ANNALES SCIENTIFIQUES DE L'É.N.S.

F. TRICERRI
L. VANHECKE

Curvature homogeneous riemannian manifolds

Annales scientifiques de l'É.N.S. 4e série, tome 22, nº 4 (1989), p. 535-554
<http://www.numdam.org/item?id=ASENS_1989_4_22_4_535_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 22, 1989, p. 535 à 554.

CURVATURE HOMOGENEOUS RIEMANNIAN MANIFOLDS ( )

BY F. TRICERRI AND L. VANHECKE

1. Introduction

Let (M, g) be a Riemannian manifold and denote by R its Riemannian curvature tensor. We suppose that (M, g) is of class C^∞ and, if necessary, that (M, g) is connected. The Riemannian manifold is said to be *curvature homogeneous* if for all points p and q of (M, g) there exists an isometry f of the tangent space $T_p M$ on the tangent space $T_q M$ which preserves the curvature tensor, that is $f^* R_q = R_p$ ([17], [6], p. 109).

Of course, a (locally) homogeneous Riemannian manifold is curvature homogeneous but the converse is only true in special cases. (*See* Section 5 and [5], [19] for counterexamples.) For example, let $(N, \bar{g})$ be an irreducible symmetric space and let (M, g) be a curvature homogeneous space whose Riemannian curvature tensor is the same as that of $(N, \bar{g})$. It is proved in [21] that in this case (M, g) is locally symmetric and hence locally isometric to the so-called model space $(N, \bar{g})$. We note that this result will also follow from a more general theorem which will be proved in Section 3 and it can be considered as a particular case of a theorem of Z. I. Szabó [18], Theorem 4.3. (*See* also the remarks at the end of section 4.)

This property shows that, up to local isometry, there is a unique curvature homogeneous Riemannian manifold whose Riemannian curvature tensor is that of an irreducible symmetric space. In Section 4 we will show that this result still holds when the model space $(N, \bar{g})$ is a *reducible* symmetric space with a de Rham decomposition without Euclidean factor. This means that the curvature tensor R has vanishing *index of nullity*. In what follows we will call the set of the germs of Riemannian metrics which are locally isometric, an *isometry class*. Then our considerations lead to the following

(*) This work was partially supported by contract Nr. 86.02130.01, C.N.R., Italy.

problem, stated by M. Gromov:

PROBLEM 1. — *Do the isometry classes of the germs of Riemannian metrics which have the same Riemann curvature tensor as that of a given homogeneous Riemannian manifold depend on a finite number of parameters?*

Of course, there are examples of curvature homogeneous manifolds which are not locally isometric and whose curvature tensor is that of a homogeneous space. Such examples are given in the work of Ferus, Karcher and Münzner about isoparametric hypersurfaces [5] and for these examples there are only a finite number of isometry classes.

In Section 5 we shall study an *infinite* family of isometry classes of irreducible complete Riemannian metrics on $\mathbb{R}^3$ which are curvature homogeneous but not locally homogeneous. For these examples, the curvature tensor R is that of $\mathbb{R} \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the hyperbolic plane of constant curvature -1.

These examples have been introduced by K. Sekigawa [15]. Their isometry classes are in one-to-one correspondence with $\mathbb{R}^2/\mathbb{Z}_2$; so they depend on two parameters. Nevertheless, the metrics of Sekigawa are not the only metrics on $\mathbb{R}^3$ which have the same curvature tensor as $\mathbb{R} \times \mathbb{H}^2$. Indeed, O. Kowalski and the two authors constructed in [12] some non-homogeneous metrics on $\mathbb{R}^3$ which have the same curvature tensor as $\mathbb{R} \times \mathbb{H}^2$ and whose isometry classes depend on two arbitrary functions. So we get a negative answer to Problem 1, but we are unable to give an answer to:

PROBLEM 2. — *Do the isometry classes of the germs of Riemannian metrics which have the same Riemann curvature tensor as that of a given "irreducible" homogeneous Riemannian manifold depend on a finite number of parameters?*

Beside these local problems it is also possible to consider a global one. In fact, let M be a *compact* manifold of dimension n and $\mathscr{M}_0(\mathrm{M})$ the set of Riemannian metrics on M which have the same curvature tensor as a homogeneous space $(\mathrm{N}, \bar{g})$. The diffeomorphism group $\mathrm{Diff}(\mathrm{M})$ of M acts naturally on $\mathscr{M}_0(\mathrm{M})$. So we can consider the *moduli space* $\mathscr{M}_0(\mathrm{M})/\mathrm{Diff}(\mathrm{M})$ and state the following.

CONJECTURE OF GROMOV (see [2]). — *The moduli space* $\mathscr{M}_0(\mathrm{M})/\mathrm{Diff}(\mathrm{M})$ *is finite dimensional.*

For example, this conjecture is true if the model space $(\mathrm{N}, \bar{g})$ is a symmetric space of *non-positive* sectional curvature and, when $\dim \mathrm{M} \geqq 3$, its de Rham decomposition does not contain any Euclidean factor nor any factor isometric to the hyperbolic plane $\mathbb{H}^2$. (Compare with the examples of section 5.) In this case, the index of nullity of $(\mathrm{N}, \bar{g})$ is zero. Therefore, all the metrics of $\mathscr{M}_0(\mathrm{M})$ are locally isometric to $\bar{g}$ and locally symmetric. If $\dim \mathrm{M}=2$, they have the same constant negative Gauss curvature and the result follows from the classical theory about the moduli space on a Riemann surface. If $\dim \mathrm{M} \geqq 3$, the *Mostow rigidity theorem* [13] applies and $\mathscr{M}_0(\mathrm{M})/\mathrm{Diff}(\mathrm{M})$ reduces to a point. [Recall that all the metrics of $\mathscr{M}_0(\mathrm{M})$ have the same curvature tensor and this removes any indetermination.]

Motivated by these problems we will study in Section 2 and Section 3 the case where the so-called model space $(\mathrm{N}, \bar{g})$ is a general homogeneous Riemannian manifold. Contrary to the case of symmetric spaces, isometry classes of such manifolds $(N, \bar{g})$ are not determined by the Riemannian curvature tensor but we now need the curvature and the torsion tensor of a particular invariant connection, namely the canonical connection of a reductive homogeneous space. In what follows we call this the *Ambrose-Singer connection* ([1], [8], [20]). As is well-known, this connection is not uniquely determined but it depends on the group G of isometries acting transitively on the manifold and also on the reductive decomposition of the Lie algebra of G. On the other hand, all these connections determine the same (M, g). (We refer to [20] for more details.) Note that for a symmetric Riemannian space, the Riemannian connection D is an Ambrose-Singer connection.

In Section 2 we give a brief survey about the curvature and torsion of an Ambrose-Singer connection. In particular we consider the notion of an *infinitesimal model* of a Riemannian homogeneous space. This notion is implicitly contained in [14] and may be seen as a generalization of that of a *holonomy system* introduced by J. Simons in [16]. (*See* also [3].) Infinitesimal models have also been used by O. Kowalski in his book on generalized symmetric spaces [9].

