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KENSHÔ TAKEGOSHI

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ENERGY ESTIMATES AND LIOUVILLE THEOREMS FOR HARMONIC MAPS

BY KENSHO TAKEGOSHI

This article consists of two sections. In the first section, we shall establish a method to estimate *the energy of harmonic maps* from a non-compact Kähler manifold provided with certain *hyperconvex exhaustion function* into other Kähler manifolds and induce two *monotonicity formulae* for those maps. In spite of the elementary importance of establishing such a method in function theory of several complex variables, up to now not much is known about the general method to estimate the energy of harmonic maps or even holomorphic maps of Kähler manifolds. As a by-product of this method, we can show the same monotonicity formulae for smooth non-negative plurisubharmonic functions on such a non-compact Kähler manifold.

To estimate the energy of those global solutions of elliptic differential equations of second order, our method requires that a given non-compact Kähler manifold (M, ds^2_M) is provided with a non-negative exhaustion function Φ such that Φ *is uniformly Lipschitz continuous* and Φ^2 is C^∞ *strongly hyper* $m-1$ *convex* on M relative to the Kähler metric ds^2_M respectively and the complex dimension m of M is greater than or equal to two.

Fortunately there are several classes of non-compact Kähler manifolds provided with such a hyperconvex exhaustion function.

For a given non-constant differentiable map $f: (M, ds^2_M) \to (N, ds^2_N)$ from a non-compact Kähler manifold (M, ds^2_M) provided with the hyperconvex exhaustion function Φ as above into a Kähler manifold (N, ds^2_N) and any non-critical value r of Φ, we induce an *integral inequality* involving the *energy* $E(f, r)$ of f on a sublevel set $M(r)=\{\Phi<r\}$ of Φ (*cf.* (1.13)), its derivative $\frac{\partial}{\partial r}E(f, r)$ and the integral $B(f, r)$ of the component of normal direction of the differential df of f on the boundary $\partial M(r)=\{\Phi=r\}$ (*cf.* (1.14) and Lemma 1.22, (1.23)) when f is either *pluriharmonic* or *harmonic* and the Riemannian curvature of ds^2_M is semi-negative in the sense of Siu. This integral inequality plays the crucial role in this article. In fact, from this integral inequality we can derive two energy estimates for the above f which imply the monotone increasing property of the function $E(f, r)/r^\mu$. Here μ is the positive constant determined by the ratio of the lower bound

of the strong hyper $m-1$ convexity of Φ^2 and the uniform Lipschitz constant of Φ relative to the Kähler metric ds_M^2 respectively.

This integral inequality is induced from an *integral formula* for vector bundle-valued differential forms on bounded domains with smooth boundary which was produced by Donnelly and Xavier (*cf.* [6] and Proposition 1.10). Using this integral formula, we can show the same formulae for the function $G(f, r) = \int_{M(r)} \operatorname{Trace}_{ds_M^2} \partial\bar{\partial} f^2 \, dv_M / r^\mu$ if f is a smooth *non-negative plurisubharmonic* function on M.

Here it should be noted that this integral formula can be applied to show the *analyticity of harmonic maps of Kähler manifolds* (*cf.* Remark 1.39).

For instance, we can obtain the following result as a corollary of our general result (*cf.* Theorem 1.18).

THEOREM 1. — *Let* $A \subsetneq \mathbb{C}^n$ *be an* $m \geqq 2$ *dimensional connected closed submanifold of* $\mathbb{C}^n$ *and let* Φ *be the restriction of the function* $\|z\| = \sqrt{\sum_{i=1}^{n} |z^i|^2}$, $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$ *onto* A ($0 \notin A$).

Suppose for a given Kähler metric ds_A^2 *on* A *the number* ρ_1 *defined by*

$$(*) \qquad \rho_1 := \inf_{x \in A} \sum_{i=2}^{m} \varepsilon_i(x)$$

is positive where $\varepsilon_1 \geqq \varepsilon_2 \geqq \ldots \geqq \varepsilon_m$ *are the eigenvalues of the Levi form of* Φ^2 *relative to* ds_A^2 *and the number* ρ_2 *defined by*

$$(**) \qquad \rho_2 := \sup_{x \in A} |\partial \Phi|^2_{ds_A^2}(x)$$

is finite (For instance, if ds_A^2 *is the induced metric of Euclidean metric* ds_e^2 *of* $\mathbb{C}^n$, *then we can take* $\rho_1 = m-1$ *and* $\rho_2 = 1/2$) *and a given non-constant differentiable map* $f: (A, ds_A^2) \to (N, ds_N^2)$ *into a Kähler manifold* (N, ds_N^2) *is either pluriharmonic or harmonic and the Riemannian curvature of* ds_M^2 *is semi-negative in the sense of Siu.*

Then the energy $E(f, r)$ *of* f *on* $A(r) = \{\Phi < r\}$ *satisfies the following properties:*

The function $H(f, r) := E(f, r)/r^\mu$ ($\mu = \rho_1/\rho_2$) *is an increasing function of* r *and the following estimates hold*

$$H(f, r_2) - H(f, r_1) \geqq \int_{r_1}^{r_2} \frac{B(f, t)}{t^\mu} \, dt$$

and

$$\mathrm{H}(f, r_2) \geqq \mathrm{H}(f, r_1) \exp\left(\int_{r_1}^{r_2} \frac{\mathrm{B}(f, t)}{\mathrm{E}(f, t)}\, dt\right)$$

for any $r_2 > r_1 > \inf_{x \in \mathrm{A}} \Phi(x)$.

The same formulae hold for the function

$$\mathrm{G}(f, r) := \mathrm{F}(f, r)/r^{\mu}, \qquad \mathrm{F}(f, r) := \int_{\mathrm{A}(r)} \mathrm{Trace}_{ds_{\mathrm{A}}^2}\, \partial\bar{\partial} f^2\, dv_{\mathrm{A}},$$

if f *is a smooth non-negative plurisubharmonic function on* A.

Remark 1. — When $\dim_{\mathbb{C}} \mathrm{A} = 1$, the condition (*) in Theorem 1 is meaningless. But setting $\rho_1 = 0$ and assuming the condition (**), we can obtain the above estimates for $\mu = 0$ and any non-constant differentiable map $f : (\mathrm{A}, ds_{\mathrm{A}}^2) \to (\mathrm{N}, ds_{\mathrm{N}}^2)$ into any complex manifold $(\mathrm{N}, ds_{\mathrm{N}}^2)$ because $\frac{\partial}{\partial r} \mathrm{E}(f, r)$ (or $\frac{\partial}{\partial r} \mathrm{F}(f, r)) \geqq \mathrm{B}(f, r)$ for almost all r (*cf.* (1.14)).

The former estimate in Theorem 1 is called the *monotonicity formula* in [19]. The same statement as Theorem 1 holds for several classes of non-compact Kähler manifolds (*cf.* § 1, Examples).

In the second section, as an application of the result obtained in the first section, we shall show *Liouville theorems for harmonic maps and plurisubharmonic functions* on a non-compact Kähler manifold provided with the hyperconvex exhaustion function Φ as above under certain *slow volume growth condition.*

Up to now several authors have investigated Liouville theorem for those maps on various non-compact manifolds (*cf.* [2], [3], [7], [8], [9], [11], [12], [13], [18], [33] and so on). One of the typical methods to study Liouville theorem for those objects is what we call *Bochner technique* which shows the vanishing of certain function theoretic or geometric object by coupling Weitzenböck formula with either a curvature condition or a maximum principle (*cf.* [31]). In particular this method plays an important role to study Liouville theorem on a non-compact manifold whose curvature is non-negative or at most has a lower bound (*cf.* [4], [16], [32]). But this method is *useless* to study Liouville theorem on a non-compact manifold whose curvature is *non-positive* and *not bounded from below.*

The following theorems show that our method based on the energy estimate for harmonic maps can be applied to study Liouville theorem for those maps on non-compact Kähler manifolds with (asymptotically) *non-positive curvature.*

THEOREM 2. — *Let* $(\mathrm{A}, ds_{\mathrm{A}}^2) \overset{\iota}{\hookrightarrow} (\mathbb{C}^n, ds_e^2)$ *be an* $m \geqq 1$ *dimensional connected closed submanifold of* $\mathbb{C}^n$ *provided with the induced metric* $ds_{\mathrm{A}}^2 = \iota^* ds_e^2$ *and let* Φ *be the restriction of the norm* $\|z\|$, $z \in \mathbb{C}^n$, *onto* A.

Suppose the function $n(\mathrm{A}, r) := \mathrm{Vol}(\mathrm{A}(r))/r^{2m}$ *satisfies*

$$\int_{\delta}^{\infty} \frac{dt}{tn(\mathrm{A}, t)} = \infty \quad \textit{for some } \delta > 0$$

Then α) $(\mathrm{A}, ds_{\mathrm{A}}^2)$ *admits no non-constant bounded harmonic functions*

β) *Let* $f: \mathrm{A} \to \mathrm{N}$ *be a holomorphic map into a projective algebraic variety* N *with a very ample line bundle* L. *If the set* $\mathrm{E}_f(\mathrm{L}) = \{\sigma \in \mathbb{P}(\Gamma(\mathrm{N}, \mathrm{L})) : \mathrm{Im} f \cap \mathrm{supp}(\sigma) = \varnothing\}$ ((σ) *is the divisor defined by the section* σ) *has positive measure, then f is a constant map.*

γ) *A admists no non-constant negative plurisubharmonic functions*

Remark 2. — It is known that $n(\mathrm{A}, r)$ is a *non-decreasing* function of r (*cf.* [21]). Theorem 2 is partially known in the following cases:

(1) A *is affine algebraic or equivalently* $n(\mathrm{A}, r)$ *is bounded* (*cf.* [26], [27]).

(2) $\dim_{\mathbb{C}} \mathrm{A} = 1$ *and* $\int_{\delta}^{\infty} (tn(\mathrm{A}, t))^{-1} dt = \infty$ (*cf.* [13], [27]).

The class of connected closed submanifolds of $\mathbb{C}^n$ satisfying the condition $\int_{\delta}^{\infty} (tn(\mathrm{A}, t))^{-1} dt = \infty$ contains *smooth affine algebraic varieties* properly (*cf.* [10], § 1).

The holomorphic sectional curvature of the induced metric ds_{A}^2 is non-positive and not necessarily bounded from below even if A is affine algebraic (*cf.* [29] Examples 3 and 4).

When A is *singular* and *transcendental i.e.* $n(\mathrm{A}, r)$ is *unbounded*, up to now there is only one result obtained by Sibony and Wong [22] in this direction *i.e.* they proved that $\mathrm{A} \subsetneq \mathbb{C}^n$ *admits no non-constant bounded holomorphic function if* A *is a pure* $m \geqq 1$ *dimensional irreducible closed subvariety and satisfies* $\liminf_{r \to \infty} \frac{n(\mathrm{A}, r)}{\log r} < \infty$. Though we omit details here, using the *plurisubharmonicity* of $\log \| z \|$, $z \in \mathbb{C}^n$, and the method used in the proof of Theorem 2.1 in paragraph 2, we can generalize their result as follows:

γ)′ $A \subsetneq \mathbb{C}^n$ *admists no non-constant negative plurisubharmonic functions which are smooth on the non-singular part of* A *if* A *is a pure* $m \geqq 1$ *dimensional irreducible closed subvariety and satisfies* $\int_{\delta}^{\infty} (tn(\mathrm{A}, t))^{-1} dt = \infty$.

Anyway to show these two assertions, the plurisubharmonicity of $\log \| z \|$ plays an important role. However it should be noted that Theorem 2 is proved independent of the existence of such a *plurisubharmonic* or *parabolic* exhaustion function (*cf.* The proof of Theorem 2.1 in paragraph 2 and [27]).

