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F-ISOCRYSTALS AND THEIR MONODROMY GROUPS

BY RICHARD CREW

Introduction

The main object of this paper is to prove a p-adic analogue of Grothendieck's Global Monodromy Theorem. The theorem in its l-adic form is the following: let k be a finite field of characteristic p, k^{alg} the algebraic closure of k, X a normal geometrically connected k-scheme of finite type, and suppose that $\rho : \pi_1(X) \to GL(V)$ is an l-adic representation of its fundamental group, where l is a prime different from p. Finally let G denote the Zariski closure in GL(V) of the image under ρ of the geometric monodromy group $\pi_1(X \oplus k^{alg})$. The monodromy theorem says that the radical of the connected component G^{conn} is unipotent. One of the more important applications of this theorem is Deligne's construction [6] of the "determinantal weights" associated to an l-adic representation, a fundamental step in his second proof of the Weil conjectures. It is at first disconcerting to note that there is no corresponding result for p-adic representations of π_1. Katz and Lang [10] prove an analogue of the monodromy theorem for p-adic representations when X/k is smooth and *proper*, but their result does not hold for nonproper varieties; there are well-known examples of p-adic characters of $\pi_1(X)$ of infinite order when X/k is a smooth affine curve.

On the other hand, p-adic representations do not arise from geometry in the same way as l-adic representations do (e.g. as the monodromy representation coming from the relative l-adic cohomology of a smooth family); what one gets directly from a geometric situation is a more general kind of object, an F-*isocrystal*. The p-adic representations can be identified with a full subcategory of the category of F-isocrystals, namely the *unit-root* F-isocrystals. Thus there can be no analogue of the global monodromy theorem for general F-isocrystals, but then not all F-isocrystals come from geometry. The interesting class seems to be Berthelot's category of *overconvergent* F-isocrystals ([1], [2]), and it is this category for which we will prove an analogue of the monodromy theorem.

The monodromy groups themselves will be defined by means of the theory of Tannakian categories, following the example of Katz [9]. Once we have set up all the requisite machinery, the proof of the monodromy theorem can proceed along the same lines as in [6], which we do in 4. As in the l-adic situation, the basic case is that of an F-isocrystal of rank one, which we treated in [5]. A key part of the rest of the argument is the Tannakian description of the category of F-isocrystals on a scheme over a perfect field which we give in 2. The problem is that the category of F-isocrystals on X/K is a *non-neutral* Tannakian category, and so cannot be described as a category of representations of some group. Of the several ways around this, we have decided to give a description in terms of groups endowed with a "Frobenius structure". This is not really necessary if k is a finite field, for then the category can be "linearized" by replacing the Frobenius by a suitable power. But we will need this technique in other situations, as for example in 3, when we have to study the category of unit-root F-isocrystals in the case when k is algebraically closed. Another possibility is to brutally linearize the category by extension of scalars "à la Saavedra"; this was in fact my original approach to the problem, abandoned for the reasons given in 3. In 5 however we consider the case when k is finite, and show that the methods of 2 amount to the construction of a "Weil group" attached to an F-isocrystal (or to a category of F-isocrystals). The results of 4 then enable us to set up a p-adic theorey of determinantal weights, and to prove some simple results about them.

The monodromy groups defined here do not seem to be any easier to compute than the differential galois groups computed by Katz [9]. In 4 we treat one of the simpler examples, the overconvergent F-isocrystal coming from the relative H^1_{cris} of a family of elliptic curves. The corresponding monodromy group turns out to be SL(2), just as in the l-adic case – which might at first seem surprising, for if the family is totally ordinary, the presence of the "unit-root" sub-F-isocrystal makes it appear that the monodromy group should be solvable. In fact, however, the unit-root sub-F-isocrysal is *not* overconvergent, as we pointed out in [5]. That the p-monodromy group should turn out to be the "same" as the l-adic is very suggestive, although it is not yet clear to what extent the monodromy group of an overconvergent F-isocrystal should resemble the geometric monodromy group of an l-adic representation that is "compatible" with it (for that matter, it is not known whether the members of a compatible system of l-adic representation have the "same" mondromy groups). In a subsequent paper, I will treat another interesting F-isocrystal, the p-adic hypergeometric equation studied by Dwork and Sperber in connection with the theory of Kloosterman sums, and will show that in most cases its (overconvergent) monodromy group is the same as the corresponding l-adic one.

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1. F-isocrystals

1.1. Fix a perfect field k of characteristic $p>0$. In what follows, k-schemes will be assumed to be separated and of finite type. We denote by K(k) the fraction field of the ring of Witt vectors W(k) of k. Letters like K, L will usually denote finite extensions of K (*k*). Our basic references for the theory of *convergent* and *overconvergent isocrystals* are [12] and [2]. There are useful summaries in [1], [5], and the sketch which follows is merely meant to establish notation. For any *k*-scheme X (separated and of infinite type) the basic constructions give us categories Isoc (X/K), $\mathrm{Isoc}^{\dagger}$ (X/K) known as the categories of *convergent* (resp. *overconvergent*) *isocrystals* on (X|K). In what follows, the term "isocrystal" by itself will mean an object of either category. These categories are of local nature on X and functorial in the pair X/K (we will explain this in a moment). If X/*k* is smooth and a formally smooth lifting $\mathfrak{X}$/R, R being a discrete valuation ring with fraction field K and residue field *k*, then Isoc (X/K) is equivalent to the category $\mathrm{Diff}_{\mathrm{conv}}(\mathfrak{X}^{\mathrm{an}})$ of locally free sheaves on the rigid-analytic space $\mathfrak{X}^{\mathrm{an}}$ endowed with a convergent connection; the term *convergent* means, roughly speaking, that the Taylor series associated to the connection ([5], 1.2.2, [2]) converges in every opeen unit disk in $\mathfrak{X}^{\mathrm{an}}$. For more general X/*k*, one has to resort to cutting, pasting, embedding, etc. as in [2], or else to a "site-theoretic" definition (as in [12]). The category $\mathrm{Isoc}^{\dagger}$ (X/K) is more difficult to describe, and for the moment we shall just say that for the objects and morphisms of $\mathrm{Isoc}^{\dagger}$ (X/K) one imposes additional conditions of "overconvergence at infinity". Forgetting the additional conditions gives rise to a faithful functor ([2] 2.3.10 (i))

$$(1.1.1)\qquad \begin{cases} \mathrm{Isoc}^{\dagger}(X/K) \to \mathrm{Isoc}(X/K) \\ \quad M \mapsto \hat{M}. \end{cases}$$

It is not known whether this functor is *fully* faithful. The notation is meant to suggest "restriction to the completion" of the weakly complete (in the sense of Washnitzer-Monsky) algebras on which live the representatives of an object of $\mathrm{Isoc}^{\dagger}$ (X/K).

To describe what is meant by "functoriality in thee pair X/K" we introduce a category $\mathscr{B}$ whose objects are pairs (X, K), where X is a separated *k*-scheme of finite type and K is an extension of K (*k*); we will prefer to write the pairs as X/K rather than as (X, K). A morphism X′/K′ → X/K is a commutative diagram

$$(1.1.2)\qquad \begin{cases} X' & \to & \mathrm{Spec}(k') & \leftarrow & \mathrm{Spec}(R'/m') & \to & \mathrm{Spec}(R') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \to & \mathrm{Spec}(k) & \leftarrow & \mathrm{Spec}(R/m) & \to & \mathrm{Spec}(R) \end{cases}$$

in which R has maximal ideal *m*, fraction field K and residue field an extension of *k*, and similarly for R′, *k*′, K′. Then the functoriality of the categories Isoc (X/K), $\mathrm{Isoc}^{\dagger}$ (X/K) can be described by saying that there are fibered categories **Isoc**, $\mathbf{Isoc}^{\dagger}$ over $\mathscr{B}$ such that the fiber of **Isoc** (resp. $\mathbf{Isoc}^{\dagger}$) any over Z/L is Isoc (Z/L) (resp. $\mathrm{Isoc}^{\dagger}$ (Z/L)). Thus for any morphism $f: X/K \to X'/K'$ there is a functor $f^*: \mathrm{Isoc}(X'/K') \to \mathrm{Isoc}(X/K)$ [resp. $f^*: \mathrm{Isoc}^{\dagger}(X'/K') \to \mathrm{Isoc}^{\dagger}(X/K)$] satisfying the usual compatibilities (*cf.* [2], 2.3.6). Finally, there is an evident Grothendieck topology on $\mathscr{B}$ for which the open coverings of X/K are those induced by the (zariski) open covers of X, and the assertion that Isoc (X/K) and $\mathrm{Isoc}^{\dagger}$ (X/K) are of local nature on X means that **Isoc** and $\mathbf{Isoc}^{\dagger}$ are stacks with respect to this topology (*cf.* [2], 2.3.3 (i)).

When $f: X/K \to X'/K'$ is the morphism 1.1.2 in $\mathscr{B}$, we will say that f *covers* the morphism $X \to X'$. In a number of situations there are obvious (or nearly obvious) morphisms in $\mathscr{B}$ covering a morphism $X \to X'$; for example, if $U \to X$ is an open immersion, there is an obvious $U/K \to X/K$ covering it and we will denote the corresponding functor by $M \to M \setminus U$. If $x \to X$ is a point of X and $f: x/K' \to X/K$ covers $x \to X$, we will denote $f^* M$ by M_x, if the reference to K' is not essential. If M is an overconvergent isocrystal on X/K and x is a closed point of X, then there is a canonical isomorphism $M_x \simeq \hat{M}_x$, as one can see immediately from the constructions. Finally, if k'/k is an extension and $\mathrm{Spec}(k')/K' \to \mathrm{Spec}(k)/K$ covers the natural map, then there is a natural extension of scalars functor $\mathrm{Isoc}(X/K) \to (X \otimes k'/K')$, which we will write $M \mapsto M \otimes K'$.

When $X = \mathrm{Spec}(k)$, both Isoc (X/K) and $\mathrm{Isoc}^\dagger(X/K)$ can be identified with the category Vec_K of K-vector spaces. In general, they are K-linear abelian $\otimes$-categories (*cf.* [2], 2.3.3 (iii)). Furthermore if $x \to X$ is a point of X with values in a perfect field and $x/K' \to X/K$ covers $x \to X$, then the pullback functor $\mathrm{Isoc}(X/K) \to \mathrm{Vec}_{K'}$ is faithful and exact (*see* 1.9 below).

We will say that an isocrystal on X/K is *constant* if it is (possibly after passing to extensions of k and K) a pullback via the structure map $X \to \mathrm{Spec}(k)$ of an isocrystal on Spec (k).

From now on we will denote by $F: X \to X$ a fixed power of the absolute Frobenius morphism, say $F(x) = x^q$ for some $q = p^f$; then for any lifting $\sigma: K \to K$ of the q-*th* power automorphism of k, there is a unique morphism $F_\sigma: X/K \to X/K$ covering F and inducing σ on K. We will now assume that there is a σ which fixes a uniformizer of K; this can always be guaranteed in practice, at the cost of replacing K by a finite unramified extension. We will now fix such a σ and drop the subscript on F_σ whenever it is convenient. A *convergent* (resp. *overconvergent*) F-*isocrystal* on X/K is an object M of Isoc (X/K) [resp. $\mathrm{Isoc}^+(X/K)$] endowed with an isomorphism

$$\Phi: \quad F_\sigma^* M \xrightarrow{\sim} M \tag{1.1.3}$$

(strictly speaking we should call these $\sigma -$F-isocrystals, or F_σ-crysals...). We shall call an isomorphism such as 1.1.3 a *Frobenius structure* on M. Morphisms of course are morphisms of isocrystals compatible with the Frobenius structure. The category of convergent (resp. overconvergent) F-isocrystals on X/K will be denoted by F-Isoc (X/K) (resp. $\mathrm{F\text{-}Isoc}^\dagger(X/K)$), and "F-isocrystal" by itself will mean an object of either category. When $X = \mathrm{Spec}(k)$, both categories are equivalent to the category $\mathrm{F\text{-}criso}_K$ of F-isocrystals on K, *i.e.* of K-vector spaces endowed with a σ-linear automorphism. In general they are abelian $\otimes$-categories, though not K-linear; rather they are K_0-linear, where K_0 denotes the fixed field of σ on K.

Denote by F-Cris (X/W) the category of F-crystals on X/W. Berthelot constructs (*cf.* [2], 2.3.11) a functor

$$\begin{cases} \text{F-Cris}(X/W \to \text{F-Isoc}(X/K) \\ \qquad M \mapsto M \otimes \mathbf{Q} \end{cases} \tag{1.1.4}$$

which is fully faithful up to isogeny; *i. e.* 1.1.4 induces an isomorphism

$$\text{(1.1.5)} \qquad \mathrm{Hom}_{\mathrm{F-Cris}}(\mathrm{M}, \mathrm{N})\otimes \mathrm{K}_0 \tilde{\to} \mathrm{Hom}_{\mathrm{F-Isoc}}(\mathrm{M}\otimes \mathbf{Q}, \mathrm{N}\otimes \mathbf{Q}).$$

If X/*k* is smooth, then 1.1.4 is essentially surjective up to Tate twists ([2], 2.3.12). One would like to know which objects in the essential image of 1.1.4 are overconvergent. Furthermore if X/*k* is smooth, it is not difficult to show that 1.1.4 is fully faithful up to isogeny in the category of crystals, *i. e.*

$$\text{(1.1.6)} \qquad \mathrm{Hom}_{\mathrm{Cris}}(\mathrm{M}, \mathrm{N})\otimes \mathrm{K} \tilde{\to} \mathrm{Hom}_{\mathrm{Isoc}}(\mathrm{M}\otimes \mathbf{Q}, \mathrm{N}\otimes \mathbf{Q})$$

for any crystals M, N endowed with a Frobenius structure. In fact, since the assertion is local on X, we can assume X = Spec (A) is affine, and that $\mathfrak{U}$/R lifts A. An F-crystal M on X/W can be identified with a locally free $\mathfrak{U}$-module $\mathscr{M}$ endowed with a integrable nilpotent connection, and the isocrystal M⊗**Q** is $\mathscr{M}\otimes$K endowed with the corresponding connection, which is convergent since M has a Frobenius. Morphisms in either category are just horizontal maps (not necessarily compatible with Frobenius), so that 1.1.6 is clear.

Since we have chosen a σ which fixes a uniformizer of K, Manin's structure theorem for F-isocrystals on a field is applicable ([11], 2.1), and one can define in the usual way the *Newton polygon* of a F-isocrystal on K, or of an F-isocrystal on X/K at a closed point *x* of X. A *unit-root* F-isocrystal on X/K is an F-isocrystals whose Newton polygon at every point of X is purely of slope zero. The unit-root F-isocrystals in F-Isoc (X/F) and F-Isoc† (X/Y) constitute full subcategories of F-Isoc (X/K), resp. F-Isoc† (X/Y) which we will denote by UR (X/K) resp. UR† (X/K). We will need the following consequence of the specialization theorem for Newton polygons of F-isocrystals ([4], Theorem 2.1):

1.2. Proposition. – *Suppose that* X/*k* is geometrically connected and let (M, Φ) *be an* F-*isocrystal on* X/K. *If, for some geometric generic point* $x \to$ X, *the fiber* $(\mathrm{M}, \Phi)_x$ *is unit-root, then* (M, Φ) *is a unit-root* F-*isocrystal on* X/K.