In Section 3 we consider Riemannian manifolds equipped with a metric connection ∇ whose curvature tensor R_∇ and torsion tensor T_∇ are the same as those of a given infinitesimal model $\mathfrak{m}$. When $\mathfrak{m}$ is supposed to be naturally reductive we obtain that T_∇ is parallel with respect to ∇. Moreover, when $\mathfrak{m}$ is in addition Einsteinian, R_∇ is also parallel and hence, for naturally reductive Einstein models $\mathfrak{m}$, ∇ is an Ambrose-Singer connection. Using standard results, we obtain from this that the Riemannian manifolds with such a model are locally homogeneous and locally isometric to that infinitesimal model. Note that there is no scarcity of examples of naturally reductive homogeneous spaces (*see* for example [4]). All normal homogeneous Riemannian manifolds are naturally reductive ([8], [22]) and all isotropy irreducible homogeneous Riemannian manifolds have the same property [23]. (For further references and examples we refer to [11], [20].)

We return to symmetric infinitesimal models in Section 4 to prove the already mentioned extension of the result proved in [21].

We are grateful to M. Gromov for his interest in this work and for several useful discussions.

2. Ambrose-Singer connections

Let (M, g) be a Riemannian homogeneous space. As is well-known, there exists on (M, g) a metric connection ∇ such that its curvature tensor R_∇ and its torsion tensor T_∇ are parallel with respect to ∇. In what follows, a connection on a Riemannian manifold which satisfies these three properties will be called an *Ambrose-Singer connection.* This is motivated by:

THEOREM 2.1 [1]. — *Let* (M, g) *be a connected, simply connected and complete Riemannian manifold equipped with an Ambrose-Singer connection. Then* (M, g) *is homogeneous.*

(*See* also [20].)

The curvature and torsion tensor of an Ambrose-Singer connection satisfy a collection of algebraic identities which we recall first:

$$(R_\nabla)_{XY} = -(R_\nabla)_{YX}, \tag{2.1}$$

$$(T_\nabla)_X Y = -(T_\nabla)_Y X, \tag{2.2}$$

$$(R_\nabla)_{XY} \cdot g = 0, \tag{2.3}$$

$$(R_\nabla)_{XY} \cdot T_\nabla = 0, \tag{2.4}$$

$$(R_\nabla)_{XY} \cdot R_\nabla = 0, \tag{2.5}$$

$$\sum_{X, Y, Z} \{ (R_\nabla)_{XY} Z + (T_\nabla)_{(T_\nabla)_X Y} Z \} = 0, \tag{2.6}$$

$$\sum_{X, Y, Z} \{ (R_\nabla)_{(T_\nabla)_X YZ} \} = 0. \tag{2.7}$$

Here X, Y, Z are C^∞ vector fields on (M, g) and $\sum$ denotes the cyclic sum. Moreover, R_∇ and T_∇ are defined by:

$$(R_\nabla)_{XY} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y] \tag{2.8}$$

and

$$(T_\nabla)_X Y = \nabla_X Y - \nabla_Y X - [X, Y]. \tag{2.9}$$

Further, in (2.3), (2.4) and (2.5), $(R_\nabla)_{XY}$ acts as a derivation on the tensor algebra. Hence, these identities may be rewritten as follows:

$$g((R_\nabla)_{XY} Z, W) + g(Z, (R_\nabla)_{XY} W) = 0, \tag{2.10}$$

$$[(R_\nabla)_{XY}, (T_\nabla)_Z] - (T_\nabla)_{(R_\nabla)_{XY} Z} = 0, \tag{2.11}$$

$$[(R_\nabla)_{XY}, (R_\nabla)_{ZW}] - (R_\nabla)_{(R_\nabla)_{XY} ZW} - (R_\nabla)_{Z\,(R_\nabla)_{XY} W} = 0. \tag{2.12}$$

The identities follow at once from the fact that g, R_∇ and T_∇ are parallel with respect to ∇ and from the *Ricci identity*

$$\nabla^2_{XY} - \nabla^2_{YX} = -(R_\nabla)_{XY} - \nabla_{(T_\nabla)_X Y}. \tag{2.13}$$

(2.6) is an immediate consequence of the *first Bianchi identity*

$$\sum_{X, Y, Z} \{ (R_\nabla)_{XY} Z + (T_\nabla)_{(T_\nabla)_X Y} Z + (\nabla_X (T_\nabla))_Y Z \} = 0. \tag{2.14}$$

Further, (2.7) follows at once from the *second Bianchi identity*

$$\sum_{X, Y, Z} \{ \nabla_X (R_\nabla)_{YZ} + (R_\nabla)_{(T_\nabla)_X YZ} \} = 0. \tag{2.15}$$

Next we consider the notion of an infinitesimal model. Let $(V, \langle , \rangle)$ be a Euclidean vector space of dimension n. Let

$$T: \quad V \to \operatorname{Hom}(V, V) : X \mapsto T_X$$

and

$$R: \quad V \times V \to \operatorname{Hom}(V, V) : (X, Y) \mapsto R_{XY}$$

be tensors of type (1,2) and (1,3) on V. Then we state

DEFINITION 2.1. — (T, R) is said to be an *infinitesimal model* (of a homogeneous Riemannian manifold) *on* V if T and R satisfy the identities (2.1)-(2.7).

Two infinitesimal models, (T, R) on $(V, \langle , \rangle)$ and (T', R') on $(V', \langle , \rangle')$, are said to be *isomorphic* if there exists an isometry a: $(V, \langle , \rangle) \to (V', \langle , \rangle')$ such that $aT = T'$ and $aR = R'$, that is

$$(2.16) \qquad (aT)_{X'} = a \circ T_{a^{-1}X'} \circ a^{-1},$$

$$(2.17) \qquad (aR)_{X'Y'} = a \circ R_{a^{-1}X'\,a^{-1}Y'} \circ a^{-1}.$$

It is clear that when (M, g) is a homogeneous Riemannian manifold, ∇ an Ambrose-Singer connection on it and 0 an arbitrary point of M, then $T = T_{\nabla|0}$ and $R = R_{\nabla|0}$ determine an infinitesimal model on $(V = T_0 M, \langle , \rangle = g|_0)$. Since (M, g) is (locally) homogeneous, the choice of another point 0 gives an isomorphic infinitesimal model.

Conversely, let (T, R) be an infinitesimal model on $(V, \langle , \rangle)$. Following Nomizu [14] (*see* also [9], [10]), let $\mathfrak{h}$ be the Lie subalgebra of the Lie algebra $\mathfrak{so}(V)$ of skew-symmetric endomorphisms of V defined by

$$(2.18) \qquad \mathfrak{h} = \{ A \in \mathfrak{so}(V) \mid A \cdot T = 0, A \cdot R = 0 \}.$$

It follows from (2.3), (2.4) and (2.5) that $R_{XY} \in \mathfrak{h}$. Further, let $\mathfrak{g}$ be the direct sum of V and $\mathfrak{h}$ and put

$$(2.19) \qquad [X, Y] = -T_X Y - R_X Y,$$

$$(2.20) \qquad [A, X] = A(X),$$

$$(2.21) \qquad [A, B] = A \circ B - B \circ A$$

for all $X, Y \in V$ and $A, B \in \mathfrak{h}$. Then, the identities (2.1)-(2.7) yield that $\mathfrak{g}$, with this bracket, becomes a Lie algebra. Now, let G be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$ and let H be the connected subgroup corresponding to $\mathfrak{h}$. When H is *closed*, then $M = G/H$ is a smooth manifold and the inner product $\langle , \rangle$ extends to a G-invariant Riemannian metric g on M. The canonical connection associated with the reductive decomposition $\mathfrak{g} = V \oplus \mathfrak{h}$ is an Ambrose-Singer connection with curvature tensor R and torsion tensor T at the origin.