The volume growth condition in Theorem 2 depends on the choice of metrics and the above growth condition is *optimal* in the following sense:

Let $(\mathrm{M}, p, ds_{\mathrm{M}}^2)$ be a real two dimensional *model* (*i.e.* the metric ds_{M}^2 is *rotationally symmetric* relative to p) whose metric in polar coordinates centered at p is $ds_{\mathrm{M}}^2 = dr^2 + f(r)^2 d\theta^2$. Then $f: [0, \infty) \to [0, \infty)$ satisfies that $f(0) = 0$, $f'(0) = 1$, $f(r) > 0$ if $r > 0$ and $f''(r) = \mathrm{K}(r) f(r)$ ($\mathrm{K}(r)$ is called the radial curvature function). For a given

$\varepsilon > 1$, let f be convex and let $f(r) = r(\log r)^{\varepsilon}$ outside a compact subset $[0, b_{\varepsilon}]$. The convexity of f assures non-positive curvature. Hence the function $n(\mathrm{M}, r) := \mathrm{Vol}(\mathrm{B}(r))/r^2$ (here $\mathrm{B}(r)$ is the geodesic ball of radius r centered at p) is a non-decreasing function of r. It is easily verified that $n(\mathrm{M}, r) \sim (\log r)^{2\varepsilon}$ for any $r > 3 b_{\varepsilon}$ (For details, the reader should be refered to the proof of Theorem 2.4 in paragraph 2 and [30]). Since $\varepsilon > 1$, $\int_{\delta}^{\infty} (tn(\mathrm{M}, t))^{-1}\, dt < \infty$. On the other hand, M has the conformal type of the unit disk since $\int_{1}^{\infty} dt/f(t) < \infty$ (*cf.* [9] Proposition 5.13). Therefore M admits many non-constant *bounded* harmonic functions.

However in the case of Theorem 2, it is not so clear whether the volume growth condition relative to the induced metric is optimal or not.

THEOREM 3. — *Let* $(\mathrm{M}, ds_{\mathrm{M}}^2)$ *be an* $m \geqq 1$ *dimensional complete Kähler manifold with a pole* $0 \in \mathrm{M}$ *and let* Φ *be the distance function from* $0 \in \mathrm{M}$ *relative to* ds_{M}^2. *Then the assertions* α), β) *and* γ) *in Theorem* 2 *hold for* $(\mathrm{M}, ds_{\mathrm{M}}^2)$ *if the radial curvature of* ds_{M}^2 *satisfies one of the following two conditions:*

(i) $|$*radial curvature at* $x| < \dfrac{\varepsilon}{(\Phi(x)+\eta)^2 \log(\Phi(x)+\eta)}$ *for a sufficiently small* ε, $0 < \varepsilon = \varepsilon_{m,n} < 1$, $\eta > e$ *and any* $x \in \mathrm{M}$.

(ii) *the radial curvature of* ds_{M}^2 *is non-positive on* M *and* $0 \geqq$ *radial curvature at* $x \geqq -\dfrac{\varepsilon}{\Phi(x)^2 \log \Phi(x)}$ *for a sufficiently small* ε, $0 < \varepsilon = \varepsilon_m < 1$ *and any* $x \in \mathrm{M} \setminus \mathrm{M}(r_0)$, $r_0 \gg 1$

Remark 3. – In Theorem 3, if $\dim_{\mathbb{C}} \mathrm{M} = 1$, then it is known that $(\mathrm{M}, ds_{\mathrm{M}}^2)$ satisfying the condition (i) or (ii) is conformally equivalent to the complex plane $(\mathbb{C}, dz\, d\bar{z})$ (*cf.* [9] Proposition 7.6). But in the case $\dim_{\mathbb{C}} \mathrm{M} \geqq 2$, we do not know whether $(\mathrm{M}, ds_{\mathrm{M}}^2)$ satisfying the condition (i) or (ii) is biholomorphic to the m dimensional complex Euclidean space $(\mathbb{C}^m, ds_e^2)$ (*cf.* [9], [15], [17], [25]).

In any case, *by Hessian comparison theorem i.e.* the estimate of solutions of Jacobi equations, we may say that Theorem 3 contains the case treated by Greene and Wu in [9], Theorem C (Quasi-isometry Theorem) (*cf.* [30] and Theorem 2.4 in § 2).

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1. Energy estimates for harmonic maps

Let $(\mathrm{M}, ds^2_{\mathrm{M}})$ be an m dimensional Kähler manifold with the metric tensor

$$ds^2_{\mathrm{M}} = 2\,\mathrm{Re} \sum_{i,j=1}^{m} g_{i\bar{j}}\, dz^i\, d\overline{z^j}.$$

From now on, we always assume that M is *connected* and *non-compact*.

On the space $\mathrm{C}^{p,q}(\mathrm{M})$ of C^∞ differential forms of (p,q) type on M, the pointwise inner product is defined by

$$\langle u, v \rangle = 2^{p+q} \sum_{\mathrm{A}_p, \mathrm{B}_q} u_{\mathrm{A}_p \bar{\mathrm{B}}_q} \overline{v^{\bar{\mathrm{A}}_p \mathrm{B}_q}} \quad \text{for } u \text{ and } v \in \mathrm{C}^{p,q}(\mathrm{M})$$

The star operator $*$: $\mathrm{C}^{p,q}(\mathrm{M}) \to \mathrm{C}^{m-q,m-p}(\mathrm{M})$ relative to ds^2_{M} is defined by

$$* u = \mathrm{C}(m,p,q) \sum_{\mathrm{A}_q, \mathrm{B}_p} \mathrm{sign}\begin{pmatrix} 1, \dots, m \\ \mathrm{A}_q \mathrm{A}_{m-q} \end{pmatrix} \mathrm{sign}\begin{pmatrix} 1, \dots, m \\ \mathrm{B}_p \mathrm{B}_{m-p} \end{pmatrix} \det(g_{i\bar{j}})\, u^{\bar{\mathrm{B}}_p \mathrm{A}_q}\, dz^{\mathrm{A}_{m-q}} \wedge d\overline{z}^{\mathrm{B}_{m-p}}$$

for $\mathrm{C}(m,p,q) = (\sqrt{-1})^m (-1)^{(1/2)m(m-1)+pm} 2^{p+q-m}$ and $u \in \mathrm{C}^{p,q}(\mathrm{M})$. Using the star operator, the inner product on $\mathrm{C}^{p,q}(\mathrm{M})$ is defined by

$$(u, v) = \int_{\mathrm{M}} u \wedge * \bar{v} \quad \text{for } u \text{ and } v \in \mathrm{C}^{p,q}(\mathrm{M}).$$

The following relation holds

$$u \wedge * \bar{v} = \langle u, v \rangle\, dv_{\mathrm{M}}.$$

Here dv_{M} is the volume form of M relative to ds^2_{M} and is defined by

$$dv_{\mathrm{M}} = \frac{\overset{m}{\wedge}\, \omega_{\mathrm{M}}}{2^m\, m!}$$

for the Kähler form $\omega_{\mathrm{M}} = \sqrt{-1} \sum_{i,j=1}^{m} g_{i\bar{j}}\, dz^i \wedge d\overline{z^j}$ of ds^2_{M}. These formulae are used to determine the numerical coefficients of several integrals and operators which appear in this article.

Let Φ be a *continuous* function on M. Throughout this section, we assume the following conditions on Φ.

(1.1) $\Phi \geqq 0$ *and* $\Psi := \Phi^2$ *is of class* C^∞

(1.2) Φ *is an exhaustion function of* M *i.e. each sublevel set* $\mathrm{M}(r) := \{\Phi < r\}$ *is relatively compact in* M *for* $r \geqq 0$

(1.3) Φ *has only non-degenerate critical points outside a compact subset* K_* *of* M

Remark 1.4. – The condition (1.3) is assumed to avoid complicated discussions and is sufficient for our purpose.

Under the condition (1.3), all critical points of Φ on $M \setminus K_*$ are *isolated.* Moreover if r is a critical value of Φ, $r > r_* := \sup_{x \in K_*} \Phi(x)$, then by (1.3), $\partial M(r) := \{\Phi = r\}$ is the union of a $2m-1$ dimensional submanifold made up of all the non-critical points in $\partial M(r)$ and a finite set of critical points. Let $x \in \partial M(r)$ be a non-critical point of Φ. The volume element dS_r of $\partial M(r)$ near x is defined by

$$dv_M = \frac{d\Phi}{|d\Phi|_{ds^2_M}} \wedge dS_r \tag{1.5}$$

We set

$$\omega_r := \frac{dS}{|d\Phi|_{ds^2_M}} \tag{1.6}$$

For $u \in C^{s,t}(M)$, we denote by $\mathbf{e}(u) : C^{p,q}(M) \to C^{p+s,q+t}(M)$ the left multiplication operator by u and denote by $\mathbf{e}(u)^* : C^{p,q}(M) \to C^{p-s,q-t}(M)$ the *adjoint* operator of $\mathbf{e}(u)$ relative to the inner product (,) *i.e.* $\mathbf{e}(u)^* = (-1)^{(p+q)(s+t-1)} * \mathbf{e}(\bar{u}) *$ on $C^{p,q}(M)$.

Since Φ has only non-degenerate critical points on $M \setminus K_*$, Stokes theorem holds on $M[r] := \{\Phi \leq r\}$ for any $r > r_*$.

For a C^∞ differential 1-form φ on M, we have from (1.5) and (1.6)

$$\int_{M(r)} d * \varphi = \int_{\partial M(r)} \mathbf{e}(d\Phi)^* \varphi\, \omega_r \quad \text{for any } r > r_*. \tag{1.7}$$

Here if r is a critical value of Φ, then the integral on the right hand-side is taken over the non-critical points of $\partial M(r)$.

For a given C^∞ differential 1 form

$$\varphi = \sum_{i=1}^{m} \varphi_i\, dz^i + \varphi_{\bar{i}}\, d\overline{z^i}$$

on M, we consider the tangent vector $\Theta = \{\theta^i, \theta^{\bar{i}}\}$ on M defined by $\theta^i = \sum_{i=1}^{m} g^{\bar{j}i} \varphi_{\bar{j}}$ and $\theta^{\bar{i}} = \sum_{i=1}^{m} g^{\bar{i}j} \varphi_j$.

We denote by ∇_i (resp. $\nabla_{\bar{i}}$) the i-th component of the covariant differentiation of type (1, 0) (resp. (0, 1)) relative to ds^2_M. Since $d * \varphi = 2\left(\sum_{i=1}^{m} \nabla_i \theta^i + \nabla_{\bar{i}} \theta^{\bar{i}}\right) dv_M$, we have from (1.7)

$$(1.8)\qquad 2\int_{M(r)}\left(\sum_{i=1}^{m}\nabla_{\bar{i}}\theta^{i}+\nabla_{i}\theta^{\bar{i}}\right)dv_{M}=\int_{\partial M(r)}\mathbf{e}(d\Phi)^{*}\varphi\,\omega_{r}$$

for any $r>r_{*}$.

Let $f:(M, ds_M^2)\to(N, ds_N^2)$ be a differentiable map into an n dimensional Kähler manifold (N, ds_N^2) with the metric tensor

$$ds_N^2=2\,\mathrm{Re}\sum_{\alpha,\beta=1}^{n}h_{\alpha\bar{\beta}}\,dw^{\alpha}\,dw^{\bar{\beta}}.$$

We always regard *any real-valued* C^{∞} *function on* M *as a differentiable map from* (M, ds_M^2) *into the Kähler manifold* $(\mathbb{C}, dz\,d\bar{z})$ *by composing* f: $(M, ds_M^2)\to(\mathbb{R}, dx^2)$ *with the inclusion map* $\iota:(\mathbb{R}, dx^2)\hookrightarrow(\mathbb{C}, dz\,d\bar{z})$ $(z=x+\sqrt{-1}\,y)$.

Let TM and TN be the complex tangent bundle of M and N respectively. Since the complexified differential df of f is regarded as an f^{*}TN-valued differential 1-form, we obtain an $f^{*}\mathrm{TN}^{1,0}$-valued differential (1.0) form ∂f and an $f^{*}\mathrm{TN}^{1,0}$-valued differential (0, 1) form $\bar{\partial} f$ by composing the map $\prod^{1,0}\circ df:\mathrm{TM}\to\mathrm{TN}^{1,0}$, $\prod^{1,0}:\mathrm{TN}\to\mathrm{TN}^{1,0}$ being the projection, with the inclusions $\mathrm{TM}^{1,0}$ into TM and $\mathrm{TM}^{0,1}$ into TM respectively (*cf.* [7]). Then the form ∂f (resp. $\bar{\partial} f$) is represented by (f_i^{α}) (resp. $(f_{\bar{i}}^{\alpha})$) locally where $f_i^{\alpha}=\dfrac{\partial f^{\alpha}}{\partial z^{i}}$ and so on.

