A. S. DE MEDEIROS

Smooth singular solutions of hyperplane fields. I

Annales scientifiques de l'É.N.S. 4e série, tome 23, nº 4 (1990), p. 657-666

<http://www.numdam.org/item?id=ASENS_1990_4_23_4_657_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 23, 1990, p. 657 à 666.

SMOOTH SINGULAR SOLUTIONS OF HYPERPLANE FIELDS (I)

By A. S. de MEDEIROS (*)

0. Introduction

The search of solutions of the total differential equations in general has been, along the time, a subject of great interest to many mathematicians. As a matter of fact, the first evidence of such interest reports to the beginning of the 19th century, with the work of Pfaff, where special attention to the non-integrable 1-forms was given by the first time (*see* [3]).

In the more general context of exterior systems, an extensive analysis of the existence of analytic solutions passing through a regular point of the system, is presented by E. Cartan in [2]. On the other hand, although Cartan had considered the singular solutions, *i. e.* solutions passing through nonregular points, no general result is established that ensure their existence. In this paper we establish such kind of results for a single holomorphic total differential equation on $\mathbf{C}^n$.

We prove (Theorem B) that given such an equation, say $\omega=0$, and a singularity x_0 of ω, *i. e.* $\omega(x_0)=0$, there exists a holomorphic solution of dimension r passing through x_0, where $2r$ is the rank of $d\omega(x_0)$. In fact, Theorem B is a direct consequence of our main result (Theorem A) that assures the existence of a holomorphic solution of dimension $[n/2]$ passing through x_0, if $d\omega(x_0)$ has maximal rank. (Under this hypothesis, when n is even, the number $[n/2]$ is an upper bound for the dimensions of the solutions of $\omega=0$.)

We point out that, when x_0 is a regular point, Theorem A is a mere consequence of the canonical forms established in [6]. Unfortunately, as it is explicitly stated in [4], such canonical forms, around the singular points, are not anymore available. Thus, considering that establishing canonical forms is a much deeper result than the one we are interested on, we were urged to try the direct geometric methods, developed here, to treat the problem.

The development of the subject is carried out into three paragraphs, according as described below.

* The author was partially supported by $\mathrm{CNP_q}$.

In paragraph 1 we settle the basic facts and definitions we make use of throughout the paper. Some results established in this paragraph, such as Corollary 1.3.2 and all the references to the real field, are actually not used in the sequel. However, they play an important role in the analysis of the real case. (Not included here to avoid an extremely long paper.)

Paragraph 2 consists essentially of the proof of Theorem A, besides some considerations about a variant of it (Theorem B).

Finally, in paragraph 3, we discuss informally some results we have already established in the real case.

The definitions are given along the various sections and are printed in a different character to make them more easily accessible.

1. Preliminaries

1.1. General elementary results

We shall denote by $\Lambda^1(\mathbf{K}^n)$ (resp. $\mathscr{X}(\mathbf{K}^n)$) the set of germs of analytic differential 1-forms (resp. analytic vector fields) on $\mathbf{K}^n$ vanishing at the origin, where $\mathbf{K}=\mathbf{R}, \mathbf{C}$.

Given $\omega \in \Lambda^1(\mathbf{K}^n)$ we shall use indistinctly the expressions: *a singular solution of the total differential equation* $\omega=0$, *a singular integral manifold of* ω, *and a singular solution of the hyperplane field defined by* ω, to mean a germ of differentiable submanifold, at the origin of $\mathbf{K}^n$, such that ω pulls back to zero on it.

If ω is linear and E is a linear subspace of $\mathbf{K}^n$, which is itself a singular solution of $\omega=0$, we say that E is an *isotropic subspace of* ω. This terminology, borrowed from the multilinear algebra, seems to be, conceptually, very appropriate. In fact, the linear subspace E is a singular integral manifold of the linear form ω if, and only if, it is an isotropic subspace of the bilinear form $\omega(x).y$.