These considerations show that the study of Riemannian homogeneous spaces G/H is equivalent to the study of a class of infinitesimal models.

In what follows we denote by $\mathfrak{m}(V)$ the set of infinitesimal models on $(V, \langle , \rangle)$. It is the subset of $(\overset{3}{\otimes} V^*) \oplus (\overset{4}{\otimes} V^*)$ determined by the algebraic conditions (2.1)-(2.7). Note that we do not make a distinction between covariant and contravariant notation. We pass from the one to the other by using the inner product $\langle , \rangle$.

In [20] we introduced the notion of a *homogeneous Riemannian structure* and gave a classification of these structures. A homogeneous structure on a Riemannian manifold is a tensor field S determined by

$$S = D - \nabla$$

where D denotes the Riemannian connection and ∇ an Ambrose-Singer connection. Since ∇ is metric, T_∇ determines S completely by

$$2g(S_X Y, Z) = -g(T_\nabla)_X Y, Z) + g((T_\nabla)_Y Z, X) - g((T_\nabla)_Z X, Y). \tag{2.22}$$

Conversely, we have

$$(T_\nabla)_X Y = S_Y X - S_X Y.$$

In this context, we say that an *infinitesimal model* (T, R) is *of type* $\mathscr{J}$ when the tensor S determined by

$$2g(S_X Y, Z) = -T_{XYZ} + T_{YZX} - T_{ZXY}, \tag{2.23}$$

where $T_{XYZ} = \langle T_X Y, Z \rangle$, is of type $\mathscr{J}$ according to [20], p. 40. For example, we say that the model (T, R) is of type $\mathscr{T}_3$ or is a *naturally reductive model* if

$$S_X X = 0 \quad \text{or, equivalently,} \quad T_{YXX} = 0$$

for all X, $Y \in V$. It is clear that the *symmetric infinitesimal models* are obtained by putting $T = 0$.

We note that in this last case Nomizu's construction is the classical construction of E. Cartan of a symmetric space by using the curvature tensor. Moreover, in this case, the subgroup H of G, corresponding to the Lie algebra determined by (2.18), is always closed and compact (*see* [7], p. 218-223). Hence, in this case there is a one-one correspondence between the symmetric spaces and the symmetric infinitesimal models.

3. Naturally reductive homogeneous spaces

Let OM be the bundle of orthonormal frames of (M, g). A point $u = (q; u_1, \ldots, u_n)$ of OM determines an isometry between $V = \mathbb{R}^n$, equipped with the standard inner product, and $(T_q M, g_{|q})$ by

$$u(\xi) = u(\xi^1, \ldots, \xi^n) = \sum_{i=1}^{n} \xi^i u_i. \tag{3.1}$$

Further, let ∇ be a metric connection on M. Then its torsion and its curvature tensor define a map Φ of OM in $(\overset{3}{\otimes} V^*) \oplus (\overset{4}{\otimes} V^*)$ given by

$$\Phi(u) = (T_\nabla(u), R_\nabla(u)) \tag{3.2}$$

where

$$T_\nabla(u)_{\xi_1 \xi_2 \xi_3} = (T_{\nabla \mid q})_{u\xi_1\, u\xi_2\, u\xi_3}, \tag{3.3}$$

$$R_\nabla(u)_{\xi_1 \xi_2 \xi_3 \xi_4} = (R_{\nabla \mid q})_{u\xi_1\, u\xi_2\, u\xi_3\, u\xi_4}, \tag{3.4}$$

and $\xi_1, \xi_2, \xi_3, \xi_4 \in V$.

The orthogonal group acts on the right on OM and also on $(\overset{3}{\otimes} V^*) \oplus (\overset{4}{\otimes} V^*)$. This last action is given by

$$(T, R)\, a = (T\, a, R\, a) \tag{3.5}$$

where

$$(T\, a)_{\xi_1 \xi_2 \xi_3} = T_{a\xi_1\, a\xi_2\, a\xi_3}, \tag{3.6}$$

$$(R\, a)_{\xi_1 \xi_2 \xi_3 \xi_4} = R_{a\xi_1\, a\xi_2\, a\xi_3\, a\xi_4} \tag{3.7}$$

and $\xi_1, \ldots, \xi_4 \in V$, $a \in O(n)$.

Further, the set $\mathfrak{m}(V)$ of infinitesimal models on V is *invariant* under this action of the orthogonal group and moreover, since $(ua)\xi = u(a\xi)$, we have

$$\Phi(ua) = \Phi(u)\, a. \tag{3.8}$$

Hence Φ is *equivariant*.

DEFINITION 3.1. — Let $\mathfrak{m} = (T, R)$ be an infinitesimal model on $V = \mathbb{R}^n$ corresponding to a homogeneous Riemannian manifold (M^0, g^0). We say that a *metric connection* ∇ *on* M *has the torsion and the curvature of* $\mathfrak{m}$ if $\Phi(OM)$ is contained in the orbit of $\mathfrak{m}$ under the action of the orthogonal group.

The definition above is equivalent to the following fact: for all points p of M, there exists an isometry a_p of $V = \mathbb{R}^n$ on T_p M such that

$$(T_{\nabla \mid p})\, a_p = T, \qquad (R_{\nabla \mid p})\, a_p = R.$$

Or, equivalently, there exists an orthonormal frame $(u_1, \ldots, u_n)$ of T_p M such that

$$(T_\nabla)_{ijh} = (T_\nabla)_{u_i u_j u_h} = T_{ijh},$$

$$(R_\nabla)_{ijh} = (R_\nabla)_{u_i u_j u_h u_k} = R_{ijhk},$$

where T_{ijh} and R_{ijhk} are the components of T and R with respect to the natural frame $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$ of $V = \mathbb{R}^n$.

Using these remarks we obtain:

PROPOSITION 3.2. — *Let* ∇ *be a metric connection on* (M, g) *which has the torsion and the curvature of an infinitesimal model* $\mathfrak{m}=(T, R)$. *Then*

(i) T_∇ *and* R_∇ *satisfy the identities* (2.1)-(2.7);

(ii) $\sum_{X, Y, Z} (\nabla_X(T_\nabla))_Y Z=0$;

(iii) $\sum_{X, Y, Z} (\nabla_X(R_\nabla))_{YZ}=0$;

(iv) $\|T_\nabla\|=\|T\|$ *and* $\|R_\nabla\|=\|R\|$ *are constant on* M.

Now, let $\mathfrak{m}$ be a naturally reductive infinitesimal model, that is $T_{XYZ}=\langle T_X Y, Z\rangle$ is skew-symmetric. Hence, if ∇ has the torsion and the curvature of $\mathfrak{m}$,

$$(T_\nabla)_{XYZ}=g((T_\nabla)_X Y, Z)$$

is a three-form on (M, g). With this, we are ready to prove

LEMMA 3.3. — *Let* ∇ *be a metric connection on* (M, g) *which has the torsion and curvature of a naturally reductive infinitesimal model* $\mathfrak{m}=(T, R)$. *Then* T_∇ *is parallel with respect to* ∇.

