ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 23, nº 4 (1990), p. 593-624
<http://www.numdam.org/item?id=ASENS_1990_4_23_4_593_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 23, 1990, p. 593 à 624.

REDUCED UNSTABLE A-MODULES AND THE MODULAR REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

By V. FRANJOU and L. SCHWARTZ

ABSTRACT. — One studies the category, $\mathcal{U}$, of unstable modules over the Steenrod algebra modulo the sub-category, $\mathcal{N}il$, of nilpotent modules. One shows that the quotient category $\mathcal{U}/\mathcal{N}il$ is filtered in such a way that the n-th sub-quotient is equivalent to the category of modules over $\mathbb{F}_2[\mathfrak{S}_n]$, the group algebra of the symmetric group $\mathfrak{S}_n$. One gives applications.

RÉSUMÉ. — On étudie la catégorie, $\mathcal{U}$, des modules instables sur l'algèbre de Steenrod modulo la sous-catégorie, $\mathcal{N}il$, des modules nilpotents. On montre que la catégorie quotient $\mathcal{U}/\mathcal{N}il$ possède une filtration dont le n-ième sous-quotient est équivalent à la catégorie des modules sur $\mathbb{F}_2[\mathfrak{S}_n]$, l'algèbre du groupe symétrique $\mathfrak{S}_n$. On donne des applications.

0. Introduction

Modules over the mod 2 Steenrod algebra have been known for a long time to have a close relationship with the symmetric groups. The object of this paper is to study one aspect of this phenomenon. Specifically we study unstable modules over the mod-2 Steenrod algebra denoted by A, modulo nilpotent elements. Let us recall that an A-module M is unstable if, for any x in M, $Sq^i x$ is zero as soon as i is greater than $|x|$, the degree of x. The mod-2 cohomology of a space X, $H^*(X;\mathbb{F}_2)$ is an unstable A-module.

In an unstable A-module M, an element x is said to be nilpotent if $Sq^{2^n|x|}\ldots Sq^{|x|}(x)=0$ as soon as n is large enough; in the case where $M=H^*(X;\mathbb{F}_2)$, this reads $x^{2^{n+1}}=0$ as soon as n is large enough, this justifies the terminology.

Let $\mathcal{U}$ be the (abelian) category of unstable A-modules, $\mathcal{N}il$ the full subcategory whose objects are the nilpotent unstable A-modules, *i.e.* unstable A-modules such that any element in M is nilpotent, $\mathcal{N}il$ is an abelian category.

The idea is to consider the quotient category $\mathcal{U}/\mathcal{N}il$ [Ga], it has the same objects as $\mathcal{U}$ but morphisms differ; let us just say for the moment that "one inverts" morphisms φ in $\mathcal{U}$ such that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are in $\mathcal{N}il$. Generalities about abelian categories imply that the forgetful functor $r:\mathcal{U}\to\mathcal{U}/\mathcal{N}il$ has a right adjoint s.

Before going on, let us give now three reasons why one is interested in studying $\mathcal{U}/\mathcal{N}il$.

The first one is Quillen's Theorem [Qui] on the cohomology of a finite group G. It says that the canonical map q_G from $H^*(BG; \mathbb{F}_p)$ into the inverse limit $\varprojlim_{C(G)} H^*(BV; \mathbb{F}_p)$ [where C(G) is the category of elementary abelian p-subgroups V of G, morphisms being subconjugacies] is an "F-isomorphism". In our terminology, this exactly means that q_G is an isomorphism in $\mathscr{U}/\mathscr{N}il$ (*see* also [LS1]).

The second one is Lannes' Theorem [La] computing the set of homotopy classes of maps from the classifying space BV, of an elementary abelian p-group V, into a space X. Under reasonable assumptions on X, it is shown to be a functor of $H^*(X; \mathbb{F}_p)$ modulo the radical (the ideal of nilpotent elements).

We insist on the fact that, in $H^*(X; \mathbb{F}_2)$, an element is nilpotent in the ordinary sense if it is in the sense defined above.

The third reason is the work of Adams and Wilkerson [AW]. For the condition of "quadratic closure" introduced by these authors, together with the condition that $x \to Sq^{|x|}x$, $M \to M$ is injective, implies that M is of the form sN for some object N in $\mathscr{U}/\mathscr{N}il$. Therefore the study of $\mathscr{U}/\mathscr{N}il$ is closely related to the study of these modules.

Let us now describe the main result of the paper. The category $\mathscr{U}/\mathscr{N}il$ is filtered by full subcategories $\mathscr{V}_n$;

$$\mathscr{V}_0 \subset \mathscr{V}_1 \subset \ldots \subset \mathscr{V}_n \subset \mathscr{V}_{n+1} \ldots$$

defined as follows: an object in $\mathscr{U}/\mathscr{N}il$ is in $\mathscr{V}_n$ *if and only if the integer* $\sup \alpha(d)$ ($\alpha(d)$ is the number of 1 in the diadic expansion of d), where d runs through integers such that $(sM)^d$ is non zero, *is less or equal to n*. In this definition, sM could be replaced by any $N \in \mathscr{U}$ such that

(i) rN is isomorphic to M,

(ii) the operation $x \to Sq^{|x|}x$, $N \to N$ is injective.

An unstable A-module satisfying (ii) is called reduced.

The categories $\mathscr{V}_n$ are abelian.

THEOREM 0.1. — *The quotient category $\mathscr{V}_n/\mathscr{V}_{n-1}$ is equivalent to the categoy of (right) $\mathbb{F}_2[\mathfrak{S}_n]$-modules, $\mathfrak{S}_n$ being the symmetric group.*

We observe moreover that the smallest abelian subcategory of $\mathscr{U}/\mathscr{N}il$, which contains the $\mathscr{V}_n$ for all n and is stable under direct limits, is $\mathscr{U}/\mathscr{N}il$ itself.

Theorem 0.1 was first proved by J. Lannes and the second author using a completely different approach ([HLS], [LS2]) which depends on the fact that $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ is injective in the category $\mathscr{U}$ ([Ca], [Mi], [LZ2]), on the properties of the functor T of J. Lannes [La] and on the classification result, for injectives in the category $\mathscr{U}$, of [LS1]. The result proved in [LS2] is in fact stronger for it identifies $\mathscr{U}/\mathscr{N}il$ with a certain category of "analytic functors" (see [HLS]) in the sense defined by I. Macdonald in [Mc] and then the type of techniques of [Mc] again gives (0.1).

There are two key ingredients in the proof of Theorem 0.1.

The first one, which is proved in Chapter 4 and is Theorem 3.1.4 of the paper, gives a criteria for an object M in $\mathscr{U}/\mathscr{N}il$ to be in $\mathscr{V}_n$ in terms of the action of the Steenrod algebra on $s\mathrm{M}$. In fact we proceed in the reverse order: we give in Chapter 2 the definition of the filtration $\mathcal{V}_n$ which depends on the action of the Steenrod algebra, introduce the second characterization in Chapter 3 and prove equivalence in Chapter 4.

The second one, proved in Chapter 3, is that *if* $\mathrm{M} \in \mathcal{V}_n$, *then* $(s\mathrm{M})^d$ *for* d *such that* $\alpha(d)=n$, *is in a natural way a right* $\mathbb{F}_2[\mathfrak{S}_n]$*-module which does not depend on the choice of* d (among those satisfying $\alpha(d)=n$). This is essentially Lemma 3.2.4.

A large part of Chapter 2 is concerned with a study of sub-A-modules of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^d; \mathbb{F}_2)$, which is needed in Chapter 3 and later.

As a related result to Theorem 0.1, we get in Chapter 3.

THEOREM 0.2. — *Let* S *be a simple object in* $\mathcal{V}_n - \mathcal{V}_{n-1}$, *then there exists a uniquely defined (up to isomorphism) simple right ideal* I *of* $\mathbb{F}_2[\mathfrak{S}_n]$ *such that* S *is isomorphic to* $r(\mathrm{I}.\mathrm{F}(1)^{\otimes n})$. *Conversely, for any simple right ideal* J *of* $\mathbb{F}_2[\mathfrak{S}_n]$, $r(\mathrm{J}.\mathrm{F}(1)^{\otimes n})$ *is a simple object in* $\mathcal{V}_n$.

In fact, Theorem 0.2 was proved before Theorem 0.1, along a closely related line.

Here F(1) is the free unstable A-module on one generator of degree 1, F(1) identifies with the A-span of u in $\mathrm{H}^*(\mathrm{B}\,\mathbb{Z}/2; \mathbb{F}_2) = \mathbb{F}^2[u]$ (therefore it has $\mathbb{F}_2$-basis $u^{2^n}, n \geqq 0$). The group $\mathfrak{S}_n$ acts on $\mathrm{F}(1)^{\otimes n}$, and if I is a subset of $\mathbb{F}_2[\mathfrak{S}_n]$, $\mathrm{I}.\mathrm{F}(1)^{\otimes n}$ denotes the A-module $\{sv/s \in \mathrm{I}, v \in \mathrm{F}(1)^{\otimes n}\}$.

In other words, isomorphism classes of simple objects in $\mathcal{U}/\mathcal{N}il$ are in bijection with isomorphism classes of simple $\mathbb{F}_2[\mathfrak{S}_n]$-modules, $n>0$ [this is implicit in (0.1) and (0.2) gives the rule].

Before going on, let us make two remarks:

(i) The category $\mathcal{N}il$ is the first step of a filtration on $\mathcal{U}$, namely there is a decreasing filtration

$$\mathcal{N}il_{-1} = \mathcal{U} \supset \mathcal{N}il_0 = \mathcal{N}il \supset \mathcal{N}il_1 \supset \ldots \supset \mathcal{N}il_i \ldots$$

such that $\mathcal{N}il_i/\mathcal{N}il_{i+1}$ is equivalent to $\mathcal{N}il_{i+1}/\mathcal{N}il_{i+2}$ and $\bigcap_i \mathcal{N}il_i$ is trivial (*see* [Sc]).

(ii) $\mathcal{U}$ being a category of graded modules over a connected $\mathbb{F}_2$-algebra, there is little hope to get anything interesting by looking at simple objects in $\mathcal{U}$! The situation is much better in $\mathcal{U}/\mathcal{N}il$.

In Chapter 5, we note various easy consequences of Theorems (0.1) and (0.2), among which it is worth to note that one gets [using injectivity in $\mathcal{U}$ of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^d; \mathbb{F}_2)$!] a new proof of the main part of the classification result of [LS1]:

THEOREM 0.3. — *An indecomposable unstable* A*-module which is injective and reduced is isomorphic to a direct summand of* $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^d; \mathbb{F}_2)$ *for some* d.

Chapters 6 and 7 give applications of the preceding results. The main result of Chapter 6 is a criteria on a simple object of $\mathcal{U}/\mathcal{N}il$ telling when it is a subquotient of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^d; \mathbb{F}_2)$. It is known that (isomorphism classes of) simple objects in $\mathcal{U}/\mathcal{N}il$ are in bijection with (isomorphism classes) of simple $\mathbb{F}_2$-representations of $\mathfrak{S}_n \geqq 0$. Such isomorphisms classes are classified by column 2-regular partitions $(\lambda_1, \ldots, \lambda_t)$ of the integer n. ($\lambda_1 \geqq \ldots \geqq \lambda_t$ and "regularity" means that $\lambda_i - \lambda_{i+1} \leqq 1$). Let us denote S_λ a simple object of $\mathcal{U}/\mathcal{N}il$ associated to λ.

THEOREM 0.4. — *The simple object* S_λ *occurs in the composition series of* $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ *if and only if* $\lambda_1 \leqq d$.

This theorem implies earlier results of N. Kuhn [K] and is equivalent (modulo $\mathcal{U}/\mathcal{N}il$ technicalities) to earlier results of D. Carlisle and N. Kuhn ([CK], Chapter 2 and 3).

The first part of Chapter 6 is concerned in making explicit the construction of S_λ.

It is shown in Chapter 5 that isomorphism classes of indecomposable summands of $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$, $d \geqq 0$, are indexed by column 2-regular partitions, as are isomorphism classes of simple objects of $\mathcal{U}/\mathcal{N}il$. Let us denote E_λ a representative in the isomorphism class associated to λ. Let λ' be the conjugate partition of λ.

Except special cases, one has very little information about the E_λ, one would like to know for example the Poincaré series. But outside the case of the Steinberg module $L(n)$ ([Mit P]), it is, in general, out of reach. Here is a formula for the connectivity of the E_λ.

THEOREM 0.5. — *The first degree where* E_λ *is non zero is* $\lambda'_1 + 2\lambda'_2 + \ldots + 2^{j-1}\lambda'_j + \ldots$.

This has been proved also (later) by Carlisle and Kuhn.

This result gives also the lowest degree where a simple $\mathbb{F}_2$-representation of $GL_d \mathbb{F}_2$ occurs as a composition factor of $\mathbb{F}_2[x_1, \ldots, x_d]$. Such isomorphism classes are classified by column 2-regular partitions $(\lambda_1, \ldots, \lambda_d)$ such that $\lambda_d > 0$. Let F_λ be a representative in the isomorphism class associated to λ.

THEOREM 0.6. — *The representation* F_λ *occurs for the first time in the composition series of* $\mathbb{F}_2[x_1, \ldots, x_d]$ *in degree* $\lambda'_2 + 2\lambda'_3 + \ldots$.

The two preceding results were also proved by D. Carlisle and N. Kuhn [CK]: Their approach uses less the Steenrod algebra (in fact the Steenrod algebras appears in their paper through the use of the "hyperalgebra" $\mathcal{U}_{\mathbb{F}_p}$) and much more combinatorics.

To finish this introduction we note that most of our results extend to the category of unstable modules over the mop p Steenrod algebra, p odd. However we do not know a formula for the connectivity result of chapter 7 in this case; it should be pointed out that the $\mathcal{N}il$-closure is much more delicate to handle when p is odd.

