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The indecomposable $K_3$ of fields


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THE INDECOMPOSABLE $K_3$ OF FIELDS (*)

BY MARC LEVINE

Introduction

In this paper, we extend the theorem of Merkurjev-Suslin (Hilbert's Theorem 90 for $K_2$) to the relative $K_2$ of semi-local principal ideal rings (PIR) containing a field. Most of the results Suslin proves for $K_2$ of fields in [S] then carry over to the relative $K_2$ of a semi-local PIR, e.g. computation of the torsion subgroup, and the isomorphism $K_3(F)/n \rightarrow H^2_{et}(F, \mu_n^{o^{2}})$. Applying this to the semi-local ring of $\{0, 1\}$ in $A^1_E$, for a field $E$, gives a computation of the torsion and co-torsion in $K^E_F / K_3(E)^{\text{ind}} = K_3(E)/K_3(E)^{\text{dec}}$, where $K_3(E)^{\text{dec}}$ is the subgroup of $K_3(E)$ generated by products from $K_1(E)$. Specifically we show

1. The $l$-primary torsion subgroup of $K_3(E)^{\text{ind}}$ is $H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$ for $(l, \text{char}(E)) = 1$; $K_3(E)^{\text{ind}}$ has no $p$-torsion if char($E$) = $p > 0$.

2. $K_3(E; \mathbb{Z}/n)^{\text{ind}} \sim H^1(E, \mathbb{Z}_p^{o^{2}})$ for $(n, \text{char}(E)) = 1$, so $\lim_{\leftarrow} K_3(E)^{\text{ind}}/p \sim H^1(E, \mathbb{Z}_l(2))$

for $l \neq \text{char}(E)$.

3. $K_3(E)^{\text{ind}}$ satisfies Galois descent for extensions of degree prime to char($E$).

4. Bloch's group $B(E)$ is uniquely $l$-divisible if $E$ contains an algebraically closed field, and $l \neq \text{char}(E)$.

Let $F$ be a number field, $l$ an odd prime number, $S$ the set of places of $F$ lying over $l$, and $\mathfrak{o}_S$ the ring of $S$-integers in $F$. Quillen (see [Li2]) has conjectured

(Q) There are isomorphisms

$$c_{q, 2} : K_{2 q - 2} (\mathfrak{o}_S) \otimes \mathbb{Z}_l \rightarrow H^2 (\text{Spec } \mathfrak{o}_S, Z_l(q))$$

$$c_{q, 1} : K_{2 q - 1} (\mathfrak{o}_S) \otimes \mathbb{Z}_l \rightarrow H^1 (\text{Spec } \mathfrak{o}_S, Z_l(q)).$$

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Borel [Borel] has computed the ranks of the K-groups $K_\bullet(\mathcal{O}_F)$ as

$$K_{2q}(\mathcal{O}_F) \otimes \mathbb{Q} = 0; \quad \text{for } q \geq 1$$
$$K_{2q-1}(\mathcal{O}_F) \otimes \mathbb{Q} = \mathbb{Q}^{r_2}; \quad \text{for } q = 2n, \quad n \geq 1$$
$$K_{2q-1}(\mathcal{O}_F) \otimes \mathbb{Q} = \mathbb{Q}^{1+r_2}; \quad \text{for } q = 2n+1, \quad n \geq 1.$$  

Quillen [Q3] has shown that the groups $K_\bullet(\mathcal{O}_F)$ are finitely generated. Soule [So] has constructed Chern classes

$$c_{q,2} : K_{2q-2}(\mathcal{O}_S, \mathbb{Z}/l^r) \to H^2(Spec \mathcal{O}_S, (\mu_l)^{\otimes q})$$
$$c_{q,1} : K_{2q-1}(\mathcal{O}_S, \mathbb{Z}/l^r) \to H^1(Spec \mathcal{O}_S, (\mu_l)^{\otimes q}),$$

for any set of places $S'$ containing $S$, and has verified the surjectivity part of the conjecture (Q), at least for $l > q$, as well as the injectivity modulo torsion.

Soule has also shown that $K_{2q-1}(F) = K_{2q-1}(\mathcal{O}_F)$ for $q \geq 2$; it is easily seen that the natural map $H^1(Spec \mathcal{O}_S, \mathcal{O}_l(q)) \to H^1(F, \mathcal{O}_l(q))$ is an isomorphism for $q \geq 2$. Bass and Tate ([B—T]) have computed the Milnor K-groups of number fields; they show in particular that $K^W(\mathbb{Z}) = (\mathbb{Z}/2)^r$. This, together with (1) and (2), proves Quillen's conjecture for $K_3$. In fact, for all prime $l$, (1) and (2) imply that the Chern class

$$c_{2,1} : K_3(F)^{ind} \otimes \mathbb{Z}/l \to H^1_{et}(F, \mathcal{O}_l(2))$$

is an isomorphism.

For the case $F = \mathbb{Q}$ this gives a new proof of the result of Lee and Szczarba [L-S] that $K_3(\mathbb{Z}) = \mathbb{Z}/48$. Indeed, it follows from our results that $K_3(\mathbb{Q})^{ind} = \mathbb{Z}/24$; to complete the computation one need only show that the symbol $\{-1, -1, -1\}$ of $K_3(\mathbb{R})$ is non-zero and divisible by 2 in $K_3(\mathbb{Z})$. This is done, for example, in [Igusa]. More generally, this gives the complete determination of $K_3(\mathcal{O}_F)$, $F$ a number field, as

5. $K_3(\mathcal{O}_F) = K_3(\mathcal{O}_F)_{tor} \otimes \mathbb{Z}^{r_2};$

$$K_3(\mathcal{O}_F)_{tor} = \begin{cases} \mathbb{Z}/2^{r_1-1} \otimes \mathbb{Z}/2w_2(F); & \text{if } r_1 > 0 \\ \mathbb{Z}/2w_2(F); & \text{if } r_1 = 0 \end{cases}$$

where $w_q(F)$ denotes the order of the group $H^0_{et}(F, \mathcal{O}_l(q))$.

Lichtenbaum [Li2] has conjectured that, for $F$ a totally real number field, and $q$ a positive even number,

$$\zeta_F(1-q) = \#(K_{2q-2}(\mathcal{O}_F))/\#(K_{2q-1}(\mathcal{O}_F)),$$

at least up to powers of 2. This follows from the conjecture (Q), and the conjecture of Lichtenbaum [Li2]:

(Li2) Let $F$ be a totally real number field, $l$ an odd prime, $q$ an even positive number. Then

(i) the groups $H^1(Spec \mathcal{O}_S, j_* \mathcal{O}_l/Z_l(q))$ and $H^0(Spec \mathcal{O}_S, j_* \mathcal{O}_l/Z_l(q))$ are finite.
(ii) the groups $H^k(\text{Spec } \mathcal{O}_k, j_* \mathbb{Q}_l/\mathbb{Z}_l(q))$ are zero for $k \geq 2$.

(iii) $|\zeta_p(1-q)| = \#(H^1(\text{Spec } \mathcal{O}_k, j_* \mathbb{Q}_l/\mathbb{Z}_l(q)))/\#(H^0(\text{Spec } \mathcal{O}_k, j_* \mathbb{Q}_l/\mathbb{Z}_l(q)))$.

Here $| \cdot |$ denotes the $l$-primary part of a rational number, and $j: \text{Spec } F \to \text{Spec } \mathcal{O}_k$ is the inclusion.

Here is a brief history of this conjecture and its proof:

Birch and Tate ([B], [T2]) conjectured that, for all totally real fields $F$, the

(BT) $\#(K_2(\mathcal{O}_F)) = w_2(F) \zeta_F(-1)$.

Tate’s computation of $K_2(\mathcal{O}_n)$ [T] shows this is equivalent to (Li2)(iii) for $q=2$. Coates and Lichtenbaum ([Li] and [C-L]) then showed conjecture (Li2) follows from the Main Conjecture in Iwasawa theory relating the $p$-adic interpolation of classical $L$-functions with Iwasawa’s $p$-adic $L$-functions constructed from Galois representations arising from the cyclotomic $\mathbb{Z}_p$ extension of $F$. They also verified the Main Conjecture in some cases. Mazur and Wiles ([M-W]) proved the Main Conjecture (for odd primes) for abelian number fields. Recent work of Wiles has extended this to all totally real fields, completing the proof of (Li2).

Our formula (1) shows that $w_2(F) = \#(K_3(F)^{\text{ind}})_{\text{tor}}$, which proves (Li1) for $q=2$. We can also write this as

$$\zeta_F(-1) = 2 \#(K_2(\mathcal{O}_F))/\#(K_3(F)^{\text{ind}}).$$

The work of Serre [Se] shows that the exponent $\gamma$ is non-negative; $\gamma$ has been shown by Hurrelbrink and Kolster [H-K] to be 0 for the fields

(i) $\mathbb{Q}((\sqrt{d}), d=2, p, \text{ or } 2p \text{ with } p \text{ prime, } p \equiv \pm 3 \text{ mod } 8$

(ii) $\mathbb{Q}((\sqrt{d}), d=pq, \text{ with } p \text{ and } q \text{ distinct primes } p, q \equiv 3 \text{ mod } 8, \text{ or } d=p \text{ with } p \text{ prime, } p=u^2-2w^2, u>0, u \equiv 3 \text{ mod } 4, w \equiv 0 \text{ mod } 4$

(iii) $\mathbb{Q}(\zeta_{2^m})$

(iv) $\mathbb{Q}(\zeta_p)^+$, if $p$ and $q=(p-1)/2$ are prime, and 2 is a primitive root mod $q$.

The conjectures of Lichtenbaum and Quillen were made “up to powers of 2”. From (1) we see that the “correct” group for $q=2$ having a good relation with Galois cohomology, including the prime 2, is $K_3(\mathbb{Z})^{\text{ind}}$. Let $\text{gr}_n^\gamma$ denote the associated graded with respect to the gamma filtration. As $K_2(\mathcal{O}_n)$ agrees with $\text{gr}_1^\gamma K_2(\mathcal{O}_n)$ and $K_3(\mathbb{Z})^{\text{ind}}$ agrees with $\text{gr}_1^\gamma K_3(\mathcal{O}_n)$, at least up to 2-torsion, our results suggest that (Li1) should perhaps be weakened as follows: for $F$ totally real, the value $\zeta_F(1-q)$ is given by the formula

$$\zeta_F(1-q) = a_q \prod_{n=0}^{2q-1} \#(\text{gr}_n^\gamma K_n(\mathcal{O}_F))^{(-1)^n}$$

where $a_q$ is a rational number involving only primes less than $2q+1$. More optimistically, Lichtenbaum [Li3] and Beilinson [Be] conjecture the existence of a “bigraded arithmetic cohomology theory over $\mathbb{Z}$”, $H^p_n(\mathbb{Z}(q))$, which computes $\text{gr}_n^\gamma K_2(\mathbb{Z}(q), p)$ up to primes less than $2q-p$, and which has a precise relationship with Galois cohomology. This
cohomology theory arises as the hypercohomology of a complex of sheaves $\Gamma(q)$ (for the etale topology in Lichtenbaum's theory, for the Zariski topology in Beilinson's). The value $\zeta_F(1-q)$ should then be given as the Euler characteristic

$$\zeta_F(1-q) = \prod (H^r(F, \mathbb{Z}(q)))^{1-p}.$$ 

This is the motivation for the formula (1). Lichtenbaum [Li4] has constructed the weight two arithmetic complex $\Gamma(2)$ for fields, which gives

$$H^2_{\text{et}}(F, \mathbb{Z}(2)) = K_2(F)$$

$$H^1_{\text{et}}(F, \mathbb{Z}(2)) = [K_3(F)]_{\text{ind}}^{\text{Gal}(F_s/F)}.$$ 

From (3), we have $H^1_{\text{et}}(F, \mathbb{Z}(2)) = K_3(F)_{\text{ind}}$, at least after inverting $\text{char}(F)$. This gives some evidence for the interpretation of $\zeta_F(1-q)$ as an Euler characteristic. One can also unite our results, the Merkurjev-Suslin theorem for $K_2$, and Suslin's computation of the torsion in $K_2$ in a way that is suggestive of an arithmetic cohomology theory. In fact, we have the exact sequence

$$0 \to H^0_\text{et}(E, \mu_n^{\otimes2}) \to K_3(E)_{\text{ind}} \xrightarrow{\times n} K_5(E)_{\text{ind}} \to H^1_\text{et}(E, \mu_n^{\otimes2})$$

$$\to K_2(E) \to K_2(E) \to H^2_{\text{et}}(E, \mu_n^{\otimes2}) \to 0$$

where $E$ is an arbitrary field, and $n$ is prime to the characteristic of $E$. This exact sequence arises from the exact triangle

$$\Gamma(2) \to \Gamma(2) \to \mu_n^{\otimes2}$$

$$\xrightarrow{\times n}$$

together with the computation of $H^*_\text{et}(E, \mathbb{Z}(2))$ above. This formulation was pointed out to me by Bruno Kahn.

The proof of Hilbert's Theorem 90 is a modification of the proof used by Suslin in [S]. The analysis of the $H^1(X, \mathcal{X}_2)$ for $X$ a Brauer-Severi scheme over a (equicharacteristic) semi-local PIR $R$ is essentially the same as in the case $R$ a field. Suppose $R$ contains $\mu$. Let $\alpha$ be a unit in $R$, $R^\alpha$ the extension $R[X]/X^l-\alpha$, or the extension $R[X]/X^l-X-\alpha$ if $l = \text{char}(R)$, and $J^\alpha$ the Jacobson radical of $R^\alpha$. The next step is to show the relation

$$\langle x, 1-\text{Norm}(x) \rangle \in (1-\sigma)K_2(R^\alpha, J^\alpha)$$

for $x \in (1+J^\alpha)^\times$, $\text{Norm}(x) \neq 1$. This is done by the "generic element" method first, where one can assume that $R$ is local, in which case the relative $K_2$ is a subgroup of the usual $K_2$. One then makes a specialization argument, which is the main technical difficulty. After this point, the proof proceeds essentially as in [S].
The first chapter gives a discussion of the properties of relative K-theory. This is essentially an extension of most of the results of Quillen's *Higher Algebraic K-theory I* to the setting of relative K-theory. Most of this chapter is quite straightforward, but as there is no reference for the material in the literature we include it here. The second chapter gives a description of relative $K_2$ via the symbols of Keune and Loday, and the symbols of Bloch. We also prove some preliminary results required for the construction of the specialization subgroup and homomorphism, which is the main technical construction of the chapter. In chapter three, we apply the generic element method to get the relation described above. We also get simplified generators for $K_2(R^*, J^*)$, assuming as in [S] that $R$ has no prime to $l$ extensions and that the norm map $N: (1 + J)^* \to (1 + J)^*$ is surjective. In chapter four we prove Hilbert's Theorem 90 for relative $K_2$, and the other results above. In chapter five, we use the continuous cohomology of Jannsen to extend the results of Tate and Merkurjev-Suslin on Galois symbols to the case of relative $K_2$.

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Hilbert's theorem 90 for relative $K_2$, and its consequences for $K_3$ have been proven independently by Merkurjev and Suslin [M-S2]. Their approach differs from ours in that they derive the behavior of the relative $K_2$ under extension by a Brauer-Severi scheme by the Galois-theoretic properties of $K_2$, rather than redoing the argument for fields in the relative case, as we have done. They also achieve some simplification by a judicious use of Karoubi-Villamayor K-theory to define norms. Finally, they prove the relation $(\star)$ by a more direct method, avoiding our use of specialization. In addition to the results given here, they show that $K_3(E)^{\text{ind}}$ is uniquely $p$-divisible in characteristic $p > 0$, and they prove a part of a conjecture of Milnor on the relation between Milnor K-theory and the Witt ring of a field.

1. Relative K-theory

1.1. Here we recall the definition and some basic properties of relative K-theory. For a more detailed discussion, see [Coombes].

Let $f: \mathcal{A} \to \mathcal{B}$ be an exact functor on exact categories $\mathcal{A}$ and $\mathcal{B}$. Define $(K(f), \ast)$ to be the homotopy fiber of $BQf: BQ\mathcal{A} \to BQ\mathcal{B}$ with basepoint $\ast$ coming from the
zero objects of \( \mathcal{A} \) and \( \mathcal{B} \). The K-groups of \( f \) are then the homotopy groups of \( \text{K}(f) \):

\[
K_p(f) := \pi_{p+1}(\text{K}(f), \ast).
\]

One gets a long exact sequence

\[
\rightarrow K_p(f) \rightarrow K_p(\mathcal{A}) \rightarrow K_p(\mathcal{B}) \rightarrow K_{p-1}(f) \rightarrow
\]

from the fibration \( \text{K}(f) \rightarrow BQ \mathcal{A} \rightarrow BQ \mathcal{B} \).

Let \( f_0: \mathcal{A}_0 \rightarrow \mathcal{B}_0 \) be another exact functor on exact categories. Suppose we have a pair of exact functors \( G: \mathcal{A}_0 \rightarrow \mathcal{A} \), \( H: \mathcal{B}_0 \rightarrow \mathcal{B} \), and a natural isomorphism \( \theta: fG \rightarrow Hf_0 \). Then \( \theta \) induces a homotopy \( BQ\theta \) between \( BQf \circ BQG \) and \( BQH \circ BQf_0 \), hence the triple \( (G, H, \theta) \) gives a map \( BQ(G, H, \theta): \text{K}(f_0) \rightarrow \text{K}(f) \). This induces a homomorphism \( (G, H, \theta) \ast: K_p(f_0) \rightarrow K_p(f) \), and a commutative ladder

\[
\rightarrow K_p(f_0) \rightarrow K_p(\mathcal{A}_0) \rightarrow K_p(\mathcal{B}_0) \rightarrow K_{p-1}(f_0) \rightarrow
\]

\[
(G, H, \theta)^* \downarrow \quad \ast^* \downarrow \quad \ast^* \downarrow \quad (G, H, \theta)^* \downarrow
\]

\[
\rightarrow K_p(f) \rightarrow K_p(\mathcal{A}) \rightarrow K_p(\mathcal{B}) \rightarrow K_{p-1}(f) \rightarrow
\]

Now let \( X \) be a scheme over a ring \( R \), \( Y \) a closed subscheme \( j_Y: Y \rightarrow X \) the inclusion. Let \( \mathcal{P}_X \) (resp. \( \mathcal{P}_Y \)) be the exact category of locally free sheaves on \( X \) (resp. \( Y \)) of finite rank. Then \( j_Y^*: \mathcal{P}_X \rightarrow \mathcal{P}_Y \) is exact; let \( \text{K}(X, Y) \) denote the homotopy fiber \( \text{K}(/_Y) \), and \( \text{K}_p(X, Y) \) the \( p \)-th \( \text{K} \)-group of \( X \) relative to \( Y \). One defines a relative \( \text{K} \)' similarly: let \( \mathcal{M}(X, Y) \) be the exact subcategory of coherent sheaves on \( X \), \( \mathcal{M}_X \) consisting of sheaves \( \mathcal{F} \) with \( \text{Tor}^i_X(\mathcal{F}, \mathcal{O}_Y) = 0 \) for \( i > 0 \). Then \( j_Y^*: \mathcal{M}(X, Y) \rightarrow \mathcal{M}_Y \) is exact. We let \( \text{K}'(X, Y) \) denote \( \text{K}(j_Y^*: \mathcal{M}(X, Y) \rightarrow \mathcal{M}_Y) \), and \( \text{K}'_p(X, Y) \) the \( p \)-th \( \text{K} \)-group of \( j_Y^* \). The inclusions \( i_X: \mathcal{P}_X \rightarrow \mathcal{M}(X, Y) \), \( i_Y: \mathcal{P}_Y \rightarrow \mathcal{M}_Y \) induce \( i: \text{K}_p(X, Y) \rightarrow \text{K}'_p(X, Y) \), and we have the commutative ladder

\[
\rightarrow \text{K}_p(X, Y) \rightarrow \text{K}_p(X) \rightarrow \text{K}_p(Y) \rightarrow
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\rightarrow \text{K}'_p(X, Y) \rightarrow \text{K}'_p(X) \rightarrow \text{K}'_p(Y) \rightarrow
\]

Thus, if \( X \) and \( Y \) are regular, the resolution theorem shows that \( \text{K}'_p(X, Y) \rightarrow \text{K}_p(X, Y) \) is an isomorphism. Similarly, if \( \mathcal{O}_Y \) has finite Tor dimension over \( \mathcal{O}_X \) (e.g. \( X \) regular or \( Y \) locally principal) the resolution theorem shows that the inclusion \( \mathcal{M}(X, Y) \rightarrow \mathcal{M}_X \) induces an isomorphism \( \text{K}_p(\mathcal{M}(X, Y)) \rightarrow \text{K}_p(\mathcal{M}_X) \).

Let \( h: X' \rightarrow X \) be a morphism of schemes, and let \( Y' \) be a closed subscheme of \( X' \) contained in \( h^{-1}(Y) \). Then the pair of exact functors \( (h^*, h|_{Y'}) : (\mathcal{P}_X, \mathcal{P}_Y) \rightarrow (\mathcal{P}_{X'}, \mathcal{P}_{Y'}) \), together with the natural isomorphism \( \theta(h^*): j_{Y'}^* \circ h^* \rightarrow h|_{Y'}^* \circ j_{Y'}^* \), gives a map \( (h^*, h|_{Y'}) : \text{K}_p(X, Y) \rightarrow \text{K}_p(X', Y') \). We denote this map by \( h^* \). If \( h \) is flat, we get a similar functorial pull-back \( h^*: \text{K}_p(X, Y) \rightarrow \text{K}_p(X', Y') \). The functoriality of \( h^* \) is rather difficult to show directly; we briefly describe the method used in [Coombes], as this will also be applicable when we discuss multiply-relative \( K \)-theory.
Let \( \mathcal{C} \) be a small category. For an object \( c \) of \( \mathcal{C} \), we let \( /c \) denote the category of objects over \( c \), i.e., objects are morphisms \( f: c' \to c \) in \( \mathcal{C} \) and morphisms are commutative triangles and \( X: \mathcal{C} \to \text{Schemes} \) a functor, so \( \{X(c) \mid c \in \mathcal{C}\} \) is a set of schemes indexed by the category \( \mathcal{C} \). Let \( \mathcal{P}/X(c) \) be the category in which an object is a set indexed by \( /c \):

\[
\{ \mathcal{P}(f, c') \in \text{Obj}(\mathcal{P}_{X(c)}) \mid f: c' \to c \text{ is a morphism} \in \mathcal{C} \}
\]

together with a choice of isomorphism \( j_h: \mathcal{P}(f, c') \to X(h)^* (\mathcal{P}(f', c'')) \) for each \( h: (f, c') \to (f'', c''') \). In addition, we require for each \( h: (c', f) \to (c'', f') \) and \( k: (c'', f') \to (c''', f'') \) the diagram

\[
\begin{array}{c}
X(h)^* \mathcal{P}(f', c'') \\
\downarrow j_h \\
\mathcal{P}(f, c') \\
\downarrow_{j_{kh}} \\
X(kh)^* \mathcal{P}(f'', c''')
\end{array}
\]

commutes.

Morphisms in \( \mathcal{C} \) are maps \( g(f, c'): \mathcal{P}(f, c') \to \mathcal{Q}(f, c') \) so that the obvious diagram commutes. Given a morphism \( g: b \to c \) in \( \mathcal{C} \), we get a functor

\[ g^*: \mathcal{P}/X(c) \to \mathcal{P}/X(b) \]

by restricting to the subcategory \( /b \) of \( /c \). Coombs then shows that \( (gh)^* = h^* g^* \), and that the projection \( \mathcal{P}/X(c) \to \mathcal{P}_{X(c)} \) is an equivalence of categories. In addition, enlarging the indexing category is compatible with this equivalence. Thus, replacing the spaces \( BQ \mathcal{P}_{X(c)} \) with \( BQ \mathcal{P}/X(c) \), we get a functor from \( \mathcal{C} \) to \( \text{Top} \), which makes that functoriality of the homotopy fibers obvious. To avoid overburdening the notation, we will henceforth assume that we have made this construction wherever necessary. A similar construction works for the categories \( \mathcal{M}_K \).

If \( g: (X', Y') \to (X, Y) \) is finite and \( Y' = g^{-1}(Y) \), using the above construction defines a functorial \( g_*: K'_p(X', Y') \to K'_p(X, Y) \). Given such a \( g \), and a flat map \( h: (Z, W) \to (X, Y) \) with \( W \) contained in \( h^{-1}(Y) \), let \( Z' = Z \times_X X' \), \( W' = W \times_Y Y' \), and form the cartesian square

\[
\begin{array}{c}
(Z', W') \to (X', Y') \\
\downarrow \quad \downarrow \\
(Z, W) \to (X, Y)
\end{array}
\]

Then \( h^{-1}(Y') \) contains \( W' \), and \( g^{-1}(W) = W' \), so \( h^*: K'_p(X', Y') \to K'_p(Z', W') \) and \( g^*: K'_p(Z', W') \to K'_p(Z, W) \) are defined and the diagram

\[
\begin{array}{ccc}
K'_p(X', Y') & \to & K'_p(Z', W') \\
\downarrow s_* & & \downarrow s^* \\
K'_p(X, Y) & \to & K'_p(Z, W)
\end{array}
\]

commutes. If \( X, Y, Z, \) and \( W \) are smooth, we get a similar commutative diagram for the relative K-theories, for \( g \) finite as above, and \( h \) an arbitrary morphism. To see this,
let $\mathcal{M}^X(Y, Z)$ be the subcategory of $\mathcal{M}(X, Y)$ consisting of sheaves $\mathcal{F}$ such that $\text{Tor}^X_f(\mathcal{F}, \mathcal{O}_Z) = \text{Tor}^X_f(\mathcal{F}, \mathcal{O}_W) = 0$ for $i > 0$, and similarly let $\mathcal{M}_Y$ be the subcategory of $\mathcal{M}_Y$ consisting of sheaves $\mathcal{F}$ such that $\text{Tor}^Y_f(\mathcal{F}, \mathcal{O}_Z) = \text{Tor}^Y_f(\mathcal{F}, \mathcal{O}_W) = 0$ for $i > 0$. Then $f^*_X: \mathcal{M}^X(Y, Z) \rightarrow \mathcal{M}_Y$ restricts to $f^*_X: \mathcal{M}^X(Y, Z) \rightarrow \mathcal{M}_Y$ and $g^*_Y: (\mathcal{P}_X, \mathcal{P}_Y) \rightarrow (\mathcal{M}_X, \mathcal{M}_Y)$ factors through $(\mathcal{M}_X, \mathcal{M}_Y)$. Letting $K^+_p(X, Y)$ be the homotopy group $\pi_{p+1}(K(j^+_Y))$, we get as above a commutative diagram

$$
\begin{array}{c}
K^+_p(Z, W) \xrightarrow{K^+_p} K^+_p(Z', W') \\
\downarrow \phi^+ \quad \downarrow \phi^+
\end{array}
$$

$$
\begin{array}{c}
K^+_p(X, Y) \xrightarrow{K^+_p} K^+_p(Z, W) \\
\uparrow \quad \uparrow
\end{array}
$$

$$
\begin{array}{c}
K^+_p(X, Y) \xrightarrow{K^+_p} K^+_p(Z, W) \\
\end{array}
$$

where the bottom two isomorphisms come from the resolution theorem, and the five lemma.

