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ON THE REAL COHOMOLOGY OF ARITHMETIC GROUPS AND THE RANK CONJECTURE FOR NUMBER FIELDS

BY JUN YANG (1)

Let G be a connected semi-simple algebraic group defined over the rational number field $\mathbb{Q}$, and Γ an arithmetic subgroup of G. Denote the symmetric space of $G(\mathbb{R})$ with respect to a maximal compact subgroup K by X. The real cohomology $H^*(\Gamma)$ of Γ is isomorphic to I_G^*, the group of G-invariant forms on X, in a certain range depending on the rank of the algebraic group in question, (*cf.* [B3]). In fact, this is how Borel computed the ranks of K-groups of algebraic number fields in [B3].

In general, there is a homomorphism

$$j^*: \quad I_G^* \to H^*(\Gamma)$$

which was proven to be surjective in dimension no greater than a constant $m(G)$ by a series of works of H. Garland [Ga], W. C. Hsiang [GH] and A. Borel [B3]. Furthermore, Borel proved that the map is also injective in dimension no greater than a constant $c(G)$, which we denote by $c(G/\mathbb{Q})$ in this paper. We will improve this injectivity result, our new constant is roughly twice that of Borel's. The main theorem of the paper is

THEOREM A. — *The map j^q is injective for $q \leqslant l(G/\mathbb{Q})$, where $l(G/\mathbb{Q}) = 2c(G/\mathbb{Q}) + 1$ is a constant which can be computed in terms of the absolute root structure and the $\mathbb{Q}$-rank of* G. *In particular, if k is a number field, and $G/\mathbb{Q} = R_{k/\mathbb{Q}} SL_n$, then $l(G/\mathbb{Q}) \geqslant d(n-1)$, where $R_{k/\mathbb{Q}}$ is the restriction of scalar functor, and $d = [k:\mathbb{Q}]$.*

The reason we wanted to improve the constant $c(G)$ of Borel is that we are interested in the so-called rank conjecture in algebraic K-theory, which is, I believe, due to Suslin. Little is known about the rank conjecture for general fields. Our purpose here is to prove the rank conjecture for all non-trivial algebraic number fields, *i.e.*, for number fields with degree $\geqslant 2$ over $\mathbb{Q}$. To prove the rank conjecture for a number field k, it

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suffices to establish the following simple statements (*see* § 4):

For k, the image of the natural map

$$H_{2n-1}(GL_m(k);\mathbb{Q}) \to H_{2n-1}(GL(k);\mathbb{Q})$$

contains $PH_{2n-1}(GL(F);\mathbb{Q})$ for $m \geqslant n \geqslant 1$, and

$$\{\mathrm{im}: H_{2n-1}(GL_m(k);\mathbb{Q}) \to H_{2n-1}(GL(k);\mathbb{Q})\} \cap PH_{2n-1}(GL(k);\mathbb{Q}) = 0$$

if $m < n$, where PH denotes the primitive homology. As a corollary of Theorem A, we get the main result of our paper

THEOREM B. — *If k is an algebraic number field not equal to $\mathbb{Q}$, then the rank conjecture is true for all* K*-groups of k.*

The following corollary follows immediately from the fact that an arbitrary algebraic extension field of $\mathbb{Q}$ is the direct limit of certain algebraic number fields and from the fact that K-theory commutes with direct limits.

COROLLARY. — *The rank conjecture holds for all non-trivial algebraic extension fields of $\mathbb{Q}$. In particular, it holds for $\bar{\mathbb{Q}}$, the field of algebraic numbers.*

Our work was inspired by the observation that Borel's work [B3] implies the rank conjecture for all number fields of degree $\geqslant 6$. The proof of Theorem A is an elaboration of ideas in [B3]. More precisely, Borel has constructed a complex of "logarithmic forms" C^*, which computes the cohomology of X/Γ. The constant $c(G)$ is precisely the upper limit of the range in which forms in C^* are L^2; the L^2 condition is normally required to prove some sort of "Hodge Theorem" on a noncompact Riemannian manifold. We have observed however, in the special setting we are considering, that it is possible to do Hodge-type of argument without all the forms being L^2. That is essentially why we can improve Borel's constant.

Let's have a brief look at the main idea behind the proof of Theorem A. It is well-known that every form in I_G^* is harmonic. In the very special case when X/Γ is compact, I_G^* then maps injectively into $H^*(X/\Gamma) \cong H^*(\Gamma)$ by the famous Hodge Theorem. But even on a non-compact oriented complete Riemannian manifold M, there is a weak form of Hodge Theorem which we now recall. Let $(.,.)_x$ denote the inner product on $\oplus_q \Lambda^q T_x^*(M)$ induced from the Riemannian metric. A scalar product on $\Omega^*(M)$ can then be defined as follows

$$(\alpha, \beta)_M = \int_M (\alpha, \beta)_x \, dV_M.$$

Let $\delta = (-1)^{n(p+1)+1} * d * : \Omega_M^p \to \Omega_M^{p-1}$ be the formal adjoint operator of the differential operator d. Then one has the following

PROPOSITION. — *Let $\alpha \in \Omega_M^p$, $\beta \in \Omega_M^{p+1}$. If α, $d\alpha$, β, and $d\beta$ are all square integrable on* M, *then we have*

$$(\alpha, \delta\beta)_M = (d\alpha, \beta)_M.$$

It is instructive to recall the proof. We follow Borel ([B3], § 2). On M, there is a family of exhaustion functions $\{\sigma_r\}_{r\in\mathbf{R}_+^*}$ with compact supports, which satisfy $(d\sigma_r, d\sigma_r)^{1/2} < \mathrm{D}/r$, for a constant D and $\lim_{r\to\infty}\sigma_r = 1$. Integrating by parts, we have

$$(\sigma_r\alpha, \delta\beta)_{\mathrm{M}} = (d(\sigma_r\alpha), \beta)_{\mathrm{M}} = (d\sigma_r \wedge \alpha, \beta)_{\mathrm{M}} + (\sigma_r d\alpha, \beta)_{\mathrm{M}}.$$

The proposition therefore follows from the assertion that

$$\lim_{r\to\infty}(d\sigma_r \wedge \alpha, \beta)_{\mathrm{M}} = 0,$$

which is easily verified under the assumptions that α, β be L^2.

