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CONVERGENCE OF RIEMANNIAN MANIFOLDS WITH INTEGRAL BOUNDS ON CURVATURE. II

BY DEANE YANG

1. Introduction

In this paper I obtain the existence of harmonic co-ordinates on a geodesic ball of a given radius in a Riemannian manifold satisfying certain assumptions described below. In particular, the results here generalize those obtained by L. Z. Gao [4]. More recently, M. Anderson [1] has also generalized Gao's results.

Both Gao and Anderson proceed by blow-up arguments and use convergence theorems. Although the estimates obtained here are similar to those in [4], the proof given here is more direct and uses the local Ricci flow studied in [11] instead of a convergence theorem.

Combining the harmonic co-ordinates with the results in [11] yields a stronger version of Gao's Lipschitz convergence theorem ([5], [4], [1]). In particular, weaker assumptions on the Ricci curvature are needed in the version here. On the other hand, the approach taken in [1], combined with the isoperimetric inequality proved in [11] and the estimate described in Appendix D, also yields the convergence theorem presented here.

Various pinching theorems can be obtained as corollaries of the convergence theorem. I describe only one here, an almost-Einstein pinching theorem. Unlike pinching theorems obtained by others, it requires no pointwise curvature bound or diameter bound.

The notation in this paper follows [11].

In the appendices I have included standard formulas satisfied by harmonic functions on a Riemannian manifold and various versions of standard L^p estimates for elliptic equations. Most of the elliptic estimates are obtained by Moser iteration, which is convenient because it does not require the use of co-ordinates. These estimates suffice if the dimension of the manifold is 4 or larger. To obtain optimal regularity for the metric in dimension 3, it is necessary to use a little bit of harmonic analysis.

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2. Ideas

Given a Riemannian manifold M, co-ordinates $x^1, \ldots, x^n$ on some open set are called *harmonic co-ordinates* if each x^i is a harmonic function, *i.e.*

$$\Delta_g x^i = 0, \qquad i = 1, \ldots, n.$$

To obtain harmonic co-ordinates, we start by applying results developed in [11]. First, we can obtain a geodesic ball on which there are bounds on the Sobolev constant, on the L^p norm of Ricci curvature, and on the $L^{n/2}$ norm of Riemann curvature. Therefore, we can solve the local Ricci flow with a uniform time estimate. Restricting to a smaller ball, we obtain a 1-parameter family of metrics $g(t)$, $0 \leqq t \leqq T$, solving the Ricci flow.

Bounds on the Sobolev constant and on the sectional curvature then imply a lower bound on the injectivity radius of $g(T)$, on a smaller ball. Restricting to an even smaller ball if necessary, we apply results of Jost-Karcher [7] to obtain harmonic co-ordinates with respect to $g(T)$.

Using the harmonic co-ordinates for $g(T)$ as Dirichlet boundary data, we obtain for each t, n functions that are harmonic with respect to $g(t)$. To show that the functions are co-ordinates, we apply standard elliptic estimates (*see* Appendix) to the derivative with respect to time of the gradients of the functions. The elliptic estimates also yield the expected regularity for the metric tensor written with respect to harmonic co-ordinates.

Using the regularity of the metric in harmonic co-ordinates, it follows that the Riemann curvature tensor is bounded in L^p. Moreover, the Sobolev constant is bounded. The Lipschitz convergence theorem now follows from an L^p convergence theorem proven in [11].

3. Lipschitz convergence theorem

The following theorem has the same assumptions as in Theorem 2.1 in [11], but obtains a stronger conclusion.

THEOREM 3.1. — *Let* $n \geqq 3$, $p > n/2$, *and* $0 < \eta < 1$. *There exist constants* $\varepsilon(n) > 0$ *and* $\kappa(n, p, \eta) > 0$ *such that the following hold:*

Let $M_1, \ldots$ *be a sequence of complete n-dimensional Riemannian manifolds,* $\Omega_i \subset M_i$ *open subsets, and* D, $\rho > 0$, $K \geqq 0$ *constants satisfying the following:*

$$\text{vol}(B(x, \rho)) \geqq \eta^n n^{-1} \omega \rho^n, \qquad x \in \Omega_i \tag{3.2}$$

$$\text{diam}(\Omega_i) < D \tag{3.3}$$

$$\| Rm \|_{n/2, B(x, \rho)} \leqq \varepsilon(n) \eta^{2(n+1)} \tag{3.4}$$

$$\rho^{2-(n/p)} \| Rc \|_{p, \Omega_i} \leqq \kappa(n, p, \eta)^2 \tag{3.5}$$

Given $\varepsilon > 0$, *assume that there is a* $v > 0$ *such that* $\text{vol}(\Omega_{i,\varepsilon}) > v$. *Then there exists a subsequence* $\Omega_{i,\varepsilon}$ *and diffeomorphisms* $\Phi_i : \Omega \to \Omega_{i,\varepsilon}$ *such that the sequence* $\Phi_i^* g_i$ *of Riemannian metrics converge uniformly. Moreover, given any* $x \in \Omega$, *the* Φ_i *can be defined so that there exists a ball* B *containing* x *on which the metrics* $\Phi_i^* g_i$ *converge weakly in* L_2^p.

Proof. — By Theorem 7.1 each manifold M_i can be covered by a fixed number N of balls on which there are harmonic co-ordinates. The estimates for the metric and its derivatives up to second order imply an L^p bound on the Riemann curvature on each ball and therefore on the entire manifold. The local Sobolev constants can also be patched together to yield a global Sobolev inequality. The theorem then follows from Theorem 12.1 of [11]. The L_2^p convergence is obtained by simply defining Φ_i so that there is a fixed set of co-ordinates on B that are harmonic for each $\Phi_i^* g_i$. □

Remark. — The general discussion on Lipschitz convergence of Riemannian manifolds given in the beginning of [2] shows that the diffeomorphisms Φ_i can be chosen so that the metrics $\Phi_i^* g_i$ converge weakly in L_2^p globally on Ω.

Corollary 2.6 of [11] also has a corresponding version here.