1.2. ADDITIVITY FOR RELATIVE K-THEORY. — The additivity theorem of Quillen for an exact sequence of exact functors extends to relative K-theory. To see this, it is convenient to use Waldhausen's [W] construction of the homotopy fiber of an exact functor $f: \mathcal{A} \rightarrow \mathcal{B}$. This is the simplicial set $F(f)$, with $n$-simplices

$$F_n(f) = \{ (A_0 \hookrightarrow \ldots \hookrightarrow A_n, B_0 \hookrightarrow \ldots \hookrightarrow B_n, \omega) \}
$$

where the $A_i$'s are objects of $\mathcal{A}$, the $B_j$'s are objects of $\mathcal{B}$, and $\omega$ is an isomorphism

$$f(A_i/A_0 \hookrightarrow \ldots \hookrightarrow A_n/A_0) \rightarrow (B_0 \hookrightarrow \ldots \hookrightarrow B_n).
$$

Included in this is the data of compatible choices of the quotients $A_i/A_j$ and $B_j/B_j$ for $i > j$. The boundary maps $d_i$ are "omit the $i$-th term" for $i \geq 1$, and $d_0$ is "mod out by $A_0$ (resp. $B_0$)". Given an exact functor $f_0: \mathcal{A}_0 \rightarrow \mathcal{B}_0$, a pair of exact functors $G: \mathcal{A}_0 \rightarrow \mathcal{A}$, $H: \mathcal{B}_0 \rightarrow \mathcal{B}$, and a natural isomorphism $\theta: f \circ G \rightarrow H \circ f_0$, we get a map of simplicial sets

$$(G, H, \theta): F(f_0) \rightarrow F(f)
$$

by

$$(G, H, \theta)((A, B, \omega)) = (G(A), H(B), H(\omega) \circ \theta(d_0 A)).
$$

In addition, Waldhausen shows that $\Omega BQF(f)$ is a natural model for the homotopy fiber of $BQf: BQ\mathcal{A} \rightarrow BQ\mathcal{B}$. We now show

**Proposition 1.1.** — Let $f_0: \mathcal{A}_0 \rightarrow \mathcal{B}_0$, $f: \mathcal{A} \rightarrow \mathcal{B}$ be exact functors, and let

$$0 \rightarrow (G', H', \theta') \rightarrow (G' H' \theta) \rightarrow (G'', H'', \theta'') \rightarrow 0
$$

be an exact sequence of functors from $(\mathcal{A}_0, \mathcal{B}_0)$ to $(\mathcal{A}, \mathcal{B})$, with compatible natural isomorphisms. Then

$$(G, H, \theta)^* = (G', H', \theta')^* + (G'', H'', \theta'')^*
$$

as maps $K^+_p(f_0) \rightarrow K^+_p(f)$. 

Proof. — Let $E(f)$ be the simplicial set with $E_n(f)$ consisting of short exact sequences $E$ in $F_n(f)$:

$$E = 0 \to sE \to tE \to qE \to 0.$$ 

The exact functor $(s, q): E_n(f) \to (F_n(f))^2$ induces by [Quillen] a homotopy equivalence $Q(s, q): QC_n(f) \to Q(F_n(f))^2$, hence a homotopy equivalence $BQ_n(f) \to (BQF_n(f))^2$. Let $\oplus: (BQF_n(f))^2 \to BQ(f)$ be the section $(P, Q) \to (0 \to P \to P \oplus Q \to Q \to 0)$ to $(s, q)$. An exact sequence of pairs of exact functors with compatible natural isomorphisms

$$0 \to (G', H', \theta') \to (G' H' \theta) \to (G'', H'', \theta'') \to 0$$

from $(A, B)$ to $(A, B)$ gives an exact functor $\sigma: F.(f) \to E.(f)$. A choice of homotopy from $id_{BQ_n(f)}$ to $\oplus \circ (s, q)$ gives a homotopy from $BQ(G, H, \theta)$ to $BQ(G', H', \theta') \oplus BQ(G'', H'', \theta'')$. This gives the desired additivity.

1.3. PRODUCTS. — In [Weibel] products in relative $K$-theory are constructed using the Waldhausen construction above. More specifically, there are functorial products

$$\bigcup: K_p(X, Y) \otimes K_q(X) \to K_{p+q}(X, Y).$$

Moreover, if $f: X' \to X$ is a finite morphism with $\mathcal{O}_{X'}$ projective as an $\mathcal{O}_X$ module, and $Y' = f^{-1}(Y)$, then we have the projection formulae:

$$\begin{align*}
(f_*(\alpha \cup f^*(\beta))) &= f_*(\alpha) \cup \beta; & \alpha \in K_p(X', Y'), \beta \in K_q(X) \\
(f_*(\alpha) \cup f^*(\beta)) &= \alpha \cup f_*(\beta); & \alpha \in K_p(X, Y), \beta \in K_q(X').
\end{align*}$$

(1.4)

1.4. The five lemma gives the homotopy property: Let $\pi: \mathbb{A}^1_X \to X$ be the projection, $Y$ a closed subscheme of $X$ with $\mathcal{O}_Y$ having finite Tor dimension over $\mathcal{O}_X$. Then $\pi^*: K'_p(X, Y) \to K'_p(\mathbb{A}^1_X, \mathbb{A}^1_Y)$ is an isomorphism. If $X$ and $Y$ are smooth, then $\pi^*: K_p(X, Y) \to K_p(\mathbb{A}^1_X, \mathbb{A}^1_Y)$ is an isomorphism.

1.5. THE LOCALIZATION SEQUENCE. — Let $(X, Y)$ be as above with $\mathcal{O}_Y$ of finite Tor dimension over $\mathcal{O}_X$. Let $Z$ be a closed subscheme of $X$ with $\mathcal{O}_Z$ in $\mathcal{M}(X, Y)$. Let $U = X - Z$, $Y_U = Y \cap U$, $Y_Z = Y \cap Z$. By the resolution theorem, the inclusions $\mathcal{M}(X, Y) \to \mathcal{M}_X$, $\mathcal{M}(U, Y_U) \to \mathcal{M}_U$, and $\mathcal{M}(Z, Y_Z) \to \mathcal{M}_Z$ induce homotopy equivalences on the $Q$ constructions. In addition, the localization theorem of Quillen shows that

$$\begin{align*}
K'(Z, Y_Z) \to K'(X, Y) \to K'(U, Y_U) \\
\downarrow \quad \downarrow \quad \downarrow \\
BQ_M(Z, Y_Z) \to BQ_M(X, Y) \to BQ_M(U, Y_U) \\
\downarrow \quad \downarrow \quad \downarrow \\
BQ_M(Y_Z) \to BQ_M(Y) \to BQ_M(Y_U)
\end{align*}$$

is a commutative diagram of homotopy fiber sequences. The Quatzcoatl lemma then shows that the natural map

$$K'(Z, Y_Z) \to \text{fiber}(K'(X, Y) \to K'(U, Y_U))$$

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is a homotopy equivalence. This gives a long exact localization sequence

\[
(1.5) \quad K'_p(Z, Y_2) \to K'_p(X, Y) \to K'_p(U, Y \cup) \to K'_{p-1}(Z, Y_2) \to \ldots
\]

Swan has shown in [Swan] that the localization sequence for K' theory is natural; the
same argument applied to the Waldhausen construction for K'(X, Y), K'(U, Y \cup) and
K'(Z, Y_2) shows that (1.5) is natural for pullbacks by flat maps, and pushforward for
finite maps.

1.6. QUILLEN SPECTRAL SEQUENCE. — In this section we suppose that Y is a locally
principal subscheme of X, defined locally by a non zero-divisor. Then \( \mathcal{M}(X, Y) \) is just the
category of coherent sheaves having no \( I_Y \)-torsion. In particular, if Z is a reduced closed
subscheme, then \( \mathcal{O}_Z \) is in \( \mathcal{M}(X, Y) \) if and only if Z intersects Y properly. Furthermore, if
\( \mathcal{F} \) is in \( \mathcal{M}(X, Y) \), then \( \text{supp}(\mathcal{F}) \) intersects Y properly.

Let \( \mathcal{M}_{(i)}(X, Y) \) be the subcategory of \( \mathcal{M}(X, Y) \) of sheaves \( \mathcal{F} \) with \( \text{codim}_X \text{supp}(\mathcal{F}) \geq i \).
Then \( j^* \) maps \( \mathcal{M}(X, Y) \) to \( \mathcal{M}_Y \); let \( K'(X^i, Y^i) \) be the homotopy fiber of
\( BQ_{j^*} : \mathcal{M}_{(i)}(X, Y) \to \mathcal{M}_Y \). By the remarks above, the map

\[
(1.6) \quad \lim_{Z \subset X, \ Z \text{ reduced, closed}} K'(Z, Z_Y) \to K'(X^i, Y^i)
\]

is a homotopy equivalence. Let \( \mathcal{M}_{(X, Y)}^{(k)} \) be the direct limit

\[
\mathcal{M}_{(X, Y)}^{(k)} = \lim_{\to} \mathcal{M}_{(X-Z, Y-Z)}^{(i)}
\]

and let \( \mathcal{M}_Y^{(k)} \) be the direct limit

\[
\mathcal{M}_Y^{(k)} = \lim_{\to} \mathcal{M}_Y^{(i)}
\]

Let \( K'(X^{(i)}, Y^{(i)}) \) be the homotopy fiber of \( BQ_{j^*} : \mathcal{M}_{(X, Y)}^{(k)} \to BQ, \mathcal{M}_Y^{(k)} \). Then (1.5), (1.6),
and a limit argument shows that

\[
K'(X^k, Y^k) \to K'(X^i, Y^i) \to K'(X^{(i)}, Y^{(i)})
\]
is a homotopy fiber sequence. Let

\[ K'_p(X^i, Y^i) = \pi_{p+1}(K'(X^i, Y^i)), \]
\[ \overline{K}'_0(X^i, Y^i) = \text{Im}(K'_0(X^i, Y^i) \to K'_0(X^i, Y^i)), \]
\[ \overline{K}'_0(\mathcal{M}^i_{(X, Y)}) = \text{Im}(K'_0(\mathcal{M}^i_{(X, Y)}) \to K'_0(\mathcal{M}^i_{(X, Y)})). \]

The method of the exact couple gives a spectral sequence

\[ E^p_1(X, Y) \Rightarrow K^p_*(X, Y), \]
\[ E^p_1(X \mid Y) = \begin{cases} K^p_*(X^p, Y^p); & -p+1 > 0, p \leq \dim Y, \\ \overline{K}'_0(X^p, Y^p); & -p+1 = 0, \\ 0; & \text{otherwise.} \end{cases} \]

The filtration on \( K'_*(X, Y) \) is the "topological" filtration:

\[ F^p K'_*(X, Y) = \text{Im}(K'_*(X^p, Y^p) \to K'_*(X, Y)). \]

We denote \( F^p K'_*(X, Y) \) by \( K'_*(X, Y)^p \) and the \( E_\infty \) term \( \text{Gr}^p K'_*(X, Y) \) by \( K'_*(X, Y)^{p+1} \).

We can similarly form an \( E_1 \) spectral sequence converging to \( K'_*(X) \):

\[ E^p_1(X \mid Y) \Rightarrow K^p_*(X), \]
\[ E^p_1(X \mid Y) = \begin{cases} K^p_*(\mathcal{M}^p_{(X, Y)}); & -p+1 > 0, p \leq \dim Y, \\ \overline{K}'_0(\mathcal{M}^p_{(X, Y)}); & -p+1 = 0, \\ 0; & \text{otherwise,} \end{cases} \]

and an \( E_1 \) spectral sequence converging to \( K'_*(Y) \):

\[ E^p_1(Y) \Rightarrow K^p_*(Y), \]
\[ E^p_1(Y) = K^p_*(\mathcal{M}^p_{(X, Y)}). \]

If the maps \( K'_0(X^p, Y^p) \to K'_0(X^p, Y^p + 1) \) and \( K'_0(\mathcal{M}^p_{(X, Y)}) \to K'_0(\mathcal{M}^p_{(X^p, Y^p)}) \) are surjective for \( p = 0, \ldots, \dim Y \), then we get a long exact sequence of \( E_1 \) terms:

\[ \to E^p_1(Y) \to E^p_1(X, Y) \to E^p_1(X \mid Y) \to E^p_1(Y) \to \]

compatible with the differentials. We also have the usual Quillen spectral sequence on \( X \):

\[ E^p_1(X) \Rightarrow K^p_*(X), \]
\[ E^p_1(X) = K^p_*(\mathcal{M}^p_{(X, Y)}). \]

The inclusion \( \mathcal{M}^p_{(X, Y)} \to \mathcal{M}^p_{(X^p, Y^p)} \) gives a map of \( E_1 \) terms compatible with the differentials.
Lemma 1.2. — Suppose $X$ is quasi-projective over a Noetherian ring, and regular in a neighborhood of $Y$. Then the mapping $$(i_{X-Y})^*: \mathcal{M}_{X,Y}^{p+1} \to \mathcal{M}_{X-Y}^{p+1}$$ induces an isomorphism

$$K_0(\mathcal{M}_{X,Y}^{p+1}) \iso K_0(\mathcal{M}_{X-Y}^{p+1}) \oplus \bigoplus_{x \in (X-Y)^P} \mathbb{Z}.$$

In addition, if $X$ is a scheme over a field $k$, then $(i_{X-Y})^*$ gives rise to short exact sequences $(q \geq 1)$:

$$0 \to K_q(\mathcal{M}_{X,Y}^{p+1}) \to K_q(\mathcal{M}_{X-Y}^{p+1}) \to K_{q-1}(\mathcal{M}_{Y}^{p+1}) \to 0 \tag{\delta}$$

Here, $\delta$ is the composition

$$K_q(\mathcal{M}_{X,Y}^{p+1}) \xrightarrow{inc} K_q(\mathcal{M}_{X-Y}^{p+1/p+2}) \xrightarrow{\delta} K_{q-1}(\mathcal{M}_{X}^{p+1/p+2}) \xrightarrow{proj} K_{q-1}(\mathcal{M}_{Y}^{p+1}) $$

Proof: For $Z$ a closed subset of $X$, let $\mathcal{M}_X(Z)$ denote the category of $\mathcal{O}_X$-modules $\mathcal{F}$ with supp$(\mathcal{F}) \subset Z$. Let $\mathcal{M}_{X|Y}^{p+1}$ be the direct limit

$$\mathcal{M}_{X|Y}^{p+1} = \lim_{\rightarrow} \mathcal{M}_{X-W}(Z-W)$$

with $W \subset Z \subset X$, codim$_X Z = p$, codim$_X W = p + 1$, $W$, $Z$ intersect $Y$ properly.

The resolution theorem [Quillen] shows that $Q.\mathcal{M}_{X,Y}^{p+1} \to Q.\mathcal{M}_{X-Y}^{p+1}$ is a homotopy equivalence. Indeed, given $\mathcal{F}$ in $\mathcal{M}_{X-W}(Z-W)$, with $Z, W$ as above, we can find a closed subscheme $\mathcal{Z} \subset X-W$, with codim$_{X-W} \mathcal{Z} = p$, and $\mathcal{O}_\mathcal{Z}$ having no $\mathcal{I}_Y$ torsion, such that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{Z}$ module. Take a surjection

$$0 \to K \to (\mathcal{O}_\mathcal{Z}(N))^a \to \mathcal{F} \to 0,$$

then $K$ is also $\mathcal{I}_Y$ torsion free, hence $K$ and $(\mathcal{O}_\mathcal{Z}(N))^a$ determine elements of $\mathcal{M}_{X,Y}^{p+1}$, and the hypotheses of the resolution theorem are satisfied.

The localization theorem [Quillen] shows that

(\star) \quad Q.\mathcal{M}_Y^{p+1} \to Q.\mathcal{M}_{X,Y}^{p+1} \to Q.\mathcal{M}_{X-Y}^{p+1}

is a homotopy fiber sequence. On the other hand, let $x$ be a generic point of a codimension $p$ irreducible subscheme of $Y$. Take a closed codimension $p$ reduced irreducible subscheme $D$ of $X$ such that $D$ contains $x$, $D$ is regular at $x$, and $D$ intersects $Y$ properly. Let $R$ be the semi-local ring of $D \cap Y$ in $D$, $R^N$ the normalization of $R$. 

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Then \( R^N \) is regular, semi-local, and one dimensional, hence a PIR. In particular, the inclusion map \( x \to \text{Spec}(R^N) \) induces the 0 map

\[
Z \cong K_0(k(x)) \to K_0(R^N) \to K'_0(R).
\]

Thus \( K_0(\mathcal{M}_{q}^{(p+1)}|_Y \to K_0(\mathcal{M}_{X}^{(p+1)}|_Y \) is the zero map. This, together with the localization sequence derived from (\( \star \)), shows that the map \( K_0(\mathcal{M}_{X}^{(p+1)}|_Y \to K_0(\mathcal{M}_{X}^{(p+1)}|_Y \) is an isomorphism, which proves the first assertion.

For the second, as \( X \) is a scheme over a field, the ring \( R^N \) is regular, semi-local, and contains a field. Thus, by Gersten's conjecture (proved by Quillen [Quillen] in this case), the map

\[
K_q(k(x)) \to K_q(R^N) \to K'_q(R)
\]

is the zero map. Thus \( K_q(\mathcal{M}_{Y}^{(p+1)} \to K_q(\mathcal{M}_{Y}^{(p+1)} \) is the zero map, and the sequence

\[
0 \to K_q(\mathcal{M}_{X}^{(p+1)}|_Y \to K_q(\mathcal{M}_{X}^{(p+1)}|_Y \to K_q^{-1}(\mathcal{M}_{Y}^{(p+1)} \to 0
\]

\[
\uparrow
\]

\[
K_q(\mathcal{M}_{X}^{(p+1)}|_Y)
\]

derived from (\( \star \)) is exact. This proves the second assertion. \( \square \)

**Lemma 1.3.** — Suppose \( X \) is quasi-projective over a Noetherian ring and regular in a neighborhood of \( Y \). Then the map

\[
K_0(\mathcal{M}_{X,y}^{(p)} \to K_0(\mathcal{M}_{X,y}^{(p+1)}; \quad 0 \leq p \leq \dim Y
\]

is surjective. If \( Y \) is regular, then the map

\[
K'_0(\mathcal{M}_{X,y}^{(p)} \to K_0(\mathcal{M}_{X,y}^{(p+1)}; \quad 0 \leq p \leq \dim Y
\]

is surjective.

**Proof.** — By the previous lemma, we have

\[
K_0(\mathcal{M}_{X,y}^{(p+1)} \cong \bigoplus_{x \in (X-Y)^p} Z \cong \bigoplus_{x \in (X-Y)^p} K_0(k(x)).
\]

Similarly,

\[
K_0(\mathcal{M}_{Y,y}^{(p+1)} \cong \bigoplus_{y \in Y^p} Z \cong \bigoplus_{y \in Y^p} K_0(k(y)).
\]

In the commutative ladder with exact rows

\[
\begin{array}{ccc}
\to K'_0(\mathcal{M}_{X,y}^{(p+1)} \to K_0(\mathcal{M}_{X,y}^{(p+1)} \to K_0(\mathcal{M}_{X,y}^{(p+1)})
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \quad \alpha \quad \downarrow \\
K'_0(\mathcal{M}_{X,y}^{(p+1)} \to K_0(\mathcal{M}_{X,y}^{(p+1)} \to K_0(\mathcal{M}_{X,y}^{(p+1)})
\end{array}
\]

\[
\begin{array}{ccc}
\to K'_0(\mathcal{M}_{Y,y}^{(p+1)} \to K_0(\mathcal{M}_{Y,y}^{(p+1)} \to K_0(\mathcal{M}_{Y,y}^{(p+1)})
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow \quad \beta \quad \downarrow \\
K'_0(\mathcal{M}_{Y,y}^{(p+1)} \to K_0(\mathcal{M}_{Y,y}^{(p+1)} \to K_0(\mathcal{M}_{Y,y}^{(p+1)})
\end{array}
\]

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we have compatible splittings \( s_{(X, Y)} \) and \( s_Y \) to \( \alpha \) and \( \beta \), where \( s_{(X, Y)} \) is defined by sending the vector space of rank \( n \) over \( k(x) \), for \( x \) in \( (X - Y)^p \), to the free rank \( n \mathcal{O}_x \) module; \( s_Y \) is defined similarly. This proves the first assertion.

We have the exact relativization sequence

\[
\rightarrow K_1(\mathcal{M}_{(X, Y)}^{(p+1)}) \rightarrow K_1(\mathcal{I}_Y^{(p+1)}) \rightarrow K_0'(X^{(p+1)}, Y^{(p+1)}) \rightarrow \\
\bigoplus_{y \in Y^p} k(y)^* 
\]

Given \( y \) in \( Y^p \), chose a reduced irreducible subscheme \( D \) containing \( y \) as in lemma 1.2. We retain the notations of that lemma. Given \( \alpha \) in \( k(y)^* \), we can find a unit \( u \) in the semi-local ring \( R \) such that \( u \) restrict to \( \alpha \) at \( y \), and \( u \) restrict to \( 1 \) at all other closed points of \( \text{Spec}(R) \). This shows that the map \( K_1(\mathcal{M}_{(X, Y)}^{(p+1)}) \rightarrow K_1(\mathcal{I}_Y^{(p+1)}) \) is surjective, and the map \( \gamma \) in (*) is injective. The second assertion follows from this and the existence of the splittings \( s_{(X, Y)} \) and \( s_Y \).

As a consequence of the lemmas above, if \( X \) is quasi-projective, and regular in a neighborhood of \( Y \), and \( Y \) is regular, then the \( E_1^{p+1} \) terms in the spectral sequences defined above are

\[
E_1^{p+1}(X, Y) = K_0'(X^{(p+1)}, Y^{(p+1)}) \\
E_1^{p+1}(X \mid Y) = K_0(\mathcal{M}_{(X, Y)}^{(p+1)}).
\]

To end this section, we consider an important case of the above. Let \( R \) be a semi-local PIR containing a field \( k_0 \). Let \( I = (\pi) R \) be the Jacobson radical of \( R \), \( R = R/I \). If \( g: T \rightarrow \text{Spec}(R) \) is an \( R \)-scheme, we let \( \bar{T} \) denote the fiber \( g^{-1}(\text{Spec}(R)) \). \( R(T) \) with denote the semi-local ring of \( T \) in \( T \), \( I(T) \) the ideal \( (\pi) R(T) \). We will occasionally abuse standard terminology and refer to the total quotient ring of \( R \) as the quotient field of \( R \).

**Lemma 1.4.** — Suppose that \( X \) is a quasi-projective \( R \)-scheme, smooth over \( R \), and \( Y = X \). Then the map \( E_1^{p+1}(X \mid Y) \rightarrow E_1^{p+1}(X) \) is an isomorphism.

**Proof.** — We apply lemma 1.2 to describe the \( E_1 \) term \( E_1^{p+1}(X \mid Y) \):

\[
0 \rightarrow E_1^{p+1}(X \mid Y) \rightarrow \bigoplus_{x \in (X-Y)^p} K_{-p-q}(k(x)) \xrightarrow{\partial} \bigoplus_{y \in Y^p} K_{-p-q-1}(k(y)) \rightarrow 0.
\]

From this it follows that the \( E_2 \) term is given by

\[
E_2^{p+1}(X \mid Y) = \frac{\ker \left[ \bigoplus_{x \in (X-Y)^p} K_{-p-q}(k(x)) \xrightarrow{\partial} \bigoplus_{x \in X^{p+1}} K_{-p-q-1}(k(x)) \right]}{\partial \left[ \ker \left( \bigoplus_{x \in (X-Y)^{p-1}} K_{-p-q+1}(k(x)) \xrightarrow{\partial} \bigoplus_{y \in Y^{p-1}} K_{-p-q}(k(y)) \right) \right]}
\]
Similarly, the $E^2$ term $E^{p,q}_2(X)$ is given by
\[
E^{p,q}_2(X) = \ker \left[ \bigoplus_{x \in X^p} K_{-p-q}(k(x)) \rightarrow \bigoplus_{x \in X^{p+1}} K_{-p-q-1}(k(x)) \right] / \partial \left[ \bigoplus_{x \in X^{p-1}} K_{-p-q+1}(k(x)) \right].
\]

Let $\xi$ be a class in $E^{p,q}_2(X)$. Represent $\xi$ by $z$,
\[
z \in \bigoplus_{x \in X^p} K_{-p-q}(k(x)).
\]
We write $z$ as
\[
z = \sum_{x \in X^p} z_x; \quad z_x \in K_{-p-q}(k(x)).
\]

If $x$ is a codimension $p-1$ point of $Y$ with $z_x \neq 0$, take the $D$ containing $x$ as in lemma 2.1. By Gersten's conjecture applied to the regular ring $R(D)^h$, we can find $\eta$ in $K_{-p-q+1}(k_0(D))$ with
\[
\partial \eta = z_x + \tau; \quad \tau = \sum_{x \in X^{p-1} \cap D} \tau_x.
\]
so that $\tau_y = 0$ for all $y$ in $Y^{p-1}$. Then $z' := z - \partial \eta$ is a new representative for $\xi$; repeating this for all $x$ in $Y^{p-1}$ with $z_x \neq 0$, we see that $\xi$ is in the image of $E^{p,q}_2(X | Y)$. This proves surjectivity; the proof of injectivity is similar and will be left to the reader. \(\square\)

1.7. RELATIVE $K$-THEORY OF PROJECTIVE SPACES. — Let $S$ be a scheme, $\mathcal{S}$ a closed subscheme, $\mathcal{V}$ a locally free sheaf of rank $n$ on $S$, $\mathcal{V}^0$ the restriction to $S$, $X = \mathbb{P}(\mathcal{V})$, $\overline{X} = \mathbb{P}(\mathcal{V}^0)$. The pair of exact functors
\[
(F_\rho, F_\rho): (\mathcal{P}_S, \mathcal{P}_\mathcal{S}) \to (\mathcal{P}_X, \mathcal{P}_{\overline{X}})
\]
\[
((\mathcal{E}, \mathcal{E}) \to (\mathcal{E} \otimes \mathcal{O}_X(-i), \mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-i)),
\]

Together with the natural isomorphism
\[
\theta_i(\mathcal{E}): i_{\overline{X}*}(\mathcal{E} \otimes \mathcal{O}_X(-i)) \to i_{\overline{X}*}(\mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-i))
\]
gives for each $i$ a homomorphism
\[
(F_\rho, F_\rho, \theta_i)_*: K_*(S, \mathcal{S}) \to K_*(X, \overline{X}).
\]
Since the maps
\[
\sum_{i=0}^{n-1} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} F_\rho: \bigoplus_{i=0}^{n-1} K_*(S) \to K_*(X),
\]

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and

\[ \sum_{i=0}^{n-1} \sum_{r=0}^{n-1} F_i \cdot K_\ast(S) \to K_\ast(X) \]

are isomorphisms, we get the following computation of \( K_\ast(P(\nu), P(\nu)) \):

(1.9) \[ \sum_{i=0}^{n-1} (F_i, F_i, 0)_{\ast} : \oplus K_\ast(S, S) \to K_\ast(P(\nu), P(\nu)) \]

is an isomorphism.

Now suppose \( S = \text{Spec}(R) \), where \( R \) is a semi-local PIR containing an infinite field \( k_0 \). We retain the notations of the end of the previous section. C. Sherman [Sherman] has shown that the Quillen spectral sequence converging to \( K_\ast(P_\ast) \) degenerates at \( E_2 \), for \( k \) a field or a semi-local ring; we now prove an analogue for the relative situation.

**Proposition 1.5.** — Let \( \nu = R^{n+1} \). Let \((X, \tilde{X}) = (P(\nu), P(\nu))\). The spectral sequence

\[ E_{1}^{p, q}(P(\nu), P(\nu)) \Rightarrow K_{-p-q}(P(\nu), P(\nu)) \]

degenerates at \( E_2 \). The \( E_2 \) term is given by

\[ E_{2}^{p, q}(P(\nu), P(\nu)) \cong K_{-p-q}(R, \tilde{R}). \]

The isomorphism above is given by the composition

\[ K_{-p-q}(R, \tilde{R}) \to K_{-p-q}(P^{n-p}, P^{n-p}) \to K_{-p-q}(\mathcal{M}_X^{p+1}, \tilde{X}), \]

where \( P^{n-p} \) is any codimension \( p \) linear subspace of \( P(\nu) \), and

\[ \pi_p : P^{n-p} \to \text{Spec}(R) \]

is the projection. Finally, let \( \gamma \) be the class of \( \mathcal{O}_X(-1) \) in \( K_0(X) \). Then the topological filtration on \( K_\ast(X, \tilde{X}) \) is given by

\[ K_\ast(X, \tilde{X})^p = \sum_{j \geq p} (1-\gamma)^p \cup \pi^\ast(K_\ast(R, \tilde{R})). \]

**Proof.** — We denote \( P(R^{n+1}) \) by \( \mathbb{P}^n \); similarly denote the affine space \( \text{Spec}(R[X_1, \ldots, X_n]) \) by \( \mathbb{A}^n \). We first prove the following

**Claim.** — Let \( Z \) be a reduced closed codimension \( p \) subscheme of \( X \), flat over \( R \). Then

\[ \text{Im}(K_\ast(Z, Z) \to K_\ast(X^{p-1}, X^{p-1})) \subset \text{Im}(K_\ast(P^{n-p}, P^{n-p}) \to K_\ast(X^{p-1}, X^{p-1})), \]

where \( P^{n-p} \) is any codimension \( p \) linear subspace of \( \mathbb{P}^n \).

