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THE CONNECTION BETWEEN THE K-THEORY LOCALIZATION THEOREM OF THOMASON, TROBAUGH AND YAO AND THE SMASHING SUBCATEGORIES OF BOUSFIELD AND RAVENEL

BY AMNON NEEMAN

0. Introduction

Let $\mathscr{R}$ and $\mathscr{S}$ be triangulated categories, and suppose $\mathscr{R}$ is a full triangulated subcategory of $\mathscr{S}$. Then $\mathscr{R}$ is called *épaisse* if it contains all $\mathscr{S}$-direct summands of its objects.

If $\mathscr{R}$ is an épaisse subcategory of $\mathscr{S}$, there is a standard way to construct a quotient category $\mathscr{S}/\mathscr{R}=\mathscr{T}$. The construction closely parallels the passage from an abelian category $\mathscr{B}$ and a Serre subcategory $\mathscr{A}$ to be the quotient category $\mathscr{B}/\mathscr{A}=\mathscr{C}$.

For Quillen's K-theory, Quillen showed in **[Q]** that applying the functor "K-theory" to the maps of abelian categories

$$\mathscr{A}\to\mathscr{B}\to\mathscr{C}$$

yields a homotopy fibration

$$\mathrm{K}(\mathscr{A})\to\mathrm{K}(\mathscr{B})\to\mathrm{K}(\mathscr{C}).$$

One would like to make an analogous statement for triangulated categories. In a previous article [N], the author defined a K-theory for arbitrary triangulated categories. This K-theory has some nice properties; in particular if $\mathscr{A}$ is an abelian category and $\mathrm{D}^b(\mathscr{A})$ is its bounded derived category, then $\mathrm{K}(\mathrm{D}^b(\mathscr{A}))$ agrees with $\mathrm{K}(\mathscr{A})$, Quillen's K-theory of $\mathscr{A}$.

For two years now, the author has unsuccessfully been trying to prove a localization theorem for this K-theory. Precisely, given a triangulated category $\mathscr{S}$ and an épaisse subcategory $\mathscr{R}$ as above, and with $\mathscr{T}$ being the quotient as defined above, one would like to show that the sequence

$$\mathrm{K}(\mathscr{R})\to\mathrm{K}(\mathscr{S})\to\mathrm{K}(\mathscr{T})$$

is a homotopy fibration.

The results of this article concern triangulated categories. The applications is K-theoretic. But the fact that the K-theory of triangulated categories is still in an embryonic stage of its development does not present a real difficulty. It constitutes little more than a technical nuisance. In the application, one simply works with the Waldhausen K-theory of closed model categories, and uses the Walhausen Approximation Theorem and the Fibration Theorem. Precisely, let **R** be a closed model realisation or $\mathscr{R}$, **S** a closed model realisation for $\mathscr{S}$ and **T** a closed model realisation for $\mathscr{T}$. As above, suppose that $\mathscr{R}$ is an épaisse subcategory of $\mathscr{S}$ and $\mathscr{T}$ is the quotient. Suppose further that the triangulated functors $\mathscr{R} \to \mathscr{S}$ and $\mathscr{S} \to \mathscr{T}$ lift to functors on closed models $\mathbf{R} \to \mathbf{S}$ and $\mathbf{S} \to \mathbf{T}$. Then, for Waldhausen's K-theory functor on closed model categories, there is a fibration

$$\mathrm{K}(\mathbf{R}) \to \mathrm{K}(\mathbf{S}) \to \mathrm{K}(\mathbf{T}).$$

There is an excellent exposition of this in [TT], Section 1.

In the remainder of this introduction we will pretend that there is a K-theory functor defined on triangulated categories, which satisfies a localization theorem. I strongly conjecture that the functor defined in [N] works. In any case, modulo suitable technicalities involving closed model categories, everything K-theoretic we say can be translated into real theorems.
The problem with localisation sequences of triangulated categories

$$\mathscr{R} \to \mathscr{S} \to \mathscr{T}$$

(*i.e.* triples of triangulated categories as above, with $\mathscr{R}$ épaisse in $\mathscr{S}$ and $\mathscr{T} = \mathscr{S}/\mathscr{R}$ the quotient) is that the interesting ones occuring in nature tend to involve very large categories $\mathscr{R}$, $\mathscr{S}$ and $\mathscr{T}$. For instance, it is often the case that the categories are closed with respect to arbitrary small coproducts. But it is well-known that then the K-theory spectrum must be contractible. It is therefore interesting to know some construction which, starting with a localisation sequence of large triangulated categories, produces a localisation sequence of triangulated categories with interesting K-theories.

Now suppose that $\mathscr{S}$ is a triangulated category closed with respect to the formation of all small coproducts. A full triangulated subcategory $\mathscr{R}$ is called *localizing* if it is closed with respect to the formation of $\mathscr{S}$-coproducts of its objects. It is very easy to show that then $\mathscr{R}$ must be épaisse. We define $\mathscr{T}$ to be the quotient triangulated category, as above. To define suitable small categories, we consider the full subcategories of *compact* objects.

Definition 0.1. – Let $\mathscr{T}$ be any triangulated category. An object t in $\mathscr{T}$ is called *compact* if $\mathrm{Hom}(t, -)$ respects coproducts.

Definition 0.2. – The full subcategory of $\mathscr{T}$ consisting of the compact objects will be denoted $\mathscr{T}^c$. Clearly, $\mathscr{T}^c$ is an épaisse subcateogry of $\mathscr{T}$.

Suppose now that $\mathscr{S}$ is a triangulated category closed under the formation of small coproducts, $\mathscr{R}$ is a localizing subcategory, and $\mathscr{T}$ is the quotient as above. We fix this notation for the rest of the Introduction, indeed for most of the article. Then one can ask whether there is a sequence

$$\mathscr{R}^c \to \mathscr{S}^c \to \mathscr{T}^c$$

and more specifically whether one can, in reasonable circumstances, conclude that the map $\mathscr{S}^c/\mathscr{R}^c \to \mathscr{T}^c$ comes close to being an isomorphism.

THEOREM 0.3 (Thomason and Trobaugh). — *Let* X *be a quasi-compact, separated scheme. Suppose* X *admits an ample family of line bundles. Let* U *be an open subscheme, and let* $\mathrm{X}-\mathrm{U}$ *be the complement. Let* $\mathscr{A}$ *be the abelian category of quasicoherent sheaves on* X, *whose support is contained in* $\mathrm{X}-\mathrm{U}$. *Let* $\mathscr{B}$ *be the abelian category of quasicoherent sheaves on* X *and let* $\mathscr{C}$ *be the abelian category of quasicoherent sheaves on* U. *Then it is well known that* $\mathscr{A}$ *is a Serre subcategory of* $\mathscr{B}$, *and the quotient* $\mathscr{B}/\mathscr{A}$ *is* $\mathscr{C}$.

Let $\mathscr{S}=\mathrm{D}(\mathscr{B})$, $\mathscr{T}=\mathrm{D}(\mathscr{C})$ and $\mathscr{R}=\mathrm{D}_{\mathscr{A}}(\mathscr{B})$, the category of chain complexes of objects of $\mathscr{B}$ with $\mathscr{A}$-cohomology. Then it is obvious that $\mathscr{S}$ is closed with respect to coproducts, that $\mathscr{R}$ is a localizing subcategory of $\mathscr{S}$, and that $\mathscr{T}$ is the quotient as above.