Any number of pinching theorems can be inferred from Theorem 3.1. Unlike earlier pinching theorems such as those in [9], no pointwise curvature bound is needed at all. One such corollary is the following:

COROLLARY 3.6. — *Let* $n \geqq 3$, $p > n/2$, $p_0 > 1$, $0 < \eta < 1$, $s > 0$. *Then there exists* $\delta(n, p, p_0, \eta, s) > 0$ *such that any complete, connected n-dimensional Riemannian manifold satisfying for some* $\rho > 0$

$$\text{vol}(B(x, \rho) > \eta^n n^{-1} \omega \rho^n, \qquad x \in \Omega$$

$$\| Rm \|_{n/2, B(x, \rho)} \leqq \varepsilon(n) \eta^{2(n+1)}$$

$$\rho^{2-(n/p)} \| Rc \|_p \leqq \kappa(n, p, \eta)^2$$

$$\rho^{2-(n/p_0)} \left\| Rc - \frac{1}{n} S g \right\|_{p_0} \leqq \delta(n, p, p_0, \eta, s)$$

$$\rho^{2-n} \int_M S \, dV_g > s$$

where $\varepsilon(n)$ *and* $\kappa(n, p, \eta)$ *are as in Theorem* 3.1, *is compact and admits an Einstein metric with positive scalar curvature.*

Sketch of proof. — The proof is by now a standard one. Assume that no such $\delta>0$ exists. Then there exists a sequence of complete Riemannian manifolds M_i satisfying the assumptions above and

$$\rho^{2-n}\left\| \mathrm{Rc}(g_i)-\frac{1}{n}\mathrm{S}(g_i)\,g_i \right\|_{p_0} \to 0$$

but not admitting an Einstein metric with positive scalar curvature. In each manifold M_i fix an exhaustion $\Omega_{i,j}$ of the manifold by bounded open sets such that $\operatorname{diam}(\Omega_{i,j}) \leqq j$. For each j pass to a convergent subsequence, using Theorem 3.1. Now taking the limit of a subsequence of the diagonal sequence, we obtain a manifold with a complete Einstein metric with positive scalar curvature. By Myers' theorem the limiting manifold has bounded diameter. Therefore, for sufficiently large j, Theorem 3.1 implies that some of the $\Omega_{i,j}$ are diffeomorphic to the limiting manifold. This contradicts the definition of the sequence and the non-existence of $\delta>0$. □

We leave as a trivial exercise the statements and proofs of analogous pinching theorems for space forms and almost flat manifolds.

4. Estimates for harmonic co-ordinates on a geodesic ball with bounded curvature

Let $B=B(x_0, \rho)$ be a geodesic ball such that the sectional curvature K is bounded, $|K| \leqq \kappa^2$. Assume that $\rho \leqq \operatorname{inj}(B)$.

We shall fix a constant $\varepsilon(n)>0$ satisfying conditions described in the proof below and assume that $\kappa\rho \leqq \varepsilon(n)$.

First, choose ε so that

$$\frac{1}{2}\theta \leqq \sin\theta \leqq \sinh\theta \leqq 2\theta, \qquad 0 \leqq \theta \leqq \varepsilon.$$

It follows from the Bishop-Gromov volume comparison theorem [3] that

$$2^{-n+1} v_0 r^n \leqq \operatorname{vol}(B(x_0, r)) \leqq 2^{n-1} v_0 r^n,$$

where v_0 is the volume of a flat ball with unit radius.

It also follows that

$$C_s(B) \leqq A(n)$$

for some constant $A(n)>0$.

Let $r=d(x, x_0)$.

Following [8], let u be a unit vector field on B that is parallel along geodesics passing through x_0 and l its associated almost linear function. In other words, if $\gamma : (-\rho, \rho) \to B$

is the unique geodesic passing through x_0 such that $\gamma'(0)=u(x_0)$, then

$$l(x)=\frac{d(x, \gamma(-r(x)))^2-d(x, \gamma(r(x)))^2}{4r(x)}.$$

By Theorem 2.6.1 of [8], there exists a constant $C(n)$ such that the following estimates hold:

$$|\nabla l(x)-u(x)| \leqq C(n)\kappa^2 r^2$$
$$|\nabla^2 l(x)| \leqq C(n)\kappa^2 r$$

Let $\varepsilon<1/(2C(n))$. It follows that

$$|l| \leqq 2r.$$

Let $h(x)$ be the unique solution to

$$\Delta h=0, \qquad h|_{\Delta B}=l.$$

By the Hopf maximum principle [6], h achieves its extreme values on the boundary of B and therefore

$$\|h\|_{B,\infty} \leqq \|l\|_{B,\infty} \leqq 2\rho$$

Next, we want to estimate the gradient of h. First,

$$\Delta(h-l)=-\Delta l, \qquad (h-l)|_{\partial B}=0$$

Integrating by parts,

$$\begin{aligned}\int_B |\nabla(h-l)|^2 &= \int (h-l)\Delta l\\ &=(\|h\|_\infty+\|l\|_\infty)\|\Delta l\|_\infty \operatorname{vol}(B)\\ &\leqq C(n)\kappa^2\rho^{2+n}\end{aligned}$$

Now observe that

$$\Delta[\nabla(h-l)]=\mathrm{Rc}(\nabla(h-l))+\nabla(\Delta l)$$

Applying Theorem C.7 and setting $B'=B(x_0, (1/2)\rho)$,

$$\begin{aligned}\|\nabla(h-l)\|_{B',\infty} &\leqq C(n)[\rho^{-2}+\kappa^2]^{n/4}\|\nabla(h-l)\|_{B,2}+C(n)[\rho^{-2}+\kappa^2]^{-1/2}\|\Delta l\|_{B,\infty}\\ &\leqq C(n)\kappa\rho\end{aligned}$$

It follows that

(4.1) $$\|\nabla h-u\|_{B',\infty} \leqq C(n)\kappa\rho$$

From this we obtain harmonic co-ordinates (compare with Theorem 2.8.1 in [7]):

THEOREM 4.2. — *Let* ρ, $\kappa > 0$, *and* $B = B(x_0, \rho)$ *be a geodesic ball such that* $\rho < \mathrm{inj}(B)$ *and the sectional curvature* K *of* B *is bounded*, $K < \kappa^2$.

Then there exists $\varepsilon(n) > 0$ *such that given* $0 < \delta < 1$, *if*

$$\kappa \rho < \varepsilon(n) \delta$$

then there exist harmonic co-ordinates $h^1, \ldots, h^n$ *on* $B' = B(x_0, (1/2)\rho)$ *such that the metric* $g = g_{ij}\, dh^i\, dh^j$ *and its Christoffel symbols* Γ^i_{jk} *satisfy the following bounds on* B':

$$1 - \delta \leqq [g_{ij}] \leqq 1 + \delta$$
$$\| \Gamma^i_{jk} \|_\infty \leqq C(n) \rho^{-1}$$

Proof. — Let $u^1, \ldots, u^n$ an orthonormal frame of unit vector fields on B that are parallel along radial geodesics; $l^1, \ldots, l^n$ the corresponding almost linear functions; and $h^1, \ldots, h^n$ the corresponding harmonic functions.