Proof. — To simplify the notation we put $T'=T_\nabla$. First, from Proposition 3.2, we get that $\|T'\|$ is constant. Hence

$$0=\sum_m \nabla^2_{mm}\|T'\|^2=2\sum_{m, i, j, h}(\nabla^2_{mm}T')_{ijh}T'_{ijh}+2\|\nabla T'\|^2.$$

Next, using (ii) of Proposition 3.2, we obtain

$$\|\nabla T'\|^2=\sum_{m, i, j, h}(\nabla^2_{mi}T')_{jmh}T'_{ijh}+\sum_{m, i, j, h}(\nabla^2_{mj}T')_{mih}T'_{ijh}$$

or, since T' is a three-form,

$$\|\nabla T'\|^2=2\sum(\nabla^2_{mi}T')_{jmh}T'_{ijh}.$$

Now we use the Ricci identity and (2.4) to obtain

$$\begin{aligned}\|\nabla T'\|^2&=2\sum_{m, i, j, h}(\nabla^2_{im}T')_{jmh}T'_{ijh}-2\sum_{m, i, j, h, p}(\nabla_p T')_{jmh}T'_{mip}T'_{ijh}\\&=2\sum_{m, i, j, h}(\nabla^2_{im}T')_{hjm}T'_{ijh}-2\sum_{m, i, j, h, p}(\nabla_p T')_{hjm}T'_{mip}T'_{ijh}.\end{aligned}$$

Using again (ii) of Proposition 3.2 and the skew-symmetry of T′ we get

$$\|\nabla T'\|^2 = -2 \sum_{m,i,j,h,p} (\nabla_p T')_{hjm} T'_{mip} T'_{ijh}$$
$$= 2 \sum_{m,i,j,h,p} (\nabla_h T')_{jpm} T'_{mip} T'_{ijh} + 2 \sum_{m,i,j,h,p} (\nabla_j T')_{phm} T'_{mip} T'_{ijh}$$
$$= 4 \sum_{m,i,j,h,p} (\nabla_h T')_{jpm} T'_{mip} T'_{ijh}$$
$$= 4 \sum_{m,i,j,h,p} (\nabla_p T')_{mhj} T'_{jih} T'_{imp} = -2\|\nabla T'\|^2.$$

This yields the required result.

Corollary 3.4. — *With the same hypotheses as in Lemma* 3.3 *we have the following relation between* R_∇ *and the Riemann curvature tensor* R *of the Riemannian connection* D:

$$(R_\nabla)_{XY} = R_{XY} + [S_X, S_Y] + S_{(T_\nabla)_X Y}$$

where $S = D - \nabla$.

Proof. — This follows at once from the definition of the curvature tensor and from $\nabla(T_\nabla) = 0$.

Corollary 3.5. — *With the same hypotheses as in Lemma* 3.3 *we have*

$$(R_\nabla)_{XYZW} = g((R_\nabla)_{XY} Z, W) = (R_\nabla)_{ZWXY}.$$

Proof. — Use Corollary 3.4 and the skew-symmetry of S.

Corollary 3.6. — *With the same hypotheses as in Lemma* 3.3, *the Ricci tensor* ρ_∇, *given by*

$$(\rho_\nabla)_{XY} = \sum_m (R_\nabla)_{XE_m YE_m},$$

is symmetric. Moreover, if the Riemannian Ricci curvature ρ *satisfies* $\nabla\rho = 0$, *then* $\nabla\rho_\nabla = 0$.

Proof. — The symmetry follows at once from Corollary 3.5. Further, since $\nabla(T_\nabla) = 0$, (2.22) yields $\nabla S = 0$. The rest follows now easily from Corollary 3.4.

Lemma 3.7. — *With the same hypotheses as in Lemma* 3.3, $\nabla\rho = 0$ *implies* $\nabla(R_\nabla) = 0$.

Proof. — Put $R' = R_\nabla$, $T' = T_\nabla$, $\rho' = \rho_\nabla$. Since $\|R'\|$ is constant, we get

$$0 = \sum_m \nabla^2_{mm} \|R'\|^2 = 2 \sum_{m,i,j,h,k} (\nabla^2_{mm} R')_{ijhk} R'_{ijhk} + 2\|\nabla R'\|^2.$$

Using (iii) of Proposition 3.2 we get

$$\|\nabla R'\|^2 = -\sum_{m,i,j,h,k} (\nabla^2_{mi} R')_{jmhk} R'_{ijhk} - \sum_{m,i,j,h,k} (\nabla^2_{mj} R')_{mihk} R'_{ijhk}$$
$$= -2 \sum_{m,i,j,h,k} (\nabla^2_{mi} R')_{jmhk} R'_{ijhk}.$$

Now we use the Ricci identity and (2.5) to obtain

$$\|\nabla R'\|^2 = -2 \sum_{m,\,i,\,j,\,h,\,k} (\nabla^2_{im} R')_{jmhk} R'_{ijhk} + 2 \sum_{m,\,i,\,j,\,h,\,k,\,p} (\nabla_p R')_{jmhk} R'_{ijhk} T'_{mip}.$$

Corollary 3.5 and again Proposition 3.2 yield

$$\begin{aligned}\|\nabla R'\|^2 &= -2 \sum_{m,\,i,\,j,\,h,\,p} (\nabla^2_{im} R')_{hkjm} R'_{ijhk} + 2 \sum_{m,\,i,\,j,\,h,\,k,\,p} T'_{mip} (\nabla_p R')_{jmhk} R'_{ijhk} \\ &= 4 \sum_{m,\,i,\,j,\,h,\,k} (\nabla^2_{ih} R')_{kmjm} R'_{ijhk} + 2 \sum_{m,\,i,\,j,\,h,\,k,\,p} T'_{mip} (\nabla_p R')_{jmhk} R'_{ijhk} \\ &= 4 \sum_{i,\,j,\,h,\,k} (\nabla^2_{ih} \rho')_{kj} R'_{ijhk} + 2 \sum_{m,\,i,\,j,\,h,\,k,\,p} T'_{mip} (\nabla_p R')_{jmhk} R'_{ijhk} \\ &= 2 \sum_{m,\,i,\,j,\,h,\,k,\,p} T'_{mip} (\nabla_p R')_{jmhk} R'_{ijhk}.\end{aligned}$$

Now, (2.4) yields

$$\sum_m (R'_{ijhm} T'_{mkl} + R'_{ijkm} T'_{hml} + R'_{ijlm} T'_{hkm}) = 0.$$

Since $\nabla(T_\nabla) = 0$, we get from this

$$\sum_m \{ (\nabla_p R')_{ijhm} T'_{mkl} + (\nabla_p R')_{ijkm} T'_{hml} + (\nabla_p R')_{ijlm} T'_{hkm} \} = 0$$

and hence

$$\sum_{m,\,l} \{ (\nabla_l R')_{ijhm} T'_{mkl} + (\nabla_l R')_{ijkm} T'_{hml} + (\nabla_l R')_{ijlm} T'_{hkm} \} = 0.$$

On the other hand, using once again Proposition 3.2 (iii) and also $\nabla\rho' = 0$, we obtain

$$\sum_l (\nabla_l R')_{ijlm} = 0.$$

Using the last two formulas we obtain

$$\begin{aligned}\|\nabla R'\|^2 &= 2 \sum_{m,\,i,\,j,\,h,\,k,\,l} T'_{mhl} (\nabla_l R')_{kmij} R'_{hkij} \\ &= -2 \sum_{m,\,i,\,j,\,h,\,k,\,l} T'_{hml} (\nabla_l R')_{ijkm} R'_{ijhk} \\ &= 2 \sum_{m,\,i,\,j,\,h,\,k,\,l} T'_{mkl} (\nabla_l R')_{hmij} R'_{hkij} = -\|\nabla R'\|^2.\end{aligned}$$

This yields the required result.