It is easy to see that, if M is a singular integral manifold of $\omega \in \Lambda^1(\mathbf{K}^n)$ then, the tangent space, $T_0 M$, to M at 0 is an isotropic subspace of the first jet, $J_0^1(\omega)$ of ω at 0. (More generally $T_0 M$ is a singular integral manifold of the first non-zero jet of ω at 0.)

We shall denote by $b_\omega(x, y)$ and $g_\omega(x)$ the bilinear form $J_0^1(\omega)(x).y$ and the quadratic form $J_0^1(\omega)(x).x$ respectively.

Thus, $T_0 M$ beeing an isotropic subspace of b_ω (and consequently of g_ω) is a necessary condition for M to be a singular solution of $\omega=0$.

A germ of differentiable manifold satisfying this condition is said to be *isotropic* (at 0, with respect to ω).

Remark 1.1.1. — It follows immediatly from the above conclusions that:

(i) For $\mathbf{K}=\mathbf{R}$, the set of forms in $\Lambda^1(\mathbf{R}^n)$ having no singular solution has nonempty interior. As a matter of fact, $\omega_0=\sum_1^n x_i\, dx_i$ is an interior point of this set, for g_ω turns out to be definite positive for all ω sufficiently close to ω_0. Hence, no isotropic manifold can exist. In the complex case it follows from Theorem A and Proposition 2.2.1, that this set is nonempty and has empty interior. These facts illustrate the great difference between the real and complex cases.

(ii) From the generic point of view the maximal expected dimension of a singular integral manifold on $\mathbf{K}^n$ is $[n/2]$. In fact, g_ω is generically nondegenerated and, it is well known that, in this case, $[n/2]$ is an upper bound for the index of Witt of g_ω, which is, by definition, the maximal dimension of the isotropic subspaces of g_ω.

1.2. Generic necessary conditions in even dimension

Considering (ii) above, it is natural to ask whether singular integral manifolds of dimension $[n/2]$ do generically exist or not. In order to answer this question, we shall now restrict ourselves to the open and dense subset $\tilde{\Lambda}^1(\mathbf{K}^n)$ of $\Lambda^1(\mathbf{K}^n)$, consisting of those forms whose exterior differentials at 0 have maximal rank, *i.e.* $\omega \in \tilde{\Lambda}^1(\mathbf{K}^n)$ if, and only if, $r(d\omega(0)) = n$ or $n-1$ accordingly to n is even or odd respectively.

Our purpose is to establish some fundamental results about the forms in this set.

Henceforth, in this section, n is supposed to be even. Some properties inherent to the 1-forms on even dimensional spaces allow us to better analyse them. Furthermore, as far as our purpose is concerned, the results we establish in these particular dimensions give enough information about all dimensions in general.

Let $\omega \in \tilde{\Lambda}^1(\mathbf{K}^n)$, with $n = 2m$. Since $d\omega(0)$ has maximal rank, there exists a unique $X \in \mathscr{X}(\mathbf{K}^n)$ such that $\omega = i(X)\, d\omega$ (the interior product of X and $d\omega$). We shall write $X = X(\omega)$ if an explict reference to the form ω turns out to be relevant.

We notice that ω is invariant by the Lie derivative with respect to X, *i.e.* $L_X \omega = \omega$ or, equivalently, $X_t^* \omega = e^t \omega$.

PROPOSITION 1.2.1. — *An m-dimensional singular integral manifold of* $\omega \in \tilde{\Lambda}^1(\mathbf{K}^{2m})$ *is an isotropic* X-*invariant manifold.*

Proof. — We conclude easily that:

(i) The equation $X_t^* \omega = e^t \omega$ implies that the iterated $X_t(N)$ of any integral manifold N of ω, not necessarily singular, is again an integral manifold.

(ii) Since the rank of $d\omega(0)$ is maximal, any integral manifold of ω, in a neighborhood of 0, is at most m-dimensional.