The *energy density* $e(f)$ of f is defined by

$$e(f):=e'(f)+e''(f)$$
$$e'(f):=h_{\alpha\bar{\beta}}(f)\,g^{\bar{j}i}f_i^{\alpha}\overline{f_j^{\beta}}$$

and

$$e''(f):=h_{\alpha\bar{\beta}}(f)\,g^{\bar{j}i}f_{\bar{j}}^{\alpha}\overline{f_{\bar{i}}^{\beta}}$$

We denote by $\mathscr{L}(\Psi)$ the *Levi form* of $\Psi=\Phi^2$. We define an $f^{*}\mathrm{TN}^{1,0}$-valued differential (1,0) form $\mathscr{L}(\Psi)(\partial f)$ and an $f^{*}\mathrm{TN}^{1,0}$-valued differential (0, 1) form $\mathscr{L}(\Psi)(\bar{\partial} f)$ as follows:

$$(1.9)\qquad \begin{aligned}\mathscr{L}(\Psi)(\partial f)&=\left\{\sum_{i,j,k=1}^{m}g^{\bar{j}k}\Psi_{i\bar{j}}f_k^{\alpha}\,dz^{i}\right\}_{1\leqq\alpha\leqq n}\\ \mathscr{L}(\Psi)(\bar{\partial} f)&=\left\{\sum_{i,j,k=1}^{m}g^{\bar{k}j}\Psi_{j\bar{i}}f_{\bar{k}}^{\alpha}\,d\bar{z}^{i}\right\}_{1\leqq\alpha\leqq n}\end{aligned}$$

where $\Psi_{i\bar{j}}=\dfrac{\partial^2\Psi}{\partial z^{i}\,\partial\bar{z}^{j}}$.

We denote by $\nabla_{1,0}$ (resp. $\nabla_{0,1}$) the covariant differentiation of type $(1,0)$ (resp. $(0,1)$) induced from the connection on $T^*M \otimes f^*TN$ relative to ds_M^2 and $f^* ds_N^2$. The exterior differentiation

$$D_{1,0} : C^{p,q}(M, f^*TN) \to C^{p+1,q}(M, f^*TN)$$

(resp. $D_{0,1} : C^{p,q}(M, f^*TN) \to C^{p,q+1}(M, f^*TN)$) is defined by $\nabla_{1,0}$ (resp. $\nabla_{0,1}$). We denote by

$$D_{1,0}^* : C^{p,q}(M, f^*TN) \to C^{p-1,q}(M, f^*TN)$$

(resp. $D_{0,1}^* : C^{p,q}(M, f^*TN) \to C^{p,q-1}(M, f^*TN)$) the *formal adjoint* operator of $D_{1,0}$ (resp. $D_{0,1}$) (*cf.* [7]). Here $C^{p,q}(M, f^*TN)$ denotes the space of f^*TN-valued C^∞ differential forms of (p,q) type.

Let $f : (M, ds_M^2) \to (N, ds_N^2)$ be a differentiable map into a Kähler manifold (N, ds_N^2). Then the following two formulae hold (*cf.* [5], [6], [28]).

PROPOSITION 1.10. — (i) *For any non-critical value r of Φ*

$$(1.11)\quad \int_{M(r)} [2\{\mathrm{Trace}_{ds_M^2} \mathscr{L}(\Psi)\, e(f) - \langle \mathscr{L}(\Psi)(\partial f), \partial f\rangle_{f^*TN} - \langle \mathscr{L}(\Psi)(\bar\partial f), \bar\partial f\rangle_{f^*TN}\}$$
$$+ \langle \mathbf{e}(\partial\Psi)^* \partial f, D_{1,0}^* \partial f\rangle_{f^*TN} + \langle D_{0,1}^* \bar\partial f, \mathbf{e}(\bar\partial\Psi)^* \bar\partial f\rangle_{f^*TN}$$
$$+ \langle \partial f, \mathbf{e}(\bar\partial\Psi)^* D_{0,1} \partial f\rangle_{f^*TN} + \langle \mathbf{e}(\partial\Psi)^* D_{1,0} \bar\partial f, \bar\partial f\rangle_{f^*TN}]\, dv_M$$
$$= 2r\left\{\int_{\partial M(r)} [|\partial\Phi|_M^2\, e(f) - |\mathbf{e}(\partial\Phi)^* \partial f|_{f^*TN}^2 - |\mathbf{e}(\bar\partial\Phi)^* \bar\partial f|_{f^*TN}^2]\, \omega_r\right\}$$

(ii)

$$(1.12)\quad 2\langle \partial f, (D_{1,0} D_{1,0}^* - D_{0,1}^* D_{0,1})(\partial f)\rangle_{f^*TN}$$
$$= 2\langle (D_{0,1} D_{0,1}^* - D_{1,0}^* D_{1,0})(\bar\partial f), \bar\partial f\rangle_{f^*TN}$$
$$= \sum R^N_{\alpha\bar\beta\gamma\bar\delta}(f)\,(f^\alpha_{\bar i} \overline{f^\beta_{\bar j}} - f^\alpha_{\bar j} \overline{f^\beta_{\bar i}})\,\overline{(f^{\delta,i} \overline{f^{\gamma,\bar j}} - f^{\delta,j} \overline{f^{\gamma,\bar i}})}$$

where $\langle\ ,\ \rangle_{f^*TN}$ is the pointwise inner product on $C^{p,q}(M, f^*TN)$ relative to ds_M^2 and $f^* ds_N^2$ and $R^N_{\alpha\bar\beta\gamma\bar\delta}$ is the Riemannian curvature tensor of ds_N^2.

Proof. — First we show (1.11). We consider the following differential 1 forms on M:

$$\varphi_1 := e'(f)\,\bar\partial\Psi$$
$$\varphi_2 := \frac{1}{2}\sum_{\alpha,\beta,i} h_{\alpha\bar\beta}(f)\,(\mathbf{e}(\partial\Psi)^* \partial f)^\alpha\, \overline{f^\beta_i}\, d\bar z^i$$
$$\varphi_3 := e''(f)\,\partial\Psi$$
$$\varphi_4 := \frac{1}{2}\sum_{\alpha,\beta,i} h_{\alpha\bar\beta}(f)\, f^\alpha_{\bar i}\,\overline{(e(\bar\partial\Psi)^* \bar\partial f)^\beta}\, d\bar z^i$$

Using φ_k, we define the tangent vectors $\Theta_k = \{\theta_k^i, \theta_k^{\bar{i}} = 0\}$ as before. We choose holomorphic normal coordinate systems (z^i) around $x \in M$ and (w^α) around $y = f(x) \in N$ *i.e.* $g_{i\bar{j}}(x) = \delta_{ij}$, $dg_{i\bar{j}}(x) = 0$ and $h_{\alpha\bar{\beta}}(y) = \delta_{\alpha\beta}$, $dh_{\alpha\bar{\beta}}(y) = 0$ respectively. Then all the Christofell symbols ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\alpha\beta}^\gamma$ of ds_M^2 and ds_N^2 vanish at x and y respectively since ${}^M\Gamma_{ij}^k = \sum_l g^{\bar{l}k} \partial_i g_{j\bar{l}}$ and ${}^N\Gamma_{\alpha\beta}^\gamma = \sum_\delta h^{\bar{\delta}\gamma} \partial_\alpha h_{\beta\bar{\delta}}$ respectively. Using these coordinate systems, the integrand of the left hand-side of (1.11) can be obtained by calculating $\sum_{i=1}^m \nabla_i (\theta_1^i - \theta_2^j + \theta_3^i - \theta_4^i)$ pointwise (*cf.* [28] Proposition 1.14). Substituting $\theta_1 - \theta_2 + \theta_3 - \theta_4$ and $\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4$ into the formula (1.8) respectively, we obtain the formula (1.11).

To show the formula (1.12), we fix the above holomorphic normal coordinate systems for any point $x \in M$ and $y = f(x) \in N$. Then all the Christofell symbols of ds_M^2 and ds_N^2 vanish at x and y respectively and moreover it holds that $R^N_{\alpha\bar{\beta}\gamma\bar{\delta}} = \partial_\gamma \partial_{\bar{\delta}} h_{\alpha\bar{\beta}}$ and $\partial_\sigma^N \Gamma_{\alpha\beta}^\gamma = \partial_\beta {}^N\Gamma_{\alpha\sigma}^\gamma$ at y respectively. Using these properties, the formula (1.12) follows from a routine calculation.

Q.E.D.

We denote $M(r_2, r_1) = \{r_1 < \Phi < r_2\}$ for $r_2 > r_1 > 0_* := \inf_{x \in M} \Phi(x)$ and set $M(r, 0_*) = M(r)$ for $r > 0_*$.

For a differentiable map $f: (M, ds_M^2) \to (N, ds_N^2)$ of Kähler manifolds, the *energy* $E(f, r_2, r_1)$ of f on $M(r_2, r_1)$ is defined by

$$E(f, r_2, r_1) := \int_{M(r_2, r_1)} e(f)\, dv_M \tag{1.13}$$

We set $E(f, r, 0_*) = E(f, r)$ for $r > 0_*$. For some positive constant $c_0 > 0$, we define

$$B(f, r) := c_0 \int_{\partial M(r)} [|\mathbf{e}(\partial\Phi)^* \partial f|^2_{f^* TN} + |\mathbf{e}(\bar{\partial}\Phi)^* \bar{\partial} f|^2_{f^* TN}]\, \omega_r \tag{1.14}$$

for $r > r_*$.

If r is a critical value of Φ, then the integral on the right hand-side of (1.14) is taken over the non-critical points of $\partial M(r)$. It is easily verified that $B(f, r)$ is *finite* and a *continuous* function of $r > r_*$ (*cf.* [8], p. 275).

DEFINITION 1.15. — *A differentiable map $f: (M, ds_M^2) \to (N, ds_N^2)$ of Kähler manifolds is called harmonic if f satisfies the following equation*

$$\mathrm{Trace}_{ds_M^2} \nabla_{1,0} \bar{\partial} f = 0$$

and f is called pluriharmonic if

$$\nabla_{1,0} \bar{\partial} f = 0$$

Clearly any pluriharmonic map of Kähler manifolds is harmonic and any holomorphic or anti-holomorphic map of Kähler manifolds is pluriharmonic.

From now on we assume that *the complex dimension* m *of* M *is greater than or equal to two* and moreover assume the following conditions on Φ:

(1.16) *the constant* $\rho_1 := \inf_{x \in M \setminus K_{**}} \sum_{i=2}^{m} \varepsilon_i(x)$ *is positive where* $\varepsilon_1 \geqq \varepsilon_2 \geqq \ldots \geqq \varepsilon_m$ *are the eigenvalues of the Levi form of* $\Psi = \Phi^2$ *relative to* ds^2_M *and* K_{**} *is a compact subset of* M

(1.17) *the constant* $\rho_2 := \sup_{x \in M \setminus M[0_*]} |\partial\Phi|^2_{ds^2_M}(x)$ *is finite.*

The main result of this section is stated as follows.

THEOREM 1.18. — *Let* (M, ds^2_M) *be an* $m \geqq 2$ *dimensional connected non-compact Kähler manifold and let* Φ *be the function satisfying the conditions* (1.1), (1.2), (1.3), (1.16) *and* (1.17).