Proof. – By [4] 2.1, there is a dense open U⊆X such that the fiber of (M, Φ) at any point of U is unit-root. Since the Newton polygon rises under specialization, and since the location of the endpoint is the same for all points of X by [4] 2.1.3, we see that (M, Φ) is unit root at every closed point. ■

Since K_0 is the subfield of K fixed by σ, it is a local field; in fact if F is the *q-th* power map, it is a finite totally ramified extension of the fraction field of W ($\mathbf{F}_q$). If G/L is an affine group scheme over a field L, we denote by $\mathrm{Rep}_{\mathrm{L}}$(G) the category of finite-dimensional representations of G defined over L. If P is a profinite group and L is a field with a *p*-adic topology, then we denote by $\mathrm{Rep}_{\mathrm{L}}^{\mathrm{ctn}}$(P) the category of continuous finite-dimensional representations of P. One basic result that we shall need from [5] is the following:

1.3. THEOREM ([5] 2.1 and 2.2.4). — *Suppose that* X/k *is smooth and geometrically connected and that* $\mathbf{F}_q \subseteq k$. *Then there is an equivalence of* $\otimes$*-categories*

$$G: \ \mathrm{Rep}^{\mathrm{ctn}}_{K_0}(\pi_1(X)) \xrightarrow{\sim} \mathrm{UR}(X/K). \ \blacksquare \tag{1.3.1}$$

One would like to have a similar description of $\mathrm{UR}^{\dagger}(X/K)$, but at present one has (partial) results only in the case when X/k is a smoth curve. Let $X \subsetneq \bar{X}$ be a smooth compactification of X; then we say that a representation $\rho: \pi_1(X) \to GL(V)$ on a K_0-vector space has *finite local monodromy)* if the image under ρ of the inertia group at any point of $\bar{X} - X$ is finite. We will denote by $\mathrm{Rep}_{K_0}(\pi_1(X)^{\mathrm{fin}}$ the category of representations with finite local monodromy.

1.4. THEOREM ([5] 3.1 and 2.2). — *Suppose that* X/K *is a smooth geometrically connected curve and that* $\mathbf{F}_q \subseteq k$. *Then there is a fully faithful* $\otimes$*-functor*

$$G^{+}: \ \mathrm{Rep}^{\mathrm{ctn}}_{K_0}(\pi_1(X))^{\mathrm{fin}} \to \mathrm{UR}^{\dagger}(X/K) \tag{1.4.1}$$

such that

$$\left\{\begin{array}{ccc} \mathrm{Rep}^{\mathrm{ctn}}_{K_0}(\pi_1(X))^{\mathrm{fin}} & \xrightarrow{G^{\dagger}} & \mathrm{UR}^{\dagger}(X/K) \\ \downarrow & & \downarrow \\ \mathrm{Rep}^{\mathrm{ctn}}_{K_0}(\pi_1(X)) & \xrightarrow{G} & \mathrm{UR}(X/K) \end{array}\right. \tag{1.4.2}$$

is 2*-commutative. Every rank one object of* $\mathrm{UR}(X/K)$ *is in the essential image of* $G^{\dagger}$. $\blacksquare$

It is not known whether $G^{\dagger}$ is an equivalence of categories. Nonetheless, if ρ is an object of $\mathrm{Rep}_{K_0}(\pi_1(X))^{\mathrm{fin}}$, then one sees from the commutativity of 1.4.2 that the essential image of $G^{\dagger}|[\rho]$ (where $[\rho]$ is the $\otimes$-subcategory generated by ρ) is stable under the formation of subquotients. Therefore the essential image of $G^{\dagger}|[\rho]$ is the entire $\otimes$-subcategory $[M, \Phi]$ of $\mathrm{UR}^{\dagger}(X/K)$ generated by $(M, \Phi) = G^{\dagger}(\rho)$, and $G^{\dagger}$ induces an equivalence of categories

$$G^{\dagger}: \ [\rho] \xrightarrow{\sim} [M, \Phi] \tag{1.4.3}$$

1.5. COROLLARY. — *With* X/k *as in* 1.4, *suppose in addition that* k *is the perfection of an absolutely finitely generated field. Then for any rank one object* (M, Φ) *of* F-$\mathrm{Isoc}^{\dagger}(X/K)^{\mathrm{ur}}$, *some tensor power* $(M, \Phi)^{\otimes N}$ *is constant.*

Proof. — By 1.4, there is a character $\rho: \pi^1(X) \to K^{\times}$ with finite local monodromy such that $G^{\dagger}(\rho) \simeq (M, \Phi)$. Since $G^{\dagger}$ is fully faithful, it is enough to show that some tensor power of ρ is trivial on the geometric fundamental group $\pi_1(X \otimes k^{\mathrm{alg}})$. Let $X \subsetneq \bar{X}$ be a smooth compactification; then since ρ has finite local monodromy, some tensor power of ρ extends to a character of $\pi_1(\bar{X})$. So it is enough to see that the image of

$\pi_1(\bar{X}\otimes k^{alg})^{ab}$ in $\pi_1(\bar{X})^{ab}$ is finite. For this, we need a result of Katz and Lang [10]:

1.6. THEOREM. – *Suppose that X_0/k_0 is smooth and proper and that k_0 is absolutely finitely generated. Then the image of $\pi_1(X_0\otimes k_0^{alg})^{ab}$ in $\pi_1(X_0)^{ab}$ is finite.* ■

To finish the proof of 1.5, we note that since $\bar{X}/k$ is of finite type, we can find an absolutely finitely generated field k_0 whose perfection is k, and a smooth proper k_0-scheme X_0 such that $\bar{X}\simeq X_0\otimes k$. Since the projection $\bar{X}\to X_0$ is a homeomorphism in the étale topology, $\pi_1(\bar{X})$ and $\pi_1(\bar{X}\otimes k^{alg})$ are isomorphic to $\pi_1(X_0)$ and $\pi_1(X_0\otimes k^{alg})$. Thus the assertion we need for 1.5 follows from 1.6. ■

1.7. Suppose now that $\pi: Y\to X$ is a finite étale map of smooth k-schemes. We will construct a functor

$$\pi_*:\ \mathrm{Isoc}(Y/K)\to \mathrm{Isoc}(X/K) \tag{1.7.1}$$

and, if X and Y are smooth curves, a functor

$$\pi_*:\ \mathrm{Isoc}^\dagger(Y/K)\to \mathrm{Isoc}^\dagger(X/K) \tag{1.7.2}$$

left adjoint to π^*. We will discuss in detail the convergent case, as the overconvergent case is similar.

Since we can argue locally on X, we can assume that X is affine, and choose a lifting $\mathfrak{Y}/R\to\mathfrak{X}/R$ of π as in 1.1; we will use π to denote the induced morphisms $\mathfrak{Y}\to\mathfrak{X}$, $\mathfrak{Y}^{an}\to\mathfrak{X}^{an}$. We will need some notation and results from [1, 2]. Denote by $]X[_{\mathfrak{X}\times\mathfrak{X}}$ resp. $]Y[_{\mathfrak{Y}\times\mathfrak{Y}}$ the tube of the diagonal $X\hookrightarrow X\times X$ in $\mathfrak{X}^{an}\times\mathfrak{X}^{an}$ (resp. of $Y\hookrightarrow Y\times Y$ in $\mathfrak{Y}^{an}\times\mathfrak{Y}^{an}$, *cf.* [1], § 1), and by $p_i:]X[_{\mathfrak{X}\times\mathfrak{X}}\to\mathfrak{X}^{an}$, $q_i:]Y[_{\mathfrak{Y}\times\mathfrak{Y}}\to\mathfrak{Y}^{an}$ $(i=1, 2)$ the natural projections. A convergent isocrystal M on Y/K can be identified with a coherent sheaf M on $\mathfrak{Y}^{an}$ endowed with a convergent connection, *i.e.* an isomorphism $q_1^* M\simeq q_2^* M$ on $]Y[_{\mathfrak{X}\times\mathfrak{X}}$ restricting to the identity on the diagonal and satisfyind a cocycle condition [1 § 4.1]. Since $\pi: \mathfrak{Y}^{an}\to\mathfrak{X}^{an}$ is finite, the direct image $\pi_* M$ of M is coherent, and we must construct a convergent connection on $\pi_* M$, namely an isomorphism $p_1^*\pi_* M\simeq p_2^*\pi_* M$ on $]X[_{\mathfrak{X}\times\mathfrak{X}}$ restricting to the identity on the diagonal and satisfying a cocycle condition. For $i=1$ or 2 we consider the commutative diagram

$$\left\{\begin{array}{ccc}]Y[_{\mathfrak{Y}\times\mathfrak{Y}} & \xrightarrow{q_i} & \mathfrak{Y}^{an} \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ ]X[_{\mathfrak{X}\times\mathfrak{X}} & \xrightarrow{p_i} & \mathfrak{X}^{an}. \end{array}\right. \tag{1.7.3}$$

If we can show this is cartesian, then we will have an isomorphism $p_i^*\pi_* M\xrightarrow{\sim}(\pi\times\pi)_* q_i^* M$ since M is coherent, π is finite and p_i is flat. The desired isomorphism is the composite

$$p_1^*\pi_* M\xrightarrow{\sim}(\pi\times\pi)_* q_1^* M\xrightarrow{\sim}(\pi\times\pi)_* q_2^* M\xrightarrow{\sim} p_2^*\pi_* M \tag{1.7.4}$$

where the middle arrow is induced by the connection $q_1^* M \simeq q_2^* M$ on M. To see that 1.7.4 restricts to the identity on the diagonal of $]X[_{\mathfrak{X} \times \mathfrak{X}}$, it is enough to observe that the inverse image of the diagonal of $]X[_{\mathfrak{X} \times \mathfrak{X}}$ under the left-hand vertical arrow in 1.7.3 is the diagonal of $]Y[_{\mathfrak{Y} \times \mathfrak{Y}}$. The verification of the cocycle condition will be left to the reader. To show that 1.7.3 is cartesian, *i.e.* that the natural map

$$]Y[_{\mathfrak{Y} \times \mathfrak{Y}} \to \mathfrak{Y}^{an} \times_{\mathfrak{X}^{an}}]X[_{\mathfrak{X} \times \mathfrak{X}}$$

is an isomorphism, we choose $i' = 2$ or 1 so that $i \neq i'$, and consider the commutative diagram

$$(1.7.5) \qquad \begin{array}{ccc}]Y[_{\mathfrak{Y} \times \mathfrak{Y}} & \hookrightarrow & \mathfrak{Y}^{an} \times \mathfrak{Y}^{an} \\ \downarrow & & \downarrow \scriptstyle{id \times \pi} \\ \mathfrak{Y}^{an} \times_{\mathfrak{X}^{an}}]X[_{\mathfrak{X} \times \mathfrak{X}} & \xrightarrow{(id, pr_{i'})} & \mathfrak{Y}^{an} \times \mathfrak{X}^{an}. \end{array}$$

In fact the bottom arrow is an open immersion (it is the composite of the open immersion $\mathfrak{Y}^{an} \times_{\mathfrak{X}^{an}}]X[_{\mathfrak{X} \times \mathfrak{X}} \hookrightarrow \mathfrak{Y}^{an} \times_{\mathfrak{X}^{an}} (\mathfrak{X}^{an} \times \mathfrak{X}^{an})$ and the isomorphism $\mathfrak{Y}^{an} \times_{\mathfrak{X}^{an}} (\mathfrak{X}^{an} \times \mathfrak{X}^{an}) \simeq \mathfrak{Y}^{an} \times \mathfrak{X}^{an}$). One checks readily that the image of the bottom arrow is the tube $]Y[_{\mathfrak{Y} \times \mathfrak{X}}$ of the graph of π:

$$Y \xrightarrow{id, \pi} Y \times X \hookrightarrow X \to \mathfrak{Y} \times \mathfrak{X}.$$

We must therefore show that in the diagram

$$\begin{array}{ccc} Y & \xrightarrow{diag} & \mathfrak{Y} \times \mathfrak{Y} \\ \| & & \downarrow \scriptstyle{id \times \pi} \\ Y & \longrightarrow & \mathfrak{Y} \times \mathfrak{X} \end{array}$$

the right-hand vertical arrow induces an isomorphism on tubes $]Y[_{\mathfrak{Y} \times \mathfrak{Y}} \simeq]Y[_{\mathfrak{Y} \times \mathfrak{X}}$. Since $\mathfrak{Y} \times \mathfrak{Y} \xrightarrow{\pi \times id} \mathfrak{X} \times \mathfrak{Y}$ is étale, this follows from [2] 1.3.1, which says that given any pair of immersions $i: Z \hookrightarrow \mathfrak{B}$, $i': Z \hookrightarrow \mathfrak{B}'$ with Z of finite type and $\mathfrak{B}$, $\mathfrak{B}'$ formally smooth of finite type, and any $u: \mathfrak{B}' \to \mathfrak{B}$, étale in a neighborhood of Z, such that $u \circ i' = i$, then u induces an isomorphism on the tubes $]Z[_{\mathfrak{B}'} \xrightarrow{\sim}]Z[_{\mathfrak{B}}$.

The overconvergent case can be handled similarly; the basic change is that one must choose smooth compactifications $X \hookrightarrow \bar{X}$, $Y \hookrightarrow \bar{Y}$ and make use of [2] 1.3..5 in place of [2] 1 3.1. Details will again be left to the reader. ■

Suppose, finally, that M is a convergent isocrystal arising from a representation of $\pi_1(Y)$, *i.e.* we have $(M, \Phi) = G(\rho)$ where G is the functor 1.3.1. One can show, using the construction in [5], § 2, that $\pi_* M$ is the isocrystal underlying the convergent F-isocrystal $G(\operatorname{Ind} \rho)$, where $\operatorname{Ind} \rho$ is the induced representation from $\pi_1(Y)$ to $\pi_1(X)$. Similarly, if X and Y are curves and ρ has finite local monodromy, then the overconvergent isocrystal $\pi_* M$ is the underlying isocrystal of $G^\dagger(\operatorname{Ind} \rho)$.

Now let $x \to X$ be a point of X with values in a perfect field, and $x/K' \to X/K$ is a map in $\mathscr{B}$ over $x \to X$. As we have remarked, the natural pullback functors have the form

$$(1.7.6) \qquad \begin{cases} \mathrm{Isoc}(X/K) \to \mathrm{Vec}_{K'} \\ \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Vec}_{K'}. \end{cases}$$

We now want to show that when X is geometrically connected, they are fiber functors in the sense of Saavedra:

1.8. LEMMA. – *If* X/k *is geometrically connected and* x *is a point of* X *with values in a perfect field, then the functor* 1.7.6 *is a faithful exact* $\otimes$*-functor.*

Proof. – We treat the convergent case first. We can factor $x \to X$ as $x \to U \to X$, such that U is a geometrically connected smooth affine subscheme of X, and such that x maps onto thee generic point of U. We will prove 1.8 by showing that the pullback functors for the two morphisms $x \to U$, $U \hookrightarrow X$ are faithful, exact, and compatible with tensor products.

We will have to make use of some notation and terminology from [2]. If X is affine, and has an embedding $X \hookrightarrow \mathscr{P}$ over R into a formally smooth formal R-scheme, then a convergent isocrystal M on X/K can be identified with a locally free sheaf on the tube $]X[_{\mathscr{P}}$ endowed with a convergent connection (*cf.* [1] 1.3 and 4.1). The tensor product on Isoc(X/K) is that induced by the tensor product of $\mathscr{P}^{an}$-modules. Since the pullback functor for $U \hookrightarrow X$ is induced by the restriction to the tube $]U[_{\mathscr{P}} \subset]X[_{\mathscr{P}}$, it is compatible with tensor products, and its faithfulness and exactness follow from [2] 2.3.3 (iii).