The bound on the metric follows directly from setting $\varepsilon(n)$ sufficiently small and using (4.1). To bound the Christoffel symbols, it suffices to bound the Hessian of each harmonic function. First, by integrating by parts, we show that on

$$\int_{B(x_0, (3/4)\rho), 2} |\nabla^2 (h - l)|^2 \leqq C(n) \kappa^2 \rho^n.$$

It follows that

$$\| \nabla^2 h \|_{B(x_0, (3/4)\rho), 2} \leqq C(n) \kappa \rho^{n/2}$$

Now apply Theorem C.7 to (A.4) to obtain the desired estimate. □

5. Bounds on harmonic functions on a geodesic ball with L^p bounds on curvature

Fix $q > n$ and $\geqq 4$. Let $B = B(x_0, \rho)$ be a geodesic ball such that

$$C_s(B) \leqq A(n)$$
$$\alpha(n) r^n \leqq \mathrm{vol}(B(x_0, r)) \leqq \beta(n) r^n, \qquad 0 \leqq r \leqq \rho$$
$$\| \mathrm{Rm} \|_{n/2} \leqq \varepsilon(n, q) A^{-1}$$
$$\| \mathrm{Rc} \|_{q/2} \leqq \sigma^2 \rho^{-2 + (2n/q)}$$

Let h be a harmonic function on B such that the following bounds hold:

$$\| h \|_\infty \leqq 2\rho$$
$$\| \nabla h \|_\infty \leqq C(n, q)$$

Multiply (A.3) by $\chi \nabla h$, where χ is an appropriately chosen cut off function and integrate by parts to obtain

$$\int_{B'} |\nabla^2 h|^2 \leqq \frac{C}{\rho} \int_B |\nabla h| . |\nabla^2 h| + \int_B |Rc| . |\nabla h|^2$$

where $B' = B(x_0, (3/4)\rho)$. Applying Hölder's inequality and the bound for ∇h then yields

$$\int_{B'} |\nabla^2 h|^2 \leqq C \rho^{n-2} (1+\sigma^2)$$

Let

$$q' = \frac{qn}{2n-q}$$

and set $\varepsilon(n, q)$ sufficiently small so that we can apply Theorem C.10 to (A.4) and obtain

$$\|\nabla^2 h\|_{q'} \leqq C(n, q) [\tau^{(n/2q')-(n/4)} \|\nabla^2 h\|_2 + \tau^{(1/2)+(n/2q')-(n/q)} \|Rc\|_{q/2} + \tau^{(1/2)+(n/2q')-(n/4)} \|Rc\|_2]$$

where

$$\tau \leqq C(n, q) \rho^2$$

Therefore,

$$\|\nabla^2 h\|_{q'} \leqq C(n, q) \rho^{(n/q')-1} (1+\sigma^2) \tag{5.1}$$

6. Bounds on the variation of a harmonic function under the Ricci flow

Let $g(t)$, $0 \leqq t \leqq T$ satisfy the Ricci flow on B. Let $h(t)$ solve

$$\Delta_{g(t)} h(t) = 0, \qquad h(t)|_{\partial B} = l,$$

where l is a fixed function independent of t. Let

$$\dot{h} = \frac{\partial h(t)}{\partial t}$$

Then $\dot{h}$ satisfies

$$\Delta \dot{h} = -2 \nabla (Rc * \nabla^2 h), \qquad \dot{h}|_{\partial B} = 0.$$

Therefore, integrating by parts yields

$$\int |\nabla \dot{h}|^2 \leqq \|\nabla \dot{h}\|_2 \|Rc\|_{q/2} \|\nabla h\|_{2q/(q-2)}$$

implying that

$$\| \nabla \dot{h} \|_2 \leqq \kappa_1^2 \rho^{n/2}$$

Now apply Theorem C.10 to (A.6) to obtain the following estimate on $B(x_0, (1/2)\rho)$:

$$\| \nabla \dot{h} \|_\infty \leqq C(n, q)[\tau^{-(n/4)} \| \nabla \dot{h} \|_2 + \| \mathrm{Rc} \|_{qn/(q-n)} \| \nabla^2 h \|_{qn/(2n-q)} (\tau^{(1/2)-(n/2q)} + \tau^{(1/2)-(n/4)} \rho^{(n/2)-(n/q)})]$$

where

$$0 \leqq \tau \leqq C(n, q) \rho^2 [1 + \sigma^{2q/(q-n)}]^{-1}$$

Therefore, on $B(x_0, (1/2)\rho)$

$$(6.1) \quad \| \nabla \dot{h} \|_\infty \leqq C(n, q)[\sigma^2 \rho^{-2} (1 + \sigma^{2q/(q-n)})^{n/4} + \| \mathrm{Rc} \|_{qn/(q-n)} \rho^{-2+(2n/q)} (1+\sigma^2)(1 + (1+\sigma^{2q/(q-n)})^{(n/4)-(n/2q)})]$$

7. Constructing harmonic co-ordinates

THEOREM 7.1. — *Given $q > n$, there exists constants $\varepsilon(n)$, $\sigma(n, q) > 0$ such that if a geodesic ball $B = B(x, 2\rho)$ satisfies*

$$C_s(B) \| \mathrm{Rm} \|_{n/2} \leqq \varepsilon(n)$$
$$\rho^{2-(2n/q)} \| \mathrm{Rc} \|_{q/2} \leqq \sigma(n, q)^2 C_S(B)^{-(n/q)} \delta$$

where $0 < \delta < 1$, then $B(x_0, (1/2)\rho)$ admits harmonic co-ordinates $h^1, \ldots, h^n$ satisfying the following bounds:

$$\| h^i \|_\infty \leqq 2\rho$$
$$\| \nabla h^i . \nabla h^j - \delta^{ij} \|_\infty \leqq \delta$$

Moreover, with respect to these co-ordinates, the metric

$$g = g_{ij}\, dh^i\, dh^j$$

satisfies

$$\| \partial^2 g_{ij} \|_{q/2,\, B(x_0, \rho/4)} \leqq C(n, q) \rho^{-2(1-(n/q))} \| \mathrm{Rc} \|_{q/2,\, B(x_0, \rho/2)}$$

where $\| \partial^2 g_{ij} \|_{p, 2}$ denotes L^p norm of the Hessian of g_{ij} with respect to the standard flat metric defined by the co-ordinates.