(i) *If a given non-constant differentiable map* $f : (M, ds^2_M) \to (N, ds^2_N)$ *into a Kähler manifold* (N, ds^2_N) *satisfies either* (1) f *is pluriharmonic or* (2) f *is harmonic and the Riemannian curvature of* (N, ds^2_N) *is semi-negative in the sense of Siu* [23] *i.e.*

$$(1.19) \qquad R^N_{\alpha\bar\beta\gamma\bar\delta}(y)\,(A^\alpha \overline{B^\beta} - C^\alpha \overline{D^\beta})\,\overline{(A^\delta \overline{B^\gamma} - C^\delta \overline{D^\gamma})}$$

is non-negative for any $y \in N$ *and complex numbers* A^α, B^α, C^γ *and* D^δ, *then the energy* $E(f, r, r_0)$ *of on* $M(r, r_0)$ *satisfies the following properties:*

The function $H(f, r, r_0) := \dfrac{E(f, r, r_0)}{r^\mu} \left(\mu := \dfrac{\rho_1}{\rho_2}\right)$ *is an increasing function of* $r \geqq r_0$ *and the following estimates hold*

$$(1.20) \qquad H(f, r_2, r_0) - H(f, r_1, r_0) \geqq \int_{r_1}^{r_2} \frac{B(f, t)}{t^\mu}\, dt$$

$$(1.21) \qquad H(f, r_2, r_0) \geqq H(f, r_1, r_0) \exp\left(\int_{r_1}^{r_2} \frac{B(f, t)}{E(f, t, r_0)}\, dt\right)$$

for $c_0 = \dfrac{1}{\rho_2}$ *in* $B(f, r)$ (*cf.* (1.14)) *and any* $r_2 > r_1 > \max(r_0, r_*)$

(ii) *If* f *is a non-constant non-negative plurisubharmonic function of class* C^2 *on* M, *then the function*

$$G(f, r, r_0) := \frac{F(f, r, r_0)}{r^\mu},$$

$$F(f, r, r_0) := \int_{M(r, r_0)} \mathrm{Trace}_{ds^2_M}\, \partial\bar\partial f^2\, dv_M,$$

satisfies the same properties as $H(f, r, r_0)$.

Here the value r_0 *is determined as follows: in the case* (i), (1), $r_0=0_*$ *if* $K_{**}=\varnothing$ *or* r_0 *is a non-critical value of* Φ *with* $r_0>r_{**}:=\sup_{x\in K_{**}}\Phi(x)$ *if* $K_{**}\neq\varnothing$ *and in the case either* (i), (2) *or* (ii) $r_0=0_*$ *if* $K_*=K_{**}=\varnothing$ *or* r_0 *is a non-critical value of* Φ *with* $r_0>\max(r_*, r_{**})$ *if otherwise.*

Theorem 1.18 follows from the following lemma which plays the crucial role in this article.

LEMMA 1.22. — *Let* (M, ds_M^2, Φ) *be as above in Theorem* 1.18.

(i) *If a given differentiable map* $f:(M, ds_M^2)\to(N, ds_N^2)$ *into a Kähler manifold* (N, ds_N^2) *satisfies the condition either* (1) *or* (2) *in Theorem* 1.18, (i), *then the following integral inequality holds*

$$r\frac{\partial}{\partial r}E(f, r, r_0)-\mu E(f, r, r_0)\geqq r B(f, r) \tag{1.23}$$

(ii) *If* f *is a non-negative plurisubharmonic function of class* C^2 *on* M, *then the function* $F(f, r, r_0)$ *defined in Theorem* 1.18, (ii) *satisfies*

$$r\frac{\partial}{\partial r}F(f, r, r_0)-\mu F(f, r, r_0)\geqq r B(f, r) \tag{1.24}$$

where $\mu=\dfrac{\rho_1}{\rho_2}$, $c_0=\dfrac{1}{\rho_2}$ *in* $B(f, r)$ (*cf.* (1.14)) *and* r *is any non-critical value of* Φ *with* $r\geqq r_0$.

First we show Theorem 1.18 by using Lemma 1.22.

Proof of Theorem 1.18. — Here we give the proof of the case (i), (1) only because other cases are proved quite similarly.

For any non-critical value r of Φ, $r_1\leqq r\leqq r_2$, we have from (1.23)

$$\frac{\partial}{\partial r}H(f, r, r_0)\geqq\frac{B(f, r)}{r^{\mu}}\quad\text{for}\quad H(f, r, r_0)=\frac{E(f, r, r_0)}{r^{\mu}} \tag{1.25}$$

Hence $H(f, r, r_0)$ is an increasing function of $r\geqq r_0$.

Integrating (1.25), we obtain (1.20) because the set of critical values of Φ is discrete.

Since $E(f, r, r_0)>0$ for any $r>r_0$ (*cf.* [20] Theorem 1), we have from (1.23).

$$\frac{\mu}{r}+\frac{B(f, r)}{E(f, r, r_0)}\leqq\frac{\partial}{\partial r}\log E(f, r, r_0) \tag{1.26}$$

Hence we obtain (1.21) by integrating (1.26).

Q.E.D.

Proof of Lemma 1.22. — (i) In the case $r_0>0_*$, to show the inequality (1.23), we should apply the integral formula (1.11) to the domain $M(r, r_0)$ for any non-critical value r and the fixed non-critical value r_0 of Φ, $r>r_0>0_*$. Since $M(r, r_0)$ has two boundaries $\partial M(r)$ and $\partial M(r_0)$, in this case two boundary integrals appear in (1.11). But the left hand-side of (1.11) is dominated by the boundary integral on $\partial \mathrm{M}(r)$ because the boundary integral on $\partial \mathrm{M}(r_0)$ is non-negative by Cauchy-Schwarz inequality.

(1) Let $f:(\mathrm{M}, ds^2_{\mathrm{M}}) \to (\mathrm{N}, ds^2_{\mathrm{N}})$ be a non-constant *pluriharmonic* map of Kähler manifolds. Then f satisfies the following equations:

$$\mathrm{D}_{0,1}\,\partial f = \mathrm{D}^*_{1,0}\,\partial f = \mathrm{D}_{1,0}\,\bar\partial f = \mathrm{D}^*_{0,1}\,\bar\partial f = 0 \tag{1.27}$$

If the compact set K_{**} (*cf*, (1.16)) is empty, then we set $r_0 = 0_*$. Otherwise we fix a non-critical value r_0 of Φ with $r_0 > r_{**}$.

By (1.11), (1.27) and the above consideration, we have for any non-critical value $r > \max(r_0, r_*)$ of Φ

$$\begin{aligned}(1.28)\quad &\int_{\mathrm{M}(r,\, r_0)} \{ \mathrm{Trace}_{ds^2_{\mathrm{M}}} \mathscr{L}(\Psi)\, e(f) - \langle \mathscr{L}(\Psi)(\partial f), \partial f \rangle_{f^*\mathrm{TN}} - \langle \mathscr{L}(\Psi)(\bar\partial f), \bar\partial f \rangle_{f^*\mathrm{TN}} \}\, dv_{\mathrm{M}} \\ &\leqq r \int_{\partial \mathrm{M}(r)} [\, |\partial\Phi|^2_{\mathrm{M}}\, e(f) - |\mathbf{e}(\partial\Phi)^*\, \partial f|^2_{f^*\mathrm{TN}} - |\mathbf{e}(\bar\partial\Phi)^*\, \bar\partial f|^2_{f^*\mathrm{TN}}]\, \omega_r\end{aligned}$$

For any point $x \in \mathrm{M} \setminus \mathrm{K}_{**}$ and $y = f(x) \in \mathrm{N}$, we choose local coordinate systems (z^i) around x and (w^α) around y so that $g_{i\bar j}(x) = \delta_{ij}$, $\Psi_{i\bar j}(x) = \varepsilon_i(x)\, \delta_{ij}$ and $h_{\alpha\bar\beta}(y) = \delta_{\alpha\beta}$ respectively. From (1.9) and (1.16), we have at x

$$\begin{aligned}(1.29)\quad &\mathrm{Trace}_{ds^2_{\mathrm{M}}} \mathscr{L}(\Psi)\, e(f) - \langle \mathscr{L}(\Psi)(\partial f), \partial f \rangle_{f^*\mathrm{TN}} - \langle \mathscr{L}(\Psi)(\bar\partial f), \bar\partial f \rangle_{f^*\mathrm{TN}} \\ &= \sum_{\alpha=1}^{n} \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \varepsilon_j(x) - \varepsilon_i(x) \right) \{ |f^\alpha_i(x)|^2 + |f^\alpha_{\bar i}(x)|^2 \} \geqq \rho_1\, e(f)(x)\end{aligned}$$

Then the inequality (1.23) follows from (1.14) $\left(c_0 = \dfrac{1}{\rho_2} \right)$, (1.17), (1.28) and (1.29).

(2) Let $f:(\mathrm{M}, ds^2_{\mathrm{M}}) \to (\mathrm{N}, ds^2_{\mathrm{N}})$ be a differentiable map of Kähler manifolds. If the compact sets K_* and K_{**} are empty, then we set $r_0 = 0_*$. Otherwise we fix a non-critical value r_0 of Φ with $r_0 > \max(r_*, r_{**})$.

Since $\mathrm{D}_{0,1}\,\partial f = \mathrm{D}_{1,0}\,\bar\partial f$ (*cf.* [28], (1.8)), by (1.12) and integration by parts, we have for any $r \geqq r_0$

$$\begin{aligned}(1.30)\quad &2[(\mathrm{D}_{1,0}\,\bar\partial f, \mathrm{D}_{1,0}\,\bar\partial f)_{f^*\mathrm{TN},\,\mathrm{M}(r)} - (\mathrm{D}^*_{0,1}\,\bar\partial f, \mathrm{D}^*_{0,1}\,\bar\partial f)_{f^*\mathrm{TN},\,\mathrm{M}(r)}] \\ &+ \int_{\mathrm{M}(r)} \sum \mathrm{R}^{\mathrm{N}}_{\alpha\bar\beta\gamma\bar\delta}(f)\, (f^\alpha_{\bar i}\overline{f^\beta_{\bar j}} - f^\alpha_{\bar j}\overline{f^\beta_{\bar i}})\, \overline{(f^{\delta,\, i}\, \overline{f^{\gamma,\, \bar j}} - f^{\delta,\, j}\, \overline{f^{\gamma,\, \bar i}})}\, dv_{\mathrm{M}} \\ &= \int_{\partial \mathrm{M}(r)} [\langle \mathbf{e}(\partial\Phi)^*\, \partial f, \mathrm{D}^*_{1,0}\,\partial f \rangle_{f^*\mathrm{TN}} + \langle \mathrm{D}^*_{0,1}\,\bar\partial f, \mathbf{e}(\bar\partial\Phi)^*\, \bar\partial f \rangle_{f^*\mathrm{TN}} \\ &\quad + \langle \partial f, \mathbf{e}(\bar\partial\Phi)^*\, \mathrm{D}_{0,1}\,\partial f \rangle_{f^*\mathrm{TN}} + \langle \mathbf{e}(\partial\Phi)^*\, \mathrm{D}_{1,0}\,\bar\partial f, \bar\partial f \rangle_{f^*\mathrm{TN}}]\, \omega_r\end{aligned}$$

On the other hand, by (1.3) and Fubini's theorem, we have

$$(1.31)\quad (\mathbf{e}(\partial\Psi)^* \partial f, D^*_{1,0}\partial f)_{f^*\mathrm{TN},\,M(r,r_0)} + (D^*_{0,1}\bar\partial f, \mathbf{e}(\bar\partial\Psi)^*\bar\partial f)_{f^*\mathrm{TN},\,M(r,r_0)}$$
$$+(\partial f, \mathbf{e}(\bar\partial\Psi)^* D_{0,1}\partial f)_{f^*\mathrm{TN},\,M(r,r_0)} + (\mathbf{e}(\partial\Psi)^* D_{1,0}\bar\partial f, \bar\partial f)_{f^*\mathrm{TN},\,M(r,r_0)} = 2\int_{r_0}^{r} t\,\mathrm{I}(t)\,dt$$

Here we have denoted by $\mathrm{I}(r)$ the right hand-side of (1.30).

(2) If f is *harmonic*, then f satisfies the following equations:

$$(1.32)\qquad D^*_{1,0}\partial f = D^*_{0,1}\bar\partial f = 0$$

By (1.19), (1.30) and (1.32), the function $\mathrm{I}(r)$ is non-negative for any $r \geqq r_0$. Hence the left hand-side of (1.31) is non-negative by the equality (1.31).

Combining this fact with (1.11), we can obtain the inequality (1.23) similarly.

(ii) Let f be a non-constant *non-negative plurisubharmonic* function of class C^2 on M.