**Proof of claim.** — Take a codimension 1 linear subspace \( P^{n-1} \) of \( \mathbb{P}^n \) such that no component of \( Z \) is contained in \( P^{n-1} \). Let \( p \) be an \( R \)-valued point of \( \mathbb{P}^{n-1} - Z \). Then
projection from $p$ defines a linear map $f_p$

$$f_p : \mathbb{P}^n -\mathbb{P}^{n-1} \to \mathbb{A}^{n-1}.$$ 

In addition, letting $Z^0$ be the intersection $Z \cap \mathbb{A}^n$, the restriction

$$f_{p|Z^0} : Z^0 \to \mathbb{A}^n$$

is a finite morphism. Let $\eta$ be in $K_*(Z, \mathbb{Z})$, $\eta^0 = \text{res}_\eta^0(\eta)$ in $K_*(Z^0, \mathbb{Z}^0)$. We have the diagram

$$\begin{array}{ccc}
\mathbb{A}^n & \leftarrow & \mathbb{A}^n \times_{\mathbb{A}^{n-1}} Z^0 \cong \mathbb{A}^n_{Z^0} \\
\downarrow f_p & \downarrow s & \downarrow \iota \\
\mathbb{A}^{n-1} & \leftarrow & Z^0
\end{array}$$

where $s$ is the section induced by the inclusion $\iota$ of $Z^0$ in $\mathbb{A}^n$. Since $\mathbb{A}^n_{Z^0}$ is the trivial line bundle over $Z^0$, we can find a regular function $f$ on $\mathbb{A}^n_{Z^0}$ with $s(Z^0)$ defined by the ideal $(f)$. Let $W^0$ be the image $q^0(\mathbb{A}^n_{Z^0})$. Then, letting $\overline{\cdot}$ denote reduction mod $t$, we have the exact sequences of functors

$$0 \to q_\ast g^* \to q_\ast g^* \to q_\ast s_\ast \to 0; \quad q_\ast s_\ast = i_\ast,$$

and

$$0 \to \overline{q_\ast g^*} \to \overline{q_\ast g^*} \to \overline{q_\ast s_\ast} \to 0; \quad \overline{q_\ast s_\ast} = \overline{i_\ast},$$

from $\mathcal{M}(Z^0, \mathbb{Z}^0)$ to $\mathcal{M}(W^0, \mathbb{Z}^0)$ and from $\mathcal{M}(Z^0, \mathbb{Z}^0)$ to $\mathcal{M}(W^0, \mathbb{Z}^0)$ respectively. This, together with the natural isomorphisms

$$\theta : j^\ast \circ (q_\ast g^*) \to (\overline{q_\ast g^*}) \circ j^\ast$$

and

$$\theta' : j^\ast \circ i_\ast \to \overline{i_\ast} \circ j^\ast$$

gives an exact sequence

$$0 \to (q_\ast g^*, \overline{q_\ast g^*}, \theta) \xrightarrow{\times (f, \overline{i})} (q_\ast g^*, \overline{q_\ast g^*}, 0) \to (i_\ast, \overline{i_\ast}, \theta') \to 0$$

of pairs of functors with compatible natural isomorphisms. Thus

$$i^0_\ast : K_*(Z^0, \mathbb{Z}^0) \to K_*(W^0, \mathbb{Z}^0);$$

and $$i^0 : Z^0 \to W^0$$

the inclusion, is the zero map, by Proposition 1.1.

Let $W$ be the closure of $W^0$ in $\mathbb{P}^n$. From the localization sequence

$$\to K_*(W \cap \mathbb{P}^{n-1}, \mathbb{W} \cap \mathbb{P}^{n-1}) = K_*(W, \mathbb{W}) \to K_*(W^0, \mathbb{W}^0) \to,$$
we see that \( i_\ast(\eta) \) in \( K_\ast(W, W) \) is the image of a class \( \xi \) of \( K_\ast(W \cap P^{n-1}, W \cap P^{n-1}) \).

By induction on \( n \), there is a \( P^{n-p} \) in \( X_i \):= \( P^{n-1} \), and an element \( \alpha \) of \( K_\ast(P^{n-p}, P^{n-p}) \) with the image of \( \xi \) in \( K_\ast(X_i^{p-2}, X_i^{p-2}) \) equal to the image of \( \alpha \) in \( K_\ast(X_i^{p-2}, X_i^{p-2}) \). Thus, the image of \( \xi \) in \( K_\ast(X^{p-1}, X^{p-1}) \) equals the image of \( \alpha \) in \( K_\ast(X^{p-1}, X^{p-1}) \).

On the other hand, if \( Y \) and \( Y' \) are two \( P^{n-p} \)'s contained in a \( P^{n-p+1} := Y'' \), then the exact sequences

\[
0 \to J_Y \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0
\]

and

\[
0 \to J_Y' \to \mathcal{O}_{Y''} \to \mathcal{O}_{Y'} \to 0,
\]

together with the isomorphisms \( J_Y \cong \mathcal{O}_{Y'}(-1) \cong J_{Y'} \), and the isomorphism \( K_\ast(P^n, P^n) \cong \{K_\ast(R, R)\}_\ast \), shows that

\[
\text{Im}[K_\ast(P^{n-p}, P^{n-p})] \to K_\ast(X^{p-1}, X^{p-1})]
\]

is independent of the choice of the \( P^{n-p} \) in \( P^n \). This proves the claim. \( \square \)

Next, we note that \( i_\ast: K_\ast(P^{n-p}, P^{n-p}) \to K_\ast(P^n, P^n) \) is injective for all linear subspaces \( i: P^{n-p} \to P^n \). Indeed, we have the localization sequence

\[
\to K_\ast(P^{n-1}, P^{n-1}) \ast \to K_\ast(P^n, P^n) \ast \to K_\ast(\mathbb{A}^n, \mathbb{A}^n) \ast \to
\]

so \( j^* \) is split by \( \pi^*_n (\pi^*_{n-1})^{-1} \), and \( i_\ast \) is thus injective. The general case follows by induction. As a consequence, the map

\[
(\star) \quad K'_\ast(X^p, \mathbb{X}^p) \to \text{Im}[K'_\ast(X^{p/p+2}, \mathbb{X}^{p/p+2}) \to K'_\ast(X^{p/p+2}, \mathbb{X}^{p/p+2})]
\]

is surjective. Indeed, let \( \eta \) be in \( K'_\ast(X^{p/p+2}, \mathbb{X}^{p/p+2}) \). Take \( \xi \) to be the element \( \partial(\eta) \) in \( K'_{-1}(X^{p+2}, \mathbb{X}^{p+2}) \), where \( \partial \) is the boundary in the localization sequence

\[
K'_\ast(X^p, \mathbb{X}^p) \to K'_\ast(X^{p/p+2}, \mathbb{X}^{p/p+2}) \to K'_{-1}(X^{p+2}, \mathbb{X}^{p+2}) \to \cdots
\]

Then \( \xi \) goes to zero in \( K'_{-1}(X^p, \mathbb{X}^p) \), on the other hand, we can find a \( \tau \) in \( K'_{-1}(P^{n-p-2}, P^{n-p-2}) \) with

\[
\text{Im}[\xi \to K'_{-1}(X^{p+1}, \mathbb{X}^{p+1})] = \text{Im}[\tau \to K'_{-1}(X^{p+1}, \mathbb{X}^{p+1})].
\]

As \( K_{-1}(P^{n-p-2}, P^{n-p-2}) \to K'_{-1}(X^p, \mathbb{X}^p) \) is injective, this forces \( \tau \) to be zero, hence \( \xi \) goes to zero in \( K'_{-1}(X^{p+1}, \mathbb{X}^{p+1}) \). Let \( \delta \) be the boundary in the localization sequence

\[
K'_\ast(X^p, \mathbb{X}^p) \to K'_\ast(X^{p/p+1}, \mathbb{X}^{p/p+1}) \to K'_{-1}(X^{p+1}, \mathbb{X}^{p+1}) \to \cdots
\]

and let \( \eta ' \) be the image of \( \eta \) in \( K'_\ast(X^{p/p+1}, \mathbb{X}^{p/p+1}) \). Then the element \( \delta(\eta ' \) of \( K'_{-1}(X^{p+1}, \mathbb{X}^{p+1}) \) is the image of \( \xi \), hence \( \delta(\eta) = 0 \). Thus there is a \( \sigma \) in \( K'_\ast(X^p, \mathbb{X}^p) \).
with

$$\text{Im} [\sigma \to K'_q(X^{p/p+1}, R^{p/p+1})] = \eta',$$

hence (★) is surjective, as claimed.

An immediate consequence of the surjectivity of (★) is the degeneration of our spectral sequence at $E_2$. In addition, the surjectivity of (★), together with the claim proved above, shows that the map

$$s_p: K_{-p-q}(\mathbb{P}^n-\mathbb{P}^n, \mathbb{P}^n) \to E_{2-p}^{p,q} = E_{\infty}^{p,q}$$

is surjective. Since the subgroup $i_*(K_{-p-q}(\mathbb{P}^n-\mathbb{P}^n, \mathbb{P}^n))$ of $K_{-p-q}(\mathbb{P}^n-\mathbb{P}^n, \mathbb{P}^n)$ clearly goes to zero under $s_p$, for any hyperplane $i: \mathbb{P}^n \to \mathbb{P}^n$, this proves the statement about the topological filtration on $K_*(\mathbb{P}^n, \mathbb{P}^n)$. Finally, let $i_p: \mathbb{P}^n \to \mathbb{P}^n$ be the inclusion, $\pi_p: \mathbb{P}^n \to \text{Spec}(R)$ the projection. Since

$$i_p^* \pi_p^*(\alpha) \equiv (1 - \gamma)^p \mod \sum_{j > p} (1 - \gamma)^j \cup K_*(R, \overline{R})$$

for $\alpha$ in $K_*(R, \overline{R})$, we see that $i_p^* \pi_p^*(\alpha) = 0$ implies $\alpha = 0$, so $s_p$ is an isomorphism, which completes the proof. □

**Corollary 1.6.** — *The sequence*

$$0 \to K_*(R, \overline{R}) \to K_*(R(\mathbb{P}^n), \overline{R(\mathbb{P}^n)}) \to K_{* - 1}(\mathbb{P}^{n/2}, \mathbb{P}^{n/2})$$

*is exact.*

*Proof:*

$$\ker(d_1^{* 0}) = E_1^{* 0}$$

$$= E_{\infty}^{* 0}$$

$$= K_*(\mathbb{P}^n, \mathbb{P}^n)^{0/1}.$$  

On the other hand,

$$K_*(\mathbb{P}^n, \mathbb{P}^n)^{0/1} = \bigoplus_{i=0}^{n-1} \gamma^i \cup \pi^* K_*(R, \overline{R})(1 - \gamma) \cup \pi^* K_*(R, \overline{R})$$

$$\cong \pi^* K_*(R, \overline{R}).$$  □

**1.8. Relative K_1.** — We return briefly to a more general setting. Let $X$ be a smooth scheme over a field $k$, $Y$ a locally principal closed subscheme of $X$. We want to compute $K_1(X^{p/p+1}, Y^{p/p+1}).$

**Lemma 1.8.** — *Let Z be a reduced semi-local k-scheme of (Krull) dimension one, Z a principal closed subscheme defined by a non zero divisor. If $Z_i$ is an irreducible component of $Z$, let $\bar{Z}_i$ denote $Z_i \cap \bar{Z}$. Suppose for each closed point $z$ of $Z$, there is an irreducible ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPERIEURE
component $Z_i$ of $Z$ with localization $\tilde{Z}_{i(z)}$ of $\tilde{Z}$ at $z$ isomorphic to $\text{Spec}(k(z))$. Then the two sequences

$$0 \to K_i'(Z, \tilde{Z}) \to K_i'(Z) \to K_i'(Z) \to 0$$

and

$$0 \to K_0'(Z, \tilde{Z}) \to K_0'(Z) \to K_0'(Z) \to 0$$

are exact.

Proof: We need only show that $K_i'(Z) \to K_i'(\tilde{Z})$ is surjective for $i=1, 2$. By devissage,

$$K_i'(Z) \cong \bigoplus_{z \in \tilde{Z}} K_i(k(z)).$$

The assumption $\tilde{Z}_{i(z)} \cong \text{Spec}(k(z))$ implies that the composition

$$K_1(Z_i) \to K_1'(Z) \to K_1(k(z))$$

is just the restriction map

$$\Gamma(Z_0 \otimes_{\tilde{Z}} k) \to k(z)^*.$$

The result for $K_1'$ now follows from the Chinese remainder theorem. Since $K_2(k(z))$ is generated by symbols, a similar argument proves surjectivity for $K_2$, completing the proof. □

Corollary 1.9. — Let $X$ be a smooth scheme over $k$, $Y$ a locally principal closed subscheme. Then the sequence

$$0 \to K_i'(X^{p+1}, Y^{p+1}) \to K_i(\mathcal{M}_{X,Y}^{p+1}) \to K_i(\mathcal{M}_Y^{p+1}) \to 0$$

is exact for $i=0, 1$. In addition, we have the exact sequence

$$0 \to K_1(\mathcal{M}_{X,Y}^{p+1}) \to \bigoplus_{x \in (X-Y)^p} k(x)^* \to \bigoplus_{y \in Y^p} Z \to 0$$

and an isomorphism

$$K_0(\mathcal{M}_{X,Y}^{p+1}) \sim \bigoplus_{x \in (X-Y)^p} Z.$$

Proof: — The first statement follows from lemma 1.8 and a limit argument; the second is a special case of lemma 1.2. □
**Remark.** — One important consequence of corollary 1.9 is encapsulated in the commutative diagram:

\[
\begin{array}{c}
K'_2(X^{p-1/p}, X^{p-1/p}) \rightarrow \bigoplus_{x \in X^{p-1}} K_2(k(x)) \\
\downarrow \epsilon \\
0 \rightarrow K_1(X^{p+p+1}, X^{p+p+1}) \rightarrow \bigoplus_{x \in X^p} K_1(k(x))
\end{array}
\]

where \( T \) is the usual tame symbol map. In words, for \( \eta \) in \( K'_2(X^{p-1/p}, Y^{p-1/p}) \), we can compute \( \delta(\eta) \) as the tame symbol \( T(\eta') \), where \( \eta' \) is the image of \( \eta \) in \( \bigoplus_{x \in X^{p-1}} K_2(k(x)) \).

We also have

**Corollary 1.10.** — We retain the conventions immediately preceding lemma 1.4. Let \( S = \text{Spec}(R) \), where \( R \) is a semi-local PIR with infinite residue fields, \( \pi: \mathbb{A}^1_S \rightarrow S \) the affine line over \( S \). Let \( s: S \rightarrow \mathbb{A}^1_S \) be a section to \( \pi \), \( B \) the semi-local ring of \( s(\mathcal{S}) \) in \( \mathbb{A}^1_S \), and let \( L = R(\mathcal{A}_S) \) be the semi-local ring of \( \mathcal{A}_S \) in \( \mathcal{A}_S \). Then the map

\[
K_2(B, B) \rightarrow K_2(L, L)
\]

induced by the inclusion \( B \rightarrow L \) is injective.

**Proof.** — Let \( U \) be an open subset of \( \mathbb{A}^1_S \) containing each generic point of \( \mathbb{A}^1_S \). Then there is a section \( \sigma: S \rightarrow U \) to \( \pi|_U \). Since

\[
\pi^*: K_i(R, \mathring{R}) \rightarrow K_i(\mathbb{A}^1_S, \mathbb{A}^1_S)
\]

is an isomorphism by the homotopy property (§1.4), it follows that the map \( K_i(\mathbb{A}^1_S, \mathbb{A}^1_S) \rightarrow K_i(U, U) \) is injective. Passing to a suitable limit, we see that the maps

\[
K_2(\mathbb{A}_S, \mathbb{A}_S) \rightarrow K_2(B, B); \quad K_2(\mathbb{A}_S, \mathbb{A}_S) \rightarrow K_2(L, L)
\]

are injective.

Let \( I_L, I_B \) be the index sets

\[
I_L = \{ Z \subset \mathbb{A}_S^1 \mid Z \text{ is reduced, closed, codim} Z = 1, Z \text{ is flat over } S \}
\]

\[
I_B = \{ Z \subset \mathbb{A}_S^1 \mid Z \in I_L \text{ and } Z \cap s(\mathcal{S}) = \emptyset \}.
\]

We have the compatible localization sequences:

\[
0 \rightarrow K_2(\mathbb{A}_S^1, \mathbb{A}_S^1) \rightarrow K_2(L, L) \rightarrow \lim_{\rightarrow \atop Z \in I_L} K_1(Z, Z) \rightarrow 0
\]

\[
0 \rightarrow K_2(\mathbb{A}_S^1, \mathbb{A}_S^1) \rightarrow K_2(B, B) \rightarrow \lim_{\rightarrow \atop Z \in I_B} K_1(Z, Z) \rightarrow 0
\]
We can restrict the limits to be over $\mathbb{Z}$'s which satisfy the hypotheses of lemma 1.8. If $Z$ is of this type, the argument of lemma 1.8 shows that the map

$$K'_1(Z) \to K_1(k(Z)) = k(Z)^*$$

is injective. Thus the map $\beta$ is injective, hence $\alpha$ is injective, as desired. □

1.9 Relative $K$-theory of Brauer-Severi schemes. — Let $R$ be as in § 1.7, with Jacobson radical $(t)R$, and let $\mathcal{D}$ be an Azumaya algebra over $R$. Let $S = \text{Spec}(R)$, and let $\pi : X \to S$ be the Brauer-Severi scheme associated with $\mathcal{D}$. Quillen’s computation of the $K$-theory of $X$ gives pairs of functors

$$(G_0, G_1) : (\mathcal{P}_\mathcal{D} \otimes i, \mathcal{P}_\mathcal{D} \otimes i) \to (\mathcal{P}_X, \mathcal{P}_X)$$

by

$$G_1(\mathcal{E}) = \mathcal{J} \otimes \mathcal{D} \otimes i \mathcal{E}; \quad G_1(\mathcal{E}) = \mathcal{J} \otimes \mathcal{D} \otimes i \mathcal{E},$$

where $\mathcal{J}$ is a certain vector bundle on $X$, $\mathcal{J}$ the restriction to $\mathcal{X}$. Letting $\theta_i$ be the usual natural isomorphism, we get maps

$$(G_0, G_1, \theta_i)^*: K_q(\mathcal{D} \otimes i, \mathcal{D} \otimes i) \to K_q(X, \mathcal{X}).$$

From Quillen’s computation of $K_q(X)$ and $K_q(\mathcal{X})$, together with the five lemma, the map

$$\sum_{i=0}^{\dim X} (G_0, G_1, \theta_i)^*: \bigoplus_{i=0}^{\dim X} K_q(\mathcal{D} \otimes i, \mathcal{D} \otimes i) \to K_q(X, \mathcal{X})$$

is an isomorphism.

The explicit description of $K_1(\mathcal{D} \otimes i)$ as $\mathcal{D} \otimes i/[\mathcal{D} \otimes i, \mathcal{D} \otimes i]$ shows that the restriction map

$$K_1(\mathcal{D} \otimes i) \to K_1(\mathcal{D} \otimes i)$$

is surjective. This gives the exact sequence

$$0 \to K_0(X, \mathcal{X}) \to K_0(X) \to K_0(\mathcal{X}) \to 0$$

If $\mathcal{D} \otimes i$ is split, it is easy to see that the map

$$K_2(\mathcal{D} \otimes i) \to K_2(\mathcal{D} \otimes i)$$

is an isomorphism.
is also surjective, giving the exact sequence
\[ 0 \rightarrow K_1(X, X) \rightarrow K_1(X) \rightarrow K_1(\bar{X}) \rightarrow 0 \]

As the \( D^i \) are semi-local, the map
\[ K_0(D^i) \rightarrow K_0(D^i) \]
is injective, hence \( K_0(X, X) = 0 \). We now show a similar vanishing of the \( E_2 \) terms in the Quillen spectral sequence for relative \( K_n(X, X) \).

Let \( CH^p(X, \bar{X}) := E_2^{p-p}(X, X) \). The maps on the \( E_2 \) terms give a complex
\[ CH^p(X, \bar{X}) \rightarrow E_2^{p-p}(X | \bar{X}) \rightarrow E_2^{p-p}(\bar{X}). \]

By lemma 1.4, and a quick look at the \( E_2 \) terms, this is
\[ CH^p(X, \bar{X}) \rightarrow CH^p(X, \bar{X}) \]

**Lemma 1.11.** — If \( D \) is split, then \( CH^p(X, \bar{X}) \rightarrow CH^p(X) \) is injective.

**Proof.** — This is obvious for \( p=0 \), so assume that \( p \geq 1 \). Let \( z \) be in \( K_0(X^{p+p+1}, \bar{X}^{p+p+1}) \), i.e., \( z \) is a codimension \( p \) cycle on \( X \), flat over \( R \), with \( z \cdot \bar{X} = 0 \) as a cycle on \( \bar{X} \). Suppose the class of \( z \) in \( E_2^{p-p}(X | \bar{X}), [z] \), is zero, i.e.
\[ z = \sum_i \text{div}(f_i) \]
with the \( f_i \) in \( k(D_i)^* \) for codimension \( p-1 \) subvarieties \( D_i \) of \( X \), flat over \( R \). Specializing the collection \( \sum (f_i, D_i) \) to \( \bar{X} \) gives an element \( \sum (\tilde{f}_i, D_i) \)
\[ \sum (\tilde{f}_i, D_i) \in \bigoplus_{x \in \bar{X}} k(x)^*. \]

Since
\[ \sum \text{div}(\tilde{f}_i) = (\sum \text{div}(f_i)). \tilde{X} = z. \tilde{X} = 0, \]

\( \sum (\tilde{f}_i, D_i) \) determines an element \( \xi \) of \( E_2^{p-p-1}(\bar{X}) \). We note that \( E_2^{p-p-1}(\bar{X}) \) is the cohomology group \( H^p(X, \mathcal{X}_{p+1}) \). Since \( D \) is split, \( \bar{X} \) is just \( P_{R^{p-1}} [n = \text{rank}_R(\mathcal{D})] \), and \( H^p(X, \mathcal{X}_{p+1}) \) is isomorphic to the group of units \( R^* \). Let \( u \) be the element of \( R^* \) corresponding to \( \xi \), and lift \( \bar{u} \) to \( u \) in \( \mathcal{D}^* \). We may replace \( \sum (f_i, D_i) \) with \( \sum (u^{-1} f_i, D_i) \), so we may assume that \( \xi = 0 \) in \( H^p(X, \mathcal{X}_{p+1}) \); thus if \( p \geq 2 \) we can find a reduced closed subscheme \( \tilde{E} \) of \( X \) of codimension \( p-2 \), and \( \tilde{\eta} \) in \( K_2(k(\tilde{E})) \) with
\[ T(\tilde{\eta}) = \sum (\tilde{f}_i, D_i) \quad (T = \text{tame symbol}). \]
After adding components to $E$, and extending $\eta$ by 1 on these additional components, we may assume there is a reduced closed subscheme $E$ of $X$, flat over $R$, with $E = E \cap X$. Let $R(E)$ denote the semi-local ring of $E$ in $E$. Since since $K_2(k(E))$ is generated by symbols, we can lift $(\eta, E)$ to $(\eta, E)$, $\eta \in K_2(R(E))$, with $E$ flat over $R$, and of codimension $p-2$ on $X$. Modify $\sum (f_0, D_0)$ by $T(\eta)$ to get $\sum (f_0', D_0')$. If $p=1$, then our element $\sum (f_0', D_0')$ is just an element $f$ of $k(X)^*$, and we take $f' = u^{-1} f$. Then $\sum (f_0', D_0')$ gives an element of $K_1(X, X^{p+1})$, i.e., $\sum (f_0', D_0')$ gives an element $\tau$ of $K_1(X, X^{p+1})$ with $\text{div}(\tau) = z$. Thus 

$$[z] = 0 \text{ in } CH^p(X, \bar{X})$$

as desired. □

**Corollary 1.12.** — If $\mathcal{D}$ is split, and $\mathcal{D}$ has prime rank $l$ over $R$, then $CH^p(X, \bar{X}) = 0$ for all $p \geq 0$.

**Proof.** — Let $h: T \to S$ be a finite degree $l$ cover splitting $\mathcal{D}$; we may assume that $T$ is étale over $S$. Since $\dim(X) = l$, $CH^p(X)[1/(l-1)!]$ injects into $K_0(X)[1/(l-1)!]$, which injects into $K_0(X_T)[1/(l-1)!]$. Since the kernel of

$$h^*: CH^p(X) \to CH^p(X_T)$$

is $l$-torsion, $h^*$ is thus injective. Thus

$$h^*: CH^p(X, \bar{X}) \to CH^p(X_T)$$

is injective, with $h^*(CH^p(X, \bar{X}))$ contained in the kernel of

$$i^*: CH^p(X_T) \to CH^p(X_T).$$

Since $X_T$ is a projective space over $T$, $i^*$ is injective, hence $CH^p(X, \bar{X}) = 0$, as claimed. □

**Corollary 1.13.** — Assume that $\mathcal{D}$ is split, and $\mathcal{D}$ has prime rank $l$ over $R$. Suppose further that $R$ contains an infinite field $k_o$. Let $h: S' \to S$ be a finite étale cover. Then

$$E_2^{1,-2}(X, \bar{X}) \to E_2^{1,-2}(X_{S'}, \bar{X}_{S'})$$

is injective.

**Proof.** — Since $CH^p(X, \bar{X}) = E_2^p = 0$, the differentials going out of $E_2^{1,-2}(X, \bar{X})$ are all zero for $r \geq 2$. There are no differentials going into $E_2^{1,-2}$ by reasons of dimension, so

$$E_2^{1,-2}(X, \bar{X}) = E_\infty^{1,-2}(X, \bar{X}) = K_1(X, \bar{X})^{1/2},$$

and similarly for $(X_{S'}, \bar{X}_{S'})$. We may assume that $S'$ splits $\mathcal{D}$, i.e. $X_{S'}$ is $\mathbb{P}^{l-1}_{S'}$. By the computation of $K_*(X_{S'}, \bar{X}_{S'})$ in paragraph 1.7, we have

$$K_1(X_{S'}, \bar{X}_{S'}) = \bigoplus_{i=0}^{i=1} (1 + I_{S'})^* \cdot \gamma_i^{l-1}.$$
I_s the Jacobson radical of R_s := \Gamma(S', \mathcal{O}_S).

By the sequence (1.10)', the map K_1(X, \bar{X}) \to K_1(X_s, \bar{X}_s) is injective. Let N be the subgroup of R_s * of reduced norms from S. Similarly define \bar{N} as the group of reduced norms from \bar{S}. Then N is isomorphic to K_1(\mathcal{D}), \bar{N} is isomorphic to K_1(\bar{\mathcal{D}}), the kernel N^0 of N \to \bar{N} is isomorphic to K_1(\mathcal{D}, \bar{\mathcal{D}}). Thus N^0 is the subgroup of N of reduced norms \zeta = \text{Nrd}(x) from \mathcal{D} with \zeta \equiv 1 \mod(t). Furthermore, we can identify K_1(X, \bar{X}) with the subgroup

$$
\bigoplus_{l=1}^{l-1} (1+I_s)^* \oplus \bigoplus_{i=0}^{l-1} N^0. \gamma^i
$$

of \bigoplus_{l=0}^{l-1} [R_s^* \gamma]^l = K_1(X_s).