Then the theorem states that the map $\mathscr{R}\to\mathscr{S}$ takes $\mathscr{R}^c$ to $\mathscr{S}^c$, the map $\mathscr{S}\to\mathscr{T}$ takes $\mathscr{S}^c$ to $\mathscr{T}^c$, that the induced map $\mathscr{S}^c/\mathscr{R}^c\to\mathscr{T}^c$ is fully faithful, and that every object in $\mathscr{T}^c$ is a direct summand of an object in $\mathscr{S}^c/\mathscr{R}^c$ (*i.e.* the smallest épaisse subcategory of $\mathscr{T}^c$ containing the quotient $\mathscr{S}^c/\mathscr{R}^c$ is all of $\mathscr{T}^c$. Thus $\mathscr{T}^c$ is the épaisse closure of $\mathscr{S}^c/\mathscr{R}^c$ in $\mathscr{T}^c$).

There is a very nice generalization of the Thomason-Trobaugh theorem which is due to Yao.

THEOREM 0.4 (Yao). — *The conclusion of Theorem* 0.3 *rmains true when* $\mathscr{B}$ *is replaced by an admissible abelian category and* $\mathscr{A}$ *and* $\mathscr{C}$ *are a suitable subcategory and its quotient category.*

Remark 0.5. — It is a little complicated to state the hypotheses Yao needs to make on $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$. Suffice it to say that the theorem can be applied to non-commutative rings, in a way not covered by [TT].

For the K-theoretic application, one needs the following lemma:

LEMMA 0.6. — *Suppose* $\mathscr{T}$ *is a triangulated category, and* $\mathscr{S}$ *is a full triangulated subcategory whose épaisse closure is all of* $\mathscr{T}$. *Then a delooping of the map* $\mathrm{K}(\mathscr{S})\to\mathrm{K}(\mathscr{T})$ *is a covering space, and is a homotopy equivalence if and only if* $\mathscr{S}\cong\mathscr{T}$, i.e. *the inclusion* $\mathscr{S}\hookrightarrow\mathscr{T}$ *is an equivalence of categories.*

Remark 0.7. — The proof of Lemma 0.6 is so simple that it generalizes without difficulty to the K-theory of triangulated categories in [N].

(Wiebel pointed out to me that in the Waldhausen framework the statement of Lemma 0.6 is a little delicate; I was very happy to hear this. It provides yet another example of a theorem whose statement and proof become simpler in the triangulated K-theory of [N]. In the Waldhausen framework the result is due to [TT]. We give a "triangulated" proof in the Appendix).

Thus in the situation of the theorem of Thomason, Trobaugh and Yao there is a homotopy fibration

$$K(\mathscr{R}^c) \to K(\mathscr{S}^c) \to K(\mathscr{S}^c/\mathscr{R}^c)$$

and the map $K(\mathscr{S}^c/\mathscr{R}^c) \to K(\mathscr{T}^c)$ is an isomorphism on the connected component at 0, while the map on Π_0 is injective. Thus the sequence

$$K(\mathscr{R}^c) \to K(\mathscr{S}^c) \to K(\mathscr{T}^c)$$

is almost a homotopy fibration. With a suitable definition of a non-connective K-theory spectrum, one can make it a genuine homotopy fibration. Once again, there are very good discussions of this in [TT] and [Y].

It seems fair to say that Yao pushed the methods employed in the two proofs to the limit. His proof is an impressive technical *tour de force*. The point of this article is that, using a completely different technique, one can prove a vast generalization of the theorem of Thomason, Trobaugh and Yao. The new proof hinges on ideas of Bousfield and Ravenel. Let us begin by stating the theorem.

THEOREM 2.1. — *Suppose $\mathscr{S}$ is any triangulated category closed with respect to arbitrary coproducts. Suppose that the subcategory $\mathscr{S}^c$ of compact objects is small, and that $\mathscr{S}$ is the smallest localizing subcategory containing $\mathscr{S}^c$. Suppose furthermore that there is a set* R *of objects in $\mathscr{S}^c$, and $\mathscr{R}$ is the smallest localizing category containing* R. *Let $\mathscr{T}$ be the quotient category $\mathscr{S}/\mathscr{R}$. Then the map $\mathscr{R} \to \mathscr{S}$ carries $\mathscr{R}^c$ to $\mathscr{S}^c$, the map $\mathscr{S} \to \mathscr{T}$ carries $\mathscr{S}^c$ to $\mathscr{T}^c$, the natural functor $\mathscr{S}^c/\mathscr{R}^c \to \mathscr{T}^c$ is fully faithful, and $\mathscr{T}^c$ is the épaisse closure of the image.*

Remark 0.8. — Under the hypotheses of Thomason, Trobaugh and Yao, the category $\mathscr{S}$ and $\mathscr{R}$ are in fact generated by their compact objects. In a special case, this is discussed in [BN], Section 6. The more general statement is left to the reader, but follows from essentially the same argument.

The idea of the proof is really in the work of Bousfield and of Ravenel. Bousfield shows that under hypotheses far more general than the above the functor $\mathscr{S} \to \mathscr{T}$ has a right adjoint, the so-called Bousfield localization. Ravenel shows that under the hypotheses above, the localization functor commutes with coproducts. These two statements, together with the fairly explicit construction of the localization, yield the proof of Theorem 2.1.

There is an amusing corollary of Theorem 2.1, due to Thomason and Trobaugh.

COROLLARY 0.9. — *The objects of $\mathscr{T}^c$ isomorphic to objects in the image of $\mathscr{S}^c/\mathscr{R}^c$ are precisely those for which the image in $K_0(\mathscr{T}^c)/K_0(\mathscr{S}^c)$ vanishes.*

In fact one can generalize a little.

COROLLARY 0.10. — *Let $\mathscr{T}$ be a triangulated category, $\mathscr{S}$ a full triangulated subcategory whose épaisse closure is $\mathscr{T}$. Then an object* X *in $\mathscr{T}$ is isomorphic to an object in $\mathscr{S}$ if and only if its image in $K_0(\mathscr{T})/K_0(\mathscr{S})$ vanishes.*

Proof. – Pick X an object in $\mathscr{T}$ which is not isomorphic to any object of $\mathscr{S}$. Form the smallest full, triangulated subcategory of $\mathscr{T}$ containing $\mathscr{S}$ and X. Call this category $\mathscr{S}_X$.

Clearly, the épaisse closure of $\mathscr{S}$ in $\mathscr{S}_X$ is $\mathscr{S}_X$, and the épaisse closure of $\mathscr{S}_X$ in $\mathscr{T}$ is $\mathscr{T}$. Thus lemma 0.6 applies, and we get inclusions $K_0(\mathscr{S}) \subset K_0(\mathscr{S}_X)$ and $K_0(\mathscr{S}_X) \subset K_0(\mathscr{T})$. Because X is not isomorphic to an objects in $\mathscr{S}$, the inclusion $\mathscr{S} \subset \mathscr{S}_X$ is proper, and therefore the inclusion of K-groups is proper. But $K_0(\mathscr{S}_X)$ is generated in $K_0(\mathscr{T})$ by $K_0(\mathscr{S})$ and X, and we therefore deduce that the image of X in $K_0(\mathscr{T})$ is not in $K_0(\mathscr{S})$. $\square$

The article is structured as follows. Section 1 contains a brief background sketch of the work of Bousfield and Ravenel. In particular, the Bousfield localization functor is only constructed in the special case of interest for the proof. Section 2 contains the proof of Theorem 2.1. K-theory never gets mentioned again.