1. Introduction of $\mathcal{U}/\mathcal{N}il$ and recollections about Milnor's coaction

1.1. DEFINITION AND FIRST PROPERTIES OF THE CATEGORY $\mathcal{U}/\mathcal{N}il$. — Let A be the mod 2-Steenrod algebra. As usual we denote by $\mathcal{U}$ the category of unstable A-modules. Recall that an A-module M is said to be unstable if, for any x in M, $Sq^i x = 0$ as soon as i is greater than $|x|$, the degree of x. Note that this forces M to be trivial in negative degrees.

The category $\mathcal{U}$ is abelian and contains the mod 2 cohomology of spaces. In the sequel we will often abbreviate "unstable A-module" by "unstable module".

Let M be an unstable module, and let x be an element in M. One says that x is nilpotent if:

– $Sq^{2^n|x|} \ldots Sq^{|x|} x = 0$ as soon as n is large enough [LS1].

Recall that if M happens to be the mod 2 cohomology of a space $Sq^{|x|} x$ is the cup-square x^2, therefore the condition reads as:

– $x^{2^{n+1}} = 0$ as soon as n is large enough.

LEMMA 1.1.1. — *Let* M *be an unstable module. Let* Nil(M) *be the subset of the nilpotent elements of* M. *Then* Nil(M) *is a sub* A-*module of* M.

This is classical [Kr]. It is for example a consequence of relation (1.3.3), we just recall that the fact that M is unstable is crucial.

An unstable module M is *nilpotent* if Nil(M) = M. For example the suspension of an unstable module is nilpotent. If the unstable module M is nilpotent any sub A-module is nilpotent and any quotient of M is nilpotent.

An unstable module M is *reduced* if Nil(M) = $\{0\}$.

For short, in the sequel, we will often say "nilpotent module" (resp. "reduced module") instead of "nilpotent unstable A-module" (resp. "reduced unstable A-module").

Let us denote by $\mathcal{N}il$ the full subcategory of $\mathcal{U}$ which has as objects the nilpotent modules; it is an abelian category. One checks [Sc] that $\mathcal{N}il$ is the smallest Serre class, in $\mathcal{U}$, which is stable under direct limit and which contains all suspensions.

Let us now define the quotient category $\mathcal{U}/\mathcal{N}il$ as follows:

– $\mathcal{U}/\mathcal{N}il$ has the same objects as $\mathcal{U}$;

– given two objects M, N in $\mathcal{U}$, by definition

$$\mathrm{Hom}_{\mathcal{U}/\mathcal{N}il}(\mathrm{M}, \mathrm{N}) = \operatorname*{dir\,lim}_{(\mathrm{M}', \mathrm{N}')} \mathrm{Hom}_{\mathcal{U}}(\mathrm{M}', \mathrm{N}/\mathrm{N}'),$$

where the limit is taken on all pairs (M′, N′), where M′ (resp. N′) is a sub-A-module of M (resp. N) and M/M′ (resp. N′) is nilpotent. Note that a map φ in $\mathcal{U}$ is a monomorphism (resp. an epimorphism) when considered in $\mathcal{U}/\mathcal{N}il$ if and only if $\mathrm{Ker}\,\varphi$ (resp. $\mathrm{Coker}\ \varphi$) is nilpotent (we refer to [Ga] for further details). Let us denote by $r : \mathcal{U} \to \mathcal{U}/\mathcal{N}il$ the forgetful functor, it is exact. As the category $\mathcal{U}$ has generators (see 1.3) and exact direct limits, it has injective hulls [LS1], hence r has a right adjoint ([Ga], chap. III, § 3), we will denote it by s.

DEFINITION 1.1.2. — An A-module M is $\mathcal{N}il$-closed if the unit of adjunction $\mathrm{M} \to sr\mathrm{M}$ is an isomorphism.

LEMMA 1.1.3 ([Ga], Chap. III, § 2). — *An unstable* A-*module* M *is* $\mathcal{N}il$-*closed if and only if the following holds:*

$$\mathrm{Ext}^i_{\mathcal{U}}(\mathrm{N}, \mathrm{M}) = \{0\} \qquad \text{for} \quad i = 0, 1 \qquad \text{and any N in } \mathcal{N}il.$$

Remarks 1.1.4. — In other words, M is $\mathcal{N}il$-closed if and only if:

(i) M *is reduced*;

(ii) M *has no quadratic extensions, i.e.* if M is a sub-A-module of a reduced unstable module M′ and if $x \in M$ is of the form $Sq^{|y|} y$, for some $y \in M'$, then $y \in M$.

In fact, in order to check that an unstable A-module M is $\mathcal{N}il$-closed, it is enough to verify it is reduced and to verify (ii) for a particular M′ *which is* $\mathcal{N}il$*-closed.*

We refer to [LZ2] and [GLZ] for details and recall that these conditions found their origin in the work of Adams and Wilkerson [AW] and that such modules were called "Δ-modules" by Smith and Switzer [SS].

1.2. Recollections on the coaction [Mil], [St]. — The action of the Steenrod algebra on a module M is most conveniently expressed by using the coaction $\lambda : M \to M \hat{\otimes} A_*$. We now describe it.

Let us recall that A_* is the dual of A; as A is a Hopf algebra A_* is also a Hopf algebra and [Mil], [St]:

— as an algebra, A_* is a polynomial algebra on generators ξ_i, $i \geqq 0$, such that $|\xi_i| = 2^i - 1$, $\xi_0 = 1$;

— the diagonal $\Delta : A_* \to A_* \otimes A_*$ is given by

$$\Delta(\xi_i) = \sum_{l+k=i} \xi_l^{2^k} \otimes \xi_k.$$

The completed tensor product $M \hat{\otimes} A_*$ is the graded $\mathbb{F}_2$-vector space which is given by $(M \hat{\otimes} A_*)^n = \prod_{l-k=n} M^l \otimes A_k$.

The map λ: $M \to M \hat{\otimes} A_*$ is of degree zero and completely determined by the following property:

$$\text{for any } \theta \in A, \qquad \theta x = \lambda(x)/\theta. \tag{1.2.1}$$

As usual, if I is a multi-index $(i_1, \ldots, i_r)$, we write ξ^I for the monomial $\xi_1^{i_1} \ldots \xi_r^{i_r}$.

Therefore, for $x \in M$, $\lambda(x)$ can be written as a formal sum $\sum_I x_I \otimes \xi^I$, $x_I \in M$, the sum being taken over all multi-indices I, the notation "x_I" will be used in all the paper.

The definition implies that the A-module Ax generated by x is the span over $\mathbb{F}_2$ of the x_I's.

The following fundamental example is used by Milnor to define the ξ_i's. Recall that $H^*(B\mathbb{Z}/2; \mathbb{F}_2)$ is a polynomial algebra (over $\mathbb{F}_2$) on one generator u of degree 1. Then

$$\lambda(u) = \sum_{i \geqq 0} u^{2^i} \otimes \xi_i. \tag{1.2.2}$$

The definition of λ implies the following formula:

For any x in an A*-module* M, *any* θ *in* A, *one has*

$$\lambda(\theta x) = \sum_I x_I \otimes \Delta(\xi^I)/\theta. \tag{1.2.3}$$

Applying (1.2.3) in the case where $M=H^*(B\mathbb{Z}/2;\mathbb{F}_2)$, $x=u$ and $\theta=Sq^{2^{n-1}}\dots Sq^1$, one gets

$$(1.2.4)\qquad \lambda(u^{2^n})=\sum_{i\geq 0} u^{2^{n+i}}\otimes\xi_i^{2^n}.$$

Note finally that the coaction λ on a tensor product $M\otimes N$ is given by the following formula [Mil]: for any $x\in M$, $y\in N$, one has

$$(1.2.5)\qquad \lambda(x\otimes y)=\sum_{I}\sum_{J+K=I}(x_J\otimes y_K)\otimes\xi^I,$$

where the sum $J+K$ means the sum index by index. The formula makes sense because the second summation is finite. We now give an application; let $F(1)$ denotes the A-span of u in $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$, it has $\mathbb{F}_2$-basis u^{2^n}, $n\geq 0$ (*see* 1.3 for further details), then:

LEMMA 1.2.6. — *Let $(a_i)_{1\leq i\leq n}$ be a family of pairwise distinct integers, and let v be the monomial $u^{2^{a_1}}\otimes\dots\otimes u^{2^{a_n}}$ in $F(1)^{\otimes n}$. Then, the inclusion $Av\subsetneq F(1)^{\otimes n}$ is an isomorphism in $\mathcal{U}/\mathcal{N}il$.*

Proof. — One computes $\lambda(v)$ using (1.2.4) and (1.2.5) to find that v_I is non-zero if and only if ξ^I is of the form $\xi_{\alpha_1}^{2^{a_1}}\dots\xi_{\alpha_n}^{2^{a_n}}$ for some integers α_i, $1\leq i\leq n$; and in this case $v_I=u^{2^{a_1+\alpha_1}}\otimes\dots\otimes u^{2^{a_n+\alpha_n}}$, for the a_i's are pairwise distinct.

To prove the result, it is enough to show that every monomial has a power in Av. As every monomial has a power which writes $u^{2^{a_1+\alpha_1}}\otimes\dots\otimes u^{2^{a_n+\alpha_n}}$, we are done.

1.3. THE FREE UNSTABLE MODULES $F(n)$ ([MP]). — Let $F(n)$ be the free unstable module on one generator, i_n, of degree n. It is characterized by the property that the evaluation map $\mathrm{Hom}_{\mathcal{U}}(F(n),M)\to M^n$ given by $f\mapsto f(i_n)$ is an isomorphism for any unstable module M.

The unstable module $F(1)$ is easily identified with the one introduced above: the sub A-module of $H^*(B\mathbb{Z}/2,\mathbb{F}_2)$ generated by u.

Let $\omega_n: F(n)\to F(1)^{\otimes n}$ be the unique non-trivial map, it sends i_n on $u\otimes\dots\otimes u$. The symmetric group $\mathfrak{S}_n$ acts on the left on $F(1)^{\otimes n}$ by permuting the factors: for $\sigma\in\mathfrak{S}_n$, one has

$$\sigma(u^{2^{a_1}}\otimes\dots\otimes u^{2^{a_n}})=u^{2^{a_{\sigma^{-1}(1)}}}\otimes\dots\otimes u^{2^{a_{\sigma^{-1}(n)}}}.$$

The action of $\mathfrak{S}_n$ commutes with Steenrod operations, therefore ω_n takes values in the invariants $(F(1)^{\otimes n})^{\mathfrak{S}_n}$. In fact,

LEMMA 1.3.1 (SEE [LZ1]). — *The map ω_n induces an isomorphism from $F(n)$ into $(F(1)^{\otimes n})^{\mathfrak{S}_n}$.*

To illustrate the use of (1.3.1) let us introduce a notation and state a formula.

Let $Sq_0: M\to M$ be the map defined, for any unstable module M, by $x\to Sq^{|x|}x$. Then

$$(1.3.2)\qquad \lambda(Sq_0\,x)=\sum_I Sq_0\,x_I\otimes\xi^{2I},$$

for any x in an unstable module M, where, if $\mathrm{I}=(i_1, \ldots, i_r)$, $2\mathrm{I}=(2i_1, \ldots, 2i_r)$.

Proof. – It is enough to show it for $\mathrm{M}=\mathrm{F}(n)$ and $x=i_n$. But using (1.3.1) one is reduced to the case $\mathrm{M}=\mathrm{F}(1)^{\otimes n}$ and $x=u\otimes \ldots \otimes u$. One concludes with (1.2.4) and (1.2.5). Obviously this can also be shown directly using the Adem relations and the instability property.

To conclude let us observe that, by their very definition, the F (n)'s are generators for the category $\mathcal{U}/\mathcal{N}il$ as they are for $\mathcal{U}$. However there is, outside F(0), no projective in $\mathcal{U}/\mathcal{N}il$. The following will be proved elsewhere.

PROPOSITION. — *Let* P *be a projective object in* $\mathcal{U}/\mathcal{N}il$, *then* $\mathrm{P}\cong \mathrm{F}(0)^{\oplus\alpha}$ *for some cardinal* α. *Therefore,* $\mathcal{U}/\mathcal{N}il$ *has not enough projectives.*

2. The weight of an unstable module

2.1. DEFINITION OF THE WEIGHT. — Let us recall that, if n is an integer, $\alpha(n)$ denotes the number of 1 in the diadic expansion of n. We extend this definition to a multi-index $\mathrm{I}=(i_1, \ldots, i_r)$ by declaring that $\alpha(\mathrm{I})=\alpha(i_1)+\ldots+\alpha(i_r)$.

DEFINITION 2.1.1. — Let x be an element in an A-module M. The weight of x, denoted by $\pi(x)$, is the upper bound, may be infinite, of the $\alpha(\mathrm{I})$'s over the set of those multi-indices I such that x_I is non-zero.

Examples 2.1.2. – If $\mathrm{M}=\mathrm{H}^*(\mathrm{B}\,\mathbb{Z}/2; \mathbb{F}_2)$ and $x=u$, one gets, from (1.2.2), $\pi(u)=1$;

– if $\mathrm{M}=\mathrm{F}(1)^{\otimes n}$ and $x=u\otimes \ldots \otimes u$, one gets, from (1.2.5), $\pi(u\otimes \ldots \otimes u)=n$; therefore $\pi(i_n)=n$ [recall that i_n is the generator of $\mathrm{F}(n)^n$];

– if $\varphi:\mathrm{M}\to\mathrm{N}$ is a A-linear, it is clear that $\pi(\varphi(x))\leqq\pi(x)$ for any $x\in\mathrm{M}$.

Therefore, one easily gets.

LEMMA 2.1.3. — *For any unstable module* M *and any* x *in* M, $\pi(x)$ *is finite.*

This lemma does not hold if M is not supposed unstable.

