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ANNIHILATORS AND ASSOCIATED VARIETIES
OF UNITARY HIGHEST WEIGHT MODULES

BY ANTHONY JOSEPH

1. Introduction

1.1. This paper is a sequel to [10], hereafter referred to as EJ. We shall adopt the same notation, which will nevertheless be redefined unless it is completely standard. Let \( g \) be a complex simple Lie algebra, with \( g = n^+ \oplus h \oplus n \) a triangular decomposition in the sense of [7], 1.10.14.

Fix a non-compact real form \( g_0 \) of \( g \). The classification of unitary highest weight modules has been studied by several authors (see EJ, Introduction) and was in particular completed in [9] and in [14]. In EJ we cast this into a new and quite intrinsic form simplifying both the formulae and calculations involved. Let us recall briefly some details of the classification. First one may assume that the reductive subalgebra \( \mathfrak{f} \) corresponding to a maximal compact subalgebra \( g^* \) of \( g \) is the Levi factor of a maximal parabolic subalgebra \( p^+ \) of \( g \) whose nilradical \( m^+ \) is commutative.

Let \( \alpha \) denote the simple (non-compact) root not occurring in \( \mathfrak{f} \) and \( \omega \) the corresponding fundamental weight. Let \( P^+ \) denote the set of \( \mathfrak{f} \)-dominant integral weights. For each \( \tau \in P^+ \) and each \( u \in \mathbb{R} \) we let \( V(\tau) \otimes C_u^\omega \) denote the simple finite dimensional \( p^+ \) module with highest weight \( \tau + u\omega \) and \( N(\tau + u\omega) \) the corresponding induced \( g \) module. All unitary highest weight modules occur as the unique simple quotient \( L(\tau + u\omega) \) of some \( N(\tau + u\omega) \). Indeed let \( s \) denote the level of \( \tau \) (EJ, 1.6). Then there exist real parameters \( u^*_1 < u^*_2 < \ldots < u^*_s \) such that \( N(\tau + u\omega) \) is simple and unitary if and only if \( u^*_i \leq u^*_j \), the \( L(\tau + u^*_i \omega) \), \( i = 1, 2, \ldots, s \) are unitary and this list exhausts all unitary highest weight modules. The parameters \( u^*_i \) were given in [9] but can also be derived from [14]. Here we shall use the formula (EJ, 4.2) which is both simple and intrinsic. For \( s \geq 2 \) one has that \( u^*_i - u^*_i - 1 = \varepsilon_{\mathfrak{g}, s} \omega, \forall i = 1, 2, \ldots, s - 1 \). Remarkably \( \varepsilon_{\mathfrak{g}, s} \) is independent of \( \tau \) and \( \omega \). We call this the equal spacing rule. We also use a result of M. G. Davidson, T. J. Enright and R. J. Stanke ([6], Thm. 3.1) which asserts in particular that the maximal submodule \( N(\lambda) \) of \( N(\lambda) \) is generated by a highest weight vector.

1.2. Now take \( \tau = 0 \) in 1.1 and denote \( u^0 \) simply by \( u \). One has \( u_i = -(i - 1) \varepsilon_{\mathfrak{g}, s} \) in the notation of EJ, 3.6. Let \( \mathcal{V}_i \) denote the associated variety of \( L(u_i \omega) \). These are interesting subvarieties of \( m^+ \), singular for \( i > 1 \). Set \( J_i = \text{Ann}_{u_i \omega} L(u_i \omega) \). Continuing
the work of T. Levasseur, S. P. Smith and J. T. Stafford [25] who studied the case \( i = 2 \), T. Levasseur and J. T. Stafford [26] showed for \( \mathfrak{g} \) classical that \( J_i \) is always a maximal ideal and remarkably that \( \mathcal{U}(\mathfrak{g})/J_i \) identifies with the ring \( \mathcal{D}_i \) of differential operators on \( \mathcal{V}_i \). This was important as it meant one could say rather a lot about \( \mathcal{D}_i \), a situation which is remarkable considering that \( \mathcal{V}_i \) is singular. A difficulty in the work of Levasseur-Stafford is that it involved rather long case by case analysis using in particular Howe theory. Here following mainly [18] and the analysis in EJ we shall give a short intrinsic proof of their results which furthermore applies to arbitrary \( \mathfrak{g} \) simple (Theorems 4.2, 4.5).

We remark that the varieties \( \mathcal{V}_i \) occurred earlier in the work of M. Harris and H. P. Jakobsen [12]. They describe a constant coefficient differential operator on \( m^+ \) and use it to construct the unitary highest weight modules in the case \( \tau = 0 \) ([12], Sect. 3). This may be viewed as giving the space of regular functions on \( \mathcal{V}_i \) a \( \mathcal{U}(\mathfrak{g}) \) module structure (which is furthermore a unitary highest weight module). This was a first step in [26] whose authors were unaware of this connection with unitary. However the two main problems in [26] mentioned above were not considered in [12].

1.3. The second aim of our work concerns the associated variety \( \mathcal{V}_i^\tau \) of an arbitrary unitary highest weight module \( L(\tau + u^i\omega) \) which is not induced. Assume again for the moment that \( \mathfrak{g} \) is classical. T. J. Enright pointed out to me the following remarkable result obtained in [6], Sect. 7. Let \( m \) be the subalgebra of \( \mathfrak{g} \) opposed to \( m^+ \). For any such unitary module \( L \) and any \( 0 \neq f \in L \) the ideal \( \text{Ann}_\mathcal{U}(m) f \) in the (commutative) ring \( \mathcal{U}(m) \) is prime! Although this is also true for the induced module \( N(\lambda) \) it is almost never true for any non-trivial simple quotient \( L(\lambda) \) of \( N(\lambda) \). Indeed setting \( J_i = \text{Ann}_\mathcal{U}(\mathfrak{g}) L(\tau + u^i\omega) \) the above property implies [Lemma 6.5 (iii)] that the Goldie rank \( \text{rk}(\mathcal{U}(\mathfrak{g})/J_i) \) of the quotient ring is bounded by \( \dim V(\tau) \). Recalling that for a finite dimensional simple module \( L \) one has \( \text{rk}(\mathcal{U}(\mathfrak{g})/\text{Ann} L) = \dim L \), one sees that this result never holds when \( \tau \neq 0 \) and \( u \) is chosen so that \( L(\tau + u\omega) \) is finite dimensional. This is consistent with the classical fact that a non-compact real semi-simple Lie group with trivial centre admits no non-trivial finite dimensional unitary representations.

The above result of M. G. Davidson, T. J. Enright and R. J. Stanke is obtained by a quite complicated procedure involving Howe theory and the construction of harmonic polynomials. Here we give a simple intrinsic proof (Theorem 5.16). This not only extends the result to arbitrary \( \mathfrak{g} \) but also gives a quite explicit method for determining \( \mathcal{V}_i^\tau \). In more detail, let \( t \) denote the level of the zero weight (EJ, 1.4, 1.6). This is always an upper bound on the level of any other \( \ell \) dominant weight \( \tau \). By convention we define \( \mathcal{V}_j^\tau = m^+ \) for \( j > t \). Fix \( \tau \). Then there exists \( j \in \mathbb{N}_+ \) such that \( \mathcal{V}_i^\tau = \mathcal{V}_j^\tau \). We first show (Theorem 2.5) that \( \mathcal{V}_i^\tau = \mathcal{V}_{j+i-1}^\tau \) for all \( i = 1, 2, \ldots, s \). This is a rather easy consequence of the Jakobsen-Vergne tensor product construction in [15]. In type \( A_n \), a comparison result of a similar nature but concerning annihilation by constant coefficient differential operators, can be found in [13], Introduction and Corollary 3.6. Secondly in Section 7 we explicitly compute \( \mathcal{V}_i^\tau \) for each \( \tau \). The latter depends in a quite complicated way on \( \tau \). For example we had first guessed that \( \mathcal{V}_i^\tau = \mathcal{V}_t^\tau \), but this fails badly. One has \( \mathcal{V}_i^\tau = \mathcal{V}_{l(\tau)}^\tau \) where \( l(\tau) \) is given in the Table. In type \( A_n \), we may view \( l(\tau) \) as the length of the support of \( \tau + \omega \).
1.4. Set \( Q_i = \text{Ann}_{U(m)} L(u_i \omega) \) which is a prime ideal of \( U(m) \). The result described in 1.3 can be expressed (see 2.4) as saying that for each pair \( \tau, j \) there exists \( i \in \{1, 2, \ldots, t+1\} \) such that \( L(\tau + u_j \omega) \) is a torsion-free \( U(m)/Q_i \) module. One can ask if the only non-trivial simple quotients of \( N(\tau + u_0 \omega) \) satisfying the above condition are the unitary ones. \textit{A priori} this would seem rather optimistic. However the above inequality condition on Goldie rank shows that it is generically true. This is because by [21], 5.1, the degree of the Goldie rank polynomial defined by the coherent family attached to \( L(\tau + u_0 \omega) \) strictly exceeds the degree of Goldie rank polynomial defining \( V(\tau) \) – the latter being the product of the compact positive roots. It is hence quite accidental that the higher degree polynomial takes a smaller value as in the case of the unitary quotients. Besides it would be rather exciting to have a Goldie rank criterion for unitary; but the naive inequality fails (8.3). It is also perhaps interesting to recall that despite his initial scepticism to the idea, D. A. Vogan ([29], Prop. 7.12) actually proved a result in this direction for complex groups.

1.5. Assume \( \text{Ann}_{U(m)} L(u \omega) \neq 0 \). In section 6 we give a necessary (Theorem 6.8) condition for \( J_j \) to be maximal. In section 8 we give several examples of non-maximal annihilators including in type \( D_2 \), \( l \geq 2 \) an ideal \( l-1 \) steps from being maximal. Given the truth of a certain simplicity conjecture (6.11) we also derive a sufficient condition (Proposition 6.14) for \( J_j \) to be maximal. Unfortunately this is not quite the converse to Theorem 6.8.

1.6. It could be a rather difficult matter to find an example for which \( \text{rk} U(\mathfrak{g})/J_j = 1 \), yet \( \tau \neq 0 \). Fortunately we found quite accidentally examples in type \( D_{2l+1} \) (8.9).

Acknowledgements

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2. Primeness and a tensor product reduction

2.1. Define \( Q_r, r = 1, 2, \ldots, t \) as in 1.4, equivalently as in EJ, 8.1 and set \( Q_r = \{ 0 \} \), \( r \leq t+1 \). Correspondingly (see 1.2) we set \( u_r = -(r-1) e_{\theta, a}, \) \( r \in \mathbb{N}^+ \). Our immediate aim is to prove that \( Q_r \) is a prime ideal of \( S(m) \). This is essentially well-known; but the usual proofs involve case by case analysis (cf. [12], Sect. 4; [25], Chap. II). Here we give an intrinsic proof based on the following easy and perhaps known lemma.
2.2. Define the sequence $\beta_1, \beta_2, \ldots, \beta_t$ of strongly orthogonal positive non-compact roots as in EJ, 1.4, and recalling (EJ, 2.1) set

$$\mu_i = \sum_{j=1}^{i} \beta_j.$$

**Lemma.** — Let $-\nu$ be a weight of $S(m)$. The equation $k\mu_j = \nu + \mu_i, k \in \mathbb{N}$ has no solution for $j < i$.

We must obviously have $k \geq 1$. Now $\nu$ is a sum of positive non-compact roots and we can assume of these exactly $l_s$ be in $\Gamma_n \setminus \{\beta_s\}, s = 1, 2, \ldots, t$ (notation 3.2). Assume $j < i$. Cancelling off the $\beta_s, s \leq j$ occurring in both sides of the above equation we can write

$$\sum_{s=1}^{j} k_s \beta_s = \nu + \beta_{j+1} + \ldots + \beta_i, \quad k_s \in \mathbb{Z}.$$  

Equating coefficients of the non-compact simple root $\alpha$ it follows from (*) that

$$\sum_{s=1}^{j} k_s \geq (i-j) + \sum_{s=1}^{j} l_s > \sum_{s=1}^{j} l_s.$$

Take $\gamma \in \Gamma_n \setminus \{\beta_s\}$. Then $(\beta_r, \gamma) = 0$ for $r < s$ whilst $(\beta_s, \gamma) = (1/2)(\beta_s, \beta_s)$ by [18], 2.2(iv). Finally suppose $r > s$. If $(\beta_r, \gamma) \geq 0$, then $\gamma - \beta_r \in \Gamma_n$ and so $\gamma - \beta_r - \beta_r$ cannot be a root for $r' > s$. Hence there is at most one $r > s$ such that $(\beta_r, \gamma) > 0$ and since $\gamma + \beta_r$ is not a root we further have $(\beta_r, \gamma) = (1/2) (\beta_r, \beta_r)$. Let $l_{r,s}, r > s$ denote the number of $\gamma \in \Gamma_n$ occurring in $\nu$ for which $(\beta_r, \gamma) > 0$. By the above $\sum_{r} l_{r,s} \leq l_s$. Then by (*) for all $r \leq j$ we obtain

$$2k_r = (\beta_r, \nu) \leq l_r + \sum_{s \geq r} l_{r,s}.$$

Summing over $r \leq j$ gives

$$\sum_{r=1}^{j} k_r \leq \sum_{r=1}^{j} l_r,$$

in contradiction to (**). This proves the lemma.

2.3. Recall (EJ, 8.1) that $Q_j \subset Q_i$ for $j \geq i$. For $j > i$ this inclusion is obviously strict. Furthermore this extends to the case $j = t + 1$. Let $\text{Spec}_t S(m)$ denote the set of $t$ stable prime ideals of $S(m)$.

**Proposition.** — $\text{Spec}_t S(m) = \{Q_i\}_{i=t+1}^{t+1}.$

Obviously each $Q_i$ is a $t$ stable ideal of $S(m)$. Conversely let $Q$ be a non-zero $t$ stable prime ideal of $S(m)$. By commutativity of $m$ and finiteness of $t$ action $Q^n = Q^n \neq 0,$
where \( n_\iota = n \cap \Gamma \). By semisimplicity of \( \mathfrak{h} \) action there exists \( \mu \in \mathfrak{h}^* \) such that \( Q^\mu = 0 \).

Let \( v_\jmath \) denote (EJ. 2.1) the unique up to scalars vector of weight \( -\mu_i \) in \( S(\mathfrak{m})^\mathfrak{h} \). Then by EJ. 2.1 (iv), one has \( \mu = \sum k_i \mu_i, k_i \in \mathbb{N} \) and up to a scalar a non-zero vector in \( Q^\mu = 0 \) is the product of the \( v_\jmath^i \). Certainly \( \mu \neq 0 \) for otherwise \( Q = S(\mathfrak{m}) \). Hence \( v_\jmath \in Q \) for some \( \jmath \). If \( i \) is the least integer with this property it follows by EJ. 8.1, that \( Q = Q_i \).

It remains to show that all the \( Q_i \) are prime ideals. This is proved by induction on \( i \). Since \( V_1 = \mathfrak{m} \) it follows that \( Q_1 \) is the augmentation ideal of \( S(\mathfrak{m}) \) and so it holds for \( i = 1 \). Suppose we have shown that \( Q_1, Q_2, \ldots, Q_{i-1} \) are prime and consider \( Q_i \). The radical \( \sqrt{Q_i} \) of \( Q_i \) contains \( Q_i \) for \( l \geq i \) and is an intersection of \( \mathfrak{f} \) stable prime ideals of \( S(\mathfrak{m}) \). By EJ. 8.1, we conclude that \( \sqrt{Q_i} = Q_j \) for some \( j \leq i \).

If \( j = i \) we are done. Otherwise \( j < i \) and there exists a positive integer \( k \) such that \( v_\jmath^k \in Q_i^\mu \).

Now consider a non-zero weight vector \( a \in Q_i^\mu = (V(\mathfrak{m}) S(\mathfrak{m}))^\mu \). We can write
\[
dim V_i = \sum_{r=1} \dim V_i^r \cdot c_r \]
with \( b_i \in V_i^r, c_r \in S(\mathfrak{m}) \) being weight vectors. We can assume the indexing to be chosen so that \( b_1 \) is of lowest weight amongst the \( b_i \) (and that \( b_1 c_r \neq 0 \)). Then \( [(\text{ad } x) b_1] c_r = 0 \) for all \( x \in \mathfrak{m} \) since \( a \) is \( \mathfrak{m} \) invariant. Since \( S(\mathfrak{m}) \) is an integral domain, it follows that \( b_1 \in V_i^r = C v_\jmath \). Taking \( a = v_\jmath^k \) and letting \( -v \) denote the weight of \( c_1 \), we conclude that
\[
k \mu_j = \mu_i + v.
\]
By 8.2 this equation has no solution. This contradiction proves the proposition.

**Remark.** — Let \( K \) denote the connected algebraic subgroup of \( \text{GL}(\mathfrak{m}) \) with Lie algebra \( \mathfrak{f} \). Let \( \mathcal{V} \) denote the closure of a \( K \) orbit in \( \mathfrak{m} \) and \( Q \) its ideal of definition. Obviously \( Q \in \text{Spec S}(\mathfrak{m}) \) and so \( \mathcal{V} = \mathcal{V}_i \) for some \( i \). Thus there are finitely many \( K \) orbits in \( \mathfrak{m} \) and by the irreducibility of the \( \mathcal{V}_i \) each of the latter is the closure of a \( K \) orbit. Notice that we can also deduce Proposition 2.3 if we can show that the number of \( K \) orbits in \( \mathfrak{m} \) is at most \( t+1 \). All this is well-known; but we point it out anyway.

2.4. Fix \( \tau \in \mathbb{P}_+ \) of level \( s \) and \( i = \{ 1, 2, \ldots, s \} \). Set \( \lambda = \tau + \mathfrak{u} \{ 0 \} \) and identify \( V(\lambda) := V(\tau) \otimes C_{\mathfrak{u} \{ 0 \}} \) with its image in the quotient \( L(\lambda) \) of the induced module \( N(\lambda) := U(\mathfrak{g}) \otimes U(\mathfrak{u} \{ 0 \}) V(\lambda) \). It then makes sense to consider \( \text{Ann}_{U(\mathfrak{m})} V(\lambda) \) and this identifies with a \( \mathfrak{f} \) stable ideal of \( S(\mathfrak{m}) \). We shall eventually prove the remarkable fact that this ideal is prime and hence by 2.3 one of the \( Q_i \). For the moment observe the

**Lemma.** — Assume \( \text{Ann}_{U(\mathfrak{m})} V(\lambda) = Q_j \) for some \( j \in \{ 1, 2, \ldots, t+1 \} \). Then for each \( 0 \neq f \in L(\lambda) \) one has \( \text{Ann}_{U(\mathfrak{m})} f = Q_j \). Equivalently \( L(\lambda) \) is a torsion-free \( U(\mathfrak{m})/Q_j \) module.

Since \( \mathfrak{m} \) is commutative and \( U(\mathfrak{m}) V(\lambda) = L(\lambda) \) we obtain the inclusion \( Q := \text{Ann}_{U(\mathfrak{m})} f \supseteq Q_j \). Suppose this inclusion is strict. Since \( \mathfrak{p}^+ \) acts finitely on \( f \) we have for the canonical filtration of \( U(\mathfrak{g}) \) that \( \mathfrak{p}^+ \subset \text{gr Ann}_{U(\mathfrak{g})} f \). It follows that the
associated variety of \( U(g) \) identifies with the subvariety \( \mathcal{V}(Q) \) of \( m^+ \) of zeros of \( Q \). On the other hand the associated variety of \( L(\lambda) \) is just \( \mathcal{V}(Q_j) \). Since \( Q_j \) is prime, we have a strict inclusion \( \mathcal{V}(Q) \not\supset \mathcal{V}(Q_j) \). Yet \( L(\lambda) \) is simple, so \( U(g) f = L(\lambda) \) and the resulting contradiction proves the lemma.

2.5. We now reduce primeness to the case \( i=1 \), referred to generally as the last place of unitary. For this we use the tensor product construction introduced by H. P. Jakobsen and M. Vergne, [15]. Set \( \lambda_i^o = \tau + u_i^o \omega, \xi_i^o = u_i \omega \). Recall that \( \xi_i^o = (i-1) e_{q_i-1} \omega, \forall i \leq t \) (EJ, 4.3). By the equal spacing principle we have \( \lambda_i^o + \xi_i^o = \lambda_i^o + \xi_i^o \).

**Theorem.** — Fix \( \tau \in P^+ \) of level \( s \). Suppose \( \text{Ann}_{U(m)} V(\lambda_i^o) = Q_j \) for some \( j \in \mathbb{N} \). Then \( \text{Ann}_{U(m)} V(\lambda_i^o) = Q_{j+i-1}, \forall i \in \{1, 2, \ldots, s\} \).

The positive definite forms on \( L(\lambda_i^o) \) and \( L(\xi_i^o) \) give a positive definite product form on \( L(\lambda_i^o) \otimes L(\xi_i^o) \). Let \( e_1^o \) (resp. \( f_1^o \)) denote the canonical generator of \( L(\lambda_i^o) \) [resp. \( L(\xi_i^o) \)]. The restriction of a positive definite form to a submodule is again positive definite and so we conclude that the \( U(g) \) submodule of \( L(\lambda_i^o) \otimes L(\xi_i^o) \) generated by \( e_1^o \otimes f_1^o \) is unitary. Since it is a highest weight module of highest weight \( \lambda_i^o \) we conclude that it identifies with \( L(\lambda_i^o) \). Taking account of the \( p^+ \) and \( \mathfrak{f} \) actions we conclude [noting \( V(\xi_i^o) = \mathbb{C} f_1^o \)] that

\[
U(m) (V(\lambda_i^o) \otimes V(\xi_i^o)) = L(\lambda_i^o)
\]

for this identification. Moreover \( V(\lambda_i^o) \otimes V(\xi_i^o) \) identifies with \( V(\lambda_i^o) \).