As for the pullback by $x \to U$, we can write $U = \mathrm{Spec}(A_0)$, and since U is smooth, we can find a formally smooth R-algebra A such that $A \otimes k = A_0$, and we put $\mathfrak{U} = \mathrm{Spf}(A)$. Let $x = \mathrm{Spec}(k')$, and let R′ be a complete discrete valuation ring extending R with residue field k' and fraction field K′. Since U/k is smooth, $x \to U$ lifts to $\mathrm{Spf}(R') \to \mathfrak{U}$, and the pullback functor for $x \to U$ is

$$(1.8.1) \qquad M \mapsto \Gamma(\mathfrak{U}^{an}, \mathscr{M}) \hat{\otimes}_{A_K} K'$$

where $A_K = A \otimes K$ is the affinoid algebra of $\mathfrak{U}^{an}$, and $A_K \to K'$ is obtained from $\mathrm{Spf}(R') \to \mathfrak{U}$. We can replace the completed tensor product in 1.8.1 by an ordinary tensor product, since $\mathscr{M}$ is locally free; then 1.8.1 is visibly compatible with tensor products. It is faithful, because if a horizontal section m of M vanishes at a point of $\mathfrak{U}^{an}$, then it vanishes on a neighborhood of that point; since U is connected, $\mathfrak{U}^{an}$ is connected as well, and the section m must vanish identically. Finally 1.8.1 is exact, since it is the composite of the functors $M \mapsto \Gamma(\mathfrak{U}^{an}, \mathscr{M})$ and $N \mapsto N \otimes_{A_K} K'$, which are exact on the categories of quasicoherent $\mathscr{O}_{\mathfrak{U}^{an}}$-modules, resp. locally free A_K-modules (note that we cannot use [2] 2.3.3 (iv) directly here, since it seems to be stated only for morphisms of finite type, which is not the case for $x \to U$ unless x is a closed point).

The overconvergent case can be proven in the same way; one could also deduce it from the convergent case, since the second functor in 1.7.6 is the composition of the first with 1.1.1, which is also faithful, exact, and compatible with tensor products. ■

2. Monodromy groups

2.1. To define the monodromy groups attached to objects of Isoc (X/K) or Isoc^{+} (X/K), we will use the theory of Tannakian categories, for which our references are [6], [12]. For the most part we will follow the terminology of the latter. In particular, if $(\mathscr{C}, \omega)$ is a neutral K-linear Tannakian category and g is an R-valued point of $\mathrm{Aut}^{\otimes}\omega$, then for any object M of $\mathscr{C}$ the action of g on $\omega(\mathrm{M})\otimes\mathrm{R}$ will be written g_{M}. If L/K is a (finite) extension we will denote the extension of scalars of $\mathscr{C}$ to L by $\mathscr{C}\otimes\mathrm{L}$ rather than $\mathscr{C}_{\mathrm{L}}$. If L/K is not finite, then an infinite extension of scalars can only be performed on an ind-Tannakian category, and so if $\mathscr{C}$ is Tannakian, then $\mathscr{C}\otimes\mathrm{L}$ will denote Saavedra's extension of scalars to L of the category of ind-objects ind-$\mathscr{C}$.

From now on all k-schemes are assumed to be geometrically connected. Suppose that $\mathrm{X}(k)$ is nonempty, and fix a k-point $x\in\mathrm{X}(k)$. We denote by ω either of the functors

$$(2.1.1)\qquad \begin{cases} \omega: & \mathrm{Isoc}(\mathrm{X}/\mathrm{K}) \to \mathrm{Vec}_{\mathrm{K}} \\ \omega: & \mathrm{Isoc}^{\dagger}(\mathrm{X}/\mathrm{K}) \to \mathrm{Vec}_{\mathrm{K}} \\ & \mathrm{M} \mapsto \mathrm{M}_{x} \end{cases}$$

obtained by taking fibers at x. Since X/k is geometrically connected, these are both faithful functors, the first by [12] 1.18 and the second by the faithfulness of the first and of the completion functor 1.1.1. By 1.8 they are faithful exact $\otimes$-functors, and therefore *fiber functors* in the sense of [6], [12]. One checks that the conditions of [7] 1.20 are satisfied, and since $\mathrm{End}(1)=\mathrm{K}$ for any unit object 1 of Isoc (X/K) (resp. $\mathrm{Isoc}^{\dagger}$(X/K)), it follows from [7] 2.11 that Isoc (X/K) [resp. $\mathrm{Isoc}^{\dagger}$(X/K)] is a neutral Tannakian category. We denote by $\pi_1^{\mathrm{Isoc}}(\mathrm{X}/\mathrm{K}, x)$ [resp. $\pi_1^{\mathrm{Isoc}^{\dagger}}(\mathrm{X}/\mathrm{K}, x)$] the affine K-group which represents $\mathrm{Aut}^{\otimes}\omega$, so that we have equivalences of categories

$$(2.1.2)\qquad \begin{cases} \eta_x^{\mathrm{Isoc}}: & \mathrm{Isoc}(\mathrm{X}/\mathrm{K}) \tilde{\to} \mathrm{Rep}_{\mathrm{K}}(\pi_1^{\mathrm{Isoc}}(\mathrm{X}/\mathrm{K}, x)) \\ \eta_x^{\mathrm{Isoc}^{\dagger}}: & \mathrm{Isoc}^{\dagger}(\mathrm{X}/\mathrm{K}) \tilde{\to} \mathrm{Rep}_{\mathrm{K}}(\pi_1^{\mathrm{Isoc}^{\dagger}}(\mathrm{X}/\mathrm{K}, x)) \end{cases}$$

whose composition with the obvious forgetful functor into $\mathrm{Vec}_{\mathrm{K}}$ is equal to ω. We will often suppress mention of x since these groups are independent of the choice of x, up to (non-canonical) isomorphism.

If $\mathscr{C}$ is any rigid $\otimes$-subcategory of Isoc (X/K) or $\mathrm{Isoc}^{\dagger}$(X/K), then $\mathrm{Aut}^{\otimes}(\omega|\mathscr{C})$ is represented by a quotient of π_1^{Isoc} resp. $\pi_1^{\mathrm{Isoc}^{\dagger}}$ which we shall call $\mathrm{DGal}(\mathscr{C}, x)$. As above, there is an equivalence of categories

$$\eta_x^{\mathscr{C}}: \quad \mathscr{C} \tilde{\to} \mathrm{Rep}_{\mathrm{K}}(\mathrm{DGal}(\mathscr{C}, x))$$

The basic example is the $\oplus$-subcategory [M] generated by an isocrystal M (*i.e.* the strictly full subcategory of Isoc (X/K) resp $\mathrm{Isoc}^{\dagger}$(X/K) consisting of objects isomorphic to subquotients of $\mathrm{M}^{\otimes n}\otimes\check{\mathrm{M}}^{\otimes m}$), in which case we will denote DGal ([M]) simply by

DGal(M), and the associated equivalence of categories by

$$\eta_x^{\mathrm{M}}: \quad [\mathrm{M}] \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{K}}(\mathrm{DGal}(\mathrm{M}, x)).$$

Since [M] has a finite (in fact singleton) set of ⊗-generators, DGal(M) is actually an algebraic group over K. The notation is meant to suggest the term "differential galois group", and as usual we will often suppress the x.

Recall that if $(\mathscr{C}_1, \omega_1)$, $(\mathscr{C}_2, \omega_2)$ are two K-linear neutral Tannakian categories endowed with fiber functors into $\mathrm{Vec}_{\mathrm{K}}$, then any ⊗-functor $\mathrm{F}: \mathscr{C}_2 \to \mathscr{C}_1$ such that $\omega_1 \circ \mathrm{F} = \omega_2$ induces a map of affine K-groups $\omega^{\mathrm{F}}: \mathrm{Aut}^{\otimes}\omega_1 \to \mathrm{Aut}^{\otimes}\omega_2$. Furthermore ω^{F} is faithfully flat if and only if F is fully faithful and the essential image of F is closed under the formation of subquotients, while ω^{F} is a closed immersion if and only if every object of $\mathscr{C}_1$ is isomorphic to a subquotient of an object of the form F(M) with M in $\mathscr{C}_2$. We will use these principles constantly; as elementary examples, we have:

(i) There are homomorphisms

$$(2.1.4) \qquad \begin{cases} \pi_1^{\mathrm{Isoc}} \twoheadrightarrow \mathrm{DGal}(\mathrm{M}) \\ \pi_1^{\mathrm{Isoc}^\dagger} \twoheadrightarrow \mathrm{DGal}(\mathrm{M}) \end{cases}$$

for any object M of Isoc(X/K) resp $\mathrm{Isoc}^\dagger$(X/K), induced by the obvious inclusion functors.

(ii) The completion functor 1.1.1 is a ⊗-functor, and since for any overconvergent isocrystal M there is a canonical isomorphism $\hat{\mathrm{M}}_x \simeq \mathrm{M}_x$, the completion induces a canonical homomorphism

$$(2.1.5) \qquad \pi_1^{\mathrm{Isoc}} \to \pi_1^{\mathrm{Isoc}^\dagger}.$$

Since any object of $[\hat{\mathrm{M}}]$ is a subquotient of the completion of an object of [M], 2.1.5 induces a canonical closed immersion

$$(2.1.6) \qquad \mathrm{DGal}(\hat{\mathrm{M}}, x) \subsetneq \mathrm{DGal}(\mathrm{M}, x).$$

(iii) A morphism $\pi: \mathrm{Y/L} \to \mathrm{X/K}$ in $\mathscr{B}$ induces L-linear ⊗-functors

$$(2.1.7) \qquad \begin{cases} \mathrm{Isoc}(\mathrm{X/K}) \otimes \mathrm{L} \to \mathrm{Isoc}(\mathrm{Y/L}) \\ \mathrm{Isoc}^\dagger(\mathrm{X/K}) \otimes \mathrm{L} \to \mathrm{Isoc}^\dagger(\mathrm{Y/L}) \\ [\mathrm{M}] \otimes \mathrm{L} \to [\pi^* \mathrm{M}]. \end{cases}$$

Thus if $\pi(y) = x$, we get canonical homomorphisms

$$(2.1.8) \qquad \begin{cases} \pi_1^{\mathrm{Isoc}}(\mathrm{Y/L}, y) \to \pi_1^{\mathrm{Isoc}}(\mathrm{X/K}, x) \otimes_{\mathrm{K}} \mathrm{L} \\ \pi_1^{\mathrm{Isoc}^\dagger}(\mathrm{Y/L}, y) \to \pi_1^{\mathrm{Isoc}^\dagger}(\mathrm{X/K}, x) \otimes_{\mathrm{K}} \mathrm{L} \end{cases}$$

and a canonical closed immersion

$$\mathrm{DGal}(\pi^* M, y) \hookrightarrow \mathrm{DGal}(M, x) \otimes_K L \tag{2.1.9}$$

for any isocrystal M.

(iv) If, the notation of the previous item, we have that X/k is smooth, L/K is finite, and $Y = X \otimes k'$, then we actually have an *isomorphism*

$$\mathrm{DGal}(\pi^* M) \xrightarrow{\sim} \mathrm{DGal}(M) \otimes L \tag{2.1.10}$$

for any isocrystal on X/K, as one easily sees by adapting the argument of [10] 1.3.2. In fact, the argument of [10] shows that 2.1.10 remains valid whenever there is a tower $K \subseteq L \subseteq M$ such that K (resp. L) is the fixed field of the group of continous automorphisms of M/K (resp. M/L). For example, 2.1.10 is valid if L is a finite extension of the completion of the maximal unramified extension of K.

If x is any point of X with values in any perfect field, and $x/L \to X/K$ covers $x \to X$, then one still obtains a fiber functor into the category of L-vector spaces, and one obtains an equivalence of ind-Tannakian categories

$$\eta_x^M: \quad \text{ind} - [M] \otimes L \xrightarrow{\sim} \text{ind} - \mathrm{Rep}_K(\mathrm{DGal}(M, x)) \tag{2.1.11}$$

for any object M of Isoc(X/K) or $\mathrm{Isoc}^\dagger(X/K)$. Of course if x is a closed point, then L/K can be taken to be finite and one does not need to use ind-Tannakian categories; in any case, there is a non-canonical isomorphism

$$\mathrm{DGal}(M, x) \simeq \mathrm{DGal}(M, y) \otimes L.$$

2.2. The K_0-linear categories F-Isoc(X/K), F-$\mathrm{Isoc}^\dagger$(X/K) of F-isocrystals on X/K are not, in general, neutral Tannakian categories; they have fiber functors in K but not necessarily in K_0. The theory of Saavedra shows that they are equivalent to categories of representations of affine gerbes over K_0, and their extension of scalars to K are equivalent to categories of representations of affine K-groups. However the extension of scalars destroys what might be useful information, and is not so easy to deal with. The construction which follows attempts to describe the Tannakian gerbes associated to categories like F-Isoc(X/K), F-$\mathrm{Isoc}^\dagger$(X/K) as "semilinear" extensions of **Z** by an affine K-group. When a category of F-isocrystals *does* have a fiber functor in K_0 (as in 5), we will obtain actual extensions of **Z** by an affine K-group.

Let G be an affine K-group. A *Frobenius structure* on G is a group isomorphism

$$\Phi: \quad G^{(\sigma)} \xrightarrow{\sim} G. \tag{2.2.1}$$

where as usual σ is the canonical lifting of Frobenius to K. A *morphism* $f: (G, \Phi) \to (G', \Phi')$ is of course just a morphism $f: G \to G'$ such that $f \circ \Phi = \Phi' \circ f^\sigma$.

For example, let V be a K-vector-space and let $G = GL(V)$. Since

$$GL(V)^{(\sigma)} \simeq GL(V)^{(\sigma)},$$

any F-isocrystal structure Ψ on V gives rise to a Frobenius structure $Ad(\Psi)$ by means of the formula

$$g \mapsto \Psi g \Psi^{-1} = Ad(\Psi)(g) \tag{2.2.2}$$

for any point g of $G^{(\sigma)}$. It is easy to check, by choosing a model V_0 of V over the fixed field K_0 of σ, that if $\dim V > 1$, then all Frobenius structures on $GL(V)$ arise in this way (since in this case all endomorphisms of $GL(V)$ are inner).

If G_0 is an affine group scheme over K_0, then there is an evident Frobenius structure on $G = G_0 \otimes K$ defined by $1 \otimes \sigma$. We shall call it the *trivial* Frobenius structure on G defined by the K_0-form G_0 of G.

Suppose now that (G, Φ) is a group with Frobenius structure and that (V, Ψ) is an F-isocrystal on K. A *representation of* (G, Φ) *on* (V, Ψ) is a morphism $(G, \Psi) \to (GL(V), Ad(\Psi))$, or in other words, an ordinary representation $\rho: G \to GL(V)$ such that

$$\left\{ \begin{array}{ccc} G^{(\sigma)} & \xrightarrow{\rho^{(\sigma)}} & GL(V)^{(\sigma)} \\ \Phi\downarrow & & Ad(\Psi)\downarrow \\ G & \xrightarrow{\rho} & GL(V) \end{array} \right. \tag{2.2.3}$$

commutes, or equivalently, such that

$$\rho(\Phi(g)) = \Psi \rho(g)^{\sigma} \Psi^{-1} \tag{2.2.4}$$

for any point g of G. A morphism of representations of (G, Φ) is a morphism of representations of G compatible with the Frobenius structures, or, equivalently, a morphism in $F\,criso_K$ compatible with the G-module structures. The category of representations of (G, Φ) on F-isocrystals over K will be denoted by $F\,Rep_K(G, \Phi)$. It is easy to see that for any group with Frobenius structure (G, Φ), the category $F\,Rep_K(G, \Phi)$ is an abelian category, and in fact a rigid K_0-linear abelian tensor category. Finally, there are obvious forgetful functots

$$\left\{ \begin{array}{l} F\,Rep_K(G, \Phi) \to F\,criso_K \\ F\,Rep_K(G, \Phi) \to Rep_K(G) \end{array} \right. \tag{2.2.5}$$

compatible with the $\otimes$-structures.

Note that an isomorphism 2.2.1 does not, in general, arise from descent data on G relative to K/K_0, even when $K = K(\bar{\mathbf{F}}_q)$.

Let us mention some simple examples of representations of groups with Frobenius structure. If G_0/K_0 is affine and ρ_0 is a representation of G_0 on a K_0-space V_0, then $V = V_0 \otimes K$ with trivial structure $1 \otimes \sigma$ is a representation of $(G_0 \otimes K, 1 \otimes \sigma)$. Suppose now that G/K is a constant finite group, Φ is a Frobenius structure on G, and that (V, Ψ) is a representation of (G, Φ). Since G is constant and finite, Φ has the form $\varphi\otimes\sigma$ for some automorphism φ of G; thus some power Φ^n of Φ is a trivial Frobenius structure. It follows that slope decomposition of (V, Ψ^n) is G-stable. We conclude that if G is constant and finite and K is sufficiently large, then any representation (V, Ψ) of (G, Φ) has the property that for some n, the reprsentation (V, Ψ^n) of (G, Φ^n) is a direct sum of twists of representations of (G, Φ^n) on *unit-root* isocrystals; thus V can be given a *unit-root* F-isocrystal structure Ψ' such that (V, Ψ') is a representation of (G, Φ^n).