PROPOSITION 2.1.4. — *Let* M *be an* A-*module. The elements of weight less or equal to an integer* n *form a sub*-A-*module of* M.

Proof. – Let x be in M. We prove that any element in Ax has weight less or equal to $\pi(x)$. Recall that for any $\theta\in\mathrm{A}$, $\lambda(\theta x)=\sum_{\mathrm{I}} x_\mathrm{I}\otimes\Delta(\xi^\mathrm{I})/\theta$. One observes using the formula for the diagonal Δ and arguing by induction on $\alpha(\mathrm{I})$, that $\Delta(\xi^\mathrm{I})$ can be written as a sum of elements $\xi^\mathrm{L}\otimes\xi^\mathrm{R}$ such that $\alpha(\mathrm{L})\leqq\alpha(\mathrm{I})$ and $\alpha(\mathrm{R})\leqq\alpha(\mathrm{I})$.

The lemma 2.1.3 and the proposition 2.1.4 show that an unstable module M is the direct limit, as n runs through $\mathbb{N}$, of its sub A-modules consisting of elements of weight less than n.

DEFINITION 2.1.5. — The weight of an A-module M, denoted by $\pi(\mathrm{M})$, is the upper bound, may be infinite, of the weights of its elements.

It is now time to define the weight for an object in $\mathcal{U}/\mathcal{N}il$.

DEFINITION 2.1.6. — Let M be an object in $\mathcal{U}/\mathcal{N}il$, the weight of M, denoted $\omega(\mathrm{M})$, is defined to be $\pi(s\mathrm{M})$.

We have immediately

PROPOSITION 2.1.7. — *Let* M *be an object in* $\mathcal{U}/\mathcal{N}il$, *then* $\omega(\mathrm{M})$ *is equal to*:

(i) *the minimum of the integers* $\pi(\mathrm{M}')$ *where* M' *runs over all unstable module* M' *such that* $r\mathrm{M}'$ *is isomorphic to* M;

(ii) *the weight* $\pi(\mathrm{N})$ *of any reduced unstable module* N *such that* $r\mathrm{N}$ *is isomorphic to* M.

The proof, which is a consequence from (1.3.2) and the proof of Proposition 2.1.4, is left to the reader. Statement (ii) tells us that for reduced unstable modules at least, the weight is the same in $\mathcal{U}$ and $\mathcal{U}/\mathcal{N}il$. As we will be concerned only with reduced unstable modules our terminology will cause no ambiguities.

We denote by $\mathcal{V}_n$ the full subcategory of $\mathcal{U}/\mathcal{N}il$ of objects of weight less or equal to n. The remark before Definition 2.1.5 implies that the smallest abelian subcategory of $\mathcal{U}/\mathcal{N}il$ which contains the $\mathcal{V}_n$, for all $n\geqq 0$, and is stable under direct limits is $\mathcal{U}/\mathcal{N}il$ itself. We are now able to state, in a weak form, the main theorem of the next chapter.

THEOREM. — *The quotient category* $\mathcal{V}_n/\mathcal{V}_{n-1}$, $n\geqq 1$, *is equivalent to the category of modules over the group ring* $\mathbb{F}_2[\mathfrak{S}_n]$.

2.2. THE WEIGHT IN $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n;\mathbb{F}_2)$. – For later use we now compute the weight in the mod 2 cohomology of the group $(\mathbb{Z}/2)^n$. Recall that the cohomology $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n;\mathbb{F}_2)$ is a polynomial algebra over $\mathbb{F}_2$, on n generators $x_1,\ldots,x_n$ of degree 1.

We want to determine the weight of an element in $\mathbb{F}_2[x_1,\ldots,x_n]$. We first consider the case of a monomial $x_1^{i_1}\ldots x_n^{i_n}$. If I is the n-tuple $(i_1,\ldots,i_n)$, we denote by x^{I} the preceding monomial.

LEMMA 2.2.1. — *The weight of* x^{I} *is equal to* $\alpha(\mathrm{I})$.

More generally

PROPOSITION 2.2.2. — *The weight of a homogeneous element* x *in* $\mathbb{F}_2[x_1,\ldots,x_n]$ *is the supremum of the* $\alpha(\mathrm{I})$'*s, where* I *runs through the n-tuples* I *such that, in the monomial basis,* x *has a non-zero component on* x^{I}.

In the rest of the paper when considering a vector space with a given basis, we will say that a basis element w appears in an element x, if the expression of x in the given basis has a non-zero coefficient on w. We will be mainly concerned with the case of a polynomial algebra with a given monomial basis.

Proof of (2.2.1). – The inequality $\pi(x^{\mathrm{I}})\leqq\alpha(\mathrm{I})$ follows from (1.2.5) and (1.3.2). For the reverse, let us write x^{I} in the following way

$$x^{\mathrm{I}}=\prod_{l=1}^{l=\alpha(\mathrm{I})} x_{m_l}^{2^{a_l}},$$

where, for every l, $a_l \leqq a_{l+1}$ and $m_l < m_{l+1}$ if $a_l = a_{l+1}$.

Note that the sequence of integers $(2^{a_l})_{1 \leqq l \leqq \alpha(\mathrm{I})}$ is uniquely defined, being the family of powers of 2 appearing in the diadic expansion of the i_h's.

Then, one proves that for the *multi-index* $\mathrm{J} = (2^{a_1}, \ldots, 2^{a_{\alpha(\mathrm{I})}})$, $(x^{\mathrm{I}})_{\mathrm{J}}$ is non-zero. One first note that

$$(x^{\mathrm{I}})_{\mathrm{J}} = \sum_{\sigma} \prod_{1}^{\alpha(\mathrm{I})} x_{m_{\sigma^{-1}(l)}}^{2^{a_l+l}},$$

where σ runs over those permutations of $\mathfrak{S}_{\alpha(\mathrm{I})}$ fixing the sequence $(a_l)_{1 \leqq l \leqq \alpha(\mathrm{I})}$. This formula follows from (1.2.5) and (1.3.2). As the $(a_l + l)$'s are pairwise distinct, one checks now, by routine argument, that $\prod_{1}^{\alpha(\mathrm{I})} x_{m_l}^{2^{a_l+l}}$ has non-zero coefficient in $(x^{\mathrm{I}})_{\mathrm{J}}$ which is therefore non-zero.

PROOF OF PROPOSITION 2.2.2. — The weight of x is clearly less or equal to the supremum of the weight of the monomials appearing in it.

Conversely, let x^{I} a monomial appearing in x such that $\alpha(\mathrm{I})$ is maximal. We claim, keeping the notations of the proof of (2.2.1), that x_{J} is non-zero.

It is enough to prove that among the monomials whose sum is x, x^{I} is the only one to let appear in x_{J} the term $\prod_{1}^{\alpha(\mathrm{I})} x_{m_l}^{2^{a_l+l}}$.

Let $x^{\mathrm{I}'}$ be another monomial appearing in x, and let suppose that $(x^{\mathrm{I}'})_{\mathrm{J}}$ is non-zero. Write $x^{\mathrm{I}'}$ as we did for x^{I}

$$x^{\mathrm{I}'} = \prod_{1}^{\alpha(\mathrm{I}')} x_{q_l}^{2^{b_l}},$$

with $b_l \leqq b_{l+1}$ and $q_l < q_{l+1}$ if $b_l = b_{l+1}$.

As $(x^{\mathrm{I}'})_{\mathrm{J}}$ is non-zero, one concludes first that $\alpha(\mathrm{I}') \geqq \alpha(\mathrm{J}) = \alpha(\mathrm{I})$, as $\alpha(\mathrm{I})$ is maximal, one gets $\alpha(\mathrm{I}') = \alpha(\mathrm{I})$. Secondly, one observes that the sequence $(b_l)_{1 \leqq l \leqq \alpha(\mathrm{I})}$ has to be equal to the sequence $(a_l)_{1 \leqq l \leqq \alpha(\mathrm{I})}$. Therefore, $(x^{\mathrm{I}'})_{\mathrm{J}}$ writes as

$$(x^{\mathrm{I}'})_{\mathrm{J}} = \sum_{\sigma} \prod_{1}^{\alpha(\mathrm{I})} x_{q_{\sigma^{-1}(l)}}^{2^{a_l+l}},$$

again over those permutations σ fixing $(a_l)_{1 \leqq l \leqq \alpha(\mathrm{I})}$. The exponents $(a_l + l)_{1 \leqq l \leqq \alpha(\mathrm{I})}$ are pairwise distinct, so that the sum contains the term $\prod_{1}^{\alpha(\mathrm{I})} x_{m_l}^{2^{a_l+l}}$ if and only if there is a permutation τ such that $q_{\tau^{-1}(l)} = m_l$ for every l. Hence

$$x^{\mathrm{I}'} = \prod_{1}^{\alpha(\mathrm{I})} x_{q_l}^{2^{b_l}} = \prod_{1}^{\alpha(\mathrm{I})} x_{m_{\tau(l)}}^{2^{a_l}} = \prod_{1}^{\alpha(\mathrm{I})} x_{m_l}^{2^{a_l}},$$

the last equality resulting from the fact that τ fixes (a_l). Therefore $x^{\mathrm{I}'}=x^{\mathrm{I}}$ and we are done.

We note as a corollary

COROLLARY 2.2.3. — *Let* M *be a non-zero, sub* A-*module of* $F(1)^{\otimes n}$. *Then,* M *has weight n and* M *is non-zero in a degree d such that* $\alpha(d)=n$.

Proof. — The unstable module F (1) is contained in $\mathbb{F}_2[x]$ as the $\mathbb{F}_2$-span of the classes $x^{2^h}, h\geqq 0$. In the same way, $F(1)^{\otimes n}$ is contained in $\mathbb{F}_2[x_1, \ldots, x_n]$ as the $\mathbb{F}_2$-span of the monomials $x_1^{2^{a_1}}\ldots x_n^{2^{a_n}}, a_1\geqq 0, \ldots, a_n\geqq 0$. It is enough then to apply Proposition 2.2.2 to compute the weight of a sub A-module. For the remaining part, one observes that the element x_{J} described in the proof of Proposition 2.2.2 has the required property.

3. The structure of the category $\mathscr{U}/\mathscr{N}il$

3.1. STATEMENTS. — We introduced in Chapter 2 a filtration on the category $\mathscr{U}/\mathscr{N}il$ by subcategories $\mathscr{V}_n$. Let us recall the main statement about this filtration.

THEOREM 3.1.1. — *The quotient category* $\mathscr{V}_n/\mathscr{V}_{n-1}$ *is equivalent to the category* $\mathrm{Mod}_{\mathbb{F}_2[\mathfrak{S}_n]}$ *of right modules over the group ring* $\mathbb{F}_2[\mathfrak{S}_n]$.

This result was first proved by Jean Lannes and the second author using a completely different approach [LS2]. A related result to the theorem is the classification of simple objects in $\mathscr{U}/\mathscr{N}il$ (which is also found in [LS2]).

Let us note that an unstable module M is simple as an object of $\mathscr{U}/\mathscr{N}il$, if and only if, for any non trivial sub-A-module M′ of M, $M/M'\in\mathscr{N}il$.

Let us introduce a few notations. Let I be a right ideal of $\mathbb{F}_2[\mathfrak{S}_n]$, the group $\mathfrak{S}_n$ acts on the left on $F(1)^{\otimes n}$; let $\mathrm{I}.F(1)^{\otimes n}$ the sub-A-module of $F(1)^{\otimes n}$ spanned over $\mathbb{F}_2$ by the elements $s\omega$ for $s\in\mathrm{I}$ and $\omega\in F(1)^{\otimes n}$.

Let us note first an easy and technical point.

PROPOSITION 3.1.2. — *For any right ideal* I *of* $\mathbb{F}_2\mathfrak{S}_n$, *the unstable* A-*module* $\mathrm{I}.F(1)^{\otimes n}$ *is* $\mathscr{N}il$-*closed.*

We have now

THEOREM 3.1.3. — *Let* I *be a simple right ideal of* $\mathbb{F}_2[\mathfrak{S}_n]$, *then, as an object of* $\mathscr{U}/\mathscr{N}il$, $\mathrm{I}.F(1)^{\otimes n}$ *is simple. Let* M *be a simple object of* $\mathscr{U}/\mathscr{N}il$ *of weight n, there exists a unique (up to isomorphism) simple right ideal* J *of* $\mathbb{F}_2[\mathfrak{S}_n]$ *such that* M *is isomorphic to* $\mathrm{J}.F(1)^{\otimes n}$ *(considered as an object of* $\mathscr{U}/\mathscr{N}il$).

Let us remark that (2.2.3) tells us that if J is non trivial, the weight of $\mathrm{J}.F(1)^{\otimes n}$ is n;

The proof of Theorems 3.1.1 and 3.1.3 depends on two more technical results which follow.

The first, which is the most important, gives a "computation" of the weight which depends only on the structure of sM as an $\mathbb{F}_2$-graded vector space.

THEOREM 3.1.4. — *Let* M *be an object in* $\mathcal{U}/\mathcal{N}il$. *The weight*, $\omega(\mathrm{M})$, *of* M *is the largest integer* $\alpha(d)$ *when* d *runs over those integers such that* $(s\,\mathrm{M})^d$ *is non trivial.*

One checks easily that this is equivalent to the following statement: The weight, $\pi(\mathrm{N})$, of a reduced unstable A-module N is the largest integer $\alpha(d)$ when d runs over those integers d such that N^d is non-zero. In particular in (3.1.4), one could replace $s\,\mathrm{M}$ by any reduced unstable module isomorphic to M in $\mathcal{U}/\mathcal{N}il$.

Note also that Corollary 2.2.3 implies that Theorem 3.1.4 holds for sub-A-modules of $\mathrm{F}(1)^{\otimes n}$.

The second result is a consequence of the preceding one and is as follows

PROPOSITION 3.1.5. — *The unstable A-module* $\mathrm{F}(1)^{\otimes n}$ *is projective in the category* $\mathcal{V}_n$.