This proposition is the starting point of the proof of injectivity in [B3]. Let $\mathrm{M} = \mathrm{X}/\Gamma$, β be a harmonic form on X/Γ which is the descent of an element of $\mathrm{I}_{\mathrm{G}}^*$. It is proved in [B3] that all bi-invariant forms $\beta \in \mathrm{I}_{\mathrm{G}}^*$ are square integrable on X/Γ and that if β is exact, one can always choose $\alpha \in \mathrm{C}^*$, so that $\beta = d\alpha$. If one knows that α is L^2, then by the above proposition we have

$$(\beta, \beta)_{\mathrm{M}} = (d\alpha, \beta)_{\mathrm{M}} = (\alpha, \delta\beta)_{\mathrm{M}} = 0,$$

and it follows immediately that $\beta = 0$. It is proved in [B3] that every form in C^* is square integrable up to dimension $c(\mathrm{G}/\mathbb{Q})$, hence we know Theorem A must be true at least in the same range. However, if we examine the proof of the proposition more closely, we find it is not necessary that α be square integrable for the above argument to work. Indeed, since we know β is harmonic and L^2, and that σ_r has compact support, it follows that if $\beta = d\alpha$, then

$$\begin{aligned}(\beta, \beta) = (d\alpha, \beta) &= \lim_{r\to\infty}(\sigma_r d\alpha, \beta)\\ &= \lim_{r\to\infty}[(d(\sigma_r\alpha), \beta) - (d\sigma_r \wedge \alpha, \beta)] = \lim_{r\to\infty}[(\sigma_r\alpha, \delta\beta) - (d\sigma_r \wedge \alpha, \beta)]\\ &= -\lim_{r\to\infty}(d\sigma_r \wedge \alpha, \beta).\end{aligned}$$

In the above calculation, we do not need α to be L^2! In order to prove injectivity, it suffices to prove that

$$\lim_{r\to\infty}(d\sigma_r \wedge \alpha, \beta) = 0.$$

As proved in [B3], forms in $\mathrm{I}_{\mathrm{G}}^*$ tend to 0 quickly as one approaches the boundary of $\bar{\mathrm{X}}$, the Borel-Serre compatification of X, while $\alpha \in \mathrm{C}^*$ does not grow too fast near the boundary of $\bar{\mathrm{X}}$. Since β is better than L^2, we can work in a range where α may not be L^2 but the above limit is zero. This is our basic idea.

Theorem B is not only theoretically interesting, but also has concrete applications to number theory. Many years after Borel's work on the so-called higher regulators ([B4]), people have been trying to find explicit functions to represent them. The first regulator can be written down in terms of the function log $|\ |$ by the classical Dirichlet Unit Theorem, while Bloch [Bl] showed that the second regulator can be expressed in terms of the so-called Bloch-Wigner function. Zagier [Z] has conjectured that the m-th Borel regulator can be expressed in terms of the classical m-logarithm function. In [Y], we show that the 3rd Borel regulator can be expressed in terms of the Hain-MacPherson trilogarithm [HM]. The rank conjecture is an essential ingredient. Goncharov [Go] has announced that the 3rd regulator can be expressed in terms of the classical trilogarithm.

For the readers' convenience, we use exactly the same notations as [B3] whenever possible. The first two sections are devoted to some known background material so most results are stated without proofs. The main references for the first two sections are [B3], [B2] and [BS].

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Notations and conventions. — Throughout the paper, the letter G (possibly with subscripts) will denote a connected semi-simple algebraic group defined over a field $k \subset \mathbb{R}$, while Γ (possibly with subscripts) will always denote an arithmetic subgroup of the corresponding (should be clear in the context) algebraic group. The group of k-characters of G will be denoted by $\chi(G)_k$. Lie groups (over $\mathbb{R}$, unless specified otherwise) are denoted by Roman uppercase letters while the corresponding Lie algebras are always denoted by the corresponding gothic lowercase letters. The letter K is always understood to be a maximal compact subgroup of $G(\mathbb{R})$, and X the corresponding symmetric space $K \diagdown G(\mathbb{R})$. The G-invariant real differential forms on X will be denoted by I_G^*.

All cohomology theories in this paper are supposed to have *real* coefficients unless specified otherwise.

For two real functions f, g defined on a set U, we write $f \prec g$ if there exists a constant $c>0$ such that $f(x) \leqslant c g(x)$, for all $x \in U$.

1. Decomposition of X

This section is basically a collection of results of [B3]. For the proofs of the results in this section, *cf.* [B3], [B1].

1.1. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to K, and θ the Cartan involution with respect to K. Let P be a parabolic k-subgroup of G, and M the Levi subgroup of P stable under θ. We put

$$^{0}\mathrm{M} = \bigcap_{\chi \in \mathrm{X}(\mathrm{M})_k} \ker \chi^2,$$

and let $\mathrm{S_P}$ be a maximal k-split torus of $\mathrm{R}(\mathrm{P})$, the radical of P. Then we have a decomposition

$$\mathrm{P}(\mathbb{R}) = \mathrm{A_P} \ltimes {}^{0}\mathrm{M}(\mathbb{R}) \ltimes \mathrm{U}(\mathbb{R})$$

where $\mathrm{A_P}$ denotes $\mathrm{S_P^0}(\mathbb{R})$ and U denotes the unipotent radical of P.

We let $\mathrm{Z} = (\mathrm{K} \cap \mathrm{P}) \diagdown {}^{0}\mathrm{M}(\mathbb{R})$, then we have a diffeomorphism

$$\mu_0: \quad \mathrm{Y} = \mathrm{A_P} \times \mathrm{Z} \times \mathrm{U}(\mathbb{R}) \xrightarrow{\sim} \mathrm{X}$$

The map induces an action of $\mathrm{P}(\mathbb{R})$ on Y which can be described as

$$(a, z, u)\, bmv = (ab, zm, m^{-1} b^{-1} bmv)$$

where $b \in \mathrm{A_P}$, $m \in {}^{0}\mathrm{M}(\mathbb{R})$, and $v \in \mathrm{U}(\mathbb{R})$.

Fix a maximal k-split torus S containing S_p. Denote the set of k-roots of G with respect to S by Φ. Fix a minimal parabolic k-subgroup P_0 of P. Let Δ be a base of Φ with respect to P_0. Then the set of parabolic k-subgroups containing P_0 is parametrized by the subsets of Δ. Let $\mathrm{I} = \mathrm{I}(\mathrm{P}) \subset \Delta$ be the subset corresponding to P, then there is a canonical isomorphism ([BS], § 4.1)

$$\mathrm{A_P} \xrightarrow{\sim} (\mathbb{R}_+^*)^{\Delta - \mathrm{I}}.$$

1.2. On $\mathfrak{g}$ we have a scalar product defined by

$$g_0(\xi, \eta) = -\mathrm{B}(\xi, \theta(\eta))$$

where ξ, η are vectors in γ, $\mathrm{B}(.,.)$ is the Killing form. The restriction of g_0 to $\mathfrak{p}$ defines a metric on $\mathrm{T_o}(\mathrm{X})$ via the natural projection from $\mathrm{G}(\mathbb{R})$ to X, where $\mathrm{T_o}(\mathrm{X})$ is the tangent space at the canonical base point o. This then extends to a $\mathrm{G}(\mathbb{R})$-invariant metric on X, which we denote by dx^2.