Now let G=T be a split torus over K. We have $T=\mathrm{Hom}(L, \mathbf{G}_m)$, where L is the character group of T, and one checks easily (using the rigidity of tori) that any Frobenius structure Φ on T is induced by an automorphism $\varphi : L \xrightarrow{\sim} L$ of L combined with a trivial Frobenius structure on $\mathbf{G}_m$. From this we get the following description of the representations of (T, Φ): if we identify a representation of T on a K-vector space V with an L-grading

$$V=\bigoplus_{l\in L} V_l \tag{2.2.6}$$

of V, then a representation of (T, Φ) on (V, Ψ) is an L-grading of V and a Frobenius structure Ψ on V of the form

$$\Psi=\sum_{l\in L}\Psi_l, \qquad \Psi_l : \ V_l^{(\sigma)} \xrightarrow{\sim} V_{\varphi(l)}. \tag{2.2.7}$$

We thus see that φ permutes the weights of L in V, and in particular that if V is a *faithful* representation of T, then φ is of finite order.

2.3. If $\mathscr{C}$ is a $\otimes$-category of Isoc(X/K) or Isoc†(X/K) on which F* induces an equivalence of categories, then for any point x of X with values in a perfect field, there is a natural Frobenius structure on $\mathrm{D\,Gal}(\mathscr{C})\otimes K'$, whenever $x/K'\to X/K$ covers $x\to X$. In fact, if we recall that F* on $\mathrm{Isoc}(x/K)\simeq\mathrm{Isoc}^\dagger(x/K)$ is just the functor

$$V\mapsto V^{(\sigma)}$$

then the commutative diagram 1.8.4 means that there is a natural isomorphism of functors

$$\omega^{(\sigma)} \xrightarrow{\sim} \omega\circ F^*. \tag{2.3.1}$$

This if F* is an equivalence on $\mathscr{C}$, we get an isomorphism

$$\Phi_x^{\mathscr{C}} : \ \mathrm{D\,Gal}(\mathscr{C}, x)^{(\sigma)} \xrightarrow{\sim} \mathrm{D\,Gal}(\mathscr{C}, x). \tag{2.3.2}$$

One sees immediately that $\Phi_x^{\mathscr{C}}$ has the following "explicit" description: Let R be a K-algebra, g an R-valued point of G, and N an object of $\mathscr{C}$. Since F* is an equivalence, we can choose an isomorphism

$$\alpha: \quad F^* L \xrightarrow{\sim} N \tag{2.3.3}$$

in $\mathscr{C}$, and then write

$$\omega(\alpha): \quad \omega(L)^{(\sigma)} \xrightarrow{\sim} \omega(N) \tag{2.3.4}$$

making use of 2.3.1. The $\Phi_x^{\mathscr{C}}(g)$ is the unique isomorphism making

$$\left\{\begin{array}{ccc} \omega(L)^{(\sigma)}\otimes R & \xrightarrow{g_L^{(\sigma)}} & \omega(L)^{(\sigma)}\otimes R \\ \omega(\alpha)\downarrow & & \downarrow\omega(\alpha) \\ \omega(N)\otimes R & \xrightarrow{\Phi^{\mathscr{C}}(g)_N} & \omega(N)\otimes R. \end{array}\right. \tag{2.3.5}$$

commutative.

When we are tempted to suppress the base point, we will write Φ in place of Φ_x. This is more serious than omitting the base point of a π_1, for if x and y are points of $X(k)$, it is not necessarily the case that *any* of the natural but non-canonical isomorphisms $D\,Gal(M, x) \simeq D\,Gal(M, y)$ identify Φ_x with Φ_y (*cf.* Remark 2.8). This is really what one should expect, as will become clearer in 5 when we construct Frobenius elements.

We can use the canonical Frobenius structure $\Phi^{\mathscr{C}}$ on $D\,Gal(\mathscr{C})$ to describe the possible Frobenius structures one can put on an object on $\mathscr{C}$. To see this, let $\mathscr{C}_F$ be the strictly full subcategory of F-Isoc (X/K) resp. F-Isoc† (X/K) consisting of F-isocrystals (N, Φ) whose underlying isocrystal N belongs to $\mathscr{C}$. Clearly $\mathscr{C}_F$ is a rigid K_0-linear sub-$\otimes$-category of F-Isoc (X/K) resp. F-Isoc† (X/K). Now fix an object (N, Φ) of $\mathscr{C}_F$, set $V = \omega(N)$, and let $\rho: D\,Gal(\mathscr{C}) \to GL(V)$ be the representation associated to (N, Φ) by 2.1.3. If in 2.3.5 and 2.3.6 we take $\alpha = \Phi$ and $L = N$, then $\psi \overset{\text{dfn}}{=} \omega(\alpha)$ defines a Frobenius structure on V, and 2.3.5 says that $\rho(\Phi^{\mathscr{C}}(g)) = \Psi\,\rho(g)^\sigma\,\Psi^{-1}$, which is exactly the condition 2.2.4 for (ρ, V, Ψ) to define a representation of $D\,Gal(\mathscr{C}, \Phi^{\mathscr{C}})$ on (V, Ψ). This construction is obviously natural in (N, Ψ), so we have constructed a functor

$$\eta_x^{\mathscr{C}_F}: \quad \mathscr{C}_F \to F\,Rep_K(D\,Gal(\mathscr{C}), \Phi_x^{\mathscr{C}}) \tag{2.3.6}$$

such that

$$\left\{\begin{array}{ccc} \mathscr{C}_F & \xrightarrow{\eta_x^{\mathscr{C}F}} & F\,Rep_K(D\,Gal(\mathscr{C}), \Phi_x^{\mathscr{C}}) \\ \text{forget}\downarrow & & \downarrow\text{forget} \\ \mathscr{C} & \xrightarrow{\eta_x^{\mathscr{C}}} & F\,Rep_K(D\,Gal(\mathscr{C})) \end{array}\right. \tag{2.3.7}$$

is 2-commutative.

2.4. PROPOSITION. — *The functor* 2.3.6 *is an equivalence:*

$$\eta_x^{\mathscr{C}_F}: \quad \mathscr{C}_F \xrightarrow{\sim} \mathrm{F\,Rep}_K(\mathrm{D\,Gal}(\mathscr{C}, x), \Phi_x^{\mathscr{C}}) \tag{2.4.1}$$

Proof. — (1) $\eta^{\mathscr{C}_F}$ is fully faithful: For (N, Φ), (N', Φ') in $\mathscr{C}_F$, set

$$\eta(N, \Phi) = (\rho, V, \Psi)$$
$$\eta(N', \Phi') = (\rho', V', \Psi')$$

and let

$$f: \quad (\rho, V, \Psi) \to (\rho', V', \Psi')$$

be a morphism in $\mathrm{F\,Rep}_K(G, \Phi)$. Since $\eta^{\mathscr{C}}$ is fully faithful, the morphism $f: V \to V'$ of representations of $\mathrm{D\,Gal}(\mathscr{C})$ is the fiber of a morphism $\varphi: N \to N'$ of isocrystals. We must check that φ is compatible with the Frobenius structures Φ, Φ', or in other words, that

$$\left\{\begin{array}{ccc} F^*N & \xrightarrow{F^*(\varphi)} & F^*N' \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi'} \\ N & \xrightarrow{\varphi} & N' \end{array}\right. \tag{2.4.2}$$

commutes. It is enough to check this after applying the fully faithful functor ω to 2.4.2, which becomes the commutative diagram

$$\begin{array}{ccc} V^{(\sigma)} & \xrightarrow{f^{(\sigma)}} & (V')^{(\sigma)} \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi'} \\ V & \xrightarrow{f} & V' \end{array}$$

expressing the fact that $f: (V, \Psi) \to (V', \Psi')$ is a morphism in $\mathrm{F\,criso}_K$.

(2) $\eta^{\mathscr{C}_F}$ is essentially surjective: Let (ρ, V, Ψ) be a representation of $\mathrm{D\,Gal}(\mathscr{C}), \Phi^{\mathscr{C}})$ on (V, Ψ). Then 2.3.7 and the equivalence $\eta^{\mathscr{C}}$ provides us with an object N in $\mathscr{C}$ such that $\eta^{\mathscr{C}}(N) = (\rho, V)$. Now choose an isomorphism

$$\alpha: \quad F^*L \xrightarrow{\sim} N \tag{2.4.3}$$

in $\mathscr{C}$ and consider the diagrams

$$\left\{\begin{array}{ccc} \omega(L)^{(\sigma)}\otimes R & \xrightarrow{g_L^{(\sigma)}} & \omega(L)^{(\sigma)}\otimes R \\ {\scriptstyle\omega(\alpha)}\downarrow & & \downarrow{\scriptstyle\omega(\alpha)} \\ \omega(N)\otimes R & \xrightarrow{\Phi^{\mathscr{C}}(g)_N} & \omega(N)\otimes R \\ {\scriptstyle\Psi^{-1}}\downarrow & & \downarrow{\scriptstyle -1} \\ \omega(N)^{(\sigma)}\otimes R & \xrightarrow{g_N^{(\sigma)}} & \omega(N)^{(\sigma)}\otimes R \end{array}\right. \tag{2.4.4}$$

in which R is any K-algebra and g is any R-valued point of $\mathrm{D\,Gal}(\mathscr{C})$. The top square is the commutative diagram 2.3.5 and the bottom square is commutative because $(\rho, V=\omega(N), \Psi)$ is a representation of $(\mathrm{D\,Gal}(\mathscr{C}), \Psi^{\mathscr{C}})$. Untwisting the outside square yields commutative diagrams

$$\begin{array}{ccc} \omega(L)\otimes R & \xrightarrow{g_L} & \omega(L)\otimes R \\ \downarrow & & \downarrow \\ \omega(N)\otimes R & \xrightarrow{g_N} & \omega(N)\otimes R \end{array}$$

expressing the existence of an isomorphism

(2.4.5) $$L \xrightarrow{\sim} N$$

in $\mathscr{C}$. Substituting 2.4.5 into 2.4.3 yields a Frobenius structure $\Phi : F^* N \to N$, and one checks using 2.4.4 that $\eta^{\mathscr{C}_F}(\mathscr{C}, \Phi)=(\rho, V, \Psi)$. ■

2.5. If (M, Φ) is an F-isocrystal on X/K, then the results of 2.3 and 2.4 are always applicable to [M]. We thus have a canonical Frobenius structure Φ^M on $\mathrm{D\,Gal}(M)$, and an equivalence of categories

(2.5.1) $$\eta_x^M : \quad [M]_F \xrightarrow{\sim} \mathrm{F\,Rep}_K(\mathrm{D\,Gal}(M), \Phi_x^M).$$

To see this, we note that the existence of a Frobenius structure on M at least guarantees that [M] is F*-stable:

$$F^* : \quad [M] \to [M]$$

which by 2.1.9 gives a closed immersion

(2.5.2) $$\mathrm{D\,Gal}(M) \hookrightarrow \mathrm{D\,Gal}(M)^{(\sigma)}$$

Since D Gal (M) is an algebraic group, however, a closed immersion such as 2.5.2 must be an isomorphism, and one checks easily that the inverse of 2.5.2 is exactly the canonical Frobenius structure 2.3.2.

When $\mathscr{C}$ is an inductive limit of F*-stable finitely ⊗-generated subcategories of Isoc (X/K) or $\mathrm{Isoc}^{\dagger}(X/K)$, then one can use the same argument as before to show that F* is an autoequivalence on $\mathscr{C}$, and thus that proposition 2.4 is available for $\mathscr{C}$. Again, it is known that F* is an autoequivalence on Isoc (X/K) itself ([12], 4.10), so that if Φ^{Isoc} denotes the canonical Frobenius structure on π_1^{Isoc}, we have

(2.5.3) $$\eta_x^{\mathrm{Isoc}} : \quad F-\mathrm{Isoc}(X/K) \xrightarrow{\sim} \mathrm{F\,Rep}_K(\pi_1^{\mathrm{Isoc}}, \Phi_x^{\mathrm{Isoc}}).$$

I do not know if F^* is an equivalence on $\mathrm{Isoc}^\dagger(X/K)$, but we can still give a description of $F-\mathrm{Isoc}^\dagger(X/K)$ similar to 2.5.3. Denote by $\mathrm{Isoc}(X/K)^\sim$, $\mathrm{Isoc}^\dagger(X/K)^\sim$ the full subcategories of $\mathrm{Isoc}(X/K)$ resp. $\mathrm{Isoc}^\dagger(X/K)$ $\otimes$-generated by the image of the forgetful functors

$$F-\mathrm{Isoc}(X/K)\to\mathrm{Isoc}(X/K),\qquad F-\mathrm{Isoc}^\dagger(X/K)\to\mathrm{Isoc}^\dagger(X/K).$$

If we define

$$(2.5.4)\qquad \begin{cases} \pi^{F-\mathrm{Isoc}}(X/K)=\mathrm{Aut}^{\otimes}(\omega\,|\,\mathrm{Isoc}(X/K)^\sim) \\ \pi_1^{F-\mathrm{Isoc}^\dagger}(X/K)=\mathrm{Aut}^{\otimes}(\omega\,|\,\mathrm{Isoc}^\dagger(X/K)^\sim) \end{cases}$$

and denote by $\Phi^{F-\mathrm{Isoc}}$, $\Phi^{F-\mathrm{Isoc}^\dagger}$ the correspondong canonical Frobenius structures, then we have

2.6. THEOREM. — *There are equivalences of categories*

$$(2.6.1)\qquad \begin{cases} \eta_x^{F-\mathrm{Isoc}}: & F-\mathrm{Isoc}(X/K)\xrightarrow{\sim} F\,\mathrm{Rep}_K(\pi_1^{F-\mathrm{Isoc}}(X/K,x),\Phi_x^{F-\mathrm{Isoc}}) \\ \eta_x^{F-\mathrm{Isoc}^\dagger}: & F-\mathrm{Isoc}^\dagger(X/K)\xrightarrow{\sim} F\,\mathrm{Rep}_K(\pi_1^{F-\mathrm{Isoc}^\dagger}(X/K,x),\Phi_x^{F-\mathrm{Isoc}^\dagger}). \end{cases}$$

Proof. — Both $\mathrm{Isoc}(X/K)^\sim$, $\mathrm{Isoc}^\dagger(X/K)^\sim$ are inductive limits of finitely $\otimes$-generated F^*-stable subcategories of $\mathrm{Isoc}(X/K)$, $\mathrm{Isoc}^\dagger(X/K)$, so 2.5.1 is applicable. It is then enough to note that

$$(\mathrm{Isoc}(X/K)^\sim)_F\xrightarrow{\sim} F-\mathrm{Isoc}(X/K)\qquad (\mathrm{Isoc}^\dagger(X/K)^\sim)_F\xrightarrow{\sim} F-\mathrm{Isoc}^\dagger(X/K).$$

I cannot say if 2.6.1 is a more useful description of $F-\mathrm{Isoc}(X/K)$ than 2.5.3. At any rate the obvious inclusion functor induces a surjective homomorphism $\pi_1^{\mathrm{Isoc}}\to\pi_1^{F-\mathrm{Isoc}}$ compatible with Φ^{Isoc}, $\Phi^{F-\mathrm{Isoc}}$ and fitting into a commutative diagram

$$(2.6.2)\qquad \begin{array}{ccc} \pi_1^{\mathrm{Isoc}} & \twoheadrightarrow & \pi_1^{F-\mathrm{Isoc}} \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{Isoc}^\dagger} & \twoheadrightarrow & \pi_1^{F-\mathrm{Isoc}^\dagger}. \end{array}$$

2.7. So far in this section, x has been closed point of X; nonetheless most of what we have done remains valid if x is a point of X with values in a perfect field, as long as we replace Tannakian categories by ind-Tannakian categories when necessary. In fact, if $\mathscr{C}$ is a Tannakian category of convergent or overconvergent isocrystals on which F^* is an autoequivalence, then the same is true of the ind-Tannakian category $\mathrm{ind}-\mathscr{C}$. Thus if x is a point of X with values in perfect field and $x/L\to X/K$ covers $x\to X$, there is an equivalence

$$\eta_x:\quad ([M]\otimes L)_F\xrightarrow{\sim} F\,\mathrm{Rep}_L(D\,\mathrm{Gal}(M,x),\Phi_x^M)$$

compatible with the equivalence 2.1.11.