THEOREM 3.1.4. — Will be proved in Chapter 4, the proof splits naturally in two parts:

– first, one shows that the weight as defined in chapter 2 is greater than the integer defined in theorem 3.1.4, it is easy;

– second, one proves the reverse inequality, it is much harder.

The proof of Proposition 3.1.5 is given in section 3.2.

Now we show how they imply Theorems 3.1.1 and 3.1.3. The proof of Proposition 3.1.2, which is classical, will be given at the end of the chapter.

3.2. PROOF OF THEOREM 3.1.1. – Let us define a functor $p_n: \mathcal{U}/\mathcal{N}il \to \mathrm{Mod}_{\mathbb{F}_2[\mathfrak{S}_n]}$ by the following formula: $p_n(\mathrm{M}) = \mathrm{Hom}_{\mathcal{U}/\mathcal{N}il}(\mathrm{F}(1)^{\otimes n}, \mathrm{M})$, the (right) action of $\mathfrak{S}_n$ on $p_n(\mathrm{M})$ being induced by the (left) action of $\mathfrak{S}_n$ on $\mathrm{F}(1)^{\otimes n}$; let us also denote by p_n the restriction of this functor to $\mathcal{V}_n$. The Proposition 3.1.5 tells us that p_n is exact on $\mathcal{V}_n$.

Now, let us give an example

$$p_n(\mathrm{F}(1)^{\otimes n}) \cong \mathbb{F}_2[\mathfrak{S}_n]. \tag{3.2.1}$$

This is a consequence of (1.2.6), one keeps the notations of this lemma. One has $p_n(\mathrm{F}(1)^{\otimes n}) \cong \mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, \mathrm{F}(1)^{\otimes n})$ by Lemma 1.2.6 and Proposition 3.1.2. In a degree d such that $\alpha(d)$ is equal to n, $\mathrm{F}(1)^{\otimes n}$, as left $\mathbb{F}_2[\mathfrak{S}_n]$-module, is isomorphic to $\mathbb{F}_2[\mathfrak{S}_n]$; therefore for any $\varphi \in \mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, \mathrm{F}(1)^{\otimes n})$, $\varphi(v)$ writes as $s_\varphi v$ for some uniquely defined $s_\varphi \in \mathbb{F}_2[\mathfrak{S}_n]$. A routine checking shows that $\varphi \to s_\varphi$ determines an isomorphism between $p_n(\mathrm{F}(1)^{\otimes n})$ and $\mathbb{F}_2[\mathfrak{S}_n]$. We just check it is equivariant. Let σ be in $\mathfrak{S}_n$ and φ be in $p_n(\mathrm{F}(1)^{\otimes n})$. We need to identify $s_{\varphi\sigma}$. By Lemma 1.2.6, there exists $\theta \in \mathrm{A}$ such that $\theta v = \mathrm{S}\,q_0^k\,\sigma\,v$ for some $k \geqq 0$. Therefore

$$\begin{aligned} \mathrm{S}q_0^k(\varphi\sigma(v)) &= \varphi(\theta\,v) = \theta(s_\varphi\,v) \\ &= s_\varphi\,\sigma(\mathrm{S}\,q_0^k\,v). \end{aligned}$$

As $\mathrm{S}\,q_0$ is injective, we get $s_{\varphi\sigma} = s_\varphi\,\sigma$.

More generally

LEMMA 3.2.2. — *Let* M *be an object in* $\mathcal{V}_n$, *d an integer such that* $\alpha(d)=n$. *Then, as an* $\mathbb{F}_2$*-vector space,* $p_n(\mathrm{M})$ *is naturally isomorphic to* $(s\mathrm{M})^d$.

Note that the lemma implies that $(s\mathrm{M})^d$ "does not depend" on d for those d such that $\alpha(d)=n$ and is in a natural way an $\mathbb{F}_2[\mathfrak{S}_n]$-module. In fact, inspection of the proof shows that $\mathrm{S}q_0:(s\mathrm{M})^d\to(s\mathrm{M})^{2d}$ is an isomorphism compatible with the action of $\mathfrak{S}_n$. We will give the proof of this lemma at the end of the paragraph.

Proof of (3.1.5). – We claim that Lemma 3.2.2 implies Proposition 3.1.5. It is enough to show that if $\mathrm{M}\to\mathrm{N}$ is an epimorphism in $\mathcal{V}_n$, then, in degrees d such that $\alpha(d)=n$, $(s\mathrm{M})^d\to(s\mathrm{N})^d$ is surjective. Here are two ways to proceed.

The morphism $\mathrm{M}\to\mathrm{N}$ being an epimorphism in ν_n, it is one also in $\mathcal{U}/\mathcal{N}il$. Therefore, for any x in $(s\mathrm{N})^d$, there exists h such that $\mathrm{S}q_0^h x$ is in the image of $(s\mathrm{M})^{d2^h}$ in $(s\mathrm{N})^{d2^h}$. But as $\mathrm{S}q_0^h:(s\mathrm{M})^d\to(s\mathrm{M})^{d2^h}$ is an isomorphism (3.2.2 and comments), one is done.

One can also do it as follows. One assumes first that N is noetherian (as an object of $\mathcal{U}/\mathcal{N}il$). It follows that $(s\mathrm{N})^{d2^h}$ does not depend on h as h is large enough and is finite dimensional. This is easily proved by observing that the hypothesis on N implies there is a map $\mathrm{L}\to s\mathrm{N}$ from a reduced finitely generated unstable A-module which is an isomorphism in $\mathcal{U}/\mathcal{N}il$. Therefore, in this case, one gets that $(s\mathrm{M})^{d2^h}\to(s\mathrm{N})^{d2^h}$ is surjective for h large enough. Therefore by (3.2.2), $(s\mathrm{M})^d\to(s\mathrm{N})^d$ is surjective. The general case follows using either the fact that N is the direct limit of its noetherian subobjects and the fact that s commutes with direct limits [Ga], either that $\mathrm{F}(1)^{\otimes n}$ is finitely generated as an A-module which follows from an easy generalisation of (1.2.6).

LEMMA 3.2.3. — *The functor* p_n *is trivial on* $\mathcal{V}_{n-1}$.

Proof. – Let M be an object in $\mathcal{V}_{n-1}$, one has by Lemma 1.2.6 an isomorphism $p_n(\mathrm{M})\cong\mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, s\mathrm{M})$ (keeping one more time the notations of (1.2.6)). But $s\mathrm{M}$ is trivial in degrees d such that $\alpha(d)=n$ by the easy part of Theorem 3.1.4, as $\alpha(|v|)=n$ we are done.

The functor p_n is exact on ν_n by Theorem 3.1.5, therefore it induces a functor $\tilde{p}_n:\mathcal{V}_n/\mathcal{V}_{n-1}\to\mathrm{Mod}_{\mathbb{F}_2[\mathfrak{S}_n]}$.

THEOREM 3.2.4. — *The functor* $\tilde{p}_n$ *is an equivalence.*

Proof. – Let us define a functor $m_n:\mathrm{Mod}_{\mathbb{F}_2[\mathfrak{S}_n]}\to\mathcal{V}_n$ by the following formula

$$m_n(\mathrm{R})=\mathrm{R}\underset{\mathbb{F}_2[\mathfrak{S}_n]}{\dot{\otimes}}\mathrm{F}(1)^{\otimes n},$$

where R is a (right) $\mathbb{F}_2[\mathfrak{S}_n]$-module. We are going to show that:

(i) m_n is a section of p_n (*i. e.* $p_n\circ m_n(\mathrm{R})\cong\mathrm{R}$);

(ii) for any M in $\mathcal{V}_n$, $p_n(\mathrm{M})=0$ implies that $\omega(\mathrm{M})<n$.

Direct computation along the same line as (3.2.1) shows that $p_n\circ m_n(\mathrm{R})\cong\mathrm{R}$. One proceeds as follows. One first observe that

$$p_n\circ m_n(\mathrm{R})\cong\mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, m_n(\mathrm{R}))\tag{3.2.5}$$

[keeping the notations of (1.2.6)]. This is not immediate for we do not know that $m_n(\mathrm{R})$ is $\mathcal{N}il$-closed. But one checks that the localization map $m_n(\mathrm{R}) \to sr(m_n(\mathrm{R}))$ is an isomorphism in degrees d such that $\alpha(d)=n$ (this follows from the fact that, in such a degree, $m_n(\mathrm{R})$ identifies with R and $\mathrm{S}q_0 : m_n(\mathrm{R})^d \to m_n(\mathrm{R})^{2d}$ identifies with the identity). Therefore the cokernel of the localization is trivial in these degrees and there are no non-trivial map of $\mathrm{A}v$ into it $(\alpha(|v|)=n)$. It follows that $\mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, m_n(\mathrm{R})) \cong \mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, sr(m_n(\mathrm{R})))$ and (3.2.5) follows. Note that we have not used (3.1.4).

For a given $\varphi \in \mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, m_n(\mathrm{R}))$, there is a unique $s_\varphi \in \mathbb{F}_2[\mathfrak{S}_n]$ such that $\varphi(v)=s_\varphi \otimes v$ (recall that $m_n(\mathrm{R})^{|v|} \cong \mathrm{R}$). We claim that the map $\varphi \to s_\varphi$ induces an isomorphism from $p_n \circ m_n(\mathrm{R})$ into R. Then, the proof goes as in Example 3.2.1.

Next, we are going to prove a bit more than (ii), namely one shows that the "evaluation" map

$$ev : m_n \circ p_n(\mathrm{M}) \to s\,\mathrm{M}$$

is an isomorphism in degree d such that $\alpha(d)=n$, therefore an isomorphism in $\mathcal{V}_n/\mathcal{V}_{n-1}$ by the hard part of Theorem 3.1.4. It clearly proves (ii). We first explain what we mean by "evaluation" map

$$\begin{aligned} m_n \circ p_n(\mathrm{M}) &\cong \mathrm{Hom}_{\mathcal{U}/\mathcal{N}il}(\mathrm{F}(1)^{\otimes n}, \mathrm{M}) \underset{\mathbb{F}_2[\mathfrak{S}_n]}{\otimes} \mathrm{F}(1)^{\otimes n} \\ &\cong \mathrm{Hom}_{\mathcal{U}}(\mathrm{F}(1)^{\otimes n}, s\,\mathrm{M}) \underset{\mathbb{F}_2[\mathfrak{S}_n]}{\otimes} \mathrm{F}(1)^{\otimes n}. \end{aligned}$$

Therefore there is a well defined evaluation from $m_n \circ p_n(\mathrm{M})$ into $s\,\mathrm{M}$. In a degree d such that $\alpha(d)=n$, both sides are isomorphic to $(s\,\mathrm{M})^d$ by 3.2.2 and the evaluation map is an isomorphism (*see* the proof of 3.2.2). This concludes the proof of 3.2.4.

PROOF OF LEMMA 3.2.2. — Consider an integer d such that $\alpha(d)=n$ and an element x in $(s\,\mathrm{M})^d$, write d as $2^{a_1}+\ldots+2^{a_n}$. As in Lemma 1.2.6, let us denote by v the monomial $u^{2^{a_1}} \otimes \ldots \otimes u^{2^{a_n}}$. Let us recall the isomorphism $p_n(\mathrm{M}) \cong \mathrm{Hom}_{\mathcal{U}}(\mathrm{A}\,v, s\,\mathrm{M})$.

We claim now that the assignement $v \mapsto x$ extends to an A-linear map (which is a priori unique). This proves Lemma 3.2.2.

There is an A-linear map extending the assignement $v \mapsto x$ if and only if the map which sends v_{I} to x_{I} for any multi-index I extends to a (unique) $\mathbb{F}_2$-linear map. In the proof of Lemma 1.2.6, the non-zero v_{I} are described, they form an $\mathbb{F}_2$-basis for $\mathrm{A}\,v$; therefore it is enough to show that x_{I} is zero as soon as v_{I} is zero.

Let us now introduce a notation which will be useful also later. Given a multi-index $\mathrm{J}=(j_1, \ldots, j_r)$, denote by $c(\mathrm{J})$ the sum $j_1+\ldots+j_r$. If L is an unstable module and y an element of degree k of L, one has

$$y_{\mathrm{I}} \neq 0 \Rightarrow c(\mathrm{I}) \leq k. \tag{3.2.6}$$

This is proved by considering the "universal" case: the generator $F(k)^k$. If I is a multi-index such that y_I is non-zero, define I' to be the multi-index $(k-c(I), i_1, \ldots, i_r)$, then

$$y_{I'} = Sq_0\, y_I. \tag{3.2.7}$$

This is one more time checked on the universal case.

Now assume that we have a multi-index I for which x_I is non-zero, one has to prove that v_I is non-zero. The relation 3.2.7 shows that $x_{I'}$ is non-zero (sM is reduced), therefore the hard part of Theorem 3.1.4 implies

$$\alpha(I') \leqq \pi(x) \leqq n,$$

we insist on the fact that this is really the place where the use of Theorem 3.1.4 is crucial. On the other hand

$$\alpha(I') \geqq \alpha(c(I')) = \alpha(d) = n,$$

so $\alpha(I') = n$. Therefore, the power of 2 appearing in the diadic expansion of the indices contained in I' are exactly $2^{a_1}, \ldots, 2^{a_n}$. In the proof of Lemma 1.2.6, it is noted that, in this case, $v_{I'}$ is non-zero, so is v_I by (3.2.7).