Theorem 2.1, and the more restricted theorems of Thomason Trobaugh and Yao, are highly applicable. But there is a very thorough and complete discussion of the applications in [TT], and in [Y] there are various examples of applications of the stronger theorem not covered by [TT]. I will therefore restrict myself to observing that if $\mathscr{S}$ is the topolocigal category of all spectra and $\mathscr{R}$ is a smashing subcategory generated by its compact object, Theorem 2.1 applies (*cf.* [W2]). Furthermore, this case is clearly not covered by [TT] or [Y].

I would like to thank Yao for very helpful discussions. I would like to thank Thomason, Weibel and Yao for helpful suggestions that improved the original manuscript and cleared up the presentation.

1. Bousfield localisation and smashing subcategories

We begin with a definition:

DEFINITION 1.1. – Let $\mathscr{T}$ be any triangulated category, closed with respect to the formation of all small coproducts. Suppose X is an object of $\mathscr{T}$. Then X is called *compact* if the functor Hom(X, $-$) commutes with the formation of direct sums. The full subcategory whose objects are all the compact objects of $\mathscr{T}$ is called $\mathscr{T}^c$. Clearly, $\mathscr{T}^c$ is triangulated.

Example 1.2. – If $\mathscr{T}$ is the category of all spectra, then $\mathscr{T}^c$ is the subcategory of finite spectra. If X is a quasi-compact, separated scheme and $\mathscr{T}$ is the derived category of the category of quasicoherent sheaves on X, then $\mathscr{T}^c$ is the full subcategory of all perfect complexes.

Next comes another definition:

DEFINITION 1.3 (Ravenel). – Suppose $\mathscr{S}$ is a triangulated category closed with respect to the formation of arbitrary small coproducts. Suppose that $\mathscr{R}$ is a full, triangulated subcategory of $\mathscr{S}$ which is closed with respect to the formation of arbitrary $\mathscr{S}$-coproducts.

That means that $\mathscr{R}$ is closed with respect to the formation of coproducts, and furthermore the inclusion functor $\mathscr{R} \to \mathscr{S}$ respects coproducts. Then we call $\mathscr{R}$ a *localising subcategory* of $\mathscr{S}$.

In both examples discussed in Example 1.2, it is the case that $\mathscr{T}$ is closed with respect to the formation of arbitrary coproducts and that $\mathscr{T}^c$ is a small category. Usually, it is also true that the smallest localising subcategory containing $\mathscr{T}^c$ is all of $\mathscr{T}$. In the topolocigal example it is simply true. In the algebro-geometric example, weak hypothesis on the scheme X suffice; for instance, it suffices to know that X admits an ample family of line bundles. This will be the situation that will most interest us in this article. We will make frequent use of another standard concept, namely the homotopy colimit of a sequence. Let us therefore define it.

DEFINITION 1.4. — Let $\{X_n\}$ be a sequence of objects in a triangulated category $\mathscr{T}$. Suppose for each $n \geqq 0$ we are given a map $X_n \to X_{n+1}$. Suppose the category $\mathscr{T}$ is closed with respect to the formation of arbitrary coproducts. Then we define hocolim (X_n) to be the third edge of the triangle

$$\begin{tikzcd} \oplus X_n \arrow[rr, "1-\text{shift}"] & & \oplus X_n \arrow[dl] \\ & \operatorname{hocolim}(X_n) \arrow[ul, "(1)"] & \end{tikzcd}$$

LEMMA 1.5. — *Suppose $\mathscr{T}$ is a triangulated category closed with respect to the formation of coproducts. Suppose $\{X_n\}$ is a sequence of objects in $\mathscr{T}$, together with connecting morphisms as in Definition* 1.4. *Suppose t is a compact object of $\mathscr{T}$. Then there is a natural isomorphism*

$$\operatorname{colim} \operatorname{Hom}(X_n) \to \operatorname{Hom}(t, \operatorname{hocolim}(X_n))$$

Proof. — From the triangle used to define hocolim (X_n), we get an exact sequence after applying the functor Hom $(t, -)$. Because t is compact, we get a commutative diagram where the rows are exact and the vertical maps are isomorphisms of sets

$$\begin{tikzcd} \operatorname{Hom}(t, \operatorname{hocolim}(X_n)) \arrow[r] \arrow[d] & \operatorname{Hom}(t, \oplus\Sigma X_n) \arrow[r, "1-\text{shift}"] \arrow[d] & \operatorname{Hom}(t, \oplus\Sigma X_n) \arrow[d] \\ \operatorname{Hom}(t, \operatorname{hocolim}(X_n)) \arrow[r] & \oplus\operatorname{Hom}(t, \Sigma X_n) \arrow[r, "1-\text{shift}"] & \oplus\operatorname{Hom}(t, \Sigma X_n) \end{tikzcd}$$

In the bottom row, the map 1-*shift* is clearly injective. Therefore we conclude that the map $\operatorname{Hom}(t, \operatorname{hocolim}(X_n)) \to \operatorname{Hom}(t, \oplus X_n)$ must be zero. Therefore, the rows in the following diagram are exact, and the columns are isomorphisms:

$$\begin{tikzcd} \operatorname{Hom}(t, \oplus X_n) \arrow[r, "1-\text{shift}"] \arrow[d] & \operatorname{Hom}(t, \oplus X_n) \arrow[r] \arrow[d] & \operatorname{Hom}(t, \operatorname{hocolim}(X_n)) \arrow[r] \arrow[d] & 0 \\ \oplus\operatorname{Hom}(t, X_n) \arrow[r, "1-\text{shift}"] & \oplus\operatorname{Hom}(t, X_n) \arrow[r] & \operatorname{Hom}(t, \operatorname{hocolim}(X_n)) \arrow[r] & 0 \end{tikzcd}$$

and the bottom row identifies $\mathrm{Hom}(t, \mathrm{hocolim}(X_n))$ with the direct limit $\mathrm{colim}\,\mathrm{Hom}(t, X_n)$. $\square$

CONSTRUCTION 1.6. (Adams, Bousfield). – Let $\mathscr{S}$ be a triangulated category closed with respect to the formation of arbitrary coproducts. Let R be a set of compact objects in $\mathscr{S}$, and suppose R is closed under taking suspensions. Let $\mathscr{R}$ be the smallest localising subcategory of $\mathscr{S}$ containing all of R. It is relatively trivial to show that $\mathscr{R}$ is épaisse (every localising subcategory is. *See* [BN], Remark 1.4 and Section 3). We remind the reader that an épaisse subcategory is a triangulated subcategory which is closed with respect to the formation of direct summands. To show that $\mathscr{R}$ is épaisse, (without having to look up the reference), let r be an object of $\mathscr{R}$ and suppose it decomposes in $\mathscr{S}$ as $r=r_1 \oplus r_2$. Let e be the idempotent $e: r \to r$ such that e is 1 on r_1 and 0 on r_2. Then consider the sequence

$$r \xrightarrow{e} r \xrightarrow{e} r \xrightarrow{e} r \xrightarrow{e} \dots$$

Clearly, the *hocolim* of this sequence is r_1. But since the construction of the *hocolim* involves only coproducts and triangles on objects of $\mathscr{R}$, r_1 must be in $\mathscr{R}$.