Now, let M satisfy the hypothesis of the proposition. If X is not tangent to M we use the flow of X, and (i) above, to construct an integral manifold of dimension $m+1$ in a neighborhood of a nontangency point, which is a contradiction in virtue of (ii).

The fact that M is isotropic was previously established in section 1.1. ■

1.3. Some sufficient conditions and a linearization result

In order to establish the main result of this section (Proposition 1.3.2) we make use of an important relation connecting the eigenvalues of the L_A operators with those of A.

More precisely, let n be arbitrary (even or odd) and let A be a linear vector field on $\mathbf{K}^n$. Denote by $\Lambda_k^p(\mathbf{K}^n)$ the set of homogeneous differential p-forms of degree k on $\mathbf{K}^n$. For each p and k fixed, $L_A : \Lambda_k^p(\mathbf{K}^n) \circlearrowleft$ is clearly a linear operator. We claim that

PROPOSITION 1.3.1. — *If* $\mathrm{Spect}(A)=\{\lambda_1, \ldots, \lambda_n\}$ *then,* $\mathrm{Spect}(L_A)$ *is the set of all complex numbers* μ *of the form*

$$\mu=\sum_{i=1}^{n} m_i \lambda_i+\sum_{j=1}^{p} \lambda_{i_j}$$

where $m_i, i_j \in \mathbf{N}, \sum_{i=1}^{n} m_i=k$ *and* $1 \leqq i_1 < \ldots < i_p \leqq n$.

Proof. — Routine. ■

Remark 1.3.1. — In the case we are interested on, $p=1$, and μ is given by the following simpler formula

$$\mu=\sum_{i=1}^{n} m_i \lambda_i, \qquad m_i \in \mathbf{N}, \qquad \sum_{i=1}^{n} m_i=k+1.$$

We shall now derive some useful results that are direct consequences of the above proposition.

COROLLARY 1.3.1. — *Let* ω *be a linear differential* 1-*form on* $\mathbf{K}^n$ *such that* $L_A \omega=\omega$ *for some linear vector field* A *on* $\mathbf{K}^n$. *Let* E *be an* A-*invariant linear subspace of* $\mathbf{K}^n$ *such that* $\lambda+\nu \neq 1$ *for all* $\lambda, \nu \in \mathrm{Spect}(A \mid E)$. *Then,* E *is an isotropic subspace of* ω.

Proof. — If the restriction $\omega \mid E$ is not identically zero, the equality $L_A \omega=\omega$ says that $1 \in \mathrm{Spect}(L_{A \mid E})$ (considered as an operator on $\Lambda_1^1(E)$). By Proposition 1.3.1, we must have $1=\lambda_1+\lambda_2$ for some $\lambda_1, \lambda_2 \in \mathrm{Spect}(A \mid E)$, contradicting the hypothesis of the corollary. ■

We shall say that $X \in \mathscr{X}(\mathbf{K}^n)$ is *not u-ressonant* (at 0) if there exists no relation of the type

$$\sum_{i=1}^{n} m_i \lambda_i=1 \qquad \text{with} \quad m_i \in \mathrm{N} \qquad \text{and} \qquad \sum_{i=1}^{n} m_i > 2$$

among the eigenvalues λ_i of $J_0^1(X)$. If such a relation exists we say that X *has a u-ressonance of order* $l=\sum_{i=1}^{n} m_i$.

The expression u-ressonance was chosen to emphasize the similarity between the relations defining the usual concept of ressonance and the unitary linear combinations that appear in Proposition 1.3.2.

COROLLARY 1.3.2. – *Let* $\omega \in \Lambda^1(\mathbf{K}^n)$ *and* $X \in \mathscr{X}(\mathbf{K}^n)$ *be such that* $L_X \omega = \omega$. *Suppose that* X *is locally analytically linearizable in a neighborhood of* 0. *Then, if* X *is not u-ressonant, the diffeomorphism linearizing* X *so does linearize* ω.