3.3. Proof of theorem 3.1.3. — To prove Theorem 3.1.3, we will proceed in two steps. We first show that for a simple object M in $\mathcal{U}/\mathcal{N}il$, which is of weight n, $p_n(M)$ is a simple, non-zero, right $\mathbb{F}_2[\mathfrak{S}_n]$-module J. Then one shows that M is isomorphic to $J.F(1)^{\otimes n}$, J being identified with a simple right ideal of $\mathbb{F}_2[\mathfrak{S}_n]$. The fact that $p_n(M)$ is non-zero is a consequence of Theorem 3.1.4. The simplicity of $p_n(M)$ is easy and left to the reader. We observe that if J is a simple right ideal of $\mathbb{F}_2[\mathfrak{S}_n]$, then $J.F(1)^{\otimes n}$ is a simple object in v_n. This is proved as follows; let $S \subset J.F(1)^{\otimes n}$, we know that S has weight n if it is non-zero (Corollary 2.2.3). Therefore $p_n(S)$ is equal to J if it is non-zero and we are done.

Let us identify $p_n(M)$ with a simple right ideal J of $\mathbb{F}_2[\mathfrak{S}_n]$ and consider the diagram:

$$\begin{array}{c} J \underset{\mathbb{F}_2[\mathfrak{S}_n]}{\otimes} F(1)^{\otimes n} \cong p_n(M) \underset{\mathbb{F}_2[\mathfrak{S}_n]}{\otimes} F(1)^{\otimes n} \xrightarrow{ev} sM \\ \downarrow \pi \\ J.F(1)^{\otimes n} \end{array}$$

where π is the projection which sends $s \otimes \omega$ to $s\omega$. In the degrees d such that $\alpha(d)$ is equal to n, π is an isomorphism. Therefore, $\operatorname{Ker}\pi$ has weight at most $n-1$ by the difficult part of Theorem 3.1.4. Hence $ev(\operatorname{Ker}\pi)$ is zero as sM contains no non-trivial subobjects of weight less or equal to $n-1$. Therefore, π factors through ev and there is a non-zero map $\varphi: J.F(1)^{\otimes n} \to sM$. The result follows, sM and $J.F(1)^{\otimes n}$ being simple.

3.4. Proof of Proposition 3.1.2. — We first note that $F(1)^{\otimes n}$ is $\mathcal{N}il$-closed. This is a consequence of paragraph 8 of [LZ 2], this follows also easily from 1.3.2. Now if

I is a right ideal of $\mathbb{F}_2[\mathfrak{S}_n]$, $\mathrm{I}.\mathrm{F}(1)^{\otimes n}$ is obviously reduced and quadratically closed in $\mathrm{F}(1)^{\otimes n}$, therefore (*see* 1.1), it is $\mathcal{N}il$-closed.

4. Proof of Theorem 3.1.4

Recall that we want to prove the following

THEOREM. — *Let* M *be an object in* $\mathcal{U}/\mathcal{N}il$. *The weight of* M *is the largest integer* $\alpha(d)$ *for those d such that* $(s\mathrm{M})^d$ *is non-zero.*

We first show the easy part of the theorem: the inequality $\omega(\mathrm{M}) \geqq \sup \alpha(d)$, d running over those integers such that $(s\mathrm{M})^d \neq \{0\}$. This is a consequence of the fact that $s\mathrm{M}$ is reduced. If x is a non-zero element of $s\mathrm{M}$ of degree d, one has $\mathrm{S}q_0 x = x_{(d)}$ [(d) denotes here a multi-index]; therefore, as $\mathrm{S}q_0 x$ is non-zero, $\pi(x) \geqq \alpha((d)) = \alpha(d)$.

Let us now introduce a few notations. Let x be an element of degree d. If I is a multi-index such that x_{I} is non-zero, $c(\mathrm{I}) \leqq d$, so we can define $\delta\mathrm{I}$ to be the multi-index $(d - c(\mathrm{I}), i_1, \ldots, i_r)$ [where $\mathrm{I} = (i_1, \ldots, i_r)$]. This was denoted I′ in the proof of 3.2.2. Recall that (3.2.7).

$$x_{\delta\mathrm{I}} = \mathrm{S}q_0 x_{\mathrm{I}}.$$

In the course of this paragraph, we shall say that a multi-index I *carries the weight of* x if x_{I} is non-zero and $\alpha(\mathrm{I})$ is equal to $\pi(x)$. Remark that when x is an element of a reduced unstable module, if I carries the weight of x then (because of 3.2.7) $c(\mathrm{I})$ is equal to the degree of x.

Note also that the operator δ respects the reverse lexicographical order on multi-indices

$$\mathrm{I} \leqq \mathrm{J} \Rightarrow \delta\mathrm{I} \leqq \delta\mathrm{J}.$$

Let us denote by E_u the set of multi-indices I such that I carries the weight of x and x_{I} is of degree u $(|\xi^{\mathrm{I}}| = u - |x|)$.

LEMMA 4.1. — *If h is large enough, the operator* δ *induces a bijection from* $\mathrm{E}_{2^h u}$ *into* $\mathrm{E}_{2^{h+1} u}$.

Proof. — If I carries the weight of x so does $\delta\mathrm{I}$, therefore δ sends $\mathrm{E}_{2^h u}$ into $\mathrm{E}_{2^{h+1} u}$ for any h. But δ is injective, therefore it is enough to show that the cardianal of $\mathrm{E}_{2^h u}$ is bounded independantly of h. The cardinal of $\mathrm{E}_{2^h u}$ is certainly not greater than the number of multi-indices I such that x_{I} is non-zero and is of degree $2^h u$. To get a bound it is enough to look at the universal case when $\mathrm{M} = \mathrm{F}(d)$ and $x = i_d$. In this case the problem reduces in majoring the number of non decreasing sequences of d powers of 2 adding up to $2^h d$ (to prove that use, for example, 1.3.1). Let us call this number $e_d(h)$. We claim that $e_d(h)$ is constant as soon as h is greater or equal to $d-1$. Indeed $e_d(h+1)$ is greater than $e_d(h)$ only if there is a sequence of d powers of 2 adding up to $2^{h+1} d$ and starting with 1. But such a sequence has at least $(h+2)$ terms.

We are now ready to prove the difficult part of the theorem.

To show the reverse inequality, we need to find for any x in $s\mathrm{M}$ an element y in $\mathrm{A}x$ such that y is non-zero and $\alpha(|y|)=\pi(x)$. We look among the x_I's to get such an element.

Let us choose a multi-index I carrying the weight of x. It is possible to suppose that I is the largest element in $\mathrm{E}_{|x_\mathrm{I}|}$. Recall that the order on $\mathrm{E}_{|x_\mathrm{I}|}$ is the reverse lexicographical one, and that x_I is non-zero as I carries the weight of x by hypothesis.

Let us denote $\pi(x)$ by h and write

$$\xi^\mathrm{I}=\prod_{i=1}^{i=h}\xi_{n_i}^{2^{a_i}}$$

with the convention that, for all i's, $n_i \leqq n_{i+1}$ and $a_i > a_{i+1}$ if $n_i = n_{i+1}$.

This is uniquely defined, the convention on the a_i's is not as essential in the sequel as the one on the n_i's. Note that these are not the conventions of Chapter 2.

Next consider the elements

$$\xi^\mathrm{R}=\prod_{i=1}^{i=h}\xi_{h+1-i}^{2^{a_i}}$$

$$\xi^\mathrm{L}=\prod_{i=1}^{i=h}\xi_{n_i-h-1+i}^{2^{a_i+h+1-i}}.$$

If necessary replace I by some $\delta^r\mathrm{I}$ to secure that $n_i-h-1+i$ is greater than 1 for all i, Lemma 4.1 allows us to do it for one can suppose that, for all $r \geqq 0$, $\delta^r\mathrm{I}$ is the largest element in $\mathrm{E}_{2^r|x_\mathrm{I}|}$. Let us note now that if $i<j$, then $n_i-h-1+i<n_j-h-1+j$. We now consider $\Delta\xi^\mathrm{I}$, it is given by the formula

$$\Delta\xi^\mathrm{I}=\prod_{i=1}^{i=h}\left(\sum_{\nu_i=0}^{\nu_i=n_i}\xi_{n_i-\nu_i}^{2^{a_i+\nu_i}}\otimes\xi_{\nu_i}^{2^{a_i}}\right).$$

We claim that $\xi^\mathrm{R}\otimes\xi^\mathrm{L}$ appears in $\Delta\xi^\mathrm{I}$ with a non-zero coefficient. Suppose that a sequence $(\nu_i)_{1\leqq i\leqq h}$ creates this term, it must agree with the sequence $(h, \ldots, 1)$ up to a permutation σ. So the sequence $(n_i-\nu_i)_{1\leqq i\leqq h}$ is of the form $(n_1-\sigma(h), n_2-\sigma(h-1), \ldots, n_h-\sigma(1))$. On the other side, we know it has to agree, up to a permutation, with the sequence $(n_1-h, n_2-h-1, \ldots, n_h-1)$ because the $(n_i-h-1+i)$'s air pairwise distinct.

As n_h-1 is strictly greater than n_i-j as soon as $j>1$, an equality of the form $n_h-1=n_i-\sigma(h+1-i)$ implies the following equalities: $n_h=n_i$ and $1=\sigma(h+1-i)$. Suppose $i<h$, in this case we know that $n_i=n_{i+1}=\ldots=n_h$ and $a_i>a_{i+1}>\ldots>a_h$. But the term $\xi_{n_i}^{2^{a_i}}$, which creates $\xi_{n_i-1}^{2^{a_i+1}}$ on the left hand side of $\Delta\xi^\mathrm{I}$, creates $\xi_1^{2^{a_i}}$ on the right hand side, and it is the only one which can create a power of ξ_1. But this is impossible because ξ_1 appears in ξ^R to the power 2^{a_h} and $a_h<a_i$. Therefore, $i=h$ and $\sigma(1)=1$; an induction completes the proof.

LEMMA 4.2. — *Let* K *be a multi-index such that* x_K *is non-zero. If, in the decomposition of* $\Delta\xi^K$ *in the monomial basis, the monomial* $\xi^L \otimes \xi^R$ *has a non-zero coefficient, then the multi-index* K *is equal to* I.

Proof. — First, $\alpha(K)$ is equal to $\pi(x)$ because $\alpha(K) \geqq \alpha(R) = \pi(x)$. Thus K carries the weight of x and is in $E_{|x_I|}$ where it precedes the maximal element I. Write now

$$\xi^K = \prod_{i=1}^{i=h} \xi_{n'_i}^{2^{a'_i}}$$

with $n'_{i+1} \geqq n'_i$ and $a'_{i+1} < a'_i$ if $n'_{i+1} = n'_i$.

The hypothesis tells us there is a sequence $(\nu_i)_{1 \leqq i \leqq h}$ such that

$$\xi^R = \prod_{i=1}^{i=h} \xi_{\nu_i}^{2^{a'_i}},$$

and

$$\xi^L = \prod_{i=1}^{i=h} \xi_{n'_i - \nu_i}^{2^{a'_i + \nu_i}}.$$

One first observes that, up to permutation, the sequence (ν_i) is equal to $(h, \ldots, 1)$ and that, up to permutation, the sequence $(n'_i - \nu_i)$ is equal to $(n_1 - h, \ldots, n_h - 1)$. As K precedes I, $n'_h \leqq n_h$, let us suppose that $n'_h < n_h$; one has for any j, $1 \leqq j \leqq h$,

$$n'_j - \nu_j \leqq n'_j - 1 \leqq n'_h - 1 < n_h - 1.$$

And $n_h - 1$ cannot be equal to any $n'_j - \nu_j$, a contradiction. Therefore n'_h is equal to n_h. Suppose now that $\nu_h > 1$, the preceding argument implies that, in this case, there exists $k < h$ such that $n_k = n_h$ in order to create the factor $\xi_{n_h - 1}^{2^{a_k} + 1}$ on the left, but that implies the apparition (on the right) of a factor $\xi_1^{2^{a'_k}}$; as $a'_k > a'_h$, this is impossible. Therefore ν_h is equal to 1 and a'_h is equal to a_h. One completes the proof by induction and one gets

$$1 \leqq i \leqq h, \qquad n'_i = n_i, \qquad \nu_i = h + 1 - i, \qquad a'_i = a_i.$$

END OF THE PROOF OF THE THEOREM. — Given a multi-index I, denote by Sq^I the element of A dual to ξ^I with respect to the monomial basis of A_*.

The Lemma 4.2 implies that $Sq^L Sq^R x$ is equal to x_I, this is shown as follows

$$Sq^L Sq^R x = \lambda(x)/Sq^L . Sq^R = \sum_K x_K \otimes \langle \Delta\xi^K, Sq^L \times Sq^R \rangle,$$

where the sum is over all multi-indices, with the convention that $\langle \Delta\xi^K, Sq^L \times Sq^R \rangle$ is zero if the two sides have not the same degree. The arguments above show that the only multi-index K, such that $\xi^L \times \xi^R$ appears non-trivially in $\Delta(\xi^K)$, is I.

We claim now that $\alpha(|Sq^R x|)$ is equal to $\pi(x)$. The multi-index L carries the weight of $x_R = Sq^R x$ because $\alpha(L) = \pi(x) \geqq \pi(x_R)$. Therefore $c(L)$ is equal to $|x_R|$, but

$$\alpha(|x_R|) = \alpha(c(L)) = \alpha(L) = \alpha(I) = \pi(x)$$

and we are done.

5. Complements on the structure of $\mathcal{U}/\mathcal{N}il$

The object of this chapter is to obtain further informations on the structure of $\mathcal{U}/\mathcal{N}il$, using the results of Chapter 3.