Since $\mathrm{I}(r) = -c_m \int_{M(r)} dd_c f \wedge dd_c f \wedge \omega_M^{m-2}$ ($\mathrm{I}(r)$ is the right hand-side of (1.30), $c_m = [2^{m-2}(m-2)!]^{-1}$ and $d_c = \sqrt{-1}(\bar\partial - \partial)/2$), by Stokes theorem, we have

$$(1.33)\quad 2\int_{r_0}^{r} t\,\mathrm{I}(t)\,dt = -c_m \int_{M(r,r_0)} df \wedge d_c\Phi^2 \wedge dd_c f \wedge \omega_M^{m-2}$$
$$= c_m \int_{M(r,r_0)} \{ -d[f d_c\Phi^2 \wedge dd_c f \wedge \omega_M^{m-2}] + f dd_c f \wedge dd_c \Phi^2 \wedge \omega_M^{m-2} \}$$

Next we need the following facts

(1.34) *The function* λ_r *defined by the equality*

$$\lambda_r \omega_r = f d_c \Phi^2 \wedge dd_c f \wedge \omega_M^{m-2} \quad \textit{on } \partial M(r)$$

satisfies

$$0 \leqq \lambda_r \leqq 2^m (m-2)!\, r f \mathrm{T}(f) \,|\partial\Phi|^2 \quad \textit{on } \partial M(r)$$

$$(1.35)\qquad 2^m (m-2)!\, \rho_1 f \mathrm{T}(f)\, dv_M \leqq f dd_c f \wedge dd_c \Phi^2 \wedge \omega_M^{m-2}$$

on $M(r, r_0)$

Here we denoted $\mathrm{T}(f) = \mathrm{Trace}_{ds^2_M} \partial\bar\partial f$.

For a point x of $\partial M(r)$, we choose a coordinate system (z^i) around x so that $g_{i\bar j}(x) = \delta_{ij}$ and $f_{i\bar j}(x) = v_i \delta_{ij}$. From (1.5) and (1.6), we have at x

$$\lambda_r dv_M = 2 f r d\Phi \wedge d_c\Phi \wedge dd_c f \wedge \omega_M^{m-2} = 2(m-2)!\,(\sqrt{-1})^m f r \Big(\sum_{i \neq j} v_i |\Phi_j|^2\Big) \prod_{k=1}^{m} dz^k \wedge d\bar z^k$$

by $v_i \geqq 0$

$$\leqq 2(m-2)!\,(\sqrt{-1})^m f r \Big(\sum_{i=1}^{m} v_i\Big)\Big(\sum_{j=1}^{m} |\Phi_j|^2\Big) \prod_{k=1}^{m} dz^k \wedge d\bar z^k = 2^m (m-2)!\, f r \mathrm{T}(f)\, |\partial\Phi|^2\, dv_M$$

Therefore we have (1.34).

For a point x of $M(r, r_0)$, we choose a coordinate system (z^i) around x so that $g_{i\bar{j}}(x)=\delta_{ij}$, $\Phi^2_{i\bar{j}}(x)=\varepsilon_i(x)\,\delta_{ij}$ and $T(f)(x)=\sum_{i=1}^{m} a_{i\bar{i}}$ for $a_{i\bar{j}}=f_{i\bar{j}}(x)$. We have at x

$$f\,dd_c f \wedge dd_c \Phi^2 \wedge \omega_M^{m-2}=(m-2)!\,(\sqrt{-1})^m f\Big(\sum_{i\neq j} a_{i\bar{i}}\varepsilon_j\Big)\prod_{k=1}^{m} dz^k \wedge d\bar{z}^k$$

by (1.16) and $a_{i\bar{i}} \geqq 0$

$$\geqq 2^m (m-2)!\, f \rho_1 f T(f)\, dv_M$$

Therefore we have (1.35).

By (1.33), (1.34), (1.35) and Stokes theorem, we have

$$(1.36)\qquad \int_{r_0}^{r} t\, I(t)\, dt \geqq -2r\int_{\partial M(r)} f T(f)\,|\partial\Phi|^2\,\omega_r + 2\rho_1\int_{M(r, r_0)} f T(f)\, dv_M$$

From (1.11), (1.29), (1.31) and (1.36), we have

$$\rho_1\int_{M(r, r_0)}\{e(f)+2fT(f)\}\, dv_M$$
$$\leqq r\int_{\partial M(r)}\{e(f)+2fT(f)\}\,|\partial\Phi|^2\,\omega_r - 2r\int_{\partial M(r)} |\mathbf{e}(\partial\Phi)^*\,\partial f|^2\,\omega_r$$

Therefore we have (1.24) because $T(f^2)=e(f)+2fT(f)$.

Q.E.D.

Remark 1.37. – In Theorem 1.18, when we want to estimate the energy of a given holomorphic map $f:(M, ds^2_M) \to (N, ds^2_N)$ of complex manifolds, it is sufficient to assume that ds^2_N is Kähler and ds^2_M is Kähler outside a compact subset of M from the observation in the proof of Lemma 1.22.

Moreover if $\mu > 1$ and $K_* \neq \varnothing$ or $K_{**} \neq \varnothing$, then it is easily verified that $E(f, r)/(r+1)^\mu$ (*i.e.* $r_0 = 0_*$) is an increasing function of $r > r^*$ for some sufficiently large r^*.

We call the function Φ in Theorem 1.18 a *special exhaustion function* of M relative to ds^2_M. Here we point out some examples of a Kähler manifold provided with such a special exhaustion function.

Example 1. – An $m \geqq 2$ dimensional *complex Euclidean space* $\mathbb{C}^m$ *with Euclidean metric* ds^2_e has a special exhaustion function $\Phi = \|z\|$ *i.e.* the norm function of $z \in \mathbb{C}^m$.

In this case $\rho_1 = m-1$ and $|\partial\Phi|^2_{ds^2_e} \equiv 1/2$ on $\mathbb{C}^m \setminus \{0\}$ *i.e.* $\rho_2 = 1/2$ and $K_* = K_{**} = \varnothing$. Hence $\mu = 2m-2$.

Example 2. — Let $(A, ds_A^2) \overset{\iota}{\subsetneq} (\mathbb{C}^n, ds_e^2)$ be an $m \geqq 2$ dimensional *connected closed submanifold* of $\mathbb{C}^n$ provided with the induced metric $ds_A^2 = \iota^* ds_e^2$. If necessary, translating $(z^i) = (w^i - a^i)$, $a = (a^1, \ldots, a^n) \in \mathbb{C}^n \setminus A$, we may assume that the restriction Φ of $\| z \|$ onto A has only non-degenerate critical points. Φ is a special exhaustion function of A relative to ds_A^2 *i.e.* $\rho_1 = m-1$ and $\rho_2 = 1/2$. Hence $\mu = 2m-2$.

Since every *Stein* manifold S can be realized as a closed submanifold of some $\mathbb{C}^n$ by a proper holomorphic map $h : S \subsetneq \mathbb{C}^n$, S has a special exhaustion function $\Phi = h^*(\| z \|)$ relative to the Kähler metric $ds_S^2 = h^*(ds_e^2)$ and $\mu = 2m-2$ if $\dim_{\mathbb{C}} S \geqq 2$.

Example 3. — Let M be an $m \geqq 2$ dimensional *strongly pseudo-convex* manifold and let $j : M \to R$ be the Remmert reduction of M. Since R is a normal Stein space with finitely many isolated singularities, we can embed R into some $\mathbb{C}^n$ by a proper holomorphic map $h : R \to \mathbb{C}^n$. We set $\Phi = (h \circ j)^*(\| z \|)$. Since j is biholomorphic outside a compact subset of M, we can construct a hermitian metric ds_M^2 on M whose fundamental form ω_M can be written $\omega_M = \sqrt{-1}\, \partial \bar{\partial} \Phi^2$ outside a compact subset $K_* (= K_{**})$ of M. Hence Φ is a special exhaustion function of M relative to ds_M^2 and $\mu = 2m-2$.

Example 4. — Let (M, ds_M^2) be an $m \geqq 2$ dimensional *complete Kähler manifold with a pole* $0 \in M$ *i.e.* $\exp_0 : TM_0 \to M$ is a diffeomorphism and let Φ be the distance function from $0 \in M$ relative to ds_M^2. Then Φ is an exhaustion function of M and satisfies $| \partial \Phi |_{ds_M^2}^2 \equiv 1/2$ on $M \setminus \{ 0 \}$ *i.e.* $\rho_2 = 1/2$ and $K_* = \varnothing$. If the *radial curvature* of ds_M^2 is *non-positive*, then $\Psi = \Phi^2$ is a C^∞ strictly plurisubharmonic function on M *i.e.* $K_{**} = \varnothing$ and $\rho_1 = m-1$ (*cf.* [9] Proposition 1.17 and 2.24). Hence $\mu = 2m-2$. In this way Φ is a special exhaustion function of M relative to ds_M^2 and the Kähler metric induced from $\partial \bar{\partial} \Phi^2$ respectively. In the latter case we can also take $\mu = 2m-2$.

Though according to each example, we can restate Theorem 1.18, we omit the detail here.

Remark 1.38. — Originally Donnelly and Xavier established some integral formula for differential forms with *compact supports* in [6]. But we applied their formula to vector bundle-valued differential forms on *bounded domains with smooth boundary*. By the same way we can establish the energy estimate for harmonic functions on a non-compact Riemannian manifold provided with certain exhaustion function. But to establish such an estimate for a harmonic map $f : (M, ds_M^2) \to (N, ds_N^2)$ of Riemannian manifolds, we should assume not only the non-positivity of Riemannian curvature of ds_N^2 but also the non-negativity of Ricci curvature of ds_M^2.

Remark 1.39. — From the method used to induce the integral inequality (1.23), we can also induce the following equality and inequality which are used to show the analyticity of harmonic maps of Kähler manifolds respectively.

1. *Let* $f : (M, ds_M^2) \to (N, ds_N^2)$ *be a harmonic map of compact Kähler manifolds. Then it holds that*

$$(1.40) \qquad 2 (D_{1,0} \bar{\partial} f, D_{1,0} \bar{\partial} f)_{f^* TN, M} = - \int_M \sum R^N_{\alpha \bar{\beta} \gamma \bar{\delta}}(f) \, (f^\alpha_{\bar{i}} \overline{f^\beta_{\bar{j}}} - f^\alpha_{\bar{j}} \overline{f^\beta_{\bar{i}}}) \, \overline{(f^{\delta, i} \overline{f^{\gamma, \bar{j}}} - f^{\delta, j} \overline{f^{\gamma, \bar{i}}})} \, dv_M$$

2. *Let* $D \subset\subset M$ *be a bounded domain with smooth boundary* ∂D *defined by a* C^∞ *strictly hyper* $m-1$ *convex function* Φ *on a neighborhood of* ∂D *on an* $m \geqq 2$ *dimensional Kähler manifold* (M, ds_M^2) *and let* $f: (\bar{D}, ds_M^2) \to (N, ds_N^2)$ *be a differentiable map from a neighborhood of* $\bar{D}$ *into a Kähler manifold* (N, ds_N^2) *which is harmonic on* (D, ds_M^2). *Then there exist positive constants* C *and* δ *such that*

$$\int_{-\delta}^{0} dt \int_{\{\Phi < t\}} \sum R^N_{\alpha\bar{\beta}\gamma\bar{\delta}}(f)\,(f^\alpha_{\bar{i}}\overline{f^\beta_{j}} - f^\alpha_{\bar{j}}\overline{f^\beta_{i}})\,\overline{(f^{\delta,\bar{i}}\overline{f^{\gamma,\bar{i}}} - f^{\delta,\bar{i}}\overline{f^{\gamma,\bar{i}}})}\, dv_M + \int_{\{-\delta < \Phi < 0\}} e''(f)\, dv_M \leqq C \int_{\partial D} [\bar{\partial}_b f]^2 \frac{dS}{|d\Phi|_M} \tag{1.41}$$

where

$$[\bar{\partial}_b f]^2 := |\bar{\partial}\Phi|_M^2 e''(f) - |\mathbf{e}(\bar{\partial}\Phi)^* \bar{\partial} f|_{f^*TN}^2 = 2 \sum_{i<j} h_{\alpha\bar{\beta}}(f)\,(\Phi_{\bar{i}} f^{\alpha,j} - \Phi_{\bar{j}} f^{\alpha,i})\,\overline{(\Phi_{\bar{i}} f^{\beta,j} - \Phi_{\bar{j}} f^{\beta,i})}$$

(*by Lagrange equality*).