From Lemma 3.3 and Lemma 3.7 we obtain:

THEOREM 3.8. — *Let* (M, g) *be a Riemannian manifold equipped with a metric connection* ∇ *such that its torsion and curvature tensor are the same as those of an Ambrose-Singer* *connection of a naturally reductive homogeneous Einstein space* $(N, \bar{g})$. *Then* (M, g) *is locally homogeneous and locally isometric to that model space* $(N, \bar{g})$.

What we mean here is that the curvature and the torsion of ∇ are the same as those of the infinitesimal model of $(N, \bar{g})$.

Proof. — If $(N, \bar{g})$ is an Einstein space, then $\nabla\rho = 0$. The rest follows now easily.

From this we get:

COROLLARY 3.9. — *Let* (M, g) *be a Riemannian manifold such that its Riemann curvature tensor is the same as that of a symmetric Einstein space. Then* (M, g) *is locally symmetric and locally isometric to that model space.*

Since an irreducible symmetric space is Einsteinian, Corollary 3.9 implies at once the result proved in [21].

4. Riemannian symmetric spaces

In this section we shall extend the result of Corollary 3.9.

A simply connected symmetric Riemannian manifold is, up to an isometry, determined by the Riemann curvature tensor or, equivalently, by its symmetric infinitesimal model. Putting $T = 0$ in (2.1)-(2.7) we see that the set $\mathfrak{m}_S(V)$ of these models is the subset of $\overset{4}{\otimes} V^*$ determined by the conditions

$$R_{XY} = -R_{YX}, \tag{4.1}$$

$$\langle R_{XY} Z, W \rangle + \langle Z, R_{XY} W \rangle = 0, \tag{4.2}$$

$$R_{XY} \cdot R = 0, \tag{4.3}$$

$$\sum_{X, Y, Z} R_{XY} Z = 0. \tag{4.4}$$

We say that a Riemannian manifold (M, g) *has the same curvature tensor as that of a symmetric Riemannian space* if its Levi Civita connection D has the same curvature tensor as that of the infinitesimal model corresponding to the symmetric space. This means that $R_D(OM)$, where R_D is the map defined by (3.4), is contained in the orbit of an element R of $\mathfrak{m}_S(V)$ under the action of the orthogonal group. In this case, a theorem of Singer [17], p. 688 implies that there exists a principal subbundle P of OM with structure group H where H is the connected component of the identity of the subgroup of $O(n)$ consisting of the elements leaving R invariant. This means that (M, g) has an H-*structure.*

Using Cartan's construction of a symmetric space from the curvature tensor (*see* for example [7], p. 218-223 or our remarks in Section 2), we see that H is the connected component of the identity of the *isotropy subgroup* of the connected and simply connected symmetric space (M^0, g^0) which has R as its curvature tensor.

The Lie algebra $\mathfrak{h}=\{A\in\mathfrak{so}(n)\,|\,A\cdot R=0\}$ of H contains the *holonomy algebra* K because K is generated by the operators R_{XY} and $R_{XY}\cdot R=0$. It is also easy to see that $\mathfrak{k}$ is an *ideal* of $\mathfrak{h}$.

The vector space V may be decomposed as a direct sum of orthogonal subspaces which are invariant under the action of $\mathfrak{k}$. In general we have

$$V=V_0\oplus V_1\oplus\ldots\oplus V_r$$

where $\mathfrak{k}$ acts trivially on V_0 and irreducibly on V_α for $\alpha\geqq 1$. This decomposition is also $\mathfrak{h}$-invariant. Indeed, we have that $Z\in V_0$ if and only if $R_{XY}Z=0$ for all, X, $Y\in V$. But since $A\cdot R=0$ we have also

$$R_{XY}AZ=AR_{XY}Z-R_{AXY}Z-R_{XAY}Z=0$$

and so $AZ\in V_0$. Further, the spaces V_α, $\alpha\geqq 1$, are generated by the vectors $R_{XY}Z$, where X, Y, $Z\in V_\alpha$ because the V_α are $\mathfrak{k}$-irreducible. Now, since

$$AR_{XY}Z=R_{XY}AZ+R_{AXY}Z+R_{XAY}Z=-R_{YAZ}X-R_{AZX}Y+R_{AXY}Z+R_{XAY}Z,$$

we see that $AR_{XY}Z\in V_\alpha$.

It follows from this that each element A of $\mathfrak{h}$ may be decomposed in a unique way as

$$A=A_0+A_1+\ldots+A_r$$

where

$$A_\alpha=A_{|V_\alpha},\qquad \alpha\geqq 0.$$

Now, let X_α denote the projection of X on V_α. The invariance of V_α under the action of $\mathfrak{k}$ implies that $R_{X_\alpha Y_\beta}Z_\gamma=0$ except when $\alpha=\beta=\gamma$. Hence

$$R_{XY}Z=\sum_{\alpha=1}^{r}R_{X_\alpha Y_\alpha}Z_\alpha$$

and we have $A\cdot R=0$ if and only if for all $\alpha\geqq 1$

$$A_\alpha\cdot R_\alpha=0$$

where R_α is determined by

$$(R_\alpha)_{X_\alpha Y_\alpha}Z_\alpha=R_{X_\alpha Y_\alpha}Z_\alpha.$$

This proves that

$$\mathfrak{h}=\mathfrak{so}(m)\times\mathfrak{h}_1\times\ldots\times\mathfrak{h}_r$$

where $m=\dim V_0$ and $\mathfrak{h}_\alpha$, $\alpha\geqq 1$, is the *isotropy algebra* of the connected simply connected symmetric space (M^0_α, g^0_α) with curvature tensor R_α. This space is *irreducible* since $\mathfrak{k}_\alpha=\mathfrak{k}_{|V_\alpha}$ acts irreducibly on V_α. It follows from this that $\mathfrak{h}_\alpha=\mathfrak{h}_{|V_\alpha}$ coincides with the *holonomy algebra* $\mathfrak{k}_\alpha$ of (M_α^0, g_α^0). So, if $M_\alpha^0 = G_\alpha/H_\alpha$, we have

$$H = SO(m) \times H_1 \times \ldots \times H_r.$$

Now, we return to the manifold (M, g) which has the same curvature tensor as the symmetric space (M^0, g^0). We know already that (M, g) has an H-structure. More precisely, (M, g) has an *almost product* structure, that is the tangent bundle TM can be decomposed as a direct sum of $r+1$ orthogonal vector subbundles $E_0, E_1, \ldots, E_r$ with structure groups $SO(m)$, $H_1, \ldots, H_r$ respectively.