Now take \( v' \in V(\lambda_i^o) \). We can write \( v' = v \otimes f_1^o \) for some \( v \in V(\lambda_i^o) \). By the hypothesis and 2.4 we have \( \text{Ann}_{U(m)} v = Q_p \), whilst \( \text{Ann}_{U(m)} f_1^o = Q_t \). We claim that this implies that \( \text{Ann}_{U(m)} v' = Q_{i+j-1} \). Normally such a result would be very difficult to prove as it involves analysis of a diagonal action of \( m \). However here we can obtain the result by applying the tensor product argument to the case \( \tau=0 \). Indeed the latter implies that \( L(\xi_i^o) \) is just the submodule of \( L(\xi_i^o) \otimes L(\xi_i^o) \) generated over \( U(m) \) by \( f_1^o \otimes f_1^o \). It follows that \( \text{Ann}_{U(m)} f_1^o \otimes f_1^o \) for the diagonal action of \( U(m) \), which is what we want to compute, is just \( \text{Ann}_{U(m)} f_1^o \otimes f_1^o = Q_{i+j-1} \) as required. Note that there it does not matter if \( i+j-1 \) exceeds \( t \). This is because the module \( L(\xi_i^o) \) \( \xi_r = (r-1) e_{q_i-1} \omega \) is still unitary for \( r > t \) and moreover in that case is just the induced module \( N(\xi_i^o) \) which is a free \( U(m) \) module. This proves the claim which in turn implies the assertion of the theorem.

3. Reduction to smaller rank

3.1. The proof of the main results described in the introduction obtains via a reduction technique introduced in [18], Sect. 4. The method is quite elementary, the key point being to realize the generators \( v_i \) of the \( \mathfrak{f} \) stable prime ideals of \( S(m) \) as related to the lowest weight vectors of simple Lie subalgebras of a localization of \( U(g) \). Unfortunately this is somewhat obscured by the complicated notation and induction technique that we have to introduce.
3.2. Let us recall the notation of EJ, 1.3, 1.4. Let $\Delta \subset \mathfrak{b}^*$ denote the set of non-zero roots, $\Delta^+$ (resp. $\Delta^-$) the set of positive (resp. negative) roots corresponding to the triangular decomposition of $\mathfrak{g}$ introduced in 1.1. We define subsets $\Gamma^i \subset \Delta^-$, $\Delta^i \subset \Delta$, $i=1, 2, \ldots, t$, inductively as follows. Set $\Delta^1 = \Delta$. Assume $\Delta^i$ is defined and is a simple root system. Then $\{ \gamma \in \Delta^i | (\gamma, \beta) = 0 \}$ is a root subsystem of $\Delta^i$. By definition of $t$, if $i < t$ then it admits a unique simple root subsystem containing $\alpha$, which we define to be $\Delta^{i+1}$. Observe that $\beta_i \in \Delta^i$ is the unique highest root. Finally set $\Gamma^i = \{ \gamma \in \Delta^i | (\gamma, \beta) < 0 \}$.

3.3. Recall (EJ, 1.3) that the subscript $c$ (resp. $n$) refers to compact (resp. non-compact) roots, etc. Let $a^i$ (resp. $a^i_0$) denote the subalgebras of $\mathfrak{n}$ spanned by the root vectors $x^i, \gamma \in \Gamma^i$ (resp. $\gamma \in \Gamma^i_0$). As noted in [17], 4.8, the $a^i$ are Heisenberg Lie algebras with centre $\mathbb{C} x_{-\beta_i}$. Obviously $a^0 = a^0 \cap \mathfrak{m}$ and so is commutative. It is convenient to take $a^0 = \{0\}$. Set

$$b^i = \sum_{j=0}^i a^j, \quad b^i_n = \sum_{j=0}^i a^j_n.$$

Let $\mathfrak{g}^i$ denote the simple subalgebra of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and the root vectors $x^i, \gamma \in \Delta^i$. Set $m^i = \mathfrak{g}^i \cap \mathfrak{n}$. Set $\mathfrak{l}^i = \mathfrak{m} \cap \mathfrak{g}^i$ which is the Levi factor of a maximal parabolic subalgebra $p^i$ of $\mathfrak{g}^i$ whose nilradical we denote by $\mathfrak{m}^i$. One has

$$(\ast) \quad m^i \oplus a^i_{n-1} = m^{i-1}, \quad \forall i = 2, 3, \ldots, t.$$

Again $m^i$ is a simple $\mathfrak{l}^i$ module with $x_{-\beta_i}$ as its lowest weight vector. Let $\mathfrak{l}^0$ denote the subalgebra of $\mathfrak{h}$ spanned by the coroots $0, \beta^*_1, \beta^*_2, \ldots, \beta^*_i$ and set $\mathfrak{c}^i = \mathfrak{l}^0 \oplus \mathfrak{b}^i$.

3.4. Set $y_i = 1$ and for $1 < i \leq t+1$ set $y_i = v_1 v_2 \cdots v_{i-1}$. Let $Y_i$ denote the multiplicative subset of $S(m)^n = U(m)^n$ generated by $y_i$. Since the adjoint action of $m$ and hence of each $y_i$ on $U(\mathfrak{g})$ is locally nilpotent it follows from [4], 6.1, that $Y_i$ is Ore in any subalgebra of $U(\mathfrak{g})$ containing $U(\mathfrak{b}^i_{n-1})$. [The induction argument in the lemma below gives $y_i \in U(\mathfrak{b}^i_{n-1})$.]

We apply the construction of [18], 4.1, 4.9, to the semi-direct product $\mathfrak{g}^i \oplus a^i_{n-1}$. This gives a sequence of $\mathfrak{g}^i$ modules and Lie algebra embeddings $\Theta^i = \text{Id}_{\mathfrak{b}^i}$, $\Theta^i : \mathfrak{g}^i \rightarrow Y_i^{-1} U(\mathfrak{g}^i \oplus b^i_{n-1})^{-1}$ having the form $\Theta^i(x) = x - \theta^i(x)$, where $\theta^i = 0$ and $\theta^i(\mathfrak{g}^i) \subset Y_i^{-1} U(\Theta^{i-1}(a_{n-1}^{-1}))$ for $i > 1$. The image of $\mathfrak{g}^i$ under $\Theta^i$ is a copy of $\mathfrak{g}^i$ in the localized algebra $Y_i^{-1} U(\mathfrak{g})$ commuting with the sum $b^i_{n-1}$ of Heisenberg algebras. The possibility for doing this follows from the quite general principles discussed in [7], 10.1.4. However in the present simpler situation we can give quite explicit formulae for the $\theta^i(x)$. It will be enough to analyse these in the case $i=2$, since the general situation is similar. Setting $\beta = \beta_1$, $\Gamma = \Gamma^1$ we recall ([18], 4.9) that $\theta(x_\delta) = \delta \in \Delta^2$ has denominator $x_{-\beta}$ and numerator a sum of terms of the form $x_{\gamma_1} x_{\gamma_2}$ with $\gamma_1 + \gamma_2 + \beta = 0$. In particular $\gamma_1 + \gamma_2$ is never a root and so $x_{\gamma_1}, x_{\gamma_2}$ commute. A similar result holds for $\theta^2(h), h \in \mathfrak{h}$ except that this time $\gamma_1 + \gamma_2 = -\beta$.

We now prove the result referred to in 3.1. Set $v_0 = 1$. 

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LEMMA. — For each \( i \in \{1, 2, \ldots, t\} \) one has \( \Theta^i(x_{-\beta}) = v_{i-1}^{-1} v_i \). Moreover \( v_i \in U(b_i') \).

One has \( \Theta^i(a') \subset Y_i^{-1} U(\Theta^{i-1}(a'_{\beta_{i-1}})) \). Hence \( \Theta^i(a') \subset a' + Y_i^{-1} U(\Theta^{i-1}(a'_{\beta_{i-1}})) \). It follows by an easy induction argument that \( U(\Theta^i(a')) \subset Y_i^{-1} U(b_i') \). In particular \( \Theta^i(x_{-\beta}) \subset Y_i^{-1} U(b_i') \), and has weight \( -\beta \). Although one need not have \( b_i' \subset n \) because \( \delta \in \Delta_i \) \( |(\beta_{i-1})=0| \) \( \Delta_i \) can have simple compact factors, a similar analysis taking account of these terms shows that \( \Theta(x_{-\beta}) \in n \) invariant. Since \( b_i' \subset n \) and \( \gamma_i \in Z(n) \) it follows that \( \Theta^i(x_{-\beta}) \in \text{Fract} Z(n) \) and has weight \( -\beta \). By [18], 4.12, we conclude that \( \Theta^i(x_{-\beta}) = v_{i-1}^{-1} v_i \) up to a non-zero scalar which can be absorbed in the definition of \( v_i \). In particular \( v_i \in (Y_i^{-1} U(b_i')) \cap U(m) = U(b_i') \). This implies that \( y_{i+1} \in U(b_i') \). By an easy induction argument this can be used to justify the formation of the localizations at \( y_{i+1} \).

3.5. Let \( V_i \) denote the subvariety of \( m^+ \) of zeros of \( Q_i \). This is just the associated variety of \( L(u_i, \omega) \). Let \( \mathcal{R}(V_i) \) denote the ring of regular functions on \( V_i \) and \( \mathcal{D}(V_i) \) the ring of differential operators on \( V_i \). (For general definitions see [27], Chap. 15, for example.) We have the

PROPOSITION. — For all \( i \in \{1, 2, \ldots, t\} \) one has

(i) \( Y_i^{-1} \mathcal{D}(V_i) \cong Y_i^{-1} S(b_i') \).
(ii) \( Y_i^{-1} \mathcal{D}(V_i) \cong Y_i^{-1} U(c_i') \).
(iii) \( Y_i^{-1} \mathcal{D}(V_i) \) is a simple ring.
(iv) \( \dim V_i = \sum_{j<i} |\Gamma_j'| \).

(i) Recall that \( \Theta^i \) is a \( g^i \) module homomorphism and that \( v_{i-1} \) is \( g^i \) invariant. Then the simplicity of \( m^i \) as a \( f^i \) module implies by 3.4 that

\[ \Theta^i(m^i) \subset v_{i-1}^{-1} V_i \subset Y_i^{-1} Q_i \subset Y_i^{-1} S(m). \]

Yet \( \Theta^i(m^i) \subset Y_i^{-1} U(\Theta^{i-1}(a')) \subset Y_i^{-1} U(b_i'^{-1}) \). We conclude that

\[ \Theta^i(m^i) \subset Y_i^{-1} S(b_i'^{-1}). \]

From 3.3 (*) we obtain

\[ m^i \oplus b_i'^{-1} = m. \]

From 2.3 it follows that the image of \( v_{j}, j<i \), is a non-zero divisor in \( S(m)/Q_i \) and so this ring embeds in \( Z_i := Y_i^{-1} S(m)/Q_i \). Since \( v_i \in Q_i \) it follows from 3.4 that \( \Theta^i(x_{-\beta}) \) is zero in \( Z_i \). Using the \( f^i \) action as above, it follows that \( \Theta^i(m^i) \) is also zero in \( Z_i \), equivalently that \( x = \Theta^i(x) \in Y_i^{-1} S(b_i'^{-1}), \forall x \in m^i \) in \( Z_i \). Combined with (*) it follows that \( Y_i^{-1} S(m)/Q_i = Y_i^{-1} \mathcal{D}(V_i) \) identifies with an image of \( Y_i^{-1}(b_i'^{-1}) \).

To complete the proof of (i) and indeed of the proposition we use the fact (EJ, 5.2) that \( S(m)/Q_i \) admits a \( U(g) \) module structure extending the left \( m \) module action by multiplication. This gives additional information which would be otherwise rather hard to obtain. Setting \( \xi_i = u_i^\omega \) we identify \( S(m)/Q_i \) with \( L(\xi_i) \). Note that this identification also preserves the \( f \) module structure up to a shift defined by \( u_i \).
The action of $U(g)$ on $L(\xi)$ defines an embedding of $U(g)/\text{Ann } L(\xi)$ into $\text{End}_c L(\xi) = \text{End}_c R(\mathcal{V})$. Since $\text{ad } m$ has a locally nilpotent action on $U(g)$, it follows that the image is contained in $D(\mathcal{V})$. Take $j < i$. As the image of $v_j$ is a non-zero divisor in $S(m)/Q$, it follows that $v_j m = 0$, $m \in L(\xi)$ implies $m = 0$. It follows that the image of $v_j$ (and hence of $y_j$) in $U(g)/\text{Ann } L(\xi)$ is a non-zero divisor. This gives an embedding $U(g)/\text{Ann } L(\xi) \subset Y^{-1}_i (U(g)/\text{Ann } L(\xi))$. Let $\pi : U(g) \rightarrow U(g)/\text{Ann } L(\xi)$ denote the canonical projection.

Since each $a^i$ is a Heisenberg Lie algebra with centre $C x_{-i}$, the construction in 3.4 shows that $Y^{-1}_i U((t^{-1})$ is a localized Weyl algebra and hence a simple ring. (This is discussed in further detail in [18], Sect. 6.) Now $t^{-1} \subset b \oplus n \subset f \oplus m$, and so we conclude that $Y^{-1}_i U((t^{-1})$ identifies with a subring of $Y^{-1}_i U((t^{-1}) \pi U(f \oplus m))$. Now the action of $f \oplus m$ on $L(\xi)$ results from the identification of the latter as a quotient of the induced module $N(\xi)$. From this it is easy to check that the action of the subring $Y^{-1}_i U((t^{-1})$ on the image of the map $\varphi : Y^{-1}_i U((b^{-1}_n) \rightarrow Y^{-1}_i (S(m)/Q)) = Y^{-1}_i L(\xi)$ identifies $Y^{-1}_i U((b^{-1}_n)$ with the standard (and hence simple) module over this localized Weyl algebra. (In other words $b^{-1}_n$ acts by multiplication and the remaining $\dim t^{-1} - \dim b^{-1}_n = \dim b^{-1}_n$ generators by appropriate differentiation.) This proves that $\varphi$ is injective and so completes the proof of (i). Notice that we have also proved (ii), (iii) and furthermore that the embedding $U(g)/\text{Ann } L(\xi) \subset R(\mathcal{V})$ gives rise to an isomorphism

\[(***) \quad Y^{-1}_i (U(g)/\text{Ann } L(\xi)) \cong Y^{-1}_i (D(\mathcal{V})).\]

Finally

$$\dim \mathcal{V}_i = \sum_{j=1}^{i-1} \dim a^j_n = \sum_{j=1}^{i-1} |\Gamma_n^j|$$

which is (iv).

3.6. The irreducible varieties $\mathcal{V}_i$, $i \in \{1, 2, \ldots, t+1\}$ arise as associated varieties of the highest weight modules $L(\xi)$ and so are the closures of orbital varieties (see [23], Sect. 7). To show that every orbital variety arises in such a fashion (see [21], Sect. 8.1) is a difficult and as yet unsolved problem. The present simple case is already quite subtle and has a significant history (EJ, 5.2) and [12].

### 4. The maximal ideal and surjectivity theorems

4.1. As discussed in 1.2 we now recover the results of Levasseur-Stafford in [26] and further extend them to the exceptional cases (actually only $E_7$ remained open). Our analysis is furthermore case by case free.

4.2. Retain the notation of Section 3.
THEOREM. — Fix \( i \in \{1, 2, \ldots, t\} \) and set \( \xi_i = u_i\omega \). Then \( \mathcal{J}_i := \text{Ann} L(\xi_i) \) is a maximal ideal of \( \mathcal{U}(g) \).

We can assume \( i > 1 \) for \( \mathcal{J}_i \) is just the augmentation ideal of \( \mathcal{U}(g) \). If \( \mathcal{J}_i \) were not maximal then by 3.5 (i) and 3.5 (iii) it would be contained in some maximal ideal \( \mathcal{J} \) satisfying \( \mathcal{J} \cap \mathcal{Y}_j \neq \emptyset \). Since trivially \( \mathcal{J} \cap \mathcal{Y}_i = \emptyset \) there exists a largest integer \( j \), \( 1 \leq j \leq i - 1 \) such that \( \mathcal{J} \cap \mathcal{Y}_j = \emptyset \). We recall an argument in [21], 4.4, to show that \( \nu_l \in \mathcal{J} \) for \( l \) sufficiently large.

Since a maximal ideal is primitive, Duflo’s theorem ([16], 7.4) gives a simple highest weight module \( L \) such that \( \mathcal{J} = \text{Ann} L \). Let \( e \) be a choice of highest weight vector for \( L \). Take \( k, l \). If \( \nu_l \in \mathcal{J} \) for \( \nu_l \in \mathcal{J} \) then it follows that \( \nu_l \in \mathcal{J} \). Then \( \mathcal{J} \cap \mathcal{Y}_j = \emptyset \) and so by choice of \( j \) we obtain \( k = j \). This proves the required assertion.

Take \( j \) as above. Since \( \mathcal{L}(\xi_j) \) is already \( \mathcal{Y} \) torsion-free (see proof of 3.5 for example) it is necessarily \( \mathcal{Y} \) torsion-free. Hence \( L(\xi_j) \) embeds in \( \mathcal{Y}^{-1} \mathcal{L}(\xi_j) \) which we may consider as a \( \mathcal{Y}^{-1} \mathcal{U}(g) \) module and hence (cf. 3.4) as a \( \mathcal{U}(\Theta^j(g^j)) \) module. Let \( f \) denote the image in \( \mathcal{Y}^{-1} \mathcal{L}(\xi_j) \) of a non-zero vector of weight \( \xi_j \) of \( L(\xi_j) \). Set \( \mathcal{J} = \Theta^j(g^j) \mathcal{I} \). We claim that \( \mathcal{I} \) acts on \( f \) by scalars.

The case \( j = 1 \) is trivial. Take \( j = 2 \) and recall the description of \( \Theta^2 \) given in 3.4. Take \( \delta \in \Delta^2 \). Then the numerator of \( \Theta^2(x_\gamma) \) takes the form \( x_{\gamma_1} x_{\gamma_2} \) with \( \gamma_1, \gamma_2 \in \Gamma \) and \( \gamma_1 + \gamma_2 + \beta = \delta \). It follows that either \( \gamma_1 \) or \( \gamma_2 \) is compact. Since \( x_{\gamma}f = 0 \), \( \forall \, \gamma \in \Delta \), we conclude that \( \Theta^2(x_\gamma)f = 0 \), \( \forall \, \delta \in \Delta^2 \). Take \( h \in \mathcal{H} \). Then the numerator of \( \Theta^2(h) \) takes the form \( x_{\gamma_1} x_{\gamma_2} \) with \( \gamma_1, \gamma_2 \in \Gamma \) and \( \gamma_1 + \gamma_2 + \beta = 0 \). Hence again either \( \gamma_1 \) or \( \gamma_2 \) is compact. However this time \( \{x_{\gamma_1}, x_{\gamma_2}\} \) is a multiple of \( x_{-\beta} \) and so \( \Theta^2(h)f \) can be a non-zero multiple of \( f \). Consequently \( f \) viewed as a weight vector for \( \mathcal{K} \) will have a weight which may differ from \( \xi_j \). We could in principle calculate the resulting shift of weight directly; but this would be a messy error-prone calculation. We shall find a more devious method to calculate this shift. Taking account of the stepwise nature of the construction in 3.2-3.4, repetition of the above analysis establishes the claim for arbitrary \( j \).

Given \( \gamma \in \Delta^j \), we set \( \mathcal{X}_\gamma = \Theta^j(x_\gamma) \). Now assume \( \gamma \in \Delta^+ \) and let us show that \( \mathcal{X}_\gamma f = 0 \). As above we are reduced to the case \( j = 2 \) and furthermore we can assume that \( \gamma \) is non-compact. Then the numerator of \( \Theta^2(x_{\gamma}) \) has terms of the form \( x_{\gamma_1} x_{\gamma_2} \) with \( \gamma_1, \gamma_2 \in \Gamma \) and \( \gamma_1 + \gamma_2 + \beta = \gamma \). Hence both \( \gamma_1 \) and \( \gamma_2 \) are compact and so \( \Theta^2(x_{\gamma})f = 0 \). Since \( x_{\gamma}f = 0 \), we obtain \( \mathcal{X}_\gamma f = 0 \) as required.

We conclude from the above that \( L_2 := \mathcal{U}((g^i)) f \) is an image of a module \( N_2 \) induced from a 1 dimensional representation of the parabolic subalgebra \( \mathcal{P} \) of \( g^i \) with Levi factor \( \mathcal{I} \) and nilradical \( \mathcal{M}^+ = \mathcal{C} \{ \Theta^j(x_{\gamma}) \gamma \in \Gamma^j(\mathcal{I}) \} \).

We now compute the highest weight of \( L_2 \) (which we recall differs slightly from \( \xi_j \)). Extend \( \Theta^j(\cdot) \) to an algebra homomorphism of \( \mathcal{U}(g^j) \) into \( \mathcal{U}(\mathcal{G}) \mathcal{U}(g^j \mathcal{P}^{-1})^{-1} \). Let \( \pi^j \) denote the subalgebra of \( g^j \) spanned by the \( x_{-\gamma} \) for \( \gamma \in \Delta^j \cap \Delta^+ \). The result in EJ, 2.1,
applies to the pair $\mathfrak{g}^j$, $\mathfrak{t}^j$ and so we obtain a unique up to scalars element $v_{-j+1}^{(i)} \in S(\mathfrak{m}^j)^{u_i}$ of weight $-(\beta_i + \beta_{j+1} + \ldots + \beta_i)$. Exactly as in the proof of 3.4 one checks that $v_{-j+1} := \Theta_j(v_{-j+1}^{(i)}) \in Y_j^{-1} S(\mathfrak{m})^u$ and has weight $-(\mu_i - \mu_{j+1})$ which equals the above sum. By EJ, 2.1, such an element is necessarily proportional to $v_{-j+1}^{(i)}$. Since $v_{j} \neq 0$, we conclude that $\tilde{v}_{-j+1} f = 0$. By EJ, 5.3, we conclude that $L_2$ has highest weight $\xi_{i} = u_{i-j+1} + \omega$ viewed as a weight of the simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. This does not fully calculate the highest weight of $L_2$ but is sufficient for our present purposes.

Now recall the maximal ideal $J = \text{Ann} L(\xi)$ of $U(\mathfrak{g})$. Recall further that for all $k < j$, $\nu_k$ is a non-zero divisor in $U(\mathfrak{g})/J$ and that $v_j \in J$ for some $l \in \mathbb{N}^+$. Then $J := Y_j^{-1} J \cap U(\mathfrak{g}) \supset \text{Ann} L_2$. Moreover $\tilde{v}_1 = \Theta_j(x_{-j}) = v_{-j-1} v_j$ and so $\tilde{v}_1 \in J$. Then by Borho's lemma ([18], 6.11) applied to the simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ with lowest weight vector $x_{-j} = \tilde{v}_1$, it follows that $J$ has finite codimension in $U(\mathfrak{g})$. If we let $\tilde{\rho}$ denote the half sum of roots in $\Delta^f \cap \Delta^+$, this in turn implies that $\tilde{\rho} + \xi_i$ is integral and regular for $\Delta$. The final step in the proof of the theorem consists showing that the above condition is never satisfied. Recalling that $j \leq i - 1$ it suffices to proves the lemma below.