Let $dy^2 = \mu_0^* dx^2$, and denote the right-invariant metrics on A and $\mathrm{U}(\mathbb{R})$ induced from g_0 by da^2 and du^2 respectively. The restriction of g_0 to $\mathfrak{a}$ can be written in the form

$$\sum_{\alpha, \beta \in \Delta - \mathrm{I}} c_{\alpha\beta}\, d\alpha\, d\beta.$$

Hence the metric on A can be written as

$$da^2 = \sum_{\alpha, \beta \in \Delta - \mathrm{I}} c_{\alpha\beta}\, \alpha^{-1} \beta^{-1}\, d\alpha\, d\beta.$$

Let Φ_P denote the set of roots of P with respect to S_P. The set Φ_P is the union of the set of positive roots $\Phi^+ \subset \Phi$ and the set of roots in Φ generated by I, which we denote by Φ_I. For simplicity, let $\Phi_P^c = \Phi^+ - \Phi_I$, then

$$\mathfrak{u} = \bigoplus_{\beta \in \Phi_P^c} \mathfrak{u}_\beta.$$

For $\beta \in \Phi_P^c$, let h^β be the right invariant scalar product on $\mathfrak{u}$ which is zero on $\mathfrak{u}_\alpha$ if $\alpha \neq \beta$ and equal to g_0 on $\mathfrak{u}_\beta$.

1.3. PROPOSITION. — *At any point $(a, z, u) \in Y$, the spaces $\mathfrak{a}$ at a, $T(Z)$ at z and $\mathfrak{u}^\beta$ at u are mutually orthogonal and we have the decomposition of metric as follows*

$$(dy^2)_{(a, z, u)} = (da^2)_a \oplus (dz^2)_z \oplus \bigoplus_{\beta \in \Phi_P^c} 2^{-1} a^{2\beta} h_\beta(z)$$

where $h_\beta(z) = (\operatorname{int} m)^* h_\beta$, for any $m \in {}^0M(\mathbb{R})$, such that $z = o\, m$.

Cf. Prop. 4.3 in [B3].

1.4 PROPOSITION. — *If dV_Y, dV_{A_P}, dV_Z and dV_U are the volume elements of the metrics dy^2, da^2, dz^2 and du^2, then*

$$dV_{A_P} = c \Lambda_{\alpha \in \Delta} d\alpha/\alpha$$

where $c = (\det c_{\alpha\beta})^{1/2}$ and

$$dV_Y = 2^{-\dim U/2} a^{2\rho_P} dV_{A_P} \Lambda dV_Z \Lambda dV_U.$$

Here $2\rho_P = \sum_{\beta \in \Phi_P^c} (\dim \mathfrak{u}_\beta)\, \beta$.

1.5. Let us recall more notations from [B3]. Assume $\Delta - I$ consists of roots $\alpha_1, \ldots, \alpha_s$ and let $m = \dim X$. We choose a moving frame ω^i, $1 \leqslant i \leqslant m$ on T^*Y, so that ω^i is lifted from $d \log \alpha_i$ on A_P if $i \leqslant s$, from an orthonormal frame on Z if $s < i \leqslant t$, and from a set of right invariant 1-forms on U if $t < i \leqslant m$.

Let $I_m = \{1, \ldots, m\}$, and for $\in I_m$, we put

$$\alpha(i) = \begin{cases} 0 & \text{if } 1 \leqslant i \leqslant t \\ \beta & \text{if } \omega_e^i \in \mathfrak{u}_\beta^* \end{cases}$$

For a subset J of I_m, we let

$$\omega^J = \Lambda_{i \in J} \omega^i, \qquad \alpha(J) = \sum_{i \in J} \alpha(i)$$

and denote by $|J|$ the number of elements in J. Then a q-form τ on Y can be written as

$$\tau = \sum_{|J| = q} f_J \omega^J,$$

where the f_J's are functions on Y.

1.6. Let $dy^2 = g_{ij}\,\omega^i\,\omega^j$ denote the metric on Y induced from the metric of X, and (g^{ij}) the inverse matrix of (g_{ij}). Write

$$I_m = I_1 \cup I_0 \cup \left(\bigcup_{\beta} I_\beta\right)$$

where $I_1 = \{1, \ldots, s\}$, $I_0 = \{s+1, \ldots, t\}$ and $I_\beta = \{i \in I_m \mid \omega_e^i \in \mathfrak{u}_\beta^*\}$. Let also

$$h_{\beta, z} = \sum_{i, j \in I_\beta} h_{\beta, ij}(z)\,\omega^i\,\omega^j$$

and denote the inverse matrices of $(h_{\beta, ij})$ and (c_{ij}) by (h_β^{ij}) and (c^{ij}). Then it follows immediately from Proposition 1.4 that

$$g^{ij} = \begin{cases} c^{ij} & \text{if } i, j \in I_{-1}, \\ \delta_{ij} & \text{if } i, j \in I_0, \\ 2\,a^{-2\beta}\,h_\beta^{ij}(z) & \text{if } i, j \in I_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

For two sets of indices $J = \{j_1, \ldots, j_q\}$ and $J' = \{j'_1, \ldots, j'_q\}$, it is convenient to introduce the notation $J \sim J'$ when $|J| = |J'|$ and J and J' have the same number of elements in each of I_{-1}, I_0, and each I_β. Let

$$g^{J, J'} = \det(g^{ij})_{i \in J,\, j \in J'}$$

it then follows that $g^{J, J'} = 0$ unless $J \sim J'$. Assume this is the case, then we have $\alpha(J) = \alpha(J')$ and

$$|g^{J, J'}(y)| \prec a^{-\alpha(J) - \alpha(J')}\,\varphi(z) = a^{-2\alpha(J)}\,\varphi(z),$$

where $y = (a, z, u) \in Y$, and $\varphi(z)$ is a smooth function on Z.

2. Properties of Siegel sets

2.1. Let us fix a decomposition of X with respect to a parabolic subgroup P as in Section 1.1. For $t > 0$, we define

$$A_t = \{a \in A_P \mid a^\alpha < t, \text{ for all } \alpha \in \Delta\}.$$

A Siegel set $\mathfrak{S}_{t, W}$ in X with respect to the decomposition in 1.1 is defined as

$$\mathfrak{S}_{t, W} = \mu_0(A_t \times W)$$

where W is open and relatively compact in $Z \times U(\mathbb{R})$. In case we want to emphasize that the Siegel set is given with respect to a specific decomposition with base point o and a parabolic k-subgroup P, we shall write $\mathfrak{S}_{o, P, t, W}$.

Note: one should be aware that the Siegel sets defined above are the so-called open Siegel sets of [B3].