PROPOSITION 5.1. — *Every noetherian object of $\mathcal{U}/\mathcal{N}il$ is artinian (i.e. any descending sequence of sub-objects stabilizes).*

Proof. – We first notice that a noetherian object has finite weight. This allows us to proceed by induction on the weight. Let us assume that any noetherian object of weight less than $(n-1)$ is artinian. (The case $n=0$ is left to the reader.) Let M be a noetherian object of $\mathcal{U}/\mathcal{N}il$ of weight n. Let $N_1 \supset N_2 \supset \ldots$ a descending chain of sub-objects. With the notations of Chapter 3, one gets a descending sequence of noetherian $\mathbb{F}_2[\mathfrak{S}_n]$-modules

$$p_n(N_1) \supset p_n(N_2) \supset \ldots \supset p_n(N_i) \supset \ldots .$$

Noetherian objects are artinian in $\mathrm{Mod}_{\mathbb{F}_2[\mathfrak{S}_n]}$, therefore this sequence stabilizes at a certain step, say at $p_n(N_k)$. Lemma 3.2.4 implies that N_k/N_l $(l \geqq k)$ is of weight at most $(n-1)$. A limit argument shows that $N_k/\bigcap_{l \geqq k} N_l$ has weight at most $(n-1)$. It is also noetherian being a quotient of N_k. By induction it is artinian. Therefore the descending sequence $N_j/\bigcap_{l \geqq k} N_l$ $(j \geqq k)$ stabilizes, this implies that $N_{j+1} = N_j$ as soon as j is large enough.

As a corollary, we note that this implies that $\mathcal{U}/\mathcal{N}il$ is locally finite, in the terminology of Gabriel.

Let us recall that, in an abelian category, an injective hull for an object M is an essential morphism $i: M \to E_M$ where E_M is injective (An essential morphism $i: M \to E$ is by definition a morphism such that any $f: E \to N$ is a monomorphism if and only if $f \circ i$ is a monomorphism.) An injective hull, when it exists, is unique up to isomorphism.

The category $\mathcal{U}$ has injective hulls [LS1], therefore ([Ga], chap. 3), $\mathcal{U}/\mathcal{N}il$ has also injective hulls (any object has an injective hull).

Let us recall also that the socle of an object M is the largest semi-simple (direct sum of simple objects) sub-objet of M. The socle of an object exists in the category $\mathcal{U}/\mathcal{N}il$ (*see* [Bo2]). An indecomposable injective in an abelian category has a socle which is either trivial, either simple ([Ga], [Bo1], [LS1]).

COROLLARY 5.2. — *The map taking a simple object of $\mathcal{U}/\mathcal{N}il$ to its injective hull induces a bijection between the isomorphism classes of simple objects and the isomorphism classes of indecomposable injectives of $\mathcal{U}/\mathcal{N}il$.*

The proof is classical. One gets a map in the other direction by considering the socle of an indecomposable injective. One has to check that the socle of an indecomposable injective is non trivial, but any object M of $\mathcal{U}/\mathcal{N}il$ contains non trivial noetherian subobjects by Proposition 5.1, therefore it contains non trivial semi-simple objects, hence the socle of M, soc(M), is non trivial. The rest is routine (*see* [Bo1], [LS1]).

Let us note also that the functor r determines a bijection between isomorphism classes of indecomposable injective objects of $\mathcal{U}/\mathcal{N}il$ and isomorphism classes of $\mathcal{N}il$-closed indecomposable injective objects in $\mathcal{U}$ ([Ga], Chap. III).

An indecomposable injective E is the injective hull of its socle soc(E). As E is indecomposable, soc(E) is simple, and isomorphic to $\mathrm{I}.\mathrm{F}(1)^{\otimes n}$ for a certain simple right ideal I of $\mathbb{F}_2[\mathfrak{S}_n]$, for a certain n. But $\mathrm{I}.\mathrm{F}(1)^{\otimes n}$ embeds in $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$ as $\mathrm{F}(1)^{\otimes n}$ does. The theorem of Carlsson, Miller, Lannes and Zarati ([Ca], [Mi], [LZ2]) tells us that $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$ is injective in $\mathcal{U}$. The injective hull E of $\mathrm{I}.\mathrm{F}(1)^{\otimes n}$ is therefore a direct factor inside $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$. Therefore

THEOREM 5.3. — *A reduced indecomposable injective in the category $\mathcal{U}$ is isomorphic to a direct factor of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$ for some n.*

This was first proved by Jean Lannes and the second author ([LS1]).

6. Simple subquotients of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$

It is natural to ask, given a simple object in $\mathcal{U}/\mathcal{N}il$, whether or not it appears as a subquotient of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^n; \mathbb{F}_2)$. Before stating the result and giving applications of it, we need to introduce some terminology about representations of the group $\mathfrak{S}_n$.

6.1. DESCRIBING THE SIMPLE OBJECTS OF $\mathcal{U}/\mathcal{N}il$. — We recall here some facts for the modular representation theory of the symmetric groups. This material is covered in [JK], [Ja1], [Ja2] to which we refer for further details.

We shall denote λ a partition of an integer n, this is a decreasing sequence of integers $(\lambda_1, \ldots, \lambda_t)$ whose sum is n. The conjugate partition λ' of λ is defined by

$$\lambda'_j = \mathrm{Card}\{ i/\lambda_i \geqq j \}, \forall j \in \mathbb{N}.$$

We call λ'_1 the length of λ.

A partition λ is column 2-regular if, for any i, $\lambda_i - \lambda_{i+1} \leqq 1$; it is equivalent to say that λ' is strictly decreasing.

Associated to λ, there is a Young diagram with n nodes: it is a diagram made with α_i nodes on the i-th row and α'_j nodes on the j-th column. More formally, it is defined in [Mc] to be the set of points $(i,j) \in \mathbb{N}^2$ such that $1 \leqq j \leqq \lambda_i$; *i and j are the coordinates of the node.* The most usual convention in drawing this diagram is to let increase the first coordinate i (the row index) downwards and the second one (the column index) from left to right. Let R_λ (resp. C_λ) be the group of permutations of the nodes stabilizing the rows (resp. the columns).

Example. – The Young diagram associated to the partition (2.2.1) is as follows

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array}$$

(where the nodes are replaced by "•").

In this case, the group of permutations of the Young diagram is isomorphic to $\mathfrak{S}_5$, $R_{(2,2,1)}$ to $\mathfrak{S}_2 \times \mathfrak{S}_2$ and $C_{(2,2,1)}$ to $\mathfrak{S}_3 \times \mathfrak{S}_2$.

In general, R_λ is isomorphic to $\mathfrak{S}_{\lambda_1} \times \ldots \times \mathfrak{S}_{\lambda_t} (t=\lambda'_1)$ and C_λ is isomorphic to $\mathfrak{S}_{\lambda'_1} \times \ldots \times \mathfrak{S}_{\lambda'_{t'}} (t'=\lambda_1)$.

A λ-tableau m is a map from the Young diagram associated to λ into $\mathbb{N}$.

Let us introduce two λ-tableaux,

– Let t_λ be the tableau which assigns to any node of the i-th row the integer $\lambda'_1 - i$;

– Let $\bar{t}_\lambda$ be the tableau obtained from t_λ by "reversing the order on the columns", *i.e.* it associates the integer $\lambda'_1 - \lambda'_j + i - 1$ to the node which is in the i-th row and the j-th column.

Example. – In what follows, we use the standard notations for λ-tableaux, *i.e.* we write down the integer which is the image of node (i,j) at its place. If $\lambda=(3,2,2,1)$, then

$$t_\lambda = \begin{array}{ccc} 3 & 3 & 3 \\ 2 & 2 & \\ 1 & 1 & \\ 0 & & \end{array} \quad \text{and} \quad \bar{t}_\lambda = \begin{array}{ccc} 0 & 1 & 3 \\ 1 & 2 & \\ 2 & 3 & \\ 3 & & \end{array}$$

The group of permutations $\mathfrak{S}_\lambda$ (of the nodes) of the Young diagram associated to λ acts, by composition, on the right, on λ-tableaux. Let $\bar{R}_\lambda$ (resp. $\bar{C}_\lambda$) the sum in $\mathbb{F}_2[\mathfrak{S}_\lambda]$ of the elements of R_λ (resp. of C_λ).

Let T_λ be the $\mathbb{F}_2$-vector space with basis the set of λ-tableaux.

PROPOSITION 6.1.1. — *Let* λ *be a column* 2-*regular partition, then*

(i) $\bar{t}_\lambda \bar{R}_\lambda \bar{C}_\lambda = t_\lambda \bar{C}_\lambda$.

(ii) the element $\bar{t}_\lambda \bar{R}_\lambda \bar{C}_\lambda \bar{R}_\lambda$ of T_λ is non zero;

This is proved in (JK] (Chapter 8, exercise 8.3) for (i); (ii) follows easily from (i), in fact one checks that all λ-tableaux $\bar{t}_\lambda \sigma$, for $\sigma \in R_\lambda$, are pairwise distinct and appear with coefficient 1 in $t_\lambda \bar{C}_\lambda \bar{R}_\lambda$.

This proposition can be transposed in our setting by associating a monomial of $F(1)^{\otimes n}$ with every λ-tableau t (λ is a partition of n); let us denote by $t_{i,j}$ the integer associated to the node which is on the i-th row and the j-th column, and consider the following element of $\mathrm{F}(1)^{\otimes n}$

$$\bigotimes_{j=1}^{j=\lambda_1} \bigotimes_{i=1}^{\lambda'_j} u^{2^{t_{i,j}}},$$

denoted v_t.

Example. — For $\lambda=(3,2,2,1)$, one has

$$v_{\bar{t}_\lambda}=u \otimes u^2 \otimes u^4 \otimes u^8 \otimes u^2 \otimes u^4 \otimes u^8 \otimes u^8.$$

Let us identify the Young diagram, as set, with $\{1, \ldots, n\}$ by sending the node (i,j) to $\lambda'_1+\ldots+\lambda'_{j-1}+i$ (*i.e.* one reads each columns downwards and the columns, successively, from left to right).

Example. — If $\lambda=(3,2,1)$, the numbering is read as follows

$$\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 3 & & \end{array}$$

This bijection identifies $\mathfrak{S}_\lambda$ with $\mathfrak{S}_n$ and in particular C_λ is identified with

$$\mathfrak{S}_{(1,\ldots,\lambda'_1)} \times \mathfrak{S}_{(\lambda(_1+1,\ldots,\lambda'_1+\lambda'_2)} \times \ldots \times \mathfrak{S}_{(\lambda'_1+\ldots+\lambda'_{t-1}+1,\ldots,\lambda'_1+\ldots+\lambda'_{t'})}$$

and R_λ with

$$\mathfrak{S}_{(1,\lambda'_1+1,\lambda'_2+1,\ldots)} \times \mathfrak{S}_{(2,\lambda'_1+2,\ldots)} \times \ldots \times \mathfrak{S}_{(\lambda'_1)},$$

where $(a_1, \ldots, a_k)$ being a subset of $(1, \ldots, n)$, $\mathfrak{S}_{(a_1,\ldots,a_k)}$ is the group of permutations of this subset.

With this convention, the assignement $t \to v_t$ is anti-equivariant:

$$v_{t\sigma}=\sigma^{-1} v_t.$$

The following is proved in [JK] (Chapter 7, exercise 7.2 in particular), *see* also [Ja1] and [Ja2].

THEOREM 6.1.2 ([JK], [Ja1], [Ja2]). — *The ideals* $\bar{\mathrm{R}}_\lambda \bar{\mathrm{C}}_\lambda \bar{\mathrm{R}}_\lambda \mathbb{F}_2[\mathfrak{S}_n]$ *form a complete set of inequivalent simple right ideals of* $\mathbb{F}_2[\mathfrak{S}_n]$ *when* λ *runs through the set of column* 2-*regular partitions of n.*

Let us denote by e_λ the element $\bar{\mathrm{R}}_\lambda \bar{\mathrm{C}}_\lambda \bar{\mathrm{R}}_\lambda$; then

COROLLARY 6.1.3. — *Let* S_λ *be the unstable* A-*module* $e_\lambda \mathrm{F}(1)^{\otimes n}$. *As n runs through all integers and* λ *through all column* 2-*regular partitions of n, the objects* $r\mathrm{S}_\lambda$ *form a complete set of non isomorphic simple objects in* $\mathcal{U}/\mathcal{N}il$.

Moreover, (6.1.1) tells us that the element $e_\lambda v_{\bar{t}_\lambda}$ in S_λ is non zero; in fact, it contains the monomial $v_{\bar{t}_\lambda}$ with a non zero coefficient. We will denote this element s_λ in the sequel, note that this is not the class of lowest degree of S_λ. The class of lowest degree of S_λ is described in the same way but using the tableau which associates 0 to any node of the first row, 1 to any node of the second row....

6.2. SIMPLE SUB-QUOTIENTS OF $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$. — Here is the main result of this paragraph.

THEOREM 6.2.1. — *Let λ be a column 2-regular partition of n. Then, S_λ is a subquotient of $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ in the category $\mathcal{U}/\mathcal{N}il$ if and only if $\lambda_1 \leqq d$.*

This result is in fact equivalent to one of D. Carlisle and N. Kuhn as we noted in the introduction. In the proof, we will identify $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ with $\mathbb{F}_2[x_1, \ldots, x_d]$.

PROOF OF SUFFICIENCY. — The primitive filtration (which coincides with the weight filtration) of $H^*(B\mathbb{Z}/2; \mathbb{F}_2)$ has subquotients $\Lambda^n F(1)$, $n \geqq 0$. The product filtration on $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ has as subquotients the $\Lambda^{\alpha_1} F(1) \otimes \ldots \otimes \Lambda^{\alpha_u} F(1)$ with $u \leqq d$. But S_λ is a quotient of $\Lambda^{\lambda'_1} F(1) \otimes \ldots \otimes \Lambda^{\lambda'_{t'}} F(1)$ $(t' = \lambda_1)$ by construction.

PROOF OF NECESSITY. — Suppose there is a sub A-module M of $H^*(B(\mathbb{Z}/2)^d; \mathbb{F}_2)$ and a non zero map $\varphi: M \to S_\lambda$. For k large enough, the element $Sq_0^k s_\lambda$ (which is non zero) is in the image of $\varphi: \varphi(x) = Sq_0^k s_\lambda$ for some $x \in M$.