4.3. Define $u_i \in \mathbb{R}$ as in 1.2 and EJ, 3.4, 4.3.

**Lemma.** — Take $i \in \{1, 2, \ldots, t\}$. If $\rho + u_i \omega$ is both integral and regular, then $i = 1$.

We adopt the normalization of EJ, Table, that is $(\alpha, \alpha) = 2$ or equivalently $(\alpha, \omega) = 1$. One has $u_1 = 0$ and so $u_i$ can be computed from EJ, Table. We can assume without loss of generality that $u_i$ is an integer. It is then enough to show that for $i, 2 \leq i \leq t$ there exists a non-compact positive root $\gamma$ such that $0 = (\gamma, \rho + u_i \omega) = (\gamma, \rho) + u_i$ equivalently that $\{(\gamma, \rho) | \gamma \in \Delta^+\} \supset \{1, 2, \ldots, [u_i - u_j]\}$.

Suppose all the roots in $\Delta$ have the same length. Since $\mathfrak{m}$ is a simple $\mathfrak{f}$ module, the left hand side takes all positive integer values up to $(\beta, \rho)$. Yet by EJ, 4.2, one has $u_1 - u_i = - u_i < (1/t) \leq \beta, \rho$, which concludes the proof in this case.

It remains to consider $\mathfrak{g}$ of type $C_t$. Then $u_1 - u_i = (l - 1)/2$. Using the Bourbaki convention ([5], Pl. III) we have $\alpha = 2e_i, \omega = e_i + e_{2i} + \ldots + \epsilon_i$ and so $(\alpha, \omega) = 2$. Also $2 \epsilon_i, i = 1, 2, \ldots, l$ is a non-compact positive root and $(2, \epsilon_i, \rho) = 2(l - i + 1)$. Taking account of our present normalization, this shows that the left hand side above contains the set $\{1, 2, \ldots, l\}$ which is all we require.

4.4. As in 4.2 we set $\xi_i = u_i \omega, i \in \{1, 2, \ldots, t\}$. Let $F(\xi_i)$ [resp. $A(\xi)$] denote the $\mathbb{C}$-endomorphisms of $L(\xi_i)$ which are locally finite under the diagonal action of $\mathfrak{g}$ (resp. $\mathfrak{n}$). One has embeddings $U(\mathfrak{g})/J_i \subset F(\xi_i) \subset A(\xi_i)$. Recall that $n^l = m \oplus \mathfrak{n}$. Since $\mathfrak{n}$ has a locally nilpotent action on $L(\xi_i)$ it follows that $A(\xi_i)$ contains all $\mathbb{C}$-endomorphisms of $L(\xi_i)$ which are only required to be $\mathfrak{m}$ locally finite. Since $\mathfrak{m}$ is commutative, identifying $L(\xi)$ with $S(\mathfrak{m})/Q_i$ gives the

**Lemma.** — One has $A(\xi_i) = \mathcal{D}(\mathcal{V}_i), \forall i \in \{1, 2, \ldots, t + 1\}$. In particular $A(\xi_i)$ is an integral domain and $J_i$ is completely prime.

4.5. We now extend the main surjectivity result of Levasseur-Stafford ([26], 0.3) to arbitrary $\mathfrak{g}$. 

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THEOREM. — Take \( i \in \{1, 2, \ldots, t\} \). Then the embeddings \( U(g)/J_i \subseteq F(\xi_i) \subseteq A(\xi_i) \) are all isomorphisms.

By (**) of 3.5 these are isomorphisms up to localization with respect to the Ore set \( Y_i \). By 4.2 it then follows from [20], 9.1, that the first embedding is an isomorphism.

To show that the second embedding is an isomorphism we apply [24], 5.8. Since \( L(\xi_i) \) is simple (EJ, 8.2) it remains to show that \( L(\xi_i) \) is rigid in the sense of [24], 1.2. As discussed in [24], 1.5, this last property is an immediate consequence of the fact that the associated variety of \( L(\xi_i) \) is \( \mathcal{V}_i \) and so is a proper closed subvariety of the nilradical \( m^+ \) of a maximal parabolic subalgebra \( p^+ \) of \( g \) (and hence cannot be induced). This proves the theorem.

Remark. — For \( i=t+1 \), the second embedding cannot be an isomorphism since \( A(\xi_{t+1}) \) is a Weyl algebra — this is of course the induced case (cf. [24], 7.6 and [25], 3.9).

4.6. COROLLARY. — Take \( i \in \{1, 2, \ldots, t+1\} \). The ring \( \mathcal{D}(\mathcal{V}_i) \) of differential operators on \( \mathcal{V}_i \) is simple and noetherian.

For \( i \leq t \), this follows from 4.2 and 4.5. For \( i=t+1 \), it follows from the remark in 4.5.

Remark. — The above result for \( i=2 \) is due to Levasseur-Smith-Stafford ([25], 5.3) and for \( g \) classical to Levasseur-Stafford ([26], 0.3). The only new case is \( E_7 \) for \( i=3 \). Yet our present proof is much simpler and essentially case by case free. For \( 2 \leq i \leq t \), the \( \mathcal{V}_i \) are all singular so the result is not an immediate consequence of general considerations (as in say [27], 15.3.8).

5. Transference of unitarity

5.1. We now establish the main results claimed in 1.3. For this we use the construction of 3.2-3.4 and adopt the notation there. We use \( \mathcal{C} \) to denote the well-known Bernstein-Gelfand-Gelfand category (see [7], 7.8.15 for example).

5.2. Fix \( \tau \in \mathbb{P}_e^+ \), \( u \in \mathbb{R} \) and set \( \lambda = \tau + u \omega \). Set \( N(\lambda) = U(g) \otimes_{U(\mathfrak{g}^+)} (V(\tau) \otimes \mathbb{C}_u \omega) \) and let \( L(\lambda) \) be the unique simple quotient of \( N(\lambda) \). Denote again by \( V(\tau) \) the image of \( 1 \otimes V(\tau) \otimes 1 \) in \( L(\lambda) \). Assume that \( L(\lambda) \) is not finite dimensional. By Borho's lemma ([18], 6.11) this implies that \( L(\lambda) \) in \( Y_2 \) torsion-free. Set \( \widetilde{\mathfrak{g}} = \Theta(\mathfrak{g}^+) \subseteq Y_2^{-1} U(\mathfrak{g}^2 \oplus \mathfrak{a}^I)^{\mathfrak{g}^+} \), \( \widehat{\mathfrak{f}} = \Theta^2(\mathfrak{f}^2) \), \( \tilde{x} = \Theta^2(x) \), \( \tilde{a} = \Theta^2(x) \), \( \forall x \in \mathfrak{g}^2 \). Consider \( Y_2^{-1} L(\lambda) \) as a \( \widetilde{\mathfrak{g}} \) module.

Since \( V(\lambda) \) is a simple \( \mathfrak{f} \) module, its lowest \( \mathfrak{f} \) weight space is one dimensional and generates a simple \( \mathfrak{f}^2 \) module \( V_1 \). From now on fix a highest \( \mathfrak{f}^2 \) weight vector \( f \in V_1 \).

LEMMA.

(i) \( a^I_1 V_1 = 0 \).

(ii) \( \tilde{a} \delta \chi V_1 = 0 \), \( \forall \delta \in \Lambda_2^0 \cup (\Lambda_2^0 \cap \Delta^+) \).

(iii) \( \tilde{h} \geq f \in \mathcal{C} f, \forall h \in \mathfrak{h} \).

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(iv) $f$ is a highest $\tilde{g}$ weight vector.

The restriction to $t^2$ of the adjoint action of $g^2$ on $a^1$ leaves $a^1_j$ (and $a^1_i$) invariant. By choice of $f$ this proves (i).

Recall that each term in the numerator of $x_8$ takes the form $x_{i_1} x_{i_2}$ with $\gamma_1 + \gamma_2 + \beta = \delta$. If both $\gamma_1$, $\gamma_2$ are non-compact, then $\delta \in \Delta_n^+$ which is excluded by the hypothesis of (ii). Hence (ii) follows from (i). The proof of (iii) is similar to (ii) except that we have to remember the scale factor coming from $[x_{i_1}, x_{i_2}]$ which is proportional to the denominator of $\theta^2(h)$.

Finally (iv) obtains from (ii) and (iii) since for say $x_\rho$, $\delta \in \Delta^2 \cap \Delta^+$ we have $x_\rho = x_\rho - x_8$ and $x_8 f = 0$ by choice of $f$.

5.3. Retain the notation and hypotheses of 5.2. Let $W'_c$ denote the compact Weyl group for $\Theta'(g')$. This is just the Weyl group for $t^1$ defined by the root system $A^2$. Let $w'_c$ be the unique longest element in $W'_c$. Let $\tilde{g}'$ denote the derived algebra of $\tilde{g}$. This is a simple Lie algebra with root system $\Delta^2$. Set $\tilde{h}' = \tilde{h} \cap \tilde{g}'$. Recall (EJ, 3.6) the definition of $\varepsilon_{b, a}$.

LEMMA. — As an $\tilde{h}'$ vector $f$ has weight $\lambda_2 := w^2 w^1 \lambda + \varepsilon_{b, a} \omega$.

It is immediate that $f$ is an $h$ weight vector of weight $w^2 w^1 \lambda$. This is not quite the weight of $f$ as an $\tilde{h}'$ weight vector because of the scale factor occurring in each $\theta^2(h)$, $h \in \tilde{h}$. To compute the contribution of these scale factors it is enough to compute $\lambda_1$ for a special choice of $\lambda$. In the notation of 4.2, we take $\lambda = \xi_2 = u_2 \omega = (u_2 - u_1) \omega = -\varepsilon_{b, a} \omega$. In this case $\tilde{g}' f = 0$. This is because by 5.2 we have $v_2 L(\xi_2) = 0$, so in particular $v_1^{-1} v_2 = \Theta^2(x_{-\beta_1}) = \tilde{x}_{-\beta_1} \in \text{Ann } U(\tilde{g}) f$. This implies that $U(\tilde{g}) f$ is the trivial $\tilde{g}'$ module. Consequently $f$ has zero $\tilde{h}'$ weight. We conclude that the scale factors add on the term $\varepsilon_{b, a} \omega$ when $f$ is viewed as a weight vector for $\tilde{h}'$.

5.4. Drop the superscripts on $\Gamma^1$, $\alpha^1$, $c^1$. Set $\Delta_n^0 = \Gamma_n \setminus \{-\beta\}$, and $p_\gamma = x_\gamma, \forall \gamma \in \Gamma_n^0$, $\gamma \neq -\beta$. Then $[q_\gamma, p_{-\gamma}] = 1, \forall \gamma \in \Gamma_n^0$ and all other commutators vanish. Set $A_0 = \mathbb{C}[q_\gamma, p_{-\gamma}, \gamma \in \Gamma_n^0], A' = \mathbb{C}[q_{-\beta}, q_{-\beta}, p_{-\beta}] A = A_0 \otimes A'$ which are (localized) Weyl algebras. We define an anti-automorphism $\tilde{\sigma}$ of $A$ by $\tilde{\sigma}(q_\gamma) = p_\gamma, \tilde{\sigma}(p_\gamma) = q_\gamma, \forall \gamma \in \Gamma_n^0, \tilde{\sigma}(q_{-\beta}) = q_{-\beta}, \tilde{\sigma}(p_{-\beta}) = q_{-\beta} p_{-\beta} q_{-\beta}$. Observe that

\[ (\ast) \quad \tilde{\sigma}(h_\beta) = h_\beta \quad \text{and} \quad \tilde{\sigma}(q_\gamma p_\gamma) = q_\gamma p_\gamma, \quad \forall \gamma \in \Gamma_n^0. \]

Recall that the map $\theta^2 : g^2 \to Y^{-1} U(\alpha)$ is a Lie algebra homomorphism.

LEMMA.

(i) $\tilde{\sigma}(h_\beta) = h_\beta, \forall h \in \tilde{h}$.

(ii) $\tilde{\sigma}(x_\delta) = x_{-\delta}, \forall \delta \in \Delta \tilde{e}$.

(iii) $\tilde{\sigma}(x_\delta) = -x_{-\delta}, \forall \delta \in \Delta \tilde{e}$.

(i) is an immediate consequence of (\ast) above. The proof of (ii) and (iii) which are in principle quite delicate are made enormously easier by the fact that for each $\gamma \in \Gamma_n^0$ and each $\delta \in \Delta \tilde{e}$ the $\delta$-string containing $\gamma$ has at most two elements. This is obvious if
all roots have the same length. It holds in type $B_n$ because $\delta = \pm \alpha_i$ and is necessarily a long root. In type $C_n$ the assertion is checked by explicit computation.

Take $\delta \in \Delta^2$. Recall that $[x - \bar{x}, A_0] = 0$, $\forall x \in \mathfrak{g}^2$. On easily checks that

$$\bar{x}_\delta = 0^2 (x_\delta) = \sum_{\gamma \in \Gamma^0_n} c_{\gamma + \delta}^{\delta} q_{\gamma + \delta} p_{-\gamma}$$

where

$$c_{\gamma + \delta}^{\delta} q_{\gamma + \delta} = [x_\delta, q_{\gamma}]$$

and since we have a Chevalley basis, these coefficients are integers.

Now fix $\gamma \in \Gamma^0_n$ such that $\gamma + \delta$ is a root. By the first remark $\gamma - \delta$ is not a root and $(\delta, \gamma) = -1$. Thus

$$-q_\gamma = [h_\delta, q_{\gamma}] = -[x_{-\delta}, [x_\delta, q_{\gamma}]]$$

and so

$$c_{\gamma - (\gamma + \delta)}^{\delta} q_{\gamma + \delta} = 1.$$ We conclude that these coefficients are pairwise equal. A similar result holds if $\gamma - \delta$ is a root. Combined, this gives just what we require to prove (i).

Take $\delta \in \Delta^2 \cap \Delta^+$. One has

$$\bar{x}_\delta = \sum_{\gamma \in \Gamma^0_n} d_{-\gamma, \beta + \gamma + \delta}^\delta q_{-\beta} p_{\beta + \gamma + \delta}$$

where

$$d_{-\gamma, \beta + \gamma + \delta}^\delta q_{-\beta} p_{\beta + \gamma + \delta} = [x_\delta, q_{\gamma}]$$

and as before these coefficients are integers. Again

$$\bar{x}_{-\delta} = \sum_{\gamma \in \Gamma^0_n} e_{\gamma - (\beta + \gamma + \delta)}^{\delta} q_{-\beta} q_{\gamma} q_{-(\beta + \gamma + \delta)}$$

where

$$e_{\gamma - (\beta + \gamma + \delta)}^{\delta} q_{-\beta} q_{\gamma} q_{-(\beta + \gamma + \delta)} = -[x_{-\delta}, p_{\beta + \gamma + \delta}]$$

and as before these coefficients are integers.

Now fix $\gamma \in \Gamma^0_n$ such that $\gamma + \delta$ is a root. As before

$$-q_\gamma = [h_\delta, q_{\gamma}] = -[x_{-\delta}, [x_\delta, q_{\gamma}]]$$

and so

$$-e_{\gamma - (\beta + \gamma + \delta)}^{\delta} d_{-\gamma, \beta + \gamma + \delta} = 1.$$
We conclude that these coefficients differ pairwise exactly by a sign. This is just what is required to prove (iii).

5.5. We have $a_{-\gamma}f=0$ by 5.2 (i). Hence $p_{-\gamma}f=0$, $\forall \gamma \in \Gamma^0$. Hence

$$M_0 := A_0f = \mathbb{C}[q_{\gamma}, \gamma \in \Gamma^0_0]$$

is the standard $A_0$ module in which $q_{\gamma}$, $\gamma \in \Gamma^0_0$ acts by multiplication and $p_{-\gamma}$ by differentiation with respect to $q_{\gamma}$.

Let $\varepsilon : A_0 \to \mathbb{C}$ be the projection defined by the canonical basis of $A_0$ in which the $q_{\gamma}$ (resp. $p_{\gamma}$) appear to the left (resp. right). Let $j$ denote complex conjugation. One checks that the map $(a, b)\mapsto \varepsilon ((j\tilde{\sigma}(a))b)$ of $A_0 \times A_0 \to \mathbb{C}$ factors to a sesquilinear $\tilde{\sigma}$-contravariant form $\langle \ , \ \rangle$ on $M_0$. Moreover up to scalars the monomials in the $q_{\gamma}$ form an orthonormal basis and hence this form is positive definite. Of course this construction is quite classical and known to physicists as the Fock space construction of the “unitary” representation of the Weyl algebra given by the Stone-von Neumann theorem. (One nevertheless needs to check that signs do work out correctly — for example

$$\langle q, q \rangle = \langle 1, \tilde{\sigma}(q)q \rangle = \langle 1, pq \rangle = \langle 1, [p, q] \rangle \langle 1, 1 \rangle = 1,$$

as required.)

5.6. Since $h_\beta f \in \mathbb{C}f$ we may identify $M' := A'f$ with $\mathbb{C}[q^{-1}_\beta, q_{-\beta}]$. On $M'$ we let $\langle \ , \ \rangle$ denote the unique sesquilinear form which extends $\langle q^{-1}_\beta, q_{-\beta} \rangle = \delta_{kl}$ (where $\delta$ is the Kronecker delta). It is obviously positive definite and noting that $\tilde{\sigma}(h_\beta) = h_\beta$ and $A' = \mathbb{C}[q^{-1}_\beta, q_{-\beta}, h_\beta]$, we easily conclude that it is $\tilde{\sigma}$-contravariant.

It is immediate that $M := Af = M_0 \otimes M'$. Hence the

**Lemma.** — *The product form $\langle \ , \ \rangle$ on $M$ is sesquilinear, $\tilde{\sigma}$-contravariant and positive definite.*

5.7. Let $g_0^2$ denote the centralizer of $x_1 - \bar{p}$ in $g^2$. One has $g_0^2 \otimes \mathbb{C}h_\beta = g^2$. Set $\tilde{g}_0^2 = \Theta^2 (g_0^2)$. One has $\tilde{g} \subset \tilde{g}_0^2$ and is of at most codimension 1. Set $B = U(\tilde{g}_0^2)$. Set $L_2 = Bf = U(\tilde{g}f)$. View $A$, $B$ as subrings of $\mathbb{Y}_2^{-1} U(g)$.

(i) The map $a \otimes b \mapsto ab$ of $A \otimes B$ is an isomorphism of rings.

(ii) The map of $\otimes bf \mapsto abf$ of $M \otimes L_2$ onto $L := ABf$ is an isomorphism of $AB$ modules.

(iii) $AB = \mathbb{Y}_2^{-1} U(a) U(g^2) = U(g^2) \mathbb{Y}_2^{-1} U(c)$.

(i) follows from the fact that $A$ is central, simple and $B$ is the commutant of $A$ in $AB$.

(ii) By say Quillen’s lemma ([7], 2.6.4) $\text{End}_A M$ reduces to scalars. Then a standard application of the Jacobson density theorem shows that any $A \otimes B$ submodule of $M \otimes L_2$ takes the form $M \otimes N$ where $N$ is a $B$ submodule of $L_2$. This proves (ii).

(iii) is an obvious consequence of the relation $[g^2, a] = a$ and the definition of $\Theta^2$.

**Remark.** — (i), (ii) can also be proved by an elementary direct computation.
5.8. Let $\sigma$ denote the “non-compact” Chevalley antimorphism of $\mathfrak{g}$ defined in EJ, 5.1. We define $\overline{\sigma}$ on $\mathfrak{g}^2$ by taking it to be the restriction of $\sigma$. Set 
$\overline{x} = \Theta(x) = x - \Theta^2(x) = x - \overline{x}, \quad \forall x \in \mathfrak{g}^2$. By 5.4 and the definition of $\sigma$ we obtain the remarkable fact that 5.4 holds with $x$ replaced by $\overline{x}$. By 5.2, $L_2 = U(\mathfrak{g}/f)$ is a highest weight module. From the standard construction (cf. EJ, 5.1) we obtain a sesquilinear $\overline{\sigma}$-contravariant form $\langle \cdot , \cdot \rangle$ on $L_2$.

5.9. Let $\overline{\mathfrak{g}}$ denote the image of $\mathfrak{g}^2$ under the Lie algebra map $\Theta^2: \mathfrak{g}^2 \to A$. Set $N = U(\overline{\mathfrak{g}})/f$ which is a submodule of $M$. By construction $x = \Theta^2(x) + \Theta^2(x) = \overline{x} + \overline{x}$, $\forall x \in \mathfrak{g}^2$ and moreover $[\overline{x}, \overline{y}] = 0, \forall x, y \in \mathfrak{g}^2$. Recalling 5.7 we conclude that $N \otimes L_2$ identifies with a $\overline{\mathfrak{g}} \times \overline{\mathfrak{g}}$ submodule of $L$, itself a submodule $Y^{-1}_2 L(\lambda)$. Moreover the action of $\mathfrak{g}^2$ on $Y^{-1}_2 L(\lambda)$ restricted to $N \otimes L_2$ is just the diagonal action, for $N, L_2$ viewed as $\mathfrak{g}^2$ modules. Let $n, e$ be a choice of highest weight vector for $N, L_2$. By 5.7 (ii), we may identify $n \otimes e$ with $f$ and hence $U(\mathfrak{g}_2)/f$ with the submodule of $N \otimes L_2$ generated by $n \otimes e$. It follows from 5.3 (or of course directly from the proof of 5.3) that $n$ has highest weight $-\varepsilon_0 w = \xi_1$. Moreover 5.4 just says that $\overline{\sigma}$ is a automorphism $\sigma$ defined on $\mathfrak{g}^2$ by EJ, 5.1. Thus the restriction of the contravariant form $M$ to $N$ coincides up to a scalar with that defined in EJ, 5.1. Since the former is positive definite on $M$, it is positive definite on $N$ and so this construction reproofs that $N$ is a unitary highest weight module, hence simple and isomorphic to $L(\xi_1)$. Possibly a more elegant proof of 5.4 would obtain by using our prior knowledge of the unitarity of $L(\xi_1)$. Take $\lambda = -((1 - 1) \varepsilon_0) w = 0$ and recall 3.4. Then this construction also recovers (by an essentially elementary argument) the apparently deep fact, namely that $Q_{i+1} = \text{Ann}_{U(\mathfrak{g}_n)}(f_2 \otimes f_i)$, noted during the proof of 2.5.