2.8. *Remark.* — The fiber at x of an F-isocrystal on X/K is naturally an F-isocrystal on K, whence functors

$$(2.8.1) \qquad \begin{cases} \omega^F: & F-\mathrm{Isoc}(X/K) \to F\,\mathrm{criso}_K \\ \omega^F: & F-\mathrm{Isoc}^\dagger(X/K) \to F\,\mathrm{criso}_K. \end{cases}$$

They are obviously faithful, exact, and $\otimes$-compatible, and one sees immediately from the constructions that equivalences such as 2.4.1 and 2.6.1 identify ω^F with the first forgetfull functor in 2.2.4. Thus 2.4, 2.6 can be thought of as examples of a "relative" Tannaka theory, relative, that is, to "fiber functors" such as 2.8.1. In fact if $G = \mathrm{Spec}(A)$ is an affine K-group and Φ is a Frobenius structure on G, then Φ^{-1} induces a Frobenius structure $\Psi : A^{(\sigma)} \to A$ on the affine algebra of G which is immediately seen to be compatible with the natural bialgebra structure of A. In other words A is a "bialgebra object" in $F\,\mathrm{criso}_K$, and one can show in addition that $F\,\mathrm{Rep}_K(G, \Phi)$ is equivalent to the category of "A-comodule objects" in $F\,\mathrm{criso}_K$, as in [13]. Unlike the usual "absolute" Tannaka theory, however, there are in general many different isomorphism classes of faithful exact $\otimes$-functors $[M] \to F\,\mathrm{criso}_K$ (recall the discussion following 2.3.7), and although $\mathrm{Aut}^\otimes(\omega^F \mid [M])$ can be shown to be representable by an algebraic group over K_0, it is not necessarily a K_0-form of D Gal(M). The possibility of a "relative" Tannaka theory of this sort was suggested to me by T. Ekedahl.

2.9. *Remark.* — Suppose $U \subsetneq X$ is a Zariski-dense open set on X, which is assumed to be geometrically connected. By analogy with the behavior of the (étale) fundamental group, one is led to ask whether the canonical injection $\mathrm{D\,Gal}(M \mid U) \subsetneq \mathrm{D\,Gal}(M)$ is an isomorphism. We shall shall see later (4.12) that this is *not* true in the convergent category. The question is still open in the overconvergent case.

In either category, one can still ask whether the functor $[M] \to [M \mid U]$ is fully faithful (this is a weaker property than that of $\mathrm{D\,Gal}(M \mid U) \to \mathrm{D\,Gal}(M)$ being an isomorphism). For a nontrivial example, see 4.12.

3. Unit-root F-isocrystals

3.1. If X/k is smooth and geometrically connected, then according to theorem 1.2 the category of p-adic representations of $\pi_1(X)$ is equivalent to the category of convergent unit-root F-isocrystals on X/K. In this section we shall show that if (M, Φ) is a unit-root F-isocrystal, then D Gal(M) has a simple description in terms of the p-adic representation associated to (M, Φ) (3.7 and 3.8 below). We begin with some general remarks about unit-root F-isocrystals.

Recall that UR(X/K), resp. $\mathrm{UR}^\dagger(X/K)$ denotes the category of convergent (resp. overconvergent) unit-root F-isocrystals on X/K. Let $\mathrm{Isoc}(X/K)^{\mathrm{ur}}$ denote the $\otimes$-subcategory of Isoc(X/K) generated by the essential image of the forgetful functor

$$\mathrm{UR}(X/K) \to \mathrm{Isoc}(X/K).$$

Since $\mathrm{Isoc}(X/K)^{ur}$ is an inductive limit of finitely $\otimes$-generated F^*-stable subcategories of $\mathrm{Isoc}(X/K)$, F^* induces an autoequivalence of $\mathrm{Isoc}(X/K)^{ur}$. Thus if x is any k-point of X and ω is the associated fiber functor, and if we set

$$\pi_1^{ur} = \mathrm{Aut}^{\otimes}(\omega \,|\, \mathrm{Isoc}(X/K)^{ur}) \tag{3.1.1}$$

then we have an equivalence of categories (*cf.* 2.5.1)

$$\eta_x: \quad \mathrm{Isoc}(X/K)_F^{ur} \xrightarrow{\sim} \mathrm{F\,Rep}_K(\pi_1^{ur}(X, x), \Phi_x^{ur}) \tag{3.1.2}$$

where Φ_x^{ur} is the canonical Frobenius structure on $\pi_1^{ur}(X, x)$.

Of course what we really want to describe is $\mathrm{UR}(X/K)$, which is smaller than $\mathrm{Isoc}(X/K)_F^{ur}$. If (G, Φ) is an affine K-group with Frobenius structure, we will denote by $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$ the full subcategory of $\mathrm{F\,Rep}_K(G, \Phi)$ consisting of (ρ, V, Ψ) such that (V, Ψ) is a unit-root F-isocrystal on K. Since 3.1.2 "is" the restriction of $\eta_x^{F-\mathrm{Isoc}}$ to $\mathrm{Isoc}(X/K)_F^{ur}$, it is a fully faithful functor

$$\eta_x^{ur}: \quad \mathrm{UR}(X/K) \to \mathrm{F\,Rep}_K(\pi_1^{ur}(X, x), \Phi_x^{ur})^{ur}.$$

In other words, if (M, Φ) is unit-root, then so is the fiber (M_x, Φ_x). If x is a point of X with values in a perfect field and $x/L \to X/K$ covers $x \to X$, then similar considerations show that there is a fully faithful functor

$$\eta_x^{ur}: \quad \mathrm{UR}(X/K) \otimes L \to \mathrm{F\,Rep}_L(\pi_1^{ur}(X, x), \Phi_x^{ur})^{ur}. \tag{3.1.3}$$

3.2. Lemma. — *If x is a geometric generic point, then η_x^{ur} is an equivalence.*

Proof. — We have only to show that η^{ur} is essentially surjective. Since 3.1.2 is an equivalence, this boils down to the assertion that if (M_x, Φ_x) is unit-root, then so is (M, Φ); but this is just proposition 2.1. ■

Incidentally, 1.2 is the first "purely analytic" fact we have used so far.

3.3. We see from 3.2 that the problem of understanding $\mathrm{UR}(X/K)$ reduces to a purely group-theoretic one: given an affine K-group G with Frobenius structure Φ, we want to describe the K_0-linear Tannakian category $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$, at least when $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$ is $\otimes$-generated by unit-root F-isocrystals. Until 3.7, k will be algebraically closed.

The first thing to observe is that $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$ is a neutral Tannakian category. In fact the structure theory for F-isocrystals on K [11] tells us that the category $\mathrm{F\,criso}_K^{ur}$ of unit-root F-isocrystals on F is equivalent to Vec_{K_0}, a pair of inverse equivalences being given by

$$\begin{cases} \mathrm{F\,criso}_K^{ur} \to \mathrm{Vec}_{K_0} & \mathrm{Vec}_{K_0} \to \mathrm{F\,criso}_K^{ur} \\ (V, \Phi) \mapsto \mathrm{Ker}(1-\Phi) & V_0 \mapsto (V_0 \otimes K, 1 \otimes \sigma) \end{cases} \tag{3.3.1}$$

From this we see that $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$ has a fiber functor into Vec_{K_0}:

$$\left\{\begin{array}{l}\omega_0: \quad \mathrm{F\,Rep}_K(G, \Phi)^{ur} \to \mathrm{Vec}_{K_0} \\ \quad (\rho, V, \Psi) \mapsto \mathrm{Ker}(1-\Psi).\end{array}\right. \tag{3.3.2}$$

Thus if we set $G_0 = \mathrm{Aut}^{\otimes}\,\omega_0$, we have the usual equivalence of categories

$$\mathrm{F\,Rep}_K(G, \Phi)^{ur} \xrightarrow{\sim} \mathrm{Vec}_{K_0}(G_0). \tag{3.3.3}$$

We now want to consider the forgetful functor

$$\mathrm{F\,Rep}_K(G, \Phi)^{ur} \to \mathrm{Rep}_K(G) \tag{3.3.4}$$

and its extension to the categories of ind-objects

$$\mathrm{ind}-\mathrm{F\,Rep}_K(G, \Phi)^{ur} \to \mathrm{ind}-\mathrm{Rep}_K(G). \tag{3.3.5}$$

The source of 3.3.5 is K_0-linear, while the target is K-linear, so by [13], 1.5.3.1, the functor 3.3.5 can be factored

$$\mathrm{ind}-\mathrm{F\,Rep}_K(G, \Phi)^{ur} \xrightarrow{\alpha} \mathrm{F\,Rep}_K(G, \Phi)^{ur}\otimes K \xrightarrow{\beta} \mathrm{ind}-\mathrm{Rep}_K(G).$$

By definition, $\mathrm{F\,Rep}_K(G, \Phi)^{ur}\otimes K$ is the category of ind-objects of $\mathrm{F\,Rep}_K(G, \Phi)^{ur}$ endowed with an action of K; thus if we set $L = K\otimes_{K_0}K$, then objects of $\mathrm{F\,Rep}_K(G, \Phi)^{ur}\otimes K$ can be viewed as triples (V, ρ, Ψ) where V is an L-module, ρ is a representation of G on V, and Ψ is a $\sigma\otimes 1$-linear isomorphism of V compatible with Φ in the sense of equation 2.2.4. With this identification, we can describe α and β as follows: α is the functor which Saavedra calls i_{K/K_0}, and is given by

$$\alpha: \quad (V, \rho, \Psi) \to (V\otimes_{K_0}K, \rho\otimes_{K_0}K, \Psi\otimes_{K_0}K) \tag{3.3.6}$$

while β is given by

$$\beta: \quad (V, \rho, \Psi) \to (V\otimes_L K, \rho\otimes_L K) \tag{3.3.7}$$

where K is regarded as an L-algebra by means of the multiplication map $a\otimes b \to ab$ (*cf.* [13] 1.5.2 and 1.5.3.2). By construction, it is clear that

$$\left\{\begin{array}{ccc}\mathrm{F\,Rep}_K(G, \Phi)^{ur} & \xrightarrow{3.3.4} & \mathrm{Rep}_K(G) \\ \alpha\downarrow & & \downarrow \mathrm{incl.} \\ \mathrm{F\,Rep}_K(G, \Phi)^{ur}\otimes K & \xrightarrow{\beta} & \mathrm{ind}-\mathrm{Rep}_K(G)\end{array}\right. \tag{3.3.8}$$

is 2-commutative.

The composition of β with the forgetful functor $\mathrm{ind}-\mathrm{Rep}_K(G) \to \mathrm{ind}-\mathrm{Vec}_K$ is a fiber functor

$$(\mathrm{forget})\circ\beta: \quad \mathrm{F\,Rep}_K(G, \Phi)\otimes K \to \mathrm{ind}-\mathrm{Vec}_K.$$

2.4. Lemma. — $(forget)\circ\beta\simeq\omega_0\otimes K$.

Proof. — The fiber functor $\omega_0\otimes K$ is just

$$(V,\ \rho,\ \Psi)\to \operatorname{Ker}(1-\Psi)$$

as before. By 3.3.6-8 is enough to show that for all (V, ρ, Ψ) in $F\operatorname{Rep}_K(G, \Phi)^{ur}\otimes K$, we have a canonical isomorphism

$$\operatorname{Ker}(1-\Psi)=V\otimes_L K. \tag{3.4.1}$$

Since V can be viewed as an object of ind $-$ F$\operatorname{criso}_K^{ur}$, Manin's theory [11] is applicable and one has (*cf.* 3.31) a canonical isomorphism

$$V\simeq K\otimes_{K_0}\operatorname{Ker}(1-\Psi). \tag{3.4.2}$$

But for any K-vector space N there is a canonical isomorphism

$$(K\otimes_{K_0}N)\otimes_L K\simeq N$$

so we need only take $N=\operatorname{Ker}(1-\Psi)$.

From 3.4, we see that β induces a homomorphism of K-groups

$$\gamma:\quad G\to G_0\otimes K.$$

Next, we observe that since $(G_0\otimes K)^{(\sigma)}\simeq G_0\otimes K$, the K-group $G_0\otimes K$ has an evident Frobenius structure given by $1\otimes\sigma$. By 3.3.7, 3.4.2, and 2.2.4, this Frobenius structure sits in a commutative diagram

$$\left\{\begin{array}{ccc} G^{(\sigma)} & \xrightarrow{\sigma(\gamma)} & (G_0\otimes K)^{(\sigma)} \\ \Phi\downarrow & & \downarrow 1\otimes\sigma \\ G & \xrightarrow{\gamma} & G_0\otimes K \end{array}\right. . \tag{3.4.3}$$

3.5. Lemma. — *γ is surjective.*

Proof. — It is enough to show that β is fully faithful, and that its essential image is stable under the formation of subquotients. We first show that β is fully faithful: let (ρ, V, Ψ), (ρ', V', Ψ') be objects of $F\operatorname{Rep}_K(G, \Phi)^{ur}\otimes K$ and let

$$(V\otimes_L K,\ \rho\otimes_L K)\to(V'\otimes_L K,\ \rho'\otimes_L K)$$

be a morphism in $\operatorname{Rep}_K(G)$. Then we see from 3.4.1 and 3.4.2 that there is a morphism $V\to V'$ that is *necessarily* compatible with Ψ, Ψ' on account of 3.4.2; in the same way we can check that it is also compatible with the G-actions. We now show that the essential image of β is stable under the formation of subquotients: since we have shown thar β is fully faithful, we can replace "subquotients" by "subobjects", so let $V\otimes_L K$ be a representation of G in the image of β and let $W\subsetneq V\otimes_L K$ be a subrepresentation. Then $K\otimes_{K_0}W$ is a free L-module of finite type and $\sigma\otimes 1$ defines a

Frobenius structure on $K\otimes_{K_0}W$. That this Frobenius structure is compatible with the Frobenius structure Φ on G follows from the commutativity of 3.4.3. ■

3.6. PROPOSITION. — *Suppose that* $\mathrm{Rep}_K(G)$ *is* $\otimes$*-generated by a collection* $\{V_i\}_{i\in I}$ *such that each* V_i *has a unit-root Frobenius structure* F_i *making* (V_i, F_i) *a representation of* (G, Φ). *Then* $G \xrightarrow{\sim} G_0\otimes K$.