Consider now the multi-index I consisting of the powers of 2 appearing as exponents in the element $Sq_0^k s_\lambda$, these powers being disposed in increasing order, it is

$$2^k, \ldots, \underbrace{2^{k+i-1}, \ldots, 2^{k+i-1}}_{\lambda_{t-i+1} \text{ times}}, \ldots, \underbrace{2^{k+\lambda'_1-1}, \ldots, 2^{k+\lambda'_1-1}}_{\lambda_1 \text{ times}}$$

(as usual $t = \lambda'_1$). This is (essentially) the same multi-index as the one used for the proof of lemma 2.2.1.

The same computation as for lemma 2.2.1 shows that $(Sq_0^k s_\lambda)_I$ is non zero.

Therefore, among the monomials adding up to x, there is at least one (let us denote it z) such that z_I is non zero. We are going to give an upper bound of $|z|$ in term of n, then a lower bound in term of λ_1.

Write z as $z = \prod_1^\alpha x_{m_i}^{2^{a_i}}$ with the convention that $a_i \leqq a_{i+1}$ and that $m_i < m_{i+1}$ if $a_i = a_{i+1}$. Inspection on $\lambda(z)$ shows that if z_I is non zero, then every index of I is a sum of 2^{a_i}'s. As $c(I)$ is equal to $|z|$, then each 2^{a_i} must figure in one of those sums. Thus, some index of I is at least the greatest of the 2^{a_i}'s, that is: for all i, $2^{\lambda'_1 - 1 + k} \geqq 2^{a_i}$. The majoration now follows

$$|z| = \sum_{i=1}^{i=\alpha} 2^{a_i} = \sum_{j=1}^{j=d} \sum_{m_i = j} 2^{a_i} \leqq \sum_{j=1}^{j=d} \sum_{a \leqq \sup a_i} 2^a = d(2^{s+1} - 1)$$

where s denotes $\sup a_i$, $1 \leqq i \leqq \alpha$. Therefore

$$|z| < d\, 2^{\lambda'_1 + k}.$$

We now get a minoration

$$|z| = |\mathrm{S}\, q_0^k s_\lambda| = \sum_1^{\lambda'_1} \lambda_i\, 2^{\lambda'_1 - i + k} \geqq \sum_1^{\lambda_1} (\lambda_1 - i + 1)\, 2^{\lambda'_1 - i + k}$$

$$= 2^{k + \lambda'_1 - \lambda_1} \left(\sum_1^{\lambda_1} i . 2^{i-1} \right) = 2^{k + \lambda'_1 - \lambda_1} ((\lambda_1 - 1)\, 2^{\lambda_1} + 1).$$

Therefore

$$(\lambda_1 - 1)\, 2^{\lambda'_1} + 2^{\lambda'_1 - \lambda_1} < d\, 2^{\lambda'_1},$$

so

$$\lambda_1 \leqq d \qquad \text{Q.E.D.}$$

Special cases of these results were obtained by N. Kuhn in [K]; he proved here that if $\mathrm{L}(n)$ is the "Steinberg module" of S. Mitchell and S. Priddy [MitP], then $\mathrm{Hom}_{\mathscr{U}}(\mathrm{L}(m), \mathrm{L}(n))$ is trivial as soon as $m < n$. It is possible to show that $\mathrm{L}(n)$, considered as an object of $\mathscr{U}/\mathscr{N}il$, is the injective hull of $\mathrm{S}_{(n, n-1, \ldots, 1)}$ (using [Mit] for example). As $\mathrm{L}(m)$ is a direct summand of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^m; \mathbb{F}_2)$ the existence of a non trivial map from $\mathrm{L}(m)$ into $\mathrm{L}(n)$ would imply that $\mathrm{S}_{(n, n-1, \ldots, 1)}$ is a sub-quotient of $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^m; \mathbb{F}_2)$, which is impossible as $m < n$.

More generally let us denote by E_λ the injective hull of S_λ in $\mathscr{U}/\mathscr{N}il$; by abuse of notations, denote also $s\,\mathrm{E}_\lambda$ by E_λ. The E_λ's form a set of representatives of non-equivalent reduced indecomposable injective in $\mathscr{U}$ ($\mathscr{U}/\mathscr{N}il$). We will show in Chapter 7.

Proposition 6.2.2. — *The unstable* A-*module* E_λ *embeds in* (*and is a direct factor of*) $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^d; \mathbb{F}_2)$ *as soon as* $d \geqq \lambda'_1$.

And we have

Corollary 6.2.3. — *The unstable* A-*module* E_λ *does not admit* S_μ *as sub-quotient* (*in the category* $\mathscr{U}/\mathscr{N}il$) *as soon as* $\lambda'_1 < \mu_1$.

In the category $\mathscr{U}$ that means that E_λ has not $\Phi^k \mathrm{S}_\mu$ as subquotient, for any $k \geqq 0$, as soon as $\lambda'_1 < \mu_1$. Here Φ is the "double" or Frobenius functor of [MP] (*see* also [LZ2]).

7. Connectivity of the indecomposable summands of $\mathrm{B}(\mathbb{Z}/2)^n$

7.1. This chapter computes the connectivity of the indecomposable reduced injectives in the category $\mathscr{U}$. Let us recall from chapter 5 that isomorphism classes of indecomposable reduced injectives in the category $\mathcal{U}$ are in bijection (via the socle) with isomorphism classes of simple objects in the category $\mathcal{U}/\mathcal{N}il$. These isomorphism classes are indexed by column 2-regular partitions λ (let us recall that E_λ denotes the injective hull of S_λ).

Let us now state the main result of this chapter.

THEOREM 7.1.1. — *The unstable* A-*module* E_λ *is trivial in degrees strictly less than* $\lambda'_1+2\lambda'_2+\ldots+2^{t'-1}\lambda'_{t'}$ *and it is of dimension* 1 *in this degree.*

As usual, t' is equal to λ_1 in this result.

In order to prove the theorem we need to construct an embedding of S_λ in $H^*(B(\mathbb{Z}/2)^{\lambda'_1};\mathbb{F}_2)$, allowing us to consider E_λ as a direct factor in it; this will be done in sections 7.2 and 7.3. Let us recall that the indecomposable injective E_λ embeds in (and is a direct summand of) $H^*(B(\mathbb{Z}/2)^d;\mathbb{F}_2)$ if and only if its socle S_λ does, and the following result holds whose proof will be given elsewhere [LS2].

PROPOSITION. — *The unstable* A-*module* S_λ *embeds in* $H^*(B(\mathbb{Z}/2)^d;\mathbb{F}_2)$ *if and only if* $\lambda'_1\leqq d$.

We proceed as follows. In the second section (7.2), we explain the combinatorics we need to define the embedding of S_λ into $H^*(B(\mathbb{Z}/2)^{\lambda'_1};\mathbb{F}_2)$. In the third section (7.3), we prove it is an embedding. In the fourth section (7.4), we prove Theorem 7.1.1 and give an application.

7.2. PRELIMINARIES ON COMBINATORICS. – Consider a column 2-regular partition λ, and construct a λ-tableau m_λ as follows. Let us denote, as usual, by $m_{i,j}$ the integer associated to the node on the i-th row and the j-th column. Start at the bottom row and associates zero to the only node, in other words $m_{\lambda'_1,1}$ is zero. Then one goes up row by row and, as λ is column 2-regular, $m_{i,j}$ is defined by reverse induction on i, $1\leqq i\leqq\lambda'_1$, by the two rules:

(i) if $\lambda_i=\lambda_{i+1}+1$, then $m_{i,\lambda_i}=0$ and for $j<\lambda_i$ then $m_{i,j}=m_{i+1,j}+1$;

(ii) if $\lambda_i=\lambda_{i+1}$, then $m_{i,j}=m_{i+1,j}+1$ if $m_{i+1,j}<\lambda_{i+1}-1$ and $m_{i,j}=0$ if $m_{i+1,j}=\lambda_{i+1}-1$.

The definition makes sense because one checks by induction that $m_{i+1,j}\leqq\lambda_{i+1}-1$ for all j.

Example. – For $\lambda=(3,3,2,2,1)$, with the usual conventions

$$m_\lambda=\begin{matrix}2&0&1\\1&2&0\\0&1&\\1&0&\\0&&\end{matrix}$$

An easy induction shows that m_λ is the only λ-tableau such that

– The set of values of the tableau m_λ on the i-th row is $\{0,\ldots,\lambda_i-1\}$;

– Each column of m_λ is divided in "bands of nodes" carrying consecutive integers in increasing order upwards and starting with zero; in our example there are 2 bands of length 3, 2 of length 2 and 1 of length 1.

Let us be more precise. We define subsets of the Young diagram associated to λ as follows: two nodes are in the same subset if and only if they are in the same column and $m_{i,j} - m_{l,j} = i - l$ ((i,j) and (l,j) being the coordinates of the nodes).

By induction over λ'_1, one checks this defines a partition of the Young diagram and:

– there are exactly λ'_1 subsets in this partition;

– among which, $(\lambda'_1 - \lambda'_2)$ are of cardinality $1, \ldots, (\lambda'_h - \lambda'_{h+1})$ are of cardinality $h, \ldots$;

– the set of values taken by the tableau on any subset of cardinality h is $\{0, \ldots, h-1\}$, the values being "read" in increasing order from the bottom.

Let us call these subsets $M_1, \ldots, M_{\lambda'_1}$ assuming that $\operatorname{card} M_i \leqq \operatorname{card} M_j$ as soon as $i \leqq j$.

Let us recall that we identify the Young diagram with the set $\{1, \ldots, n\}$ via the bijection which sends the nodes with coordinates (i,j) to $\lambda'_1 + \ldots + \lambda'_{j-1} + i$. In other words they are ordered (increasingly) from top to bottom on columns, and the columns are ordered from left to right. Therefore the M_i's are identified with subsets of $\{1, \ldots, n\}$.

Let ω be the map from $\{1, \ldots, n\}$ into $\{1, \ldots, \lambda'_1\}$ defined by $\omega(i) = k$ if and only if $i \in M_k$.

We define a map $\varphi\lambda : F(1)^{\otimes n} \to H^*(B(\mathbb{Z}/2)^{\lambda'_1}; \mathbb{F}_2)$ as follows: we identify $H^*(B(\mathbb{Z}/2)^{\lambda'_1}, \mathbb{F}_2)$ with $\mathbb{F}_2[x_1, \ldots, x_{\lambda'_1}]$ and

$$\varphi_\lambda(u^{2^{a_1}} \otimes \ldots \otimes u^{2^{a_n}}) = \prod_{i=1}^{i=n} x_{\omega(i)}^{2^{a_i}} = \prod_{i=1}^{i=\lambda'_1} x_i^{\sum_{j \in M_i} 2^{a_j}}.$$

Example. – Take $\lambda = (3, 2, 2, 1)$, then

$$m_\lambda = \begin{matrix} 1 & 2 & 0 \\ 0 & 1 & \\ 1 & 0 & \\ 0 & & \end{matrix}$$

The partition of the Young diagram is therefore given, identifying the diagram with $\{1, \ldots, 8\}$, by $M_1 = \{5, 6, 7\}$, $M_2 = \{1, 2\}$, $M_3 = \{3, 4\}$ and $M_4 = \{8\}$. Then

$$\varphi_\lambda(u^{2^{a_1}} \otimes \ldots \otimes u^{2^{a_8}}) = x_1^{2^{a_5} + 2^{a_6} + 2^{a_7}} x_2^{2^{a_1} + 2^{a_2}} x_3^{2^{a_3} + 2^{a_4}} x_4^{2^{a_8}}.$$

We apply immediately the preceding combinatorics and we introduce a class in $F(1)^{\otimes n}$:

$$\omega_{\lambda'} = \bar{R}_\lambda \bigotimes_{i=1}^{i=n} u^{2^{e_i}},$$

where $e_i = j - 1$ if $\lambda'_1 + \ldots + \lambda'_{j-1} < i \leqq \lambda'_1 + \ldots + \lambda'_j$. In other words, one considers the class associated to the λ-tableau u_λ taking the value $j-1$ on the j-th column and one applies $\bar{R}_\lambda$ to it.

Following [W], we now use the notion of "weight vector" for monomials in $\mathbb{F}_2[x_1, \ldots, x_d]$. The weight vector of $x_1^{a_1} \ldots x_d^{a_d}$ is the d-tuple $(\alpha(a_1), \ldots, \alpha(a_d))$. In particular the weight vector of $\prod_1^t x_i^{2^{\lambda_i}-1}$ is $(\lambda_1, \ldots, \lambda_t)$. On the set of weight vectors, we consider the partial order given by

$$(\alpha_1, \ldots, \alpha_d) \leqq (\beta_1, \ldots, \beta_d)$$

if $\alpha_1 \leqq \beta_1, \alpha_2 \leqq \beta_2, \ldots, \alpha_d \leqq \beta_d$.

LEMMA 7.2.1. — *The element $\varphi_\lambda(\omega_{\lambda'})$ is the sum of the monomial $\prod_1^t x_i^{2^{\lambda_i}-1}$ with monomials of weight vector strictly less than $(\lambda_1, \ldots, \lambda_t)$.*

The lemma results from the fact that there is a unique $r_\lambda \in \mathrm{R}_\lambda$ such that the tableau $u_\lambda r_\lambda^{-1}$ equals m_λ, and one checks easily that for all others elements r in R_λ, the tableau $u_\lambda r^{-1}$ takes at least twice the same value on at least one subset M_i. This translates as follows for the corresponding element $\omega_{\lambda'}$ in $\mathrm{F}(1)^{\otimes n}$: among the elements $\bigotimes_1^n u^{2^{b_j}}$ adding up to $\omega_{\lambda'}$ in $\mathrm{F}(1)^{\otimes n}$, there is only one that φ_λ sends to a monomial of weight vector $(\lambda_1, \ldots, \lambda_t)$ all the other elements being sent to monomials of strictly less weight vector.