The formula (1.40) yields the alternative proof of the analyticity of harmonic maps of compact Kähler manifolds and the integral inequality (1.41) implies that f is *holomorphic* on D if *the Riemannian curvature of* (N, ds_N^2) *is semi-negative in the sense of Siu and* $\bar{\partial}_b f = 0$ *on* ∂D *i.e.* f *satisfies the tangential Cauchy-Riemann equation on* ∂D. On those topics, the reader should be refered to [1], [23], [24].

2. Liouville theorems for harmonic maps

In this section, first we show two Liouville theorems for harmonic maps and plurisubharmonic functions. Later using these theorems, we give the proofs of Theorems 2 and 3 stated in the introduction.

We first state the following theorem.

THEOREM 2.1. — *Let* (M, ds_M^2) *be an* $m \geqq 1$ *dimensional connected non-compact Kähler manifold provided with a function* Φ *which satisfies the conditions* (1.1), (1.2), (1.3), (1.16) (*here we set* $\rho_1 = 0$ *if* $m = 1$) *and* (1.17) *and let* $V(r)$ *be the volume of* $M(r)$ *relative to* ds_M^2.

Suppose there exists a continuous non-decreasing function $g: [0, \infty) \to (0, \infty)$ *such that*

$$\int_{\delta}^{\infty} \frac{dt}{t g(t)} = \infty \quad \textit{for some } \delta > 0 \tag{2.2}$$

and

$$(2.3) \qquad \lim_{r \to \infty} \sup \frac{n(\mathrm{M}, r)}{g(r)} < \infty$$

for $n(\mathrm{M}, r) := \dfrac{\mathrm{V}(r)}{r^{\mu+2}}$ *and* $\mu := \dfrac{\rho_1}{\rho_2}$.

Then α) $(\mathrm{M}, ds_{\mathrm{M}}^2)$ *admits no non-constant bounded harmonic functions.*

β) *Let* $f: \mathrm{M} \to \mathrm{N}$ *be a holomorphic map into a projective algebraic variety* N *with a very ample line bundle* L. *If the set* $\mathrm{E}_f(\mathrm{L}) := \{\sigma \in \mathbb{P}(\Gamma(\mathrm{N}, \mathrm{L})) : \mathrm{Im} f \cap \mathrm{supp}(\sigma) = \varnothing\}$ ((σ) *is the divisor defined by the section* σ) *has positive measure, then f is a constant map.*

γ) A *admits no non-constant negative plurisubharmonic functions of class* C^2.

We define the following positive functions g_n $(n \geqq 0)$ defined by $g_n(r) := \prod_{i=1}^{n} \mathrm{L}_i(r)$ $(r > 0)$, $\mathrm{L}_0(r) \equiv 1$, $\mathrm{L}_1(r) = \log r$ and $\mathrm{L}_{i+1}(r) = \mathrm{L}_i(\log r)$ for any sufficiently large r and $i \geqq 1$. We should note that $\int_{l_n}^{\infty} (t g_n(t))^{-1} dt = \infty$ for some $l_n \gg 0$ and any $n \geqq 0$.

THEOREM 2.4. — *Let* $(\mathrm{M}, ds_{\mathrm{M}}^2)$ *be an* $m \geqq 1$ *dimensional complete Kähler manifold with a pole* $0 \in \mathrm{M}$ *and let* Φ *be the distance function from* 0 *relative to* ds_{M}^2.

Suppose there exists a continuous increasing function $h : [r_*, \infty) \to (1, \infty)$ *such that*

$$(2.5) \qquad \max_{1 \leqq i \leqq m} |\varepsilon_i(x) - 1| \leqq \frac{1}{h(\Phi(x))} \text{ for any } x \in \mathrm{M} \setminus \mathrm{M}(r_*)$$

where ε_i *are the eigenvalues of the Levi form of* $\Psi = \Phi^2$ *relative to* ds_{M}^2 *and*

$$(2.6) \qquad \lim_{r \to \infty} \sup \frac{\exp\left((4m-2)\displaystyle\int_{r_*}^{r} \frac{dt}{t h(t)}\right)}{g_n(r)} < \infty$$

for some $n \geqq 0$.

Then the assertions α), β) *and* γ) *of Theorem* 2.1 *hold for* $(\mathrm{M}, ds_{\mathrm{M}}^2)$.

Remark 2.7. — In Theorems 2.1 and 2.4, if M is *Stein*, then we can relax the regularity condition of plurisubharmonic function in the assertion γ) because every negative plurisubharmonic function on a Stein manifold can be approximated by a smooth one. In Theorem 2.4, from the condition (2.6), $h(r)$ is unbounded. When $\int_{r_*}^{\infty} (t h(t))^{-1} dt < \infty$ *i.e.* $n = 0$, the assertion α) has been verified in some cases (*cf.* [14], [30]).

Proof of Theorem 2.1. — α) Let f be a non-constant *bounded harmonic* function on M *i.e.* $\Delta_{\mathrm{M}} f \equiv 0$ and $0 \leqq |f| < \mathrm{C} < \infty$ for some $\mathrm{C} > 0$.

We set ourselves in the situation of Lemma 1.22, (i), (2). First we obtain the following inequality.

$$\text{(2.8)} \qquad \mathrm{E}(f, r, r_0)^2 \leqq c_* \frac{\partial}{\partial r} \mathrm{V}(r)\, \mathrm{B}(f, r)$$

for any non-critical value r of Φ, $r > r_0$ and $c_* > 0$. By the harmonicity of f and Stokes theorem, we have

$$2\,\mathrm{E}(f, r, r_0) \leqq (df, df)_{\mathrm{M}(r)} = \int_{\partial \mathrm{M}(r)} \langle f, \mathbf{e}(d\Phi)^* df \rangle\, \omega_r$$

By the boundedness of $|f|$ and applying Cauchy-Schwarz inequality to the right hand-side, we obtain (2.8). By (2.3), we have

$$\text{(2.9)} \qquad n(\mathrm{M}, r) \leqq c_1\, g(r) \text{ for any } r > r_1 > r_0$$

Setting $\mathrm{H}(r) := \dfrac{\mathrm{E}(f, r, r_0)}{r^\mu}$ for $r > r_0$, from (1.25) and (2.8), we have

$$\mathrm{H}(r)^2 \leqq c_* \frac{\dfrac{\partial}{\partial r}\mathrm{V}(r) \dfrac{\partial}{\partial r}\mathrm{H}(r)}{r^\mu}$$

Hence by Cauchy-Schwarz inequality, we have

$$\text{(2.10)} \qquad (r_2 - r_1)^2 \leqq c_* \int_{r_1}^{r_2} \frac{\dfrac{\partial}{\partial t}\mathrm{V}(t)}{t^\mu}\, dt \left(\frac{1}{\mathrm{H}(r_1)} - \frac{1}{\mathrm{H}(r_2)} \right)$$

for any $r_2 > r_1 > r_0$.

By (2.9), we have

$$\text{(2.11)} \qquad \int_{r_1}^{r_2} \frac{\dfrac{\partial}{\partial t}\mathrm{V}(t)}{t^\mu}\, dt \leqq c_2\, r_2^2\, g(r_2)$$

for any $r_2 > r_1 > r_0$.

From (2.10) and (2.11), we have

$$\frac{c_3}{g(r_2)}\left(1-\frac{r_1}{r_2}\right)^2 \leqq \frac{1}{\mathrm{H}(r_1)} - \frac{1}{\mathrm{H}(r_2)} \tag{2.12}$$

for any $r_2 > r_1 > r_0$.

We consider a sequence $\{r_n\}_{n \geqq 1}$ so that $r_{n+1} = 2r_n$ and $r_1 = 2r_0$. Substituting $r_1 = r_n$ and $r_2 = r_{n+1}$ into (2.12), we have

$$\frac{c_4}{g(r_{n+1})} \leqq \frac{1}{\mathrm{H}(r_n)} - \frac{1}{\mathrm{H}(r_{n+1})} \tag{2.13}$$

for any $n \geqq 1$.

Hence we have from (2.13).

$$\int_{r_2}^{\infty} \frac{dt}{tg(t)} \leqq \frac{c_5}{\mathrm{H}(r_1)} < \infty$$

This contradicts to (2.2).

β) Let N be a projective algebraic variety with a very ample line bundle L. We assume that N is *reduced* and *irreducible*. The space $\Gamma(\mathrm{N}, \mathrm{M})$ of global sections of L is a finite dimensional vector space. We set $\mathrm{V} := \Gamma(\mathrm{N}, \mathrm{L})$ and $\dim_{\mathbb{C}} \mathrm{L} = n+1$.

Let $h : \mathrm{M} \to \mathrm{N}$ be a holomorphic map into N so that the set

$$\mathrm{E}_h(\mathrm{L}) = \{\sigma \in \mathbb{P}_n(\mathrm{V}) : \mathrm{Im}\, h \cap \mathrm{supp}(\sigma) = \varnothing\}$$

((σ) is the divisor defined by the section σ) has positive measure in $\mathbb{P}_n(\mathrm{V})$. We shall induce a contradiction by assuming that h is non-constant.

Since L is very ample, we have an embedding $j : \mathrm{N} \subsetneq \mathbb{P}_n(\mathrm{V}^*)$ (V^* is the dual vector space of V). We consider the holomorphic map $f := j \circ h : \mathrm{M} \to \mathbb{P}_n(\mathrm{V}^*)$. Since $\mathrm{L} = j^* \mathrm{H}$ (H is the hyperplane bundle over $\mathbb{P}_n(\mathrm{V}^*)$) and $\mathbb{P}_n(\mathrm{V}) = \mathbb{P}_n(\mathrm{V}^*)^*$ (the dual projective space of $\mathbb{P}_n(\mathrm{V}^*)$), setting $\mathbb{P}_n = \mathbb{P}_n(\mathrm{V}^*)$, we may assume that $f : \mathrm{M} \to \mathbb{P}_n$ is non-constant and $\mathrm{E}_f = \{\xi \in \mathbb{P}_n^* : \mathrm{Im} f \cap \mathrm{supp}(\xi) = \varnothing\}$ has positive measure in $\mathbb{P}_n$. Under this assumption, we have only to show the estimate (2.8) in account of Lemma 1.22, (i) and (1.25).

Let $\sigma = (\sigma_0 : \sigma_1 : \ldots ; \sigma_n) \in \mathbb{P}_n$ (resp. $\xi = (\xi^0 : \xi^1 : \ldots ; \xi^n) \in \mathbb{P}_n^*$) be the homogeneous coordinates of $\mathbb{P}_n$ (resp. $\mathbb{P}_n^*$). We denote by ω (resp. ω^*) the Kähler form of the *Fubini Study* metric of $\mathbb{P}_n$ (resp. $\mathbb{P}_n^*$). We denote $\langle \sigma, \xi \rangle = \sum_{i=0}^{n} \sigma_i \xi^i$ and $\|\sigma\| = \sum_{i=0}^{n} |\sigma_i|^2$ for $\sigma \in \mathbb{P}_n$ and $\xi \in \mathbb{P}_n^*$. We define a positive function Λ on $\mathbb{P}_n \times \mathbb{P}_n^*$ by

$$\Lambda(\sigma, \xi) := \frac{\|\sigma\| \|\xi\|}{|\langle \sigma, \xi \rangle|} \text{ for } \sigma \in \mathbb{P}_n \text{ and } \xi \in \mathbb{P}_n^*$$

It is easily verified that the function Λ satisfies the following properties:

(β.1) *For any* $\sigma \in \mathbb{P}_n$, *the functions* $\log \Lambda(\sigma, \ )$ *and* $\Lambda(\sigma, \ ) \geqq 1$ *are integrable on* $\mathbb{P}_n^*$ *and*

$$A_1 := \int_{\xi \in \mathbb{P}_n^*} \log \Lambda(\sigma, \xi)\, \omega^{*n}$$

$$A_2 := \int_{\xi \in \mathbb{P}_n^*} \Lambda(\sigma, \xi)\, \omega^{*n}$$

are positive constants not depending on $\sigma \in \mathbb{P}_n$ ($\omega^k = \Lambda^k \omega$ *and so on*).