2.2. Let Γ be a torsion-free arithmetic group, and let

$$\pi: \quad X \to X/\Gamma$$

be the natural projection. For suitably chosen t and W, π maps $\mathfrak{S}_{t, W}$ injectively onto an open set V in X/Γ. Such open sets are called *special neighborhoods*. The main property of Siegel sets with respect to a torsion-free arithmetic group we need is described in the following theorem.

2.3. Theorem. — *If* G *is a connected semi-simple algebraic group defined over* $\mathbb{Q}$, *and* Γ *a torsion-free arithmetic subgroup, then there exists a finite cover of* X/Γ *consisting of special neighborhoods.*

For a proof, *see* [B3], § 6.

2.4. In [B3], it is proved that there exists a subcomplex C^* of $\Omega^*(X/\Gamma)$ which contains I_G^* with the property that the natural inclusion

$$C^* \to \Omega^*(X/\Gamma)$$

induces an isomorphism on cohomology. The complex C^* consists of forms with "logarithmic growth." More precisely, a form $\sigma \in \Omega^*(X/\Gamma)$ lies in C^* if and only if on any special neighborhood $\pi(\mathfrak{S}_{t, W})$, the pull back of σ has logarithmic growth on $\mathfrak{S}_{t, W}$, that is, if

$$\tau = (\pi\mu_0)^* (\sigma|_{\pi(\mathfrak{S}_{t, W})}) = \sum_J f_J \omega^J$$

on $A_t \times \omega$, then

$$|f_J(a, w)| \prec |P(\log a^{\alpha_1}, \ldots, \log a^{\alpha_s})| \quad (\text{where } a \in A_t, \text{ and } w \in W, \text{ for all } J),$$

for some polynomial P in s variables. In fact, to be completely rigorous, one really needs a compactification of X, which we again refer to [B3] and [BS]. For our purpose, it suffices to know the following weaker growth conditions

$$|f_J(a, w)| \prec a^{-\varepsilon\nu}, \qquad \text{where} \quad \nu = \frac{1}{2} \sum_{1 \leq i \leq s} \alpha_i,$$

for all $\varepsilon > 0$.

2.5. Let $d\mu$ denote the measure on X induced from the $G(\mathbb{R})$-invariant metric dx^2 (*cf.* § 1). The pushdown of $d\mu$ to X/Γ for Γ discret and torsion-free will also be denoted by $d\mu$. Let $(\ ,\ )_x$ be the inner product on $\oplus_q \Lambda^q T^*(X/\Gamma)$ induced from the riemannian metric. Define an inner product (,) on $\Omega^*(X/\Gamma)$ by

$$(\omega, \xi) = \int_{X/\Gamma} (\omega, \xi)_z \, d\mu,$$

where it is understood that the inner product is meant for any two forms such that the above integral converges. Denote by δ the formal ajoint operator of d on X/Γ. It is well known that every $\omega \in I_G^*$ is harmonic, *i.e.* $d\omega = 0$, $\delta\omega = 0$. A form ω is called *square integrable* on an open subset U of a Riemannian manifold M if the integral

$$\int_U (\omega_x, \omega_x)_x \, d\mu$$

converges. The restriction of an invariant form $\omega \in I_G^*$ to a Siegel set satisfies nice growth conditions.

2.6. PROPOSITION. — *Let* $\omega \in I_G^*$ *be an invariant form. If for a Siegel set* $\mathfrak{S}_{t, W}$, *we write*

$$\tau = (\pi \mu_0)^* (\sigma |_{\pi(\mathfrak{S}_{t, W})}) = \sum_{|J| = q} f_J \omega^J$$

on $A_t \times W$, *then we have*

$$|f_J(a, w)| \prec a^{\alpha(J)}.$$

In particular, ω *is square integrable on every Siegel set.*

This is essentially Corollary 5.7 in [B2].

3. Proof of theorem A

In this section, Γ always denotes a *torsion-free* arithmetic subgroup. As pointed out in [B3], it suffices to prove the theorem for such Γ.

3.1. Define a smooth regular function $\lambda : X/\Gamma \to \mathbb{R}_+^*$ on X/Γ by regularizing the distance function on X/Γ with respect to the canonical base point. Then λ satisfies

$$|\lambda(x) - \lambda(y)| \prec \operatorname{dist}(x, y)$$

which implies that

$$\| d\lambda \| \prec 1.$$

For more details, *cf.* [Rh], § 35.

Let $m : [0, \infty) \to [0, 1]$ be a smooth function which takes the value 1 on [0, 1] and 0 on $[2, \infty)$. Define a family of functions $\{\sigma_r\}_{r \in \mathbb{R}_+^*}$ on X/Γ by

$$\sigma_r : \quad X/\Gamma \to \mathbb{R}_+^*$$

$$x \mapsto m(\lambda(x)/r)$$

The following lemma is trivial to check.

3.2. LEMMA. — *The family of functions* $\{\sigma_r\}_{r \in \mathbb{R}_+^*}$ *has the following property*:

(i) $0 \leqslant \sigma_r \leqslant 1$;

(ii) *the sets* $C_r = \{x \in X/\Gamma \mid \sigma_r(x) = 1\}$ *and* $D_r = \overline{\{x \in X/\Gamma \mid \sigma_r(x) \neq 0\}}$ *are both compact*;

(iii) $C_r \subset C_{r'}$ *for* $r < r'$, *and the union of* C_r *is all of* X.

3.3. For a parabolic k-subgroup P of G, let $X(A_P)$ denote the group of continuous homomorphisms of A_P into $\mathbb{R}_+^*$. Then any element $\lambda \in X(A_P)$ can be written as

$$\lambda = \sum c_i \alpha_i.$$

(*See* Section 1 for notations). We write $\lambda \gg 0$ if each $c_i > 0$. Define the constant $l(G, P)$ by

$$l(G, P) = \max\{q \mid 2\rho_P - \alpha(J) \gg 0, \text{ for all } |J| \text{ with } |J| < q\}.$$

In particular, if P_0 is a minimal parabolic k-subgroup, then we let

$$l(G/k) = l(G, P_0).$$

Then $l(G/k)$ is independent of the choice of the minimal parabolic k-subgroup P_0, and $l(G/k) \leqslant l(G, P)$, for all parabolic k-subgroup P (*cf.* [B3], §7.1). In particular, when $k = \mathbb{Q}$, it is easy to see that $l(G/\mathbb{Q}) = 2c(G/\mathbb{Q}) + 1 = 2c(G) + 1$, where $c(G)$ is the constant defined in Section 7.1 of [B3].