Proof. – Let $A = J_0^1(X)$ and let $f \in \mathrm{Diff}(\mathbf{K}^n, 0)$ be such that $f^* X = A$.

The equation $L_A f^* \omega = f^* \omega$, and the linearity of A, imply the L_A-invariance of the k-homogeneous parts $(f^* \omega)_k$ of $f^* \omega$ at 0 *i.e.* $L_A (f^* \omega)_k = (f^* \omega)_k$ for all positive integer k. Since X is not u-ressonant we must have $(f^* \omega)_k = 0$ for $k > 1$. Hence, $f^* \omega$ is linear as promissed. ■

We finish this paragraph with

PROPOSITION 1.3.2. – *Let* $\omega \in \Lambda^1(\mathbf{K}^n)$ *and* $X \in \mathscr{X}(\mathbf{K}^n)$ *be such that* $L_X \omega = \omega$. *If* X *is not u-ressonant, every* X*-invariant isotropic manifold* M *of* ω *is a singular solution of* $\omega = 0$.

Proof. – Let $\hat{\omega} = \omega \mid M$, $\hat{X} = X \mid M$ and $\hat{A} = J_0^1(X \mid M)$. Suppose that $\hat{\omega} \neq 0$, and let $J_0^k(\hat{\omega})$ be the first non-zero jet of $\hat{\omega}$ at 0.

We claim that $L_{\hat{A}}(J_0^k(\hat{\omega})) = J_0^k(\hat{\omega})$. In fact, $L_{\hat{A}}(J_0^k(\hat{\omega}))$ is, clearly, the k-jet of $L_{\hat{X}} \hat{\omega}$ at 0, which is, in virtue of $L_{\hat{X}} \hat{\omega} = \hat{\omega}$, equal to $J_0^k(\hat{\omega})$.

On the other hand, since M isotropic means $J_0^1(\hat{\omega}) = 0$, we must have $k > 1$. These facts contradict the hypothesis that X is not u-ressonant. Thus, $\hat{\omega} = 0$ and M is a singular solution of $\omega = 0$. ■

2. The existence theorem in the complex field

2.1. Some technical preparation

Let $\tilde{\Lambda}^1(\mathbf{K}^{2m})$ be as defined in section 1.2, and let $\omega \in \tilde{\Lambda}^1(\mathbf{K}^{2m})$.

By Darboux's theorem, $d\omega$ is locally analytically equivalent to the canonical symplectic form, $\theta_0 = \sum_1^m dx_i \wedge dy_i$, of $\mathbf{K}^{2m} = \mathbf{K}^m \oplus \mathbf{K}^m$. In virtue of this, we shall suppose from now on that, $d\omega = \theta_0$.

Let $i(I)\theta_0$ denote the interior product of the radial vector field $I(x) = x$ on $\mathbf{K}^{2m}$ with the symplectic form θ_0. Since $di(I)\theta_0 = 2\theta_0$, there exists a unique analytic function H, vanishing at 0, such that $\omega = dH + \frac{1}{2} i(I)\theta_0$. Hence, the vector field $X = X(\omega)$ may be written in the form $X = X_H + \frac{1}{2} I$. Where X_H is the Hamiltonian vector field induced by the function H in the symplectic space $(\mathbf{K}^{2m}, \theta_0)$.

We define the *spectrum* of ω, denoted by $\mathrm{Spect}(\omega)$, to be the set $\mathrm{Spect}(J_0^1(X(\omega)))$.

The above decomposition of X leads to the relation

$$\mathrm{Spect}(\omega)=\frac{1}{2}+\mathrm{Spect}(\mathrm{J}_0^1(\mathrm{X}_\mathrm{H}))$$

and, since X_H is a Hamiltonian vector field, Spect (ω) turns out to be symmetric with respect to $1/2$ in the complex plane.