Example. — Ir $\lambda = (3, 3, 2, 2, 1)$

$$m_\lambda = \begin{matrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & \\ 1 & 0 & \\ 0 & & \end{matrix}, \qquad u_\lambda = \begin{matrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & \\ 0 & 1 & \\ 0 & & \end{matrix},$$

$\mathrm{M}_1 = \{1, 2, 3\}$, $\mathrm{M}_2 = \{7, 8, 9\}$, $\mathrm{M}_3 = \{4, 5\}$, $\mathrm{M}_4 = \{10, 11\}$ and $\mathrm{M}_5 = \{6\}$ (here we identify the Young diagram with $\{1, 2, \ldots, 11\}$). Then

$$r_\lambda = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \\ 6, 11, 3, 9, 5, 10, 2, 8, 4, 1, 7 \end{pmatrix}.$$

We keep the notations introduced in 6.1 (especially 6.1.1).

In chapter 6, the simple object S_λ has been identified with the unstable module $e_\lambda \mathrm{F}(1)^{\otimes n}$ where $e_\lambda = \bar{\mathrm{R}}_\lambda \bar{\mathrm{C}}_\lambda \bar{\mathrm{R}}_\lambda$, we can consider the restriction of φ_λ to S_λ. In the next section we are going to prove the following lemma using the preceding result.

LEMMA 7.2.2. — *The restriction of the map φ_λ to S_λ is injective.*

7.3. PROOF OF LEMMA 7.2.2. — As S_λ, when considered in $\mathcal{U}/\mathcal{N}il$, is simple and as $\mathrm{H}^*(\mathrm{B}(\mathbb{Z}/2)^{\lambda'_1}; \mathbb{F}_2)$ is reduced, it is enough to show that φ_λ is non zero on one element of S_λ. We proceed as follows; We introduce a certain sub-A-module $\mathrm{W}_{\lambda'}$ of $\mathrm{F}(1)^{\otimes n}$. *The socle* (in the sense of $\mathcal{U}/\mathcal{N}il$) of $\mathrm{W}_{\lambda'}$ is S_λ; although we do not need this fact here we prove it for later use. The lemma 7.2.1 will tell us that φ_λ is non zero on the class of minimal degree of $W_{\lambda'}$ which is $\omega_{\lambda'}$. To finish, we give an operation θ such that $\theta\omega_{\lambda'} \in S_\lambda$ and $\theta\varphi_\lambda(\omega_{\lambda'}) \neq 0$.

The unstable A-module $W_{\lambda'}$ is defined to be $\bar{R}_\lambda(F(1)^{\otimes n})^{C_\lambda}$ where $(F(1)^{\otimes n})^{C_\lambda}$ denotes the invariants under C_λ in $F(1)^{\otimes n}$; S_λ is included in $W_{\lambda'}$, moreover

$$S_\lambda \subset \bar{R}_\lambda \bar{C}_\lambda F(1)^{\otimes n} \subset W_{\lambda'};$$

let us call $\bar{W}_{\lambda'}$ the intermediate unstable module in this sequence of inclusions.

LEMMA 7.3.1. — *If the partition* λ *is column* 2-*regular, the socle* (*in the category* $\mathscr{U}/\mathscr{N}il$) *of the unstable* A-*module* $W_{\lambda'}$ *is isomorphic to* S_λ.

Remark. — This lemma, with lemma 7.2.2, implies that φ_λ is injective on the whole of $W_{\lambda'}$.

Proof. — In degrees d such that $\alpha(d)=n$, $W_{\lambda'}$ and $\bar{W}_{\lambda'}$ are isomorphic. Therefore the inclusion $\bar{W}_{\lambda'} \subsetneq W_{\lambda'}$ is an isomorphism in $\mathscr{V}_n/\mathscr{V}_{n-1}$. As $W_{\lambda'}$ and $\bar{W}_{\lambda'}$ are sub-A-modules of $F(1)^{\otimes n}$, their socle is a direct sum of simple objects of weight n (2.2.3), hence $\operatorname{soc}(W_{\lambda'})$ is isomorphic to $\operatorname{soc}(\bar{W}_{\lambda'})$.

Let S be a simple subobject of $\bar{W}_{\lambda'}$, it is of weight n, therefore $p_n(S)$ is in the socle of $p_n(\bar{R}_\lambda \bar{C}_\lambda \mathbb{F}_2[\mathfrak{S}_n].F(1)^{\otimes n}) \cong \bar{R}_\lambda \bar{C}_\lambda \mathbb{F}_2[\mathfrak{S}_n]$.

This latter $\mathbb{F}_2[\mathfrak{S}_n]$-module is considered in [J1] (p. 17 and 41). It is known to *have as socle the module* $\bar{R}_\lambda \bar{C}_\lambda \bar{R}_\lambda \mathbb{F}_2[\mathfrak{S}_n]$ *as soon as* λ *is column* 2-*regular*. This proves the lemma. However this terminology being unusual, we will be a bit more precise. The usual Specht module associated to λ is defined to be isomorphic to $\bar{C}_\lambda \bar{R}_\lambda \mathbb{F}_2[\mathfrak{S}_n]$. The latter is in fact the contragredient to the former (*see* [JK] Chapter 7 and the exercises). If λ is column 2-regular, the Specht module associated to λ has a unique maximal, non trivial, sub-module. The quotient of the Specht module by its maximal submodule is isomorphic to $\bar{R}_\lambda \bar{C}_\lambda \bar{R}_\lambda \mathbb{F}_2[\mathfrak{S}_n]$. The latter is a simple, self-dual module, and dualizing proves our claim.

It follows that S is isomorphic to S_λ.

Clearly, there is a unique class of lowest degree in $W_{\lambda'}$, it is the class $\omega_{\lambda'}$ introduced in the preceding section.

We keep the notations introduced in 6.1 (especially 6.1.1).

LEMMA 7.3.2. — *There exists an operation* θ *in the Steenrod algebra such that* $\theta u_\lambda = \bar{C}_\lambda t_\lambda$, *in particular* $\theta\omega_{\lambda'} \in S_\lambda$.

Here, by abuse of notations, u_λ and t_λ are the monomials in $F(1)^{\otimes n}$ associated to the λ-tableau u_λ and t_λ.

Proof. — One considers the operation Sq^I which is dual, in the monomial basis of A_*, to

$$\xi^I = \prod_{(i,j)} \xi_{\lambda_1+1-i-j}^{2^{j-1}},$$

where the product is taken over all the nodes of the tableau associated to the partition λ, the nodes being represented by their coordinates (i,j). In other words, the product is taken over all pairs (i,j) such that $1 \leqq i \leqq \lambda'_1$, $1 \leqq j \leqq \lambda_i$. The computation is the same as the one made in Chapter 2 and left to the reader.

In order to complete the proof, it remains to show that $Sq^{I} \varphi_\lambda(\omega_{\lambda'})$ is non zero.

LEMMA 7.3.3. — *If θ is an operation in the Steenrod algebra, for any monomial x in $\mathbb{F}_2[x_1, \ldots, x_d]$, θx is a sum of monomials of weight vectors smaller than the one of x.*

Proof. — Inspect the coaction or look at the action of Sq^{2^h} on a monomial.

The lemmas 7.2.1 and 7.3.3 imply that it is enough to show that $Sq^{I}(x_1^{2^{\lambda_1}-1} \ldots x_t^{2^{\lambda_t}-1})$ (which is the same as $(x_1^{2^{\lambda_1}-1} \ldots x_t^{2^{\lambda_t}-1})_{I}$) contains monomials of weight vector $(\lambda_1, \ldots, \lambda_t)$ (recall that $t = \lambda'_1$). This is one more time a computation similar to those of Chapter 2. The general formula is as follows

$$(x_1^{2^{\lambda_1}-1} \ldots x_t^{2^{\lambda_t}-1})_{I} = \left(\prod_{j=1}^{j=\lambda_1} (x_1 \ldots x_{\lambda'_j})^{2^{j-1}}\right)_{I}$$

$$= \prod_{j=1}^{j=\lambda_1} \left(\sum_{\sigma \in \mathfrak{S}_{\lambda'_j}} x_{\sigma^{-1}(1)}^{2^{\lambda'_1-1}} \ldots x_{\sigma^{-1}(\lambda_j)}^{2^{\lambda'_1-\lambda'_j}}\right).$$

Example. — If $\lambda = (2, 2, 2, 1)$, then

$$\xi^{I} = \xi_0^3 \xi_1^3 \xi_2^3 \xi_3$$

and

$$(x_1^3 x_2^3 x_3^3 x_4)_{I} = \left(\sum_{\mathfrak{S}_4} x_{\sigma^{-1}(1)} \ldots x_{\sigma^{-1}(4)}^{8}\right)\left(\sum_{\mathfrak{S}_3} x_{\sigma^{-1}(1)}^{2} \ldots x_{\sigma^{-1}(3)}^{8}\right).$$

This formula shows in particular that $(x_1^{2^{\lambda_1}-1} \ldots x_t^{2^{\lambda_t}-1})_{I}$ contains monomials of weight vector $(\lambda_1, \ldots, \lambda_t)$.

7.4. END OF THE PROOF OF THEOREM 7.1.2 WITH AN APPLICATION. — We observe that there does not exist monomials of weight vector greater or equal to $(\lambda_1, \ldots, \lambda_t)$ and of degree smaller than $\lambda'_1 + 2\lambda'_2 + \ldots + 2^{t'-1}\lambda'_{t'}$. But if x is an element of $E_\lambda \subset \mathbb{F}_2[x_1, \ldots, x_{\lambda'_1}]$, the fact that S_λ is the socle of E_λ implies that $Ax \cap S_\lambda \neq \{0\}$, therefore, for some $\theta \in A$, θx must involve monomials of weight vector $(\lambda_1, \ldots, \lambda_t)$. Therefore, $|x|$ is at least $\lambda'_1 + \ldots + 2^{t'-1}\lambda'_{t'}$.

As S_λ is the socle of $W_{\lambda'}$, $W_{\lambda'}$ embeds in E_λ; therefore E_λ is non zero in the degree of $\omega_{\lambda'}$ which is $\lambda'_1 + \ldots + 2^{t'-1}\lambda'_{t'}$. A similar argument using the weight vectors shows that E_λ is of dimension 1 in this degree.

We give an application of the preceding result. The group $GL_d(\mathbb{F}_2)$ acts by linear substitutions on the algebra $\mathbb{F}_2[x_1, \ldots, x_d]$; let us consider a simple $\mathbb{F}_2$-representation of $GL_d(\mathbb{F}_2)$, say S. One may ask what is the lowest integer k such that S is a composition factor of the representation of $GL_d(\mathbb{F}_2)$ afforded by $\mathbb{F}_2[x_1, \ldots, x_d]$ in degree k.

The simple $\mathbb{F}_2$-representations of $GL_d(\mathbb{F}_2)$ are indexed, up to isomorphism, by partitions $\lambda=(\lambda_1, \ldots, \lambda_d)$ $(\lambda_d>0)$, which are column 2-regular ([JK], chapter 8).

Example. — There are four isomorphism classes of simple representations of $GL_3 \mathbb{F}_2$, corresponding to:

$\lambda=(1, 1, 1)$, the representation is the trivial one $\mathbb{F}_2$;

$\lambda=(2, 1, 1)$, the representation is the standard one $\mathbb{F}_2^3$;

$\lambda=(2, 2, 1)$, the representation is the second exterior power of the standard one;

$\lambda=(3, 2, 1)$, the representation is the Steinberg representation.

Let us denote by F_λ the simple representation associated to λ.

THEOREM 7.4.1. — *Let* $\lambda=(\lambda_1, \ldots, \lambda_d)$ *be a column 2-regular partition,* F_λ *the corresponding simple representation of* $GL_d(\mathbb{F}_2)$. *Then,* F_λ *occurs in the composition series of* $\mathbb{F}_2[x_1, \ldots, x_d]$ *in degree n for the first time when* $n=\lambda'_2+2\lambda'_3+\ldots$.

We will be very short, for this is a consequence of Theorem 7.1.2 and [HK]. If ε_λ is a primitive idempotent of $\mathbb{F}_2[GL_d \mathbb{F}_2]$ such that $\mathbb{F}_2[GL_d \mathbb{F}_2]\varepsilon_\lambda$ is a projective cover of F_λ, it is shown in [HK] that $\mathbb{F}_2[x_1, \ldots, x_d]\varepsilon_\lambda$ is isomorphic to $E_\lambda \oplus E_{\bar{\lambda}}$ where $\bar{\lambda}=(\lambda_1-1, \lambda_2-1, \ldots, \lambda_d-1)$. The representation F_λ occurs in the composition series of $\mathbb{F}_2[x_1, \ldots, x_d]$ in a certain degree n if and only if $\mathbb{F}_2[x_1, \ldots, x_d]\varepsilon_\lambda$ is non zero in this degree [Mit]. Therefore the integer we are looking for is the smallest degree such that $E_{\bar{\lambda}}$ is non zero.

Remerciements

Le premier auteur remercie de leur soutien le F.N.R.S. et l'Université de Genève qui l'ont accueilli durant la phase finale de la rédaction de cet article.

Le deuxième auteur tient à exprimer ici ses remerciements à l'Université de Chicago et à l'Université Northwestern pour leur accueil chaleureux et leur soutien durant l'année universitaire 1987-1988 au cours de laquelle ce travail fut mis en forme.

Les auteurs tiennent à remercier : Mark Feschbach, Hans-Werner Henn, Nick Kuhn, Jean Lannes, Peter May, Haynes Miller, Stewart Priddy, Pierre Vogel, Reg Wood et Michel Zisman pour leurs conseils ou l'intérêt qu'ils ont manifesté pour ce travail. Ces conseils ont permis tant d'améliorer que de clarifier les résultats présentés ici.

Bernadette Barbichon a assuré la saisie de cet article, nous la remercions tant de la qualité de son travail que de sa patience.

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(Manuscript received July 11, 1989,
in revised form, November 10, 1989).

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