Proof. — First, solve the local Ricci flow on $B(x_0, 2\rho)$ so that on $B(x_0, \rho)$ we obtain a smooth family of metrics $g(t)$, $0 \leqq t \leqq T$, that satisfies the standard Ricci Flow

$$\frac{\partial g}{\partial t} = -2\, \mathrm{Rc}(g(t))$$

By Theorem 9.1 of [11], we can take

$$T=\rho^2\, min(1,\, C(n,\, g)\, A^{-(n/(q-n))}(\sigma^2\, \delta)^{-(q/(q-n))})$$

By choosing σ sufficiently small, we can assume that $T=\rho^2$.

Also, by Theorem 9.1 of [11], we find that

$$\| Rm(g(T)) \|_\infty\, \rho^2 \leqq \varepsilon(n)\, C(n)$$

and

$$\frac{1}{2} g_0 \leqq g(t) \leqq 2 g_0, \qquad 0 \leqq t \leqq T$$

It follows that $B(x_0, \rho)$ is contained inside a geodesically convex ball for $g(T)$ and that there exist constants $A(n)$, $\alpha(n)$, $\beta(n)>0$ such that with respect to each metric $g(t)$, $0 \leqq t \leqq T$,

$$C_S(B(x_0, \rho)) \leqq A(n)$$

$$\alpha(n)\, r^n \leqq vol(B(x_0,\, r) \leqq \beta(n)\, r^n, \qquad 0 \leqq r \leqq \rho$$

Also, by Theorem 4.2 there exist harmonic co-ordinates $h^1(T), \ldots, h^n(T)$ on $B(x_0, \rho)$ with respect to the metric $g(T)$.

Now for each $0 \leqq t \leqq T$, let $h^i(t)$, $1 \leqq i \leqq n$, satisfy

$$\Delta_{g(t)}\, h^i(t)=0, \qquad h^i(t)\,|_{\partial B(x_0,\, \rho)}=l^i$$

where l^i is defined with respect to the metric $g(T)$ as described in the proof of Theorem 4.2. We want to show that $h^1(t), \ldots, h^n(t)$ remain co-ordinates for all $0 \leqq t \leqq T$. To do this it suffices to choose δ sufficiently small in Theorem 4.2 and to show that if δ here is sufficiently small, then

$$\int_0^T \left| \frac{\partial}{\partial t} \nabla h^i(t) \right| dt$$

is also small.

On the other hand, note that

$$\frac{\partial}{\partial t} \nabla h^i(t)=\nabla \dot{h}^i,$$

where

$$\dot{h}^i=\frac{\partial h^i}{\partial t}$$

Applying the estimates in [11] yields

$$\| \mathrm{Rc}(g(t)) \|_{qn/(q-n)} \leqq \mathrm{C}(n, q)\, t^{(1/2)-(3n/2q)} \rho^{2n/q} \tau^2 \delta$$

Substiting this into (6.1) yields

$$\| \nabla \dot{h}^i \|_\infty \leqq \mathrm{C}(n, q)\, \tau^2 \delta [\rho^{-2} + \rho^{-3+(3n/q)} t^{(1/2)-(3n/2q)}]$$

Integrating, we get

$$\int_0^{\mathrm{T}} \| \nabla \dot{h}^i \|_\infty \, dt \leqq \mathrm{C}(n, q)\, \tau^2 \delta$$

Therefore, by setting $\tau(n, q) > 0$ sufficiently small, we obtain the desired conclusion.

Remark. – The L^∞ bound for $\nabla \dot{h}$ can still be obtained when $q < 4$ by using L^p bounds on $\mathrm{Rc}(g(t))$ and $\nabla^2 h^i(t)$, $p \geqq 2$, $0 < t < \mathrm{T}$. This leads to the existence of harmonic co-ordinates in dimension 3, even when $q < 4$.

The $\mathrm{L}^{q/2}$ bound on the second derivatives of g, $q > n$, follow by applying Lemma D.1, with

$$u = [g^{ij} - \delta^{ij}]$$

to the equation

$$g^{pq} \partial_p \partial_q g^{ij} = g_{pq} g^{rs} \Gamma^i_{\ pr} \Gamma^j_{\ qs} + 2 \mathrm{R}^{ij}$$

which is a rewritten version of

$$\Delta (\nabla h^i . \nabla h^j) = 2 \nabla^2 h^i . \nabla^2 h^j + 2 \mathrm{Rc}(\nabla h^i, \nabla h^j)$$

In dimensions 4 or larger, these bounds on the metric also follow easily from the L^q bound on $\nabla^2 h^i$, which is obtained from Moser iteration. □

APPENDIX A

BASIC FORMULAS FOR HARMONIC FUNCTIONS AND THEIR VARIATIONS

Given $t \in [0, \mathrm{T}]$, let $g = g(t)$ be a smooth 1-parameter family of Riemannian metrics on a smooth manifold M. Let ∇ denote covariant differentiation with respect to g and $\Delta = -\nabla^* \nabla$ the corresponding Laplace-Beltrami operator.

We shall denote differentiation of a function f with respect to t by $\dot{f}$.

Fix local co-ordinates $x^1, \ldots, x^n$. Then the Christoffel symbols are denoted

$$\Gamma^k_{ij} = \frac{1}{2} g^{kp} (\partial_i g_{pj} + \partial_j g_{ip} - \partial_p g_{ij}).$$

The Riemann curvature tensor will be denoted

$$\mathrm{Rm}(g) = \mathrm{R}_{ijkl}(dx^i \wedge dx^j)(dx^k \wedge dx^l),$$

the Ricci curvature

$$\mathrm{Rc}(g) = \mathrm{R}_{ij}\, dx^i\, dx^j,$$

and the scalar curvature $\mathrm{S} = g^{ij}\mathrm{R}_{ij}$. The Einstein or gravitational tensor will be denoted

$$\mathrm{G}_{ij} = \mathrm{R}_{ij} - \frac{1}{2}\mathrm{S}\, g_{ij}.$$

Raising and lowering indices is always done with respect to the metric. Indices that are being contracted will usually be denoted p or q, while the free indices will usually be i, j, k, l.