Proof. — By 3.5, it is enough to show that γ is injective, or equivalently, that every object of ind$-\mathrm{Rep}_K(G)$ is a subquotient of an object in the image of β. By hypothesis, this kind of property holds for the forgetful functor 3.3.4, so it holds for β as well, by the commutativity of 3.3.8. ■

We can now return to our geometric situation: X/k is smooth and geometrically connected, and x is a geometric generic point of X. Choose a morphism $x/L \to X/K$ covering $x \to X$. We denote by $\pi_1^{et}(X, x)$ the "affine group scheme hull" over K_0 of the fundamental group $\pi_1(X, x)$ of X, *i.e.* the affine K_0-group whose category of K_0-representations is equivalent to the category of *continuous* representations of π_1 on finite-dimensional K_0-vector spaces. The full subcategory of $\mathrm{Rep}_K\,\pi_1^{et}(X, x)$ consisting of representations with finite image is equivalent to the category of K-representations of $\pi_1(X, x)$ as a proalgebraic group, which allows us to identify $\pi_1(X, x)$ with the group of connected components of $\pi_1^{et}(X, x)$:

$$\pi_0(\pi_1^{et}) \xrightarrow{\sim} \pi_1. \tag{3.6.1}$$

Yet another π_1-variant is the affine K-groupe π_x^{ur} defined by 3.1.1. The relation between them is given by

3.7. PROPOSITION. — *Suppose k is algebraically closed,* X/k *is smooth and connected,* x *is a geometric generic point of* X, *and* $x/L \to X/K$ *covers* $x \to X$. *Then there is a canonical isomorphism*

$$(\pi_1^{ur}(X, x), \Phi_x^{ur}) \xrightarrow{\sim} (\pi_1^{et}(X, x)\otimes L, 1\otimes\sigma) \tag{3.7.1}$$

of groups with Frobenius structure. If ρ *is a representation of* $\pi_1(X, x)$ *and* $(M, \Phi) = G(\rho)$ *is the corresponding convergent unit-root* F*-isocrystal, then there is a canonical isomorphism*

$$(D\,\mathrm{Gal}(M, x), \Phi_x^M) \xrightarrow{\sim} (\overline{\mathrm{Im}(\rho)}\otimes L, 1\otimes\sigma). \tag{3.7.2}$$

If X/k *is a smooth connected curve and* ρ *has finite local monodromy, then* 3.7.2 *holds for* $(M, \Phi) = G^{\dagger}(\rho)$.

Proof. — The first assertion follows from the $\otimes$-equivalences 1.3.1, 3.1.3, and the isomorphism 3.3.3. If $(M, \Phi) = G(\rho)$, then the functor 1.3.1 identifies the $\otimes$-category $[\rho]$ generated by ρ with the $\otimes$-category $[M, \Phi]$ generated by (M, Φ). In the context of 3.3.3, we then have $G = D\,\mathrm{Gal}(M)$, and G_0 is the Zariski-closure of the image of ρ, from which the equivalence 3.7.2 follows. Finally if ρ has finite local monodromy, then the equivalence 1.4.3 shows that 3.7.2 is valid in this case too. ■

Finally, we can use 2.1.10 to remove the assumption that k is algebraically closed:

3.8. COROLLARY. — *Suppose* X/k *is smooth and geometrically connected, and let* ρ *be a representation of* $\pi_1(X)$. *If* $G(M, \Phi)=(\rho)$, *then*

$$\text{(3.8.1)} \qquad D\,\mathrm{Gal}(M)\otimes L \simeq \overline{\mathrm{Im}(\rho\,|\,\pi_1(X\otimes k^{\mathrm{alg}}))}\otimes L.$$

If X/k *is a smooth geometrically curve and* ρ *has finite local monodromy, then* 3.8.1 *holds for* $(M, \Phi)=G^\dagger(\rho)$.

Proof. — Let K^{ur} be the completion of the maximal unramified extension of K; then the paragraph following 2.1.10 allows us to conclude that

$$D\,\mathrm{Gal}(M, y)\otimes K^{ur} \simeq D\,\mathrm{Gal}(M\,|\,X\otimes k^{\mathrm{alg}}, y)$$

for any k-point y of X. On the other hand, 3.7 says that

$$D\,\mathrm{Gal}(M\,|\,X\otimes k^{\mathrm{alg}}, x) \simeq \overline{\mathrm{Im}(\rho\,|\,\pi_1(X\otimes k^{\mathrm{alg}}))}\otimes L,$$

so 3.8 follows from the non-canonical isomorphism of $D\,\mathrm{Gal}(M, x)$ with $D\,\mathrm{Gal}(M, y)\otimes L$. ■

3.9. *Remark.* — Suppose that y is a k-point of X, so that $D\,\mathrm{Gal}(M, y)\otimes L$ is non-canonically isomorphic to $D\,\mathrm{Gal}(M, x)$. In the context of 3.7, one can ask if there is already a non-canonical isomorphism between $D\,\mathrm{Gal}(M, y)$ and $\overline{\mathrm{Im}(\rho)}\otimes K$. The most that can be deduced from 3.7 is that these groups are inner twists of each other.

Finally, suppose that (M, F) is an F-isocrystal on a smooth geometrically connected scheme X over an algebraically closed field k, and that X/k has a lifting to a formally smooth $\mathfrak{X}/R$. Then we can view (M, F) as a locally free sheaf on $\mathfrak{X}^{an}$ endowed with a convergent connection and a Frobenius structure. It is well known that when (M, F) is unit-root, the Frobenius structure determines the connection, *i.e.* no information is lost by "forgetting the connection". On the other hand, proposition 3.7 shows that some, but not much information is lost by forgetting the Frobenius structure: one can still recover the Zariski-closure of the image of π_1, up to an inner twist.

4. Structure theory

The main result of this section (indeed, of the paper) is the global monodromy theorem 4.9. Once we have acquired the requisite tools, the proof is basically that of Deligne [6]. We conclude by calculating the monodromy groups of the convergent and overconvergent isocrystals associated to a nonisotrivial family of elliptic curves on a smooth curve.

Throughout this section X/k will be *smooth* and *geometrically connected* and (K, σ) will be as in 1. If L/K is a finite extension, then we will assume that we have chosen an extension of σ which fixes a uniformizer of L. For most of this section we will not want to keep track explicitly of base fields. Thus if we assert that a given isocrystal can be given a Frobenius structure with certain properties, it should be understood that this Frobenius structure might not be relative to the original Frobenius morphism of the base, but rather to a power of this morphism. The meaning of the symbol σ will float in the same way.

We say that a (convergent or overconvergent) isocrystal (M, F) on X/K is *isotrivial* if there is a finite étale $\pi: Y \to X$ such that the pullback $\pi^*(M, F)$ is constant. Our first task is to check that this is equivalent to D Gal(M) being finite. First, some simple remarks:

4.1. LEMMA. — *Let* M *be a* (*convergent or overconvergent*) *isocrystal on* X/K *such that* D Gal(M) *is finite. Then*

(i) *If* M *has a Frobenius structure, then it has a unit-root Frobenius structure.*

(ii) *If* M *is a subquotient of an isocrystal with a Frobenius structure, then it is a subobject of an* N *with Frobenius structure and such that* D Gal(N) *is finite.*

Proof. — It is enough to prove the corresponding statements about representations of groups with Frobenius structure. For (i), we consider a representation (V, Ψ) of (G, Φ), where G is finite. We have seen (at the end of § 2.2) that for some n there is a Frobenius structure Ψ' for F^n on V such that (V, Ψ') is a unit-root representation of (G, Φ^n), whence (i). Part (ii) reduces to the following assertion: suppose (V, Ψ) is a representation of (G, Φ), and W is a G-subrepresentation of V with finite image; then W occurs as a G-subrepresentation of a (V', Ψ') such that V' has finite image. Let H denote the quotient of G by its connected component. Since the connected component is a characteristic subgroup, the Frobenius structure Φ induces a Frobenius structure on H, which as before can be identified with a finite order automorphism φ of H. If φ has order n, then a representation of (H, Φ) can then be identified with a representation of the semidirect product H' of $\mathbf{Z}/n$ by H, where the generator of $\mathbf{Z}$ acts on H by φ. Suppose now that W is a representation of G with finite image, *i.e.* a representation of H. Then the induced representation V' of W to H' can be identified with a representation of (H, Φ), *i.e.* a representation of (G, Φ) with finite image, which contains W as a G-subrepresentation. ■

Note that the (N, Ψ) constructed for (ii) can be taken to be unit-root.

4.2. PROPOSITION. — *Suppose* X/k *is smooth and geometrically connected* (*resp. a smooth geometrically connected curve*), $\pi: Y \to X$ *is finite étale, and* N *is a constant convergent* (*resp. overconvergent*) *isocrystal on* Y/K. *Then* D Gal(π_* N) *is finite.*

Proof. — We can assume the Y is geometrically connected. Since N is constant it can be given the structure of a convergent (resp. overconvergent) unit-root F crystal, and thus corresponds to a representation 1 (the identity representation) of π_1(Y). Then by the results of 1.7, the convergent (resp. overconvergent) F-isocrystal π_* N corresponds to the representation Ind (1) of π_1(X). Since the latter has finite image, 3.7 shows that D Gal(π_* N) is finite. ■

4.3. Proposition. — *Let* X/k *be a smooth curve over an algebraically closed field,* (M, F) *an overconvergent* F-*isocrystal on* X/K, *and* N *an object of* [M]. *Then the following statements are equivalent:*

(i) N *is isotrivial*

(ii) $\hat{N}$ *is isotrivial*

(iii) $D\,\mathrm{Gal}(N)$ *is finite*

(iv) $D\,\mathrm{Gal}(\hat{N})$ *is finite*

If any of these conditions hold, there is a canonical isomorphism

$$D\,\mathrm{Gal}(\hat{N}) \xrightarrow{\sim} D\,\mathrm{Gal}(N). \tag{4.3.1}$$

Proof. — The implications (i) ⇒ (ii) and (iii) ⇒ (iv) are immediate. Next, we show (i) ⇒ (iii): if $\pi: Y \to X$ is a finite étale galois cover such that $\pi^* N$ is trivial, then we have $N \subsetneq \pi_* \pi^* N$. By 4.2, $D\,\mathrm{Gal}(\pi_* \pi^* N)$ is finite, so $D\,\mathrm{Gal}(N)$ is finite too. The same argument also shows that (ii) ⇒ (iv). Conversely, (iv) ⇒ (ii); in fact, by (ii) of lemma 4.1, we can assume that $N \subseteq N'$ where $D\,\mathrm{Gal}(N')$ is finite and N' has a unit-root Frobenius structure. It then follows from 1.3 and 3.7 that N' becomes trivial on a suitable finite étale cover of X, whence so does N. Finally, we show that (ii) ⇒ (i). Let $\pi: Y \to X$ be a finite étale galois cover such that $\pi^* \hat{N}$ is trivial; then as before we have $\hat{N} \subsetneq \pi_* \pi^* \hat{N}$. Furthermore since $\pi^* \hat{N}$ is trivial, there is a natural unit-root Frobenius structure on $\pi_* \pi^* \hat{N}$ which by 3.7 induces a unit-root Frobenius structure on $\hat{N}$ (possibly after making an extension of K). Finally, since $\pi^* \hat{N}$ is trivial, we see from 1.4 that N is of the form $G^\dagger(\rho)$, and since $G^\dagger$ is fully faithful and natural in X, we see that the triviality of $\pi^* \hat{N}$ implies the triviality of $\pi^* N$. ■

Note that the hypothesis that X is a curve is really only used in the proof of (ii) ⇒ (i).

Denote by $UR(X/K)^{\text{finite}}$, resp. $UR^\dagger(X/K)^{\text{finite}}$ the full subcategory of $UR(X/K)$, resp. $UR^\dagger(X/K)$ consisting of unit-root F-isocrystals (M, F) such that $D\,\mathrm{Gal}(M)$ is finite. The argument given above shows that in fact any object of $UR(X/K)^{\text{finite}}$ is the completion of an object of $UR^\dagger(X/K)^{\text{finite}}$. We therefore have

4.4. Corollary. — *Suppose* X/k *is a curve. Then the completion functor* $M \to \hat{M}$ *induces an equivalences of categories*

$$UR(X/K)^{\text{finite}} \xrightarrow{\sim} UR^\dagger(X/K)^{\text{finite}}. \quad ■$$

We can now determine the groups of connected components of $\pi_1^{F-\mathrm{Isoc}}(X)$ and $\pi^{F-\mathrm{Isoc}^\dagger}$:

4.4. Proposition. — *If k is algebraically closed, then there is a natural isomorphism*

$$\pi_0(\pi_1^{F-\mathrm{Isoc}}(X)) \xrightarrow{\sim} \pi_1(X). \tag{4.4.1}$$

If X *is a curve, then there is also a natural isomorphism*

$$\pi_0(\pi_1^{\mathrm{F-Isoc}^\dagger}(\mathrm{X})) \xrightarrow{\sim} \pi_1(\mathrm{X}). \tag{4.4.2}$$

Proof. — Recall that for any proalgebraic group G, the subcategory of $\mathrm{Rep}_\mathrm{K}(\mathrm{G})$ corresponding to the quotient group $\pi_0(\mathrm{G})$ of G is the full subcategory of representations of G with finite image (since for an algebraic group "finite" is equivalent to "discrete"). In the case of $\pi_0(\pi_1^{\mathrm{F-Isoc}}(\mathrm{X}))$, the category in question is the category of convergent isocrystals M such that

(i) D Gal (M) is finite

(ii) M is a subquotient of the underlying isocrystal of a convergent F-isocrystal on X/K. We want to compare this with the category of finite-dimensional K-representations of $\pi_1(\mathrm{X})$ (continuous for the discrete topology on K). After a suitable extension of scalars, the latter category is equivalent to the category of convergent isocrystals M satisfying (i) and

(iii) M is a subquotient of the underlying isocrystal of a convergent unit-root F-isocrystal on X/K.

Lemma 4.1 shows that the conjunction of (i) and (ii) is equivalent to the conjunction of (i) and (iii), which proves the assertion about $\pi_1^{\mathrm{F-Isoc}}(\mathrm{X})$. When X is a curve, the category of K-representations of $\pi_0(\pi_1^{\mathrm{F-Isoc}^\dagger}(\mathrm{X}))$ is equivalent to the category of overconvergent M satisfying (i) and

(ii′) M is a subquotient of the underlying isocrystal of an overconvergent F-isocrystal on X/K

and as before this is equivalent to the category of overconvergent M satisfying (i) and

(iii′) M is a subquotient of the underlying isocrystal of an overconvergent unit-root F-isocrystal on X/K.

Now from 4.1 (ii) and 4.4 we see that the category of overconvergent M satisfying (i) and (iii′) is equivalent to the category of convergent M satisfying (i) and (iii), from which the last part of 4.4 follows. ■

4.5. COROLLARY. — *Suppose that k is algebraically closed,* Y *is connected, and* $\pi: \mathrm{Y} \to \mathrm{X}$ *is finite étale and principal homogenous under a (finite) group* G. *Then there is an exact sequence*

$$0 \to \pi_1^{\mathrm{F-Isoc}}(\mathrm{Y}) \to \pi_1^{\mathrm{F-Isoc}}(\mathrm{X}) \to \mathrm{G} \to 0. \tag{4.5.1}$$

if X *and* Y *are curves, there is an exact sequence*

$$0 \to \pi_1^{\mathrm{F-Isoc}^\dagger}(\mathrm{Y}) \to \pi_1^{\mathrm{F-Isoc}^\dagger}(\mathrm{X}) \to \mathrm{G} \to 0. \tag{4.5.2}$$

Proof. — Let H be the cokernel of $\pi_1^{\mathrm{F-Isoc}}(\mathrm{Y}) \to \pi_1^{\mathrm{F-Isoc}}(\mathrm{X})$. Then the diagram

$$\begin{array}{ccccccccc} 0 \to & \pi_1^{\mathrm{F-Isoc}}(\mathrm{Y}) & \to & \pi_1^{\mathrm{F-Isoc}}(\mathrm{X}) & \to & \mathrm{H} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \pi_1(\mathrm{Y}) & \to & \pi_1(\mathrm{X}) & \to & \mathrm{G} & \to 0 \end{array}$$

tells us that $H \to G$ is surjective, and its kernel is the connected component of H. On the other hand, H is dual to the category of convergent isocrystals on X that become trivial on Y. By 4.3, such isocrystals have finite D Gal, so we see that H is discrete. The argument for overconvergent isocrystals on a curve is the same. ■

4.6. PROPOSITION. — *If* (M, Φ) *is a convergent* F-*isocrystal on* X/K, *or if* (M, Φ) *is overconvergent and* X/k *is a curve, then there is a finite étale cover* $\pi : Y \to X$ (*defined over a finite extension of* k) *such that*

$$\mathrm{D\,Gal}(\pi^* M) \xrightarrow{\sim} \mathrm{D\,Gal}(M)^0.$$

Proof. — If k is algebraically closed, we know by 4.5 that $\mathrm{D\,Gal}(M)/\mathrm{D\,Gal}(M)^0$ is a quotient of $\pi_1(X)$, and the assertion then follows from 4.4. In the general case, we first extend scalars to the algebraic closure k^{alg} of k. If L is an extension of K with residue field k^{alg}, then we at least have $\mathrm{D\,Gal}(\pi^* M \otimes L) \xrightarrow{\sim} \mathrm{D\,Gal}(M \otimes L)^0$. The result then follows from 2.1.10. ■

4.7. LEMMA. — *Suppose* T/K *is a torus and* $\Phi : T^{(\sigma)} \to T$ *is a Frobenius structure on T. If there is a faithful representation* (V, Ψ) *of* (T, Φ), *then there is a finite extension* L/K, *a positive integer* n, *and a torus* T_0/K_0 (*where* $K_0 = K^{\sigma^n}$) *such that*

$$(T, \Phi^n) \otimes L \simeq (T_0 \otimes K, 1 \otimes \sigma^n).$$

Proof. — By extending scalars, we can assume that T/K is split. We can then apply the discussion at the end of 2.2 to conclude that the Frobenius structure Φ on T is determined by a finite-order automorphism $\varphi : X \xrightarrow{\sim} X$ of the character group X of T. Then if we replace Φ by a suitable power, the resulting Frobenius structure will be trivial. ■

If G/K is a connected algebraic group, we will denote by R(G) the radical of G, and by $R_u(G)$ the unipotent radical. Since R(G) and $R_u(G)$ are characteristic subgroups of G, any Frobenius structure on G induces Frobenius structures on R(G) and $R_u(G)$, and on the quotient $R(G)/R_u(G)$.