(β.2) *For any subset* $E \subset \mathbb{P}_n^*$ *with* $\int_E \omega^{*n} > 0$ *if* $f(\partial M(r)) \cap \operatorname{supp}(\xi) = \varnothing$ *for any* $\xi \in E$, *then the functions* $\log f^* \Lambda$ *and* $f^* \Lambda$ *are integrable on* $\partial M(r) \times E$ (*Here* (ξ) *is the hyperplane defined by* $\langle \sigma, \xi \rangle$).

(β.3) *There exists a positive constant* C *not depending on* $(\sigma, \xi) \in \mathbb{P}_n \times \mathbb{P}_n^*$ *such that*

$$|\partial_\sigma \log \Lambda(\sigma, \xi)|_\omega \leqq C\, \Lambda(\sigma, \xi)$$

for any $\xi \in \mathbb{P}_n^*$ *and any* $\sigma \in \mathbb{P}_n \setminus \operatorname{supp}(\xi)$.

Using these properties of Λ, we show the estimate (2.8) for the above holomorphic map $f: M \to \mathbb{P}_n$.

We set $\eta := \int_{E_f} \omega^{*n}$. By the hypothesis, we have $\eta > 0$. For any $\xi \in \mathbb{P}_n^*$, it holds that

$$\omega = 2\, dd_c \log \Lambda(\sigma, \xi) \text{ on } \mathbb{P}_n \setminus \operatorname{supp}(\xi). \tag{2.14}$$

Here $d_c = \sqrt{-1}(\bar{\partial} - \partial)/2$. We set ourselves in the situation of Lemma 1.22, (i). Since f is holomorphic, for any $\xi \in E_f$ and any non-critical value r of Φ, $r > r_0$,

$$E(f, r, r_0) = c \int_{M(r, r_0)} f^* \omega \wedge \omega_M^{m-1} \quad (c = c_m > 0)$$

by (2.14) and Stokes theorem

$$\leqq 2c \int_{\partial M(r)} d_c \log f^* \Lambda(\sigma, \xi) \wedge \omega_M^{m-1}$$

by Cauchy-Schwarz inequality

$$\leqq c_1 \int_{\partial M(r)} |\partial_\sigma \log \Lambda(f(z), \xi)|_\omega \, |\mathbf{e}(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r$$

by (β.3)

$$\leqq c_2 \int_{\partial M(r)} \Lambda(f(z), \xi)\, |\mathbf{e}(\partial\Phi)^* \partial f|_{f^* T\mathbb{P}_n} \omega_r.$$

Hence by (β.2) and Fubini theorem we have

$$\eta\, \mathrm{E}(f, r, r_0) \leqq \int_{\partial \mathrm{M}(r)} \left(\int_{\xi \in \mathrm{E}_f} \Lambda(f(z), \xi)\, \omega^{*n} \right) |\mathbf{e}(\partial \Phi)^* \partial f|_{f^* \mathrm{T}\mathbb{P}_n}\, \omega_r$$

by (β.1)

$$\leqq c_3 \int_{\partial \mathrm{M}(r)} |\mathbf{e}(\partial \Phi)^* \partial f|_{f^* \mathrm{T}\mathbb{P}_n}\, \omega_r.$$

Therefore applying Cauchy-Schwarz inequality to the right hand-side, we have (2.8).

γ) Let f_0 be a non-constant *negative plurisubharmonic* function of class C^2 on M. Then $f = \exp f_0$ is a non-negative plurisubharmonic function of class C^2 on M. Since $f \Delta_{\mathrm{M}} f \leqq 0$ and $0 \leqq f < 1$, we can obtain (2.8) for the function $\mathrm{F}(f, r, r_0)$ similarly. By the integral inequality (1.24), we can show (1.25) for the function $\mathrm{G}(f, r, r_0)$. Using these two integral inequalities, we can attain the same contradiction.

Q.E.D.

Proof of Theorem 2.4. — To show this theorem, we need to modify the way of energy estimate in the first section.

We take a value r_0 of Φ with $r_0 > r_*$ so that $h(r_0) > 2$. First we show the following estimate which is trivial in the one dimensional case (*cf.* Remark 1 in the introduction).

(*i*) *If a non-constant differential map* $f: (\mathrm{M}, ds_{\mathrm{M}}^2) \to (\mathrm{N}, ds_{\mathrm{N}}^2)$ *into a Kähler manifold* $(\mathrm{N}, ds_{\mathrm{N}}^2)$ *satisfies either* (1) *or* (2) *in Theorem* 1.18, (*i*), *then it hold that*

$$r^{2m-2} \exp\left(\chi_f(r) - (2m-2) \int_{r_1}^{r} \frac{dt}{t h(t)} \right) \leqq \mathrm{C}\, \mathrm{E}(f, r, r_0) \tag{2.15}$$

for $\chi_f(r) := \displaystyle\int_{r_1}^{r} \frac{\mathrm{B}(f, t)}{\mathrm{E}(f, t, r_0)}\, dt$, $\mathrm{C} = \mathrm{C}_f > 0$ *and any* $r > r_1 > r_0$ ($c_0 = 2$ *if* $m = 1$ *or* $c_0 = 1$ *if* $m \geqq 2$).

(*ii*) *If f is a non-constant non-negative plurisubharmonic function of class* C^2 *on* M, *then setting* $\mathrm{F}(f, r, r_0) = \displaystyle\int_{\mathrm{M}(r, r_0)} \mathrm{Trace}_{ds_{\mathrm{M}}^2} \partial \bar{\partial} f^2\, dv_{\mathrm{M}}$, *the same estimate* $(2.15)_*$ *as* (2.15) *holds for* $\lambda_f(r) := \displaystyle\int_{r_1}^{r} \frac{\mathrm{B}(f, t)}{\mathrm{F}(f, t, r_0)}\, dt$.

We have only to show the case $m \geqq 2$ and here give only the proof of the case (i), (1) because other cases are proved quite similarly to this case in view of the proof of Lemma 1.22.

Let $f:(\mathrm{M}, ds_{\mathrm{M}}^2) \rightarrow (\mathrm{N}, ds_{\mathrm{N}}^2)$ be a pluriharmonic map into a Kähler manifold $(\mathrm{N}, ds_{\mathrm{N}}^2)$. We set

$$\mathrm{E}_*(f, r, r_0) := \int_{\mathrm{M}(r, r_0)} \left(1 - \frac{1}{h(\Phi)}\right) e(f)\, dv_{\mathrm{M}}$$

for any $r > r_0$

Since $|\partial\Phi|_{\mathrm{M}}^2 = \frac{1}{2}$ on $\mathrm{M} \setminus \{0\}$, by (1.28), (1.29) and (2.5), we have for any $r > r_0$

$$(2m-2)\,\mathrm{E}_*(f, r, r_0) \leqq \frac{r h(r)}{h(r)-1} \frac{\partial}{\partial \mathrm{r}} \mathrm{E}_*(f, r, r_0) - 2r\mathrm{B}(f, r) \tag{2.16}$$

Since $h(r_0) \geqq 2$, we have from (2.16)

$$(2m-2)\left(\frac{1}{r} - \frac{1}{r h(r)}\right) + \frac{\mathrm{B}(f, r)}{\mathrm{E}_*(f, r, r_0)} \leqq \frac{\partial}{\partial r} \log \mathrm{E}_*(f, r, r_0) \tag{2.17}$$

for any $r > r_0$.

Integrating (2.17) and using $\mathrm{E}_*(f, r, r_0) \leqq \mathrm{E}(f, r, r_0)$, we can obtain (2.15).

Next we need the following estimates:

$$\frac{\partial}{\partial r} \mathrm{V}(r) \leqq c_1 r^{2m-1} \exp\left(2m \int_{r_1}^{r} \frac{dt}{t h(t)}\right) \tag{2.18}$$

$$\mathrm{V}(r) \leqq c_2 r \frac{\partial}{\partial r} \mathrm{V}(r) \tag{2.19}$$

By a standard calculation (*cf.* [8], p. 273-274), we have

$$\frac{\partial}{\partial r} \int_{\partial \mathrm{M}(r)} |d\Phi|_{\mathrm{M}}^2 \omega_r = \int_{\partial \mathrm{M}(r)} -\Delta_{\mathrm{M}} \Phi\, \omega_r \tag{2.20}$$

for $\Delta_{\mathrm{M}} = -4 \sum_{i, j=1}^{m} g^{\bar{j}i} \partial_i \partial_{\bar{j}}$ by the Kählerity of ds_{M}^2. Since $|d\Phi|_{\mathrm{M}}^2 \equiv 1$ on $\mathrm{M} \setminus \{0\}$, by the assumption (2.5) and (2.20), we have for any $r > r_0$

$$\frac{\partial^2}{\partial r^2} \mathrm{V}(r) \leqq \frac{1}{r}\left(2m - 1 + \frac{2m}{h(\Phi)}\right) \frac{\partial}{\partial r} \mathrm{V}(r)$$

Hence we have (2.18) by integrating the above inequality. Applying $\varphi = d\Phi^2$ to (1.7), we have for any $r > 0$

$$\int_{\mathrm{M}(r)} -\Delta_{\mathrm{M}} \Phi^2\, dv_{\mathrm{M}} = 2r \int_{\partial \mathrm{M}(r)} |d\Phi|_{\mathrm{M}}^2 \omega_r \tag{2.21}$$

By the assumption (2.5), $-\Delta_M \Phi^2$ is bounded from below. Since $|d\Phi|_M^2 \equiv 1$ on $M \setminus \{0\}$, from (2.21) we have (2.19).

At the present stage, we can begin the proofs of α), β) and γ).

α) Let f be a non-constant *bounded harmonic* function on (M, ds_M^2). Then we can obtain the following inequalities:

$$E(f, r, r_0)^2 \leqq c_3 \frac{\partial}{\partial r} V(r) B(f, r) \tag{2.22}$$

$$\frac{E(f, r, r_0)}{r^{2m-2}} \leqq c_4 n(M, 2r) \tag{2.23}$$

for any $r > 2r_0$.

Here $n(M, r) := V(r)/r^{2m}$. The inequality (2.22) is nothing but (2.8). Hence we have only to show (2.23). Since Φ is a uniformly Lipschitz continuous exhaustion function on M, there exists a smooth function σ_r on M such that $\sigma_r \equiv 1$ on $M(r)$, $\sigma_r \equiv 0$ on $M \setminus M(2r)$ and $|d\sigma|_M \leqq c_5/r$, where c_5 is a positive constant not depending on r (*cf.* [13] Lemma 1). Using σ_r, we have the following inequality (*cf.* [13], Proofs of Theorems 2.1 and 2.2, (2.7))

$$(df, df)_{M(r)} \leqq \frac{c_6}{r^2} \int_{M(2r)} |f|^2 \, dv_M \quad \text{for any } r > 0 \tag{2.24}$$

Since $|f|$ is bounded, we have (2.23) from (2.24).

From (2.6), (2.15), (2.18) and (2.22), we have

$$\frac{c_7 e^{\chi_f(r)}}{r g_n(r)} \leqq \frac{B(f, r)}{E(f, r, r_0)} \quad \text{for any } r > r_1$$

Hence we have

$$c_7 \int_{r_1}^{r} \frac{e^{\chi_f(t)}}{t g_n(t)} dt \leqq \chi_f(r) \quad \text{for any } r > r_1 \tag{2.25}$$

From (2.25), we can obtain the following assertion inductively.

There exist positive constants $\{C_{(k)}\}_{0 \leqq k \leqq n}$ *and real numbers* $\{r_{(k)}\}_{0 \leqq k \leqq n}$, $r_{(k)} < r_{(k+1)}$ *and* $r_{(0)} = r_1$ *such that*

$$C_{(k)} \int_{r(k)}^{r} \frac{dt}{t g_{n-k}(t)} \leqq \chi_f(r)$$

for any $r > r_{(k)}$ *and* $0 \leqq k \leqq n$.

Finally we obtain

$$C_{(n)} \log r \leqq \chi_f(r) + O(r) \text{ for any } r > r_{(n)} \tag{2.26}$$

On the other hand, from (2.6), (2.15), (2.18), (2.19) and (2.23), we have

$$\chi_f(r) \leqq c_8 \log g_n(r) + \mathrm{O}(r) \text{ for any } r > r_1 \tag{2.27}$$

From (2.26) and (2.27), we obtain a contradiction.

