S(s, \chi)| \geq (1 - \sqrt{c(s_0, q)}) \varphi(q) + O\left(q^{1-\sigma + \varepsilon}\right) + O\left(\sqrt{\varphi(q) \delta'(\sigma)}\right) + O\left(q^{3/2-\sigma} (\log q)^{-1}\right). \]

On the other hand, by the Cauchy-Schwarz inequality, setting \(N(s_0, q)\) to be the number of \(\chi \pmod{q}\) with \(L(s_0, \chi) \neq 0\), we get
\[ |\sum'' S(s_0, \chi)| \leq N(s_0, q) \left(\sum |S(s_0, \chi)|^2\right). \]

From now on, we shall assume that \(t = \text{Im} \, s_0\) satisfies \(|t| \ll (\log q)^{-1}\). Suppose first that \(\sigma_0 = 1/2\). We have from Proposition (4.2)
\[ \sum_{\chi \pmod{q}} \left| S\left(\frac{1}{2} + it, \chi\right) \right|^2 = \frac{5}{2} \varphi(q) + O\left(q (\log q)^{-1/2}\right). \]

We deduce that
\[ \frac{2}{5} \varphi(q) (1 - \sqrt{c(s_0, q)})^2 + O\left(q \sqrt{\frac{\log \log q}{\log q}}\right) \leq N(s_0, q) \left(1 + O\left((\log q)^{-1/2}\right)\right). \]

Thus,
\[ N(s_0, q) \geq \frac{2}{5} \varphi(q) (1 - \sqrt{c_2})^2 + O\left(q \sqrt{\frac{\log \log q}{\log q}}\right). \]

Now let us set
\[ j = \left[\left(\sigma_0 - \frac{1}{2}\right) \log q\right] + 1. \]
Thus,
\[
\frac{1}{2} + \frac{j-1}{\log q} < \sigma_0 \leq \frac{1}{2} + \frac{j}{\log q}.
\]

We will suppose that \( q \) is sufficiently large that \( j \geq 2 \). Then Proposition (4.2) gives
\[
(11.1) \sum_{\chi \pmod{q}} |S(\sigma_0 + it_0, \chi)|^2 = \varphi(q) \left\{ 1 - \frac{e^{-j+1}}{(j-1)^2} + \frac{\log q}{(j-1)^2} - \frac{e^{-2(j-1)}}{j-1} \right. \\
+ O \left( \frac{e^{-2j}}{ \log q \left( 1 + \frac{1}{j} + \frac{1}{(\log q)^{\sigma_0}(1-\sigma_0)} \right) } \right) \right. \\
= \varphi(q) \left\{ 1 - \frac{e^{-j+1}}{(j-1)^2} + \frac{\log q}{(j-1)^2} - \frac{e^{-2(j-1)}}{j-1} + o(e^{-j}) \right\}.
\]

Also, if \( \sigma_0 \) is bounded away from 1 (say \( \sigma_0 \leq 3/4 \)) then
\[
\sum_{\chi \neq 1} |I(\sigma_0, \chi)|^2 = c(s_0, q) \varphi(q) + O(qe^{-2j}(\log q)^2).
\]

If \( 3/4 \leq \sigma_0 < 1 - (\log q)^{-1} \), Then
\[
\sum_{\chi \neq 1} |I(\sigma_0, \chi)|^2 = c(s_0, q) \varphi(q) + O(qe^{-2j}(\log q)^2).
\]

We see that under our assumption on \( |t| \), we have
\[
c(s_0, q) = \frac{2C_2}{\pi j} |\Gamma(1-\sigma_0)|^2 e^{-2j} + o \left( \frac{1}{\log q} \right).
\]

Putting all these estimates together, we deduce that
\[
\mathcal{N}(s_0, q) \geq (\alpha_j + o(1)) \varphi(q)
\]
Where
\[
\alpha_j = \frac{(1-e^{-j}) \sqrt{2C_2/\pi j} |\Gamma(1-\sigma_0)|^2}{(1-(j-1)^{-2} (e^{-j+1} - (j-1)^{-1} e^{-2j+2})}.
\]

12. NON-VANISHING AT A VARIABLE POINT. — In the previous section, we showed that a positive proportion of the \( L(s, \chi) \) are non-zero at a given real value \( \sigma \) of \( s \) in the critical strip. Now we shall refine this to a statement uniform on a line. Up to this point, we have made no significant use of the parameter \( \Gamma \). We shall now choose it to be \( \Gamma = \gamma^{1/4} \).