The topological filtration on K_1(X_s, \bar{X}_s) is given by

$$
K_1(X_s', \bar{X}_s')^p = \left( \sum_{i=p}^{l-1} (1-\gamma)^i (1+I_s)^* \right)
$$

so

$$
K_1(X_s', \bar{X}_s')^2 \cap K_1(X, \bar{X}) = \sum_{i=2}^{l-1} (1-\gamma)^i. N^0.
$$

Take \alpha in N^0, so \alpha = \text{Nrd}(x) for some x in \mathcal{D}. Since k_0 is infinite, and \mathcal{D} is split, there is an element y of \mathcal{D} with \text{Nrd}(y) = 1 (i.e. \bar{y} \in \text{SL}_l(\bar{R})), such that the characteristic polynomial of \bar{xy} is separable over \bar{R}. Since \bar{y} is in the commutator subgroup of \mathcal{D}*, we can lift \bar{y} to an element y of \mathcal{D} with \text{Nrd}(y) = 1. Then \bar{xy} is separable over the quotient field K of R; thus we may assume that x is separable over K. Let E be a maximal separable subfield of \mathcal{D}_K containing x. Let \mathcal{E} be the integral closure of R in E, so we get a finite ring extension R \to \mathcal{E}. Let \hat{\mathcal{E}} be the integral closure of R in the Galois closure \hat{E} of E over K. Then G := \text{Gal}(\hat{E}/R) is a subgroup of \Sigma, and has a subgroup H, corresponding to E, of index l. Thus there is a non-trivial l-Sylow subgroup G_l \cong \mathbb{Z}/l in G. Let \mathcal{E}_l be the subring of \mathcal{E} fixed by G_l, giving a diagram

$$
\begin{array}{ccc}
\mathcal{E} & \to & \hat{\mathcal{E}} \\
\downarrow & & \downarrow \\
\mathcal{E}_l & \to & \hat{\mathcal{E}}_l \\
\downarrow & & \downarrow \\
R & \to & \hat{R}
\end{array}
$$

As \text{[E_l: R]} is prime to l, E and \mathcal{E}_l \otimes_R K are disjoint over k, so x has the same conjugates over R and over \mathcal{E}_l. Thus

$$
a = \text{Nm}_{E/K}(x) = \text{Nm}_{\mathcal{E}/\mathcal{E}_l}(x).
$$

By applying Hilbert's theorem 90 (see lemma 1.14 below) we can modify x by an element y of \hat{\mathcal{E}}* of norm 1 so that

$$
x \in \mathcal{E}, x \equiv 1 \mod(t).
$$
In addition, we have
\[ \text{Nm}_{\mathcal{E}/R}(x) = a^d \quad \text{with} \quad d \mid (l-1)!. \]

Let \( g : \text{Spec}(\mathcal{E}) \to \text{Spec}(R) \), \( g : X_{\mathcal{E}} \to X \) be the covering maps. Then \( x \) lifts to an element \( \hat{x} \) of \( K_1(\mathcal{E}, \mathcal{E}/(t)) \) with
\[ g_*(\hat{x}) = a^d, \]
and
\[ (1 - \gamma)^i a^d = g_*(((1 - \gamma)^i \cdot \hat{x}). \]

Thus \( (1 - \gamma)^i a^d \) is in \( g_*(K_1(X, \mathcal{E})) \), which is a subgroup of \( K_1(X, \mathcal{E}) \), hence \( (1 - \gamma)^i a^d \) is in \( K_1(X, \mathcal{E}) \) for \( i \geq 2 \). Thus
\[ K_1(X_{\mathcal{E}'}, \mathcal{E}_{\mathcal{E}'}) [1/(l-1)!] \cap K_1(X, \mathcal{E}) [1/(l-1)!] = K_1(X, \mathcal{E}) [1/(l-1)!], \]
hence the kernel of the map \( K_1(X, \mathcal{E}) \to K_1(X_{\mathcal{E}'}, \mathcal{E}_{\mathcal{E}'}) \) is \( (l-1)! \) primary torsion. Since we can split \( \mathcal{E} \) by a degree \( l \) cover, the above kernel must also be \( l \)-primary torsion, hence the map
\[ K_1(X, \mathcal{E}) \to K_1(X_{\mathcal{E}'}, \mathcal{E}_{\mathcal{E}'}) \]
is injective, as desired. \( \square \)

**Lemma 1.14.** — Let \( T \) be a semi-local PIR containing an infinite field \( k_0 \). Let \( T \to T' \) be a cyclic extension of degree \( l \). Let \( I \) be the Jacobson radical of \( T \), \( K \) the quotient field of \( T \), \( K' \) the quotient field of \( T' \). Suppose \( a \in (1 + I) \) is a norm:
\[ a = N_{K'/K}(x), \quad x \in K'. \]

Then
\[ a = N_{T'/T}(y) \]
for some \( y \) in \((1 + IT')\).

**Proof.** — This is an easy consequence of Hilbert's theorem 90 (for \( K_1 \)); we leave the details to the reader.

1.10. Restrictions and Norms. — We consider the functorial properties of relative K-theory in some greater detail.

Let \( f : T \to S \) be a map of schemes \( S \subset S \) a closed subscheme. Let \( \mathcal{T} = f^{-1}(S) \), and \( \hat{T} \subset \hat{T} \) an closed subscheme. The map \( f^* \) is exact on \( \mathcal{P}_S \) and \( \mathcal{P}_S \) so the pair of functors
\[ (f^*, \mathcal{F}^*) : (\mathcal{P}_S, \mathcal{P}_S) \to (\mathcal{P}_T, \mathcal{P}_T) \]

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together with the usual natural isomorphism $\theta$ determines the pull back
$f^*: K_\varphi(S, S) \to K_\varphi(T, T)$. Similarly, the diagram
\[ \begin{array}{c}
\mathcal{P}_T \to \mathcal{P}_\tilde{T} \\
\downarrow \\
\mathcal{P}_T \to \mathcal{P}_\tilde{T}
\end{array} \]
commutes (up to a natural isomorphism), so we get a homorphism
\[ \text{res}_{(\tilde{T}, T)}: K_\varphi(T, T) \to K_\varphi(T, \tilde{T}). \]
The composition gives a pullback
\[ f^*: K_\varphi(S, S) \to K_\varphi(T, T). \]

These constructions are just special cases of the pullback discussed in paragraph 1.1, hence they are functorial.

We now suppose that $f: T \to S$ is finite and flat. Restriction of scalars then gives commutative diagrams (up to natural isomorphisms):
\[ \begin{array}{c}
\mathcal{P}_T \to \mathcal{P}_\tilde{T} \\
\downarrow \\
\mathcal{P}_S \to \mathcal{P}_{\tilde{S}}
\end{array} \]
\[ \begin{array}{c}
\mathcal{M}(T, \tilde{T}) \to \mathcal{M}_\tilde{T} \\
\downarrow \\
\mathcal{M}(S, \tilde{S}) \to \mathcal{M}_{\tilde{S}}
\end{array} \]
inducing $f_\#: K_\varphi(T, T) \to K_\varphi(S, S)$ and $f_\#: K_\varphi(T, \tilde{T}) \to K_\varphi(S, \tilde{S})$.

We now suppose further that $T, S, \tilde{T}$ and $\tilde{S}$ are regular. Write $\tilde{S}$ as a union of connected components
\[ S = \bigcup S_i, \]
and let $\tilde{T}_i, \ldots, \tilde{T}_{i_n}$ be the components of $\tilde{T}$ lying over $S_i$. We also assume that each $S_i$ and $\tilde{T}_i$ are principal:
\[ I(S_i) = (u_i), \quad I(\tilde{T}_i) = (t_{ij}), \]
and that $T$ is a thickening of $\tilde{T}$ with components $T_j$,
\[ I(T_j) = (t_{ij})^{e_{ij}}, \quad e_{ij} > 0. \]
Let $e = (e_1, \ldots, e_{ij}, \ldots)$. We call $e$ the total ramification index.

Let $\Theta_\varphi: \mathcal{P}_\varphi \to \mathcal{M}_\varphi$ be the functor whose restriction to $\mathcal{P}_\varphi$ is $M \to (M)^{e_{ij}}$. Given a module $M$ in $\mathcal{P}_\varphi$, we form the filtration
\[ \mathcal{F}^n(M): 0 = M_0 \subset (t_{ij})^{-1} M \subset \ldots \subset (t_{ij}) M \subset M. \]
Since $M$ is a projective $T^j$ module, the graded quotients $(t^i_j) M/(t^{i+1}_j) M$ are each isomorphic to $M/(t^i_j) M$ by the map

$$m \mapsto t^i_j \cdot m \mod (t^{i+1}_j) M.$$ 

The "natural filtration" theorem of Quillen then gives a homotopy $H$ between the two maps

$$BQ r_\ddagger: BQ P_\ddagger \to BQ M_\ddagger,$$

$$r_\ddagger: P_\ddagger \to M_\ddagger$$

the natural inclusion and

$$BQ(\oplus_e \circ (- \times \dagger_\ddagger)): BQ P_\ddagger \to BQ M_\ddagger.$$

We get a similar homotopy $H'$ between the two maps

$$BQ(r_\ddagger \circ (- \times \dagger_\ddagger)): BQ P_\ddagger \to BQ M_\ddagger$$

and

$$BQ(\oplus_e \circ (- \times \dagger_\ddagger)): BQ P_\ddagger \to BQ M_\ddagger.$$

In fact, replace $- \times \dagger_\ddagger$ with the composition $(- \times \dagger_\ddagger) \circ (- \times \dagger T)$, then we can take $H'$ to be $H \circ BQ(- \times \dagger T)$.

We thus get a diagram

\[
\begin{array}{ccc}
K'(T, T) & \to & BQ P_T \\
\uparrow (id, r_\ddagger) & & \uparrow r_\ddagger \\
K(T, T) & \to & BQ P_T \\
\downarrow (id, - \times \dagger T) & & \downarrow - \times \dagger T \\
K(T, \dagger T) & \to & BQ P_T
\end{array}
\]

\[
\begin{array}{ccc}
& & BQ M_\ddagger \\
& & \uparrow r_\ddagger \\
& & BQ P_\ddagger \oplus_e \\
\end{array}
\]
Here we suppress some of the BQ's, and the natural isomorphisms used to get the maps on the homotopy fibers. Since the two homotopies $H$ and $H'$ are compatible, the triangle
\[
\begin{array}{c}
(K(T, T), (id, r^T)) \\
\rightarrow \\
K'(T, T), (id, r^T))
\end{array}
\]
commutes, up to homotopy, inducing a commutative triangle
\[
\begin{array}{c}
(K_*(T, T), (id, x^f)) \\
\rightarrow \\
K_*(T, T)
\end{array}
\]
Since $S$ and $\bar{S}$ are smooth, we have the commutative diagram
\[
\begin{array}{c}
K_*(T, T) \\
\rightarrow \\
K_*(S, S)
\end{array}
\]
Define $f^H_*: K_*(T, T) \rightarrow K_*(S, S)$ to be the composition
\[
f^H_* = (r_3)^{-1} \circ f_* \circ (\oplus e)^H.
\]
Let
\[
(\times): K_*(T, \bar{T}) \rightarrow K_*(\bar{T})
\]
be the map restricting to $\times e_{ij}: K_*(T^{ij}) \rightarrow K_*(\bar{T}^{ij})$ on each factor $K_*(T^{ij})$. Let
\[
(f^{ij}_*) : K_*(T^{ij}) \rightarrow K_*(S^{ij})
\]
be the restriction of scalars. Then
\[
f_* \circ (\times) = \bigoplus_{i,j} (f^{ij}_*) \circ (\times e_{ij}) f^{ij}.
\]

**Proposition 1.15.** — *The ladder*

\[
\begin{array}{c}
K_*(T, T) \\
\rightarrow \\
K_*(S, S)
\end{array}
\]

is commutative. *Suppose that $O_T$ is a free $O_S$ module (e.g. $O_S$ semi-local), of rank $n$. Then the composition*

\[
\begin{array}{c}
K_*(S, \bar{S}) \\
\rightarrow \\
K_*(T, \bar{T}) \\
\rightarrow \\
K_*(S, \bar{S})
\end{array}
\]

*is commutative.*
is multiplication by $n$.

Proof. — We have the diagram

$$
\begin{array}{ccc}
K_*(S, S) & \rightarrow & K_*(T, T) \\
\downarrow f_\ast & & \downarrow \text{res} \\
K_*(T, \bar{T}) & \rightarrow & (\Theta_{\bar{g}})_H
\end{array}
$$

(\star)

with all triangles commuting. Suppose that $\Theta_T \cong (\Theta_{\bar{g}})^*$. Then the composite

$$(\mathcal{P}_S, \bar{\mathcal{P}}) \xrightarrow{\Theta_{\bar{g}}} (\mathcal{P}_T, \bar{\mathcal{P}}) \xrightarrow{(\mathcal{P}_T, \mathcal{M}_T)} (\mathcal{P}_S, \mathcal{M}_S)$$

sends $(M, M)$ to $(M^*, M^*)$, hence induces multiplication by $n$

$$
\times n : \quad K_*(S, \bar{S}) \rightarrow K'_*(S, \bar{S}) \cong K_*(S, \bar{S})
$$

Thus the composite

$$
K_*(S, \bar{S}) \rightarrow K'_*(T, \bar{T}) \rightarrow K'_*(S, \bar{S}) \rightarrow K_*(S, \bar{S})
$$

is multiplication by $n$. The second assertion follows from this and the commutativity of (\star).

For the first assertion, note that $\Theta_{\bar{g}} : \mathcal{P}_Gamma \rightarrow \mathcal{M}_T$ induces $\times e$ under the composition

$$
K_*(\mathcal{P}_\Gamma) \rightarrow K_*(\mathcal{M}_T) \rightarrow K_*(\mathcal{P}_\Gamma).
$$

Since the ladder

$$
\begin{array}{cccc}
\rightarrow K'_*(T, \bar{T}) & \rightarrow & K'_*(T) & \rightarrow \\
\downarrow f_* & & \downarrow f_* & \rightarrow \\
K_*(S, \bar{S}) & \rightarrow & K_*(S) & \rightarrow
\end{array}
$$

is commutative, the result follows from \(A\), (\star), and our definitions. \(\square\)

**Proposition 1.16.** Suppose $f : T \rightarrow S$ is Galois with group $G$. Then there is a homomorphism

$$
f_* : \quad K_*(T, \bar{T})[1/|G|] \rightarrow K_*(S, \bar{S})[1/|G|]
$$

satisfying

(i) $f_*(\eta^*) = f_*(\eta)$; $\eta \in K_*(T, \bar{T})$

(ii) the ladder

$$
\begin{array}{cccc}
\rightarrow K_*(T, \bar{T})[1/|G|] & \rightarrow & K_*(T)[1/|G|] & \rightarrow \\
\downarrow f_* & & \downarrow f_* & \rightarrow \\
K_*(S, \bar{S})[1/|G|] & \rightarrow & K_*(S)[1/|G|] & \rightarrow
\end{array}
$$

is commutative
(iii) $f^* \circ f_*(\eta) = \sum_{\sigma \in G} \eta^\sigma$ in $K_*(T, \hat{T})$, for $\eta$ in $K_*(T, \hat{T})$

(iv) if $\mathcal{O}_T$ is a free $\mathcal{O}_S$ module, then

$$f_* f^* : K_*(S, \hat{S}) [1/|G|] \rightarrow K_*(S, \hat{S}) [1/|G|]$$

is multiplication by $|G|$.

Proof. — The pair of functors $(\text{id}, \oplus_\sigma)$, together with the homotopy $H$ gives the commutative ladder

$$\begin{align*}
&K_{q+1}(T, \hat{T}) \rightarrow K_q(T) \rightarrow K_q(T) \\
&\uparrow \oplus_\sigma H \quad \quad \quad \uparrow \times_\sigma \\
&K_{q+1}(T, \hat{T}) \rightarrow K_q(T) \rightarrow K_q(T)
\end{align*}$$

As $T \rightarrow S$ is Galois, the ramification indices $e_{ij}$ all divide $|G|$, so $\oplus_\sigma H$ is an isomorphism. Symmetrizing $\oplus_\sigma H$ with respect to $G$ gives the map $h$:

$$h = 1/|G| \cdot \sum_{\sigma \in G} (\oplus_\sigma H)^\sigma$$

and a commutative $G$-equivariant ladder

$$\begin{align*}
&K_{q+1}(T, \hat{T}) [1/|G|] \rightarrow K_q(T) [1/|G|] \rightarrow K_q(T) [1/|G|] \\
&\uparrow h \quad \quad \quad \uparrow \times_\sigma \\
&K_{q+1}(T, \hat{T}) [1/|G|] \rightarrow K_q(T) [1/|G|] \rightarrow K_q(T) [1/|G|]
\end{align*}$$

Define $f_*$ to be the composition of $h$ with $f_* : K_q(T, \hat{T}) [1/|G|] \rightarrow K_q(S, \hat{S}) [1/|G|]$. Then (i) and (ii) are clear; (iii) follows from the isomorphisms

$$M \otimes_{\mathcal{O}_T} \mathcal{O}_T \overset{\sim}{\rightarrow} \bigoplus_{\sigma \in G} M^\sigma; \quad \quad \tilde{M} \otimes_{\mathcal{O}_T} \mathcal{O}_T \overset{\sim}{\rightarrow} \bigoplus_{\sigma \in \hat{G}} \tilde{M}^\sigma.$$

The statement (iv) follows from Proposition 1.15. □

Corollary 1.17. — Suppose $T$, $S$, and $S'$ are semi-local, one dimensional regular schemes, $f : T \rightarrow S$ a Galois cover with group $G$, and $p : S' \rightarrow S$ étale. Let $f' : T' \rightarrow S'$ be the fiber product $T \times_S S'$, let

$$f_* : K_*(T, \hat{T}) [1/|G|] \rightarrow K_*(S, \hat{S}) [1/|G|]$$

and

$$f'_* : K_*(T', \hat{T}') [1/|G|] \rightarrow K_*(S', \hat{S}') [1/|G|]$$

be given by Proposition 1.16, where $\hat{T}$ and $\hat{T}'$ are the respective reduced schemes $T_{\text{red}}$ and $T'_{\text{red}}$. Let

$$p_* : K_*(S', \hat{S}') [1/|G|] \rightarrow K_*(S, \hat{S}) [1/|G|]$$

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and

\[ p_\bullet \colon K_\bullet(T', T') [1/|G|] \to K\bullet(T, T) [1/|G|] \]

the usual pushforward. Then

\[ p_\bullet \circ f' = f_\bullet \circ p'_\bullet. \]

Proof. — Since \( f^\bullet \colon K_\bullet(S, S) \to K_\bullet(T, T) \) has \(|G|\) torsion kernel, it suffices to show that

\[ f^\bullet \circ p_\bullet \circ f' = f^\bullet \circ f_\bullet \circ p'_\bullet. \]

We have \( f^\bullet \circ p_\bullet = p'_\bullet \circ f^\bullet \), hence we need only show that

\[ p'_\bullet \circ (f^\bullet \circ f') = (f^\bullet \circ f_\bullet) \circ p'_\bullet. \]

This follows from Proposition 1.16(iii). □

1.11. ITERATED RELATIVE K-THEORY. — As if life weren't bad enough already, one can iterate the relativization of K-theory we have considered so far. More precisely, let \( X \) be a scheme, \( Y_1 \) and \( Y_2 \) closed subschemes with inclusions \( i_1 : Y_1 \to X, i_2 : Y_2 \to X \). Let \( Y_{12} \) be the intersection \( Y_1 \cap Y_2, i_{12} : Y_{12} \to Y_1 \) the inclusion. The restriction map

\[ (i_2^*, i_{12}^*) : (\mathcal{P}_X, \mathcal{P}_{Y_1}) \to (\mathcal{P}_{Y_2}, \mathcal{P}_{Y_{12}}) \]

together with the natural isomorphism \( \theta_2 \) gives a map of homotopy fibers

\[ i_2^* : K(X, Y, Y_1) \to K(Y, Y_1). \]

We let \( K(X, Y_1, Y_2) \) be the homotopy fiber of \( i_2^* \). Similarly, let \( K(X, Y_2, Y_1) \) be the homotopy fiber of \( i_1^* : K(X, Y_2) \to K(X, Y_{12}) \). The Quetzcoatl lemma shows

(a) \( K(X, Y_1, Y_2) \) and \( K(X, Y_2, Y_1) \) are naturally homeomorphic;

(b) if \( Y_1 \cap Y_2 = \emptyset \), then \( K(X, Y_1, Y_2) \) is naturally homeomorphic to \( K(X, Y_1 \cup Y_2) \).

We let \( K_{\bullet}(X, Y_1, Y_2) \) be the homotopy group \( \pi_{\bullet+1}(K(X, Y_1, Y_2), \ast) \). From (a), we get a commutative diagram

\[
\begin{array}{ccccccc}
K_{\bullet}(X, Y_1, Y_2) & \to & K_{\bullet}(X, Y_1) & \to & K_{\bullet}(Y_2, Y_{12}) \\
\downarrow & & \downarrow & & \downarrow \\
K_{\bullet}(X, Y_2) & \to & K_{\bullet}(X) & \to & K_{\bullet}(Y_2) \\
\downarrow & & \downarrow & & \downarrow \\
K_{\bullet}(Y_1, Y_2) & \to & K_{\bullet}(Y_1) & \to & K_{\bullet}(Y_{12})
\end{array}
\]

If one wants to iterate this, one runs into trouble with compatibilities between the homotopies; we therefore use the approach of paragraph 1.1 to replace the categories

\( \mathcal{P}_X, \mathcal{P}_{Y_1}, \mathcal{P}_{Y_2}, \mathcal{P}_{Y_{12}}, \) etc.
with equivalent categories so that the appropriate diagram commutes exactly, not just up to homotopy. In this case the homotopy fibers are again functorial, which enables one to define \( K_\bullet(X, Y_1, Y_2, \ldots, Y_n) \) inductively as the homotopy fiber of

\[
K(X, Y_1, Y_2, \ldots, Y_{n-1}) \to K(Y_{n-1}, Y_{1}, Y_{2}, \ldots, Y_{n-1})
\]

\( Y_{in} = Y_{i} \cap Y_{n} \).

The groups one gets are independent of the order of the \( Y_i \), and there is an isomorphism \( K_\bullet(X, Y_1, Y_2, \ldots, Y_n) \to K_\bullet(X, \cup Y_i) \) if the \( Y_i \)'s are pairwise disjoint. There is an \( n \)-dimensional commutative diagram generalizing the two-dimensional diagram above.

Returning to the case \( n = 2 \), we have the diagram

\[
\begin{array}{ccc}
K(X, Y \cup Y') & \to & \mathbb{BQ} \mathcal{P}_X \to \mathbb{BQ} \mathcal{P}_{Y \cup Y'} \\
(id_X, i_Y) & \downarrow & \downarrow (i_Y) \\
K(X, Y) & \to & \mathbb{BQ} \mathcal{P}_X \to \mathbb{BQ} \mathcal{P}_Y
\end{array}
\]

\( (j_Y: Y' \to Y \cup Y', \text{the inclusion}) \)

\[
K(Y', Y \cap Y') \to \mathbb{BQ} \mathcal{P}_{Y} \to \mathbb{BQ} \mathcal{P}_{Y \cap Y'}
\]

The map \( j_Y^* \) gives a contraction of the composition

\[
(i_Y^*, i_{Y \cup Y'}) \circ (id_X, i_Y^*): K(X, Y \cup Y') \to K(Y', Y \cap Y'),
\]

hence a lifting of \( (id_X, i_Y^*) \) to a map

\[
\Theta_{Y \cup Y'}: K(X, Y \cup Y') \to K(X, Y, Y').
\]

Similarly, we get a map

\[
\Theta_{1 \ldots n}: K(X, Y_1 \cup \ldots \cup Y_n) \to K(X, Y_1, \ldots, Y_n),
\]

inducing

\[
\Theta_{1 \ldots n}: K_\bullet(X, Y_1 \cup \ldots \cup Y_n) \to K_\bullet(X, Y_1, \ldots, Y_n)
\]

which is an isomorphism if the \( Y_i \)'s are pairwise disjoint.

1.12. Chern Classes. — In this section, we recall Gillet's construction of Chern classes [Gillet] and indicate how one constructs Chern classes for relative K-theory. In fact, Gillet has already given the details for the construction of "Chern classes with support", which is nothing more than Chern classes for the homotopy fiber of \( j^*: \mathbb{BQ} \mathcal{P}_X \to \mathbb{BQ} \mathcal{P}_U \) where \( j: U \to X \) is an open subset of \( X \). As there is no essential difference between the cases of the open immersion and the closed embedding, we will be somewhat sketchy.

We first recall some notions from the theory of sheaves of simplicial sets. We use the notations of [Gillet]; for details we refer the reader to section 1 of that work. For a
complex of sheaves of abelian groups $F^*$ on $\mathcal{S}^\text{ur}$, we let $\kappa(F^*, n)$ be the Dold-Puppe construction on $F^*$. If $\mathcal{S}$ is a sheaf of simplicial sets on $\mathcal{S}^\text{ur}$, there is the notion of the generalized cohomology groups of $\mathcal{S}$ defined by

$$H^{-p}(X, \mathcal{S}) := \pi_p(R\Gamma(X, \mathcal{S}))$$

where $R\Gamma$ is the functor described in section 1 of [Gillet]. In particular, we have

$$H^p(X, \kappa(F^*, n)) = H^p_{\mathcal{S}^\text{ur}}(X, F^*).$$

Let $\Gamma(*)$ be a twisted duality theory in the sense of [Bloch-Ogus] for schemes over a base scheme $S$. There is an injective complex of sheaves $\Gamma^*(\mathcal{S})$ on the big Zariski site over $S$ such that for each $S$-scheme $X$, we have

$$H^p(X, \Gamma(q)) = H^p_{\mathcal{S}^\text{ur}}(X, \Gamma^*(q)).$$

Let $\mathcal{B}G\ell$ be the sheaf of simplicial sets associated the presheaf

$$U \to B\Gammal(\Gamma(U, \mathcal{O}_U)).$$

Gillet constructs a map of sheaves of simplicial sets

$$c_q : Z_\infty \mathcal{B}G\ell \to Z_\infty \kappa(\Gamma^*(q), dq);$$

d $=1$ or $2$, depending on $\Gamma(-)$ which gives rise to the Chern class

$$c_{q, p} : K_{d_q-p}(X) \to H^p(X, \Gamma(q))$$

by the composition

$$K_{2q-p}(X) \to \mathcal{H}^{-d_q}(X, Z_\infty \mathcal{B}G\ell) \to \mathcal{H}^{-d_q}(X, Z_\infty \kappa(\Gamma^*(q), dq)) \xrightarrow{c_{q, p}} H^p(X, \Gamma(q)) = H^p_{\mathcal{S}^\text{ur}}(X, \Gamma^*(q)).$$

Now suppose we have a closed subscheme $i: Y \to X$ of an $S$-scheme $X$. Replacing the appropriate simplicial sheaves with weakly equivalent sheaves, we may assume that the horizontal maps in the commutative square

$$i^* : Z_\infty \mathcal{B}G\ell_X \to i_* Z_\infty \mathcal{B}G\ell_Y,$$

$$c_q \downarrow \quad \quad \quad \downarrow c_q$$

$$i^* : Z_\infty \kappa(\Gamma^*(q), dq)_X \to Z_\infty i_* \kappa(\Gamma^*(q), dq)_Y$$

are global fibrations, and all the simplicial sheaves are flasque. Let $\mathcal{X}(X, Y)$ be the fiber (hence homotopy fiber) of the first $i^*$, and let $Z_\infty \kappa(\Gamma^*(X, Y)(q), dq)$ be the fiber (hence homotopy fiber) of the second $i^*$. Let

$$\Gamma^*(X, Y)(q) = \text{Cone}(i^* : \Gamma^*(q)_X \to i_* \Gamma^*(q)_Y)[-1].$$

Then

$$\mathcal{H}^{-d_q}(X, Z_\infty \kappa(\Gamma^*(X, Y)(q), dq)) = H^p_{\mathcal{S}^\text{ur}}(X, \Gamma^*(X, Y)(q))$$
\[ H^p(X, Y, \Gamma(q)) = H^p_{graded}(X, \Gamma^*(X, Y)(q)). \]

\( C_q \) lifts to a map
\[ C_q : \mathcal{A}(X, Y) \to \mathbb{Z}_\infty \kappa(\Gamma^*(X, Y)(q), dq). \]

This defines as above the Chern class \( c_{q,p} \) by the composition
\[
\begin{align*}
K_{d_q-p}(X, Y) & \to H^{p-d_q}(X, \mathcal{A}(X, Y)) \\
& \to H^{p-d_q}(X, Z_\infty \kappa(\Gamma^*(X, Y)(q), dq)) \\
& \to H^p(X, Y, \Gamma(q)) = H^p_{graded}(X, \Gamma^*(X, Y)(q)).
\end{align*}
\]

From this construction, we see that the Chern classes are compatible with the long exact relativization sequence and the long exact cohomology sequence for the pair \((X, Y)\):
\[
\begin{array}{c}
K_{d_p}(X, Y) \to K_{d_q-p}(X) \to K_{d_q-p}(Y) \to K_{d_q-p-1}(X, Y) \\
\downarrow c_{q,p} \quad \downarrow c_{q,p} \quad \downarrow c_{q,p} \\
H^p(X, Y, \Gamma(q)) \to H^p(X, \Gamma(q)) \to H^p(Y, \Gamma(q)) \to H^{p+1}(X, Y, \Gamma(q)) \\
\end{array}
\]

Let \( x \) be in \( K_q(X, Y) \), \( y \) in \( K_k(X) \). The formula for \( c_{q,p}(xy) \) in terms of the Chern classes of \( x \) [in \( H^*(X, Y, \Gamma(\ast)) \)] and the Chern classes of \( y \) [in \( H^*(X, \Gamma(\ast)) \)] is formally the same as given by the product formula for the absolute Chern classes, using the structure of \( H^*(X, Y, \Gamma(\ast)) \) as a module over \( H^*(X, \Gamma(\ast)) \). The proof is the same as in the case of Chern classes with support, and we refer the reader to [Gillet] for details.