3.4. Now we can start the proof of Theorem A. Let $\omega \in I_G^*$ be a q-form with $q \leqslant l(G/k)$. Since Γ is discrete and torsion-free, ω descends to a form on X/Γ which we still denote by ω. To prove Theorem A, it suffices to prove that if ω is exact in X/Γ, then $\omega = 0$. Suppose that $\omega = d\xi$ for some $\xi \in \Omega^{q-1}(X/\Gamma)$. Since ω is in $I_G^* \subset C^*$, we may then choose $\xi \in C^*$. Let σ_r be as defined in 2.5. Since σ_r has compact support, we have

$$(d(\sigma_r \xi), \omega) = (\sigma_r \xi, \delta\omega) = 0.$$

Combining this with

$$d(\sigma_r \xi) = d\sigma_r \wedge \xi + \sigma_r d\xi = d\sigma_r \wedge \xi + \sigma_r \omega,$$

we have

$$0 = \lim_{r \to \infty} (d(\sigma_r \xi), \omega) = \lim_{r \to \infty} (d\sigma_r \wedge \xi, \omega) + (\omega, \omega).$$

Therefore we have

$$\|\omega\|^2 = -\lim_{r \to \infty} (d\sigma_r \wedge \xi, \omega).$$

It thus suffices to proves that

$$\lim_{r \to \infty} (d\sigma_r \wedge \xi, \omega) = 0$$

when the degree of $\omega \leqslant l(G/k)$. Since (X/Γ) is covered by special neighborhoods

$$\pi(\mathfrak{S}_{o_1, P_1, t_1, W_1}), \ldots, \pi(\mathfrak{S}_{o_n, P_1, t_n, W_n})$$

(Theorem 2.3), it suffices to prove that the pull back of

$$\int_{X/\Gamma} |(d\sigma_r \wedge \xi, W)_x| \, d\mu$$

to each Siegel set $\mathfrak{S}_{t_i, W_i}$ has limit 0 when $r \to \infty$. Let $\mathfrak{S}_{t, W}$ denote any one of the $\mathfrak{S}_{o_i, P_i, t_i, W_i}$, we need the following estimate on the pull back of λ.

3.5. LEMMA. — *Set* $\tilde{\lambda} = (\pi\mu_0)^*(\lambda)$. *If* $d\tilde{\lambda} = \sum_{i=1}^{m} s_i \omega^i$, *then on* $\mu_0^{-1}(\mathfrak{S}_{t, W})$, *we have*

$$|s_i(a, w)| \prec a^{\alpha(i)}.$$

Proof. — It is clear that $|d\tilde{\lambda}| \prec 1$ (*see* 3.1). Hence at any point $x \in X/\Gamma$, we have

$$\left(\sum_{i=1}^{m} s_i \omega^i, \sum_{i=1}^{m} s_i \omega^i\right)_x = \sum_{i, j=1}^{m} g^{i, j} s_i s_j = \sum_{i, j=1}^{s} c^{i, j} s_i s_j + \sum_{i=s+1}^{t} s_i^2 + \sum_{\beta} \sum_{i, j \in I_\beta} a^{-2\beta} h_\beta^{i, j}(z) s_i s_j \prec 1.$$

where $(a, z, u) = \mu_0^{-1}(x)$, *see* 1.1 for notations. Since the coefficient matrices $(c^{i, j})$ and $(h_\beta^{i, j}(z))$ are positive definite, and z varies on a relatively compact subset, we get

$$|s_i(a, w)| \prec 1, \qquad \text{if} \quad i \leqslant t,$$
$$|s_i(a, w)| \prec a^{\beta}, \qquad \text{if} \quad i \in I_\beta.$$

whence the lemma. □

3.6. On $\mu_0^{-1}(\mathfrak{S}_{t, W})$, let

$$\tilde{\xi} = \mu_0'^* \xi = \sum_{|J'| = q-1} t_{J'} \omega^{J'}, \qquad \omega = \sum_{|J| = q} f_J \omega^J.$$

Then by 3.4 and 2.6, have

$$|t_{J'}(a, w)| \prec a^{-\varepsilon\nu}, \qquad |f_J(a, w)| \prec a^{\alpha(J)}.$$

For an index set $J \subset I_m$, and an index $i \in I_m \setminus J$, we write $J(i) = J \cup \{i\}$. We have the following estimate:

$$\int_{\mathfrak{S}_{t,W}} |(dm(\tilde{\lambda}/r) \wedge \xi, \omega)_x| \, d\mu = \frac{1}{r} \int_{\mathfrak{S}_{t,W}} |(m'(\tilde{\lambda}/r) \, d\tilde{\lambda} \wedge \xi, \omega)_x| \, d\mu$$

$$\prec \frac{1}{r} \int_{\mathfrak{S}_{t,W}} |(\sum_{i, J'} s_i t_{J'} \omega^{J'(i)}, \sum_J f_J \omega_J)_x| \, d\mu$$

$$= \frac{1}{r} \sum_{i, J', J} \int_{\mathfrak{S}_{t,W}} |s_i t_{J'} f_J g^{J'(i), J}| \, d\mu$$

$$\prec \frac{1}{r} \sum_{i, J', J} \int_{\mathfrak{S}_{t,W}} a^{\alpha(i) - \varepsilon\nu + \alpha(J) - (\alpha(J) + \alpha(J'))} \, d\mu \prec \frac{1}{r} \sum_{i, J', J} \int_{A_t} a^{2\rho_P - \varepsilon\nu - \alpha(J')} \, dV_{A_P}.$$

Since $q \leqslant l(G/k)$, we get

$$2\rho_P - \varepsilon\nu - \alpha(J') = \sum_{j=1}^{s} m_{J',j} \alpha_j \geqslant 0,$$

for sufficiently small ε. Hence we have

$$\int_{\mathfrak{S}_{t,W}} |(dm(\tilde{\lambda}/r) \wedge \xi, \omega)_x| \, d\mu \prec \frac{1}{r} \sum_{i, J', J} \int_{A_t} a^{\sum_{j=1}^{s} m_{J',j}\alpha_j} V_{A_P} \prec \frac{1}{r} \sum_{i, J', J} \prod_{1 \leqslant j \leqslant s} \int_0^t \alpha^{m_{J,j} - 1} \, d\alpha_j \prec \frac{1}{r}.$$

Thus, when $r \to 0$, the limit of the above integral is 0. Therefore as explained in 3.4, we have $(\omega, \omega) = 0$, which then implies that $\omega = 0$. This completes the proof of Theorem A.

For applications in the next section, we need the following technical lemma.