LEMMA A.1.

$$\dot{\Gamma}^k_{ij} = \frac{1}{2} g^{kp}(\nabla_i \dot{g}_{pj} + \nabla_j \dot{g}_{ip} - \nabla_p \dot{g}_{ij}).$$

Therefore, if $\dot{g} = -2\,\mathrm{Rc}(g)$,

$$g^{ij}\dot{\Gamma}^k_{ij} = 0.$$

Now we consider harmonic functions:

LEMMA A.2. — *Given* $h : \mathrm{M} \times [0, \mathrm{T}] \to \mathbf{R}$ *such that* $\Delta h = 0$ *for all* $t \in [0, \mathrm{T}]$, h *satisfies the following equations:*

$$\Delta(\nabla_i h) = \mathrm{R}_i{}^p \nabla_p h \tag{A.3}$$

$$\Delta(\nabla_i \nabla_j h) = 2\,\mathrm{R}^p{}_{ij}{}^q \nabla_p \nabla_q h + \nabla_j(\mathrm{R}_{ip}\nabla^p h) + \nabla_i(\mathrm{R}_{jp}\nabla^p h) - \nabla_p(\mathrm{R}_{ij}\nabla^p h) \tag{A.4}$$

The variation of h, $\dot{h}$, *satisfies the following equations:*

$$\Delta \dot{h} = -2\nabla_p(\mathrm{G}^{pq}\nabla_q h) = -2\,\mathrm{R}^{pq}\nabla_p \nabla_q h \tag{A.5}$$

$$\Delta \nabla_i \dot{h} = \mathrm{R}_{ip}\nabla^p \dot{h} - 2\nabla_i(\mathrm{R}^{pq}\nabla_p \nabla_q h) \tag{A.6}$$

APPENDIX B

MOSER ITERATION FOR A HOMOGENEOUS ELLIPTIC DIFFERENTIAL EQUATION

THEOREM B.1. — *Let* M *be an open Riemannian manifold such that*

$$\left(\int_{\mathrm{M}} |f|^{2n/(n-2)}\right)^{(n-2)/n} \leqq \mathrm{A}\int_{\mathrm{M}} |\nabla f|^2,$$

for any $f \to C_0^\infty(M)$.

Let u and b be nonnegative functions on M *satisfying the following elliptic inequality:*

$$-\Delta u \leqq bu.$$

Then given $p_0 > 1$, $q > n$, *and a compactly supported, nonnegative Lipschitz function* χ,

$$|\chi^{n/p_0} u(x)| \leqq C[(A\|\chi^{q-n} b\|_{q/2})_{q/(q-n)} + A\|\nabla\chi\|_\infty^2]^{n/2p_0}\|u\|_{p_0},$$

where C *depends only on* n, p_0 *and* q.

Proof. — Throughout this proof C is a constant that may change from line to line and depends only on n, p_0, and q.

First, integration by parts and the Cauchy-Schwartz inequality leads to the following:

LEMMA B.2. — *Given* $p > 1$, $\varphi \in C_0^\infty(M)$, $f \in C^\infty(M)$, $f \geqq 0$,

$$\int |\nabla(\varphi f^{p/2})|^2 \leqq \frac{p^2}{2(p-1)} \int \varphi^2 f^{p-1}(-\Delta f) + \left[1 + \frac{1}{(p-1)^2}\right] \int |\nabla\varphi|^2 f^p.$$

Given $p' \geqq 0$, $p \geqq p_0$, it then follows that

$$\begin{aligned}\int |\nabla(\chi^{p'+1} u^{p/2})|^2 &\leqq \frac{p^2}{2(p-1)} \int \chi^{2p'+2} bu^p + (p'+1)^2\left[1 + \frac{1}{(p-1)^2}\right] \int |\nabla\chi|^2 \chi^{2p'} u^p \\ &\leqq p^2 C\left(\int \chi^{q-n} b^{q/2}\right)^{2/q} \left(\int \chi^{2p'} u^p\right)^{1-(n/q)} \left(\int (\chi^{2p'+2} u^p)^{n/(n-2)}\right)^{((n-2)/n)(n/q)} \\ &\qquad + (p'+1)^2 C\|\nabla\chi\|_\infty^2 \int \chi^{2p'} u^p \\ &\leqq \varepsilon^{1-(n/q)} A \int |\nabla(\chi^{p'+1} u^{p/2})|^2 + C[\varepsilon^{-(n/q)}(p^2\|\chi^{2(1-(n/q))} b\|_{q/2})^{q/(q-n)} \\ &\qquad + (p'+1)^2\|\nabla\chi\|_\infty^2] \int \chi^{2p'} u^p.\end{aligned}$$

Setting

$$\varepsilon = (2A)^{-q/(q-n)},$$

and applying the Sobolev inequality once more, we obtain

$$\text{(B.3)} \quad \left[\int (\chi^{2(p'+1)} u^p)^{n/(n-2)}\right]^{(n-2)/n} \leqq C[(p^2 A\|\chi^{2(1-(n/q))} b\|_{q/2})^{q/(q-n)} + (p'+1)^2 A\|\nabla\chi\|_\infty^2] \int \chi^{2p'} u^p.$$

Now for each $k \geqq 0$ set

$$p_k = p_0 \left(\frac{n}{n-2}\right)^k;$$

$$p'_k = \sum_{j=1}^{k} \left(\frac{n}{n-2}\right)^j;$$

$$\mathrm{H}_k = \left(\int \chi^{2p'_k} u^{p_k}\right)^{1/p_k}.$$

From (B.3) it follows that

$$\mathrm{H}_{k+1} \leqq [\mathrm{CA}((p_k \| \chi^{2(1-(n/q))} b \|_{q/2})^{q/(q-n)} \mathrm{A}^{n/(q-n)} + (p'_k + 1)^2 \| \nabla \chi \|^2)]^{1/p_{k+1})} \mathrm{H}_k$$

$$\leqq \left(\frac{n}{n-2}\right)^{(2q/(q-n))(k/(p_{k+1}))} [\mathrm{C}((p_0^2 \mathrm{A} \| \chi^{2(1-(n/q))} b \|_{q/2})^{q/(q-n)} + \mathrm{A} \| \nabla \chi \|_\infty^2)]^{1/p_{k+1})} \mathrm{H}_k.$$

Iterating this, we obtain

$$\mathrm{H}_k \leqq \left(\frac{n}{n-2}\right)^{(2q/(q-n))(\sigma'_k/p_0)} [\mathrm{C}((p_0^2 A \| \chi^{2(1-(n/q))} b \|_{q/2})^{q/(q-n)} + \mathrm{A} \| \nabla \chi \|_\infty^2)]^{\sigma_k/p_0} \mathrm{H}_0,$$

where

$$\sigma_k = \sum_{j=1}^{k} \left(\frac{n-2}{n}\right)^j; \qquad \sigma'_k = \sum_{j=1}^{k} j\left(\frac{n-2}{n}\right)^j.$$

Letting $k \to \infty$, we obtain the desired estimate. □

The estimate for the nonlinear elliptic equation is now easily derived:

THEOREM B.4. — *Given the same setup as in Theorem* B.1, *let u be a nonnegative function and $c>0$ a constant satisfying*

$$-\Delta u \leqq c u^2.$$

Then if

$$\| u \|_{n/2} < \frac{2(n-2)}{n^2} (c\mathrm{A})^{-1},$$

there exists a constant $c(n)$ such that if χ is a compact supported, nonnegative Lipschitz function,

$$| \chi(x)^2 u(x) | \leqq c(n) \mathrm{A} \| \nabla \chi \|_\infty^2 \| u \|_{n/2}.$$

Proof. — Applying Lemma B.2,

$$\left(\int (\chi^2 u^{n/2})^{n/(n-2)}\right)^{(n-2)/n} \leqq A\int |\nabla(\chi u^{n/4})|^2$$

$$\leqq \frac{n^2 c A}{4(n-2)}\int \chi^2 u^{(n/2)+1} + A\left[1+\left(\frac{n}{2}-1\right)^{-2}\right]\int |\nabla \chi|^2 u^{n/2}$$

$$\leqq \frac{n^2 c A}{4(n-2)}\|u\|_{n/2}\left(\int (\chi^2 u^{n/2})^{n/(n-2)}\right)^{(n-2)/n} + CA\|\nabla\chi\|_\infty^2 \int u^{n/2}.$$

Therefore,

$$\|\chi^{2n/(n-2)} u\|_{n^2/(2(n-2))} \leqq (CA\|\nabla\chi\|_\infty^2)^{2/n}\|u\|_{n/2}.$$

The theorem now follows by from Theorem C by setting

$$q=\frac{n^2}{n-2}, \qquad p_0=\frac{n}{2}. \quad \square$$

APPENDIX C

MOSER ITERATION FOR AN INHOMOGENEOUS ELLIPTIC EQUATION

Let B be an n-dimensional open Riemannian manifold and $A>0$ be such that

$$\text{(C.1)} \qquad \left(\int |f|^{2n/(n-2)}\right)^{(n-2)/n} \leqq A\int |\nabla f|^2,$$

for any $f \in C_0^\infty(M)$.

We wish to study the following elliptic inequality:

$$-\Delta u \leqq bu + \nabla . f,$$

where u, b are nonnegative functions and f is a vector-valued function.

To simplify matters, we shall assume that $b=0$, so that the equation is

$$-\Delta u \leqq \nabla . f.$$

The estimate for the general equation can then be obtained by combining the estimates obtained in Section B with those given below.

Let $q>2n/(n+2)$, $2\leqq p_0 \leqq p$, $p'\geqq 0$, and $q'>n$. The constant C will depend only on n, q, q', and p_0, and may change from line to line.

We begin with some general estimates. Given $p' \geqq 0$, $p \geqq 2$, apply Lemma B.2, integration by parts, and the Cauchy-Schwartz inequality to obtain

$$\int |\nabla(\chi^{p'+1} u^{p/2})|^2 \leqq \frac{p^2}{2(p-1)} \int \chi^{2p'+2} u^{p-1} \nabla . f + C(p'+1)^2 \int |\nabla \chi|^2 \chi^{2p'} u^p$$

$$= \frac{p(p'+1)}{p-1} \int \chi^{2p'+1} u^{p-1} \nabla \chi . f + p \int \chi^{p'+1} u^{(p/2)-1} \nabla(\chi^{p'+1} u^{p/2}) . f$$

$$+ C(p'+1)^2 \int |\nabla \chi|^2 \chi^{2p'} u^p$$

$$\leqq p^2 \int \chi^{2p'+2} u^{p-2} |f|^2 + \frac{1}{2} \int |\nabla(\chi^{p'+1} u^{p/2})|^2 + C(p'+1)^2 \int |\nabla \chi|^2 \chi^{2p'} u^p$$

It follows that

$$\int |\nabla(\chi^{p'+1} u^{p/2})|^2 \leqq 2p^2 \int \chi^{2p'+2} u^{p-2} |f|^2 + C(p'+1)^2 \int |\nabla \chi|^2 \chi^{2p'} u^p. \tag{C.2}$$

THEOREM C.3. — *Given $p_0 \geqq q > n$ and $q' > n$, there exists a constant $C(n, q, q', p_0)$ such that for any nonnegative compactly supported Lipschitz function χ and*

$$0 < \tau \leqq [\| \nabla \chi \|_\infty^2 + A^{n/(q'-n)} \| \chi^{2(1-(n/q'))} b \|_{q'/2}^{q'/(q'-n)}]^{-1},$$

$$|\chi(x)^{n/p_0} u(x)| \leqq C[(A\tau^{-1})^{n/2p_0} \| u \|_{p_0} + \tau^{1/2} (\tau A)^{n/2q} \| \chi^{1-(n/q)+(n/p_0)} f \|_q]$$

Proof. — Applying Hölder's inequality, (C.1), and the following basic inequality,

$$a^\mu b^{1-\mu} \leqq a + b, \qquad a, b \geqq 0, \quad 0 < \mu < 1,$$

to the first term on the righthand side of (C.2), we obtain

$$2p^2\int\chi^{2p'+2}u^{p-2}f^2\leqq 2\left[\tau^{-1}p^{3(1-(n/q)+((n-2)/p))-1}\int\chi^{2p'}u^p\right]^{1-(n/q)+((n-2)/p)}$$

$$\left[\varepsilon\left(\int(\chi^{2p'+2}u^p)^{n/(n-2)}\right)^{(n-2)/n}\right]^{(n/q)-(n/p)}$$

$$\times\left[\tau^{(p/2)(1-(n/q)+(n/2)-1}\varepsilon^{-(n/2)((p/q)-1)}p^{-(p/2)}\left(\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}\right]^{2/p}$$

$$\leqq 2\left[\tau^{-1}p^{3(1-(n/q)+((n-2)/p))-1}\int\chi^{2p'}u^p\right]^{(p/(p-2))(1-(n/q)+((n-2)/p)}$$

$$\times\left[\varepsilon\left(\int(\chi^{2p'+2}u^p)^{n/(n-2)}\right)^{(n-2)/n}\right]^{(p/(p-2))((n/q)-(n/p))}$$

$$+\tau^{(p/2)(1-(n/q)+(n/2)-1}\varepsilon^{-(n/2)((p/q)-1)}p^{-(p/2)}\left(\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}$$

$$\leqq 2\tau^{-1}p^{3(1-(n/q)+((n-2)/p))-1}\int\chi^{2p'}u^p+\varepsilon A\int|\nabla(\chi^{p'+1}u^{p/2})|^2$$

$$+2\tau^{(p/2)(1-(n/q)+(n/2)-1}\varepsilon^{-(n/2)((p/q)-1)}p^{-(p/2)}\left(\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}.$$