In particular, it is possible to form the category $\mathscr{T}=\mathscr{S}/\mathscr{R}$. This is a triangulated category obtained by setting all the objects of $\mathscr{R}$ isomorphic to zero. There is a natural functor $j^*: \mathscr{S} \to \mathscr{T}$. A very important fact, which is due to Sullivan, Adams and Bousfield, is that this functor has a right adjoint, which we denote by j_*, called the *Bousfield localisation functor*.

To construct j_*, one proceeds as follows. Define an object Y in $\mathscr{S}$ to be R-*local* if for every object r in R, $\mathrm{Hom}(r, Y)$ is zero. Then one proves:

LEMMA 1.7. – *Given any object* X *in* $\mathscr{S}$, *there is an* R-*local object* Y *in* $\mathscr{S}$ *and a morphism* $f: X \to Y$ *such that the mapping cone on* f *is in the subcategory* $\mathscr{R}$, *the smallest localising subcategory containing* R.

Let us first show why the Lemma 1.7 implies the existence of j_*. The point is that given any object X of $\mathscr{S}$, and any local object Y, then

$$\mathrm{Hom}_{\mathscr{S}}(X, Y) = \mathrm{Hom}_{\mathscr{T}}(X, Y)$$

is an isomorphism.

We recall that a morphism $X \to X'$ in $\mathscr{S}$ is called a quasi-isomorphism if its image in $\mathscr{T}$ is an isomorphism. $\mathscr{T}$ is the category obtained from $\mathscr{S}$ by formally inverting the quasi-isomorphisms. The notation $\mathscr{T}=\mathscr{S}/\mathscr{R}$ means that any quasi-isomorphism $X \to X'$ has for its mapping cone an object Z in $\mathscr{R}$. But for every object r in R, we known that $\mathrm{Hom}(r, Y)$ vanishes; this is because Y is local. Thus it follows that $\mathrm{Hom}(-, Y)$ vanishes on the entire localising subcategory generated by R. In particular, $\mathrm{Hom}(Z, Y)=0$. Thus $\mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y)$ must be an isomorphism. Since this is true for all quasi-isomorphisms, it follows immediately, by the definition of morphisms in $\mathscr{T}$, that

$$\mathrm{Hom}_{\mathscr{S}}(X, Y) = \mathrm{Hom}_{\mathscr{T}}(X, Y)$$

is indeed an isomorphism.

Coupled with the fact that every object in $\mathscr{S}$ is quasi-isomorphic to a local object, this immediately gives that the category of local objects is equivalent, via the projection $j^*: \mathscr{S} \to \mathscr{T}$, to the category $\mathscr{T}$. With this equivalence, we let j_* be the inclusion. The counit of the adjunction is then just the map $X \to Y$, given in Lemma 1.7.

Remark 1.8. – Everything above is completely standard, so I summarised it as succintly as I could, giving the barest sketch for the unitiated. What I really want to observe is that Bousfield gives a very explicit description of the counit of adjunction $\eta: X \to j_* j^*(X)$. Therefore we will recall the proof of Lemma 1.7 in some detail. The real virtue of Bousfield's construction will become obvious when we use it to prove the theorem of Thomason, Trobaugh and Yao.

Proof of Lemma 1.7 (Adams, Bousfield). – Let X be an object of $\mathscr{S}$, and suppose R is a set of compact objects in $\mathscr{S}$ as above. We define inductively objects X_n of $\mathscr{S}$, with $n \geqq 0$, and quasi-isomorphisms $X_n \to X_{n+1}$. By definition, we set $X_0 = X$. Let I be the set of all morphisms $\alpha_i: r_i \to X_n$ with r_i in R. Then define X_{n+1} to be the mapping cone on the morphism

$$\bigoplus_{i \in I} r_i \to X_n$$

Now define Y to be $\operatorname{hocolim}(X_n)$. Then firstly we observe that the canonical map $X = X_0 \to \operatorname{hocolim}(X_n) = Y$ is an isomorphism in $\mathscr{T}$, since each $X_n \to X_{n+1}$ is. Thus the mapping cone on $X \to Y$ is indeed an object of $\mathscr{R}$.

Next we need to show that Y is local. Pick any $r \to R$. Then r is a compact object of the category $\mathscr{S}$, and therefore by Lemma 1.5, there is an isomorphism

$$\operatorname{colim} \operatorname{Hom}(r, X_n) \to \operatorname{Hom}(r, \operatorname{hocolim}(X_n))$$

But by the construction of X_n from X_{n+1}, it is clear that the map

$$\operatorname{Hom}(r, X_n) \to \operatorname{Hom}(r, X_{n+1})$$

is the zero map. Therefore $\operatorname{Hom}(r, Y) = \operatorname{colim} \operatorname{Hom}(r, X_n)$ must vanish. □

There is one more important observation.

PROPOSITION 1.9 (Ravenel). – *Let all the notation be as in Construction* 1.6. *Then the functor* $j_*: \mathscr{T} \to \mathscr{S}$ *preserves coproducts*).

Proof. – It suffices to prove that the full subcategory of $\mathscr{S}$ consisting of local objects (which is equivalent, via j_*, to $\mathscr{T}$) is closed under the formation of $\mathscr{S}$ coproducts. Therefore let I be any index set, X_i a collection of local objects in $\mathscr{S}$ indexed by I. we need to show that $\oplus X_i$ is local.

Consider therefore any object $r \in R$. Then, because r is a compact object of $\mathscr{S}$,

$$\operatorname{Hom}(r, \oplus X_i) = \oplus \operatorname{Hom}(r, X_i)$$

and because each X_i is local, $\operatorname{Hom}(r, X_i)$ vanishes for all i. Thus the right hand side of the equation is zero, and $\oplus X_i$ is indeed local. □

Remark 1.10. – It is possible to construct localization functors in far greater generality. This was extensively studied by Bousfield. Ravenel conjectured that the localization functors which preserve coproducts all arise from a set R as above. The corresponding subcategories $\mathscr{R}$ are called *smashing*.

2. Proof of the Main Theorem

We begin by stating the main theorem:

THEOREM 2.1 (Generalized from the work of Thomason, Trobaugh and Yao). – *Let $\mathscr{S}$ be a triangulated category closed with respect to the formation of small coproducts. Let $\mathscr{S}^c$ be the subcategory of compact objects, as in Definition* 1.1. *Suppose that $\mathscr{S}^c$ is a small category, and that the smallest localising subcategory of $\mathscr{S}$ containing $\mathscr{S}^c$ is the whole of $\mathscr{S}$ (we recall that this is true in both examples of Example* 1.2, *and should somehow be viewed as "normal"). Let* R *be a subset of the objects of $\mathscr{S}^c$, closed with respect to the suspension functor. Let $\mathscr{R}$ be the smallest localising subcategory of $\mathscr{S}$ containing* R. *Let $\mathscr{T}$, j^* and j_* be as in Construction* 1.4. *Then the sequence of triangulated functors*

$$\mathscr{R} \to \mathscr{S} \to \mathscr{T}$$

yields, by restriction to compact subobjects, a sequence of functors

$$\mathscr{R}^c \to \mathscr{S}^c \to \mathscr{T}^c$$

There is therefore an induced functor

$$\mathscr{S}^c/\mathscr{R}^c \xrightarrow{F} \mathscr{T}^c$$

The functor F *is fully faithful, and identifies $\mathscr{S}^c/\mathscr{R}^c$ with a subcategory of $\mathscr{T}^c$ whose épaisse closure is all of $\mathscr{T}^c$. In order words, any object in $\mathscr{T}^c$ is a direct summand of some object in $\mathscr{S}^c/\mathscr{R}^c$.*

We will break up the proof into a sequence of easy lemmas.