Soulé [So] has defined Chern classes for \( K \)-theory with coefficients. Specifically, let \( n \) be a positive integer, and \( X \) an affine scheme over \( \mathbb{Z}[1/n] \). Then there are Chern classes
\[ c_{q,p} : K_{2q-p}(X, \mathbb{Z}/n) \to H^p_{et}(X, (\mu_n)^{\otimes q}), \]

compatible with Gillet's Chern classes
\[ c_{q,p} : K_{2q-p}(X) \to H^p_{et}(X, (\mu_n)^{\otimes q}) \]

via the natural map \( K_{2q-p}(X) \to K_{2q-p}(X; \mathbb{Z}/n) \).

2. Specialization in relative \( K_2 \)

2.1. Symbols for relative \( K_2 \). We first recall the work of Dennis-Stein, Keune and Loday on \( K_2(A, I) \). Let \( A \) be a ring (with unit), \( I \subset A \) an ideal. Keune and Loday define a relative Steinberg group \( St(A, I) \), and group of elementary matrices \( E(A, I) \) with a surjection
\[ St(A, I) \to E(A, I) \to 1. \]

\( K_2(A, I) \) is then defined to be the kernel of \( \pi \). \( St(A, I) \) and \( E(A, I) \) maps to the usual \( St(A) \) and \( E(A) \), and the resulting map of \( K_2(A, I) \) to \( K_2(A) \) fits into a long exact
Here $K_1(A, I) = \ker(Gl(A) \to Gl(A/I)/E(A, I))$.

Loday [Loday] has shown that there are natural isomorphisms of the $K_i(A, I)$ defined above with the groups $K_i(A, A/I)$ defined earlier via homotopy theory, so that the maps in the above sequence correspond with the long exact homotopy sequence for relative $K$-theory.

Keune [Keune] constructs certain explicit elements in $K_2(A, I)$ analogous to the symbols $\{a, b\}$ in $K_2$ of a field. Let $D(A, I)$ be the group with generators

$$\langle a, b \rangle, \quad \text{with } (a, b) \in A \times I \cup I \times A, \text{ and } 1 + ab \in A^*.$$ 

and relations

\begin{align*}
(D1) & \quad \langle a, b \rangle = \langle -b, -a \rangle^{-1} \\
(D2) & \quad \langle a, b \rangle \langle a, c \rangle = \langle a, b + c + abc \rangle \\
(D3) & \quad \langle a, bc \rangle = \langle ab, c \rangle \langle ac, b \rangle \quad \text{if } a, b, \text{ or } c \text{ is in } I.
\end{align*}

There is a functorial group homomorphism

$$\Phi_{A, I}: D(A, I) \to K_2(A, I);$$

if $A \to B$ is a homomorphism of $A$ to a ring $B$ in which $b$ is a unit, then $\Phi(\langle a, b \rangle)$ goes to the symbol $\{1 + ab, b\}$ in $K_2(B)$. We will often denote $\Phi(\langle a, b \rangle)$ by $\langle a, b \rangle$ if there is no cause for confusion. Keune has shown the following result:

**THEOREM.** — Let $A$ be commutative ring. Suppose $I \subset A$ is a radical ideal ($I \subset \text{Jac}(A)$).

Then

$$\Phi_{A, I}: D(A, I) \to K_2(A, I)$$

is an isomorphism.

2.2. **BLOCH’S SYMBOLS FOR RELATIVE $K_2$.** — Suppose now that $A$ is a commutative ring without nilpotents, $I$ an ideal of $A$, and let $L$ be the quotient field of $A$. Let $\bar{K}_2(A, I)$ be the group

$$\bar{K}_2(A, I) = (1 + I)^* \otimes_2 L^* / \{ f \otimes (1 - f) \mid f \in (1 + I)^*, \ f \neq 1 \}.$$ 

We denote the image of $a \otimes b$ in $\bar{K}_2(A, I)$ by $\{a, b\}$. Bloch has defined a map

$$\Psi_{A, I}: D(A, I) \to \bar{K}_2(A, I)$$

by

$$\Psi(\langle a, b \rangle) = \begin{cases} 
\{1 + ab, b\} & \text{for } b \neq 0 \\
0 & \text{if } b = 0.
\end{cases}$$
Weibel [W2] has constructed an inverse to $\psi_{A,1}$ in certain cases, which include the case

(2.1) $A$ a semi-local PIR, $I = \text{Jac}(A)$.

Our purpose here is to analyze the case where $A$ is regular and semi-local, and

$$I = \left( \prod_{i=1}^{m} t_i^{t_i} \right) \subseteq \text{Jac}(A)$$

with the ideals $(t_i)$ relatively prime. We first assume that $A$ contains a field of characteristic zero; the case of positive characteristic is actually easier to handle.

We will not show that $\psi_{A,1}$ is an isomorphism, rather we content ourselves with exhibiting a specific element of $D(A, I)$ mapping to certain symbols $\{a, b\}$, with suitable conditions on $a$ and $b$. This lifting will be compatible with the inverse to $\psi_{A,1}$ given by Weibel if $(A, I)$ satisfies (2.1).

By the Chinese Remainder Theorem, we may choose the $t_i$'s so that

$$t_i \equiv 1 \mod t_j^{t_j} \quad \text{for} \quad i \neq j.$$ 

Let

$$s = \prod t_i^{t_i}; \quad q_i = \prod_{j \neq i} t_j^{t_j}$$

and let $a$ be in $A$. Then $1 + sa$ is in $(1 + I)^*$. Fix an $i$, and let $q = q_i$, $t = t_i$, etc. Then $1 + qas$ is in $(1 + I)^*$ and

$$(1 + as)(1 + e^{-1} qas)^{-e} = 1 + as(1 - q) + s^2 c; \quad \text{for some } c \in A.$$ 

Since $q \equiv 1 \mod t^s$, we get

$$(1 + as) = (1 + t^s a' s)(1 + e^{-1} qas)^s;$$

$a' \in A$ uniquely determined. We now define a function $\tau = \tau_{A, (a_1, \ldots, t_m)}$ on pairs of the form

$$(1 + as, u; \prod t_i^{t_i}); \quad a \in A, \quad u \in A^*,$$

with values in $D(A, I)$, as follows:

1. Since $\{1 + as, u\} = \psi(\langle asu^{-1}, u \rangle)$, let

$$\tau(1 + as, u) = \langle asu^{-1}, u \rangle.$$

2. To define $\tau(1 + as, t), t = t_i$, write $1 + as$ as

$$1 + as = (1 + t^s a' s)(1 + e^{-1} qas)^s,$$
so
\[\{1 + as, t\} = \{1 + t^a' s, t\} \{1 + e^{-1} qas, t\}\]
\[= \{1 + t^a' s, -t^{-e-1} a' s\}^{-1} \{1 + e^{-1} qas, -e^{-1} as^2\}\]
\[= \{1 + t^a' s, -t^{-e-1} a' s\}^{-1} \{1 + e^{-1} qas, -e^{-1} as\} \{1 + e^{-1} qas, s\}\]
\[= \psi(\{ -t, -t^{-e-1} a' s\}^{-1} \langle -q, -e^{-1} as \rangle^e \langle e^{-1} qa, s \rangle^e).

Thus we set
\[\tau(1 + as, t) = \langle -t, -t^{-e-1} a' s\}^{-1} \langle -q, -e^{-1} as \rangle^e \langle e^{-1} qa, s \rangle^e.

3. Define \(\tau(1 + as, u, \prod t_i^n)\) by
\[\tau(1 + as, u, \prod t_i^n) = (\prod \tau(1 + as, t_i^n)) \cdot \tau(1 + as, u).

Let \(G\) be the subgroup of \(L^*\) generated by \(A^*\) and the \(t_i\)'s, and let \(Z[(1 + I)^* \times G]\) be the free abelian group on \((1 + I)^* \times G\). The above defines \(\tau\) as a map

\[\tau: Z[(1 + I)^* \times G] \to D(A, I)\]

which makes the diagram

\[Z[(1 + I)^* \times G] \to D(A, I)
\]

\[\downarrow \Psi_{A, I}
\]

\[K_2(A, I)
\]

commute. Composing \(\tau\) with \(\Phi_{A, I}: D(A, I) \to K_2(A, I)\), we get a map

\[(2.2) \quad \eta = \eta_{A, u_1, \ldots, u_m}: Z[(1 + I)^* \times G_A] \to K_2(A, I)
\]

(note the dependence on the choice of the \(t_i\)'s). \(\eta_A\) is functorial for ring homomorphisms \(\pi: A \to A'\) such that \(A'\) is semi-local, and where we use the \(\pi(t_i)\) for \(\eta_{A, t}\). In addition, suppose that \(A'\) is a semi-local PIR with Jacobson radical \(I'\), and \(\pi: A \to A'\) is a ring homomorphism with \(\pi(I) \subset I'\). Then using 2.1,

\[(2.3) \quad \pi(\eta(f, g)) = \Phi_{A', r} \circ \psi_{A', r}^{-1}(\{\pi(f), \pi(g)\}) \in K_2(A', I').
\]

The above construction also works in arbitrary characteristic if we assume that all the \(e_i\)'s are 1.

If \(A\) contains a field of characteristic \(p > 0\), there are in general obstructions to lifting a symbol \(\{a, b\}\) in \(K_2(A, I)\) to \(D(A, I)\). However, for our purposes it suffices to work in \(K_2(A, I)[1/p]\), where the lifting problem is easy to solve. In fact, suppose that \(A\) is semi-local, and \(I = (t) A\). Let \(a\) be in \(A\), so \(1 + ta\) is in \((1 + I)^*\), and let \(b\) be in \(A\) with \(|\text{div}(b)| \subset \text{supp}(A/I)\). Then for \(n \gg 0\), letting \(q = p^n\), \(b\) divides \((ta)^n\). As

\[\{1 + ta, b\}^q = \{1 + (ta)^n, b\} = \psi_{A, I}(\langle (ta)^n/b, b \rangle),
\]
we define $\eta_{A,1}$ by

\begin{equation}
\eta_{A,1}(1+ta, b) = \eta^{-1} \Phi_{A,1}(\langle (ta)\eta/b, b \rangle) \quad \text{in} \quad K_2(A, I)[1/p].
\end{equation}

It is easy to check that $\eta^{-1} \langle (ta)\eta/b, b \rangle$ in $D(A, I)[1/p]$ is independent of the choice of $n$. This defines

$$\eta_{A,1}: \mathbb{Z}[(1+I)^* \times G] \to K_2(A, I)[1/p]$$

with functorial properties as above.

2.3. PRODUCTS AND SYMBOLS. — If $A$ is a commutative ring, $u$ and $v$ units in $A$, then the symbol $\{u, v\}$ in $K_2(A)$ agrees with the cup product $u \cup v$, where we consider $u$ and $v$ as elements of $K_1(A)$ via the canonical inclusion $A^* \to K_1(A)$. We proceed to derive a similar relationship between the symbol $\{f, g\}$ in $K_2(A, I)$, $f \in (1+I)^*$, $g \in A^*$, and the cup product $f \cup g$,

$$\cup: K_1(A, I) \otimes K_1(A) \to K_2(A, I).$$

Since $K_1(A, I) = \ker(G_1(A) \to G_1(A/I)/E(A, I))$, the map $G_1(A) \to G_1(A)$ induces a homomorphism

$$\iota: (1+I)^* \to K_1(A, I).$$

This is split by the determinant map $\det: K_1(A, I) \to (1+I)^*$, so $\iota$ is injective.

**Proposition 2.1.** — Let $a$ be in $I$, let $b$ be a unit in $A$, and suppose that $1 + ab$ is a unit. Then

$$\Phi_{A,1}(\langle a, b \rangle) = (\iota(1+ab)) \cup b.$$

**Proof.** — Let $R$ be the ring $\mathbb{Z}[u, u^{-1}, t, (1+tu)^{-1}]$. Define a ring homomorphism $\pi: R \to A$ by

$$\pi(u) = b, \quad \pi(t) = a.$$ 

Since $R \to R/(t)$ is split by the inclusion $\mathbb{Z}[u, u^{-1}] \to R$, we have the exact sequences

$$0 \to K_2(R, (t) R) \to K_2(R) \to K_2(R/(t)) \to 0$$

and

$$0 \to K_1(R, (t) R) \to K_1(R) \to K_1(R/(t)) \to 0.$$ 

As $\Phi_{R,10}(\langle t, u \rangle)$ maps to the symbol $\{1+tu, u\}$ in $K_2(R)$, we get

$$\Phi(\langle t, u \rangle) = (1+tu) \cup u \quad \text{in} \quad K_2(R, (t) R).$$

The result then follows from the functoriality of $\Phi$, $\langle \ , \rangle$ and $\cup$. □
Corollary 2.2. — Let $A$ a semi-local PIR, $I \subseteq A$ the Jacobson radical, $f \in (1+I)^*$, $g \in A^*$. Then

\[
\{f, g\} = f \cup g \quad \text{in } K_2(A, I),
\]

where we identify $(1+I)^*$ with $K_1(A, I)$, $A^*$ with $K_1(A)$, and $K_2(A, I)$ with $K_2(A, I)$.

Proof. — Write $f=1+a$, $a \in I$. Then

\[
\{f, g\} = \psi_{A, 1}(\langle ag^{-1}, g \rangle) = 1(1+a) \cup g = f \cup g. \quad \square
\]

2.4. Milnor $K_3$. — Let $F$ be a field. Bass and Tate have considered the Milnor ring $K_*^M(F)$ of $F$. This is the tensor algebra on $F^*$, modulo the ideal generated by tensors $a \otimes (1-a)$, $a \neq 1$. The image of a tensor $a_1 \otimes \ldots \otimes a_n$ in $K_*^M(F)$ is denoted $\{a_1 \ldots a_n\}$. Let $R$ be a Dedekind domain with quotient field $F$. There is a tame symbol map

\[
T_p: \quad K_p(F) \to \bigoplus_{P \in \text{prime}} K_{p-1}(k(P))
\]

where $k(P)$ is the residue field of $P$. One can then define $K_*^M(R)$ to be the kernel of $T_p$. This is not really the “correct” definition in general; however, if $R$ is semi-local it is reasonable to force a “Gersten’s conjecture” for Milnor K-theory by taking this as a definition.

An obvious subgroup of $K_*^M(R)$ is the subgroup generated by the tensor algebra on $R^*$. Dennis and Stein [D-S] have shown

Theorem. — $K_*^M(R) \cong K_2(R)$ if $R$ is a semi-local PIR. Furthermore, $K_2(R)$ is generated by $R^* \otimes R^*$.

We now give an extension of the latter statement to $K_*^M$, with some additional hypotheses on $R$.

Proposition 2.3. — Let $R$ be a semi-local PIR with infinite residue fields. Then $K_*^M(R)$ is generated by $R^* \otimes R^* \otimes R^*$.

Proof. — Let $(t_1), \ldots, (t_r)$ be the maximal ideals of $R$. If $R$ is a field there is nothing to prove; we therefore assume the result for $R[t_1^{-1}]$ and proceed by induction. Let $t=t_1$ and let $\tau$ be in $K_*^M(R)$. By induction we can write $\tau$ as

\[
\tau = \prod \{a_0, b_0, c_0\}; \quad a_0, b_0, c_0 \quad \text{in } R[t_1^{-1}].
\]

As $\{t, t\} = \{t, -1\}$, we may also assume that $b_0, c_0$ are in $R^*$ and $a_0 = t$ for all $i$. Let $\eta = \prod \{b_i, c_i\}$, so $\tau = \{t, \eta\}$. Then

\[
T_\omega(\tau) = (\eta)^{\pm 1},
\]
where $\tilde{\eta}$ is the image of $\eta$ in $K_2(R/(t))$ and $T(t)$ is the $(t)$ component of $T$. Since $\tau$ is in $K^+_n(R)$, $\tilde{\eta} = 1$.

Suppose that $b'_i b''_i = b_i$ with $b'_i$ and $b''_i$ in $R^*$. Then

$$\{b'_i, c_i\} = \{b''_i, c_i\} \{b'_i, c_i\}$$

so we may assume that

(*) for every $t_j$, $j = 1, \ldots, r$, $t_j \not= 1 \mod t$.

Let $u_i, v_i$ be units in $R$ such that

1. $u_i \equiv v_i \equiv 1 \mod t$.
2. $u_i \equiv b_j \mod t$; $v_i \equiv c_j \mod t$ for $j = 2, \ldots, r$.
3. $u_i = 1 - d_i t$ with $d_i$ in $R$, $d_i \not= 0 \mod t$.

We may thus assume that $t \equiv 1 \mod t$ for $j = 2, \ldots, r$. The (2), (3) and (*) imply

4. $d_i$ is in $R^*$.

We have

$$\{t, u_i, v_i\} = \{t, 1 - d_i t, v_i\} = \{d_i^{-1}, 1 - d_i t, v_i\},$$

which is in the image of $(R^*)^{\otimes 3}$, so we may multiply $\tau$ by $\Pi \{t, u_i, v_i\}^{-1}$. Thus we may assume that

$$\eta \to 1 \in K_2(R/(t)) \quad \text{for } i = 1, \ldots, r.$$

Let $s = \prod t_i$. Then $\eta$ lifts to an element of $K_2(R, (s)) = D(R, (s))$, i.e. we can write $\eta$ as a product

$$\eta = \prod \{1 - e_i s, f_i\} \quad \text{with } e_i, f_i \in R.$$

This reduces to two types of symbols

(a) $\tau = \{t, 1 - e_i s, t_i\}, \quad i \not= 1, e \in R.$

(b) $\tau = \{t, 1 - e_i s, u\}, \quad u \in R^*, e \in R.$

For symbols of type (a), write $1 - e_i s$ as a product

$$1 - e_i s = (1 - e' t_i) (1 - e'' t_i)$$

with $e'$, $e''$, $(1 - e' t_i)$, and $(1 - e'' t_i)$ in $R^*$, which reduces us to symbols of the form

(a') $\tau = \{t, 1 - e t_i, t_i\} \quad i \not= 1, e \in R^*, 1 - e t_i$ in $R^*.$

But

$$\{t, 1 - e t_i, t_i\} = \{t, 1 - e t_i, e\}^{-1} \{t, 1 - e t_i, -1\}^{-1}$$

reducing us to symbols of the form

(a'') $\tau = \{t, 1 - e t_i, u\}, \quad e, 1 - e t_i, u \in R^*.$
Writing 1 - \( e \) as
\[
1 - e = (1 - e') (1 - e'') \quad \text{with} \quad e', e'', (1 - e') (1 - e'') \in \mathbb{R}^*
\]
reduces to symbols \( \{ t, 1 - e, u \} = \{ e, 1 - e, u \} \) which is in the image of \((\mathbb{R}^*)^3\).

For symbols of type \((b)\), write 1 - \( e \) as
\[
1 - e = (1 - e') (1 - e'') \quad \text{with} \quad e', e'', (1 - e') (1 - e'') \in \mathbb{R}^*
\]
reducing us to symbols \( \{ t, 1 - ct, u \} \) as above. This completes the proof. □

2.5. \( K_2 \) RELATIVE TO RATIONAL CURVES. — The results in this section are preparation for the specialization homomorphism to be defined in paragraph 2.6.

Let \( X \) be a regular scheme over an infinite field \( k \), and let \( Y_1, \ldots, Y_s \) be smooth irreducible curves on \( X \). Let \( Y^j \) be the connected component of \( \bigcup Y_i \) containing \( Y_p \). We say that the \( Y_i \)’s form a simple rational chain if

(2.4) (1) The \( Y_i \)’s form a divisor with normal crossing on \( X \).

(2) The dual graph of the \( Y_i \)’s is a (not necessarily connected) tree.

(3) Each connected component \( Y_i \) is a \( k \)-scheme \((k \rightarrow k)\) a field), and each node \( p \) of \( Y_i \) is \( k \)-rational. We have \( k_i = k(p) \).

(4) Each irreducible component \( Y_i \) of \( Y_i \) is absolutely irreducible and rational over \( k_i \).

If \( X = \text{Spec}(A) \), and \( Y_i \) is defined by the ideal \( I_i \), we say that \( I_1, \ldots, I_s \) form a simple rational chain of ideals in \( A \). Let \( s \) be the number of connected components of \( \bigcup Y_i \).

We will always order the \( Y_i \)’s so that
\[
Y_i \cap \left( \bigcup_{j=1}^{i-1} Y_j \right)
\]
is empty for \( i = 1, \ldots, s \), and is a single point \( p_i \) for \( i > s \). We call \( p_j \) the \( j^{th} \) node of the \( Y_i \)’s.

2.4. LEMMA. — Let \( A \) be a semi-local \( k \)-algebra, \( X = \text{Spec}(A) \), and \( Y_1, \ldots, Y_N \) smooth irreducible subvarieties of \( X \). Let \( Y \) be the union of the \( Y_i \)’s. Let \( W \) be a smooth absolutely irreducible curve on \( X \), \( p \) a point of \( W \). Let \( Z \) be the connected component of \( Y \) containing \( p \). Suppose

(i) \( W \cap Y = p \) (scheme theoretically).

(ii) There is a \( k(p) \) map \( Z \cup W \rightarrow p \).

(iii) \( W \) is rational over \( k(p) \).

Then
\[
\delta: \quad K_3(X, Y_1, \ldots, Y_N) \rightarrow K_3(W, p)
\]
is surjective.
Proof. — We have the commutative diagram
\[
\begin{array}{ccc}
K_3(X, Y) & \xrightarrow{\delta} & K_2(\bar{k} (x)) \\
\downarrow & & \\
K_3(X, Y_1, \ldots, Y_N) & \rightarrow & K_3(W, p)
\end{array}
\]
so it suffices to show that
\[K_3(X, Y) \rightarrow K_3(W, p),\]
is surjective.

We first note that
\[0 \rightarrow K_3(W, p) \rightarrow K_3(W) \rightarrow K_3(p) \rightarrow 0\]
is exact. Indeed, our assumption (ii) implies that W is a \(k(p)\) scheme, and the inclusion \(p \rightarrow W\) is split by a projection \(g: W \rightarrow p\).

Next, we claim that \(K^1(W, p)\) is contained in the image of \(K^1(W)\) in \(K^1(W)\). In fact, let \(\bar{k} = k(p)\). Since W is smooth and rational, and X is semi-local, W is the localization of \(\mathbb{A}^1_{\bar{k}}\) at a finite set of points \(S\),
\[S = \{p = q_1, \ldots, q_r\}.\]
We may assume that \(q_1 = 0\). We have the localization sequence
\[0 \rightarrow K_3(\mathbb{A}^1_{\bar{k}}) \rightarrow K_3(W) \rightarrow \bigoplus_{x \in (\mathbb{A}^1_{\bar{k}})^{\times} - S} K_2(\bar{k} (x)) \rightarrow 0\]
where \(\pi: \mathbb{A}^1_{\bar{k}} \rightarrow \text{Spec}(\bar{k}) = p\) is the projection. By [Bass-Tate], there is a similar localization sequence for Milnor K-theory:
\[0 \rightarrow K^M_3(\bar{k}) \rightarrow K^M_3(W) \rightarrow \bigoplus_{x \in (\mathbb{A}^1_{\bar{k}})^{\times} - S} K^M_2(\bar{k} (x)) \rightarrow 0\]
compatible with the map of Milnor K-theory to Quillen K-theory. Let \(K_3(W)^M\) denote the image of \(K_3(W)\) in \(K_3(W)\).

Now suppose \(\eta \in K_3(W)\) restricts to 1 in \(K_3(p)\), i.e. \(\eta\) is in \(K_3(W, p)\). Since \(K^2(F) = K_2(F)\) for \(F\) a field, we can find \(\eta^*\) in \(K_3(W)^M\) with
\[\delta(\eta) = \delta(\eta^*).\]
Then \(\eta\) and \(\eta^*\) differ by an element \(\tau\) of \(K_3(\bar{k})\):
\[\pi^* (\tau) \eta = \eta^*.\]

But then restricting to \(p\) gives
\[\tau = \eta^* \big|_p.\]
which is in $K_3(\bar{k})^M$. Modifying $\eta^*$ by $\pi^*(\tau)$ gives $\eta = \eta^*$, hence $\eta$ is in $K_3(W)^M$, as claimed.

By Proposition 2.3, given $\eta$ in $K_3(W, p)$, we can then write $\eta$ as

$$\eta = \prod \{\alpha_i, \beta_i, \gamma_i\}; \quad \alpha_i, \beta_i, \gamma_i \text{ units on } W.$$

Let $\alpha_i$, $\beta_i$, and $\gamma_i$ denote the restrictions to $p$. By (ii), the $\bar{k}$-morphism $g : W \to p$ extends to a $\bar{k}$-morphism $f : Z \to p = \text{Spec}(\bar{k})$. Let

$$\bar{a}_i = f^*(\tilde{a}_i); \quad \bar{b}_i = f^*(\tilde{b}_i); \quad \bar{c}_i = f^*(\tilde{c}_i).$$

By abuse of notation, we let $\{\bar{a}_i, \bar{b}_i, \bar{c}_i\}$ denote the element $\bar{a}_i \cup \bar{b}_i \cup \bar{c}_i$ of $K_3(Z)$. Then

$$\prod \{\bar{a}_i, \bar{b}_i, \bar{c}_i\} = 1 \text{ in } K_3(Z).$$

Lift $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ to units $a_i$, $b_i$, $c_i$ on $X$ with value 1 on $Y - Z$, and value $\alpha_i$, $\beta_i$, $\gamma_i$ on $W$. We can do this since $Z \cap W = p$ (scheme theoretically). Let $\Delta = \prod \{a_i, b_i, c_i\} \in K_3(X)$. Then

(a) $\Delta|_W = \eta$.

(b) $\Delta|_Y = 1$.

By (b), $\Delta$ lifts to an element $\Delta^*$ of $K_3(X, Y)$ which by (a) restricts to $\eta$, completing the proof. □

**Corollary 2.5.** — Let $X$ be a regular semi-local $k$-scheme, $Y_1, \ldots, Y_n$ a simple rational chain on $X$. Let $s$ be the number of connected components of $\bigcup Y_i, p_{s+1}, \ldots, p_n$ the nodes. Then for $j > s$ the sequence

$$0 \to K_2(X, Y_1, \ldots, Y_j) \to K_2(X, Y_1, \ldots, Y_{j-1}) \to K_2(Y_j, p_j)$$

is exact.

**Proof.** — Taking $N = j - 1$, $W = Y_j, p = p_j$, the subvarieties $Y_1, \ldots, Y_N$ and $W$ satisfy the hypotheses of the above lemma, since the $Y_i$'s form a simple rational chain. Since $Y_i \cap Y_j$ is either empty, or is $p_i$ for $i = 1, \ldots, j - 1$, we have the long exact relativization sequence

$$\to K_p(X, Y_1, \ldots, Y_j) \to K_p(X, Y_1, \ldots, Y_{j-1}) \to K_p(Y_j, p_j) \to.$$

The corollary follows from this and lemma 2.4. □

**Corollary 2.6.** — Let $X$ be a semi-local $k$-scheme, $Y_1, \ldots, Y_N$ subvarieties of $X$. Let $W$ be a smooth curve on $X$, disjoint from the $Y_i$'s, and $p$ a closed point of $W$ such that $k(W)$ contains $k(p)$ and $W$ is absolutely irreducible and rational over $k(p)$. Then the restriction map

$$K_2(X, Y_1, \ldots, Y_N, W) \to K_2(X, Y_1, \ldots, Y_N, p)$$

is injective.
Proof. — By lemma 2.4, the map
\[ K_3(X, Y_1, \ldots, Y_N, p) \to K_3(W, p) \]
is surjective. We have the commutative diagram
\[
\begin{array}{ccc}
K_3(X, Y_1, \ldots, Y_N) & \to & K_3(p) \\
\uparrow & & \uparrow \\
K_3(X, Y_1, \ldots, Y_N) & \to & K_2(X, Y_1, \ldots, Y_N) \\
\uparrow & & \uparrow \\
K_2(X, Y_1, \ldots, Y_N) & \to & K_2(W) \\
\end{array}
\]
The map \( \alpha \) is surjective since \( p \to W \) is split. The surjectivity of \( \varepsilon \) shows that \( \beta \) is injective. \( \square \)

Lemma 2.7. — Let \( X \) be a regular, irreducible, semi-local \( k \)-scheme, essentially of finite type over \( k \); \( X = \text{Spec}(A) \). Let \( S = \{p_1, \ldots, p_n\} \) be a set of closed points of \( X \), \( U \) an open neighborhood of \( S \) in \( X \). Then the map
\[ K_2(X, S) \to K_2(U, S) \]
is injective.