5.10. Let $\langle \cdot , \cdot \rangle$ denote the form on $N \otimes L_2$ which is the product of the $\overline{\sigma}$-contravariant forms on $N, L_2$ given by 5.6 and 5.8. We should like to compare this with the $\sigma$-contravariant form on $L(\lambda)$. In particular to show that if $L(\lambda)$ is unitary then $\langle \cdot , \cdot \rangle$ is positive definite and hence that $L_2$ is unitary. Unfortunately, we have been unable to do this, so in fact 5.4-5.6 will not be used in the sequel. We have the

**Lemma.** — Suppose $L(\lambda)$ is unitary (as a $\mathfrak{g}$ module). Then $Y^{-1}_2 L(\lambda)$ is a direct sum of unitary highest weight $\mathfrak{g}_2$ modules and hence so is $N \otimes L_2$.

It is sufficient and convenient to prove the corresponding assertion for $\mathfrak{g}_2^0$. Obviously $L(\lambda)$ is unitary as a $\mathfrak{g}_2^0$ module. Since the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}_2^0$ of $\mathfrak{g}_2^0$ acts locally finitely on $L(\lambda)$, then $Z(\mathfrak{g}_2^0)$ primary decomposition splits $L(\lambda)$ into a direct sum of modules in the $\mathfrak{c}$ category and the latter all have finite length ([7], 7.8.15). Thus $L(\lambda)$ is a direct sum of unitary highest weight $\mathfrak{g}_2^0$ module. Since $\varepsilon_0$ commutes with $\mathfrak{g}_2^0$ the same holds for $x^{-k}_0 L(\lambda), \forall k \in \mathbb{N}_+$. Then a standard argument on semisimplicity proves the assertion for $Y^{-1}_2 L(\lambda)$ and then for the submodule $N \otimes L_2$.

5.11. We are now faced with the following question. Let $N_1, N_2$ be highest weight (not necessarily simple) $\mathfrak{g}$ modules such that $N_2 \otimes N_1$ is unitary with respect to the diagonal action of $\mathfrak{g}$. Then are $N_1, N_2$ unitary? The first unpleasant fact is that this
can fail even if we impose that $N_2$ be unitary. For example, take $g$ of type $\mathfrak{sl}(2)$ with $\alpha$ the non-compact simple root (in the conventions of EJ). Take $N_1 = L(u\omega)$, $u \in ]0,1[, N_2 = L(-\omega)$. Use of $Z(\mathfrak{g})$ shows that $N_1 \otimes N_2$ is a direct sum of the modules $L((u-(2k+1))\omega)$, $k \in \mathbb{N}$ and is hence unitary. Yet $N_1$ is not unitary. The second unpleasant fact is that this can fail even if all simple factors of $N_1$ and $N_2$ are unitary. For example, take $N_1 = N(0)$ which has length 2 having the trivial module $L(0)$ as quotient and $L(-\alpha)$ as a submodule. Both factors are unitary. Take $N_2$ as above. Then every simple factor of $N_2 \otimes N_1$ is unitary. Yet this module is semisimple because its submodule $L(-\alpha) \otimes L(-\omega)$ is unitary and the quotient $L(0)$ can only be non-trivially extended (from below) by a submodule isomorphic to $L(-\alpha)$ and obviously no such factor occurs in $L(-\alpha) \otimes L(-\omega)$.

Two further unpleasant facts are uncovered by this second example. This first is non-trivial extensions between unitary modules can exist and the second that such extensions can be annihilated by tensor product. Further examples are given by the following. Retain the notation of EJ, 1.6.

**Proposition.** Let $\lambda$ be a first reduction point. Then $\overline{N(\lambda)}$ is unitary.

Let $\mu$ be a highest $\mathfrak{f}$ weight occurring in $N(\lambda)$ and suppose that $\|\mu + \rho\| \leq \|\lambda + \rho\|$. Then from the calculation in EJ, 5.2, using EJ, 3.9, 4.3, 4.4, we see that either $\mu = \lambda$, or $\mu$ is the highest $\mathfrak{f}$ weight of $\overline{N(\lambda)}$. By [9], 3.9, this proves the required assertion.

The reader may now easily check that taking $g = \mathfrak{sl}(3)$, for which 0 is again a first reduction point, one also obtains that $N(0) \otimes L(-\omega)$ is semisimple via $Z(\mathfrak{g})$ primary decomposition and is hence unitary.

**Remark.** In general the unique simple quotient of $\overline{N(\lambda)}$ need not be unitary—see section 8.6. This destroys a possible approach to establishing the main result of [8].

5.12. As before we set $\xi_1 = -e_{\mathfrak{g},s}\omega$. We call $L(\lambda)$ quasi-unitary if $\lambda = \tau + u\omega$ with $\tau \in P^+$ of level $s$ and $u < u'_i + e_{\mathfrak{g},s}\omega$ or $u = u'_i$, $1 \leq i \leq s$.

**Lemma.** Suppose $L(\lambda) \otimes L(\xi_1)$ is unitary. Then $L(\lambda)$ is quasi-unitary.

Let $e$ (resp. $f$) be a choice of highest weight vector for $L(\lambda)$ [resp. $L(\xi_1)$]. Then $L = U(\mathfrak{g})(e \otimes f)$ is a submodule of $L(\lambda) \otimes L(\xi_1)$ and so by the hypothesis is a direct sum of simple highest weight modules (recall argument in proof of 5.10). Yet $L$ is indecomposable, because it is a highest weight module. We conclude that $L$ is a simple unitary module of highest weight $\lambda + \xi_1$. From the equal spacing rule we conclude that either $L(\lambda)$ is quasi-unitary, or $\lambda + \xi_1$ is a last place of unitary. Suppose the latter holds and set $g = e \otimes f$. We claim that there exists $a \in \mathfrak{m} \otimes U(\mathfrak{f})$ such that $ag = 0$. This again follows from the classification theory and can be expressed by saying that there is a component, namely the PRV component $P$—see EJ, 1.5, such that $P$ is a relation in $\mathfrak{m} \otimes V(\lambda)$.

We can write

$$a = \sum_{i=1}^{r} x_i \otimes y_i$$
\[ x_i \in m, \ y_i \in U(\mathfrak{f})/\text{Ann}_U(\mathfrak{f})g \text{ satisfying the usual linear independence. Since } f \text{ generates a one dimension } \mathfrak{f} \text{ module we can find } \tilde{y}_i \in U(\mathfrak{f})/\text{Ann}_U(\mathfrak{f}) e \text{ such that } y_i (e \otimes f) = \tilde{y}_i e \otimes f. \] 

Since \( x_i \in m \) we obtain

\[ 0 = ag = \sum (x_i \tilde{y}_i) e \otimes f + \sum \tilde{y}_i e \otimes x_i f. \]

Now as a \( U(m) \) module, \( L(\xi_1) \) identifies with \( S(m)/Q_2 \). We recall that \( Q_2 \) is a

(homogeneous) ideal generated by quadratic elements. Then \( f, \{ x_i f \} \) are linearly independent in \( L(\xi_1) \). Consequently, \( \tilde{y}_i e = 0 \), and so \( y_i g = 0, \forall i \). This is clearly absurd and the contradiction proves the lemma.

5.13. Corollary. — Take \( M \in \text{Ob } \mathcal{O} \) (for example, a highest weight module). Suppose \( M \otimes L(\xi_1) \) is unitary. Then every simple subquotient of \( M \) is quasi-unitary.

5.14. Let \( H(\lambda) \) denote a not necessarily simple, highest weight module of highest weight \( \lambda \). Recall that if \( \lambda \in P_c^+ \) then \( N(\lambda) \) is defined EJ, 1.5.

Proposition. — Suppose \( H(\lambda) \otimes L(\xi_1) \) is unitary. Then

(i) \( \lambda \in P_c^+ \) and \( H(\lambda) \) is a quotient of \( N(\lambda) \).

Write \( \lambda = \lambda_0 + u \omega \) with \( \lambda_0 \) the first reduction point. Assume \( u \notin \mathbb{Z}, e_\alpha, \mathfrak{a} \).

(ii) If \( \lambda \neq \lambda_0 \) then \( H(\lambda) \) is unitary.

(iii) If \( H(\lambda) \) is not unitary, then \( H(\lambda) = N(\lambda) \).

By 5.13 every simple factor of \( H(\lambda) \) is quasi-unitary. By identification of \( \mathfrak{f} \) with the complexification of a maximal compact subalgebra of the real form \( g_0 \) of \( g \) it follows that every such factor is \( \mathfrak{f} \) locally finite and hence so is \( H(\lambda) \). This proves (i).

By 5.13 again, \( L(\lambda) \) is unitary. Let \( e \) (resp. \( f \)) denote the canonical generator of \( H(\lambda) \) [resp. \( L(\xi_1) \)]. Then \( L := U(\mathfrak{g}) (e \otimes f) \) is a submodule of a unitary module, hence unitary. It is also a highest weight module. Consequently \( L \cong L(\lambda + \xi_1) \). Set \( g = e \otimes f \).

Suppose the hypothesis of (ii) holds. Then \( N(\lambda + \xi_1) \) is not simple and hence for some \( i < \) level of \( j \) the PRV component of \( V(\lambda + \xi_1) \otimes V_{i+1} \) (notation EJ, 2.5) is a relation in \( L(\lambda + \xi_1) \). As in 5.12 we can choose an \( a \) of the form

\[ a = \sum x_j \otimes y_j \]

\( x_j \in S(m) \) homogeneous of degree \( i+1 \) and \( y_j \in U(\mathfrak{f})/\text{Ann}_U(\mathfrak{f}) V(\lambda + \xi_1) \) such that \( ag = 0 \). Then as in 5.12 we obtain

\[ 0 = a(e \otimes f) = \sum_{i,k} (x'_{jk} \tilde{y}_j e) \otimes x''_{jk} f \]

where

\[ \sum_{k=0}^{i+1} x'_{jk} \otimes x''_{jk} \]

is the image of \( x \) under the diagonal map using the usual Hopf algebra convention on sums and taking the \( x'_{jk} \) (resp. \( x''_{jk} \)) to be homogeneous of degree \( i+1-k \) (resp. \( k \)).
Now as in 5.12 using also that \( Q^2 \) is homogeneous we conclude that
\[
\sum_j x_j y_j e = 0.
\]
This means that \( H(\lambda) \) has level of reduction \( \leq i \) (cf. EJ, 6.4). Yet again by the classification theory (EJ, 6.4) the induced module \( N(\lambda + \xi_1) \) has level of reduction \( i + 1 \) and so \( N(\lambda) \) and hence \( H(\lambda) \) has level of reduction \( i \). Now we use the even harder fact (cf. EJ, 6.6, 8.3) that \( \overline{N(\lambda)} \) is generated by the PRV component (which is simple as a \( \mathfrak{f} \) module) of \( V(\lambda) \otimes V_i \). This forces \( H(\lambda) \) to be the simple quotient of \( N(\lambda) \) proving (ii).

(iii) follows from (ii) and 5.11.

Remark. — One may also give an elementary proof of (i) using only \( \mathfrak{f} \) structure.

5.15. Now return to the situation of 5.1-5.8. In particular define \( f \) as in 5.2 and \( L_2 \) as in 5.7.

Theorem. — Suppose that \( L(\lambda) \) is unitary and that \( \text{Ann}_{U(m)} L(\lambda) \neq 0 \). Then \( L_2 \) is unitary.

Let \( L_2(\lambda_2) \) denote the simple quotient of \( L_2 \). Unfortunately 5.12 is not quite strong enough to say that \( L_2(\lambda_2) \) is unitary. Yet \( \lambda_2 \) is given by 5.3 so this can be checked from the classification of unitary highest weight modules. This is a case by case analysis which we relegate to Section 7, so now we assume \( L_2(\lambda_2) \) to be unitary.

The assertion now follows from 5.9 and 5.14 unless \( L_2 \) is the induced module \( N_2(\lambda_2) \) defined relative to \( \tilde{g} \). The latter means that \( L_2 \) is a free \( U(\tilde{m}) \) module. Since the Weyl algebra module \( M \) is free over \( U(a_a) \) we conclude that \( L \cong M \otimes L_2 \) is free over \( U(\tilde{m} \times a_a) \). In particular \( \text{Ann}_{U(\tilde{m} \times a_a)} f = 0 \). Now \( a_x f = 0 \) by 5.2 (i) so this just means that \( \text{Ann}_{U(m^2 + a_a)} f = 0 \). Recalling that \( m^2 + a_a = m \), we obtain \( \text{Ann}_{U(m^2 + a_a)} f = 0 \) contradicting the hypothesis.

Remark. — We could obviously do better; but not quite that \( L(\lambda) \) itself is induced. This is because we have no control over the compact roots not in \( \Delta^2 \).

5.6. We now prove the remarkable result promised in 1.3.

Theorem. — Suppose \( L(\lambda) \) is unitary. Then \( Q := \text{Ann}_{U(m)} L(\lambda) \) is a prime ideal.

The proof is by induction on rank \( g \). It is trivial if rank \( g = 0 \). If \( \tau = 0 \), then the assertion is just a consequence of the classification (EJ, 8.2) of the unitary modules in this case and 2.3. If \( \tau \neq 0 \), then \( L(\lambda) \) is infinite dimensional and hence \( Y_2 \) torsionfree. Define \( L_2 \) as in 5.7. We can obviously assume \( Q \neq 0 \). Then by 5.15 \( L_2 \) is a unitary module for the strictly lower rank simple Lie algebra \( \tilde{g}^* \), so we can assume that the assertion holds for \( L_2 \).

Define \( \tilde{v}_k := \Theta^2(v_k^{(2)})v_k^{-1}v_{k+1} \) as in the proof of 4.2. Then by 2.3, 2.4 the assertion for \( L_2 \) means that there exists \( j \), \( 1 \leq j \leq t \) such that \( L_2 \) is torsion-free with respect to the \( \tilde{v}_k \), \( k < j \) and if \( j \leq t - 1 \) that \( \tilde{v}_j L_2 = 0 \). Define \( A \) as in 5.4 and \( L \) as in 5.7 (i). Then \( [v_k, A] = 0 \), whereas \( L = AL_2 \) by 5.7. We conclude that \( L \) is \( v_{k+1} \) torsion-free for \( k < j \) and if \( j \leq t - 1 \) that \( v_{j+1} L = 0 \).
We claim that the above assertions hold with \( L \) replaced by \( L(\lambda) \). Suppose first that there exists \( 0 \neq m \in L(\lambda) \) such that \( v_{k+1} \neq 0 \) for some \( k \in \mathbb{N}^+ \). Since \( \{ v_{k+1} \}_{k \in \mathbb{N}} \) is Ore in \( U(q) \) and \( L(\lambda) \) is a simple \( U(q) \) module if follows that \( v_{k} f = 0 \) for some \( f \in \mathbb{N}^+ \). We conclude that \( k \geq j \). It remains to show that \( v_{j+1} L(\lambda) = 0 \) when \( j \leq t-1 \).

Let \( e \) be a choice of highest weight for \( L(\lambda) \). It is clearly enough to show that \( v_{j+1} e = 0 \). Unfortunately \( e \notin L \) in general, so this is far from obvious. However since \( U(f) f = V(\lambda) \) there is one case when this assertion does hold, namely when \( C v_{j+1} \) is \( f \) stable. This arises (except possibly in types \( A_n, E_6 \)) when \( f = t-1 \). Remarkably we can reduce to this case.

Set \( v = v_{j+1} \) and let \( u \) be a highest weight vector in the simple (EJ, 2.1) \( f \) submodule of \( S(m) \) generated by \( v \). By our first argument it follows that there exists \( v \in \mathbb{N}^+ \) such that \( v \) annihilates every vector in the finite dimensional subspace \( V(\lambda) \) of \( L(\lambda) \). Hence \( v \) is \( m \) invariant. Consequently \( u = u(L(\lambda)) = 0 \).

Recall that \( v \) has weight \( -\mu_{j+1} \) and so \( u \) has weight \( -w_{\alpha} \mu_{j+1} \). Suppose first that \( \beta = \alpha \). We claim that \( -w_{\alpha} \mu_{j+1} = -(\beta_{j+1} + \beta_{j} + \beta_{j-1} + \ldots + \beta_{1}) \). First observe that (EJ, 2.3) this weight does in fact belong to \( W_{\alpha} \mu_{j+1} \), so it is enough to prove it to be \( f \) dominant. Let \( \alpha, \alpha' \) be the possibly two simple roots non-orthogonal to \( \beta \) (cf. (EJ, 2.1 or [18], 2.2(vi))). Since \( \beta_{s+1} \in \Delta' \) for \( s \geq s \), we have \( (\alpha, \beta) \leq 0 \), \( (\alpha', \beta) \leq 0 \), \( \forall \alpha \neq \alpha' \). It follows we can assume \( i \geq t-j \) without loss of generality. However in this case \( \alpha, \alpha' \in \Delta' \). Yet we know (cf. (EJ, 2.1 or [16], 2.8)) that \( \beta_{j+1} + \beta_{j} + \ldots + \beta_{1} \) is orthogonal to every simple root of \( \Delta' \) except those non-orthogonal to \( \beta \). Thus \( -w_{\alpha} \mu_{j+1} \) is only non-dominant with respect to the non-compact simple root \( \alpha \) and hence is \( f \) dominant.

Now because the weight of \( u \) lies in \( \Delta' \) and \( u \in m \), it follows that \( u \in S(m \cap \mathfrak{g}^{t-j}) \). Applying (EJ, 2.1) to \( \mathfrak{g}^{t-j} \) we may find \( \mathfrak{g}^{t-j} \) lowest weight vectors \( v_{1}, v_{2}, \ldots, v_{k+1} \in S(m \cap \mathfrak{g}^{t-j}) \) of weights

\[
-\beta_{r-j} - (\beta_{r-j} + \beta_{r-j+1}) - \ldots - (\beta_{r-j} + \beta_{r-j+1} + \ldots + \beta_{r}).
\]

By EJ, 2.3, the simple \( f \) modules generated by \( v_{1}, v_{2}, \ldots, v_{k+1} \) are respectively \( V_{1}, V_{2}, \ldots, V_{k+1} \). We conclude that \( L(\lambda) \) is \( v_{k+1} \) torsion-free for all \( k < j \). Observe that \( C u = C v_{k+1} \).

Now consider \( L(\lambda) \) as a \( \mathfrak{g}^{t-j} \) module. It is unitary and by the argument in 5.9, a direct sum of unitary highest weight modules \( L(\lambda_{i}) \) each of which satisfy the hypothesis of 5.15. Fix \( i \) and let \( f_{i} \in L(\lambda_{i}) \) be defined as in 5.2. By the previous paragraph \( f_{i} \) is \( v_{k+1} \) torsion free for all \( k < j \). Hence \( v_{k+1} f_{i} = 0 \) by 5.15 (as in the first step). Yet \( v_{j+1} f_{i} = u \) up to scalar and so \( u f_{i} = 0 \). It follows that \( u \) also annihilates a highest weight vector \( e_{t} \) of \( V(\lambda_{i}) \) which may also be identified with the highest weight vector \( L(\lambda_{i}) \). Hence \( u L(\lambda_{i}) = 0 \). Since \( i \) was arbitrary we conclude that \( u L(\lambda) = 0 \). Recalling that \( u \) generates \( V_{j+1} \) as a \( f \) module we obtain \( Q_{j+1} L(\lambda) = U(m) V_{j+1} L(\lambda) = 0 \). Recalling 2.3, this proves that \( Q = Q_{j+1} \) which is prime.

The cases \( \alpha \neq \beta \) can only occur in types \( A_{n}, E_{6} \). The argument is essentially the same for these cases.
First assume $g$ of type $A_n$ and $\alpha = \alpha_r$ in the Bourbaki notation ([5], Pl. I). We can assume $t \leq (n + 1)/2$ without loss of generality and then this definition of $t$ coincides with that used above. If $t = (n + 1)/2$, then $\beta_1 = \alpha_1$ and so the above argument applies. Otherwise let $g_0$ denote the Levi factor of $g$ defined by the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{t-1}$. Observing that $u \in S(g_0 \cap m)$ we see the above analysis with $g_0$ replacing $g$ applies and proves the theorem in this case.

Finally suppose $g$ of type $E_6$. We can assume $\alpha = \alpha_1$ without loss of generality. Then $t = 2$ and we can assume $j = 1$ without loss of generality. One checks that $w_0(\beta_1 + \beta_2) = -(2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ in the Bourbaki notation ([5], Pl. V). Let $g_0$ denote the Levi factor of $g$ defined by $\alpha_1, \alpha_2, \ldots, \alpha_5$. Then $u \in S(g_0 \cap m)$, so we can replace $g$ by $g_0$ in the above. Finally $w_0(\beta_1 + \beta_2)$ is orthogonal to the compact roots $\alpha_2, \alpha_3, \ldots, \alpha_5$ and so $C u = \mathfrak{f}_0 := \mathfrak{f} \cap g_0$ stable. Thus our previous analysis applies and proves the theorem in type $E_6$.

5.17. The argument in [29], 7.13 suggests an easy proof of 5.16 and it is perhaps worth mentioning why this cannot work. Define $Q = \text{Ann}_\mathfrak{u}(m) L(\lambda)$. Obviously $Q$ is stable. Hence by EJ, 8.1, and 2.3 we conclude that the radical $\sqrt{Q}$ of $Q$ is prime. It therefore suffices to show that $a^2 \in Q$ implies $a \in Q$. Suppose $a^2 \in Q$ and that we have $m \in L(\lambda)$ such that $am \neq 0$. Assume for simplicity that $a$ is real, that is $a = j(a)$. Then $\langle am, am \rangle \neq 0$ by unitary and so $\sigma(a) am \neq 0$. Repeating this argument we conclude $\sigma(a) a \sigma(a) m \neq 0$. Had we been able to push $a$ past $\sigma(a)$ then we would have got the desired contradiction. We can see why such an analysis is hopeless by taking unitary highest weight modules relative to the compact real form of $g$. All such modules are finite dimensional. If we let $\sigma_0$ denote the “compact” Chevalley antiautomorphism (EJ, 2.4) they are just those modules which admit a positive definite $\sigma_0$-contravariant form. Yet $\sigma_0$ hardly differs from $\sigma$ and in any case such an argument is hardly likely to show up the different. Of course for a finite dimensional module, $\sqrt{Q}$ is the augmentation ideal of $S(m)$ and so will coincide with $Q$ only if $\dim L(\lambda) = \dim V(\lambda)$. Since $L(\lambda)$ is a simple $g$ module and $V(\lambda)$ is a simple $\mathfrak{f}$ module the latter only holds for the trivial module.