LEMMA 2.2. – *The category $\mathscr{R}^c$ is contained in $\mathscr{S}^c$. In fact, more is true. The category $\mathscr{R}^c$ is the smallest épaisse subcategory containing* R.

Proof. – Given an object X in $\mathscr{R}^c$, then $\mathrm{Hom}(X, -)$ commutes with coproducts *in $\mathscr{R}$*, but it is not so clear that it also respects arbitrary coproducts in $\mathscr{S}$.

However, it is clear that the Bousfield localisation functor sends X to zero, since X is an object of $\mathscr{R}$. But then, by the proof of Lemma 1.5, we construct a sequence of objects X_n, and the construction clearly shows that each X_n must be in $\mathscr{R}$, and $\mathrm{hocolim}\,(X_n)$ is zero. It makes no difference whether we take homotopy colimits in $\mathscr{R}$ or in $\mathscr{S}$, since the inclusion functor preserves triangles and coproducts.

If follows therefore, by applying Lemma 1.5 to the compact object X in the category $\mathscr{R}$, that

$$O = \mathrm{Hom}(X, \mathrm{hocolim}(X_n)) = \mathrm{colim}\,\mathrm{Hom}(X, X_n)$$

and, hence for some $n>0$, the natural map $X=X_0 \to X_n$ is the zero map. But by construction, the mapping cone on $X_0 \to X_n$ is a finite extension of direct sums of objects of R. Since it is the mapping cone on the zero map, it is also the direct sum $X_n \oplus \Sigma X_0$. Thus we have proved that X_0 is a direct summand of a finite extension of coproducts of elements of R. It remains to show that the coproducts can be taken to be finite. Then we will have proved that $\mathscr{R}^c$ is in the épaisse subcategory of $\mathscr{S}$ generated by R, and this is clearly contained in $\mathscr{S}^c$.

Thus Lemma 2.2 is an immediate consequence of

LEMMA 2.3. – *Let $\mathscr{S}$ be a triangulated category closed under the formation of coproducts. Let* R *be a set of objects in $\mathscr{S}^c$, closed under suspension. Let $\mathscr{R}$ be the localising subcategory of $\mathscr{S}$ generated by* R. *Let $\langle R \rangle$ be the épaisse subcategory of $\mathscr{S}$ generated by* R.

Suppose we are given two objects of $\mathscr{S}$, and a morphism $X \to Y$ between them. Suppose that X *is a compact object in $\mathscr{R}$,* and suppose we are given a map $Y' \to Y$ *in $\mathscr{S}$ such that the mapping cone on $Y' \to Y$ is a finite extension of direct sums of objects of* R. *Then there is a map $X' \to X$ whose mapping cone is in $\langle R \rangle$, such that the composite $X' \to X \to Y$ factors through Y'.*

Proof that Lemma 2.2 *follows from Lemma* 2.3. – In the proof of Lemma 2.2 we began with a compact object X in $\mathscr{R}$, and constructed a split monomorphism $X \to Y$ where Y is a finite extension of direct sums of objects in R. So let $Y'=0$ in Lemma 2.3. It follows that there exists an $X' \to X$ with mapping cone in $\langle R \rangle$, such that the composite $X' \to X \to Y$ factors through $Y'=0$. But when we compose with the splitting $Y \to X$, this implies that the map $X' \to X$ vanishes. Thus X is a direct summand of the mapping cone, which by hypothesis lies in $\langle R \rangle$. □

Proof of Lemma 2.3. – Complete the map $Y' \to Y$ to a triangle $Y' \to Y \to E \to \Sigma Y'$. The proof is by induction on the length of E. If the length of E (*i.e.* the number of extensions needed to express E as an extension of coproducts of objects in R) is one, then the mapping cone E on the map $Y' \to Y$ is a coproduct of elements of R. Now consider the composite map $X \to Y \to E$. Because X is a compact object of $\mathscr{R}$ and E is a coproduct of objects in R, the map $X \to E$ factors through a finite direct sum of objects of R, which is a direct summand F of E. Now we may complete the commutative square

$$\begin{array}{ccc} X & \longrightarrow & F \\ \downarrow & & \downarrow \\ Y & \longrightarrow & E \end{array}$$

to a morphism of triangles

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & F & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & \Sigma Y' \end{array}$$

In particular, the cone on the map $X' \to X$ is F, which is clearly in $\langle R \rangle$, and the composite $X' \to X \to Y$ factors through Y', as required.

If the length of E is an integer $n > 1$, it is possible to factor the map $Y' \to Y$ as $Y' \to Y'' \to Y$, where the mapping cones on $Y' \to Y''$ and on $Y'' \to Y$ are both extensions of coproducts of objects of R whose length is strictly less than n. By induction we may assume that there are morphisms $X'' \to X$ and $X' \to X''$, both with mapping cones in $\langle R \rangle$, so that $X'' \to X \to Y$ factors through Y'', and $X' \to X'' \to Y''$ factors through Y'. Then we obtain a commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X'' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y'' & \longrightarrow & Y \end{array}$$

which shows us how to factor the composite $X' \to X \to Y$ through Y'. Furthermore, the mapping cone on the composite $X' \to X'' \to X$ is in $\langle R \rangle$, by the octahedral axiom. □

LEMMA 2.4. — *The functor $j^* : \mathscr{S} \to \mathscr{T}$ sends objects in $\mathscr{S}^c$ to objects in $\mathscr{T}^c$.*

Proof. — Let X be an object in $\mathscr{S}^c$. We need to show that $j^*(X)$ is a compact object in $\mathscr{T}$. Therefore let I be an arbitrary index set, and let Y_i be a collection of objects in $\mathscr{T}$ indexed by I. Then

$$\begin{aligned} \operatorname{Hom}(j^*(X), \oplus Y_i) &= \operatorname{Hom}(X, j_*(\oplus Y_i)) = \operatorname{Hom}(X, \oplus j_*(Y_i)) \\ &= \oplus \operatorname{Hom}(X, j_*(Y_i)) = \oplus \operatorname{Hom}(j^*(X), Y_i) \end{aligned}$$

The first and last equality are by adjunction. The second equality is because the functor j_* respects coproducts (Lemma 1.9). The third equality is because X is a compact object of $\mathscr{S}$; $\operatorname{Hom}(X, -)$ commutes with direct sums. □

LEMMA 2.5. — *The induced functor $j^* : \mathscr{S}^c/\mathscr{R}^c \to \mathscr{T}^c$ is fully faithful.*

Proof. — We will prove a slightly stronger statement. Given any compact object X in $\mathscr{S}$, and an arbitrary object Y in $\mathscr{S}$, then the map α

$$\operatorname{Hom}_{\mathscr{S}/\mathscr{R}^b}(X, Y) \xrightarrow{\alpha} \operatorname{Hom}_{\mathscr{T}}(X, Y)$$

is an isomorphism.