2.2. The main result

The theorem of Poincaré-Dulac and the well known result on Dynamical Systems, stated below as Theorem $*$, are the fundamental tools we make use of in the proof of Theorem A. Besides, of course, the results we have established in the foregoing sections.

For reference purpose, we remark that Theorem $*$ follows from [5], and is, as well, discussed in [7].

If A is a linear vector field on $\mathbf{K}^n$ and if $\sigma \subset \mathrm{Spect}(\mathrm{A})$, we shall denote by $\mathrm{E}(\sigma)$ the A-invariant subspace obtained by taking the direct sum of all root spaces associated with the eigenvalues of A that lie in σ.

THEOREM $*$. — *Let* X *be a* C^r-*vector field on* $\mathbf{K}^n$, *vanishing at* 0 ($r\in\mathbf{Z}^+\cup\{\infty\}\cup\{\omega\}$). *Given* $a\in\mathbf{R}^+$, *let* $\sigma_a=\{\lambda\in\mathrm{Spect}(\mathrm{J}_0^1(\mathrm{X}))\mid \mathrm{R}_e\lambda\geqq a\}$. *Then, there exists a unique* C^r X-*invariant submanifold* M (*called the strong unstable manifold correspondent to* a) *such that* $\mathrm{T}_0\mathrm{M}$ *is the invariant subspace* $\mathrm{E}(\sigma_a)$ *of* $\mathrm{J}_0^1(\mathrm{X})$.

Remark 2.2.1. — In the complex case, the real axis, in Theorem $*$, may be chosen to be any straight line through the origin. In fact, for any $\theta\in\mathbf{R}$, $\tilde{\mathrm{X}}=e^{i\theta}\mathrm{X}$ and X, have both the same invariant submanifolds, while $\mathrm{Spect}(\mathrm{J}_0^1(\tilde{\mathrm{X}}))$ is a rotation of $\mathrm{Spect}(\mathrm{J}_0^1(\mathrm{X}))$ by the angle θ.

THEOREM A. — *Let* $\omega\in\Lambda^1(\mathbf{C}^n)$ *be such that* $d\omega(0)$ *has maximal rank. Then, there exists a holomorphic singular solution of* $\omega=0$ *of dimension* $[n/2]$.

Proof. — (i) n even. Let $\mathrm{X}=\mathrm{X}(\omega)$ and $\sigma=\left\{\lambda\in\mathrm{Spect}(\omega)\mid \mathrm{R}_e\lambda\geqq\frac{1}{2}\right\}$. By Theorem $*$ there exists an X-invariant submanifold N such that $\mathrm{T}_0\mathrm{N}$ is the invariant subspace $\mathrm{E}(\sigma)$ of $\mathrm{J}_0^1(\mathrm{X})$.

We set $\bar{\omega}=\omega\mid\mathrm{N}$ and $\bar{\mathrm{X}}=\mathrm{X}\mid\mathrm{N}$, so we have $\bar{\omega}=i(\bar{\mathrm{X}})\,d\bar{\omega}$. Once Spect $(\mathrm{J}_0^1(\bar{\mathrm{X}}))=\sigma$, we are in the domain of Poincaré and then, by Poincaré-Dulac's theorem, $\bar{\mathrm{X}}$ is locally biholomorphically equivalent to a polynomial vector field.

Since there can exist no ressonance relation of the form

$$\lambda=\Sigma\, m_i\lambda_i \qquad \text{with}\quad \mathrm{R}_e\lambda=\frac{1}{2},\quad \Sigma\, m_i\geqq 2,\quad m_i\in\mathbf{N}$$

the resulting polynomial vector field turns out to have the following particular normal form

$$J_0^1(\bar{X}) + \begin{pmatrix} 0 \\ Q \end{pmatrix}$$

where Q is polynomial, the linear part $J_0^1(\bar{X})$ is in its Jordan canonical form and the coordinates refer to the decomposition

$$T_0 N = E(\sigma') \oplus E(\sigma - \sigma') \qquad \text{with} \quad \sigma' = \left\{ \lambda \in \sigma \mid R_e \lambda = \frac{1}{2} \right\}.$$

We shall write, for short, $E = E(\sigma')$ and $F = E(\sigma - \sigma')$.