6. Maximality and Goldie rank

6.1. Let $L(\lambda)$ be a unitary highest weight module and set $J(\lambda) = \text{Ann} L(\lambda)$. If $\lambda$ is a multiple of $\omega$ then $J(\lambda)$ is maximal (Theorem 4.2) and completely prime (Lemma 4.4). Here we study how these conclusions are modified in the general case.

6.2. Define $V(\lambda)$ as in 5.1. Recall that $U(m)$ is commutative and so identifies with $S(m)$. Set $Q = \text{Ann}_U(m) L(\lambda)$. By 5.16, $Q$ is a prime ideal and furthermore by 2.3, $L(\lambda)$ is torsion-free over the integral domain $U(m)/Q$. Since $L(\lambda) = U(m) V(\lambda)$ we conclude that $L(\lambda)$ has finite rank $r(\lambda) \leq \dim V(\lambda)$ as a $U(m)/Q$ module. Furthermore we can choose $0 \neq x \in U(m)/Q$ such that $X^{-1} L(\lambda)$ is a free rank $r(\lambda)$ module over $X^{-1} (U(m)/Q)$, where $X$ denotes the multiplicative set generated by $x$. 

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6.3. As in 4.4 we let \( F(\lambda) \) [resp. \( A(\lambda) \)] denote the subring of \( \text{End}_c L(\lambda) \) on which the diagonal action of \( g \) (resp. \( n \)) is locally finite. It is immediate that we have embeddings \( U(g)/J(\lambda) \subset F(\lambda) \subset A(\lambda) \). Since the action of \( f \) on \( L(\lambda) \) is locally finite, the diagonal action of \( f \) on \( \text{End}_c L(\lambda) \) is also locally finite. Consequently \( A(\lambda) \) identifies with the subring of \( \text{End}_c A(\lambda) \) on which the diagonal action of just \( m \) is locally finite.

6.4. Choose a preimage \( x \in U(m) \) of \( \tilde{x} \) and set \( X = \{ x^k \}_{k \in \mathbb{N}} \). Obviously \( X^{-1} L(\lambda) \) is isomorphic to \( X^{-1} L(\lambda) \) as a \( U(m) \) module. Since \( m \) is commutative and its diagonal action on \( A(\lambda) \) is locally nilpotent, then the diagonal action of \( x \) on \( A(\lambda) \) is again locally nilpotent and so \( X \) is Ore in \( A(\lambda) \). This gives \( X^{-1} L(\lambda) \) the structure of an \( X^{-1} A(\lambda) \) module. Moreover it is clear that we have a commutative square

\[
\begin{array}{ccc}
U(g)/J(\lambda) & \subset & A(\lambda) \\
\downarrow & & \downarrow \\
X^{-1}(U(g)/J(\lambda)) & \subset & X^{-1}A(\lambda)
\end{array}
\]

of ring embeddings.

6.5. Identify \( g \) with its dual \( g^* \) through the Killing form. Then \( m^* \) identifies with \( m^+ \). Let \( \mathcal{V} \) denote the subvariety of \( m^+ \) of zeros of \( Q \) and set \( \mathcal{V}_o = \mathcal{V} \setminus \{ x = 0 \} \). Let \( \mathcal{R} \) denote the ring of regular functions on \( \mathcal{V} \). Then \( X^{-1} \mathcal{R} \) identifies with the ring \( \mathcal{R}_o \) of regular functions on \( \mathcal{V}_o \). Again let \( \mathcal{D} \) (resp. \( \mathcal{D}_o \)) denote the ring of differential operators on \( \mathcal{V} \) (resp. \( \mathcal{V}_o \)). Then \( \mathcal{D}_o \) identifies with \( X^{-1} \mathcal{D} \) ([27], 15.1.25). Set \( r = r(\lambda) \) and let \( M_r(\mathcal{D}_o) \) denote the ring of \( r \times r \) matrices over \( \mathcal{D}_o \). Let \( \text{rk} \) denote Goldie rank.

**Lemma.**

(i) \( X^{-1} A(\lambda) \cong M_r(\mathcal{D}_o) \).

(ii) \( X^{-1} A(\lambda) \) is a simple ring.

(iii) \( \text{rk} (U(g)/J(\lambda)) \) divides \( r \). In particular it is bounded by \( \dim V(\lambda) \).

(i) It is clear that \( X^{-1} A(\lambda) \) identifies with the subring of \( \text{End}_c X^{-1} L(\lambda) \) on which the diagonal action of \( m \) is locally finite. Yet \( X^{-1} L(\lambda) \) is just \( \mathcal{R}_o \) as an \( \mathcal{R}_o \) module and so this proves (i).

(ii) By 4.6, \( \mathcal{D} \) is a simple ring. Hence so are \( \mathcal{D}_o \) and \( M_r(\mathcal{D}_o) \).

(iii) By [22], 7.11, the embedding \( F(\lambda) \subset A(\lambda) \) localizes to an isomorphism of rings of fractions. Hence \( \text{rk} F(\lambda) = \text{rk} A(\lambda) = r \). Then (iii) results from [19], I. 5.12.(ii).

**Remark.** — In 8.8 we give an example of \( \text{rk} (U(g)/J(\lambda)) = 1 \), when \( \dim V(\lambda) > 1 \).

6.6. We need the following fact which holds for any simple highest weight module \( L(\mu) \). Recall ([16], 6.31) that \( F(\mu) \) has finite length as a \( U(g) \) bimodule.

**Lemma.** — *The socle Soc F(\mu) of F(\mu) as a U(g) bimodule is an ideal of F(\mu) considered as a ring.*

Set \( F = F(\mu), S = \text{Soc} F(\mu) \). Let \( J \) denote the annihilator of \( F/S \) considered as a left \( U(g) \) module. By definition \( JF \subset S \). Yet \( JF \) is a \( U(g) \) bisubmodule of \( S \) and
hence a direct summand of $S$. Suppose $JF \subsetneq S$. Then $J$ annihilates a non-zero direct summand of $S$. This is excluded by [16], 10.9, 10.12, concerning Gelfand-Kirillov dimension $d(\cdot)$ and the remarkable fact that by [21], II, 4.13, and the truth of the Kazhdan-Lusztig conjectures one has $d(F/S) < d(F)$. Hence $JF = S$. Then $SF = JFF = JF = S$. Similarly $FS = S$.

6.7. Recall that we are assuming $L(\lambda)$ to be unitary.

**Proposition.** Suppose $Q = \text{Ann}_{U(m)} L(\lambda) \neq 0$. Then $X^{\text{op}} J(\lambda)$ is a maximal ideal of $X^{-1} U(g)$.

The hypothesis $Q \neq 0$ implies by 2.3 and 5.15 that $Q = Q_i$ for some $i \leq t$. Then $\tau^i = \tau_i$ and as discussed in the proof of 4.5 we may apply ([22], 5.8) to conclude that $F(\lambda) = A(\lambda)$. Then by 6.5, $X^{-1} F(\lambda)$ is a simple ring and so by 6.6 we have $X^{-1} F(\lambda)/X^{-1} \text{Soc} F(\lambda) = X^{-1} F(\lambda)/\text{Soc} F(\lambda) = 0$.

We conclude that $X^{-1} F(\lambda)$ is semisimple as an $X^{-1} (U(g)/J(\lambda))$ bimodule. It contains the latter as an indecomposable direct summand and hence as a simple bimodule. However the latter conclusion is just what is required for the assertion of the proposition.

6.8. We may combine 6.7 with the analysis of 4.2 to give a simple combinatorial condition for $J(\lambda)$ to be maximal. Let $\rho_i$ denote the half sum of the roots of $\Delta^i \cap \Delta^+$. 

**Theorem.** Assume $L(\lambda)$ unitary and $Q = \text{Ann}_{U(m)} L(\lambda) \neq 0$ (so then $Q = Q_i$ for some $i \in \{1, 2, \ldots, t\}$). Suppose $J(\lambda)$ is not maximal. Then there exists $j, 1 \leq j \leq t - 1$ such that $w_i^j w_i^1 \lambda + (j - 1) \rho_i \omega + \rho_j$ is regular, integral for $A^j$.

Let $J$ be a maximal ideal of $U(g)$ properly containing $J(\lambda)$. Then $J \cap X \neq \emptyset$ by 6.5. Suppose $x' \in J$ and set $Z = U(f), x' \subset U(m)$. Obviously $Z \subset J$ and in particular $Z^{x'} \subset J$. By the commutativity of $m$ one has $Z^n = Z^{x'}$. By 2.1, the weight vectors of $Z^n$ are products of the $v_k, k \in \{1, 2, \ldots, t\}$. If every such product has a factor of $v_k$ with $k \leq i$ we conclude by 8.1, that $Z^n \subset Q_i$ and so $x' \in Z \subset Q_i$, which contradicts that $x'$ has a non-zero image in $U(m)/Q_i$. This proves that there exists $j, 1 \leq j \leq i - 1$ such that $J \cap Y_j = \emptyset$ and $J \cap Y_{j+1} \neq \emptyset$.

Since $i \leq j$ we have an embedding $L(\lambda) \subsetneq Y_j^{-1} L(\lambda)$. Set $\tilde{g} = \Theta^j(q')$. By the repeated application of the construction of Section 5 we obtain a highest weight $U(\tilde{g})$ submodule $L_j$ of $Y_j^{-1} L(\lambda)$ of highest weight $\lambda_j$. Noting the combinatorial fact that $v_{\rho_i, \omega} = v_{\rho_i, \omega}$, $\forall k = 1, 2, \ldots, t$ which can for example be checked using EJ, Table, and that $w_{c}^{k+1} w_{c}^1 \lambda + (k - 2) e_{\rho_i, \omega} \omega + e_{\rho_i, \omega} \omega = w_{c}^{k} w_{c}^1 \lambda + (k - 1) e_{\rho_i, \omega} \omega$, we conclude from 5.3 that $\lambda_j = w_{c}^{j} w_{c}^1 \lambda + (j - 1) e_{\rho_i, \omega} \omega$ on $\Delta^j$. Set $J = Y_j^{-1} J \cap U(\tilde{g})$. Since $J \supset J(\lambda) = \text{Ann}_{U(g)} L(\lambda)$ and because $Y_1$ is Ore in $U(g)$ we obtain $\tilde{L}_j = 0$. Yet by 3.4 $\tilde{x}_{-\rho_j} = \Theta^j(x_{-\rho_j}) = v_j^{\sim 1} v_j$ and so $\tilde{x}_{-\rho_j} \in \tilde{J}$ for some $\ell \in \mathbb{N}$ by the hypothesis on $j$. By Borho's lemma ([18], 6.11) $\tilde{J}$ has finite codimension in $U(\tilde{g})$. Since $\tilde{J}_L = 0$, this forces $\lambda_j + \rho_j$ to be integral, regular for $\Delta^j$. Hence the theorem.

6.9. One can ask if the converse to 6.8 holds. For $j = 1$ the criterion is just that $\lambda + \rho$ be regular, integral. Since $i \geq 2$, we cannot have $\lambda = 0$. One has $(\lambda + \rho, \alpha) < 1$ for
the non-compact simple root $\alpha$ at the last place of unitarity and hence for all $L(\lambda)$ for otherwise by EJ, 7.9, one should have $\dim L(\lambda) < \infty$ which implies $\lambda = 0$. Since $\lambda$ is assumed integral this forces $(\lambda + \rho, \alpha) \leq 0$. Consequently $\lambda + \rho$ is not dominant. Then $(\lambda + \rho)$ being regular implies that $J(\lambda)$ is not maximal. We shall eventually obtain non maximal $J(\lambda)$ satisfying the hypotheses of the theorem in this fashion (Sect. 8). In such cases $L(\lambda)$ is not free over $U(g)/\mathbb{Q}$ because this would then contradict 6.7.

6.10. There are two difficulties in extending 6.9 to the case $j > 1$. The first is a combinatorial result which is rather strange.

**Lemma.** — Take $j \in \{1, 2, \ldots, t\}$. Then

$$w_\alpha^j w_\beta^j \rho_1 - \rho_j = 2(j - 1) \varepsilon_{\beta, \alpha} \omega$$
on A^j.

It is obvious that $w_\alpha^j w_\beta^j$ sends a simple root of $\Delta^j$ to a simple root of $\Delta^j$ and so the left hand side restricts to zero on $\Delta^j$. It remains to show that both sides agree on $\alpha$. Observe that $w_\alpha^j w_\beta^j \beta_j = w_\alpha^j \alpha = \beta_1$. Recall that $\alpha, \beta_1$ are both long roots. Then by the first result $(\alpha^j, w_\alpha^j w_\beta^j \rho_1 - \rho_j) = (\beta_1^j, w_\alpha^j w_\beta^j \rho_1 - \rho_j) = (\beta_1^j - \beta_1^j, \rho)$. By EJ, 3.6, one has $\varepsilon_{\beta_1^j, \alpha} = (1/2)(\beta_1^j - \beta_1^j + 1, \rho)$ and we already remarked in 6.8 that they are all equal. We conclude that $(\beta_1^j - \beta_1^j, \rho) = 2(j - 1) \varepsilon_{\beta, \alpha}$ as required.

**Remark.** — The consequence of this unfortunate fact is the following. Let $\lambda_j$ denote the highest weight of the $\mathfrak{g}$-module $L_j$ considered in the proof of 6.8. Let $L_j'$ denote a second such module obtained from some $L(\lambda')$. Then $\lambda_j + \rho_j = w_\alpha^j w_\beta^j (\lambda + \rho) - (j - 1) \varepsilon_{\beta, \alpha} \omega$, whereas had it not been for the presence of the factor of $2$ above the second term would not have appeared. This in turn would have meant that if $\text{Ann}_{\mathfrak{g}} L(\lambda) = \text{Ann}_{\mathfrak{g}} L(\lambda')$, then $\text{Ann}_{\mathfrak{g}} L_j = \text{Ann}_{\mathfrak{g}} L_j'$. In fact this pleasant conclusion does not necessarily hold. Perhaps this is because Section 5 ignored the contribution of the opposite copy $\sigma(\alpha)$ of $\alpha$. In any case the latter leads to the second of our difficulties noted below.

6.11. As in Section 5 we let $N(\mu)$ denote the module induced from a finite dimensional simple $p$ module $V(\mu)$. Let $L(\mu)$ be the unique simple quotient of $N(\mu)$. Set $L_1 = L(\mu)$ and let $j$ be the largest integer $\leq t + 1$ such that $L_1$ is $Y_j$ torsion-free. Let $L_j$ be the $\mathfrak{g} := \mathfrak{g}^j$ submodule of $Y_j^{-1} L_1$ obtained by a repeated application of the construction in Section 5. Our second difficulty in proving the converse to 6.8 comes from not knowing if the following holds.

$(\mathfrak{g}_{\alpha})$ $L_j$ is a simple $\mathfrak{g}$ module.

We let $\mathfrak{g}$ denote the corresponding question when we further impose that $\text{Ann}_{U(\mu)} L(\mu) \neq 0$. By 5.15, $\mathfrak{g}$ holds for unitary modules, since a unitary highest weight module is necessarily simple.

6.12. We first need the following fact which holds for any simple highest weight module $L(\mu)$. Let $d_A(M)$ denote the Gelfand-Kirillov dimension of a module $M$ over a $\mathbb{C}$-algebra $A$. Let $L$ be a non-zero $U(n)$ submodule of $L(\mu)$ and set $d = d_{U(n)}$. 

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**Lemma.** — *One has $d(L) = d(L(\mu))$.*

We can assume without loss of generality that $L$ is cyclic, say $L = U(n) f$. The Borel subalgebra $b = n^+ \oplus \mathfrak{h}$ acts locally finitely on $L(\mu)$. Thus $U(b) f$ is finite dimensional and so by Lie’s theorem admits a one-dimensional submodule $C e$. Obviously $C e$ is the unique highest weight space of the simple module $L(\mu)$ and so $U(\mathfrak{n}) e = L(\mu)$. Again we can choose a finite dimensional subspace $V_0$ of $U(b)$ such that $V_0 f = U(b) f$. Set $V = U(ad n) V_0 \subset U(\mathfrak{g})$ which is again finite dimensional. Since $V$ is $ad n$ stable we obtain $L(\mu) = U(\mathfrak{n}) e = U(\mathfrak{n}) V f = VU(\mathfrak{n}) f = V L$. Thus $L(\mu)$ is an image of $V \otimes L$ viewed as a $U(\mathfrak{n})$ module for diagonal action. Then $d(L(\mu)) \leq d(V \otimes L) = d(L) \leq d(L(\mu))$, as required.

6.13. Now consider the situation described in 6.10 and set $J_j = Y_j^{-1} J(\mu) \cap U(\mathfrak{g})$. It is clear that $J_j L_j = 0$; but it is not obvious if $J_j = \text{Ann} L_j$. For any $\mathbb{C}$-algebra $A$, set $d(A) = d(A)$ and recall $d(V(\mathfrak{g})/\text{Ann} L(\mu)) = 2 d_U(\mathfrak{g})(L(\mu)) = 2 d_U(\mathfrak{n})(L(\mu))$ by [16], 10.9.

**Lemma.**

(i) $d(U(\mathfrak{g})/J_j) = d(U(\mathfrak{g})/\text{Ann} L_j)$.

(ii) $d(U(\mathfrak{g})/J(\mu)) = 2 \left( \sum_{j=1}^{j-1} |\Gamma_j^j| \right) + d(U(\mathfrak{g})/\text{Ann} L_j)$.

Let $A$ be the localized Weyl algebra $Y_j^{-1} U(\mathfrak{e})$. Set $K = Y_j^{-1} U(\mathfrak{a})$ which can be considered both as a ring and a standard $A$ module. The construction of Section 5 gives a ring embedding $B := A \otimes U(\mathfrak{g})/J_j \subset Y_j^{-1} (U(\mathfrak{g})/J(\mu))$ and a $B$ module embedding $K \otimes L_j \subset Y_j^{-1} L(\mu)$. The first inclusion gives us

$$d(A) + d(U(\mathfrak{g})/J_j) \leq d(Y_j^{-1} (U(\mathfrak{g})/J(\mu))) = d(U(\mathfrak{g})/J(\mu)), \quad \text{by [4], 6.1}$$

$$= 2 d_U(\mathfrak{g})(L(\mu)).$$

Now

$$d(A) = 2 d_A(K) = 2 \dim \mathfrak{a}^o = 2 \sum_{i=1}^{j-1} |\Gamma_i^j|$$

whereas by [16], 10.9

$$d(U(\mathfrak{g})/J_j) \geq d(U(\mathfrak{g})/\text{Ann} L_j) = 2 d(L_j).$$

Yet $L' := L(\mu) \cap (K \otimes L_j)$ is a non-zero $U(\mathfrak{n})$ submodule of $L(\mu)$ and so by 6.12 we obtain

$$d_U(\mathfrak{n})(L(\mu)) = d_U(\mathfrak{n})(L(\mu)) = d_U(\mathfrak{n})(L') \leq d_U(\mathfrak{n})(K \otimes L_j) \leq d_A(K) + d(L_j).$$

Hence

$$d(A) + d(U(\mathfrak{g})/J_j) \leq 2 (d_A(K) + d(L_j))$$

$$\leq d(A) + d(U(\mathfrak{g})/J_j).$$
This forces all the above inequalities to be equalities and then inspection verifies (i), (ii).

6.14. Let $L(\mu)$ denote a simple but not necessarily unitary quotient of the induced module $N(\mu)$. Set $e = e_{\mu, \rho}$. Suppose for some $j$, $1 \leq j \leq t$ that $y w_i^j w_i^j (\mu + \rho) - (j - 1) \omega_0$ is dominant, regular and integral on $\Delta^j$, for some $y \in W$ such that $w_i^j w_i^j (\mu + \rho) - \rho$ is \textit{f} dominant.

**Proposition.** — Let $j$ be the least positive integer with the above property. Assume that $\mu_i + \rho := w_i^j w_i^j (\mu + \rho) - (i - 1) \omega_0$ is not dominant, regular and integral on $\Delta^i$ for all $i$, $1 \leq i \leq j$. Then if $\mathcal{E}_0$ holds, $J(\mu)$ is not maximal.

Let $i$ be the largest positive integer $\leq j$ such that $L(\mu)$ is $Y_i$ torsion-free. Then we can define $L_i \subset Y_i^{-1} L(\mu)$ as in 6.10. By our calculation in the remark following 6.9, we find that $L_i$ has highest weight $\mu_i$. Thus the second hypothesis just means that $L_i$ is not $\Theta^i(x_{-\rho})$ torsion and so $L(\mu)$ is not $Y_{i+1}$ torsion. We conclude that $L_j$ is defined and has no $\Theta^j(x_{-\rho})$ torsion. In particular $L_j$ is not finite dimensional.

Set $x = w_i^j w_i^j$ and $\mu' = x^{-1} y x (\mu + \rho) - \rho$. Then by the hypothesis on $y$ the construction of Section 5 applies to $L(\mu')$. Suppose $L(\mu')$ has $Y_j$ torsion for some positive integer $j \leq j$ and let $i$ be the least integer with this property. Then $L_i' \subset Y_i^{-1} L(\mu')$ constructed as above [but with respect to $L(\mu')$] has $\Theta^i(x_{-\rho})$ torsion and so is finite dimensional. Applying 6.13 to this and our previous assertion we obtain $d(U(g)/J(\mu)) > d(U(g)/J(\mu'))$. Since $J(\mu) \cap Z(g) = J(\mu') \cap Z(g) \in \text{Max } Z(g)$ we conclude from say [16], 5.21, that $J(\mu)$ is not maximal [but not necessarily contained in $J(\mu')$].

Now assume that $L(\mu')$ has no $Y_j$ torsion. Then $L_j'$ is defined and has highest weight $x \mu' + (j - 1) \omega_0 = y w_i^j w_i^j (\mu + \rho) - (j - 1) \omega_0 - \rho_j$. Thus the first hypothesis exactly means that $L_j'$ has a finite dimensional quotient. Now given the $\mathcal{E}_0$ holds we obtain that $L_j'$ itself is finite dimensional. Finally we apply 6.13 as above to obtain the required conclusion.

6.15. Even admitting $\mathcal{E}_0$, the above result is not a precise converse to 6.8. We remark that if $\text{Ann}_{U(\mu)} L(\mu) \neq 0$, then we need only assume $\mathcal{E}$ holds. This is not quite trivial, but follows by the reasoning in 5.16. Finally suppose $\text{Ann}_{U(\mu)} L(\lambda) = 0$. Then even for $L(\lambda)$ unitary one would expect that $J(\lambda)$ could fail to be maximal without the conclusion of 6.8 being satisfied.