Now recall that $\operatorname{Hom}_{\mathscr{T}}(X, Y) = \operatorname{Hom}_{\mathscr{T}}(j^*(X), j^*(Y)) = \operatorname{Hom}_{\mathscr{S}}(X, j_* j^*(Y))$. The first equality is just a definition, explaining our slightly sloppy notation of considering the morphisms in $\mathscr{T}$ between two objects in $\mathscr{S}$. The second equality is the adjunction. But the point is that the proof of Lemma 1.7 gives a very explicit construction for the Bousfield localization $j_* j^*(Y)$. We construct a sequence of objects $\{Y_n\}$ is $\mathscr{S}$, such that

$Y_0 = Y$ and there are morphisms $Y_n \to Y_{n+1}$ whose mapping cones are direct sums of objects in R, and $j_* j^*(Y) = \operatorname{hocolim}(Y_n)$. Therefore, any morphism from a compact X to $j_* j^*(Y)$ must factor through Y_n for some n.

But now Lemma 2.3 applies. Given a map $X \to Y_n$ as above, we can choose a morphism $X' \to X$ whose mapping cone is in $\langle R \rangle$, so that the composite $X' \to X \to Y_n$ factors through $Y = Y_0$. This prove that α is onto.

Suppose now that we are given a morphism $X \to Y$ in $\mathscr{S}$ which becomes zero in $\mathscr{T}$. Then the composite $X \to Y \to j_* j^*(Y)$ is zero. This means that $X \to \operatorname{hocolim}(Y_n)$ is the zero map, but because X is compact, the composite $X \to Y \to Y_n$ must vanish for some n.

Complete $Y \to Y_n$ to a triangle $E \to Y \to Y_n \to \Sigma E$. Then E is a finite extension of coproducts of elements of R, and the map $X \to Y$ must factor through $E \to Y$. Now we apply Lemma 2.3 to the maps $X \to E$ and $0 \to E$. By Lemma 2.3, there exists a map $X' \to X$ whose cone is in $\langle R \rangle$, such that the composite $X' \to X \to E$ factors through $0 \to E$. Then the composite $X' \to E \to Y$ is clearly zero, and the map $X \to Y$ vanishes in $\operatorname{Hom}_{\mathscr{S}/\mathscr{R}^b}(X, Y)$. This is enough to prove that α is faithful. □

LEMMA 2.6. — *Every object in $\mathscr{T}^c$ is a direct summand of an object in the image of $\mathscr{S}^c$; in other words, $\mathscr{T}_c$ is the smallest épaisse subcategory of $\mathscr{T}$ containing the image of $\mathscr{S}^c$.*

Proof. — Recall that we are assuming that the smallest full, triangulated subcategory of $\mathscr{S}$ containing $\mathscr{S}^c$ and closed with respect to the formation of coproducts is all of $\mathscr{S}$. We are further assuming that $\mathscr{S}^c$ is small. If we let S be the set of isomorphism classes of objects in $\mathscr{S}^c$, then after choosing a representative in each isomorphism class S satisfies all the hypotheses we have made on R. In particular one can talk of a Bousfield localisation with respect to S.

Furthermore, by Lemma 2.4 the image of S in $\mathscr{T}$, denoted $j^*(S)$, is a set of compact objects in $\mathscr{T}$. Therefore we can form Bousfield localisations in $\mathscr{T}$ with respect to $j^*(S)$. Our hypothesis is that the smallest full, triangulated subcategory of $\mathscr{S}$ closed with respect to the formation of coproducts (=localizing subcategory, Definition 1.3) containing S is all of $\mathscr{S}$. But then the smallest localizing subcategory of $\mathscr{T}$ containing $j^*(S)$ is all of $\mathscr{T}$. So Lemma 2.2 applies, with $\mathscr{R}$ replaced with $\mathscr{T}$, $\mathscr{S}$ replaced with $\mathscr{T}$, and R replaced with $j^*(S)$. And the conclusion of Lemma 2.2 asserts that $\mathscr{T}^c$ is the épaisse closure of $j^*(S) = j^*(\mathscr{S}^c)$ in $\mathscr{T}^c$. □

APPENDIX

A proof of Lemma 0.6

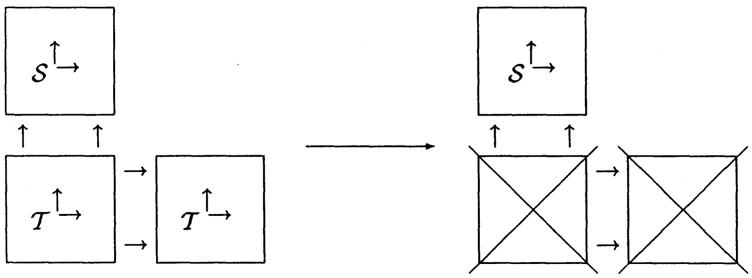
Let us begin by reminding the reader of the statement of Lemma 0.6.

LEMMA 0.6. — *Suppose $\mathscr{T}$ is a triangulated category, and $\mathscr{S}$ is a full, triangulated subcategory of $\mathscr{T}$ whose épaisse closure is $\mathscr{T}$. Then a delooping of the map*

$K(\mathscr{S}) \to K(\mathscr{T})$ *is a covering space. It is a homotopy equivalence if and only if the map* $\mathscr{S} \subsetneq \mathscr{T}$ *is an equivalence of categories.*

For the reader's convenience, we break the proof into a sequence of trivial steps. The proof is in the notation of [N]; it is assumed the reader is familiar with the notation.

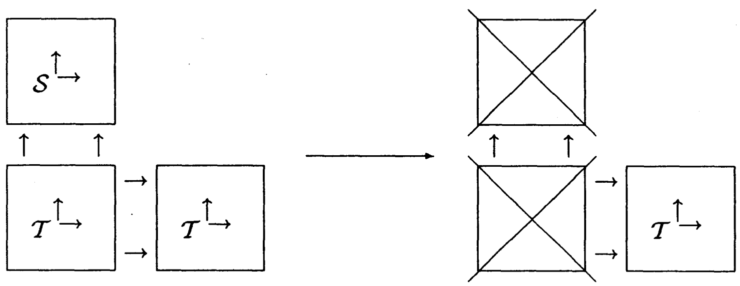
LEMMA A.1. – *The projection*



induces a homotopy equivalence.

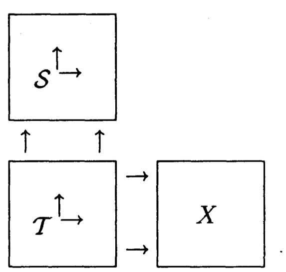
Proof. – Trivial.

LEMMA A.2. – *The projection:*

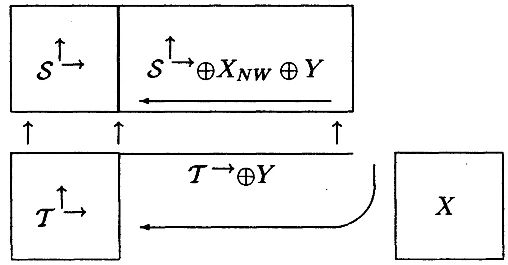


induces a quasi-fibration.