Setting

$$\varepsilon=(4A)^{-1},$$

we get

$$\text{(C.4)}\quad \int|\nabla(\chi^{p'+1}u^{p/2})|^2\leqq C\tau^{-1}p^{3(1-(n/q)+((n-2)/p))-1}+(p'+1)^2\|\nabla\chi\|_\infty^2]\int\chi^{2p'}u^p$$

$$+4\tau^{(p/2)(1-(n/q)+(n/2)-1}p^{-(p/2)}(4A)^{(n/2)((p/q)-1)}\left(\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}.$$

Now let

$$H(p,p')=\int\chi^{2p'}u^p$$

Applying (C.1) one more time and setting

$$\nu=\frac{n}{n-2}$$

we obtain

$$H(p\nu,(p'+1)\nu)^{(n-2)/n}\leqq CA[\tau^{-1}p^{3(1-(n/q)+((n-2)/p))-1}+(p'+1)^2\|\nabla\chi\|_\infty^2]H(p,p')$$

$$+4\tau^{(p/2)(1-(n/q)+(n/2)-1}p^{-(p/2)}(4A)^{(n/2)((p/q)-1)}\left(\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}.$$

Now set

$$p_k = p_0 \nu^k$$

$$p'_k = \sum_{j=1}^{k} \nu^j = \frac{n}{2}[\nu^k - 1]$$

$$\Phi_k = H(p_k, p'_k)^{1/p_k}$$

$$\mu = \frac{q}{q-n}.$$

It follows that

$$\Phi_{k+1} \leqq \nu^{(3\mu/p_0)(k/\nu^k)} C^{1/p_k} A^{1/p_k} [\tau^{-1} + \|\nabla\chi\|_\infty^2]^{1/p_k} \Phi_k + \nu^{-(k/2)} p_0^{1/2} C^{1/p_k} [p_0^{-\mu} \tau (4A)^{-1}]^{(n-2)/2p_k} \tau^{(1/2)(1-(n/q))} (4A)^{n/2q} \|\chi^{(n/p_0)-(n/q)+1} f\|_q.$$

The desired estimate then follows from lemma:

LEMMA C.5. — *Let a_k, b_k, α_k be nonnegative sequences satisfying the following inequality*

$$a_{k+1} \leqq \alpha_k a_k + b_k.$$

Then

$$\lim_{k\to\infty} a_k \leqq a_0 \prod_{k=0}^{\infty} \alpha_k + \sum_{k=0}^{\infty} b_k \prod_{j=k+1}^{\infty} \alpha_j.$$

This lemma is easily proven by induction. □

When $q<n$, we get only an integral bound:

THEOREM C.6. — *Given $2 \leqq q < n$, $q \leqq p_0 \leqq qn/(n-q)$, $q' > n$, and*

$$0 < \tau \leqq [\|\nabla\chi\|_\infty^2] + A^{n/(q'-n)} \|\chi^{2(1-(n/q'))} b\|_{q'/2}^{q'/(q'-n)}]^{-1},$$

there exists a constant $C(n, q, q', p_0) > 0$ such that the following estimate holds:

$$\|\chi^{(n/p_0)+1-(n/q)} u\|_{qn/(n-q)} \leqq C[(A\tau^{-1})^{(n/2)((1/p_0)-(1/q)+(1/n))} \|u\|_{p_0} + A^{1/2} \|\chi^{(n/p_0)-(n/q)+1} f\|_q]$$

Proof. — Let $0 \leqq \delta \leqq 2$. Observe that (C.4) still holds under the assumptions of Theorem C.6. Thus,

$$\left(\int \chi^{2p'+\delta} u^p)^{n/(n-\delta)}\right)^{(n-\delta)/n} \leqq \left((\tau^{-1}A)^{\delta/2} \int \chi^{2p'} u^p\right)^{1-(\delta/2)} \left[(\tau^{-1}A)^{-1+(\delta/2)} \left(\int \chi^{2p'+2} u^p)^{n/(n-2)}\right)^{(n-2)/n}\right]^{\delta/2}$$

$$\leqq 2(\tau^{-1}A)^{\delta/2}\int \chi^{2p'}u^p + \tau^{1-(\delta/2)}A^{\delta/2}\int |\nabla(\chi^{p'+1}u^{p/2})|^2$$

$$\leqq C(\tau^{-1}A)^{\delta/2}[1+\tau\|\nabla\chi\|_\infty^2]\int \chi^{2p'}u^p$$

$$+(\tau^{-1}A)^{-((n-\delta)/2)}\left(\tau^{(1/2)(q-n)}A^{n/2}\left(\int \chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}\right.$$

This time, let

$$\nu=\frac{n}{n-\delta}$$

$$p_k=p_0\nu^k$$

$$p'_k=\frac{\delta}{2}\sum_{j=1}^{k}\nu^j=\frac{n}{2}(\nu^k-1)$$

$$\Phi_k=H(p_k, p'_k)^{1/p_k}$$

Choose δ so that for some positive integer K,

$$p_{K+1}=\frac{qn}{n-q}.$$

The desired estimate then follows by iterating the estimates for Φ_k, $1\leqq k\leqq K+1$. □

Next, we derive the estimates for $p_0<q$:

THEOREM C.7. — *Let* $2\leqq p_0\leqq 2n/(n-2)$ *and* q, $q'>n$. *Then there exists a constant* $C(n, q, q', p_0)$ *such that for any compactly supported Lipschitz function* χ *and*

$$0<\tau\leqq[\|\nabla\chi\|_\infty^2+A^{n/(q'-n)}\|\chi^{2(1-(n/q))}b\|_{q'/2}^{q'/(q'-n)}]^{-1},$$

(C.8) $$|\chi(x)^{n/p_0}u(x)|\leqq C[(\tau^{-1}A)^{n/2p_0}\|u\|_{p_0}$$
$$+\tau^{1/2}(\tau^{-1}A)^{n/2q}\|\chi^{1-(n/q)+(n/p_0)}f\|_q+\tau^{1/2}(\tau^{-1}A)^{n/4}\|\chi^{(n/p_0)-(n/2)+1}f\|_2]$$

THEOREM C.9. — *Let* $2\leqq q<n$. *Then for any* τ *satisfying* (C.8),

$$\|\chi^{(n/p_0)+1-(n/q)}u\|_{qn/(n-q)}\leqq C[(A\tau^{-1})^{(n/2)((1/p_0)-(1/q)+(1/n))}\|u\|_{p_0}$$
$$+A^{1/2}(\|\chi^{(n/p_0)-(n/q)+1}f\|_q+(\tau^{-1}A)^{-(n/2)((1/2)-(1/q))}\|\chi^{(n/p_0)-(n/2)+1}f\|_2)]$$

Proof of both theorems. — We shall do a finite number of iterative estimates to obtain a bound for $\|u\|_q$ in terms of $\|u\|_{p_0}$. Theorem C.7 will then follow from Theorem C.3. The second theorem follows directly from the estimates below.