It follows from this particular normal form that, for each invariant subspace $G \subset E$ of $J_0^1(\bar{X})$ there corresponds an X-invariant submanifold M such that $T_0 M = G \oplus F$. In fact, $G \oplus F$ is itself an invariant subspace for the polynomial normal form. Furthermore, $X \mid M$ is not *u*-ressonant for this is already true for $X \mid N$.

In view of this, our purpose is attained if we exhibit an invariant subspace $G \subset E$ such that $G \oplus F$ has the required dimension $[n/2]$ and fulfils the remaining hypothesis of Proposition 1.3.2. For, according to this proposition and our previous conclusions, the manifold M, referred above, such that $T_0 M = G \oplus F$, will be the sought singular solution.

In order to get the subspace G we consider linear Hamiltonian vector fields A_k, having no zero eigenvalues, obtained by solely modifying $A = J_0^1(X_H(\omega))$ in its invariant subspace $E(\{0\})$. The viability of making such pertubations follows, at once, from the canonical forms, the quadratic Hamiltonian can be reduced to (*see* [1] and [8]).

Now, let

$$\sigma_k = \left\{ \lambda \in (\mathrm{Spect}(\omega_k) - \mathrm{Spect}(\omega)) \mid R_e \lambda > \frac{1}{2} \right\} \cup \{ \lambda \in \sigma' \mid \mathrm{Im}\, \lambda > 0 \}$$

and $G_k = E(\sigma_k)$, where $\omega_k = \boldsymbol{i}(A_k)\theta_0 + \frac{1}{2}\boldsymbol{i}(I)\theta_0$. We notice that $\omega_k \to J_0^1(\omega)$ and that, passing to a subsequence if necessary, G_k converges to a subspace G which is necessarily contained in E.

By Corollary 1.3.1, $G_k \oplus F$ is an invariant isotropic subspace of ω_k and, since it has dimension $[n/2]$, it follows, by an obvious taking limits argument, that G has the desired properties.

(ii) n odd. Since $d\omega(0)$ has maximal rank, there exists an $(n-1)$-dimensional subspace V of $\mathbf{C}^n$ such that $d\omega(0) \mid V$ still has maximal rank. Hence, part (i) just proved, applies to $\hat{\omega} = \omega \mid V$, and since every integral of $\hat{\omega}$ is yet an integral of ω, the proof is completed. ■

REMARK 2.2.2. — Theorem A assures the generic existence of singular solutions having the uniform maximal dimension $[n/2]$. In fact, let $\mathcal{G} = \tilde{\Lambda}^1(\mathbf{C}^n)$ if n is even, and $\mathcal{G} = \{\omega \in \tilde{\Lambda}^1(\mathbf{C}^n) \mid g_\omega \text{ is nondegenerated}\}$ if n is odd.

Then, by Theorem A, the forms lying in the open and dense subset $\mathscr{G}$ of $\Lambda^1(\mathbf{C}^n)$ justify the above statement.

An immediate consequence of Theorem A is the following generalized version of it

THEOREM B. — *Let* $\omega \in \Lambda^1(\mathbf{C}^n)$. *If* $r(d\omega(0)) = 2r$ *there exists an r-dimensional holomorphic singular solution of* $\omega = 0$.

In order to complete this paragraph we give, in Proposition 2.2.1, examples that suggest that, as far as the rank of $d\omega(0)$ is concerned, Theorem B is as sharp as possible.