3.7. LEMMA. — *Let* H *be a connected semi-simple algebraic group defined over field* k'. *If* k' *is a separable extension of* k *of finite degree* $d = [k' : k]$, *and* $G = R_{k'/k} H$, *where* $R_{k'/k}$ *stands for restriction of scalars* [W], *then*

$$l(G/k) \geqslant d \cdot l(H/k').$$

Proof. — Let $\Phi_k(G)$, $\Phi_{k'}(H)$ be the root system of G, H over k, k' respectively. There is an isomorphism ([BT], 6.21)

$$\Psi : \quad \Phi_{k'}(H) \to \Phi_k(G),$$

with the following property: if α is an arbitrary root of $\Phi_{k'}(H)$, then the dimension of the weight space of $\Psi(\alpha)$ in $\Phi_k(G)$ is d times the dimension of the weight space of α in $\Phi_{k'}(H)$. The lemma then follows immediately from the definition of $l(G/k)$ (§ 3.3). □

4. The rank conjecture for number fields

4.1. Recall for any commutative ring R, the n-th K-group of R is defined as follows,

$$K_n(R) = \pi_n(BGL^+(R)) \qquad \text{for} \quad n \geqslant 1$$

where $BGL^+(R)$ is an H-space which has the same homology as $BGL(R)$, and $\pi_1(BGL^+(R)) = GL(R)^{ab}$, the abelianization of $BGL(R)$. In fact, $BGL^+(R)$ and likewise, $BSL^+(R)$ are associative and commutative H-spaces (*cf.* [L]). By a theorem of Milnor and Moore [MM], we have the following isomorphism

$$K_n(R) \otimes \mathbb{Q} = \pi_n(BGL^+(R)) \otimes \mathbb{Q} \xrightarrow{\text{Hurewicz}} PH_n(GL(R); \mathbb{Q}),$$

where PH denotes the group of primitive homology classes. From now on, we denote the rational K-groups $K_n(R) \otimes \mathbb{Q}$ by $K_n(R)_{\mathbb{Q}}$. Using the above isomorphism, we can define the so-called rank filtration of the rational K-groups of R by

$$r_i K_n(R)_{\mathbb{Q}} = \{ \mathrm{im} : H_n(GL_i(R); \mathbb{Q}) \to H_n(GL(R); \mathbb{Q}) \} \cap PH_n(GL(R); \mathbb{Q}).$$

On the other hand, the K-groups are known to be special λ-rings, which then have the so-called γ-filtration which we denote by $\gamma^i K_n(R)$, *cf.* [H]. The γ-filtration is a decreasing filtration as opposed to the rank filtration which is obviously increasing. The rank conjecture of algebraic K-theory claims that these two filtrations complement each other when R is an infinite field. More precisely, it is conjectured that

$$K_n(R)_{\mathbb{Q}} = r_i K_n(R)_{\mathbb{Q}} \oplus \gamma^{i+1} K_n(R)_{\mathbb{Q}},$$

when R is an infinite field. Even though the rank conjecture is well known among K-theorists, its origin seems to be quite a mystery even among experts. To the best of my knowledge, the rank conjecture was first formulated by Suslin.

4.2. Now assume that k is a number field. From Borel's work [B3], one knows that $K_{2n}(k)_{\mathbb{Q}} = 0$, hence the rank conjecture is trivial for $K_{2n}(k)$. While from Beilinson's work [Bei], one knows that the γ-filtrations on $K_{2n-1}(k)_{\mathbb{Q}}$ are as follows,

$$\gamma^i K_{2n-1}(k)_{\mathbb{Q}} = \begin{cases} K_{2n-1}(k)_{\mathbb{Q}} & \text{if} \quad i \leqslant n, \\ 0 & \text{if} \quad i > n. \end{cases}$$

We sketch a proof which was carefully explained to me by R. Hain. There exists a sequence of the so-called Adams operations $\{\psi^j\}_{j=1,2,\ldots}$ on K groups of k [H]. Hence, $K_m(k)_{\mathbb{Q}}$ breaks into a direct sum of the eigenspaces of ψ^j.

$$K_m(k)_{\mathbb{Q}} = \oplus K_m^{(l)}(k),$$

where each $K_m^{(l)}(k)$ is the eigenspace of ψ^j corresponding to eigenvalue j^l (also called the weight space with weight l). One has the so-called Chern character map ch from K-groups to a decent cohomology theory, which in this case we take to be the so-called

Deligne-Beilinson cohomology $H^*_{\mathscr{D}}$ ([Bei], [Gi]). By formal properties of the Chern character, we have the following natural homomorphism

$$\mathrm{ch}: \quad K_m^{(l)}(k) \to H_{\mathscr{D}}^{2l-m}(\mathrm{spec}(k); \mathbb{R}(l)),$$

where $\mathbb{R}(l)$ denotes $(2\pi i)^l \mathbb{R}$. Beilinson proved that the Borel regulators are in fact the Chern character maps (at least up to Q*) from K-groups to the Deligne-Beilinson cohomology. Since

$$H_{\mathscr{D}}^q(\mathrm{spec}(k); \mathbb{R}(l)) \cong \begin{cases} 0, & \text{if } q \neq 1; \\ \mathbb{R}^{r_1+r_2}, & \text{if } q=1, \text{ and } l \text{ is even}; \\ \mathbb{R}^{r_2}, & \text{if } q=1, \text{ and } l \text{ is odd}, \end{cases}$$

where r_1 and r_2 are the number of real and non-conjugate complex embeddings of k respectively. Since Borel regulator maps are injective mod torsion, it follows that $K_{2n-1}(k)$ is pure of weight n. Since the rational γ-filtration can be defined by

$$\gamma^i K_m(k)_{\mathbb{Q}} = \bigoplus_{l \geq i} K_m^{(l)}(k),$$

the above results follow immediately.

4.3. Hence, the rank conjecture of a number field k will follow from the following statement: The image of the natural map

$$H_{2n-1}(GL_m(k); \mathbb{Q}) \to H_{2n-1}(GL(k); \mathbb{Q})$$

contains $PH_{2n-1}(GL(k))$ if $m \geq n$, and the image does not contain any primitive homology class if $m<n$. Hence the rank conjecture for number fields consists of two parts. One part is to prove the surjectivity in certain range (called the *upper* rank conjecture in the sequel), the other part is to prove the triviality in certain range (called *lower* rank conjecture in the sequel). It is well known that for $n>1$, one has

$$PH_n(GL(k), \mathbb{Q}) \cong PH_n(SL(k), \mathbb{Q}).$$

Hence for upper rank conjecture, when $n>1$, it suffices to prove the above statement where $GL_m(k)$ and $GL(k)$ are replaced by $SL_m(k)$ and $SL(k)$ respectively. When $n \leq 1$, the rank conjecture is trivially true. So we now assume that $n>1$.

First, we need to compute I_G^*. We have

$$I_G^* \otimes \mathbb{C} = H^*(\mathfrak{g}, \mathfrak{k}; \mathbb{C}),$$

where $H^*(\mathfrak{g}, \mathfrak{l}; \mathbb{C})$ denotes the Lie algebra cohomology. Lie algebra cohomology is computable using the compact form trick which we now recall briefly.