4.8. LEMMA. — *Let* G/K *be a connected affine algebraic group,* Φ *a Frobenius structure on* G, *and* (V, Ψ) *a faithful representation of* (G, Φ). *Then there is a finite extension* L/K, *a positive integer* n, *and a unit-root representation* (W, Ψ') *of* (G, Φ^n) *defined over* L, *such that the image of* G *is isogenous to* $R(G)/R_u(G)$. *Furthermore* (W, Ψ') *can be chosen to be a sum of unit-root representations of rank one.*

Proof. — The argument is basically that of [6] 1.3.8 and 1.3.9. We first reduce to the case when G is reductive, by showing that the semisimplification (V^{ss}, Ψ) of (V, Ψ) as a representation of (G, Φ) is semisimple as a G-module. In fact, if we choose a simple sub-G-module W of V, then the sum of the $\Psi^n(W)$ is a semisimple G-submodule of V stable under Ψ, and the assertion follows by induction on dim V. Furthermore V^{ss} is faithful as a representation of the quotient $G/R_u(G)$: if $\rho : G \to V$ is the original representation and ρ^{ss} the semisimplication, then the kernel of ρ^{ss} is mapped injectively by ρ to a unipotent subgroup of $GL(V)$, so that the kernel of ρ^{ss} is contained in $R_u(G)$ (and is therefore equal to it).

We can suppose, then, that G is reductive, so that $R(G)$ is the maximal central torus T of G. Since V is faithful as a representation of G, the previous lemma shows that there is an L/K and an $n>0$ such that Φ^n is a trivial Frobenius structure on T. If H is a semisimple subgroup of G such that $G=TH$, then $M=T\cap H$ is a finite central Φ^n-stable subgroup of G, and the homomorphism $G\twoheadrightarrow G/M\twoheadrightarrow T/M$ is compatible with Φ^n. Evidently any faithful representation W of T/M can be endowed with a unit-root Frobenius structure, such that the representation of (G, Φ^n) arising from it satisfies the conclusion of 4.8 ■

We can now prove our main result:

4.9. THEOREM. — *Suppose k is the perfection of an absolutely finitely generated field and* X/k *is a smooth geometrically connected curve. If* (M, Ψ) *is an overconvergent* F-*isocrystal on* X/K, *then the radical of* $\mathrm{D\,Gal}(M)^0$ *is unipotent.*

Proof. — By 4.6 we may assume, after extending k and passing to a finite étale cover of X if necessary, that $\mathrm{D\,Gal}(M)$ is connected. Let Φ be the canonical Frobenius structure on $G=\mathrm{D\,Gal}(M)$ and let (V, Ψ) be the representation corresponding to (M, Ψ). By 4.8 and 2.5.1 there is a unit-root overconvergent F-isocrystal (N, Ψ') with N an object of [M], such that $\mathrm{D\,Gal}(N)$ is isogenous to $R(G)/R_u(G)$, and such that (N, Ψ') is a sum of rank one objects of $\mathrm{UR}^{\dagger}(X/K)$. From 1.5, it follows that $\mathrm{D\,Gal}(N)$ is finite; it is therefore trivial, as is $R(G)/R_u(G)$. ■

4.10. COROLLARY. — *With the hypotheses and notation of* 4.9, *suppose in addition that* (M, Φ) *is a semisimple object of* $\text{F-Isoc}^{\dagger}(X/X)$. *Then* $\mathrm{D\,Gal}(M)^0$ *is a semisimple group.*

Proof. — The argument of the first paragraph of the proof of 4.8 shows that if (M, Φ) is semisimple in $\text{F-Isoc}^{\dagger}(X/K)$, then M is semisimple in $\text{Isoc}^{\dagger}(X/K)$. As we reduce to the case when $\mathrm{D\,Gal}(M)$ is connected, the conclusion follows from 4.9. ■

We will now compute a D Gal in a simple case. Let X/k be a smooth geometrically connected curve over a perfect field, and denote by W, K the ring of Witt vectors of k and its field of fractions. For convenience we will assume that k is the perfection of an absolutely finitely generated field. Let $\pi: E\to X$ be a proper smooth morphism all of whose fibers are elliptic curves. The relative crystalline cohomology $R^1\pi_{\mathrm{cris}*}(\mathcal{O}_{E/W})$ is an F-crystal of rank two, and so gives rise via 1.1.4 to a convergent F-isocrystal M of rank two on X/K, where K is the fraction field of $R=W(k)$ (one can identify M with the relative *rigid cohomology* of E/X, in the sence of [1], [2]). In certain situations M is known to be an overconvergent F-isocrystal ([1], théorème 5); we will assume this to be the case, and compute $\mathrm{D\,Gal}(M)$ and $\mathrm{D\,Gal}(\hat{M})$. Since 1.1.4 is natural in X/K, and the relative crystalline crystalline cohomology of an abelian scheme is compatible with base change, we see that formation of M commutes with base change. Thus, for example, if $E\to X$ is isotrivial, then so is M, and by 4.3, $\mathrm{D\,Gal}(M)=\mathrm{D\,Gal}(\hat{M})$ is finite. The nonisotrivial case is more interesting:

4.11. PROPOSITION. — *Let* X/k *be a smooth geometrically connected curve and let* $\pi: E\to X$ *be a family of elliptic curves over* X. *If* $E\to X$ *is nonisotrivial and the relative* *rigid cohomology* M *is overconvergent, then* $\mathrm{D\,Gal(M)} \simeq \mathrm{SL}(2)$. *If all of the fibers of* $E \to X$ *are ordinary elliptic curves, then* $\mathrm{D\,Gal}(\hat{M})$ *is a Borel subgroup of* SL (2)*; otherwise* $\mathrm{D\,Gal}(\hat{M}) \simeq \mathrm{SL}(2)$.

Proof. — The first thing to observe is that D Gal (M) and $\mathrm{D\,Gal}(\hat{M})$ are contained in SL (2). In fact, the trace map in crystalline cohomology gives an isomorphism $\Lambda^2 \hat{M} \simeq \mathcal{O}_{X/K}$, and this must give rise to an isomorphism in the overconvergent category as well, since both objects have rank one (*cf.* [5], 4.10). Furthermore, if E is ordinary, then the slope filtration gives a rank one unit-root F-isocrystal $L \subsetneq \hat{M}$, implying that $\mathrm{D\,Gal}(\hat{M})$ is contained in the Borel subgroup $B \subsetneq \mathrm{SL}(2)$ stabilizing the fiber of L.

Next, we can assume that D Gal (M) and $\mathrm{D\,Gal}(\hat{M})$ are connected. In fact, by 4.6 there is a finite étale cover $f: Y \to X$ such that the pullbacks $f^* M$, $f^* \hat{M}$ have connected monodromy group. Since the formation of M commutes with base change and E_Y/Y is not isotrivial, the hypotheses of 4.11 apply to E_Y. Since $\mathrm{D\,Gal}(f^* \hat{M}) \subsetneq \mathrm{D\,Gal}(\hat{M})$, the conclusions of 4.11 will hold for E/X if they hold for E_Y/Y.

On examining the list of isomorphism classes of connected subgroups of SL (2), we see that the only possibilities for D Gal (M) allowed by 4.9 are SL (2), the unipotent radical of a Borel, or the identity. Since $\mathrm{D\,Gal}(\hat{M}) \subsetneq \mathrm{D\,Gal(M)}$ by 2.1.6, the last two possibilities are excluded if we can show that $\mathrm{D\,Gal}(\hat{M})$ is SL (2) or a Borel subgroup.

Suppose first that all of the fibers of $E \to X$ are ordinary, so that $\mathrm{D\,Gal}(\hat{M}) \subseteqq B$. Since $\mathrm{D\,Gal}(\hat{M})$ is connected and nontrivial, it is either a torus, the unipotent subgroup $U \subset B$, or all of B. If $\mathrm{D\,Gal}(\hat{M}) = U$, then the rank one subisocrystal $L \subset \hat{M}$ would be trivial as an isocrystal, and so by 3.8, th *p*-adic representation ρ associated to L (with its unit-root Frobenius structure) would be trivial on $\pi_1(X \otimes k^{alg})$. On the other hand, ρ is just the natural representation of $\pi_1(X)$ on the étale quotient of the Tate module $T_p(E)^{et}$ of E ([8] 4.2.2), and it follows easily from a theorem of Igusa ([8] 4.3) that if $E \to X$ is not isotrivial, then no power of ρ is geometrically trivial. Thus L is not trivial as an isocrystal, and we cannot have $\mathrm{D\,Gal}(\hat{M}) = U$. To conclude, we show that if $\mathrm{D\,Gal}(\hat{M})$ is a torus, then E is isotrivial. In fact if $\mathrm{D\,Gal}(\hat{M})$ were a torus, then $\hat{M}$ would split: $\hat{M} \simeq L \oplus N$, and $N \simeq \check{L}$ since $\Lambda^2 \hat{M}$ is trivial. In the notation of 1.1.4 we have $\hat{M} = M_0 \otimes \mathbf{Q}$, where $M_0 = R^1 f_{\mathrm{cris}*}(\mathcal{O}_{E/W})$ is the relative crystalline cohomology. From 1.1.6 it follows that M_0 splits, as a crystal, into a sum of two crystals of rank one: $M_0 = L_0 \oplus \check{L}_0$, where $L = L_0 \otimes \mathbf{Q}$. To show that this is impossible, we pick a closed point $x \in X$ and consider the completion Z of X at *x*; then the restriction $L_0 | Z$ is constant since L is unit-root and Z is the formal spectrum of a complete discrete valuation ring. Since $M_0 = L_0 \oplus \check{L}_0$, it follows that $M_0 | Z$ is constant as a crystal, and thus constant as an F-crystal. On the other hand $M_0 | Z$ is, as an F-crystal, the relative crystalline cohomology of the restriction E_Z/Z, and since the crystalline R^1 of an abelian scheme can be identified with the Dieudonné crystal of its *p*-divible group, it follows that this Dieudonné crystal of the *p*-divisible group of E_Z is constant. Since Z has a *p*-base, the Dieudonné functor is fully faithful (by [3]) and thus E_Z is constant. Thus the *j*-invariant of E is constant, and E is isotrivial.

We now treat the case when there is a supersingular fiber of E/X. There must be an ordinary fiber too, or else $E \to X$ would be isotrivial; thus the ordinary locus $U \subset X$ of E is open and dense, and is not equal to X. We have $B = D\,Gal(\hat{M} \mid U) \subsetneq D\,Gal(\hat{M})$ by the last paragraph, and since $D\,Gal(\hat{M})$ is a connected subgroup of SL (2), it is enough to show that we cannot have $D\,Gal(\hat{M}) = B$. If it were, then the restriction functor $[\hat{M}] \to [\hat{M} \mid U]$ would be an equivalence of categories (since the corresponding homorphism of groups $D\,Gal(\hat{M} \mid U) \to D\,Gal(\hat{M})$ would be an isomorphism). Then since $[\hat{M}]$ is F*-stable, the sub-F-isocrystal $L \subset \hat{M} \mid U$ would have an extension L′ to all of X as a sub-F-isocrystal of $\hat{M}$, and $\hat{M}$ would, as an F-isocrystal, be an extension of $\check{L}'(1)$ by L′. But L′, being an F-isocrystal of rank one, has constant Newton polygon by [4] 2.6; then $\hat{M}$ would have constant Newton polygon as well, which contradicts the hypothesis that E is not everywhere ordinary. ∎

4.12. *Remarks.* − (i) If the ordinary locus $U \subset X$ is not all of X, then 4.10 shows that the canonical immersion $D\,Gal(\hat{M} \mid U) \subsetneq D\,Gal(M)$ is not an isomorphism, *i.e.* $D\,Gal(\hat{M})$ is *not* a "birational invariant" of (X, M).

(ii) One can deduce, from the fact that the quotient

$$D\,Gal(\hat{M})/D\,Gal(\hat{M} \mid U) \simeq \mathbf{P}^1 \quad \text{or} \quad 1$$

is a *complete* variety, that the natural inclusion functor $[\hat{M}] \to [\hat{M} \mid U]$ is fully faithful (*cf.* 2.8).

The main idea underlying this paper is that the category of overconvergent F-isocrystals on a smooth scheme is the correct *p*-adic analogue of the category of lisse *l*-adic sheaves. Theorem 4.9 and proposition 4.11, I think, make this claim at least credible. One is therefore tempted of offer the following conjecture, which refines part of a conjecture of Deligne ([6] 1.2.10):

4.13. CONJECTURE. − *Suppose that* $X/\mathbf{F}_q$ *is normal, geometrically connected, and of finite type, and let* $\mathscr{F}$ *be an irreducible lisse* $\mathbf{Q}_l$*-sheaf on* X *whose determinant is defined by a finite order character of* $\pi_1(X)$.

(i) $\mathscr{F}$ *is pure of weight zero, and there is a number fied* E *such that for any* $x \in |X|$, *the characteristic polynomial* $\det(1 - TF_x \mid \mathscr{F}_x)$ *has coefficients in* E.

(ii) *For any place* $\mathscr{P}$ *of* E *dividing p, there is an overconvergent* F*-isocrystal* $(M_{\mathscr{P}}, \Phi)$ *on* $X/E_{\mathscr{P}}$ *compatible with* $\mathscr{F}$, i.e. *such that* $\det(1 - T\,\Phi_x^{\deg x} \mid (M_{\mathscr{P}})_x) = \det(1 - TF_x \mid \mathscr{F}_x)$ *for all* $x \in |X|$.

Deligne's actual conjecture includes (i) above and, among other things, asserts that $\mathscr{F}$ belongs to a compatible family of *l*-adic representations. We have sought merely to offer a specific candidate for the "petit camarade cristallin" of [6] 1.2.10 (vi).

Given this conjecture, one might ask how D Gal of the F-isocrystal $(M_{\mathscr{P}}, \Phi)$ is related to the Zariski-closur of the image of the geometric fundamental group in the corresponding *l*-adic representation $\mathscr{F}$. The simplest assertion would be that they are always isomorphic, but if $\mathscr{F}$ belongs to a compatible system of *l*-adic representations, one would have to conclude that all of the *l*-adic monodromy groups are isomorphic, and this is not known to be true. Of course this *is* true in the case of the systems of *l*-adic representations coming from families of elliptic curves, and we have just seen that the monodromy group of the overconvergent F-isocrystal coming from such families is the same as any of the

l-adic monodromy groups. In a forthcoming paper I will show that the same holds for the l-acid and p-adic (overconvergent) monodromy groups arising from the l-adic sheaves and overconvergent F-isocrystals associated to Kloosterman sums in odd characteristic.