PROPOSITION 2.2.1. — *For every integer* $n \geqq 2$ *there exists* $\omega \in \Lambda^1(\mathbf{C}^n)$ *with* $r(d\omega(0)) = 0$ (resp. $r(d\omega(0)) = 2$), *possessing no* 1-*dimensional* (*resp.* 2-*dimensional*) *holomorphic singular integral manifold.*

Proof. — In the case $r(d\omega(0)) = 0$ we simply define $\omega = dg$, where $g: \mathbf{C}^n \to \mathbf{C}$ is any holomorphic function, vanishing at 0, such that $g^{-1}(0)$ contains no regular curve through the origin.

Examples of such functions are given by

(i) $g(x_1, \ldots, x_n) = \sum_1^n a_i x_i^{k_i}$; $a_i \in \mathbf{C}^*$ and the $k_i \in \mathbf{N}$ satisfy

$$(*) \qquad k_1 < \ldots < k_n \quad \text{and} \quad 2k_1 > k_n.$$

(ii) $h(x_1, \ldots, x_n) = x_1^2 + \sum_2^n a_i x_i^{p_i}$; $a_i \in \mathbf{C}^*$ and the p_i are odd numbers satisfying $(*)$.

Now suppose $r(d\omega(0)) = 2$. The cases $n = 2$, 3 are immediate, for they correspond to the maximal rank situation.

Given $n \geqq 4$ we consider $\mathbf{C}^n = \mathbf{C}^s \oplus \mathbf{C}^2$ $(s = n - 2 \geqq 2)$ and define, $g: \mathbf{C}^s \to \mathbf{C}$ and $\alpha \in \Lambda^1(\mathbf{C}^2)$ by:

$$g(x_1, \ldots, x_s) = \sum_1^s \frac{x_i^{k_i+1}}{k_i+1}, \qquad k_i \text{ satisfying } (*)$$

$$\alpha = u\,dv - v\,du, \qquad \text{where } u, v \text{ denote the coordinates in } \mathbf{C}^2.$$

We claim that $\omega = dg \oplus \alpha$ fulfils the requirements of the proposition.

In fact, suppose that S is a 2-dimensional singular solution of $\omega = 0$. If $f = f_s \oplus f_2$ is a parametrization of S we must have $f_2'(0) = 0$, otherwise $u = 0$ or $v = 0$ would define a 1-dimensional singular solution of $dg = 0$, which is impossible. In particular, we may suppose that the parameters in f are a couple x_i, x_j of the x_i's and that $i < j$ is the smallest pair of indices such that (x_i, x_j) parametrizes S. These assumptions imply that the $(k_j - 1)$-jet of $f_s^* dg$ depends only on x_i and dx_i. Using this fact, and computing successively the jets of $f_2^* \alpha$, taking on account the equality $f_2^* \alpha = -f_s^* dg$, we arrive in a contradiction: the k_j-jet of $f_2^* \alpha$ at 0, must vanish for $x_i = 0$. Clearly, this finishes the proof. ■

Remark 2.2.3. — For $n=4$, 5 the above proof would be much more simplified if, instead of a general g, we had taken $g(x)=x_1^2+x_2^3$ for $n=4$, and $g(x)=x_1^2+x_2^3+x_3^5$, like in (ii) above, if $n=5$. As a matter of fact, a direct computation of the first non-zero jets of $f_s^* dg$ and $f_2^* \alpha$ shows that their degrees do not fit. The latter is of degree greater than two, while the former is not.

As a consequence of Proposition 2.2.1 we obtain the following dual result.