β) We set ourselves in the situation of the proof of Theorem 2.1, β). In account of the proofs of Theorem 2.1, α) and Theorem 2.4, α), we have only to show the estimate (2.23) for the holomorphic map $f: \mathrm{M} \to \mathbb{P}_n$ in the proof of Theorem 2.1, β).

By $h(r_0) \geqq 0$ and (2.5), Φ is subharmonic on $\mathrm{M} \setminus \mathrm{M}(r_0)$. Hence for any $\xi \in \mathrm{E}_f$ and any $r > r_0$, we have

$$\begin{aligned}\int_{r_0}^{r} \mathrm{E}(f, r, r_0)\, dt &= c_m \int_{r_0}^{r} dt \int_{\mathrm{M}(t, r_0)} dd_c \log f^* \Lambda(\sigma, \xi) \wedge \omega_{\mathrm{M}}^{m-1} \\ &\leqq c_m \int_{\mathrm{M}(r, r_0)} d\Phi \wedge d_c \log f^* \Lambda(\sigma, \xi) \wedge \omega_{\mathrm{M}}^{m-1} \\ &= c_m \int_{\mathrm{M}(r, r_0)} d \log f^* \Lambda(\sigma, \xi) \wedge d_c \Phi \wedge \omega_{\mathrm{M}}^{m-1} \\ &\leqq c_1 \int_{\partial \mathrm{M}(r)} \log f^* \Lambda(\sigma, \xi) \, |\partial \Phi|_{\mathrm{M}}^2 \, \omega_r\end{aligned}$$

The last step is done by Stokes theorem and using the subharmonicity of Φ on $\mathrm{M}(r, r_0)$. Using (β.1) and (β.2), we have

$$\int_{r_0}^{r} \mathrm{E}(f, t, r_0)\, dt \leqq c_2 \int_{\mathrm{M}(r)} |\partial \Phi|_{\mathrm{M}}^2 \, \omega_r \tag{2.28}$$

for any $r > r_0$.

Since $-\Delta_{\mathrm{M}} \Phi^2$ is bounded from above by (2.5), from (2.21) and (2.28), we can obtain (2.23) for $f: \mathrm{M} \to \mathbb{P}_n$.

γ) We set ourselves in the situation of the proof of Theorem 2.1, γ).

Let f be as in the proof of Theorem 2.1, γ). In view of the estimate (2.15) and the proof of α), we have only to show the same inequalities $(2.22)_*$ and $(2.23)_*$ as (2.22) and (2.23) for $\mathrm{F}(f, r, r_0)$ respectively. Since $(2.22)_*$ is nothing but (2.8), we have only to show $(2.23)_*$. Using σ_r and integration by parts, we have

$$(\sigma_r^2, -\Box_{\mathrm{M}} f^2) = (\bar\partial \sigma_r^2, -\bar\partial f^2) \leqq \frac{c_1}{r} \sqrt{\mathrm{V}(2r)(\sigma_r^2, -\Box_{\mathrm{M}} f^2)}$$

Hence we have (2.23) because $-\Box_{\mathrm{M}} f^2 = 2 \operatorname{Trace}_{ds_{\mathrm{M}}^2} \partial \bar\partial f^2$.

Q.E.D.

Proof of Theorem 2. — Since $n(\mathrm{A}, r) = \mathrm{V}(\mathrm{A}(r))/r^{2m}$ ($\mu = 2m - 2$) is a continuous non-decreasing function and every negative plurisubharmonic function on a Stein manifold can be approximated by a smooth one, Theorem 2 follows from Theorem 2.1 immediately.

Proof of Theorem 3. — To prove this theorem, we should estimate the eigenvalues of the Levi form of $\Psi=\Phi^2$ relative to ds_M^2 by using Hessian comparison theorem.

(i) We put $\eta=e^4$ and fix a positive number ε_* with $0<8\varepsilon_*<\dfrac{1}{(4m-2)(\eta+1)}$. We set $\varepsilon=8\varepsilon_1$ for some constant ε_1 with $0<\varepsilon_1\leqq\varepsilon_*$. We consider a C^∞ function f_1: $[0,\infty)\to(0,\infty)$ defined by

$$k_1(r)=\frac{\varepsilon}{8(r+\eta)^2\log(r+\eta)}$$

We assume

$$(2.29)\qquad |\textit{radial curvature at } x\in M,\ \Phi(x)=r|\leqq k_1(r)$$

for any $r\geqq 0$.

Next we consider a C^∞ function $k_2:[0,\infty)\to(0,\infty)$ defined by

$$k_2(r)=\frac{\varepsilon}{2(r+\eta)^2\log(r+\eta)}\left(1-\frac{1}{\log(r+\eta)}\right)$$

We consider the solutions f_1 and f_2 of the following Jacobi equations:

$$f_1''(r)=-k_1(r)f_1(r), f_1(0)=0 \text{ and } f_1'(0)=1$$
$$f_2''(r)=k_2(r)f_2(r),\quad f_2(0)=0 \text{ and } f_2'(0)=1$$

Then the solutions f_1 and f_2 satisfy the following properties respectively

$$(2.30)\qquad f_1(r)>0 \quad\text{and}\quad f_1'(r)>0 \quad\text{for } r>0$$

$$(2.31)\qquad f_2(r)>0 \quad\text{and}\quad f_2'(r)>0 \quad\text{for } r>0$$

The property (2.31) follows from [9], Proposition 4.2. We should show (2.30).

We consider a C^∞ function $f_3:[0,\infty)\to[0,\infty)$ defined by

$$f_3(r)=r(\log(r+\eta))^{-\varepsilon}\quad\text{for } r\geqq 0$$

Then it holds that $f_1(0)=f_3(0)=0$, $f_3'(0)<f_1'(0)$ and $f_3''(r)/f_3(r)<f_1''(r)/f_1(r)$ for $r>0$. Hence we have $f_1(r)>f_3(r)>0$ for $r>0$ and moreover

$$(2.32)\qquad 0<f_3'(r)<f_1'(r)\quad\text{for } r>0$$

Hence we have (2.30).

Let $(M_i, ds_{M_i}^2)$ be a $2m$ dimensional *model* whose radial curvature function is k_i (*cf.* [9] Proposition 4.2) and let r_i be the distance function of M_i from some fixed point in $M_i (i=1, 2)$. By (2.29) and $-k_2\leqq -k_1$, we obtain the following assertion from *Hessian comparison theorem* concerning r_i and Ψ (*cf.* [9] Theorem A, Lemma 1.13, Proposition

2.20 and [30]).

$$\frac{rf'_1(r)}{f_1(r)} \leqq \mathscr{L}(\Psi)(v, \bar{v}) \leqq \frac{rf'_2(r)}{f_2(r)} \tag{2.33}$$

for any $v \in \mathrm{TM}_x^{1,0}$, $\Phi(x)=r>0$ and $|v|_{\mathrm{M}}=1$. Using (2.33), we shall show the following inequality:

$$-\frac{\varepsilon}{\log(r+\eta)} < \mathscr{L}(\Psi)(v, \bar{v}) - 1 < \frac{\varepsilon}{\log(r+\eta)} \tag{2.34}$$

for any $v \in \mathrm{TM}_x^{1,0}$, $\Phi(x)=r>0$ and $|v|_{\mathrm{M}}=1$. If (2.34) was proved, then setting $h(r)=\log(r+\eta)/\varepsilon$, the conditions (2.5) and (2.6) of Theorem 2.4 are verified.

Since $f_1(r) \leqq r$, setting $\varphi_1(r)=f_1(r)/f'_1(r)$, we have from (2.32)

$$\varphi_1(r) < 2(\log(r+\eta))^{\varepsilon} r \quad \text{for } r>0 \tag{2.35}$$

By (2.35) and $\varphi'_1(r)=1+k_1(r)\varphi_1(r)^2$, we have

$$r \leqq \varphi_1(r) \leqq r + \frac{\varepsilon}{2}\mathrm{I}_1(r) \quad \text{for } r>0$$

Here $\mathrm{I}_1(r)=\int_0^r (\log(t+\eta))^{2\varepsilon-1}\,dt$. Since $\mathrm{I}_1(r) \leqq (4/3)r$ for $r \geqq 0$, we have

$$r \leqq \varphi_1(r) \leqq c_1 r \quad \text{for } r>0 \tag{2.36}$$

Here $c_1=1+(2/3)\varepsilon$. Again by $\varphi'_1(r)=1+k_1(r)\varphi_1(r)^2$ and (2.36), we have

$$\varphi_1(r) \leqq r + \frac{\varepsilon c_1^2}{8}\mathrm{I}_2(r) \quad \text{for } r>0$$

Here $\mathrm{I}_2(r)=\int_0^r (\log(t+\eta))^{-1}\,dt$. Since $\mathrm{I}_2(r) \leqq 4r/3\log(r+\eta)$, we have

$$\varphi_1(r) \leqq r + \frac{\varepsilon c_1^2 r}{6\log(r+\eta)} \quad \text{for } r>0$$

Hence we have

$$r \leqq \varphi_1(r) \leqq r + \frac{\varepsilon r}{\log(r+\eta)} \quad \text{for any } r \geqq 0$$

Finally we have

$$1-\frac{rf_1'(r)}{f_1(r)}<\frac{\varepsilon}{\log(r+\eta)} \quad \text{for } r>0$$

By (2.33), this means the left hand-side of (2.34).

Next we show the right hand-side of (2.34). Setting $\varphi_2(r)=f_2(r)/f_2'(r)$, we have $\varphi_2'(r)=1-k_2(r)\varphi_2(r)^2$. So we have

$$r\geqq\varphi_2(r)\geqq r-\frac{\varepsilon r}{2\log(r+\eta)} \text{ for } \quad r\geqq 0.$$

Hence we have

$$\frac{rf_2'(r)}{f_2(r)}-1<\frac{\varepsilon}{\log(r+\eta)} \quad \text{for } r>0$$

Therefore the proof of (2.34) is complete.

(ii) We fix a number $r_1>\max(r_0, e^2)$. For some fixed positive constant ε, $0<\varepsilon<1/12(2m-1)$, we consider the following C^∞ function $k_2:[0,\infty)\to[0,\infty)$ defined by

$$k_2(r)=\frac{2\varepsilon}{r^2\log r}\left(1-\frac{1}{\log r}\right) \quad \text{for } r>r_1$$

and assume

$0\geqq$ *radial curvature* on $\partial M(r)\geqq -k_2(r)$ for any $r\geqq 0$.

We consider the solutions f_1 and f_2 of the following Jacobi equations.

$$f_1''(r)\equiv 0, f_1(0)=0 \text{ and } f_1''(0)=1$$
$$f_2''(r)=k_2(r)f_2(r), f_2(0)=0 \text{ and } f_2''(0)=1.$$

Here we consider $k_1(r)\equiv 0$ because the radial curvature of ds_M^2 is non-positive on M. Clearly $f_1(r)\equiv r$, $f_2(r)>0$ and $f_2'(r)>0$ for $r>0$. Since $-k_2(r)\leqq -\varepsilon/r^2\log r$ for $r>r_1$; by the same procedure as (i), we have only to estimate $\dfrac{rf_2'(r)}{f_2(r)}-1$.

Setting $\varphi_2(r)=f_2(r)/f_2'(r)$, $r\geqq 0$, we have

$$r\geqq\varphi_2(r)\geqq r-\frac{2\varepsilon r}{\log r}-c_2 \quad \text{for } r>r_1$$

Here $c_2=r_1-\varphi_2(r_1)\geqq 0$. We take a number $r_2>r_1$ so that

$$c_2\frac{\log r}{r}<\varepsilon<\frac{\log r}{6}$$

for $r > r_2$. Hence we have for $r > r_2$

$$\frac{r f_2'(r)}{f_2(r)} - 1 < \frac{6\varepsilon}{\log r}$$

Hence setting $h(r) = \log r / 6\varepsilon$, the conditions (2.5) and (2.6) of Theorem 2.4 are verified.

Q.E.D.

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Kensho TAKEGOSHI,
Department of Mathematics,
College of General Education,
Osaka University, Toyonaka,
Osaka 560, Japan.