Proof. — Consider the commutative ladder
\[
\begin{array}{ccc}
K_2(U) & \to & K_3(U) \\
\uparrow & & \uparrow \\
K_2(U, S) & \to & K_3(U, S) \\
\uparrow & & \uparrow \\
K_3(U) & \to & K_3(S) \\
\end{array}
\]
By Gersten's conjecture [Quillen], \( \gamma \) and \( \alpha \) are injective, hence we need to show that
\[ \text{Im}(\text{res}) = \text{Im}(\text{res'}) \]
Let \( C = X - U \). Let \( \eta \) be in \( K_3(U) \), and let \( \tau \in K'_2(C) \) be \( \partial(\eta) \), where \( \partial \) is the boundary in the localization sequence
\[ \cdots \to K_3(X) \to K_3(U) \to K_3(C) \to \cdots \]
Let \( Y \) be an affine regular irreducible \( k \)-scheme of finite type, \( V \) an open subset of \( Y \) such that
1. \( X \) is a localization of \( Y \).
2. \( U = X \cap V \).
3. There is an element \( \xi \) of \( K_3(V) \) restricting to \( \eta \).
Let \( W = Y - V \), and let \( \nu \in K'_2(W) \) be \( \partial(\xi) \), where \( \partial \) is the boundary in
\[ \cdots \to K_3(Y) \to K_3(V) \to K'_2(W) \to \cdots \]
Consider the restriction \( v^0 \) of \( v \) to \( K'_2(k(W)) = K_2(k(W)) \). Since \( K_2(k(W)) \) is generated by symbols, there is an open subset \( W^0 \) of \( W \) such that
\[
v|_{W^0} = \prod \{u_i, v_i\}; \quad u_i, v_i \text{ units on } W^0.
\]

Let \( D \) be the complement \( W - W^0 \).

Let \( n = \dim_k Y \). Take a morphism \( \pi: Y \to \mathbb{A}^{n-1}_k \) such that

(a) the fibers of \( \pi \) are curves

(b) \( \pi \) is smooth in a neighborhood of all the closed points of \( X \)

(c) \( \pi|_W \) is finite

and

(d) \( \pi^{-1}(D) \cap S = \emptyset \).

(As \( k \) is infinite, we can take \( \pi \) to be a linear projection.)

Form the fiber square
\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & \mathbb{A}^{n-1}_k W = Z \\
\downarrow & & \downarrow p \uparrow s \\
\mathbb{A}^{n-1}_k & \xrightarrow{s} & W
\end{array}
\]

where \( s \) is the section induced by the inclusion of \( W \) in \( Y \). Then \( q \) is finite; passing to a suitable open subset \( Y^0 \) of \( Y \) containing the closed points of \( X \) gives a diagram
\[
\begin{array}{ccc}
Y^0 & \xrightarrow{q} & Y^0 \times \mathbb{A}^{n-1}_k W = Z^0 \\
\downarrow & & \downarrow p \uparrow s \\
\mathbb{A}^{n-1}_k & \xrightarrow{s} & W
\end{array}
\]

with \( s(W) \) principal on \( Z^0 \), defined by an ideal \( (t) \). We may assume that \( t = 1 \) at all points of \( Z^0 \) lying over \( S \), shrinking \( Y^0 \) if necessary.

Let
\[
\sigma = t \cup p^* (v) \in K'_3(Z^0).
\]

Let
\[
\delta: K'_3(Y^0 - W) \to K'_2(W), \quad \delta': K'_3(Z^0 - s(W)) \to K'_2(s(W))
\]

be the boundary maps in the relevant localization sequences. Then \( q^*_*(\sigma) \) in \( K'_3(Y^0 - W) = K'_3(Y^0 - W) \) has boundary \( \delta(q^*_*(\sigma)) \):
\[
\delta(q^*_*(\sigma)) = q^*_*(\delta'(\sigma)) = q^*_*(s^*_*(v)) = v|_{Y^0 \cap W}.
\]
Thus $\eta \cdot q_\bullet (\sigma)^{-1}$ extends to an element $K_3(Y^0)$ which restricts to an element $\eta^*$ of $K_3(X)$. Since $t=1$ over $S$, $q_\bullet (\sigma)$ restricts to 0 on $S$, hence

$$\text{res}'(\eta) = \text{res}(\eta^*)$$

completing the proof. \(\square\)

**Corollary 2.8.** — *Let $X$ be a regular semi-local $k$-scheme, essentially of finite type over $k$. Let $W$ be a smooth absolutely irreducible curve on $X$, $p$ a point of $W$ such that $k(W)$ contains $k(p)$ and $W$ is rational over $k(p)$. Let $p=p_1, \ldots, p_n$ be closed points of $X$, and $w$ the generic point of $W$. Let $A'$ be the semi-local ring of $S=\{w, p_2, \ldots, p_n\}$ on $X$, $X'=\text{Spec}(A')$. Then the map*

$$\text{res}: K_2(X, W \cup p_2 \cup \ldots \cup p_n) \rightarrow K_2(X', w \cup p_2 \cup \ldots \cup p_n)$$

is injective.*

**Proof.** — Let $\eta$ be in the kernel of $\text{res}$. Let $X'$ be a finite type regular $k$-scheme such that $X$ is a localization of $X'$; let $W$ be the closure of $W$ in $X'$, $p_i$ the closure of $p_i$. We may choose $X'$, and an open neighborhood $U$ of $w \cup p_2 \cup \ldots \cup p_n$ in $X$ so that $\eta$ lifts to an element of the kernel of

$$K_2(X, W \cup P_2 \cup \ldots \cup P_n) \rightarrow K_2(U, (W \cup P_2 \cup \ldots \cup P_n) \cap U).$$

Since $W$ is $k(p)$ rational, the $k(p)$ rational points of $W \cap U$ are Zariski dense in $W$. Let $V=X \cap U$. Then, replacing $X$ with a larger localization of $X$ and changing notation if necessary, we may assume that $W$ contains a closed $k(p)$ point $q$ of $V$, not among the $p_i$'s, and that $\eta$ is in the kernel of

$$\text{res}: K_2(X, W \cup p_2 \cup \ldots \cup p_n) \rightarrow K_2(V, (W \cup p_2 \cup \ldots \cup p_n) \cap V).$$

We have the commutative diagram

$$\begin{align*}
K_2(X, W \cup p_2 \cup \ldots \cup p_n) & \rightarrow K_2(V, (W \cup p_2 \cup \ldots \cup p_n) \cap V) \\
\downarrow \alpha & \downarrow \gamma \\
K_2(X, q \cup p_2 \cup \ldots \cup p_n) & \rightarrow K_2(V, q \cup p_2 \cup \ldots \cup p_n)
\end{align*}$$

By corollary 2.6, $\alpha$ is injective, and $\gamma$ is injective by the previous lemma. Thus $\eta=0$, as desired. \(\square\)

**Proposition 2.9.** — *Let $A$ be a semi-local $k$-algebra, $x, y, t_1, \ldots, t_r$ in $\text{Jac}(A)$ with $t_i \equiv 1 \mod t_j$ for $i \neq j$, and $t_i \equiv 1 \mod xy$ for all $i$. We suppose that $(t_1), \ldots, (t_r)$, and $(x)+(y)$ are maximal. Let $s=\prod t_i$. Take $f, g$ in $A$ with $f \equiv 1 \mod sxy$, and

$$g=ux^*y^* \prod t_i^u; \quad u \in A^*.$$*

*Then there are elements $\tau_x$ in $K_2(A, \{x\})$, $\tau_y$ in $K_2(A, \{y\})$ such that

(a) $\tau_x = \{f, g\}$ in $K_2(A_{(sxy)}, \{x\})$

(b) $\tau_y = \{f, g\}$ in $K_2(A_{(sxy)}, \{y\})$.*
(c) $\tau_x = \tau_y$ in $K_2(A, ((x) + (y)) \cap (s))$

**Proof.** - If $n = m = 0$, let $T$ be the element of $K_2(A, (xy \prod t_i))$

$$\tau = \eta_{A, (x_1, \ldots, x_r)}(f, u \prod t_i^p)$$

and let $\tau_x, \tau_y$ be the respective images of $\tau$ in $K_2(A, xs), K_2(A, ys)$. This reduces us to the two cases $g = x, g = y$. We treat the case $g = y$.

Let $f = 1 + axys, a \in A$. Write $f$ as a product

$$1 + axys = (1 + axys^2)(1 + bxy^2 s) \quad b \in A.$$ 

Then in $D(A, ((x) + (y)) \cap (s))$, we have

$$\langle axs, y \rangle = \langle axs^2, y \rangle \langle bxys, y \rangle \quad (D2)$$

$$\langle axs^2, y \rangle = \langle -y, -axs^2 \rangle^{-1} \quad (D1)$$

$$= \langle axs, s \rangle^{-1} \langle -ys, -axs \rangle^{-1} \quad (D3)$$

$$= \langle -s, -axs \rangle \langle axs, ys \rangle \quad (D1)$$

$$\langle -s, -axs \rangle = \langle axs, ys \rangle \langle -ys^2, -ax \rangle \quad (D3)$$

$$= \langle axs, y \rangle \langle ax, ys^2 \rangle^{-1} \quad (D1).$$

Thus

$$\langle axs, y \rangle = \langle -ys^2, -ax \rangle \langle axs, ys \rangle^2 \langle -y, -bxys \rangle^{-1}$$

in $D(A, ((x) + (y)) \cap (s))$. The LHS above lifts to $\tilde{\tau}_x$ in $D(A, xs)$, the RHS to $\tilde{\tau}_y$ in $D(A, ys)$. One easily checks that

$$\Phi_{A, xs}(\tilde{\tau}_x) = \{1 + axs, y\} \quad \text{in} \quad K_2(A_{xs}, (xs))$$

$$\Phi_{A, ys}(\tilde{\tau}_y) = \{1 + axs, y\} \quad \text{in} \quad K_2(A_{ys}, (ys)).$$

Letting $\tau_x = \Phi_{A, xs}(\tilde{\tau}_x) \in K_2(A, xs), \tau_y = \Phi_{A, ys}(\tilde{\tau}_y) \in K_2(A, ys)$ completes the proof. □

2.6. THE SPECIALIZATION SUBGROUP. - Let $R$ be a semi-local PIR containing a field $k_0$, $I = \text{Jac}(R), S = \text{Spec}(R), \pi: A_5^1 \to S$ the affine line over $S$, with a section $s: S \to A_5^1$. Let $L$ be the semi-local PIR $R(A_5^1), i.e.,$ the semi-local ring of $A_5^1$ in $A_5^1$.

Let $\mu = \mu^p: A_{S, p}^1 \to A_5^1$ be a sequence of blow-ups of $A_5^1$ at points $(p_1, \ldots, p_s) := p$ lying over $s(S)$. For each partial blow-up $\mu^{(p_1, \ldots, p_i)}$, $i \leq s$, there is a section $s^{(p_1, \ldots, p_i)}: S \to A_{5, p}^1$ induced by the section $s$. We call $\mu^p$ allowable if each $p_{i+1}$ is one of the following two types

(a) $p_{i+1} \in s^{(p_1, \ldots, p_i)}(S)$

(b) $p_{i+1}$ is one of the nodes of the exceptional divisor $E(p_1, \ldots, p_i)$ of $\mu^{(p_1, \ldots, p_i)}$.

Suppose that $\mu^p: A_{S, p}^1 \to A_5^1$ is an allowable blow-up. Let $F$ be the proper transform of $A_{5, p}^1$, and let $E_1, \ldots, E_r$ be the irreducible components of the exceptional divisor of $A_{S, p}^1$. 

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\( \mu^p \). Then the curves

\[ F, E_1, \ldots, E_r \]

form a simple rational chain on \( \mathbb{A}^1_{\mathbb{A}^1_p} \).

Let \( F^* \) be the union of the \( E_i \)'s which meet \( s^p(S) \). Then \( F^* \) is a disjoint union of at most \( m P^1 \)'s, where \( m \) is the number of closed points of \( S \). Let \( Q = \{ q_1, \ldots, q_m \} \) be the set of closed points of \( S \), and let \( N = \{ n_1, \ldots, n_r \} \) be the set of nodes of \( F \cup E_1 \cup \ldots \cup E_r \). We let \( B^p \) denote the semi-local ring of \( N \cup s^p(Q) \) on \( \mathbb{A}^1_{\mathbb{A}^1_p} \) and \( B^* \) the semi-local ring of \( s^p(Q) \) on \( \mathbb{A}^1_{\mathbb{A}^1_p} \). We identify \( L \) with the semi-local ring of \( F \) on \( \mathbb{A}^1_{\mathbb{A}^1_p} \); this gives a homomorphism

\[ \xi^p: B^p \to L. \]

**Proposition 2.10.** — The map

\[ \xi^p: K_2(B^p, F, E_1, \ldots, E_r) \to K_2(L, L) \]

is injective.

**Proof.** — By lemma 2.7, the map

\[ K_2(B^p, F) \to K_2(L, L) \]

is injective. The map

\[ K_2(B^p, F, E_1, \ldots, E_r) \to K_2(B^p, F) \]

is injective by corollary 2.5. \( \square \)

For each allowable blow-up \( \mu^p: \mathbb{A}^1_{\mathbb{A}^1_p} \to \mathbb{A}^1_{\mathbb{A}^1_p} \), denote the group \( K_2(B^p, F, E_1, \ldots, E_r) \) by \( K_2(B^p, F, E^p) \). If \( \mu^p: \mathbb{A}^1_{\mathbb{A}^1_p} \to \mathbb{A}^1_{\mathbb{A}^1_p} \) is a blow-up of \( \mathbb{A}^1_{\mathbb{A}^1_p} \) factoring through \( \mu^p \):

\[ \begin{array}{ccc}
\mathbb{A}^1_{\mathbb{A}^1_p} & \xrightarrow{\mu^p} & \mathbb{A}^1_{\mathbb{A}^1_p} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_{\mathbb{A}^1_p} & \xrightarrow{\mu^p} & \mathbb{A}^1_{\mathbb{A}^1_p} \\
\end{array} \]

then we get a commutative diagram

\[ \begin{array}{ccc}
K_2(B^p, F, E^p) & \xrightarrow{\eta^*} & K_2(L, E^p) \\
\downarrow \quad \xi^p & & \downarrow \\
K_2(B^p, F, E^p) & \to & K_2(L, L). \\
\end{array} \]

This enables us to define the **specialization subgroup** \( K_2(L, L)_s \) of \( K_2(L, L) \) by

\[ K_2(L, L)_s = \bigcup_{\mu^p} \xi^p[K_2(B^p, F, E^p)]. \]

\( \mu^p \) allowable.
We define a homomorphism $\Psi_s: K_2(L, L) \to K_2(S, S)$ by

$$(2.5) \quad \Psi_s|_{K_2(B^p, F, E^p)} = \delta^{g*}: K_2(B^p, F, E^p) \to K_2(S, S).$$

Since the $\xi^p$ are all injective, and each two allowable blow-ups can be dominated by a third, $\Psi_s$ is well-defined. We now give a simple sufficient criterion for an element of $K_2(L, L)$ to be in $K_2(L, L)$. We now give a simple sufficient criterion for an element of $K_2(L, L)$ to be in $K_2(L, L)$.

**Proposition 2.11.** Let $s: S \to A^n$ be a section and let $\eta$ be in $K_2(L, L)$. Let $B$ be the semi-local ring of $s(S)$ in $A^n$, $U = \text{Spec}(B)$. Suppose there is a reduced closed curve $Z \subset U$, and an element $z$ of $B$, with $z \equiv 1 \text{ mod } IB$, such that the tame symbol $T(\eta)|_U$ is given by

$$T(\eta)|_U = z|_Z.$$

Then $\eta$ is in $K_2(L, L)$.

**Proof.** Take an allowable blow-up $\mu^p: A^n_{\mu^p} \to A^n_{\mu}$ so that the proper transform $Z^p := \mu^p[Z]$ is disjoint from the nodes of the exceptional divisor $E^p$ of $\mu^p$, and disjoint from $s^p(S)$. We may assume that each component of $\mu^p[A^n_{\mu}]$ intersects $E^p$. Blowing-up points away from the nodes of $E^p$ and away from $s^p(S)$, by

$$\mu^q: A^n_{\mu^q} \to A^n_{\mu},$$

we can separate $Z^p$ from $E^p$, i.e.

$$\mu^q[A^n_{\mu}] \cap \mu^q[A^n_{\mu}] = \emptyset.$$ 

Let $\mu: A^n_{\mu} \to A^n_{\mu}$ be the composition $\mu^p \circ \mu^q$. Let $F_1$ be the proper transform $\mu^{-1}[A^n_{\mu}]$, $Z_1$ the proper transform $\mu^q[A^n_{\mu}]$. By the Remark following Corollary 1.9, we can compute the tame symbol of

$$\mu^*(\eta) \in K_2(A^n_{\mu}, F_1)$$

as the tame symbol of the image $\bar{\mu}^*(\eta)$ of $\mu^*(\eta)$ in $K_2(A^n_{\mu}, F_1)$. Let $\bar{\eta}$ be the image of $\eta$ in $K_2(A^n_{\mu})$. Then

$$T(\bar{\mu}^*(\eta)) = \mu^*(T(\bar{\eta})) = \mu^*(z) \text{ on the cycle } \mu^*(Z).$$

Now, $\mu^*(Z) = Z_1 + Z_{\text{exc}}$, where $Z_{\text{exc}}$ is a cycle supported on the exceptional divisor of $\mu$. Since

$$\mu^*(z)|_{Z_{\text{exc}}} \equiv 1,$$

it follows that $T(\mu^*(\eta)) = 0$ in a neighborhood of $E_1 := \mu^q[A^n_{\mu}]$.

First of all, this implies by a localization sequence that $\mu^*(\eta)$ extends to an element $\eta_1$ of $K_2(U_1, F_1 \cap U_1)$, for some neighborhood $U_1$ of $E_1$. Next, $E_1$ is a union of $\mathbb{P}^1$s, and each connected component of $E_1$ intersects $F_1$. Since $\eta_1$ restricted to the generic
points of $F_1$ is zero, the restriction of $\eta_1$ to each irreducible component $E^i$ of $E_1$ goes to zero in $H^0(E^i, \mathcal{X}_2) \cong K_2(k)$, where $k$ is the field of constants of $E^i$.

Let $U_2 \subset U_1$ be a neighborhood of $E_1 \cap F_1$ such that

(i) $U_2$ contains $s^\#(S)$

(ii) $U_2$ contains all the nodes of $E_1$

(iii) $\mu^s$ maps $U_2$ isomorphically onto $\mu^s(U_2)$

(iv) $U_2 \cap E^i$ is not complete, for each irreducible component $E^i$ of $E_1$.

By (iv), we see that the restriction of $\eta_1$ to $U_2 \cap E^i$ is zero. Thus, by lemma 2.6, $\eta_1$ determines a unique element $\eta_2$ of $K_2(B^p, F, E^p)$. Clearly $\xi^p_\eta(\eta_2) = \eta$, which completes the proof. \hfill $\square$

**Proposition 2.12.** — Let $s$, $S$, $L$, $B$ and $U$ be as in Proposition 2.11. Let $f$ be in $B$ with $f \equiv 1 \mod I$. Let $M$ denote the quotient field of $L$, and let $g$ be in $M^*$ such that $s(S)$ is not a component of $\text{div}(g)$. Then $\{f, g\} \in K_2(L, L)$ is in $K_2(L, L)_s$, and

$$\Psi_s(\{f, g\}) = \{s^\#(f), s^\#(g)\}.$$ 

**Proof.** — Let $\mu^p : \mathbb{A}^{1, p}_s \to \mathbb{A}^1_s$ be an allowable blow-up such that $\mu^{-1}[\text{div}(g)]$ is disjoint from the nodes of the exceptional divisor $E$ of $\mu^p$, disjoint from $s^\#(S)$, and disjoint from $E \cap F$, where $F$ is the proper transform of $A^p$. We may assume that each connected component of $F \cup E$ has dual graph a straight line, i.e. a tree with exactly two end vertices, and that one end lies in $F$, and the other end is the unique irreducible component of $E$ passing through $s^\#(S)$.

As the tame symbol of $\mu^p_*(\{f, g\})$ in a neighborhood of $E$ is

$$T(\mu^p_*(\eta)) = f \mid_{\mu^p_*(\text{div}(g))}$$

we see as in the proof of Proposition 2.11 that $\{f, g\}$ extends to an element $\eta$ of $K_2(B^p, F, E^p)$.

Write $F$ as a union of components:

$$F = F_1 \sqcup \ldots \sqcup F_m$$

with $F_i$ lying over $q_i \in S$. We can write $E$ as a disjoint union:

$$E = E_1 \sqcup \ldots \sqcup E_m$$

where $E_i$ is the component of $E$ intersecting $F_i$. Then $F_i \cup E_i$ is a connected simple rational chain on $\mathbb{A}^{1, p}_s$. Let $E^i_1, \ldots, E^i_\ell$ be the irreducible components of $E_i$. Since there is the section $s^p$ to $\pi \circ \mu^p$ the unique component, say $E^i_1$, passing through $s^\#(q_i)$ appears with multiplicity 1 in $\mu^{p^{-1}}(\mathbb{A}^1_s)$.

Let $\nu^p : \mathbb{P}^{1, p}_s \to \mathbb{P}^1_s$ be the extension of $\mu^p$ to a blow-up of $\mathbb{P}^1_s$. Let $F_i$ denote the closure of $F_i$ in $\mathbb{P}^{1, p}_s$. Let $q = q_i$. Then the total transform $\nu^{p^{-1}}(\mathbb{P}^{1}_q)$ is connected, both $F_i$ and $E^i_1$ appear with multiplicity 1 in this divisor, and are at the "ends", i.e.

$$E^i_1, (\nu^{p^{-1}}(\mathbb{P}^{1}_q) - E^i_1) = F_i, (\nu^{p^{-1}}(\mathbb{P}^{1}_q) - F_i) = 1.$$
In particular, \((v^p - 1)E_i\) is an exceptional curve of the first kind, and can be blown down to form a regular surface

\[ \mu^+: Y \to \text{Spec}(S) \]

flat over \(S\) and having a smooth \(\mathbb{P}^1\) as fiber over \(q\). In particular, \(Y\) is a \(\mathbb{P}^1\) bundle over a neighborhood of \(q\) in \(\text{Spec}(S)\). Thus, if we let

\[ \mu_i : A_{\mathcal{S}, i}^1 \to A_{\mathcal{S}}^1 \]

be the blow-up of all the \(p_j\)'s in \(p\) lying over \(q\), we see that

\[ A_{\mathcal{S}, i}^1 - (F_1 \cup E_1^1 \cup \ldots \cup E_{i-1}^1) \]

is isomorphic to \(A_{\mathcal{S}}^1\), with fiber \(E_i^1 - E_i^{1-1}\) over \(q\). Let \(L_i\) be the semi-local ring of \(E_1^1 \sqcup F_2 \sqcup \ldots \sqcup F_m\) in \(A_{\mathcal{S}, p}\). We claim it suffices to show that

\((\star_i)\)

the image of \(\eta\) in \(K_2(L_1, L_1)\) is given by the symbol \(\{\mu^p(f), \mu^p(g)\}\).

Indeed, it then follows by induction that, letting \(L_m\) be the semi-local ring of \(E_1^1 \sqcup \ldots \sqcup E_m^1\) in \(A_{\mathcal{S}, p}\), we have

\((\star_m)\)

the image of \(\eta\) in \(K_2(L_m, L_m)\) is given by the symbol \(\{\mu^p(f), \mu^p(g)\}\).

Let \(B_m\) be the semi-local ring of \(s^p(S)\) in \(A_{\mathcal{S}, p}\). Arguing as above, \(B_m\) is isomorphic to the semi-local ring of the zero section in \(A_{\mathcal{S}}^1\), hence the map

\[ K_2(B_m, \bar{B}_m) \to K_2(L_m, \bar{L}_m) \]

is injective, by corollary 2.8. In addition, letting \(t_i\) be a generator for the ideal of \(E_i^1\) in \(B_m\) with \(t_i \equiv 1 \mod t_j\) for \(i \neq j\), we can write \(\mu^p(f)\) and \(\mu^p(g)\) as

\[ \mu^p(f) = 1 + a \prod t_i, \quad \mu^p(g) = u \prod t_i^{-1}, \quad \text{with} \quad a \in B_m, \quad u \in B_m^* \]

since \(\mu^{-1}[\text{div}(g)]\) is disjoint from \(\text{Spec}(B_m)\). Let \(\eta_m\) be the image of \(\eta\) in \(K_2(B_m, \bar{B}_m)\), and let \(\eta'\) be the element of \(K_2(B_m, \bar{B}_m)\):

\[ \eta' = \eta_{\mu^p(f), \mu^p(g)} \]

Then \(\eta' = \eta_m\) since both have the same image in \(K_2(L_m, \bar{L}_m)\). Finally,

\[ s^{\mu^p}(\eta') = \{s^{\mu^p}(\mu^p(f)), s^{\mu^p}(\mu^p(g))\} \]

the first equality following from the functorial properties of the maps \(\eta_{*,*}\) defined in paragraph 2.2. Thus \(\Psi_2(\eta) = \{s^*(f), s^*(g)\}\), as desired. We now prove \((\star)\).

We order the \(E_i^1\)'s so that \(F_1\) intersects \(E_1^1\), at say \(n^1\), and \(E_i^1\) intersects \(E_{i-1}^1\) at \(n^i\), for \(i = 2, \ldots, r_1\). Let \(N\) be the set of nodes \(n^1, \ldots, n^{r_1}\), and let \(A\) be the semi-local ring of \(N \sqcup \{s^p(q_i)\} \sqcup F_2 \sqcup \ldots \sqcup F_m\) in \(A_{\mathcal{S}, p}\). Let \(A_i\) be the semi-local ring of \(E_1^1 \sqcup F_2 \sqcup \ldots \sqcup F_m\) in \(A_{\mathcal{S}, p}\), and let \(A_0 = L, E_1^0 = F_1\). We will show by induction on \(i\) that
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The image $\eta_i$ of $\eta$ in $K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n)$ has image $\{\mu^{p_1}(f), \mu^{p_2}(g)\}$ in $K_2(A, \mathbb{A})$.

This is true for $i=0$ by construction of $\eta$. Assume $\mathbb{S}$ for $i < r_1$; then by proposition 2.9 there exist elements $\tau_i, \tau_{i+1}$

$$\tau_i \in K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n); \quad \tau_{i+1} \in K_2(A, \mathbb{E}_{i+1} \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n)$$

such that

(a) $\tau_i$ has image $\{\mu^{p_1}(f), \mu^{p_2}(g)\}$ in $K_2(A, \mathbb{A})$

(b) $\tau_{i+1}$ has image $\{\mu^{p_1}(f), \mu^{p_2}(g)\}$ in $K_2(A_{i+1}, \mathbb{A}_{i+1})$

and

(c) $\text{Im}(\tau_i) = \text{Im}(\tau_{i+1})$ in $K_2(A, n \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n); n = n^{i+1}$.

By corollary 2.8, the map

$$K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n) \to K_2(A, \mathbb{A})$$

is injective, so (a) and our inductive assumption implies

(d) $\tau_i = \eta_i$ in $K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n)$.