Finally, we note that in this particular case, the image of the canonical embedding $\mathrm{D\,Gal}(\hat{\mathrm{M}}) \subsetneq \mathrm{D\,Gal}(\mathrm{M})$ contains a Borel subgroup of $\mathrm{D\,Gal}(\mathrm{M})$. Is this always the case, say, if $\mathrm{D\,Gal}(\mathrm{M})$ is semisimple?. This is, of course, a stronger property than that of the completion functor $[\mathrm{M}] \to [\hat{\mathrm{M}}]$ being fully faithful.

5. The Weil group. Determinantal weights

5.1. When k is a finite field, we can replace the semilinear theory of section 2 with a linear theory, and get an extension of $\mathbf{Z}$ by the monodromy group which will function as an analogue of the Weil group. We assume in this section that k is the field $\mathbf{F}_q$ with cardinality q, and that F is the q-th power morphism. Since F is the identity on k, we can take $\sigma = 1$ on K. If X/k is a k-scheme, we assume that X is geometrically connected, and that $\mathrm{X}(k)$ is nonempty.

Let $\mathscr{C}$ be a category of isocrystals on X/K such that $\mathrm{F}: \mathscr{C} \xrightarrow{\sim} \mathscr{C}$. Since there is a k-rational point $x \in \mathrm{X}(k)$, $\mathscr{C}$ has a fiber functor $\omega_x : \mathscr{C} \to \mathrm{Vec}_{\mathrm{K}}$ and we set $\mathrm{G}_x = \mathrm{Aut}^{\otimes}(\omega_x)$. Since $\sigma = 1$, the canonical Frobenius structure $\Phi_x^{\mathscr{C}}$ on G_x is an automorphism $\Phi : \mathrm{G}_x \xrightarrow{\sim} \mathrm{G}_x$ of G_x. We denote by $\mathrm{W}(\bar{k}/k)$ the infinite cyclic group $\mathbf{Z}$ endowed with a distinguished generator F_x, thought of as the absolute Weil group of k endowed with its canonical geometric Frobenius. Define the "$\mathscr{C}$-Weil group" $\mathrm{W}^{\mathscr{C}}(\mathrm{X}, x)$ to be the semidirect product of G_x and $\mathrm{W}(\bar{k}/k)$, where F_x acts on G_x by means of $\Phi_x^{\mathscr{C}}$, *i.e.*

$$\mathrm{F}_x \, g \, \mathrm{F}_x^{-1} = \Phi_x^{\mathscr{C}}(g) \tag{5.1.1}$$

for any point g of G; there is then an exact sequence

$$0 \to \mathrm{G}_x \to \mathrm{W}^{\mathscr{C}}(\mathrm{X}/\mathrm{K}, x) \to \mathrm{W}(\bar{k}/k) \to 0. \tag{5.1.2}$$

If $\mathscr{C} = [\mathrm{M}]$ we will write W^{M} instead of $\mathrm{W}^{[\mathrm{M}]}$. The two universal cases are when $\mathscr{C}$ is $\mathrm{Isoc}(\mathrm{X}/\mathrm{K})^{\sim}$ or $\mathrm{Isoc}^{\dagger}(\mathrm{X}/\mathrm{K})^{\sim}$, which give us the absolute convergent and overconvergent Weil groups of X/K:

$$0 \to \pi_1^{\mathrm{F-Isoc}}(\mathrm{X}, x) \to \mathrm{W}^{\mathrm{F-Isoc}}(\mathrm{X}/\mathrm{K}, x) \to \mathrm{W}(\bar{k}/k) \to 0. \tag{5.1.3}$$

$$0 \to \pi_1^{\mathrm{F-Isoc}^{\dagger}}(\mathrm{X}, x) \to \mathrm{W}^{\mathrm{F-Isoc}^{\dagger}}(\mathrm{X}/\mathrm{K}, x) \to \mathrm{W}(\bar{k}/k) \to 0. \tag{5.1.4}$$

If we compare 5.1.1 with 2.2.4, we see that a representation (ρ, V, Ψ) of $(\mathrm{G}_x, \Phi_x^{\mathscr{C}})$ extends to a representation of $\mathrm{W}^{\mathscr{C}}(\mathrm{X}/\mathrm{K}, x)$ for which $\rho(\mathrm{F}) = \Psi$.

We should check that the groups just defined are independent of the choice of base point, up to inner isomorphism. In fact, we can be more precise. Recall that if we have two fiber functors

$$\omega_x, \omega_y: \quad \mathscr{C} \to \mathrm{Vec}_{\mathrm{K}}$$

then any choice of an isomorphism $p_{xy} : \omega_x \simeq \omega_y$ gives rise canonically to an isomorphism $P_{xy} : G_x \simeq G_y$, where we have set $G_x = \mathrm{Aut}^{\otimes}(\omega_x)$, $G_y = \mathrm{Aut}^{\otimes}(\omega_y)$. Denote by W_x, W_y the corresponding extensions defined by 5.1.2.

5.2. PROPOSITION. — *For any $p_{xy} : \omega_x \simeq \omega_y$, the isomorphism $P_{xy} : G_x \simeq G_y$ extends canonically to an isomorphism $P_{xy} : W_x \simeq W_y$.*

Proof. — We first note that since $\sigma = 1$, the isomorphism 2.3.1 becomes just $\omega_x \simeq \omega_x \circ F^*$, $\omega_y \simeq \omega_y \circ F^*$ and the canonical Frobenius structures are automorphisms $\Phi_x : G_x \xrightarrow{\sim} G_x$, $\Phi_y : G_y \xrightarrow{\sim} G_y$. So if we have any "path" $p_{xy} \in \mathrm{Isom}^{\otimes}(\omega_x, \omega_y)$ giving rise to a $P_{xy} : G_x \xrightarrow{\sim} G_y$, it induces an isomorphism $\omega_x \circ F^* \to \omega_y \circ F^*$ which is easily seen to yield the *same* isomorphism of group $P_{xy} : G_x \xrightarrow{\sim} G_y$ as p_{xy}. If we then recall that $\mathrm{Isom}^{\otimes}(\omega_x, \omega_y)$ is principal homogenous under G_x and G_y, we see that there is a unique element $g_y \in G_y(K)$ making

(5.2.1)
$$\begin{array}{ccc} \omega_x & \xrightarrow{g_y \circ p_{xy}} & \omega_y \\ \downarrow & & \downarrow \\ \omega_x \circ F^* & \xrightarrow{p_{xy}} & \omega_y \circ F^* \end{array}$$

a commutative diagram of isomorphisms. Applying $\mathrm{Aut}^{\otimes}$ to 5.2.1 yields a commutative diagram

(5.2.2)
$$\begin{array}{ccc} G_x & \xrightarrow{\mathrm{Ad}(g_y) \circ P_{xy}} & G_y \\ \Phi_x \downarrow & & \downarrow \Phi_y \\ G_x & \xrightarrow{P_{xy}} & G_y \end{array}$$

Now W_x, W_y are the extensions

$$0 \to G_x \to W_x \to \langle F_x \rangle \to 0$$
$$0 \to G_y \to W_x \to \langle F_y \rangle \to 0$$

defined by the relations

(5.2.3)
$$\begin{cases} F_x g F_x^{-1} = \Phi_2(g) \\ F_y g F_y^{-1} = \Phi_y(g). \end{cases}$$

To extend P_{xy} to an isomorphism $W_x \xrightarrow{\sim} W_y$, it is enough to come up with a value for $P_{xy}(F_x)$ consistent with the relations 5.2.3. We claim that setting $P_{xy}(F_x) = F_y g_y$ will work if g_y is defined by 5.2.1; in fact, applying P_{xy} to the first member of 5.2.3 yields

(5.2.4)
$$P_{xy}(\Phi_x(g)) = P_{xy}(F_x) P_{xy}(g) P_{xy}(F_x)^{-1}$$

while on the other hand, we have

$$(5.2.5)\qquad \begin{aligned} P_{xy}(\Phi_x(g)) &= \Phi_y(g_y P_{xy}(g) g_y^{-1}) \quad \text{by } 5.2.2 \\ &= F_y g_y P_{xy}(g) g_y^{-1} F_y^{-1}. \end{aligned}$$

The right-hand sides of 5.2.4 and 5.2.5 will agree if we set $P_{xy}(F_x) = F_y g_y$. ■

5.3. *Remark*. — If we change p_{xy} by an element of G_y, say $p_{xy} \mapsto h \circ p_{xy}$, then P_{xy}, then $P_{xy}: G_{x} \to G_y$ gets replaced by $\mathrm{Ad}(h) \circ P_{xy}$ and g_y gets replaced by $\Phi_y^{-1}(h) g_y h^{-1}$, as we see from 5.2.1. Since $F_y \Phi_y^{-1}(h) g_y h^{-1} = h F_y g_y h^{-1}$, we see that the extension $P_{xy}: W_x \to W_y$ to W_x also changes by $\mathrm{Ad}(h)$.

5.4. We can use proposition 5.2 to attach Frobenius conjugacy classes in W to closed points of X. Denote by k_n the extension of k of degree n, by K_n the unramified extension of K of degree n, and by $\bar{K}$ the union of the K_n, *i.e.* the maximal unramifiel extension of K (not complete). We choose compatible liftings σ of the q-th power automorphism to each K_n. Any closed point of X of degree n can then be identified with a $\mathrm{Gal}(k_n/k)$-orbit of $X(k_n)$. Then if (M, F) is any (convergent or overconvergent) F-isocrystal on X/K and y is a closed point of X of degree n, we will associate to any point of $X(k_n)$ lying above y a conjugacy class in $W^M(K_n)$, and then show that this class depends only on the $\mathrm{Gal}(k_n/k)$-orbit of the point. This we will define to be the Frobenius class Frob_y attached to y. We can (and will) of course do the same with any $\otimes$-category of F-isocrystals on which F* is an autoequivalence. Note that as the degree of the closed point increases, the field of definition of Frob_y grows as well, so that while all the Frobenius classes live in $W^M(\bar{K})$, only finitely many of them live in any locally compact group (compare this with the case of an l-adic representation, for which *all* of the Frobenius classes lie in a locally compact group).

As before x is a fixed k-point of X, while y will be a point in $X(k_n)$. We have evident fiber functors

$$\omega_x:\ \mathscr{C} \to \mathrm{Vec}_K \qquad \omega_y:\ \mathscr{C}\otimes K_n \to \mathrm{Vec}_{K_n}$$

and an isomorphism

$$(5.4.1)\qquad \mathrm{Aut}^{\otimes}(\omega \otimes K_n) \simeq G_x \otimes K_n.$$

Let G_y be the K_n-group $\mathrm{Aut}^{\otimes}(\omega_y)$. As F is no longer linear on $\mathscr{C}\otimes K_n$ we must replace it by F^n; then we can apply the construction of 5.1 to obtain group extensions

$$0 \to G_x \otimes K_n \to W^{\mathscr{C}\otimes K_n}(X/K_n, x) \to \langle F_x^n \rangle \to 0$$

$$0 \to G_y \to W^{\mathscr{C}\otimes K_n}(X/K_n, y) \to \langle F_y \rangle \to 0.$$

By 5.2, any isomorphism $p_{yx}: \omega_y \xrightarrow{\sim} \omega_x \otimes K_n$ canonically induces an isomorphism

$$P_{xy}:\ W^{\mathscr{C}\otimes K_n}(X/K_n, y) \xrightarrow{\sim} W^{\mathscr{C}\otimes K_n}(X/K_n, x)$$

while on the other hand there is an obvious inclusion map

$$W^{\mathscr{C}\otimes K_n}(X/K_n, x)(K_n) \subsetneq W^{\mathscr{C}}(X/K, x)(K_n)$$

(arising via pushout from the inclusion $\langle F_x^n \rangle \to \langle F_x \rangle$) by means of which we may define

$$\text{(5.4.2)} \qquad \mathrm{Frob}_y = P_{yx}(F_y) \in W^{\mathscr{C}}(X/K, x)(K_n).$$

Any other point of $X(k_n)$ defining the same closed point of X as y is of the form $F^a(y)$, so if we want to show that Frob_y depends only on the underlying closed point, it is enough to show that $\mathrm{Frob}_y = \mathrm{Frob}_{F(y)}$. A glance at 2.3.1 shows that replacing y by $F(y)$ has the effect of replacing ω_y by $\omega_y \circ F^*$. Form 5.2.1 we see that we must then replace g_v by $\Phi_y(g_y)$. Thus, finally, $P_{xy}(F_x^n) = F_y g_y$ gets replaced by $F_y \Phi_y(g_y) = F_y(F_y g_y) F_y^{-1}$; in other words, P_{xy} changes by $\mathrm{Ad}(F_y)$.

5.5. *Remark.* — Let (M, F) be an F-isocrystal and let $(\rho, V = M_x, \Psi = F_x)$ be the corresponding representation of $(\mathrm{D\,Gal}(M, x), \Phi_x)$. We have already remarked that (ρ, V, F_x) extends naturally to a representation ρ of W_x^M. It is clear that the class of F_x in W_x^M is exactly Frob_x. More generally, if y is a closed point of X of degree n, then it is immediate from the construction of Frob_y that $(V \otimes K_n, \rho(\mathrm{Frob}_y))$ is isomorphic, as a vector space with endomorphism, to (M_y, F_y).

We can now prove some basic results on determinantal weights ([6] I.3). Fix an isomorphism $\iota : K^{alg} \xrightarrow{\sim} \mathbf{C}$, and for any closed point $x \in |X|$ write $q_x = q^{\deg x}$. If (M, Φ) is an F-isocrystal on X/K, corresponding to a representation ρ of W^M, then we say that (M, Φ) is *pointwise* ι*-pure of weight n* if for any closed point x we have $|\iota(\alpha)| = q_x^{n/2}$ for any eigenvalue α of $\rho(\mathrm{Frob}_x)$. It follows from 4.9 that an overconvergent (M, Φ) of rank one is ι-pure of some weight. This makes possible.

5.6. Definition. — *Let* (M, Φ) *be an overconvergent* F*-isocrystal on a smooth curve. The* ι-determinantal weights *of* (M, Φ) *are the numbers* $(1/d)$ (ι-weights of $\wedge^d N$), *where* N *runs through the irreducible constituents of rank d of* (M, Φ) *in* $\mathrm{F\text{-}Isoc}^{\dagger}(X/K)$.

The basic result on determinantal weights says that they behave in the same way as "actual" weights with respect to simple operations of linear algebra (*cf.* [6], 1.3.13):

5.7. Proposition. — *Let* X/k *be a smooth curve*

(i) *If* $\pi : Y \to X$ *is dominant and* (M, Φ) *is an overconvergent* F*-isocrystal on* (X/K), *then* (M, Φ) *is purely of* ι*-determinantal weight* β *if and only if* $\pi^*(M, \Phi)$ *is.*

(ii) *If the overconvergent* F*-isocrystals* (M, Φ), (N, Ψ) *are purely of* ι*-determinantal weights* β *and* γ, *then* $(M, \Phi) \otimes (N, \Psi)$ *is purely of* ι*-determinantal weights* $\beta + \gamma$.

(iii) *If* (M, Φ) *is overconvergent and* $n(\beta)$ *is the sum of the ranks of the constituents of* (M, Φ) *of* ι*-determinantal weight* β, *then the determinantal weights of* $\wedge^d M$ *are the numbers* $\sum a(\beta)\beta$, *with* $\sum a(\beta) = d$ *and* $a(\beta) \leqq n(\beta)$.

The proof is exactly the same as in [6] 1.3. The key point is that if M is semisimple and Z denotes the center of W^M, then the restriction of Z of the degree map $W \to W(\bar{k}/k)$

has *finite* kernel and cokernel, which in turn can be deduced from 4.10 (*cf.* [6], 1.3.10, 1.3.11). ■

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