4.4. Let G be a reductive linear algebraic group defined over $\mathbb{Q}$ and $\mathfrak{g}$ = Lie algebra of $G(\mathbb{R})$. Again we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{k}$ = Lie algebra of a maximal compact subgroup K. Let

$$\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\,\mathfrak{p} \subset \mathfrak{g} \otimes \mathbb{C},$$

then $\mathfrak{g}_u$ is the Lie algebra of a maximal compact subgroup G_u of $G(\mathbb{C})$. Evidently, $\mathfrak{g}_u \otimes \mathbb{C} \cong \mathfrak{g} \otimes \mathbb{C}$. Hence we have

$$H^*(\mathfrak{g}, \mathfrak{k}; \mathbb{C}) \cong H^*(\mathfrak{g} \otimes \mathbb{C}, \mathfrak{k} \otimes \mathbb{C}; \mathbb{C}) \cong H^*(\mathfrak{g}_u \otimes \mathbb{C}, \mathfrak{k} \otimes \mathbb{C}; \mathbb{C}) \cong H^*(\mathfrak{g}_u, \mathfrak{k}; \mathbb{C}) \cong H^*(G_u/K; \mathbb{C}),$$

where the last isomorphism follows from the fact that the cohomology of a compact homogeneous space is isomorphic to the complex of invariant forms.

4.5. Now fix a number field k and let $G_n = R_{k/\mathbb{Q}} SL_n(k)$, where $R_{k/\mathbb{Q}}$ again is the restriction of scalars functor. We have

$$G_n(\mathbb{R}) = SL_n(\mathbb{R})^{r_1} \times SL_n(\mathbb{C})^{r_2}, \qquad G_n(\mathbb{C}) = SL_n(\mathbb{C})^d,$$

where r_1 (resp. r_2) is the number of the real (resp. non-conjugate complex) embeddings of k and $d = [k:\mathbb{Q}] r_1 + 2r_2$. Then

$$K_n = SO(n)^{r_1} \times SU(n)^{r_2}, \qquad G_{n,u} = SU(n)^d,$$

are maximal compact subgroups of $G_n(\mathbb{R})$ and $G_n(\mathbb{C})$ respectively. Let

$$X_{n,u} = K_n \backslash G_{n,u} = (SO(n) \backslash SU(n))^{r_1} \times SU(n)^{r_2}.$$

Then by 4.4, we have

$$I^*_{G_n} \otimes \mathbb{C} \cong H^*(X_{n,u}; \mathbb{C}).$$

The right handside can be readily computed from the following proposition.

4.6. Proposition. — *The rational cohomology of* SU(n) *and* SO$(n)\backslash$SU(n) *are as follows*

$$H^*(SU(n); \mathbb{Q}) \cong E\langle x_3, \ldots, x_{2n-1} \rangle,$$
$$H^*(SO(2n-1) \backslash SU(2n-1); \mathbb{Q}) \cong E\langle x_5, \ldots, x_{4n-3} \rangle,$$
$$H^*(SO(2n) \backslash SU(2n); \mathbb{Q}) \cong E\langle x_5, \ldots, x_{4n-3}, e_{2n} \rangle,$$

where E *stands for the exterior algebra, and* x_i, e_i *denote generators of degree* i. *Furthermore, if one takes* $n \to \infty$, *then each has a natural Hopf algebra structure, and the* x_i*'s are primitive generators in their corresponding Hopf algebras. By slight abuse of language, we will call* x_i*'s primitive, even if we are only talking about finite* n.

The cohomology of SU(n) can be found in a standard topology book, while the cohomology of SO$(n)\backslash$SU(n) is computed in [B1] (2).

(2) I would like to thank Professor Stephen Mitchell for showing me his well-written notes on this subject.

If instead in 4.5, we take $G'_n = R_{k/\mathbb{Q}} GL_n(k)$, then one can compute $I_{G'_n}$ in a way similar to what we did for G_n. We have the following analogue of 4.6 [B1].

4.7. Proposition. — *The rational cohomology of* U(n) *and* $O(n)\backslash$U(n) *are as follows*

$$H^*(U(n); \mathbb{Q}) \cong E\langle x_1, \ldots, x_{2n-1}\rangle,$$
$$H^*(O(2n-1)\backslash U(2n-1); \mathbb{Q}) \cong E\langle x_1, \ldots, x_{4n-3}\rangle,$$
$$H^*(O(2n)\backslash U(2n); \mathbb{Q}) \cong E\langle x_1, \ldots, x_{4n-3}, e_{2n}\rangle,$$

where x_i, e_i are generators of degree i, and they correspond to the x_i, e_i in Proposition 4.6 *under the natural restriction map* (x_1 *is mapped to* 0). *Furthermore, if one takes $n \to \infty$, then each has a natural Hopf algebra structure and the x_i's are primitive generators.*

4.8. We now begin the proof of Theorem B. We will prove for a number field the rank conjecture for real coefficients which is of course equivalent to that for rational coefficients. As we have seen, the rank conjecture for number fields consists of two parts. We first establish the lower rank conjecture which we can do without the hypothesis that $[k:\mathbb{Q}] \geqslant 2$. For this we need a slightly different interpretation of the map

$$j^*: \quad I^*_G \to H^*(\Gamma).$$

By Van Est's theorem,

$$I^*_G \cong H^*_{ct}(G(\mathbb{R})),$$

where H^*_{ct} stands for continuous group cohomology. Via this isomorphism, the map j^* can be interpreted as the composition

$$H^*_{ct}(G(\mathbb{R})) \to H^*(G(\mathbb{R})) \to H^*(\Gamma),$$

where the first map is the "forget the topology" map, and the second map is simply the restriction map. Then, by [B3], Theorem 6.4, we have the following isomorphism (same notations as in 4.5)

$$H^{2n-1}_{ct}(G_N(\mathbb{R})) \cong H^{2n-1}(SL_N(k)) \cong H^{2n-1}(SL(k))$$

for N sufficiently large. Note $SL_n(k)$ is embedded diagonally in

$$G_n(\mathbb{R}) = SL_n(\mathbb{R})^{r_1} \times SL_n(\mathbb{C})^{r_2}$$

via different real and non-conjugate complex embeddings of k. Similarly, we have

$$H^{2n-1}_{ct}(G'_N(\mathbb{R})) \to H^{2n-1}(GL_N(k)) \cong H^{2n-1}(GL(k))$$

for N sufficiently large. Although in general this is not an isomorphism, its image does contain $PH^{2n-1}(GL(k)) \cong PH^{2n-1}(SL(k))$ when $n > 1$, as follows immediately from the commutative diagram

$$\begin{array}{ccccc} H^{2n-1}_{ct}(G'_N(\mathbb{R})) & \to & H^{2n-1}(GL_N(k)) & \cong & H^{2n-1}(GL(k)) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2n-1}_{ct}(G_N(\mathbb{R})) & \cong & H^{2n-1}(SL_N(k)) & \cong & H^{2n-1}(SL(k)) \end{array}$$

where the left hand arrow is an epimorphism because of Prop. 4.7.