On the other hand, we have the commutative diagram

$$
\begin{array}{cccc}
K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n) & \xrightarrow{\text{res}_n} & K_2(A, \mathbb{E}_{i+1} \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n) \\
\downarrow \text{res}_n & & \downarrow \text{res}_{i+1} \\
K_2(A, n \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n) & \xrightarrow{\text{res}_n} & K_2(A, \mathbb{A})
\end{array}
$$

Thus, letting $\eta_{i, i+1}$ be the image of $\eta$ in $K_2(A, \mathbb{E}_1 \mathbb{F}_2 \mathbb{F}_3 \ldots \mathbb{F}_n)$, we have

$$\text{res}_n(\eta_{i, i+1}) = \text{res}_n(\eta_i) = \text{res}_n^{i+1}(\eta_{i+1})$$

so

$$\text{res}_n(\tau_i) = \text{res}_n^{i+1}(\eta_{i+1}); \quad \text{since } \tau_i = \eta_i,$$

and

$$\text{res}_n(\tau_i) = \text{res}_n^{i+1}(\tau_{i+1}); \quad \text{by construction of } \tau_i \text{ and } \tau_{i+1}.$$  

Thus

$$\text{res}_n^{i+1}(\eta_{i+1}) = \text{res}_n^{i+1}(\tau_{i+1}).$$

Since $\text{res}_n^{i+1}$ is injective by corollary 2.6, this gives

$$\eta_{i+1} = \tau_{i+1},$$

hence

$$\text{Im}(\eta_{i+1}) = \text{Im}(\tau_{i+1}) = \{\mu^{p_1}(f), \mu^{p_2}(g)\} \text{ in } K_2(A_{i+1}, \mathbb{A}_{i+1}),$$

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and the induction goes through. This completes the proof. □

Let \( S \to S' \) be a finite étale extension of semi-local PIR’s; \( J \) the Jacobson radical of \( S \). Given a section \( s: S \to \mathbb{A}_S^1 \) let \( s' \) be the induced section \( s': S \to \mathbb{A}_{S'}^1 \). Each allowable blow-up of \( \mathbb{A}_S^1 \) gives by pull-back an allowable blow-up of \( \mathbb{A}_{S'}^1 \); conversely, each allowable blow-up of \( \mathbb{A}_{S'}^1 \) is dominated by an allowable blow-up which is pulled back from \( \mathbb{A}_S^1 \). This shows that the norm map

\[
\text{N}: K_2(R(\mathbb{A}_S^1), \bar{R}(\mathbb{A}_S^1)) \to K_2(R(\mathbb{A}_{S'}^1), \bar{R}(\mathbb{A}_{S'}^1))
\]

restricts to

\[
\text{N}: K_2(R(\mathbb{A}_S^1), \bar{R}(\mathbb{A}_S^1)) \to K_2(R(\mathbb{A}_{S'}^1), \bar{R}(\mathbb{A}_{S'}^1))
\]

satisfying

\[
\Psi_s(\text{N}(\eta)) = \text{N}(\Psi_s(\eta))
\]

for \( \eta \) in \( K_2(R(\mathbb{A}_S^1), \bar{R}(\mathbb{A}_S^1)) \).

Let \( \{x, b\} \) be in \( K_2(S', \bar{S}) \) with \( x \) in \( (1 + J')^* \), \( b \) in \( L^* \) where \( L \) is the quotient field of \( S \). We claim that

\[(2.6) \quad \text{N}((\{x, b\})) = \{\text{N}(x), b\} \quad \text{in} \quad K_2(S, \bar{S}).\]

Indeed, we may assume that \( b \) is in \( S \). Write \( \mathbb{A}_S^1 = \text{Spec}(S[u]) \) and consider the symbol \( \{x, u\} \) in \( K_2(R(\mathbb{A}_S^1), \bar{R}(\mathbb{A}_S^1)) \). Since \( u \) is a unit in \( R(\mathbb{A}_S^1) \), we have

\[
\text{N}(\{x, u\}) = \{\text{N}(x), u\} \quad \text{in} \quad K_2(R(\mathbb{A}_S^1), \bar{R}(\mathbb{A}_S^1)),
\]

by Corollary 2.2, and the projection formula. Specializing via the section \( s \) with \( s^*(u) = b \), and applying Proposition 2.12 proves (2.6).

3. Some relations in relative \( K_2 \)

Let \( S \) be a semi-local PIR with Jacobson radical \( I \). We suppose that \( S \) contains a field \( k_0 \) containing \( \mu_l \). Let \( \alpha \) be in \( S^* \), let \( S_\alpha = S[X]/X^l - \alpha \), if \( \text{char}(k_0) \neq l \); if \( \text{char}(k_0) = l \), let \( S_\alpha = S[X]/X^l - X - \alpha \). Let

\[
\text{N}: K_*(S_\alpha, \bar{S}_\alpha) \to K_*(S, \bar{S})
\]

be the norm map, and let \( \sigma \) be a generator of \( \text{Gal}(S_\alpha/S) \). Our first object is to show

\[(A) \quad \{x, 1-\text{N}(x)\} \text{ is in } (1-\sigma)K_2(S_\alpha, \bar{S}_\alpha) \quad \text{for all } x \in (1+IS_\alpha)^*.
\]

In [M] and [S], this is done by an easy direct computation. We proceed here by a "generic element" method, coupled with the specialization techniques developed in chapter 2.
3.1. THE GENERIC ELEMENT. — Fix a prime \( l \), and let \( F_0 \) be the prime field. If \( F_0 = \mathbb{Q} \), let \( R = \mathbb{Q}(\zeta_l)[t]_{\mathbb{Q}[t]} \); if \( F_0 = \mathbb{F}_p \), let \( R = \mathbb{F}_p(\zeta_l, t_0)[t]_{\mathbb{F}_p[t]} \), with \( t_0 \) and \( t \) independent variables. Let \( I = (t) \), and let \( k \) be the quotient field of \( R \). If \( E \) is an extension ring of \( \mathbb{Q}(\zeta_l) \), or of \( \mathbb{F}_p(\zeta_l) \), let \( \mathbb{R}_E = E[\zeta_l] \), \( I_E = (t) \), and \( k_E \) the quotient field of \( \mathbb{R}_E \). We let \( k_0 \) be the ground field \( \mathbb{Q}(\zeta_l) \) or \( \mathbb{F}_p(\zeta_l, t_0) \), and let \( p \) be the characteristic of \( k_0 \).

**Lemma 3.1.** — Let \( E \) be an extension field of \( k_0 \). If \( l \neq p \), then \( K_2(\mathbb{R}_E, I_E) \) is generated by the symbols \( \{f, \zeta_l\} \) with \( f \) in \( (1 + I_E)^* \). \( K_2(\mathbb{R}_E, I_E) = 0 \) if \( p > 0 \).

*Proof.* — Since the surjection \( \mathbb{R}_E \to \mathbb{R}_E/I_E = \mathbb{E} \) is split, we have the short exact sequence

\[
0 \to K_2(\mathbb{R}_E, I_E) \to K_2(\mathbb{R}_E) \to K_2(\mathbb{E}) \to 0.
\]

In addition, the map

\[
K_2(\mathbb{R}_E) \to K_2(\mathbb{k}_E)
\]

is injective. Suppose \( l \neq p \). Let \( \eta \) be an \( l \)-torsion element of \( K_2(\mathbb{R}_E, I_E) \), so

\[
\text{Im}(\eta) = \{g, \zeta_l\} \quad \text{in} \quad K_2(\mathbb{k}_E),
\]

for some \( g \) in \( \mathbb{k}_E^* \), by Suslin [S]. Since \( \eta \) maps to \( K_2(\mathbb{k}_E) \) via \( K_2(\mathbb{R}_E) \), the tame symbol \( T_{\eta l}(\text{Im}(\eta)) \) vanishes, i.e.

\[
\zeta_l^{\text{ord}_{\eta l}(\eta)} = 1.
\]

Thus \( g = t^a u \), for some integer \( a \), and some unit \( u \) in \( \mathbb{R}_E \). Then \( \{g, \zeta_l\} = \{u, \zeta_l\} \). In addition, since \( \eta \) is in \( K_2(\mathbb{R}_E, I_E) \), \( \{u, \zeta_l\} \) restricts to \( 1 \) in \( K_2(\mathbb{R}_E/I_E) = K_2(\mathbb{E}) \). Let \( \text{res}: \mathbb{R}_E \to \mathbb{E} \) be the canonical surjection; then \( \{\text{res}(u), \zeta_l\} = 1 \) in \( K_2(\mathbb{E}) \). Thus

\[
\text{Im}(\eta) = \{u/\text{res}(u), \zeta_l\} \quad \text{in} \quad K_2(\mathbb{R}_E).
\]

Letting \( f = u/\text{res}(u) \) completes the proof in this case. If \( l = p \), we use the same proof, together with the result of Suslin that \( pK_2(\mathbb{k}_E) = 0 \). \( \square \)

Let \( x_0, \ldots, x_{l-1}, v \) be independent variables over \( k \), let \( u = v^l \) if \( l \neq p \); if \( l = p \), let \( u = v^p - v \). Let \( A \) and \( B \) be the rings

\[
A = k_0[x_0, x_1, \ldots, x_{l-1}, x_{l-1}^{-1}, u],
\]

\[
B = k_0[x_0, x_1, \ldots, x_{l-1}, x_{l-1}^{-1}, v],
\]

so \( B = A[v] \). Let \( x \) be the element

\[
x = 1 + t \sum x_i v^i \in \mathbb{R}_B,
\]

so \( x \) is the "generic element" of the universal Kummer extension (or Artin-Schreier extension if \( l = p \)) \( \mathbb{R}_B/\mathbb{R}_A \) having \( x_{l-1}^{-1} \) invertible, and with \( x \equiv 1 \mod t \). Let \( L \) be the quotient field of \( B \), \( E \) the quotient field of \( A \).

Let \( N: \mathbb{R}_B \to \mathbb{R}_A \) be the norm, \( \sigma \) the generator of \( \text{Gal}(\mathbb{R}_B/\mathbb{R}_A) \) with \( \sigma(v) = \zeta_l v \) for \( l \neq p \), \( \sigma(v) = v + 1 \) for \( l = p \). Let \( X^{1/l} = \text{Spec}(\mathbb{R}_B), X = \text{Spec}(\mathbb{R}_A) \). Let \( W \) be the closed subscheme

\[
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\]
of \( X^{1/t} \) defined by the ideal \(((1-N(x))/t)\), \( W' \) the subscheme defined by \((x)\). We note that \( W \) and \( W' \) are reduced and irreducible. Write \( x = 1 + ty \).

On \( X^{1/t} - (W \cup W') \), both \((1-N(x))/t\) and \( x \) are units, so the symbols \( \langle ty [(1-N(x))/t]^{-1}, (1-N(x))/t \rangle \) and \( \langle y, t \rangle \) define elements

\[
\eta_1 = \Phi(\langle ty [(1-N(x))/t]^{-1}, (1-N(x))/t \rangle) \\
\eta_2 = \Phi(\langle y, t \rangle)
\]

of \( K_2(\mathbb{X}^{1/t}) \), satisfying

\[
(3.1) \quad \{x, 1-N(x)\} = \text{Im}(\eta_1 + \eta_2) \quad \text{in} \quad K_2(R_{\mathbb{L}}, I_2).
\]

Abusing notation, we will denote the element \( \eta_1 + \eta_2 \) of \( K_2(X^{1/t} - (W \cup W'), \mathbb{X}^{1/t} - W) \) by \( \{x, 1-N(x)\} \).

Let \( z \) be the regular function on \( W \) defined by

\[
(3.2) \quad z = (1/l) \left( 1 + \sum_{i=1}^{l-1} (x_w)^{\sigma^{-1} + \ldots + \sigma^{-i}} \right), \quad \text{for} \quad l \neq p
\]

\[
(3.2)' \quad z = -v^{p-1} + \sum_{i=1}^{p-1} (-v^{p-1})^{\sigma^{-1}} (x_w)^{\sigma^{-1} + \ldots + \sigma^{-i}} \quad \text{for} \quad l = p
\]

where \( x_w \) is the restriction of \( x \) to \( W \). Then \( z \equiv 1 \mod t \); in particular \( z \) is not identically zero. Let \( Z \subset W \) be the locus \( \{z = 0\} \). Then

\[
(3.3) \quad x_w = z^{\sigma}/z \quad \text{on} \quad W - Z.
\]

Let \( A^0 = k_0[x_0, \ldots, x_{l-1}, x_{l-1}^{-1}] \), \( L^0 \) the quotient field of \( A^0 \), and let \( X^0 = \text{Spec}(R_{A^0}) \). Then

\[
X^{1/t} = \text{Spec}(R_{A^0}[v]), \quad X = \text{Spec}(R_{A^0}[u]).
\]

We form relative compactifications of \( X^{1/t} \) and \( X \) over \( X^0 \) by introducing new variables \( v_0 \) and \( u_0 \) \( = (v_0)^i \), and defining

\[
X^* = \text{Proj}_{R_{A^0}} R_{A^0}[u \mu_0, u_0]; \\
X^{1/t*} = \text{Proj}_{R_{A^0}} R_{A^0}[v \nu_0, v_0].
\]

Let \( W^1 \) be the closure of \( W \) in \( X^{1/t} \), and \( W^* \) the normalization of \( W^1 \).

**Lemma 3.2.** — The element \( x_w \) of \( k_0[W] \) extends to a regular function \( x_w^* \) on \( W^* \), with \( x_w^* \equiv 1 \mod t \).

**Proof.** — Let \( X^{1/t*}(\infty) \) be the locus \( \{v_0 = 0\} \). \( W \subset X^{1/t} \) is defined by the equation

\[
(3.4) \quad 0 = (N(x) - 1)/t = \alpha_0 + \sum_{i=1}^{l-1} \alpha_i u_i; \quad \alpha_i \text{ in } A^0,
\]
with
\[ \alpha_0 = l x_0 \mod \mathfrak{a}, \quad \alpha_{l-1} = (-1)^{l-1} \cdot l^{1-l} (x_l^{-1})^l, \quad \text{for } l \neq p \]
\[ \alpha_0 = x_{p-1} \mod \mathfrak{a}, \quad \alpha_{p-1} = t^{p-1} \cdot (x_{p-1})^p, \quad \text{for } l = p. \]

Since \( t x_l^{-1} \) is a unit on \( X^0 - X^0 \), \( W - \tilde{W} \) is finite over \( X^0 - X^0 \), so \( W^1 - W \) is contained in \( X^{1/t}(\infty) \). Extend \( x_w \) to a rational function \( x_{W*} \) on \( W^* \). Then each component \( C \) of \( (x_{W*})_w \) dominates \( X^0 \). Pass to the semi-local ring \( R^0 : = R_{t_0} \) of \( X^0 \) in \( X^0 \). Then \( W^*_0 = W^* \times X^0 \text{Spec}(R_{t_0}) \) is proper over \( R_{t_0} \), and is irreducible; since \( R_{t_0} \) is a DVR, this implies that \( W^*_0 \) is finite over \( R^0 \).

Let \( \nu \) be an extension of the valuation of \( R^0 \) defined by \( (t) \) to a valuation on \( k_0(W) \). One easily checks that
\[ \nu(\alpha_i) \geq 1 \quad \text{for } i = 0, \ldots, l-1, \quad \nu(\alpha_{l-1}) = l-1. \]

Let \( a = -\nu(\nu) \). Since \( x = 1 + t \sum x_l \nu_l \), we need only show that \( a \leq 1/l \). Assume that \( a > 1/l \). As
\[ \nu(u^l \alpha_i) \geq -ila + i \quad \text{and} \quad \nu(u^{l-1} \alpha^{l-1}) = -(l-1) ila + l-1 < 0, \]
we see that
\[ \nu(u^{l-1} \alpha_{l-1}) < \nu(u^l \alpha_i) \quad \text{for } i = 0, \ldots, l-2. \]

But then (3.4) shows that
\[ 0 \leq \nu(u^{l-1} \alpha_{l-1}) = -a(l-1) ila + l-1, \]
hence \( a \leq 1/l \), contrary to our assumption. \( \square \)

**Corollary 3.3.** — The function \( z \) defined in (3.2) extends to a regular function \( z_{W*} \) on \( W^* \), with \( z_{W*} \equiv 1 \mod t \). The divisor \( (z_{W*}) \) is disjoint from \( \tilde{W}^* \), so the divisor \( Z = (z) \) on \( W \) is proper over \( X^0 \).

**Proof.** — Immediate from the lemma and (3.2). \( \square \)

We now proceed to explicitly solve the equation
\[ \{x, 1 - N(x)\} = a^e/a \quad \text{in } K_2(U, U) \]
for a particular open subset \( U \) of \( X^{1/t} \). The final result (3.7) includes an additional term of the form \( \{g, \xi_U\} \), but we will absorb this factor later on.

Let \( W^* \to W^0 \to X^0 \) be the Stein factorization of \( W^* \to X^0 \). Then, as \( W \) is finite over \( X^0 - X^0 \), we have
\[ W^0 - \tilde{W}^0 \to W - \tilde{W}. \]

Since \( z_{W*} \) is identically 1 on \( W^* \), \( z_{W*} \) defines a regular function \( z_{W^0} \) on \( W^0 \), with \( z_{W^0} \equiv 1 \mod t \). Let \( Z^0 \subset W^0 \) be the locus \( \{z_{W^0} = 0\} \); then \( Z^0 \cap \tilde{W}^0 = \emptyset \).
Form the pullbacks
\[ X_{W_0}^{1/\ell} = X^{1/\ell} \times_{X^0} W^0; \quad X_{W_0}^{1/\ell'} = X^{1/\ell'} \times_{X^0} W^0. \]

Then
\[ p_1: X_{W_0}^{1/\ell} \to X^{1/\ell} \quad \text{and} \quad p_1: X_{W_0}^{1/\ell'} \to X^{1/\ell'} \]
are finite, \( X_{W_0}^{1/\ell} \) is isomorphic over \( W^0 \) to \( A_{W_0}^{1/\ell} \), and \( X_{W_0}^{1/\ell'} \) is isomorphic over \( W^0 \) to \( P_{W_0}^{1/\ell'} \).

Homogenizing the equation (3.4) gives the equation for \( W^1 \) in \( X^{1/\ell} \). Thus \( \tilde{W}^1 \to \tilde{X}^0 \) is finite over the locus \( x_0 \neq 0 \) (if \( l=p \), \( W^1 \to X^0 \) is finite). In particular, there is a codimension two subset \( T^0 \) of \( W^0 \), \( T^0 \subset W^0 \), and a closed subset \( T \) of \( W^1 \), \( T \subset W^1 \), such that the birational map \( W^0 \to W^1 \) defines a birational finite morphism
\[ p: W^0 - T^0 \to W^1 - T. \]

The map \( p \) composed with the inclusion \( W^1 - T \to X^{1/\ell'} \) gives a section
\[ s: W^0 - T^0 \to X_{W_0}^{1/\ell'} \cong P_{W_0}^{1/\ell}. \]

Let \( F' \) be a section of \( \mathcal{O}(1) \) on \( P_{W_0}^{1/\ell} - T^0 \) with divisor
\[ \text{div}(F') = s(W^0 - T^0). \]

Since \( T^0 \) has codimension at least two, and \( W^0 \) is normal, \( F' \) extends to a section \( F \) of \( \mathcal{O}(1) \) on \( P_{W_0}^{1/\ell} \) with
\[ \text{div}(F) = \text{closure of } s(W^0 - T^0). \]

Let \( f \) be the restriction of \( F \) to \( X_{W_0}^{1/\ell} \cong A_{W_0}^{1/\ell} \), considered as a regular function on \( A_{W_0}^{1/\ell} \). Let \( Y^0 \) be the closure in \( A_{W_0}^{1/\ell} \) of \( s(W^0 - T^0) \cap A_{W_0}^{1/\ell} \). Then
\[ \text{div}(f) = Y^0. \]

In addition, the restriction of \( p_1 \)
\[ p_{1|Y^0}: Y^0 \to X^{1/\ell} \]
gives a finite birational morphism from \( Y^0 \) onto \( W \). Let \( p_2: X_{W_0}^{1/\ell} \to W^0 \) be the second projection and let \( D^0 = p_2^{-1}(Z^0) \); we note that \( D^0 \cap A_{W_0}^{1/\ell} = \emptyset \).

Consider the symbol
\[ \{p_2^* (z_{W_0}), f\} \in K_2(R(A_{W_0}^{1/\ell}), I(A_{W_0}^{1/\ell})). \]

On \( A_{W_0}^{1/\ell} - Y^0 - D^0 \), \( p_2^* (z_{W_0}) \) is a unit, \( p_2^* (z_{W_0}) \equiv 1 \mod t \), and \( f \) is a unit. Writing \( p_2^* (z_{W_0}) \) as \( 1 + ta \), for some regular function \( a \) on \( A_{W_0}^{1/\ell} \), the symbol
\[ \mu^0 := \Phi(\langle ta f^{-1}, f \rangle) \in K_2(A_{W_0}^{1/\ell} - Y^0 - D^0, A_{W_0}^{1/\ell} - Y^0). \]
has image \( \{ p^*(z_\omega), f \} \) in \( K_2(\mathbb{R}(\Delta_{\omega}^0), I(\Lambda_{\omega}^0)) \). The tame symbol of \( \mu^0 \) is
\[
T(\mu^0) = z_{\omega}^0 \text{ on } Y^0.
\]

Let \( D = p_1(D^0) \), and let \( \mu \) be the element \( p_{1*}(\mu^0) \) of \( K_2(X^{1/2} - W - D, \tilde{X}^{1/2} - \tilde{W}) \). Then \( \mu \) has tame symbol
\[
(3.5) \quad T(\mu) = p_{1*}(T(\mu^0)) = (z \text{ on } W)
\]
in \( K_1'(X^{1/2} - D, \tilde{X}^{1/2})^{1/2} \). In addition, \( D = p_{X^0\rightarrow 1} (p_{X^0}(D)) \), i.e., \( D \) consists of fibers of \( p_{X^0} : X^{1/2} \rightarrow X^0 \).

We now pass to \( R^0 = R_{L^0} \). \( X_{R^0}^{1/2} \) is isomorphic to \( \Lambda_{R^0}^1 \). The element
\[
\{ x, 1 - N(x) \} \cdot \mu^0 / \mu \in K_2(\Lambda_{R^0}^{1,0} - (W \cup W'), \Lambda_{R^0}^{1,0} - \tilde{W})
\]
has trivial tame symbol by construction, hence determines a unique element \( \alpha \) of \( K_2(\Lambda_{R^0}^{1,0}, \Lambda_{R^0}^{1,0}) \cong K_2(R^0, I^0) \). On the other hand, under the norm map
\[
N : K_2(R_L, I_L) \rightarrow K_2(R_{E^0}, I_{E^0})
\]
(recall that \( E \) is the quotient field of \( A = k_0[x_0, \ldots, u] \) and \( L \) is the quotient field of \( A[x] \)) we have
\[
\alpha' = N(\alpha) = N(\{ x, 1 - N(x) \} \cdot \mu^0 / \mu) = \{ N(x), 1 - N(x) \} = 1 \text{ in } K_2(R_{E^0}, I_{E^0}).
\]

Since the map
\[
K_2(R_{L^0}, I_{L^0}) \rightarrow K_2(R_{E^0}, I_{E^0})
\]
is injective, we have
\[
\alpha' = 0 \text{ in } K_2(R_{L^0}, I_{L^0}).
\]

By lemma 3.1, we have
\[
(3.6) \quad \left\{ \begin{array}{ll}
\{ x, 1 - N(x) \} \cdot \mu^0 / \mu = \{ g, \zeta_l \} \text{ in } K_2(\Lambda_{R^0}^{1,0}, \Lambda_{R^0}^{1,0}), & \text{if } l \neq p, \\
\{ x, 1 - N(x) \} \cdot \mu^0 / \mu = 1 \text{ in } K_2(\Lambda_{R^0}^{1,0}, \Lambda_{R^0}^{1,0}), & \text{if } l = p.
\end{array} \right.
\]

Here \( g \) is an element of \((1 + I_{L^0})^* \), \( g = 1 + tb \).

Since \( K \)-theory commutes with direct limits, we have proved the

**Proposition 3.4.** — There is an affine open subset \( U \) of \( X^{1/2} - W - W' \), containing the generic point of \( X^{1/2} \), such that \( b \) is regular on \( U \), \( \mu \) extends to an element of \( K_2(U, \tilde{U}) \).
and

\[
(3.7)
\begin{cases}
  \{x, 1-N(x)\} \mu^\alpha/\mu = \{g, \zeta_i\} \quad \text{in } K_2(U, \bar{U}), & \text{if } l \neq p, \\
  \{x, 1-N(x)\} \mu^\alpha/\mu = 1 \quad \text{in } K_2(U, \bar{U}), & \text{if } l = p.
\end{cases}
\]

Here \(\{g, \zeta_i\}\) stands for the element \(\Phi(\langle tb^1 \zeta_i, \zeta_i \rangle)\) of \(K_2(U, \bar{U})\), and as explained above, \(\{x, 1-N(x)\}\) stands for the image of the element \(n_1 + n_2\) of \(K_2(X^{1/\ell} - W - W', X^{1/\ell} - W)\) defined in (3.1).

3.2. THE ELEMENT \(\{x, 1-N(x)\}\). — We retain the notations \(X^0, X, X^{1/\ell}, U, W, W', x, \mu\) and \(g\) of the previous section. Let \(S\) be a semi-local PIR containing \(k_0\), \(J\) the Jacobson radical of \(S\), \(J = (t)S\). Extend the inclusion \(k_0 \to S\) to a ring homomorphism \(g^*: R \to S\) by \(g^*(t) = t\). Letting \(T = \text{Spec}(S)\), we get a smooth \(R\) scheme \(g: T \to \text{Spec}(R)\) with \(T = \text{Spec}(S/J)\). Let \(t = \prod_{i=1}^r t_i\) be a prime factorization of \(t\) in \(S\).

If \(A \to \text{Spec}(R), B \to \text{Spec}(R)\) are \(R\)-schemes, we let \(A_B \to B\) be the \(B\)-scheme \(p_2: A \times_{\text{Spec}(R)} B \to B\).

Take \(\alpha\) in \(S^*\), fix a prime \(l\), and let \(S^\alpha\) be the étale cyclic extension of \(S\): \(S^\alpha(S[X]/(X^l - \alpha) = S[\beta]\), \(l \neq p\)

\[
= S[X]/(X^p - X - \alpha) = S[\beta], \quad \text{if } l = p.
\]

Let \(J^\alpha = JS^\alpha, T^\alpha = \text{Spec}(S^\alpha)\). Let \(X_T(\alpha)\) be the subscheme of \(X_T\) defined by the ideal \((u - \alpha)\), and let \(X_T^{1/\ell}(\alpha)\) be the subscheme of \(X_T^{1/\ell}\) defined by the ideal \((v - \beta)\). We get the commutative diagram

\[
\begin{array}{c}
X_T^{1/\ell}(\alpha) \to X_T(\alpha) \\
\downarrow \quad \quad \quad \downarrow \\
X_T^0 \quad \quad T^\alpha \\
\quad \quad \quad \quad T
\end{array}
\]

We note that \(X_T^{1/\ell}(\alpha)\) is isomorphic to the fiber product \((X_T^{1/\ell}) \times_T (X_T(\alpha))\). If we modify \(\alpha\) by the \(l\)-th power of a unit in \(S\), \(\alpha' = v^l\alpha\) (or \(\alpha' = \alpha + a^p + a\) if \(l = p\)) then \(T^\alpha\) and \(T^\sigma\) are isomorphic as \(T\)-schemes, thus we may assume that

\[(3.10)_T \quad \text{each component of } X_T^{1/\ell}(\alpha) \text{ has non-empty intersection with } U_{T^\alpha}.
\]

If \(Z \to T\) is a finite étale \(T\)-scheme, let \(Z^\alpha = T^\alpha \times_T Z\). For all such \(Z\), the condition \((3.10)_Z\) is satisfied.
If \( y = 1 + t \sum y_i \beta^i \) is in \((1+J)^*\), \( y_i \) in \( S^e \) and \( y_{i-1} \) in \( S^e^* \), then \( y \) determines a pair of compatible sections \( \mathcal{Y} \) and \( \mathcal{Y}^* \):

\[
\begin{align*}
X^1/\mathcal{Y}(\alpha) & \to X^0/\mathcal{Y} \\
\downarrow & \uparrow \mathcal{Y} \quad \downarrow \uparrow \mathcal{Y} \\
T^q & \to T
\end{align*}
\]

by \( \mathcal{Y} = (y_0, \ldots, y_{i-1}) \), \( \mathcal{Y}^* = (y_0, \ldots, y_{i-1}, \beta) \).

Let \( p: \mathbb{A}_{1}^e \to \mathbb{A}_{1}^e \) be the map induced by \( T^q \to T \). For \( Z \) a closed subset of \( \mathbb{A}_{1}^e \), we let \( Z^a \) denote \( p^{-1}(Z) \).

**Lemma 3.5.** — Let \( g = g(z) \) be in \( S^e[z]_{(z)} \); \( z \) an indeterminant, with \( g \equiv 1 \mod t \), such that \( \text{ord}_i(1 - N(g)) = \text{ord}_i(1 - N(g(0))) \); for \( i = 1, \ldots, r \)

where we consider \( g \) as an element of \( S^e[z]_{(z)} \) in the LHS.

Let \( C \) be a closed subset of \( \mathbb{A}_{1}^e \). Then there is a closed subset \( D \) of \( \mathbb{A}_{1}^e \), with \( D \cap \{z = 0\} = \emptyset \), and a proper map

\[
F: \mathbb{A}_{1}^e - D \to \mathbb{A}_{1}^e
\]

such that

(a) \( F: \mathbb{A}_{1}^e - D \to \mathbb{A}_{1}^e \) is étale at each generic point

(b) \( F^{-1}(0) = \{z = 0\} \cup Z \), with \( Z \cap (C \cup \{z = 0\}) = \emptyset \)

(c) \( \pi_e: Z \to T \) is finite and étale

(d) \( g \) is regular on \( \mathbb{A}_{1}^e - D^a \).