This structure may be obtained as follows: Let P be the subbundle of OM with structure group H and let $u = (q; u_1, \ldots, u_n)$ be an element of P. Put

$$E_\alpha(q) = u(V_\alpha), \qquad 0 \leqq \alpha \leqq r.$$

$E_\alpha(q)$ is a vector subspace which does not depend on u but only on q. Indeed, let $v = (q; v_1, \ldots, v_n)$. Then $v = ua$ where $a \in H$. Hence

$$v(V_\alpha) = u(a V_\alpha) = u(V_\alpha) = E_\alpha(q).$$

Our hypothesis implies that

$$(R_{D\,|\,q})_{XY} \circ u = u \circ R_{u^{-1}X\,u^{-1}Y}$$

for all X, $Y \in T_q M$. Hence

$$(R_{D\,|\,q})_{XY}(E_\alpha) = (R_{D\,|\,q})_{XY}(u V_\alpha) = u(R_{u^{-1}X\,u^{-1}Y}(V_\alpha)) = u V_\alpha = E_\alpha,$$

since V_α is invariant under the action of $\mathfrak{k}$. This shows that the subspaces $E_\alpha(q)$ of $T_q M$ are invariant under the action of $\mathfrak{h}(q)$ where $\mathfrak{h}(q)$ is the subalgebra of $\mathfrak{so}(T_q M)$ generated by the operators $(R_{|\,q})_{XY}$, X, $Y \in T_q M$. This algebra acts trivially on $E_0(q)$ and irreducibly on $E_\alpha(q)$ for all $\alpha \geqq 1$.

By following [18], p. 543-544 we can prove, using the Bianchi identities, that when X_α, Y_α are sections of E_α, [*i. e.* X_α, $Y_\alpha \in \Gamma(E_\alpha)$], we have

(4.5) $$D_{X_0} Y_0 \in \Gamma(E_0),$$

(4.6) $$D_{X_\alpha} Y_0 \in \Gamma(E_0 \oplus E_\alpha),$$

(4.7) $$D_{X_\alpha} Y_\alpha \in \Gamma(E_0 \oplus E_\alpha),$$

(4.8) $$D_{X_0} Y_\alpha \in \Gamma(E_\alpha),$$

(4.9) $$D_{X_\alpha} Y_\beta \in \Gamma(E_\beta),$$

for all α and $\beta \geqq 1$ and $\alpha \neq \beta$. Hence, the subbundles E_α, $0 \leqq \alpha \leqq r$ are *totally geodesic* if and only if

(4.10) $$D_X \Gamma(E_0) \subseteq \Gamma(E_0)$$

for all $X \in \Gamma(TM)$. Indeed, in this case we have for all $Z_0 \in \Gamma(E_0)$ and all $\alpha \geqq 1$

$$g(D_{X_\alpha} Y_\alpha, Z_0) = -g(Y_\alpha, D_{X_\alpha} Z_0) = 0.$$

This means that $D_{X_\alpha} Y_\alpha \in \Gamma(E_\alpha)$.

So, if (4.10) is satisfied, then (M, g) is locally a Riemannian product $(M_0, g_0) \times \ldots \times (M_r, g_r)$ where the M_α are integral submanifolds of E_α equipped with the metric g_α induced from g. (M_0, g_0) is flat but the other manifolds (M_α, g_α) have the same curvature tensor as (M^0_α, g^0_α). Since the (M^0_α, g^0_α) are irreducible, the manifolds (M_α, g_α) are locally symmetric and locally isometric to (M^0_α, g^0_α). This proves

THEOREM 4.1. — *Let (M, g) be a Riemannian manifold with the same curvature tensor as a Riemannian symmetric space (M^0, g^0). Let E_0 be the subbundle of* TM *where the fibre at $q \in M$ is determined by*

$$E_0(q) = \{ Z \in T_q M \,|\, (R_D)_{XY} Z = 0 \textit{ for all } X, Y \in T_q M \}.$$

If $D_X \Gamma(E_0) \subseteq \Gamma(E_0)$ for all vector fields X, *then (M, g) is locally symmetric and locally isometric to (M^0, g^0).*

Note that $E_0(q)$ is the *nullity space* of R_D at q ([8], vol. II, p. 347). Indeed, $Z \in E_0(q)$ if and only if

$$(R_D)_{XYZW} = g((R_D)_{XY} Z, W) = 0$$

for all X, Y, $W \in T_q M$. This is equivalent to $(R_D)_{ZW} = 0$ for all $W \in T_q M$.

Hence, dim $E_0(q)$ is the index of nullity of (M, g) at the point q. Since (M, g) is supposed to be curvature homogeneous, this index is constant.

Further we have

COROLLARY 4.2. — *Let (M, g) be a semi-symmetric space [that is $(R_D)_{XY} \cdot R_D = 0$] which is curvature homogeneous and such that $D_X \Gamma(E_0) \subseteq \Gamma(E_0)$. Then (M, g) is locally symmetric.*

Proof. — It suffices to note that (M, g) has the same curvature tensor as a symmetric Riemannian space.

Because a symmetric Einstein space is either flat or does not have a Euclidean factor (it has vanishing index of nullity), Corollary 3.9 follows also at once from Theorem 4.1.

In the proof of Theorem 4.1 we used some ideas also used in [18] by Z. I. Szabó, in particular in his proof of [18], Theorem 1.3. Of course, it would be possible to deduce our theorem for Szabó's results instead of the direct approach we used here. The scheme of this alternative proof is as follows: Theorem 1.3 of [18] gives an interesting decomposition of a Riemannian manifold by using the infinitesimal holonomy group. From this it follows that each curvature homogeneous semi-symmetric Riemannian manifold is locally isometric to the product of a locally Euclidean space M_0 and of r "infinitesimally irreducible" semi-symmetric spaces $M_1, M_2, \ldots, M_r$. The index of nullity of M is just the sum of the dimension of M_0 and the indexes of nullity of the irreducible factors. Moreover, the nullity distribution of M is *parallel* if and only if those of $M_1, \ldots, M_r$ are parallel. Since these factors are irreducible, this happens if and only if these distributions have dimension zero. Then Theorem 4.3 of [18] applies and from the previous remarks we get our Theorem 4.1.

In a forthcoming paper [12] we will come back to these problems and in particular to the relations of [18] with the examples treated in the next section.

5. A family of counterexamples

We consider $\mathbb{R}^3$ equipped with the metrics $g_{a,b}$ where

$$g_{a,b} = du^2 + dv^2 + f^2\,dw^2 \tag{5.1}$$

and

$$\begin{gathered} f = ae^t + be^{-t}, \\ t = u\cos w - v\sin w, \\ a, b \in \mathbb{R}^+. \end{gathered} \tag{5.2}$$

These metrics are complete and irreducible [15].