6.16. One may ask if one can have a strict inclusion $J(\lambda) \subsetneq J(\mu)$ with both $L(\lambda)$, $L(\mu)$ unitary. The above methods essentially reduce such questions to the case when $\lambda = 0$ and so $\mu = y^{-1} \rho - \rho$ for some $y \in W$. Notice however that $y^{-1} \rho - \rho$ can be a unitary parameter. Indeed write $\mu := y^{-1} \rho - \tau + u \omega$ in the conventions of EJ, Sect. 3. Then $u = (\beta^\vee, y^{-1} \rho) - (\beta^\vee, \tau + \rho)$, whereas by EJ, 4.1, we require $u = 1 + 2(\beta^\vee, \rho_j) - (\beta^\vee, \tau + \rho)$ for $\mu$ to be a (last) place of unitarity. Now take $g$ simple of type $A_n$ and $y = s_{a_1} s_{a_2} \ldots s_{a_t}$ where $a_i$ is the non-compact simple root. Then $y a_i = a_{i+1}, i < t$, $y a_{t+1} = a_1 + a_2 + \ldots + a_{t+1}$, $y a_j = a_j, j > t + 1$. Thus $\tau$ is a multiple of $\alpha_{t+1}$ and so $2(\beta^\vee, \rho_j) = n - 2$. Again $(\beta^\vee, \rho - y^{-1} \rho) = (\beta^\vee, a_1 + a_2 + \ldots + a_t) = 1$, so $(\beta^\vee, y^{-1} \rho) = n - 1$. Thus the required identity is satisfied.
7. Computation of varieties

7.1. Let $L(\lambda)$ be a unitary highest weight module. In this section we compute the associated variety $Y(L(\lambda))$ of $L(\lambda)$. Here it is perhaps helpful to recall some definitions. If $L$ is a $U(\mathfrak{g})$ module generated by a finite dimensional subspace $L^0$, then we let $Y(L^0)$ denote the subvariety of $\mathfrak{g}^*$ of zeros of the graded ideal $\text{gr} \text{Ann}_{U(\mathfrak{g})} L^0$. By an old result of I. N. Bernstein ([16], 17.2) this is independent of the choice of $L^0$ and so we may define the associated variety $Y(L)$ of $L$ to be $Y(L^0)$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. If $L$ is the image of a module induced from a finite dimensional module of a parabolic subalgebra $p=\mathfrak{f} \oplus \mathfrak{m}^+$, then it is easy to see that $Y(L)$ is a closed $\mathfrak{f}$ stable subvariety of $\mathfrak{m}^+$. In general it is false that $Y(L)$ is irreducible even for $L$ simple ([23], 10.1 and [24] note added in proof). However in our present situation this holds by EJ, 8.1, and 2.3 which in the notation of 3.5 implies that $Y(L(\lambda))=Y_{ij}$ for some $i=\{1,2,\ldots, t+1\}$. Our aim is to calculate $i$ as a function of $\lambda$. By 2.4 and 5.16 it is sufficient to do this when $\lambda$ is at a last place of unitarity, that is when $\lambda=\tau+u_i$ in the notation of EJ, 1.6.

7.2. Set $L_i=L(\lambda_i)$ with $\lambda_i=\tau+u_i \omega$, $\tau_i \in P_+^*$ and $u_i \omega$ as given by EJ, 7.1. Assume $\lambda \neq 0$, so that $L_1$ has no $Y_2$ torsion, equivalently that $\tau_1 \neq 0$. Set $\tilde{\mathfrak{g}}=\Theta^2(\mathfrak{g}^2)$ and let $L_2$ be the highest weight $\tilde{\mathfrak{g}}$ module constructed in Section 5. Then $L_2$ has highest weight $\lambda_2$ which equals $w_2 w_1 \lambda+\epsilon_{\mathfrak{g}_2}$ on $\Delta^2$. Clearly we can write

$$\lambda_2=\tau_2+u \omega, u \in \mathbb{R}$$

where $\tau_2$ is a dominant $t^2$ weight which we can choose so that $(\tau_2, \alpha)=0$. We show that if $s_2$ denotes the level of $\tau_2$ then either $u<u_2^2$ or $u=u_2^2$ for some positive integer $i \leq s_2$. (Both situations can arise.) To compute $u$ it is enough to compare it with $u_2^2$. Here we set $\epsilon=\epsilon_{\mathfrak{g}_2}$ which we recall also equals $\epsilon_{\mathfrak{g}}$. One has the

**Lemma** (Notation, EJ, 3.4, 4.2)

(i) $u_2^2=\frac{1}{2} \left| S_{1,1} \right| + 2 \left( \rho_{11}, \beta_2^\vee \right) - \left( \rho_1 + \tau_1, \beta_2^\vee \right)$

(ii) $u_2^2=\frac{1}{2} \left| S_{1,2} \right| + 2 \left( \rho_{12}, \beta_2^\vee \right) - \left( \rho_1 + \tau_2, \beta_2^\vee \right)$

(i) is immediate [noting that $r$ is chosen so that $(\tau_2, \alpha)=0$]. For (ii) we recall from (EJ, 4.2) that

$$u_2^1=1+\frac{1}{2} \left| S_{1,1} \right| + 2 \left( \rho_{11}, \beta_1^\vee \right) - \left( \rho_1 + \tau_1, \beta_1^\vee \right)$$

$$u_2^2=1+\frac{1}{2} \left| S_{1,2} \right| + 2 \left( \rho_{12}, \beta_2^\vee \right) - \left( \rho_1 + \tau_2, \beta_2^\vee \right).$$

By definition $u=\epsilon+u_2^1+r$ and so $u-u_2^2=\epsilon+u_2^1-(u_2^2-r)$. Hence (ii) results from the above if we note that $(\tau_2, \beta_2^\vee)+r=(w_2 w_1 \tau_1, \beta_2^\vee)=(w_1 \tau_1, \alpha^\vee)=(\tau_1, \beta_1^\vee)$ and $(\rho_1, \beta_1^\vee)-(\rho_2, \beta_2^\vee)=(\rho_1, \beta_1^\vee)-(\rho_2, \beta_2^\vee)=2\epsilon$ by EJ, 3.6.

7.3. Define $\alpha_i$, $\alpha_i'$, $i=1,2,\ldots, t$ as in EJ, 2.1. By [18], 2.2 (iv), one has $2(\beta_1^\vee, \alpha_i)=2(\beta_1^\vee, \alpha_i')=1$ whilst $\beta_1$ vanishes on the remaining simple roots. This leads
to the following simple rule. Let \( c_i, c'_i, i = 1, 2 \) denote the coefficients of \( \alpha_i, \alpha'_i \) in \( 2\rho_i \) expanded in terms of the simple roots of \( \Delta_i \). Then

**Lemma.**

\[
2(\rho_{i1}, \beta_{11}^i) - 2(\rho_{i2}, \beta_{21}^i) = (c_1 + c'_1) - (c_2 + c'_2).
\]

7.4. We compute the right hand side of 7.2 (ii) for each simple Lie algebra \( g \) and each choice of non-compact simple root \( \alpha \). First assume \( g \) of type \( \Delta_i \). Adopt the Bourbaki notation ([5], Pl. I) and take \( \alpha = \alpha_i \) where we can assume \( 2t \leq n+1 \) and \( t > 1 \). (This assures that \( t \) is as defined in EJ, 1.4.) One has \( S_{1,t} = \emptyset \) and \( \varepsilon = 1 \) in this case. Set

\[
\pi' = \{ \alpha_1, \alpha_2, \ldots, \alpha_{t-1} \}, \quad \pi'' = \{ \alpha_{t+1}, \ldots, \alpha_i \}.
\]

Obviously \( \pi_c = \pi' \cup \pi'' \). Given \( \tau_1 \in P_c^+ \) we set \( \text{Supp} \tau_1 = \{ \gamma \in \pi_c \mid (\tau, \gamma) \neq \emptyset \} \) and define \( \tau_2 \in P_{c,2}^+ \) by 7.2 (i).

**Lemma.** — Assume \( g \) of type \( \Delta_n \) and \( \alpha = \alpha_i \) (as above). Then for all \( 0 \neq \tau_1 \in P_c^+ \) one has

\[
u - u_{i1}^2 = \begin{cases} -e_{\beta, \gamma} & (\text{Supp } \tau_1) \cap \pi' \neq \emptyset, \\ 0 & (\text{Supp } \tau_1) \cap \pi' = \emptyset. \end{cases}
\]

One easily checks for all \( j \in \{1, 2, \ldots, l\} \setminus \{t\} \) that

\[-w_c^i \alpha_j = \begin{cases} \alpha_{t-j}, & j \leq t-1, \\ \alpha_{t+1-j}, & j \geq t+1. \end{cases}\]

Combined with a similar result for \( w_c^2 \alpha_j \) in the \( \Delta_{i-2} \) system \( \{\alpha_2, \ldots, \alpha_{i-1}\} \) this gives

\[(*) \quad w_c^1 w_c^2 \alpha_j = \begin{cases} \alpha_{j-1}, & 1 < j < t, \\ \alpha_{j+1}, & t < j < l. \end{cases}\]

Now let \( \omega_j, j = 1, 2, \ldots, l \) denote the fundamental weights corresponding to \( \pi' \) and \( \omega'_j, j = 2, 3, \ldots, l-1 \) the fundamental weights corresponding to \( \pi'' \). Then from \((*)\) we obtain

\[
w_c^2 w_c^1 \omega_j = \begin{cases} \omega_{j+1}^2, & 1 \leq j < t-1, \\ \omega_{j-1}^2, & t+1 < j \leq n, \\ 0, & j = t-1 \text{ or } t+1. \end{cases}\]

This allows us to compute \( \tau_2 \) from \( \tau_1 \). If we view \( \tau_1 \) as given by a Dynkin diagram weighted by the coefficients of \( \omega_j^1 \), then \( \tau_2 \) is obtained from \( \tau_1 \) by deleting the extreme vertices and letting the weights move by one step towards \( \alpha_i \). It easily follows from this in the notation of 7.3 that

\[
c_1 - c_2 = \begin{cases} 1, & \pi' \cap \text{Supp} \tau_1 = \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]
with a similar expression for the primed quantities but replacing \( \pi \) by \( \pi' \). Then by 7.3 we obtain the assertion of the lemma.

7.5. Assume \( g \) simple of type \( B_1 \). In the Bourbaki notation ([5], Pl. II) we can only have \( \alpha = \alpha_1 \). Again \( \alpha_2 \) is the unique simple root not orthogonal to \( \beta_1 \). Furthermore \( \Delta^2 = \{ \pm \alpha \} \) and so we always have \( \tau_2 = 0 \). From EJ, Table, we obtain \( \varepsilon = -3/2 \). Let \( k \in \{ 2, \ldots, l \} \) be the smallest integer such that \( \alpha_k \in \text{Supp} \tau_1 \) (recall \( \tau_1 \neq 0 \)). Then the connected component of \( \Delta_{\tau_1} \) containing \( \alpha_2 \) is the \( A_{k-2} \) system \( \{ \alpha_2, \ldots, \alpha_{k-1} \} \) and so by 7.3 we obtain \( 2(\rho_{\tau_1}, \beta_1^\vee) - 2(\rho_{\tau_2}, \beta_2^\vee) = k - 2 \).

Finally \( S_{\tau_1} \neq \emptyset \) exactly when \( \tau_1 = \omega_1 \) and then \( |S_{\tau_1}| = 1 \) and \( k = l \) above. Putting all this together we obtain the

**Lemma.** Assume \( g \) of type \( B_1 \) and \( \alpha = \alpha_1 \). For all \( 0 \neq \tau \in P_+^* \) one has

(i) \( \tau_2 = 0 \).

(ii) \( u = \begin{cases} l - k - (1/2) < 0, & \tau_1 \neq \omega_1 \\ 0, & \tau_1 = \omega_1. \end{cases} \)

7.6. Assume \( g \) simple of type \( C_l \). In the Bourbaki notation ([5], Pl. III) we can only have \( \alpha = \alpha_n \). Again \( \alpha_1 \) is the unique simple root not orthogonal to \( \beta_1 \). Let \( k = \{ 1, 2, \ldots, l - 1 \} \) be the smallest integer such that \( \alpha_k \in \text{Supp} \tau_1 \). Then the connected component of \( \Delta_{\tau_1} \) containing \( \alpha_1 \) is the \( A_{k-1} \) system \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \} \). Now \( \Delta^1_0 \) (resp. \( \Delta^2_0 \)) is the \( A_{l-1} \) (resp. \( A_{l-2} \)) system \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{l-1} \} \) (resp. \( \{ \alpha_2, \alpha_3, \ldots, \alpha_{l-1} \} \)). Exactly as in 7.4 this gives

\[
(*) \quad w^1_c w^2_c \alpha_j = \alpha_{j-1}, \quad \forall j, \quad 1 < j < l.
\]

Define \( \omega^j_p, i = 1, 2; j \in \{ i, i+1, \ldots, l-1 \} \) as in 7.4. Then from (*) we obtain

\[
w^2_c w^1_c \omega^j_c = \begin{cases} \omega^{j+1}_c, & i \leq j < l - 1 \\ 0, & j = l - 1. \end{cases}
\]

on \( \Delta^2 \). View \( \tau_1, \tau_2 \) as weighted Dynkin diagrams. Then \( \tau_2 \) is obtained from \( \tau_1 \) by removing the left hand vertex and letting the weights move one step towards \( \alpha_i \). It easily follows that the connected component of \( \Delta_{\tau_2} \) containing \( \alpha_2 \) is the \( A_{k-1} \) system \( \{ \alpha_2, \alpha_3, \ldots, \alpha_k \} \). From 7.3, we conclude that

\[
2(\rho_{\tau_1}, \beta_1^\vee) - 2(\rho_{\tau_2}, \beta_2^\vee) = 0.
\]

Now assume that \( \tau_1 \) is not generic in the sense of EJ, 3.4. Then as noted in EJ, 4.3, either \( \tau_1 = \omega_m \) for some positive integer \( m < l \) (in this case set \( n = l \)) or there exists \( n, m \leq n < l \) such that

\[
\tau_1 = \omega_m + \omega_n + \sum_{i=n}^{l-1} r_i \omega_i, \quad r_i \in \mathbb{N}.
\]
Moreover

\[ |S_{1, \tau_1}| = n - m. \]

We conclude that

\[ |S_{1, \tau_1} - |S_{1, \tau_2}| = \begin{cases} 1, & \tau_1 = \omega_m, \ 1 \leq m < l. \\ 0, & \text{otherwise}. \end{cases} \]

Finally from EJ, Table, we obtain \( \varepsilon = 1/2. \) Putting all this together we obtain the

**Lemma.** — Assume \( g \) of type \( C \) and \( \alpha = \alpha_i \). Then for all \( 0 \neq \tau_1 \in P_\varepsilon^+ \) one has

\[ u - u_{\tau_1}^2 = \begin{cases} 0, & \tau_1 = \omega_m, \ 1 \leq m < l. \\ -\varepsilon_{\alpha_i}, & \text{otherwise}. \end{cases} \]

7.7. Assume \( g \) of type \( D \) with \( \alpha = \alpha_i \). This case is very similar to type \( B \). We get \( \tau_2 = 0 \) and \( \varepsilon = l - 2 \). We can choose \( k \in \{1, \ldots, l\} \) so that the connected component of \( \Delta \) containing \( \alpha_2 \) is a system of type \( A_{k-2} \). Then \( 2(\rho_{\tau_1}, \beta_1) - 2(\rho_{\tau_2}, \beta_1) = k - 2 \). Consequently

**Lemma.** — Assume \( g \) of type \( D \) with \( \alpha = \alpha_i \). For all \( 0 \neq \tau_1 \in P_\varepsilon^+ \) one has

(i) \( \tau_2 = 0. \)

(ii) \( u = k - l. \)

In particular \( u < 0 \) unless \( \text{Supp } \tau_1 = \{ \alpha_{k-1} \} \) or \( \{ \alpha_i \} \).

7.8. Assume \( g \) of type \( D \) with \( \alpha = \alpha_{i-1} \) or \( \alpha_i \). These cases are equivalent so we shall assume \( \alpha = \alpha_i \). Again \( \alpha_2 \) is the unique simple root orthogonal to \( \beta_1 \). As in 7.4 one checks that

\[ w_1^i \omega_j = \alpha_j, \quad \forall j, \ 2 < j < l - 1. \]

Define \( \omega_j^i, i = 1, 2; j \in \{2 i - 1, 2 i, \ldots, l - 1\} \) as in 7.4. Then (\( *) \) gives

\[ w_2^i w_1^j \omega_j^i = \begin{cases} \omega_j^2, & 1 \leq j \leq l - 3. \\ 0, & j = l - 2, l - 1. \end{cases} \]

on \( \Delta^2 \). Thus \( \tau_2 \) is obtained from \( \tau_1 \) as weighted Dynkin diagrams by removing the vertices at \( \alpha_1, \alpha_2 \) and moving weights by two steps towards \( \alpha_{i-1} \).

If \( \text{Supp } \tau_1 = \{ \alpha_i \} \) set \( k = l \). Otherwise let \( k \in \{2, \ldots, l - 1\} \) be the smallest integer such that \( \alpha_k \in \text{Supp } \tau_1 \). We must distinguish four cases

1) \( \alpha_1 \in \text{Supp } \tau_1, \ k < l - 1. \)

In this case \( \Delta_1 \) (resp. \( \Delta_2 \)) is the \( A_{k-2} \) system \( \{ \alpha_2, \ldots, \alpha_{k-2} \} \) (resp. \( \{ \alpha_4, \ldots, \alpha_{k+1} \} \)). Then by 7.3 we obtain

(\( \star \))

\[ 2(\rho_{\tau_1}, \beta_1^i) - 2(\rho_{\tau_2}, \beta_1^i) = 0. \]

2) \( \alpha_1 \notin \text{Supp } \tau_1, \ k < l - 1. \)
Here the only difference is we adjoin $\alpha_1$ (resp. $\alpha_3$) to the above description of $\Delta_{\pm}$ (resp. $\Delta_{-\pm}$). Thus ($\ast$) also holds in this case.

3) $\alpha_1 \in \text{Supp} \tau_1$, $k=1$ or $l$.

In this case $\Delta_{\pm}$ (resp. $\Delta_{-\pm}$) is the $A_k$ (resp. $A_{-k}$) system $\{\alpha_2, \ldots, \alpha_{k-1}\}$ (resp. $\{\alpha_4, \ldots, \alpha_{l-1}\}$). Then by 7.3 recalling that $(\alpha_2, \beta_1) \neq 0$ we obtain

$$2(\rho_{\beta_1}, \beta_1') - 2(\rho_{\beta_2}, \beta_2') = (k-2) - (l-4) = 2 - l + k.$$

4) $\alpha_1 \notin \text{Supp} \tau_2$, $k=1$.

In this case $\Delta_{\pm}$ (resp. $\Delta_{-\pm}$) is the $A_{k-2}$ (resp. $A_{-k-2}$) system $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-3}\}$ (resp. $\{\alpha_3, \alpha_4, \ldots, \alpha_{l-3}\}$). Then by 7.3 recalling that $(\alpha_2, \beta_1) \neq 0$ we obtain

$$2(\rho_{\beta_1}, \beta_1') - 2(\rho_{\beta_2}, \beta_2') = 2(l-3) - 2(l-4) = 2.$$

Finally $\varepsilon = 2$. Summarizing we obtain the

**Lemma.** — Suppose $g$ of type $D_4$ with $\alpha = \alpha_i$. Then for all $0 \neq \tau_1 \in P^+_c$ one has

$$u - u_1^2 = \begin{cases} 0, & \text{Supp } \tau_1 = \{\alpha_1\} \text{ or } \{\alpha_{i-1}\}, \\ -\frac{1}{2} \varepsilon_{b, a}, & \text{Supp } \tau_1 = \{\alpha_1, \alpha_{i-1}\}, \\ -\varepsilon_{b, a}, & \text{otherwise}. \end{cases}$$

7.8. Now assume $g$ of type $E_6$. We can assume $\alpha = \alpha_i$ without loss of generality. Taking

$$\tau_1 = \sum_{i=2}^{6} k_i \omega_i^1$$

we obtain

$$\tau_2 = k_5 \omega_2^3 + k_3 \omega_4^3 + k_4 \omega_2^3 + k_2 \omega_6^3.$$

Finally $\varepsilon = 3$ for type $E_6$. An easy computation gives the

**Lemma.** — Suppose $g$ of type $E_6$ with $\alpha = \alpha_1$. Then for all $0 \neq \tau_1 \in P^+_c$ one has

$$u - u_1^2 = \begin{cases} 0, & \text{Supp } \tau_1 = \{\alpha_6\}, \\ -\frac{2}{3} \varepsilon_{b, a}, & \text{Supp } \tau_1 = \{\alpha_3\} \text{ or } \{\alpha_3, \alpha_6\}, \\ -\varepsilon_{b, a}, & \text{otherwise}. \end{cases}$$

7.9. Finally assume $g$ of type $E_7$. We must have $\alpha = \alpha_7$. Taking

$$\tau_1 = \sum_{i=1}^{6} k_i \omega_i^1$$
we obtain
\[ \tau_2 = k_5 \omega_2^2 + k_2 \omega_3^2 + k_4 \omega_4^2 + k_3 \omega_5^2 + k_1 \omega_6^2. \]

Finally \( \varepsilon = 4 \) for type \( E_7 \). An easy computation gives the

**Lemma.** — *Suppose \( g \) of type \( E_7 \) with \( \alpha = \alpha_7. \) Then for all \( 0 \neq \tau_1 \in P_\varepsilon^* \) one has

\[
\begin{aligned}
    u - u_1^{\tau_1} &= \begin{cases} 
        - \frac{3}{4} \varepsilon_{\mu, \alpha}, & \text{Supp} \tau_1 = \{ \alpha_2 \}, \\
        - \varepsilon_{\mu, \alpha}, & \text{otherwise}. 
    \end{cases}
\end{aligned}
\]

7.10. We can now complete the one verification needed to establish 5.15. Let \( L(\lambda_1) \) be unitary and define \( \lambda_2 \) considered as weight of \( A_2 \) as in 5.3.