THEOREM (12.1). — Suppose that \( q \) is a sufficiently large prime. For a positive proportion of the \( \chi \pmod{q} \), \( L(s, \chi) \) does not have a real zero in the region \( 1/2 + c|\log q| \leq \sigma < 1 \). Here, \( c > 0 \) is an absolute constant.
Proof. By the functional equation, it suffices to concentrate attention on the region $\sigma \geq 1/2$. It is well known that there is at most one $\chi$ with a real zero in the range

$$1 - (\log q)^{-1} \leq \sigma \leq 1.$$ 

Thus we consider

$$\frac{1}{2} + \frac{2}{\log q} \leq \sigma < 1 - (\log q)^{-1}$$

and split it into intervals

$$I_j: \frac{1}{2} + \frac{j}{\log q} < \sigma \leq \frac{1}{2} + \frac{j+1}{\log q},$$

of length $1/\log q$. Here $2 \leq j \leq (1/2) \log q - 2$. We count the number $Z(j, q)$ of $\chi \pmod{q}$ for which $L(s, \chi)$ has a zero in $I_j$. For each $\chi$, let $\sigma(\chi)$ denote a point in $I_j$, and let $\sigma = \sigma_j = (1/2) + (j/\log q)$. Let $C=C_j$ denote the circle of radius $r=r_j = 2/\log q$ about $\sigma$. We have by Cauchy’s theorem

$$\sum_{n \geq Y} \frac{a(n)}{n^\sigma} e^{-\eta/q} - \sum_{n \geq Y} \frac{a(n)}{n^\sigma} e^{-\eta/q} = \frac{1}{2\pi i} \int_C \left\{ \sum_{n \geq Y} \frac{a(n)\chi(n)}{n^w} e^{-\eta/q} \right\} \left( \frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma} \right) \, dw.$$ 

Let us denote the left hand side by $S_{\text{diff}}(\sigma, \sigma_\chi, \chi)$ and let us write $w=\zeta + iv$. By (a variant of) Proposition (4.2).

$$\left(12.1\right) \left| \sum_{\chi} \left| \sum_{n \geq Y} a(n)\chi(n) n^{-w} \exp \left( -\frac{n}{q} \right) \right|^2 \right|$$

$$= \varphi(q) \sum_{Y \leq n \leq q} \frac{a(n)^2}{n^{2u}} + O \left( \frac{q^{(3/2)-u}}{(1-2u)(\log q)^2} \right) + O \left( q^{1-u}(\log q)^{3/2} + O \left( q^{2-2u}(\log q)^{3u-2} \frac{1}{1-u} \right) \right)$$

for $1 > \sigma > (1/2) - (1/\log q)$. Now

$$\sum_{\chi} \left| S_{\text{diff}}(\sigma, \sigma_\chi, \chi) \right|^2 \leq \sum_{\chi} \frac{1}{4\pi^2} \left( \int_C \left| \sum_{n \geq Y} \frac{a(n)\chi(n)}{n^w} e^{-\eta/q} \right|^2 \, dw \right)$$

$$\times \left( \int_C \left| \frac{1}{w-\sigma(\chi)} - \frac{1}{w-\sigma} \right|^2 \, dw \right)$$

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and
\[
\int_{\mathcal{C}} \left| \frac{1}{w - \sigma} - \frac{1}{w - \sigma} \right|^2 |dw| = \int_{\mathcal{C}} \left| \frac{\sigma(\chi) - \sigma}{(w - \sigma)(w - \sigma)} \right|^2 |dw| \\
\leq \frac{(r/2)^2}{r^2 (r/2)^2} \cdot 2 \pi r \\
= \frac{2 \pi}{r}.
\]

Therefore, for \( j \geq 1 \), we have by (12.1) (see also (11.1)) that
\[
\sum_{\chi} \left| S_{\text{diff}}(\sigma, \sigma_\chi, \chi) \right|^2 \leq \frac{1}{2 \pi r} \int_{\mathcal{C}} \left| \sum_{\sigma > \gamma} \sum_{n > Y} \frac{a(n) \chi(n)}{n^\sigma} e^{-\sigma} \right|^2 |dw| \\
\leq \varphi(q) \left( \sum_{\gamma \leq n \leq q} \frac{a(n)^2}{n^2 ((1/2) + ((j-2)/\log q))} + o(e^{-j}) \right).
\]

We observe that
\[
\sum_{\gamma \leq n \leq q} \frac{a(n)^2}{n^2 ((1/2) + ((j-2)/\log q))} \\
\sim \sum_{\gamma \leq n \leq z} \frac{\log n/Y}{(\log Z/Y)^2} \frac{1}{n^{1+((2j-4)/\log q)}} + \sum_{z \leq n \leq q} \frac{1}{(\log Z/Y)} \frac{1}{n^{1+((2j-4)/\log q)}}
\]
and this is seen to be
\[
\sim \frac{4}{(j-2)^2} \left\{ Y^{-(2j-4)/\log q} - Z^{-(2j-4)/\log q} \right\} - \frac{2}{j-2} q^{-(2j-4)/\log q}
\]
and this is
\[
= \frac{4}{(j-2)^2} (e^{-1/2 (j-2)} - e^{-(j-2)}) - \frac{2}{j-2} e^{-(2j-4)}.
\]