It is easy to see that

$$\begin{gathered} e_1 = f^{-1}\frac{\partial}{\partial w}, \\ e_2 = \cos w\frac{\partial}{\partial u} - \sin w\frac{\partial}{\partial v}, \\ e_3 = \sin w\frac{\partial}{\partial u} + \cos w\frac{\partial}{\partial v}, \end{gathered} \tag{5.3}$$

is a global orthonormal frame and the dual frame is given by

$$\begin{gathered} \omega^1 = f\,dw, \\ \omega^2 = \cos w\,du - \sin w\,dv, \\ \omega^3 = \sin w\,du + \cos w\,dv. \end{gathered} \tag{5.4}$$

An easy calculation leads to

$$\begin{gathered} d\omega^1 = -gf^{-1}\,\omega^1 \wedge \omega^2, \\ d\omega^2 = -f^{-1}\,\omega^1 \wedge \omega^3, \\ d\omega^3 = f^{-1}\,\omega^1 \wedge \omega^2, \end{gathered} \tag{5.5}$$

where

$$g = ae^t - be^{-t}. \tag{5.6}$$

From the structure equations

$$\begin{gathered} \omega^i_j + \omega^j_i = 0, \\ d\omega^i + \sum \omega^i_k \wedge \omega^k = 0 \end{gathered} \tag{5.7}$$

we get

$$\begin{gathered} \omega^1_2 = -\omega^2_1 = gf^{-1}\,\omega^1, \\ \omega^1_3 = -\omega^3_1 = 0, \\ \omega^2_3 = -\omega^3_2 = f^{-1}\,\omega^1. \end{gathered} \tag{5.8}$$

From this we obtain

$$\begin{gathered} \mathrm{D_X}\,\omega^1 = -gf^{-1}\,\omega^1(\mathrm{X})\,\omega^2, \\ \mathrm{D_X}\,\omega^2 = f^{-1}\,\omega^1(\mathrm{X})\,(g\,\omega^1 - \omega^3), \\ \mathrm{D_X}\,\omega^3 = f^{-1}\,\omega^1(\mathrm{X})\,\omega^2. \end{gathered} \tag{5.9}$$

Further, the curvature of $(\mathbb{R}^3, g_{a,b})$ can be determined by using

$$\Omega^i_j = d\omega^i_j + \sum \omega^i_k \wedge \omega^k_j. \tag{5.10}$$

With (5.8) we get

$$\begin{gathered} \Omega^1_2 = -\Omega^2_1 = -\omega^1 \wedge \omega^2, \\ \Omega^1_3 = -\Omega^3_1 = 0, \\ \Omega^2_3 = -\Omega^3_2 = 0. \end{gathered} \tag{5.11}$$

From this we obtain:

PROPOSITION 5.1. — *The manifold $(\mathbb{R}^3, g_{a,b})$ is curvature homogeneous and has the same curvature tensor as $\mathbb{R} \times \mathbb{H}^2$ where $\mathbb{H}^2$ is the hyperbolic plane of constant curvature -1.*

The Ricci tensor of $(\mathbb{R}^3, g_{a,b})$ is given by the general formula

$$\rho_{\mathrm{XY}} = -2 \sum \Omega^m_j(\mathrm{X}, e_m)\,\omega^j(\mathrm{Y}).$$

Using (5.11) we get at once

$$\rho = -\omega^1 \otimes \omega^1 - \omega^2 \otimes \omega^2. \tag{5.12}$$

Further, (5.9) and (5.12) yield

$$\mathrm{D_X}\,\rho = f^{-1}\,\omega^1(\mathrm{X})\,(\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2).$$

Hence

$$D\rho = f^{-1}\,\omega^1 \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2). \tag{5.13}$$

From this we see that the only non-vanishing components of $D\rho$ are

$$(D_1\,\rho)_{23} = (D_1\,\rho)_{32} = f^{-1}$$

and hence

$$\|D\rho\|^2 = 2f^{-2}. \tag{5.14}$$

Since $\|D\rho\|$ is a Riemannian invariant which in this case is not constant, we get another proof of

PROPOSITION 5.2 [19]. — *The manifolds* $(\mathbb{R}^3, g_{a,b})$ *are not locally homogeneous.*

Now we determine under which conditions $g_{a,b}$ and $g_{\bar{a},\bar{b}}$ are isometric. Therefore, let φ be an isometry between $g_{a,b}$ and $g_{\bar{a},\bar{b}}$. Then we must have

$$\varphi^*\,\bar{\omega}^i = \sum_j a^i_j\,\omega^j,$$

where the one-forms $\bar{\omega}^i$ are given by (5.4). $a = (a^i_j)$ is a function with values in $O(3)$.

Since φ is an isometry we also have

$$\varphi^*\,\bar{\rho} = \rho, \qquad \varphi^*(\bar{D}\bar{\rho}) = D\rho$$

and $\bar{f} \circ \varphi = f$ since $f = \|D\rho\|^{-1}$. From (5.12) it then follows that

$$a^1_3 = a^2_3 = a^3_1 = a^3_2 = 0$$

and from (5.13) we get

$$\sum_{i,j,k} a^1_i\,a^2_j\,a^3_k\,\omega^i \otimes (\omega^j \otimes \omega^k + \omega^k \otimes \omega^j) = \omega^1 \otimes (\omega^2 \otimes \omega^3 + \omega^3 \otimes \omega^2).$$

Hence we must have

$$a^1_1\,a^2_2\,a^3_3 = 1, \qquad a^1_1\,a^2_1\,a^3_3 = a^1_2\,a^2_1\,a^3_3 = a^1_2\,a^2_2\,a^3_3 = 0$$

and so we obtain

$$a^1_2 = a^2_1 = 0.$$

Since a is orthogonal we then have $a^i_i = \pm 1$ with a sign combination such that

$$a^1_1\,a^2_2\,a^3_3 = 1.$$

It follows that we have to consider the following cases:

$$a=\mathrm{I},\qquad a=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\qquad a=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\qquad a=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}.$$

First we consider the case $a=\mathrm{I}$. Then

$$\begin{aligned}&\varphi^*\bar{\omega}^1=\omega^1,\\ &\varphi^*\bar{\omega}^2=\omega^2,\\ &\varphi^*\bar{\omega}^3=\omega^3.\end{aligned}\tag{5.15}$$

Put $\varphi(u, v, w)=(\bar{u}, \bar{v}, \bar{w})$. Then (5.15) yields, since $\bar{f}\circ\varphi=f$,

$$\begin{gathered}d\bar{w}=dw,\\ \cos\bar{w}\,d\bar{u}-\sin\bar{w}\,d\bar{v}=\cos w\,du-\sin w\,dv,\\ \sin\bar{w}\,d\bar{u}+\cos\bar{w}\,d\bar{v}=\sin w\,du+\cos w\,dv.\end{gathered}$$

Hence $\bar{w}=w+\alpha$, α being constants, and

$$\begin{cases}d\bar{u}=\cos\alpha\,du+\sin\alpha\,dv,\\ d\bar{v}=-\sin\alpha\,du+\cos\alpha\,dv.\end{cases}$$

From this we get

$$\begin{cases}\bar{u}=u\cos\alpha+v\sin\alpha+\beta,\\ \bar{v}=-u\sin\alpha+v\cos\alpha+\gamma\end{cases}$$

where β and γ are constant. Further, since $\bar{f}\circ\varphi=f$, we must have

$$\bar{a}e^{\bar{t}}+\bar{b}e^{-\bar{t}}=ae^t+be^{-t}\tag{5.16}$$

where

$$\begin{aligned}\bar{t}&=\bar{u}\cos\bar{w}-\bar{v}\sin\bar{w}\\ &=t+\beta\cos(w+\alpha)+\gamma\sin(w+\alpha).\end{aligned}$$

Hence it follows easily that (5.16) is satisfied if and only if $\beta=\gamma=0$ and $\bar{a}=a$, $\bar{b}=b$.