As before the crucial term is the first one on the righthand side of (C.2). First,

$$p\int \chi^{2p'+2}u^{p-2}f^2\leqq 2\tau^{-1}\int \chi^{2p'}u^p+2\tau^{(p/2)-1}\int \chi^{2p'+p}f^p.$$

Next, we estimate the second term:

$$\tau^{(p/2)-1}\int\chi^{2p'+p}f^p\leqq\left(\tau^{-(1-(2/p))((n/2)-1)}\int\chi^{(2/p)(2p'+n)-n+2}f^p\right)^{(q-p)/(q-2)}$$
$$\times\left(\tau^{(1/2)(q-n)+(q/p)((n/2)-1)}\int\chi^{q-n+(q/p)(2p'+n)}f^q\right)^{(p-2)/(q-2)}$$

Let $0\leqq\delta\leqq 2$. Using the same estimates as in Theorem C.6, we obtain

$$\left(\int\chi^{2p'+\delta}u^p)^{n/(n-\delta)}\right)^{(n-\delta)/n}\leqq C(\tau^{-1}A)^{\delta/2}\int\chi^{2p'}u^p$$
$$+(\tau^{-1}A)^{-((n-\delta)/2)}\left(\tau^{(q-n)/2}A^{n/2}\int\chi^{q-n+q((2p'+n)/p)}f^q\right)^{p/q}$$
$$+(\tau^{-1}A)^{-((n-\delta)/2)}\left(\tau^{1-(n/2)}A^{n/2}\int\chi^{2-n+2((2p'+n)/p)}f^2\right)^{p/2}$$

Define ν, p_k, p'_k, Φ_k as in Theorem C.6. Then iterate the estimates for Φ_k until $p_k>q$. Then apply Theorem C.3 to obtain Theorem C.7 and apply Theorem C.6 to obtain the other theorem. □

Using the same arguments as above, we also have the following:

THEOREM C.10. — *Let* $2\leqq p_0\leqq 2n/(n-2)$ *and* $2\leqq q<n$. *Assume that*

$$\|b\|_{n/2}<\left(\frac{1}{q}-\frac{1}{n}\right)\left(1-\frac{1}{q}+\frac{1}{n}\right)A^{-1}.$$

Then given any

$$0\leqq\tau\leqq C(n,q)\rho^2$$

u satisfies the estimate given in Theorem C.9.

APPENDIX D

REGULARITY FOR A NONLINEAR ELLIPTIC PDE

Throught this section B denotes on open ball in $\mathbf{R}^n$, ∂ represents partial differentiation with respect to the standard co-ordinates $x^1, \ldots, x^n$, all norms use the standard inner product and volume form on $\mathbf{R}^n$. Let Δ_0 denote the flat Laplacian.

The following is a standard result in nonlinear elliptic PDE:

LEMMA D.1. — *Let u and f be smooth functions on* B *satisfying*

$$-a^{ij}\partial_i\partial_j u=b^{kl}\partial_k u\,\partial_l u+f$$

where

$$\| a_{ij} - \delta_{ij} \|_\infty < \frac{\varepsilon}{n}, \qquad 1 \leqq i, j \leqq n$$

Then given $p > 1$, *there exists* $\delta(n, p, \varepsilon) > 0$ *and* $C(n, p, \varepsilon) > 0$ *such that if*

$$\| b \|_\infty \| u \|_\infty < \delta(n, p, \varepsilon)$$

then given any smooth compactly supported function χ,

$$\| \chi^2 \partial^2 u \|_p \leqq C(n, p, \varepsilon) [(\| \partial\chi \|_{2p}^2 + \| \partial^2 (\chi^2) \|_p) \| u \|_\infty + \| \chi^2 f \|_p]$$

Proof. — First, by the Calderon-Zygmund inequality ([6], [10])

$$\| \partial^2 (\chi^2 u) \|_p \leqq C(n) \| \Delta_0 (\chi^2 u) \|_p$$

We also need the following interpolation inequality:

LEMMA D.2. — *Given* $p > 1$ *there exists a constant* $C(p) > 0$ *such that for any smooth function u and smooth, compactly supported function* χ,

$$\| \chi \partial u \|_{2p}^2 \leqq C(p) \| u \|_\infty [\| \chi^2 \partial^2 u \|_p + \| (\partial\chi)^2 u \|_p]$$

In particular, given $\tau > 0$,

$$\| \chi \partial u \|_{2p} \leqq C(p) [\tau \| \chi^2 \partial^2 u \|_p + (\tau^{-1} + \tau \| \partial\chi \|_{2p}^2) \| u \|_\infty]$$

The lemma is proved by integrating by parts and applying Hölder's inequality. Therefore,

$$\begin{aligned}
\| \chi^2 \partial^2 u \|_p &\leqq \| \partial^2 (\chi^2 u) \|_p + 4 \| \partial\chi \|_{2p} \| \chi \partial u \|_{2p} + \| \partial^2 (\chi^2) \|_p \| u \|_\infty \\
&\leqq C(n) \| \Delta_0 (\chi^2 u) \|_p + 4 \| \partial \chi \|_{2p} \| \chi \partial u \|_{2p} + \| \partial^2 (\chi^2) \|_p \| u \|_\infty \\
&\leqq C(n, p) [\| \partial\chi \|_{2p} \| \chi \partial u \|_{2p} + \varepsilon \| \chi^2 \partial^2 u \|_p + \| b \|_\infty \| \chi \partial u \|_{2p}^2 + \| \chi^2 f \|_p] \\
&\leqq C(n, p) [(\tau \| \partial\chi \|_{2p} + \varepsilon + \| b \|_\infty \| u \|_\infty) \| \chi^2 \partial^2 u \|_p \\
&\quad + (\tau^{-1} + \tau \| \partial\chi \|_{2p}^2 + \| b \|_\infty \| u \|_\infty \| \partial\chi \|_{2p}^2 + \| \partial^2 (\chi^2) \|_p) \| u \|_\infty + \| \chi^2 f \|_p]
\end{aligned}$$

Choosing $\tau \| \partial\chi \|_{2p} > 0$ sufficiently small yields the desired conclusion. □

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