(Here, we have used the fact that \( Y = q^{1/4} \).) Let us denote the above expression by \( f(j-2) \). If \( j = 2 \) we have to replace the above by
\[
\frac{5}{2} + O((\log q)^{-1}).
\]
Then,

\[(12.2) \sum_x \left| \sum_{n \geq Y} \frac{a(n) \chi(n)}{n^s} e^{-n/q} \right|^2 \leq 2 \left\{ \sum_x \left| \sum_{n \geq Y} \frac{a(n) \chi(n)}{n^s} e^{-n/q} \right|^2 + \sum_x |S_{\text{diff}}(\sigma, \sigma', \chi)|^2 \right\} \leq 8 \varphi(q) (f(j) + f(j-2) + o(e^{-j})).\]

It is convenient to introduce here the notation

\[S^*(s, \chi) = \sum_{n \geq Y} \frac{a(n) \chi(n)}{n^s} e^{-n/q}.\]

Clearly, it is equal to \(S(x, \chi) - 1\).

Similarly, we have

\[\sum_x |I(\sigma, \chi)|^2 \leq 2 \left\{ \sum_x |I(\sigma, \chi)|^2 + \sum_x |I_{\text{diff}}(\sigma, \sigma', \chi)|^2 \right\}\]

where now,

\[I_{\text{diff}}(\sigma, \sigma', \chi) = \frac{1}{2\pi i} \int_C I(w, \chi) \left( \frac{1}{w-\sigma} - \frac{1}{w-\sigma'} \right) dw.\]

As before, if \(j \leq \log q / \log \log q\), then by Proposition (10.1), we see that

\[\sum_x |I_{\text{diff}}(\sigma, \sigma', \chi)|^2 \leq \frac{1}{2\pi r} \int_C |I(w, \chi)|^2 |dw| \]

\[\leq \frac{1}{2\pi r} \varphi(q) \left( c(s_0, q) + O \left( qe^{-2j} \min \left( \frac{1}{j}, \frac{\log \log q}{\log q} \right) \right) \right) 2\pi r \]

\[= \varphi(q) \left( \frac{C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + O \left( e^{-2j(\log q)^2} \right) \right).\]

If \(j \geq \log q / \log \log q\), the last estimate above is replaced by

\[\leq \varphi(q) \left( \frac{C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + O \left( e^{-2j(\log q)^2} \right) \right).\]

The same estimate holds for

\[\sum_x |I(\sigma, \chi)|^2.\]

Hence we deduce that

\[\frac{1}{\varphi(q)} \sum_x |I(\sigma, \chi)|^2 \leq \frac{4C_2}{\pi j} |\Gamma(1 - \sigma_0)|^2 e^{-2j} + \begin{cases} O \left( e^{-2j(\log q)^2} \right) & \text{if } j \leq 3/4 \log q, \\ O \left( e^{-2j(\log q)^2} \right) & \text{otherwise}. \end{cases}\]
Finally, a calculation similar to the one above and in Proposition (3.1) shows that
\[ \sum_{\chi} |J(\sigma, \chi)|^2 \leq q^{2-2\alpha}. \]

With that established, we return to our basic equation
\[ S(\sigma, \chi) = L(\sigma, \chi)M(\sigma, \chi) + I(\sigma, \chi) + J(\sigma, \chi) \]
and deduce that
\[ (12.3) \sum_{\chi} |L(\sigma, \chi)M(\sigma, \chi) - 1|^2 \leq 3 \sum_{\chi} (|S^*(\sigma, \chi)|^2 + |I(\sigma, \chi)|^2 + |J(\sigma, \chi)|^2). \]

Using the estimates established above, we see that the right hand side is
\[ (12.4) \leq 3\varphi(q) \left( 8(f(j)+f(j-2)) + \frac{4C_2}{\pi j} \left| \Gamma \left( \frac{1}{2} - j + \frac{1}{2} \right) \log q \right| \right)^2 e^{-2j} + o(e^{-j}). \]

Now let us set \( Z(j, q) \) to be the number of characters \( \chi \pmod{q} \) such that \( L(s, \chi) \) has a real zero in the circle \( C_j \). It follows immediately from (12.3) and (12.4) that
\[ Z(j, q) \leq 3\varphi(q) \left( 8(f(j)+f(j-2)) + \frac{4C_2}{\pi j} \left| \Gamma \left( \frac{1}{2} - j + \frac{1}{2} \right) \log q \right| \right)^2 e^{-2j} + o(e^{-j}). \]

If we sum this over \( j \geq j_0 \), for some absolute constant \( j_0 \) we see that we have
\[ \frac{1}{\varphi(q)} \sum_{j \geq j_0} \sum_{j \geq j_0} \left( \frac{32}{(j-2)^2} e^{-(j-1)^2} e^{-(j-2)} + \frac{32}{j^2} (e^{-(1/2)} e^{-j} + o(e^{-j})) \right). \]

If we choose \( j_0 \) sufficiently large, we see that the right hand side is <1. This completes the proof.

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(Manuscript received November 6, 1990, revised November 19, 1991.)

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