Next we consider the case where

$$a=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}.$$

In this case we have

$$d\bar{w}=dw,$$
$$\cos\bar{w}\,d\bar{u}-\sin\bar{w}\,d\bar{v}=-\cos w\,dw+\sin w\,dv,$$
$$\sin\bar{w}\,d\bar{u}+\cos\bar{w}\,d\bar{v}=\sin w\,du+\cos w\,dv,$$

and hence

$$\bar{w}=w+\alpha,$$
$$\bar{u}=-u\cos\alpha-v\sin\alpha-\beta,$$
$$\bar{v}=u\sin\alpha-v\cos\alpha-\gamma,$$

α, β, γ being constant. The invariance of f now yields $\beta=\gamma=0$ and $\bar{a}=b$, $\bar{b}=a$. Hence $g_{a,b}$ and $g_{b,a}$ are isometric.

The third case leads again to $\bar{a}=a$, $\bar{b}=b$ and in the last case we get $\bar{a}=b$, $\bar{b}=a$. Hence

PROPOSITION 5.3. — *The isometry classes of* $(\mathbb{R}^3, g_{a,b})$ *are in one-to-one correspondence with* $\mathbb{R}^2/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *is the permutation group of two elements.*

Remarks. — A. From (5.11) we derive that the nullity distribution E_0 of $(\mathbb{R}^3, g_{a,b})$ is the subbundle generated by e_3. Since

$$D_X e_3=\sum\omega_3^k(X)\,e_k=\omega_3^2(X)\,e_2=f^{-1}\,\omega^1(X)\,e_2,$$

we see that the condition $D_X\Gamma(E_0)\subseteq\Gamma(E_0)$ is not satisfied.

B. From the previous calculations we see that the group of isometries of $(\mathbb{R}^3, g_{a,b})$ is the group of transformations of $\mathbb{R}^3$ determined by

$$\bar{u}=u\cos\alpha+v\sin\alpha,$$
$$\bar{v}=-u\sin\alpha+v\cos\alpha,$$
$$\bar{w}=w+\alpha$$

where α is constant.

REFERENCES

[1] W. AMBROSE and I. M. SINGER, *On Homogeneous Riemannian Manifolds* (*Duke Math. J.*, Vol. 25, 1958, pp. 657-669).

[2] M. BERGER, *L'œuvre d'André Lichnerowicz en géométrie riemannienne*, in *Physique quantique et géométrie*, D. BERNARD and Y. CHOQUET-BRUHAT Eds., Colloque Géométrie et Physique 1986 en l'honneur de André Lichnerowicz, Travaux en cours, Hermann, Paris, 1988.

[3] A. L. BESSE, *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[4] J. E. D'ATRI and W. ZILLER, *Naturally Reductive Metrics and Einstein Metrics on Compact Lie groups* (*Mem. Amer. Math. Soc.*, Vol. 215, 1979).

[5] D. FERUS, H. KARCHER and H. F. MÜNZNER, *Cliffordalgebren und neue Isoparametrische Hyperflächen* (*Math. Z.*, Vol. 177, 1981, pp. 479-502).

[6] M. GROMOV, *Structures métriques pour les variétés riemanniennes*, Cedic, Paris, 1981.
[7] S. HELGASON, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
[8] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry*, Vol. I and II, Interscience Publishers, New York, 1963 and 1969.
[9] O. KOWALSKI, *Generalized Symmetric Spaces* (*Lecture Notes in Math.*, No. 805, Springer-Verlag, Berlin, Heidelberg, New York, 1980).
[10] O. KOWALSKI and F. TRICERRI, *Riemannian Manifolds of Dimension* $n \leqq 4$ *Admitting a Homogeneous Structure of Class* $\mathscr{T}_2$ (*Conferenze del Seminario di Matematica*, Univ. Bari, Vol. 222, Laterza, Bari, 1987).
[11] O. KOWALSKI and L. VANHECKE, *Riemannian Manifolds with Homogeneous Geodesics*, preprint.
[12] O. KOWALSKI, F. TRICERRI and L. VANHECKE, *Curvature Homogeneous Riemannian Manifolds*, in preparation.
[13] G. D. MOSTOW, *Strong Rigidity of Locally Symmetric Spaces* (*Ann. of Math. Studies*, Vol. 78, Princeton Univ. Press, 1973).
[14] K. NOMIZU, *Invariant Affine Connections on Homogeneous Spaces* (*Amer. J. Math.*, Vol. 76, 1954, pp. 33-65).
[15] K. SEKIGAWA, *On the Riemannian Manifolds of the Form* $\mathrm{B} \times_f \mathrm{F}$ (*Kōdai Math. Sem. Rep.*, Vol. 26, 1975, pp. 343-347).
[16] J. SIMONS, *On the Transitivity of Holonomy Systems* (*Ann. of Math.*, Vol. 76, 1962, pp. 213-234).
[17] I. M. SINGER, *Infinitesimally Homogeneous Spaces* (*Comm. Pure Appl. Math.*, Vol. 13, 1960, pp. 685-697).
[18] Z. I. SZABÓ, *Structure Theorems on Riemannian Spaces Satisfying* $\mathrm{R}(\mathrm{X}, \mathrm{Y}) \cdot \mathrm{R} = 0$. I. *The local version* (*J. Differential Geometry*, Vol. 17, 1982, pp. 531-582).
[19] H. TAKAGI, *On Curvature Homogeneity of Riemannian Manifolds* (*Tôhoku Math., J.*, Vol. 26, 1974, pp. 581-585).
[20] F. TRICERRI and L. VANHECKE, *Homogeneous Structures on Riemannian Manifolds* (*London Math. Soc. Lecture Note Series*, Vol. 83, Cambridge Univ. Press, London, 1983).
[21] F. TRICERRI and L. VANHECKE, *Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symétrique irréductible* (*C. R. Acad. Sci. Paris*, T. 302, 1986, pp. 233-235).
[22] M. Y. WANG and W. ZILLER, *On Normal Homogeneous Einstein Manifolds* (*Ann. Scient. Éc. Norm. Sup.*, 4e série, Vol. 18, 1985, pp. 563-633).
[23] J. WOLF, *The Geometry and Structures of Isotropy Irreducible Homogeneous Spaces* (*Acta Math.*, Vol. 120, 1968, pp. 59-148); Correction (*Acta Math.*, Vol. 152, 1984, pp. 141-142).

(Manuscript received April 11, 1988,
in revised form April 24, 1989).

F. TRICERRI
Istituto di Matematica "U. Dini",
Università di Firenze,
Viale Morgagni 67/A,
I-50134 Firenze, Italy,

L. VANHECKE
Department of Mathematics,
Katholieke Universiteit Leuven,
Celestijnenlaan 200B,
B-3030 Leuven, Belgium.