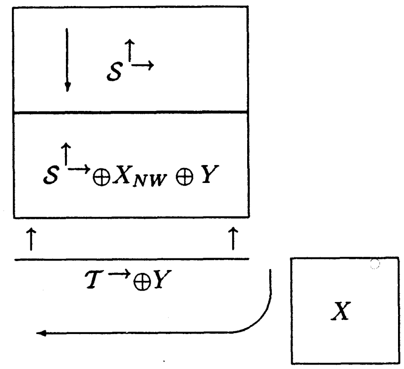
Proof. – In the notation of Section 9 of [N], this is practically the Prototype Quasifibration 9.2. Precisely, we need to study the fiber



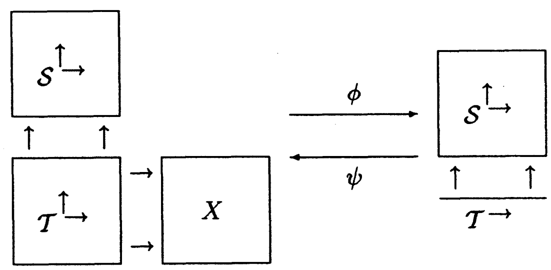
Let X_{NW} be the North-West corner object in X, as in the notation of [N], Section 9. Choose an object Y in $\mathscr{T}$, so that $Y \oplus X_{NW}$ is an object of $\mathscr{S}$. The existence of Y is guaranteed because the épaisse closure of $\mathscr{S}$ is all of $\mathscr{T}$; every object of $\mathscr{T}$ is a direct summand of an object in $\mathscr{S}$. Now, applying first the homotopy



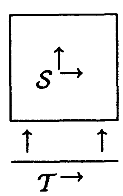
and following with the homotopy



we may deduce that up to homotopy, the identity on the fiber factors through the fiber over 0. Concretely, we have two maps



The homotopies we have just seen establish that $\psi\circ\varphi$ is homotopic to the identity. The composite $\varphi\circ\psi$ is just the translation in the K-space structure on



by the 0-cell

$$\begin{matrix} 0 \\ \uparrow \\ X_{NW}\oplus Y \end{matrix}$$

and since this 0-cell is in the connected component of the 0-cell

$$\begin{array}{c} 0 \\ \uparrow \\ 0 \end{array}$$

it follows that $\varphi \circ \psi$ is also homotopic to the identity. Now let ∂ be a face map on X, and it can easily be computed that $\varphi \circ \partial \circ \psi$ is translation in the H-space structure with respect to the 0-cell

$$\begin{array}{c} 0 \\ \uparrow \\ X_{ij} \oplus Y \end{array}$$

and this is in general not in the connected component of the neutral element. However, we may choose an object Z in $\mathcal{T}$ such that $X_{ij} \oplus Y \oplus Z$ is in $\mathcal{S}$, and then translation in the H-space structure with respect to the 0-cell

$$\begin{array}{c} 0 \\ \uparrow \\ Z \end{array}$$

is a homotopy inverse to $\varphi \circ \partial \circ \psi$. Thus ∂ is a homotopy equivalence. □

COROLLARY A.3. — *The homotopy fiber of the simplicial map*

$$\boxed{\mathcal{S}^{\uparrow\!\llcorner\!\to}} \longrightarrow \boxed{\mathcal{T}^{\uparrow\!\llcorner\!\to}}$$

can be identified with the simplicial set

$$\dfrac{\begin{array}{c} \boxed{\mathcal{S}^{\uparrow\!\llcorner\!\to}} \\ \uparrow \qquad \uparrow \end{array}}{\mathcal{T}^{\to}}$$

Remark A.4. — From now on, we will assume that the subcategory $\mathcal{S} \subset \mathcal{T}$ is *replete*. This means that every object in $\mathcal{T}$ isomorphic to an object in $\mathcal{S}$ should lie in $\mathcal{S}$. Replacing $\mathcal{S}$ by its repletion (*i.e.* the smallest full subcategory of $\mathcal{T}$ containing all objects isomorphic to objects in $\mathcal{S}$) does not change the K-theory.

LEMMA A.5. — *The simplicial subset*

$$\begin{array}{c}\boxed{\mathcal{S}^{\uparrow\rightarrow}}\\ \uparrow\qquad\uparrow\\ \hline \mathcal{S}^{\rightarrow}\end{array} \quad\hookrightarrow\quad \begin{array}{c}\boxed{\mathcal{S}^{\uparrow\rightarrow}}\\ \uparrow\qquad\uparrow\\ \hline \mathcal{T}^{\rightarrow}\end{array}$$

is open and closed.

Proof. — We need to show that for any 1-simplex, if one of its faces is in the subset, then so is the other. There are two types of 1-simplices; for simplices of the form

$$\begin{array}{c} Y' \\ \uparrow \\ Y \\ \uparrow \\ X \end{array}$$

there is nothing to prove;

$$\begin{array}{c} Y \\ \uparrow \\ X \end{array} \quad \text{and} \quad \begin{array}{c} Y' \\ \uparrow \\ X \end{array}$$

both lie in the simplicial subset

$$\begin{array}{c}\boxed{\mathcal{S}^{\uparrow\rightarrow}}\\ \uparrow\qquad\uparrow\\ \hline \mathcal{S}^{\rightarrow}\end{array}$$

if and only if X is an object of $\mathcal{S}$.

Slightly less clear is the 1-simplex

$$\begin{array}{ccc} Y & \rightarrow & Y' \\ \uparrow & & \uparrow \\ X & \rightarrow & X' \end{array}$$

Just by definition, we known that Y and Y′ are objects of $\mathcal{S}$. We need to show that X is in $\mathcal{S}$ if and only if X′ is.

Of course, we have a triangle (or semi-triangle, but I will not consider these and treat only the simplest case)

$$X \rightarrow X' \oplus Y \rightarrow Y' \rightarrow \Sigma X$$

If X′ lies in $\mathscr{S}$, then so do X′⊕Y and Y′. Because $\mathscr{S}$ is triangulated and replete, it follows that X is also an object of $\mathscr{S}$.

If X lies in $\mathscr{S}$, then because so does Y′, the third edge on the triangle on $Y' \to \Sigma X$ must also lie in $\mathscr{S}$. Thus X′⊕Y must be an object of $\mathscr{S}$. But then the projection $X' \oplus Y \to Y$ is a morphism in $\mathscr{S}$, and X′, being the third edge of the triangle on this morphism, must also be in $\mathscr{S}$. □

Proof of Lemma 0.6. – We need to show that the map

$$\boxed{\mathcal{S}^{\uparrow\llcorner\rightarrow}} \longrightarrow \boxed{\mathcal{T}^{\uparrow\llcorner\rightarrow}}$$

is a covering map; in other words, that the homotopy fiber is discrete. But Corollary A.3 identifies the fiber to be

$$\frac{\begin{matrix}\boxed{\mathcal{S}^{\uparrow\llcorner\rightarrow}}\\ \uparrow \quad \uparrow\end{matrix}}{\mathcal{T}^{\rightarrow}}$$

and Lemma A.5 shows that inside the fiber, the simplicial subset

$$\frac{\begin{matrix}\boxed{\mathcal{S}^{\uparrow\llcorner\rightarrow}}\\ \uparrow \quad \uparrow\end{matrix}}{\mathcal{S}^{\rightarrow}}$$

is open and closed. But the above is clearly contractible. Thus the connected component of

$$\frac{\begin{matrix}\boxed{\mathcal{S}^{\uparrow\llcorner\rightarrow}}\\ \uparrow \quad \uparrow\end{matrix}}{\mathcal{T}^{\rightarrow}}$$

at the neutral element is contractible. However, the space is an H-space with inverses, and hence all the components must be contractible, and the fiber is indeed discrete.