**Proposition.** — *The \( g^2 \) module \( L(\lambda_2) \) is unitary.*

By 5.3, \( \lambda^2 = w_\varepsilon^2 \lambda_1 + \varepsilon_{\mu, \alpha} \omega \) which we must show is a unitary place for \( g^2 \). [By viewing \( L(\lambda_1) \) as a unitary \( g^2 \) module it is trivial that \( w_\varepsilon^2 \lambda_1 \) is a unitary place for \( g^2 \); but this is not quite enough!]

Recall that \( \varepsilon_{\mu, \alpha} = \varepsilon_{\mu, \alpha^2} \) and that \( W_\varepsilon^2 \) stabilizes \( \omega \). By the equal-spacing principle, it follows that it is enough to establish the assertion when \( \lambda_1 \) is a last place of unitarity. Write \( \lambda_2 = \tau_2 + u \omega \) as in 7.2. By the equal spacing principle it is then enough to check that \( u - u_1^{\tau_2} \) is always a non-positive integer. From 7.4 to 7.8 we see that this is sometimes false! However in all the bad cases (i.e. when \( u - u_1^{\tau_2} < 0 \) and is not integer) we check below that \( \tau_2 + u_1^{\tau_2} \omega \) is always the first reduction point, so in fact \( \lambda_2 \) is a unitary place. This verification is quite trivial in type \( B_7 \) since \( g^2 \) is of type \( A_1 \) in that case. In the remaining bad cases it is enough to observe that \( \tau_2 \) is always of level 1. Thus for \( D_7 \) with \( \alpha = \alpha_i \) (or \( \alpha_{i-1} \)) we have \( \alpha_i \in \text{Supp} \tau_1 \). Then \( \alpha_3 \in \text{Supp} \tau_2 \) so \( \tau_2 \) is of level 1. In type \( E_6 \) with \( \alpha = \alpha \) we have \( \alpha_4 \in \text{Supp} \tau_1 \). Thus \( \alpha_4 \in \text{Supp} \tau_2 \), so \( \tau_2 \) is of level 1. In type \( E_7 \) with \( \alpha = \alpha \) we have \( \alpha_2 \in \text{Supp} \tau_1 \). Thus \( \alpha_3 \in \text{Supp} \tau_2 \). Recalling that \( \Delta^2 \) is a \( D_6 \) system with non-compact simple root \( \alpha \), in the \( E_7 \) labels (using \( D_6 \) labels the non-compact simple root becomes \( \alpha_1 \) and \( \text{Supp} \tau_2 \equiv \alpha_3 \) we again see that \( \tau_2 \) has level 1. This proves the proposition and completes the proof of 5.15.

7.11. Fix \( \tau \in P_\varepsilon^* \) and let \( \mathcal{V}_\tau^{-1} \) denote the associated variety of the unitary module \( L(\tau + u_1^\omega) \). By 2.5 and 5.16 it is enough to compute \( \mathcal{V}_\tau^{-1} \). We are now almost ready to do this but there is one more catch. In 5.15 we need to make the hypothesis that \( Q := \text{Ann}_{\mu(\omega)} L(\lambda) \neq 0 \). Assuming this holds we can then compute \( Q \). Should our computed value of this ideal be zero, there is a contradiction; but there is no difficulty. Quite simply the correct hypothesis was that \( Q = 0 \) from the start. However should the assumption that \( L_2 \) and so on be unitary lead to a computed value of \( Q \) being different from zero, we cannot conclude that \( Q \neq 0 \); because we have to make this hypothesis to conclude that \( L_2 \) is unitary! Now by 7.9, \( \lambda_2 \) is a unitary place; but the trouble is that \( L_2 \) need not be simple, through it is a highest weight module of highest weight \( \lambda_2 \). Now by 5.10, \( L_2 \otimes L(-\varepsilon_{\mu, \alpha} \omega) \) is unitary as a \( g^2 \) module and so by 5.14(iii) the only difficulty occurs if \( \lambda_2 \) is a first reduction point. Of course we must
repeat this procedure constructing $L_3$ with highest weight $\lambda_3$ and so on. Now let $j$ be the smallest positive integer (which we can assume $\leq t$) such that $\lambda_j$ is a first reduction point. Consider the simple $\Theta^j(g^j)$ quotient $L(\lambda_j)$ of $L_j$. By 7.9, $L(\lambda_j)$ is unitary. By 2.5 and 5.16, either $Q' := \text{Ann}_{U_{1}(\omega_{1}(m_j))} L(\lambda_j) = 0$ or it is generated [through the adjoint action of $I$ and multiplication by $U(m_j)$] by the highest degree fundamental invariant, namely $\Theta^j(v_{j+1})$ in the notation of 4.2. This in turn equals $v_{j+1}$ and so either $Q' = 0$ (if $Q' = 0$ or $L_j$ is not simple) or $Q = Q'$. This means that we have proved the following result. Let $l(\tau)$ denote the value assigned to $i$ satisfying $\lambda_i = \text{Ann}_{U_{1}(\omega_{1}(m_j))} L(\tau + u_{1} \omega_{1})$ calculated under the assumption that the conclusion of 5.15 holds, but without the hypothesis that this annihilator be non-zero. (That is by using 7.4-7.8 as discussed below) and let $l'(\tau)$ denote the true value of $i$ above.

**Lemma.**

(i) If $l(\tau) \neq t$, then $l(\tau) = l'(\tau)$.

(ii) If $l(\tau) = t$, either $l'(\tau) = t$ or $l'(\tau) > t$ [i.e. $\text{Ann}_{U_{1}(\omega_{1}(m_j))} L(\tau + u_{1} \omega_{1}) = 0$].

7.12. We shall eventually prove that $l'(\tau) = l(\tau)$. However first let us clarify how $l(\tau)$ is computed. First take $g$ of type $A_{1}$ and adopt the notation of 7.4. Fix $\tau_1 \in P^+_1$. If $\text{Supp} \tau_1 \cap \pi' = \emptyset$ set $s_1 = t$. Otherwise let $s_1$ be the smallest positive integer $< t$ such that $\alpha_{s_1} \in \text{Supp} \tau_1$. If $\text{Supp} \tau_1 \cap \pi' = \emptyset$ set $s'_1 = t$. Otherwise let $s'_1$ be the largest positive integer $> t$ such that $\alpha_{s'_1} \in \text{Supp} \tau_1 \cap \pi'$. We claim that $l(\tau_1) = s'_1 - s_1 + 1$ (one may also remark that $\tau_1$ has level equal to min $\{ s_1, l- s'_1 + 1 \}$). If the reader has absorbed all that has been said so far this will be completely obvious to him. Otherwise suppose for example that $s_1 < t < s'_1$. Recall 7.4 how $\tau_2$ is obtained from $\tau_1$ by shifting weights in the Dynkin diagram. Defining $s_2, s'_2$ in a like fashion for $\tau_2$ we obtain $s'_2 - s_2 + 1 = s'_1 - s_1 - 1$. It remains to show that $l(\tau_1) = l(\tau_2) + 2$. Now from 7.4 we have $u - u_1^2 = - \varepsilon_{u_{1,2}}$, so $\lambda_2$ is a second to last place of unitarity. By 2.5 this adds 1 to the degree of the fundamental invariant generating $\text{Ann}_{U_{1}(\omega_{1}(m_j))} L(\lambda_{j+1})$. Since $\Theta^2(v_{j+1}) = v_{j+1} - v_1$ a further increase of degree by 1 is obtained on passing from $\Theta^2(g^j)$ back to $g^j$. This all means that $l(\tau_1) = l(\tau_2) + 2$ as required. (By our convention in 2.1 this still holds if $i > t$.) Hopefully this is now all clear and we can state the following lemma which the above analysis proves. Define $t$ as in EJ, 1.3.

**Lemma.** — Fix $\tau_1 \in P^+_1$ and define $\tau_2$ by 7.2 ($\ast$). If $u - u_1^2 \notin \mathbb{Z} \varepsilon_{u_{1,2}}$, then $l(\tau_1) > t$. Otherwise

$$l(\tau_1) - l(\tau_2) = 1 + (u_1^2 - u)/\varepsilon_{u_{1,2}}.$$  

7.13. One can easily compute $l(\tau)$ in all cases using 7.12 and the result is given in the Table. In type $A_{1}$, $l(\tau)$ coincides in an obvious sense with the length of the support of $\tau + \omega$. If $\Delta$ has two roots lengths, $l(\tau)$ does not only depend on $\text{Supp} \tau$; but one can obtain a similar formula by splitting $\tau$ into two pieces, assigned respectively to Dynkin diagrams joined at the non-compact vertex. In types $D_n, E_6, E_7$ a good interpretation of $l(\tau)$ is less obvious.
This Table describes $l(\tau)$ for all $\tau \in \mathfrak{p}_t^+$ except that we may omit some cases for which $\tau = 0$, or $l(\tau) > t$, or when in type $D_t$ the result can be deduced from the symmetry of the Dynkin diagram. For this reason $E_s$ disappears from our Table. By convention $s \leq s' \leq \ell$ and the notation $\omega_1 + \ldots + \omega_s$ means any sum $\sum_{i=\tau}^k \omega_i$ with $k_\alpha, k_\beta$ non-zero, $k \in \mathbb{N}$.

<table>
<thead>
<tr>
<th>Type</th>
<th>$\alpha$</th>
<th>$t$</th>
<th>$\tau + \omega$</th>
<th>$l(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_t$</td>
<td>$\alpha_t$</td>
<td>$\lfloor \frac{l+1}{2} \rfloor$</td>
<td>$\omega_1 + \ldots + \omega_s$</td>
<td>$s' - s + 1$</td>
</tr>
<tr>
<td>$B_t$</td>
<td>$\alpha$</td>
<td>2</td>
<td>$\omega_1 + \omega_s$</td>
<td>2</td>
</tr>
<tr>
<td>$C_t$</td>
<td>$\alpha_t$</td>
<td>1</td>
<td>$\omega_1 + \omega_s + \ldots + \omega_l$</td>
<td>$2l - s - s' + 1$</td>
</tr>
<tr>
<td>$D_t$</td>
<td>$\alpha_t$</td>
<td>2</td>
<td>$\omega_1 + k \omega_s, k \in \mathbb{N}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$D_t$</td>
<td>1</td>
<td>$\lfloor \frac{l}{2} \rfloor$</td>
<td>$\omega_1 + \ldots + \omega_l$</td>
<td>$l - s + 1$</td>
</tr>
<tr>
<td>$D_{2t}$</td>
<td>$\alpha_{2t}$</td>
<td>1</td>
<td>$k \omega_1 + \omega_{2t}, k \in \mathbb{N}$</td>
<td>1</td>
</tr>
<tr>
<td>$E_t$</td>
<td>$\alpha_t$</td>
<td>3</td>
<td>$k \omega_s + \omega_{t-1}, k \in \mathbb{N}$</td>
<td>3</td>
</tr>
</tbody>
</table>

7.14. We now show how to prove that $l(\tau) = l'(\tau)$ when $l(\tau) = t$. For this we embed $\mathfrak{g}$ in a larger simple Lie algebra $\mathfrak{g}^0$ (if it exists) such that $(\mathfrak{g}^0)^2$ (defined as in 3.2) coincides with $\mathfrak{g}$ and so that the non-compact simple root of its root system $\Delta^0$ already lies in $\Delta^1$. Thus in type $A_t$ add one further vertex at each end. In type $C_t$ add one further vertex on the left and in type $D_t$ (with $\alpha = \alpha_{t-1}$ or $\alpha_t$) add two further vertices on the left. We use a zero subscript of superscript (put in parentheses if it is necessary to avoid ambiguity) to denote the objects for $\mathfrak{g}^0$ defined as for $\mathfrak{g}^1$. It is trivial to verify that $t^0(t) = t + 1$. Furthermore as shown in the proof of 5.16 the first highest weight vector in the $S(m^0)$ module generated by $\gamma^0$ is up to a non-zero scalar the highest weight vector in the $S(m)$ module generated by $\gamma$.

Now fix $\tau_t \in \mathfrak{p}_t^+$ for which $l(\tau_t) = t$. Extend $\tau_t$ to a first highest weight $\tau_0$ by taking $\tau_t$ to vanish on the new compact simple roots. Set $\lambda_0 = \tau_0 + u_1^0 \omega$. Then by definition $L(\lambda_0)$ is a unitary highest weight $\mathfrak{g}^0$ module and so its highest weight vector generates a unitary $\mathfrak{g}^1$ submodule of highest weight $\lambda_1 = \tau_0 + u_1^0 \omega = \tau_t + u_t^0 \omega$. One checks from the Table that $l(\tau_0) = l(\tau_t)$ in all cases, except in type $D_{2t}$ with $\alpha = \{ \alpha_t \}$. (We stress that this is not immediate and anyway not always true because the second term defined by 5.3 in the sequence obtained from $\tau_0$ is not $\tau_t$.) Yet $l(\tau_0) = l(\tau_t) = t < t^0(t)$ and so by 7.10(i) we obtain that $\text{Ann}_{u_t^0} L(\lambda_0) = Q^0_t$. Yet $\gamma_t \in Q^0_t$ and so $\gamma_t L(\lambda_1) = 0$ as required.

Apart from a few special cases which we shall analyse in the next section the above result proves the following

**Theorem.** — Let $L(\tau + u_1^0 \omega)$ be a unitary highest weight module with $\tau$ of level $s$. If $u < u_t^0$, then $\text{Ann}_{u_t^0} L(\tau + u_1^0 \omega) = 0$. Otherwise $u = u_t^0$ for some $i \in \{ 1, 2, \ldots, s \}$ and

$$\text{Ann}_{u_t^0} L(\tau + u_1^0 \omega) = Q_{l(t)} + i - 1$$
where \( l(\tau) \) is as given in the Table.

7.15. It may seem strange that the construction of 7.13 does give the required additional information. However in this I had been encouraged by a remark of W. M. McGovern that he has also used in [28], see discussion following Proposition 6.1, a sly trick of this nature which he has drawn from work of D. Barbasch [1].

8. The exceptional cases. Examples

8.1. We now complete the proof of 7.14 by analyzing in detail the few remaining cases not covered there. Recall that we must show that at a last point of unitarity \( \tau + u_1 \omega \) with \( l(\tau) = t \) one has \( Q := \text{Ann}_{U(m)} L(\tau + u_1 \omega) \neq 0 \). From 7.10 we already know that \( Q = 0 \) or \( Q = Q^\perp \). In all remaining cases (or in general by using the trick discussed in 5.16), \( Q \) is a principle ideal generated by \( \psi \), which happens to be \( \Gamma \) invariant and is of degree \( t \). All we need to show is that \( \psi V(\tau) \subset S_{t,-1}(m) P_1 \), where \( P_1 \) is the PRV component of \( m \otimes V(\tau) \). This is a question in elementary linear algebra. One can easily check it in simple examples. Thus in type \( A_3 \) we have \( t = 2 \), and \( \psi = ad - bc \) for a suitable basis of \( m \). Taking \( \tau = \alpha \), we have \( l(\tau) = 2 \). Again \( V(\tau) \) is the 2 dimensional simple \( A_1 \times A_1 \) module with basis \( (x, y) \) which we can choose so that \( ay - bx, cy - dx \) is a basis for \( P_1 \). Then the required assertion follows from the identities \((ad - bc) x = c(ay - bx) - a(cy - dx)\) and \((ad - bc) y = d(ay - bx) - b(cy - dx)\). However in say \( E_7 \) it can be that \( \psi \) is a polynomial of degree 3 in 27 variables which itself is not too easy to write down [11].

8.2. We have to consider \( g \) of type \( B_n, l \geq 3 \), type \( D_n, l \geq 4 \) with \( \alpha = \alpha_1 \), type \( E_7 \) with \( \alpha = \alpha_2 \), and type \( D_{2l-1} \) with \( \alpha = \alpha_{2l-1} \). In the first two cases \( t = 2 \). Let \( G \) denote the adjoint group of \( g \). It is well-known that \( G \mathcal{V}_2 \) is just the closure of the so-called minimal non-zero nilpotent orbit \( \mathcal{O} \). This makes these two cases a little easier.

What we have to show is equivalent to the estimate \( d_u(W)(L(\lambda)) < \dim m \). However this need not be too easy. Let \( \mathcal{V}(J(\lambda)) \) denote the associated variety of \( U(g)/J(\lambda) \). Since

\[
\dim \mathcal{V}(L(\lambda)) = d_u(W)(L(\lambda)) = \frac{1}{2} d(U(g)/J(\lambda)) = \frac{1}{2} \dim \mathcal{V}(J(\lambda))
\]

we shall be able to achieve our aim by some rudimentary primitive ideal theory (at least in the first three cases). This will compare \( J(\lambda) \) to \( J(\xi_\alpha) \) when \( \xi_\alpha := -(t-1) e_{\alpha_2} \). Below we let \( J(\mu) \) denote the annihilator of the \( (\text{not necessarily unitary}) \) simple highest weight module \( L(\mu) \). Given \( w \in W \) we set \( w.\mu = w(\mu + \rho) - \rho \).

8.3. First assume \( g \) of type \( B_l, l \geq 3 \). Then \( t = 2 \). From the Table we see that \( l(\tau) = 2 \) in just one case, namely when \( \tau = \omega_1 \). Set \( \lambda = \tau + u_1 \omega \). By EJ, 4.2, we have

\[
(*) \quad u_1^* = 1 + \frac{1}{2} |S_{1, \tau}| + 2(\beta^*, \rho) - (\beta^*, \tau + \rho).
\]
As noted in EJ, 7.1, the first three terms sum to \(-1/2\). Yet \((\beta^\vee, \rho) = 2(l-1)\) and \((\tau, \beta^\vee) = 1\) so in the Bourbaki convention ([5], Pl. II) we obtain

\[
\lambda + \rho = -\left(l - \frac{3}{2}\right) \omega_1 + \omega_2 + \ldots + \omega_{l-1} + 2 \omega_l.
\]

Let \(s_\gamma\) denote the reflection corresponding to \(\gamma \in \Delta\). An old result of Duflo ([16], 5.14) asserts that if \(\gamma \in \pi\) and \((\mu + \rho, \gamma^\vee) \notin \mathbb{Z}\), then \(J(s_\gamma, \mu) = J(\mu)\). Taking \(s = s_{\alpha_1}\) and setting

\[
\mu + \rho = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_l} (\lambda + \rho)
\]

\[
= \omega_1 + \omega_2 + \ldots + \omega_{l-1} + \frac{1}{2} \omega_{l-1} + \omega_l
\]

we thus obtain \(J(\lambda) = J(\mu)\). Yet \(\mu\) is dominant, so \(J(\mu)\) is a maximal ideal. Let \(\Delta(\mu)\) denote the subset of \(\Delta\) of roots integral with respect to \(\mu\). Since \(\mu\) is also regular we obtain from [19], 3.5, that

\[
d(U(\mathfrak{g})/J(\mu)) = |\Delta| - |\Delta(\mu)|.
\]

Now consider \(L(\xi_2)\). By definition \(\text{Ann}_{U(m)} L(\xi_2) = \mathbb{Q}_2\). From EJ, Table, we have \(\xi_{\alpha_1} = l - 3/2\) and so

\[
\xi_2 + \rho = \left(l - \frac{5}{2}\right) \omega_1 + \omega_2 + \ldots + \omega_l.
\]

One may check that

\[
\mu' + \rho = s_{\alpha_2} s_{\alpha_3} \ldots s_{\alpha_l} (\xi_2 + \rho)
\]

\[
= \omega_1 + \omega_2 + \ldots + \omega_{l-3} + \frac{1}{2} (\omega_{l-2} + \omega_{l-1}) + \omega_l.
\]

As before \(J(\xi_2) = J(\mu')\). The ideal is an old friend (see [17], Sect. 6, Table) being the unique completely prime, primitive ideal whose associated variety is \(O\). Now obviously \(\Delta(\lambda) = \Delta(\xi_2)\) whereas \(|\Delta(\mu)| = |\Delta(\lambda)|\) and \(|\Delta(\mu')| = |\Delta(\xi_2)|\). From (*) we conclude that

\[
d(U(\mathfrak{g})/J(\lambda)) = d(U(\mathfrak{g})/J(\xi_2)) < 2 \dim m
\]

as required. By say 7.10 this further gives that \(\gamma^\vee(L(\lambda)) = \gamma^\vee\) and hence that \(G \gamma^\vee\) is also the associated variety of \(J(\lambda)\). Since \(\mu + \rho, \mu' + \rho\) are both dominant, regular but distinct, so are \(J(\mu), J(\mu')\). By the above remarks \(\text{rk}(U(\mathfrak{g})/J(\lambda)) > 1\). On the other hand \(J(\lambda)\) is maximal. Actually we can compute \(\text{rk}(U(\mathfrak{g})/J(\lambda))\) explicitly from [21], II, 6.1. Indeed this is given by a polynomial \(p\) which is exactly the product of the positive roots in \(\Delta(\lambda)\), normalized to take the value 1 when \(J(\lambda)\) is replaced by the completely prime ideal \(J(\xi_2)\). Since \(\Delta(\lambda)\) is generated by \(\alpha_1 + \alpha_2 + \ldots + \alpha_l\) and the compact root system \(\Delta_c\) we conclude that

\[
\text{rk } U(\mathfrak{g})/J(\lambda) = \frac{1}{2} \dim V(\omega_l)
\]
where the \((1/2)\) factor comes from the normalization. One should check that \((1/2) \dim V(\omega_0)\) is an integer \(> 1\) for \(l \geq 3\). In fact its value is \(2^{l-2}\). We may recognize \(V(\omega_0)\) as the spin representation of \(so(2l-1)\).

It is perhaps worth mentioning that for \(\lambda_{r,s} = -(l+s-(1/2))\omega_1 + r\omega_l\) with \(s \geq 0\) and \(r-2s > 0\) a similar reasoning proves that

\[
\text{rk}(U(g)/J(\lambda_{r,s})) = \frac{1}{2}(r-2s) \dim V(r\omega_0).
\]

Here \(L(\lambda)\) is a non-trivial quotient of the induced module \(N(\lambda)\) which has rank equal to \(\dim V(r\omega_0)\) as a free \(U(m)\) module. Hence the bound in 6.5(iii) is not necessarily satisfied if \(\lambda_{r,s}\) is not a unitary place. On the other hand we can choose say \(s=1, r=3\) and then the bound of 6.5 (iii) is satisfied even though \(\lambda_{r,s}\) is not a unitary place. This completes the analysis in type \(B_l\).