We have the following commutative diagram

$$\begin{array}{ccccc} I^{2n-1}_{G_N} & \cong H^{2n-1}_{ct}(G'_N(\mathbb{R})) & \to & H^{2n-1}(GL_N(k)) \\ \downarrow & \downarrow & & \downarrow \\ I^{2n-1}_{G_m} & \cong H^{2n-1}_{ct}(G'_m(\mathbb{R})) & \to & H^{2n-1}(GL_m(k)) \end{array}$$

for $m<N$. It follows immediately from Proposition 4.7 that the *primitive* elements are mapped to 0 under the map r if $m<n$. Here to be completely rigorous, one need to take $N\to\infty$, which is a minor point we ignor. Hence from the above commutative diagram we get that the map

$$PH^{2n-1}(GL(k)) \subsetneq H^{2n-1}(GL(k)) \to H^{2n-1}(GL_m(k))$$

is trivial for $m<n$. The lower rank conjecture follows immediately from the next technical proposition.

Let A be a connected bicommutative (*i.e.* commutative and cocommutative) Hopf algebra with augmentation ideal IA. Denote its space of indecomposibles IA/IA.IA by QA. Then by [MM], Cor. 4.18, the natural map $PA\to QA$ is an isomorphism. In particular, there is a natural projection $A\to PA$. Since the cohomology ring of a connected commutative H-space has a natural bicommutative Hopf algebra structure, the following proposition follows easily.

4.9. Proposition. — *Let $f: X\to Y$ be a continuous map between two path-connected topological spaces, where* Y *is an associative and commutative* H-*space. Consider the map*

$$g^q: \quad PH^q(Y) \subsetneq H^q(Y) \to H^q(X)$$

and its dual map

$$g_q: \quad H_q(X) \to H_q(Y) \xrightarrow{p} PH_q(Y).$$

where p is the natural projection map. Then $g^q=0$ if and only if $g_q=0$; and g^q is injective if and only if g_q is surjective.

4.10. Now let us assume that $d=[k:\mathbb{Q}]\geqslant 3$. We keep the same notations as in 4.5. We have the following commutative diagram

$$\begin{array}{ccc} I^{2n-1}_{G_N} & \cong & H^{2n-1}(\Gamma_N) \\ \downarrow r & & \downarrow \\ I^{2n-1}_{G_m} & \xrightarrow{j^{2n-1}} & H^{2n-1}(\Gamma_m) \end{array}$$

where again $N \geqslant m$, and the arithmetic subgroups Γ_N, Γ_n are chosen so that the diagram commutes. From Proposition 4.6, we see that r is a monomorphism when $m \geqslant n$. Recall the definition of $l(G/k)$. From the well known root system of SL_m, one readily has

$$l(SL_m/k) = m-1.$$

By lemma 3.8, one therefore has $l(G_m/\mathbb{Q}) = d.l(SL_m(k)) \geqslant 3(m-1) \geqslant 2m-1$ for $m \geqslant 2$. In particular, we have $l(G_m/\mathbb{Q}) \geqslant 2n-1$ for all $m \geqslant n$. From Theorem A, it follows that j^{2n-1} is injective when $m \geqslant n$. From the above diagram the following theorem follows.

4.11. THEOREM. — *The natural map*

$$H^{2n-1}(SL_N(k)) \to H^{2n-1}(SL_m(k))$$

is injective for $N \gg 0$, $m \geqslant n$ *where k is a number field such that* $[k:\mathbb{Q}] \geqslant 3$.

The upper rank conjecture for number field k with $[k:\mathbb{Q}] \geqslant 3$ then follows immediately from the dual statement for homology groups. Theorem B is therefore proved for such k.

4.12. The rank conjecture for quadratic fields is more subtle. Let us see how the above argument fails for quadratic fields. Let k be a quadratic field, and let $G_n = R_{k/\mathbb{Q}} SL_n(k)$. We know that the $\mathbb{Q}$-root system of G_n consists of the following roots

$$\pm\lambda_1, \ldots, \pm\lambda_{n-1}, \pm(\lambda_1+\lambda_2), \ldots, \pm(\lambda_{n-2}+\lambda_{n-1}), \ldots, \pm(\lambda_1+\ldots+\lambda_{n-1}).$$

Therefore we have $2\rho = 2(n-1)(\lambda_1+\ldots+\lambda_{n-1})$ + other positive terms not involving λ_1 and λ_{n-1} where $\lambda_1, \ldots, \lambda_{n-1}$ are the simple $\mathbb{Q}$-roots of G_n. In order for j^{2n-1} to be injective, we need

$$2\rho - \alpha(J) \gg 0, \quad \text{for all } J \text{ satisfying } |J| \leqslant 2n-2.$$

But apparently, we may have

$$\alpha(J) = 2(n-1)(\lambda_1+\ldots+\lambda_{n-1}) + \text{other positive terms},$$

for some $|J| = 2n-2$, so we can't prove that j^{2n-1} is injective for G_n. But j^{2n-1} being injective is slightly stronger than the rank conjecture. Indeed, in view of Proposition 4.9, we need only prove that

$$p^{2n-1}: \quad PH^{2n-1}(\Gamma_N) \to H^{2n-1}(\Gamma_N) \to H^{2n-1}(\Gamma_n)$$

is injective. By Proposition 4.6, it follows that the map

$$P_{G_N}^{2n-1} \to P_{G_n}^{2n-1}$$

is an isomorphism, where $P_{G_n}^{2n-1}$ is the subgroup of $I^{2n-1}(G_n)$ consisting of canonical primitive generators (*cf.* 4.6). So by the commutative diagram in 4.10, it suffices to prove that

$$j^{2n-1}: \quad P_{G_n}^{2n-1} \to H^{2n-1}(\Gamma_n)$$

is injective. Now

$$X_n = K_n \backslash G_n(\mathbb{R}) = \begin{cases} (SO_n \backslash SL_n(\mathbb{R}))^2 & \text{if } k \text{ is real quadratic,} \\ SU_n \backslash SL_n(\mathbb{C}) & \text{if } k \text{ is imaginary quadratic.} \end{cases}$$

The point we want to make here is that $P_{G_n}^{2n-1}$ is generated by primitive elements in $SO_n \backslash SL_n(\mathbb{R})$ or $SU_n \backslash SL_n(\mathbb{C})$. In either case, $\alpha(J) \leqslant n(\lambda_1 + \ldots + \lambda_{n-1})$, for any form in $P_{G_n}^{2n-1}$. Hence, by the proof of Theorem A, we know that

$$j^{2n-1}: \quad P_{G_n}^{2n-1} \to IH^{2n-1}(\Gamma_n)$$

is injective. That is, the upper rank conjecture for quadratic fields is also true. Theorem B is proved.

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