COROLLARY 2.2.1. — *For every $n=2(r+1)$ or $2(r+1)+1$ ($r \geqq 0$) there exists $\omega \in \Lambda^1(\mathbf{C}^n)$ with $r(d\omega(0))=2r$ possessing no $(r+1)$-dimensional holomorphic singular integral manifold.*

Proof. — By entirely analogous arguments to those used in the proof of Proposition 2.2.1, and induction on r, we show that

$$\omega = dg \oplus \sum_{i=1}^{r} (u_i\, dv_i - v_i\, du_i)$$

has the desired properties. Where g is either one of the functions defined in Remark 2.2.3, accordingly to $n=2(r+1)$ or $n=2(r+1)+1$ respectively. ■

3. Some remarks on the real case

As it was pointed out in Remark 1.1.1, there is a great difference between this case and the complex one. The generic existence of singular solutions having a uniform maximal dimension fails to be true. Also, the singular solutions of maximal dimension are in general not invariant by the vector field $X(\omega)$.

We remark, however, that Theorem A is still valid if $J_0^1(X_H(\omega))$ has no purely imaginary eigenvalues.

The adequate generic approach to the real case seems to be the following:

To exhibit a generic set $\mathcal{G} \subset \Lambda^1(\mathbf{R}^n)$, and to describe precisely what are the maximal dimensions of the singular integral manifolds of any ω in $\mathcal{G}$.

We comment briefly some partial results we have established in this direction, for even dimensions.

There is a residual subsset $\mathcal{G} \subset \tilde{\Lambda}^1(\mathbf{R}^{2m})$ such that, for any $\omega \in \mathcal{G}$, there exists an analytic singular integral manifold of dimension $m-\nu+r$. Where 2ν is the number of blocks of odd order, corresponding to purely imaginary eigenvalues, in the Jordan canonical form of $J_0^1(X_H(\omega))$, and $r \leqq \nu$ is related to the following.

PROPOSITION. — *Denote by $(x_1, \ldots, x_\nu, y_1, \ldots, y_\nu)$ the coordinates in $\mathbf{R}^{2\nu}$ and let $\alpha_1 \leqq \ldots \leqq \alpha_\nu$ be real numbers. Then, the linear differential 1-form*

$$\omega = \sum_{i=1}^{\nu} \alpha_i (x_i\, dx_i + y_i\, dy_i) + \frac{1}{2} i(\mathrm{I})\, \theta_0$$

has an r-dimensional isotropic subspace if, and only if, the following inequalities hold

$$\alpha_{\nu-k+1} + \alpha_{\nu-r+k} \geqq 0 \qquad \textit{and} \qquad \alpha_k + \alpha_{r-k+1} \leqq 0$$

for all $k=1, \ldots, r$.

Finally, we remark that all the results announced here generalize, straightforwardly to C^∞ 1-forms, and even to C^l 1-forms, if l is sufficiently large and if some loss of differentiability in the results is allowed.

REFERENCES

[1] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978, pp. 381-384.
[2] E. CARTAN, *Les Systèmes Différentiels Extérieurs et Leurs Applications Géométriques*, Hermann, Paris, 1971.
[3] A. R. FORSYTH, *Theory of Differential Equations*, Vol. I, Dover Publ. Inc., New York, pp. 80-81.
[4] C. GODBILLON, *Géométrie Différentielle et Mécanique Analytique*, Hermann, Paris, 1969, pp. 116-121.
[5] W. HIRSCH, C. C. PUGH and M. SHUB, *Invariant Manifolds*, Lect. Notes Math., 583, Springer-Verlag, 1977.
[6] J. MARTINET, *Sur les Singularités des Formes Différentielles Extérieures*. Ann. Inst. Fourier, Grenoble, 20, 1970, pp. 95-178.
[7] J. PALIS and F. TAKENS, *Theory of Homoclinic Bifurcations*. Cambridge University Press (to appear).
[8] J. WILLIAMSON, *On an Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems*, Am. J. Math., 58, 1936, pp. 141-163.

(Manuscript received November 6, 1989).

A. S. DE MEDEIROS,
Instituto de Matemática Pura e Aplicada,
Estrada Dona Castorina, 110,
Jardim Botánico,
22460 Rio de Janeiro, R. J., Brésil.