8.4. In the remaining cases \(\lambda\) is integral so we need a slightly finer comparison result. Fix \(\lambda + \rho \in P^+\) regular. For all \(w \in W\) set \(\tau(w) = \{ \gamma \in \pi \mid ws < w \}\), where \(\prec\) denotes the Bruhat order with the identity \(e \in W\) being the unique smallest element. By [22], Thm. 15, one has

\[ (*) \quad \text{Fix } \gamma \in \tau(w^{-1}). \quad \text{Then } J(s_\gamma w, \lambda) \supseteq J(w, \lambda) \text{ with equality unless } \tau((s_\gamma w)^{-1}) \subset \tau(w^{-1}).\]

By the translation principle in [16], 5.16, this also holds if only \(\lambda + \rho \in P^+\). A proof of \((*)\) for equal root lengths (which is all we need here) appears in [16], 5.18.

To begin with we use the following immediate consequence of \((*)\).

\textbf{Corollary.} — Suppose \(\lambda + \rho \in P^+\). Suppose \(w \in W\) has a unique reduced decomposition \(s_{s_1} s_{s_2} \cdots s_{s_{r'}} \alpha_i \in \pi\) (with any ordering and repetitions of simple roots allowed). Then \(J(w, \lambda) = J(s_{s_1} \alpha_i, \lambda)\).

\textbf{Remarks.} — Suppose \(\lambda + \rho \in P^+\) and regular. Then \(J(\lambda)\) is maximal (and of finite codimension). By [16], 5.20, 5.21, one has \(J(\lambda) \supseteq J(s_\gamma \lambda), \forall \gamma \in \pi\) with no primitive ideals between. Suppose \(\gamma, \delta \in \pi\) do not commute. Then \(J(s_\gamma s_\delta \lambda) = J(s_\delta \lambda)\), whilst \(J((s_\gamma s_\delta)^{-1} \lambda) = J(s_\gamma \lambda)\). From this last equality and (i), (ii) below we conclude that \(\forall J(s_\gamma \lambda) = \forall J((s_\gamma s_\delta \lambda)) = \forall J(s_\delta \lambda)\). Recalling that \(q\) is simple this checks the (known) fact that the \(\forall J(s_\gamma \lambda), \gamma \in \pi\) all coincide. We have used [16], 17.12 (7), that

\[
\forall J(x, \lambda) \subseteq \forall J(y, \lambda) \iff J(x^{-1}, \lambda) \supseteq J(y^{-1}, \lambda), \forall x, y \in W,
\]

and [3], 4.10, that

\[
\forall J(x, \lambda) = \exists \forall J(x, \lambda), \forall x \in W.
\]

All this may be put in the language of left, right and two-sided cells. Though we don’t need this a complete description of cells are given implicitly by the Kazhdan-Lusztig polynomials and in almost all cases explicitly by the work of D. Barbasch and D. A. Vogan [2].
The above results do not quite go through if $\lambda$ is replaced by $\mu$ where $\mu + \rho \in \mathcal{P}^+$ is not regular owing to some subtleties in the translation principle. Yet by [16], 17.13 (4), one has $\varphi^r(L(x, \lambda)) = \varphi^r(L(x, \mu))$ if $x$ is maximal in the right coset $xW(\mu)$ where $W(\mu)$ denotes the stabilizer of $\mu + \rho$ in $W$. Since $\mu + \rho$ is dominant, this just means that $xs < x, \forall \gamma \in \pi$ for which $(\gamma, \mu + \rho) = 0$. In general we will have to check this stabilizer condition. Finally we remark that $J(\mu)$ is always the unique maximal ideal in the set $J(w, \mu), w \in W$.

8.5. Now take $g$ simple of type $D_\alpha$, $l \geq 4$ with $\alpha = \alpha_1$. Then $r = 2$. From the Table we see that $l(\tau) = 2$ only if $\text{Supp} \tau = \{\alpha_{l-1}\} \cup \{\alpha_l\}$. Both cases are equivalent so we shall just take $\tau_k = k\omega_k, k \in \mathbb{N}^+$. Set $\lambda_k = \tau + \mu_1^k \omega$. By (\(*\)) of 7.3 we have $u_k^3 = l - 2 - 2l - 3 - k = 2 - l - k$. Thus in the Bourbaki notation ([5], Pl. IV) we obtain

$$\lambda_k + \rho = (3 - k - l)\omega_1 + \omega_2 + \ldots + \omega_{l-1} + (k + 1)\omega_l,$$

On the other hand by EJ, Table, we have $\varepsilon_{\phi, z} = l - 2$ in this case, so

$$\xi_2 + \rho = (3 - l)\omega_1 + \omega_2 + \ldots + \omega_{l-1} + \omega_l.$$

Set $w = s_1 s_2 \ldots s_{l-2} s_{l-1}$. Then

$$\mu_k + \rho := w^{-1}(\lambda_k + \rho) = \omega_1 + \omega_2 + \ldots + \omega_{l-2} + (k - 1)\omega_{l-1} + \omega_l.$$

By 8.4 we conclude that $J(\lambda_k) = J(w, \mu_k) = J(s_{l-1}, \mu_k)$. If $k > 1$, then $\mu_k$ is regular and so $J(\lambda_k)$ is not maximal. When $k = 1$, then $J(\lambda_k) = J(\mu_k)$ is maximal. Again set $y = s_1 s_2 \ldots s_{l-2}$. Then

$$\mu' + \rho := y^{-1}(\xi_2 + \rho) = \omega_1 + \omega_2 + \ldots + \omega_{l-2} + \omega_{l-1} + \omega_l.$$

By 8.4 we obtain $J(\xi_2) = J(y, \mu') = J(s_{l-2}, \mu') = J(\mu')$. From the corollary and (i), (ii) above we obtain $\varphi^r(L(w, \mu_k)) = \varphi^r(L(y, \rho)) = \varphi^r(L(y, \mu')) = \varphi^r(L(\xi_2)) = \varphi^r(\xi_2)$, for all $k \in \mathbb{N}^+$ as required. Again $J(\xi_2) = J(\mu')$ is completely prime with associated variety $G \varphi^r = 0$ and by [17] it is the unique ideal with these properties. Consequently $\text{rk}(U(\mathfrak{g})/J(\lambda_k)) > 1$, $\forall k \in \mathbb{N}$. Notice the above gives examples of unitary modules $L(\lambda_k)$ satisfying $\text{Ann}_{U(\mathfrak{g})} L(\lambda_k) \neq 0$, yet do not (for $k > 1$) have maximal annihilators.

8.6. We digress slightly to consider an example in type $E_6$. Take $\alpha = \alpha_1$. Then $r = 2$. From the Table we see that $l(\tau) = 2, \tau \in \mathcal{P}^+$ has no solution. Consider nevertheless $\tau_k = k\omega_6$ with $k \in \mathbb{N}^+$. Set $\lambda_k = \tau_k + u_k^3 \omega$. One finds that $u_k^3 = 1 + 6 - 11 - k = -4 - k$. Now $\tau_k$ has level 2 so $\lambda_k$ is not a first reduction point. We can nevertheless compute the highest weight $\lambda_{k, 2}$ of $N(\lambda_k)$ from EJ, 7.2. One finds that

$$\lambda_{k, 2} = (k - 1)\omega_6 + \omega_2 - (k + 5)\omega_1.$$
This is not a unitary place, because by EJ, 4.2
\[ \lambda'_{k, 2} = (k - 1) \omega_6 + \omega_2 - (k + 11) \omega_1 \]
is a last place of unitarity. This establishes the remark made in 5.11.

This example also illustrates nicely why \( \text{Ann}_{L} (\lambda_k) = 0 \). By the above
\[ \lambda_k + \rho = -(3 + k) \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + (k + 1) \omega_6. \]

Yet \( \varepsilon_{n, 2} = 3 \) by EJ, Table, so
\[ \xi_2^{j} + \rho = -2 \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6. \]

Set \( w_0 = s_1 s_3 s_4, w_1 = w_0 s_5 s_2 s_4, w_2 = w_1 s_3, w_k = w_2 s_1, \forall k \geq 3. \) Then
\[ \mu_0 + \rho := w_0^{j} (\xi_2 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6, \]
which is dominant. Again one checks that \( \mu_k + \rho := w_k^{j} (\lambda_k + \rho) \) is dominant and also regular if \( k > 3 \). Yet \( w_0 \) does not have a unique reduced decomposition for \( k \geq 1 \). Thus from 8.4 we only obtain an inclusion \( \mathcal{V} (L (\lambda_k)) \supset \mathcal{V} (L (\xi_2)) = \mathcal{V}_{2}. \) In fact this inclusion is strict. This is obtained by the following reasoning. For simplicity we assume \( k > 3 \). Set \( \nu = s_5 s_2 s_4 s_3 s_1, \mu = \mu_k \) which is regular. Then by 8.4 (*) one has \( J (w_k, \mu) = J (\nu, \mu), \) whilst again by 8.4 (**) one has \( J (\nu^{-1}, \mu) = J (s_2 s_5, \mu). \) By [16], 5.7, the inclusions \( J (\mu) \supset J (s_2, \mu) \supset J (s_2 s_5, \mu) \) are strict. Then by [4], 3.6, the dimensions of their associated varieties decrease strictly. This implies \( \dim m \geq \dim \mathcal{V} (L (\lambda_k)) > \dim \mathcal{V} (L (\xi_2)) = \dim m - 1. \) Hence \( \text{Ann}_{L} (\lambda_k) = 0 \) as required.

8.7. Now assume \( \mathfrak{g} \) of type \( E_7 \). We have \( \alpha = \alpha_7 \) and \( t = 3 \). From the Table we see that \( \tau (\tau) = 3 \) only if \( \tau_k = k \omega_6 \) with \( k \in \mathbb{N}^{+} \). Set \( \lambda_k = \tau_k + u_{1}^{j} \omega. \) One finds that \( u_{1}^{j} = 1 + 8 - 2 k - 17 = -2 k - 8. \) Thus
\[ \lambda_k + \rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + (k + 1) \omega_6 - (2 k + 7) \omega_7. \]

By EJ, Table, we have \( \varepsilon_{n, 4} = 4 \) so
\[ \xi_3 + \rho = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 - 7 \omega_7. \]

Set \( w_{-2} = s_7 s_6 s_5 s_4 s_3 s_2, w_{-1} = w_{-2} s_4 s_3 \) which we remark have unique reduced decompositions. Set \( w_0 = w_{-1} s_5 s_6 s_4 \) which no longer has a unique reduced decomposition because \( s_5, s_3 \) can be interchanged. Thus \( w_{-1}, w_{-2} \) belong to the unique submaximal two-sided cell, whereas \( w_0 \) need not and does not already by the reasoning of 8.6. One has
\[ \mu_0 + \rho := w_0^{-1} (\xi_3 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_7, \]
which is dominant. Again
\[ w_{-1}^{-1} (\xi_2 + \rho) = \omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6 + \omega_7. \]
is also dominant. Since the associated varieties of \( J(\xi_2) \), \( J(\xi_3) \) being respectively \( G \, V^{-2} \), \( G \, V^{-3} \) do not coincide we see again that \( w_0 \) is not in the submaximal two-sided cell. Now set \( w_1 = w_0 s_2 s_5, w_2 = w_1 s_4, w_3 = w_2 s_3, w_k = w_3 s_1 \), \( \forall k \geq 4 \). One checks that
\[
\mu_k + \rho = w_k^{-1} (\lambda_k + \rho)
\]
is dominant. It is regular if and only if \( k \geq 5 \).

Set \( \lambda_0 = \xi_3 \). To show that \( V^{-1} (L(\lambda_0)) = V^{-1} (L(\xi_3)) \), \( \forall k \in \mathbb{N} \) it is enough by the same reasoning in 8.5 to show that \( J(w_k^{-1}, \mu) \) is independent of \( k \) for some and hence all \( \mu + \rho \in P^+ \) (and to check the stabilizer condition, for example; \( s_4, s_6 \) stabilize \( \mu_0 + \rho \) yet \( w_0 s_4 < w_0 \) and \( w_0 s_2 < w_0 \) as required). Fortunately this can be done by just using

8.4 (\ast). The calculation is indicated in the Figure. The vertices correspond to elements of \( W \), with length increasing downwards. The top vertex is \( w_0 \), and subsequent vertices are computed by right multiplication by \( s_i \) where \( i \) labels the corresponding edge. The labels on the vertex corresponding to \( y \) designate \( \tau(y) \). The thick lines join a pair of
vertices \(x, y\) when \(J(x, \mu) = J(y, \mu)\) by 8.4 (\(*\)). The fact that there is an unbroken thick chain from \(w_0\) to \(w_k\) for all \(k\) proves the required assertion. We remark that \(J(\lambda_k)\) for \(k \geq 5\) is two steps away from being maximal.

8.8. Now assume \(g\) of type \(D_n\), \(l \geq 4\) with \(\alpha = \alpha_i\) and \(\text{Supp } \tau = \{ \alpha_i \}\). Then \(l = [(l+1)/2]\), whereas \(l(\tau) = [(l+1)/2]\). This case is rather delicate and we should even like to see why \(\text{Ann}_{U(m)}(\tau + u_i^\lambda \omega) \neq 0\) exactly for \(l\) even. The analysis of the previous sections becomes extremely messy though could probably be carried out with the complete description of cells for the classical groups due to D. Barbasch and D. A. Vogan [2], but such calculations are only for masochists. We shall use a different approach. The first step is the following combinatorial lemma.

Fix \(i, k, r \in \mathbb{N}\) with \(0 \leq i \leq r \leq k\). Let \(V^{k, i}_r\) denote the simple finite dimensional module for \(g = gl(r+1)\) with highest weight \((k-i)\omega_i + \omega_{r-i+1}\) with the convention that \(\omega_j = 0\) for \(j > r\).

A straightforward application of Weyl's dimension formula gives

\[
\dim V^{k, i}_r = \frac{1}{(k-2i+r+1)} \cdot \frac{(k-i+r+1)!}{(k-i)! (r-i)! i!} =: d_r^{k, i}.
\]

Set

\[
e^*_r = \sum_{i=0}^{r} (-1)^i d_r^{k, i}.
\]

**Lemma.** — One has \(e^*_r = (1/2)(1 + (-1)^r)\).

Writing \(k-i+r+1\) as \(k-2i+r+1+i\) we obtain

\[
e^*_r = \sum_{i=0}^{r} (-1)^i (k-i+r)! \frac{(k-i+r)!}{(k-i)! (r-i)! i!} + \sum_{i=1}^{r} (-1)^i \frac{(k-i+r)!}{(k-2i+r+1)! (k-i)! (r-i)! (i-1)!}.
\]

The first term is a standard binomial sum and can for example be identified with

\[
\frac{1}{r!} \left. \left( \frac{d^r}{dy^r} y^k (x-y)^r \right) \right|_{x=y=1} = 1.
\]

The second term is just \(-e^*_{r-1}^{-1}\), so we have the recurrence relation \(e^*_r = 1 - e^*_{r-1}^{-1}\). Finally one observes that \(e^*_0^{-r} = 1\).

**Remark.** — Given that \(0 \leq i \leq r\) one may note that \(d_r^{k, i}\) is defined for all \(k \in \mathbb{Z}\) and the above result is also valid for such \(k\).

8.9. We take \(\tau_k = k \omega_1\) in 8.8, \(\lambda_k = \tau_k + u_i^\lambda \omega\). Let \(r(\lambda_k)\) denote the rank of \(L(\lambda_k)\) considered as a \(U(m)\) module.

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Lemma. — Suppose $k \geq l-1$. Then

$$r(\lambda_k) = \begin{cases} 0, & l \text{ even} \\ 1, & l \text{ odd.} \end{cases}$$

In particular $\text{Ann}_{U(m)} L(\lambda_k) \neq 0$ exactly when $l$ is even. Moreover

$$\text{rk} \left( U(\mathfrak{g})/\text{Ann}_{U(m)} L(\lambda_k) \right) = 1,$$

if $l$ is odd.

One checks that

$$\lambda_{k,0} := \lambda_k = k \omega_1 - (k + l - 2) \omega_l.$$ 

Now for $1 \leq i < l$, set

$$\lambda_{k,i} := (k-i) \omega_1 + \omega_{i-1} - (k + l - \delta_{i-1}) \omega_l,$$

where $\delta$ is the Kronecker delta. As above one checks that $\lambda_{k,i}, 1 \leq i < l-1$ is a last place of unitarity. Yet by the hypothesis on $k$, we find that $(k-i) \omega_1 + \omega_{i-1}$ is of level 1. Hence each such $\lambda_{k,i}$ is a first reduction point. Then $N(\lambda_{k,i})$ is unitary by 5.11, and from EJ, 7.2, one checks that its highest weight is just $\lambda_{k,i+1}$. Similarly $N(\lambda_{k,i-1})$ is unitary. Now $N(\lambda_{k,i})$ is a free $U(m)$ module of rank $\dim \mathcal{V}(l-i)\omega_1 + \omega_{i-1}$) defined with respect to $\Delta$ which is of type $\Delta_{l-1}$. Hence the assertion results from 8.8 taking $r = l-1$ and using additivity of rank.

8.10. Assume $l$ even. We now extend the conclusion $\text{Ann}_{U(m)} L(\lambda_k) \neq 0$ for all $k \in \mathbb{N}^+$. As we note below the hypothesis $k \geq l-1$ implies that $\lambda_k + \rho$ is regular and so there exists a unique $w \in W$ such that $w(\lambda_k + \rho)$ is dominant. Now fix $1 \leq k < l-1$ and assume we can find $w', w'' \in W$ such that $w = w'' w'$ where lengths add, $\mu_k + \rho := w'(\lambda_k + \rho)$ is dominant and $s_{w'} w' < w'$, whenever $\gamma \in \Pi$ satisfies $(\gamma, \mu_k + \rho) = 0$. Then by 8.4 we obtain

$$\varphi(\mathcal{V}(\lambda_k)) = \varphi(\mathcal{V}(w^{-1} \cdot \mu_k)) = \varphi(\mathcal{V}(w'^{-1} \cdot \rho)) \subset \varphi(\mathcal{V}(w^{-1} \cdot \rho)) = \mathcal{V}(w^{-1} \cdot \mu_{l-1})) = \varphi_{l/2},$$

where the last step follows by 8.9. Then the opposite inclusion follows from 7.12 and proves the required assertion.

Set $\varepsilon_i := (1/2)(1 + (-1)^i)$, $x_i = s_{i+1} s_{i+2} \ldots s_{l-1} x_i$, $i = 1, 2, \ldots, l-3$ with $x_{l-2} = s_{l-1}$. Set $w_i := x_i x_{i-1} \ldots x_1$. One checks that

$$w_i(\lambda_k + \rho) = \omega_1 + \omega_2 + \ldots + \omega_l + (k + 1 - i) \omega_{l+1} + \omega_{l+2} + \ldots + \omega_{l-2} + \omega_{l-1} - (k + l - 2 - 2i) \omega_{l}$$

for all $i \leq \min \{k + 1, l-3\}$. Thus if $k - l + 3 \geq 0$ we obtain

$$w_{l-2}(\lambda_k + \rho) = \omega_1 + \omega_2 + \ldots + \omega_{l-1} + (k - l + 3) \omega_l$$

which is dominant and also regular if $k - l + 2 \geq 0$. Observe that the expression for $w := w_{l-2}$ is reduced. Suppose now that $k - l + 3 < 0$. Then $k + 1 - l = 1$ for some $i,$
0 \leq i < l - 3. Inspection of the above formula shows that we may (by a relabelling) assume \( k = i = 1 \) without loss of generality. Set \( y_i = s_{2i} s_{2i+1} \ldots s_{i-4i} i = 1, 2, \ldots, (l/2) - 1, w_i' = y_i y_i' \). Setting \( w' = w_{l/2 - 1} \) one checks that \( \mu_0 + \rho = w' (\lambda_0 + \rho) \) is dominant. Moreover \( (\gamma, \mu_0 + \rho) = 0 \) if and only if \( \gamma = \alpha_{2i}, 1 \leq i \leq l/2 - 1 \). Yet \( s_{2i} w' < w' \), \( \forall 1 \leq i \leq l/2 - 1 \) so the stabilizer condition is satisfied. Finally set \( y_i = 1 \) for \( i \geq l/2 \) and \( z_i = x_i y_i^{-1} \). One checks that \( y_j z_i = z_i y_j \) for \( j > i \). Hence

\[
\begin{align*}
\quad w &= x_{i-2} \ldots x_1 = z_{i-2} y_{i-2} \ldots z_1 y_1 \\
&= z_{i-1} z_{i-2} \ldots z_1 y_{i-1} \ldots y_1 \\
&= w'' w', \text{ where } w'' = z_{i-2} \ldots z_1.
\end{align*}
\]

Since the expression for \( w \) was reduced, the lengths add. This completes the proof.

\section*{Index of notation}

(see also EJ, index of notation)

Symbols appearing frequently throughout the text are listed below where they are defined.

1.1. \( g, \; n^+, \; h, \; n, \; g_0, \; t, \; p^+, \; m^+, \; \alpha, \; \omega, \; P^+_c, \; V(\tau), \; N(\lambda), \; L(\lambda), \; s, \; u^i_t, \; \epsilon_\alpha, \; \overline{N}(\lambda) \).

1.2. \( u_t, \; \nu_t, \; \nu_t' \).

1.3. \( \nu_t', \; m, \; l, \; l(\tau) \).

1.4. \( Q_\nu \).

2.1. \( \beta_t, \; \mu_t \).

2.2. \( \beta_t, \; \mu_t \).

2.3. \( \text{Spec}_\nu(S(m), v_t) \).

2.4. \( \lambda_{t_\nu}^T, \; \xi_{t_\nu} \).

3.1. \( \Delta, \; \Delta^+, \; \Delta^-, \; \Gamma^l, \; \Delta^l \).

3.2. \( \alpha^l_t, \; \alpha^l_{t_\nu}, \; b^l, \; g^l_t, \; n^l, \; l, \; p^l, \; m^l, \; c^l \).

3.3. \( \gamma, \; \gamma, \; Y_t, \; \Theta^l, \; \Theta^l \).

3.4. \( \mathcal{R}(V'_t), \; \mathcal{B}(V'_t) \).

4.1. \( g_{t_\nu} \).

4.2. \( g_{t_\nu}' \).

4.3. \( F(\lambda), \; A(\lambda) \).

5.1. \( \mathcal{O} \).

5.2. \( W_{t_\nu}, \; w_t^e \).

5.3. \( \overline{\sigma}, \; \sigma, \; p_t^r, \; p_t^r \).

5.4. \( \sigma, \; p_t^r \).

6.11. \( d_A(M) \).

7.10. \( l(\tau) \).

8.1. \( w, \; \mu \).

We recall again that the subscripts \( c \) and \( n \) mean compact and non-compact respectively.
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