The fiber will be contractible if and only if

$$\begin{array}{c}\boxed{\;\mathcal{S}^{\overset{\uparrow}{\llcorner}\!\to}\;}\\ \uparrow\qquad\uparrow\\ \hline \mathcal{T}^{\to}\end{array}$$

is connected, *i.e.* if and only if it is equal to the open and closed subset

$$\begin{array}{c}\boxed{\;\mathcal{S}^{\overset{\uparrow}{\llcorner}\!\to}\;}\\ \uparrow\qquad\uparrow\\ \hline \mathcal{S}^{\to}\end{array}$$

This can only happen if $\mathscr{S}=\mathscr{T}$. $\square$

We have completed the proof of Lemma 0.6. It seems right, however, to say something about the framework of Waldhausen's K-theory of model categories.

Remark A.6. – Although Lemmas A.1 and A.2 can easily be modified to extend to the K-theory of model categories, the same is not true of Lemma A.5. Given a functor of model categories $\mathbf{S}\to\mathbf{T}$, it is not often that one would expect the connected component of

$$\begin{array}{c}\boxed{\;\mathbf{S}^{\overset{\uparrow}{\llcorner}\!\to}\;}\\ \uparrow\qquad\uparrow\\ \hline \mathbf{T}^{\to}\end{array}$$

to be the set

$$\begin{array}{c}\boxed{\;\mathbf{S}^{\overset{\uparrow}{\llcorner}\!\to}\;}\\ \uparrow\qquad\uparrow\\ \hline \mathbf{S}^{\to}\end{array}$$

For simplicity, suppose **S** and **T** to be exact categories. For the argument of Lemma A.5 to go through, one would require that.

(A.6.1) **S** be a full subcategory of **T** closed under extensions

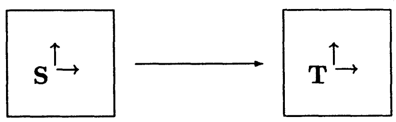
(A.6.2) Any morphism in **S** which is an admissible epi in **T** is already an admissible epi in **S**.

Nevertheless, [TT] prove Lemma 0.6. That is, given a morphism of model categories $\mathbf{S} \to \mathbf{T}$, such that the associated map of triangulated categories $\mathscr{S} \to \mathscr{T}$ satisfies the hypothesis of Lemma 0.6, then the induced map on Waldhausen K-theory is a covering space, and a homotopy equivalence if and only if $\mathscr{S}$ is equivalent to $\mathscr{T}$.

A rough outline of the argument goes as follows. By Lemma 0.6 and Corollary 01.0 (which have now been established in triangulated K-theory) we know that the category $\mathscr{S}$ contains all the objects of $\mathscr{T}$ in the image of $\mathrm{K}_0(\mathscr{S}) \subsetneq \mathrm{K}_0(\mathscr{T})$. Let $\mathbf{S}'$ be the full subcategory of $\mathbf{T}$ containing all the objects of $\mathbf{T}$ which vanish in $\mathrm{K}_0(\mathscr{T})/\mathrm{K}_0(\mathscr{S})$. Using Corollary 0.10 it is easy to show that the inclusion of $\mathbf{S}$ in $\mathbf{S}'$ gives an equivalence on the associated triangulated categories. By Waldhausen's Approximation Theorem, the Waldhausen K-theories of $\mathbf{S}$ and $\mathbf{S}'$ are homotopy equivalent, and it suffices to prove the theorem for the inclusion of $\mathbf{S}'$ in $\mathbf{T}$. But this inclusion is so nice that it satisfies (A.6.1) and (A.6.2), and in particular the proof of Lemma A.5 works.

Remark A.7. – Of course, the argument in [TT] is slightly different from what I just gave. At the time [TT] was written, triangulated K-theory did not exist. So in order to prove Corollary 0.10 (which is quite crucial to the argument in Remark A.6, and not just an amusing aside as suggested in the introduction) [TT] uses a computation in K_0 of the associated triangulated categories, the "Grayson trick". See [G], Section 1 or [TT], 5.5.4 the reader should note that although the higher K-theory of triangulated categories is quite new, the Grothendieck group K_0 was defined by Grothendieck many years ago.

The proof of Lemma A.5 is also slightly different in [TT]. Whereas we considered the homotopy fiber of the map



it is more natural in Waldhausen's K-theory to consider a delooping of the fiber. In [TT], 1.10.1 it is shown that delooping of the fiber is a $\mathrm{K}(\pi, 1)$. This is actually a major difference between Waldhausen's K-theory and the newer, triangulated theo-jyry. In the new theory, the major way to prove a map a quasi-fibration relies on Prototype Quasifibration 9.1. In Waldhausen's K-theory, the basic tool is the additivity theorem. To apply the additivity theorem, one needs to iterate the Q-construction at least once. Thus, one is always led to delooping of the maps one wishes to consider.

Nevertheless, apart from superficial differences, the proof in [TT] really is as in Remark A.6.

REFERENCES

[BN] M. Bökstedt and A. Neeman, *Homotopy Limits in Triangulated Categories*, Preprint.

[B1] A. K. Bousfield, *The Localization of Spaces with Respect to Homology*, (*Topology*, vol. **14**, 1975, pp. 133-150).

[B2] A. K. Bousfield, *The Localization of Spectra with Respect to Homology* (*Topology*, vol. **18**, 1979, pp. 257-281).

[G] D. Grayson, *Localization for Flat Modules in Algebraic* K-*theory* (*J. Alg.*, vol. **61**, 1979, pp. 463-496).

[N] A. Neeman, K-*Theory for Triangulated Categories II: Homologidal Functors*, Preprint.

[R] D. C. Ravenel, *Localization with Respect to Certain Periodic Homology Theories* (*Amer. J. of Math.*, vol. **106**, 1984, pp. 351-414).

[TT] R. Thomason and T. Trobaugh, *Higher Algebraic* K-*Theory of Schemes and of Derived Categories*, in *The Grothendieck Festschrift* (*a collection of papers to honor Grothendieck's* 60*'th birthday*), Birkhäuser, vol. **3**, 1990, pp. 347-435.

[W1] F. Waldhausen, *Algebraic* K-*Theory of Spaces* (*Lect. Notes in Math.*, vol. **1126**, 1985, pp. 318-419).

[W2] F. Waldhausen, *Algebraic* K-*Theory of Spaces, Localization, and the Chromatic Filtration of Stable Homotopy*, in *Proceedings of Alg. Top. Conf. in Aarhus*, 1982, SLN 1051, 1984, pp. 173-195.

[Y] D. Yao, *Higher Algebraic* K-*Theory of Admissible Abelian Categories and Localization Theorems*, Preprint.

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