Let \( i: Z \to \mathbb{A}_{1}^e \), \( j: Z^a \to \mathbb{A}_{1}^e \) be the inclusions. For each section \( s: T \to \mathbb{A}_{1}^e \), let \( Z(s) = F^{-1}(s(T)) \), and let \( i_s: Z(s) \to \mathbb{A}_{1}^e - D \) be the inclusion. Let \( j_s: Z(s)^a \to \mathbb{A}_{1}^e - D^a \) denote the inclusion of \( Z(s)^a \), and let

\[
\begin{align*}
\pi_s: & \quad Z(s) \to T; \quad \pi_s^a: Z(s)^a \to T^a; \\
\pi: & \quad Z \to T; \quad \pi^a: Z^a \to T^a
\end{align*}
\]

denote the projections. Then

(e) If \( F \) is étale over a neighborhood of \( s(T) \), then

\[
\{g(0), 1 - N(g(0))\} = (\pi^a_s)((j^*_s(g), 1 - N(j^*_s(g))) \cdot \pi^a_s((j^*_s(g), 1 - N(j^*_s(g)))^{-1}).
\]

**Proof.** — Our assumption on \( N(g) - 1 \) can be rephased as \( 1 - N(g) = v \cdot \Pi t_i n_i \); \( v \) a unit on an affine neighborhood \( V \) of \( \{z = 0\} \) in \( \mathbb{A}_{1}^e \).

Then

\[
\Xi := \eta_{a, t_1} \ldots \eta_v ((g, 1 - N(g)) \in K_2(V, V); \quad V = \text{Spec}(A),
\]

is defined, and for each map \( q: R \to V \), with \( R \) a regular, semi-local curve with \( \bar{R} \) reduced,

\[
q^* (\Xi) = \{q^*(g), 1 - q^*(N(g))\} \quad \text{in} \; K_2(R, \bar{R}).
\]

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Extend $p: \mathbb{A}^1_\mathbb{T} \to \mathbb{A}^1_\mathbb{T}$ to $p: \mathbb{P}^1_\mathbb{T} \to \mathbb{P}^1_\mathbb{T}$. Let $D^* = \mathbb{P}^1_\mathbb{T} - V$, $D^* = p(D^*)$. Shrinking $V$ if necessary, we may assume that $D^* = p^{-1}(D^*)$. Let $m$ be the degree of $D^*$ over $T$. We note that $D^*$ contains no component of $\mathbb{P}^1_\mathbb{T}$. Let $s_\infty$ be a section of $\mathcal{O}(m)$ on $\mathbb{P}^1_\mathbb{T}$ with $(s_\infty) = D^*$. Let $s_0$ be a section of $\mathcal{O}(m)$ such that

$$(s_0) = 1, \{z = 0\} + Z,$$

with

$$Z \subset \mathbb{A}^1_\mathbb{T} - (D^* \cup C \cup \{z = 0\}), \quad Z \to T \text{ étale, and } Z \text{ reduced.}$$

Let $D = D^* \cap \mathbb{A}^1_\mathbb{T}$.

Let $F: \mathbb{A}^1_\mathbb{T} - D \to \mathbb{A}^1_\mathbb{T}$ be the restriction to $\mathbb{A}^1_\mathbb{T} - D$ of the map

$$(id, (s_0': s_\infty)); \mathbb{P}^1_\mathbb{T} \to \mathbb{P}^1_\mathbb{T}.$$ Then $F$ is a finite degree $m$ map with $F^{-1}(0) = (s_0)$. From $(\star)$ it follows that $F$ satisfies $(a)$-$d$.

Let $F^*: \mathbb{A}^1_\mathbb{T} - D^* \to \mathbb{A}^1_\mathbb{T}$ be the map induced by $F$.

Let $\tau = F^*(\tilde{\Xi}) \in K_2(\mathbb{A}^1_\mathbb{T}, \mathbb{A}^1_\mathbb{T})$. Let

$$q: \mathbb{A}^1_\mathbb{T} \to T; \quad q^*: \mathbb{A}^1_\mathbb{T} \to T^*$$

be the projections. By the homotopy property, $\tau = q^*(\xi)$ for some $\xi$ in $K_2(T^*, T^*)$.

Then for sections $s, s^*: T \to \mathbb{A}^1_\mathbb{T}$, we get induced sections $s^*, s^*: T^* \to \mathbb{A}^1_\mathbb{T}$ and

$$\pi^*_s(j^*_s(\Xi)) = s^* \circ (F|_{Z(s)}^*)_s (j^*_s(\mu)) = s^* (F^*(\mu)) = s^* \circ q^*(\xi) = \xi,$$

and similarly for $s^*$. Taking $s^*$ to be the zero section, we find

$$\pi^*_s(j^*_s(\Xi)) = \pi^*_s(j^*(\Xi)) + s^*(\Xi) = \pi^*_s(j^*(g), 1 - N(j^*(g))). \{g(0), 1 - N(g(0))\}$$

which completes the proof. □

**Lemma 3.6.** — The subgroup of $K_2(T^*, T^*)/(1 - \sigma) \mathbb{K}_2(T^*, T^*)$ generated by the symbols

$\{y, 1 - N(y)\}$ with $y = 1 + t \Sigma y_1^j 1^j \; \text{in} \; (1 + J^*)^*$ is the same as the subgroup generated by elements of the form

$$N_{Q\cap T^*}((w, 1 - N(w))),$$

where $Q \to T$ range over finite étale $T$-schemes, and $w \in (1 + J^*)^*$, $w = 1 + t \Sigma w_1 1^j$, $w_1 \in \Gamma(Q, \mathcal{C}_Q)$ satisfies

(a) $w_{i-1}$ and $w_0$ are units in $S^*$

(b) $\mathcal{W}(x)$ is in $U_Q$ for each generic point $x$ of $Q$. 

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Here $\mathcal{W} : Q \to X^{1/l}_Q$ is the section determined by $w$.

Proof. - Given an open subset $V$ of $A^1_T$, we can find elements $y' = 1 + t \Sigma y'_i \beta^i$ and $z$ of $(1 + J^*)^*$ with

$$y'_i(T) \subseteq V \quad \text{for } i = 1, \ldots, l-1; \quad y' = z^q/z,$$

so $N(y') = 1$. Take $V$ so that $a \in V$ implies that $a + y_0$ and $a + y_{l-1}$ are units in $S^q$. Then $y'' = yy'$ satisfies (a), and

$$\{ y, 1 - N(y) \} = \{ y', 1 - N(y') \}$$

$$= \{ y''', 1 - N(y'') \} \{ y', 1 - N(y) \}$$

$$= \{ y''', 1 - N(y'') \} \{ z, 1 - N(y) \}^{(l-q)}$$

which proves (a), with $Q = T$ and $w = y''$.

For (b), let $N$ be the maximum of $\text{ord}_b (1 - N(y))$, as $b$ ranges over the closed points of $T$. Let $P_0(z), \ldots, P_{l-1}(z)$ be in $S[z]$, $z$ an indeterminate, such that

$$\left\{ \begin{array}{l}
P_i(z) \equiv 0 \text{ mod } t^N \\
P_i(0) = 0
\end{array} \right. \quad \text{for } i = 0, \ldots, l-1,$$

Let $g(z) = y + \Sigma P_i(z) \beta^i$. Let $M$ denote the total quotient field of $S$. Choosing the $P_i$ sufficiently general, we may assume that the $S[z]$-valued point $\mathcal{G}$ of $X^{1/l}$ determined by $g$ satisfies

$$\mathcal{G}(x) \in U_{M(z)} \quad \text{for each generic point } x \text{ of } \text{Spec}(M(z)).$$

By (★), $g$ satisfies the hypotheses of Lemma 3.2; applying that lemma, with $C$ being the closure of the points of $\mathcal{G}^{-1}(X^{1/l} - U)$ lying over $\text{Spec}(M)$, and $s$ being any sufficiently general section, proves (b). \hfill \square

Lemma 3.7. — Let $\mathcal{S} : T \to A^1_T$ be a section, $\mathcal{S}^a : T^a \to A^1_{T^a}$ the induced section. Let $Q$ be a neighborhood of $\mathcal{S}(T)$ in $A^1_T$, $Q^a$ the neighborhood of $\mathcal{S}^a(T^a)$ lying over $Q$. Let $R(Q)$ (resp. $R(Q^a)$), be the semi-local ring of $Q$ in $Q$ (resp. $Q^a$ in $Q^a$). Suppose there are elements $\beta \in K_2(\overline{Q^a}, \overline{Q^a})$ and $f \in (1 + JR(Q^a))^*$ with

$$\text{Im}(\beta) = \{ f, \zeta \} \text{ in } K_2(R(Q^a), R(Q^a)).$$

Then $\mathcal{S}^a(\beta) = \{ g, \zeta \}$ in $K_2(T^a, \overline{T^a})$, for some $g \in (1 + J^*)^*$.

Proof. — Let $p : A^1_{T^a} \to A^1_T$ be the obvious map. As in lemma 3.5, we construct a proper map

$$F : Q \to A^1_T$$

with $F^{-1}(0) = s(T) + Z$, $Z \to T$ étale, and $Z \subseteq Q - p(\text{div}(f)) - s(T)$. Since $f = 1 + at$ on $Q^a$, with a regular on $Q^a$, the symbol $\{ f, \zeta \}$ is the image in $K_2(R(Q^a), R(Q^a))$ of the element $\Phi(\langle at \zeta^{-1}, \zeta \rangle)$ of $K_2(Q^a, \overline{Q^a})$. Then, retaining the notations of lemma 3.5, we
have
\[ \pi^*(f^*(\beta)) = \pi^*(f^*(R), \zeta), \quad \text{by functoriality of } \Phi \]
\[ = \{\pi^*(f^*) (f), \zeta\} \quad \text{(projection formula)} \]
\[ = \{g', \zeta\}, \quad g' \in (1 + J^*)^*. \]

Similarly, for a sufficiently general section \( s: T \to A_{1}, \) we have
\[ \pi_s^*(j^*(\beta)) = \pi_s^*(j^*(f), \zeta), \quad \text{by functoriality of } \Phi \]
\[ = \{\pi_s^*(j^*) f, \zeta\} \quad \text{(projection formula)} \]
\[ = \{g'', \zeta\}, \quad g'' \in (1 + J^*)^*. \]

Using the homotopy property applied to \( \tau = F^*(\beta) \) as in the proof of lemma 3.5, we have
\[ \mathcal{E}^* \tau = \{g'', \zeta\} \{g', \zeta\}^{-1} \]
\[ = \{g''/g', \zeta\}, \]
as desired. \( \square \)

**Proposition 3.8.** Let \( y \) be in \((1 + J)^*\). Then \( \{y, 1 - N(y)\} \) is in \((1 - \alpha) K_1(T^*, T^*)\).

**Proof.** By lemma 3.6, replacing \( T \) with a finite étale cover and changing notation, we may assume that \( y = 1 + \tau \sum y_i B_i \) with the \( y_i \) in \( S \), and \( (a) \) and \( (b) \) of that lemma satisfied. We have the diagram
\[
\begin{array}{ccc}
X_{1/2}(x) & \to & X_0 \\
\downarrow \phi & & \downarrow \phi \\
T^* & \to & T
\end{array}
\]
Take a linear projection \( p: A_{1/2}^* \to A_{1/2}^* \) which then induces projections \( p_0: X_0^* \to A_{1/2}^* \), and \( p^*: X_{1/2}(x) \to A_{1/2}^* \). Choose \( p \) so that the fiber \( Q^* = p^{-1} p^*(\Phi(T)) \) satisfies
\[ (3.12) \text{ each component of } Q^* \text{ intersects } U_{1/2}. \]

Let \( Q \) be the fiber \( p_0^{-1} p_0(\Phi(T)) \). Let \( q: Q \to X, q^* : Q^* \to X_{1/2}^* \) be the respective compositions
\[ Q \to X^0 \to X_0, \quad Q^* \to X_{1/2}^* \to X_{1/2}^*. \]

Let \( x, \mu \) and \( g \) be as given in \((3.7)\), and let \( x^*, \mu^* \) and \( g^* \) be the pullbacks of \( x, \mu \) and \( g \) via \( q \) to \((1 + J R (Q^*))^*\), \( K_2(R(Q^*), R(Q^*)) \) and \((1 + J^*)^*\), respectively. From \((3.7)\) we get the equation
\[ (3.13) \quad \{x^*, 1 - N(q^*(x))\} \mu^*/\mu = \{g^*, \zeta\} \quad \text{in } K_2(R(Q^*), R(Q^*)) \]
where we take \( g = 1 \) if \( l = p \). Here we have used the functoriality of the map \( \Phi_* \) and the functoriality of the symbols \( \langle , \rangle \). In addition, letting \( W^* \) be the divisor \( q^*(W) \), \( W = ((1 - N(x))/t = 0) \), and \( x^* \) the pullback \( q^*(x) \), where \( z \) is the function constructed in
paragraph 3.1, we have from (3.5) the computation of the tame symbol of $\mu_q$:

$$T(\mu_q) = z^q \quad \text{on } W^q.$$  

In particular, $z^q$ the restriction to $W^q$ of the regular function

$$Z^q = b + \sum b^{q-1} (x^q)^{q-1} + \ldots + x^q$$

where $b = 1/l$ if $l \neq p$, and $b = (-\beta^{q-1})$ if $l = p$. $Z^q$ is thus defined in a neighborhood of $\mathcal{O}^q(T^q)$, with $Z^q \equiv 1 \mod t$. By Proposition 2.11, $\mu_q$ is in the specialization subgroup $K_2(R(Q^p), R(Q^q))_{\mathcal{O}^q}$. By Proposition 2.12, $\{x^q, 1 - N(x^q)\}$ is also in $K_2(R(Q^q), R(Q^q))_{\mathcal{O}^q}$ and

$$\Psi_{\mathcal{O}^q}(\{x^q, 1 - N(x^q)\}) = \{y, 1 - N(y)\}. $$

Thus $\{g^q, \zeta^q\}$ is also in $K_2(R(Q^p), R(Q^q))_{\mathcal{O}^q}$, and we have

$$\Psi_{\mathcal{O}^q}(\{g^q, \zeta^q\}) = \{y, 1 - N(y)\} \sqrt{v/v} \quad \text{in } K_2(T^q, T^q),$$

where $v = \Psi(\mu_q)$. This completes the proof in case $l = p$.

For $l \neq p$, the tame symbol of $\{x^q, 1 - N(q^*(x))\}_{\mu_q/\mu_q}$ vanishes in a neighborhood of $\mathcal{O}^q(T^q)$, so $\{g^q, \zeta^q\}$ extends to an element $\gamma$ of $K_2(V, \tilde{V})$, for some neighborhood $V$ of $\mathcal{O}^q(T^q)$ in $Q^q$. By lemma 3.7, we have

$$\Psi_{\mathcal{O}^q}(\{g^q, \zeta^q\}) = \{h, \zeta^q\}, \quad \text{for some } h \in (1 + J^q)^*.$$ 

But

$$\{h, \beta\}^*/\{h, \beta\} = \{h, \beta^*/\beta\}$$

which completes the proof.$\Box$

3.3. GENERATORS FOR RELATIVE $K_2$. — We now consider a filtering direct system of semi-local PIR’s: $\{S_i \mid i \in I\}$, where each $S_i$ contains $k_0$. We assume there is an initial element 0 of $I$. Let $J_i$ be the Jacobson radical of $S_i$. Let $S_\infty, J_\infty$ be the direct limits. Since K-theory commutes with direct limits, we have

$$K_2(S_\infty, J_\infty) = \lim_{\rightarrow} K_2(S_i, J_i).$$

Similarly, for $\alpha$ in $S_0^\alpha$, let $S^\alpha_i$ be $S_i[X]/X^\alpha - \alpha$ (or $S_i[X]/X^\alpha - X - \alpha$ if $l = p$), so $S^\alpha_i$ is étale over $S_i$ and has Jacobson radical $J^\alpha_i = J_i S^\alpha_i$. Also, letting $S^\alpha_\infty, J^\alpha_\infty$ be the direct limits, we have

$$K_2(S^\alpha_\infty, J^\alpha_\infty) = \lim_{\rightarrow} K_2(S^\alpha_i, J^\alpha_i).$$

Let $L$ denote the quotient field of $S_\alpha$, $L_\infty$ the direct limit of the $L_\alpha$, and similarly define $L_\alpha$ and $L_\infty$. Let $\sigma$ be the generator of $\text{Gal}(S_\alpha/S_\infty)$, $\sigma(\beta) = \zeta_0 \beta$ (or $\beta + 1$ if $l = p$), where $\beta$ is the image of $X$ in $S^\alpha_i$. 

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We suppose that \( \{S_i \mid i \in I\} \) satisfies

(I) Every \( x \) in \( 1 + J_\infty \) is a norm from \( S^*_\infty \).

(II) If \( P(u) \) is a separable polynomial with coefficients in \( S_\infty \) and has degree \( d < l \), then \( P(u) \) factors completely in \( S_\infty [u] \).

Using Hilbert's theorem 90, we can replace (I) with

(I)' Every \( x \) in \( 1 + J_\infty \) is a norm from \( 1 + J^*_\infty \).

Our object here is to show

**Proposition 3.9.** — Assuming (I) and (II), the quotient group

\[
K_2(S^*_\infty, J^*_\infty)/(1 - \sigma) K_2(S^*_\infty, J^*_\infty)
\]

is generated via symbols by \((1 + J^*_\infty)^* \otimes L^*_\infty\).

The proof proceeds in a series of steps:

Let \( G_1 \subset K_2(S^*_\infty, J^*_\infty) =: G \) be the subgroup generated by \((1 + J^*_\infty)^* \otimes L^*_\infty\), \( G_2 \) the subgroup \((1 - \sigma) G\).

**Step 1.** — \( G/G_1 \) is generated by symbols of the form

\[
\{1 + a \beta, b + c \beta\} \quad \text{and} \quad \{1 + a, b + c \beta\}
\]

with \( a \) in \( J_\infty \) and \( b, c \) in \( S^*_\infty \).

**Proof.** — \( G \) is generated by symbols \( \{f, g\} \) with \( f \) in \((1 + J^*_\infty)^*\), \( g \) in \( L^*_\infty \). Write \( g \) as

\[
g = \sum g_i \beta^i \quad \text{with} \quad g_i \text{ in } L_j \text{ for some } j.
\]

Let \( p_1, \ldots, p_r \) be the closed points of \( S_p \), \( v_1, \ldots, v_r \) the associated valuations. Take \( h \) in \( L_j \) so that

\[
v_s(h) = \min_i \{v_s(g_i)\} \quad \text{for} \quad s = 1, \ldots, r.
\]

Then \( g_i/h \) is in \( S_j \) for each \( i \), and at each \( p_s \) at least one of the \( g_i/h \) is a unit. Since \( \{f, g\} \equiv \{f, g/h\} \mod G_1 \), we may replace \( g \) with \( g/h \), and changing notation, assume that each \( p_s \) at least one of the \( g_i \) is a unit.

Assume that \( l \neq p \). Take units \( u_1, u_2 \) in \( S_p \). Note that \( g (u_1 + u_2 \beta^j) = \sum g_i \beta^i \) with

\[
g_i' = \begin{cases} u_1 g_i + u_2 g_{i-j} & \text{for} \quad i \geq j \\ u_1 g_i + u_2 \alpha g_{j-i} & \text{for} \quad i < j. \end{cases}
\]

Since \( \alpha \) is a unit, it is easy to see from this that we can find units \( u_{1,k}, u_{2,k} \) and integers \( j_k, 0 < j_k < l \) so that

\[
g' = g \cdot \prod_k (u_{1,k} + u_{2,k} \beta^{j_k})
\]
is of the form \( g' = \sum g'_i \beta^i \) with \( g'_i \) in \( S_j^* \). Replacing \( u_{1,k} + u_{2,k} \beta^k \) with \( u_{1,k} + u_{2,k} \beta^k + v_k \beta^{i-1} + w_k \beta \), it follows that

\[
g'' = g \cdot \prod (u_{1,k} + u_{2,k} \beta^k + v_k \beta^{i-1} + w_k \beta)
\]

is also of the form

\[
g'' = \sum g''_i \beta^i \quad \text{with} \quad g''_i \text{ in } S_j^*,
\]

if we choose the \( v_k \) and \( w_k \) to be sufficiently general units in \( S_p \). Changing notation, we may therefore assume that \( g_0 \) and \( g_{i-1} \) are units in \( S_p \). The proof of this fact in case \( l = p \) is similar and will be left to the reader.

Write \( f \) as

\[
f = 1 + \sum f_i \beta^i \quad \text{with the } f_i \text{ in } J_j^*,
\]

increasing \( j \) if necessary. Arguing as above, we may assume the polynomials

\[
P(u) = \sum g_i u^i; \quad Q(u) = 1 + \sum f_i u^i
\]

are separable, hence by (II) factor in \( S_a \) for some \( n > j \); changing notation we may assume that \( n = j \), and

\[
P(u) = \prod (a_i + b_i u); \quad Q(u) = \prod (c_i + d_i u)
\]

with \( a_i, b_i, c_i \), and \( d_i \) in \( S_p \). Since \( g_0 = \prod a_i, \quad g_{i-1} = \prod b_i, \quad \text{and} \quad 1 + f_0 = \prod c_i \), the elements \( a_i, b_i, \) and \( c_i \) are all units in \( S_p \). Let \( c = \prod c_i, \quad d_i = d_i/c_i \), so \( c \) is in \( (1 + J_j) \) and

\[
Q(u)/c = \prod (1 + d_i u).
\]

As the coefficient \( f_i/c \) of \( u^i \) in \( Q(u)/c \) is the \( i \)-th symmetric function of \( d'_1, \ldots, d'_{i-1} \), and \( f_i(p) \) vanishes for all closed points \( p \) of \( \text{Spec}(S_j) \), it follows that \( d'_i(p) \) also vanishes for all \( i \), and all closed points \( p \). Thus \( d'_i \) is in \( J_p \) and the symbol \( \{f, g\} \) can be written as

\[
\{f, g\} = \prod \{1 + d'_i \beta, c_j + d_j \beta\} \{c, c_j + d_j \beta\}
\]

completing the proof of step 1.

**Step 2.** \( G/G_1 \) is generated by symbols of the form \( \{1 + a, b + c \beta\} \), with \( a \) in \( J_{\infty} \), and \( b, c \) in \( S_a^* \).

**Proof.** By step 1, we need only consider symbols of the form \( \{1 + a \beta, b + c \beta\} \); \( a \) in \( J_{\infty} \), \( b \) and \( c \) in \( S_a^* \), some \( n \).

Let \( u \) be an indeterminant, let \( S = S_a, \quad J = J_n, \) let \( \bar{S} = S/J \) and let \( v \) be the indeterminant with

\[
v^l = u \quad \text{if } l \neq p; \quad v^l + v = u \quad \text{if } l = p.
\]
Let $L(v)$ denote the semi-local ring of $A^1_S$ in $A^1_S = \text{Spec}(S[v])$, $L(v) = L(v)/JL(v)$. Consider the symbol

$$\eta = \{1 + av, b + cv\} \quad \text{in} \quad K_2(L(v), L(v)).$$

Let $Z_1, Z_2 \subset A^1_S$ be the curves defined by the ideals $(1 + av), (b + cv)$ respectively. Then the projection

$$\pi: A^1_S \to T = \text{Spec}(S)$$

restricts to an isomorphism $Z_2 \to T$ and a generically $1-1$ map $Z_1 \to T$. In addition $Z_1 \cap A^1_S = \emptyset$, so $\pi: Z_1 \to T$ defines an isomorphism

$$Z_1 \to \text{Spec}(L),$$

where $L$ is the quotient field of $S$. Furthermore $Z_2 \cap \{v = 0\}$ is empty, since $b$ and $c$ are units.

The tame symbol $T(\eta)$ is given by

$$T(\eta) = (1 + av) \text{ on } Z_2 - (b + cv) \text{ on } Z_1.$$

Then we can find $f$ in $1 + J, g$ in $L^*$ such that

$$\pi^*(f)|_{Z_2} = (1 + av)|_{Z_2}, \quad \pi^*(g)|_{Z_1} = (b + cv)|_{Z_1}.$$

Thus the product $\{\pi^*(f), b + cv\} \{1 + av, \pi^*(g)\}$ has the same tame symbol as $\eta$, so there is a $\tau$ in $K_2(S, J)$ with

$$(\star) \quad \eta = \{\pi^*(f), b + cv\} \{1 + av, \pi^*(g)\} \pi^*(\tau) \quad \text{in} \quad K_2(L(v), L(v)).$$

Let $s: \text{Spec}(S^p) \to A^1_S$ be the section (over $S$) with $s^*(v) = \beta$, $p: A^1_S \to A^1_S$ the obvious map. Since $\pi^*(f)$ and $1 + av$ are units in a neighborhood of $p(s(\text{Spec}(S^p)))$, Proposition 2.11 implies that the terms in $(\star)$ pulled back to $A^1_S$ are in the specialization subgroup $K_2(L^*(v), L^*(v))$. Thus, using Proposition 2.12,

$$\{1 + a \beta, b + c \beta\} = \{f, b + c \beta\} \{1 + a \beta, g\} \cdot \tau \quad \text{in} \quad K_2(S^p, J^p).$$

This completes the proof of Step 2. $\square$

**Step 3.** $\{1 + a, b + c \beta\}$ is in $G_1 G_2$ for $a$ in $J$, $b$, $c$ in $S^*_S$.

**Proof.** By $(\Gamma)'$, we can find $x$ in $(1 + J^p_\infty)^*$ with $1 + a = N(x)$.

We claim that

$$(\star) \quad \{1 + a, b + c \beta\} = \{x, N(b + c \beta)\} \mod G_1 G_2.$$

Since $\{x, N(b + c \beta)\}$ is in $G_1$, this would complete the proof.

Write $x$ as

$$x = 1 + \sum f_i \beta_i.$$
As in step 1, we can factor this as 
\[ x = (1 + d) \prod (1 + d_i \beta^i) \quad \text{with} \quad d, d_i \in J_{\infty}. \]

Since \( 1+a = N(x) = (1 + d) \prod N(1 + d_i \beta^i) \), we need only show that

\[ \{N(y), b + c \beta\} \equiv \{y, N(b + c \beta)\} \mod G_1 G_2, \tag{\star \star} \]

for \( y = 1 + a_0 + a_1 \beta, a_0 \) and \( a_1 \) in \( J_{\infty} \).

We proceed as in step 2, retaining the notations from that step. We assume all the elements defined above lie in \( S_0^* = S^* \).

Let \( W^1, W^2 \) be the curves on \( \mathbb{A}^1_S \) defined by ideals \((1 + a_0 + a_1 v), (b + cv)\), respectively (note that \( W^1 = \emptyset \) if \( a_1 = 0 \)). As above, \( \pi: W^2 \to \text{Spec}(S) \) is an isomorphism, and \( \pi: W^1 \to \text{Spec}(L) \) is an isomorphism if \( a_1 \neq 0 \). Let \( V^1 \) and \( V^2 \) be the subschemes of \( \mathbb{A}^1_S \) defined by ideals \((N(1 + a_0 + a_1 v))\) and \((N(b + cv))\), respectively. Then \( V^1 \) and \( V^2 \) are the unions

\[ V^1 = \bigcup \sigma^i(W^1); \quad V^2 = \bigcup \sigma^j(W^2) \]

Since \( b \) and \( c \) are units,

\[ \sigma^i(W^2) \cap \sigma^j(W^2) = \emptyset \quad \text{if} \quad i \neq j. \]

Since \( W^1 \) is disjoint from \( \mathbb{A}^1_S \),

\[ \sigma^i(W^1) \cap \sigma^j(W^2) = \emptyset \quad \text{for all} \quad i \quad \text{and} \quad j. \]

Thus \( V^1 \) and \( V^2 \) are regular, disjoint, and étale over \( \text{Spec}(S) \).

Let \( \eta \) be the element of \( K_2(L(v), L(v)):\n\]

\[ \eta = \{N(1 + a_0 + a_1 v), b + cv\} \{1 + a_0 + a_1 v, N(b + cv)\}^{-1}. \]

Then \( \eta \) has tame symbol

\[ T(\eta) = (h_2 \text{ on } V^2) - (h_1 \text{ on } V^1), \]

with

\[ h_1 \in k(V^1)^*; \quad h_2 \in \Gamma(V^2, \mathcal{O}_{V^2}); \quad h_2 \equiv 1 \mod J. \]

Clearly

\[ N(h_1) = N(h_2) = 1, \]

so we can write

\[ h_1 = z_1^2/z_1; \quad h_2 = z_2^2/z_2 \quad \text{with} \quad z_1 \in k(V^1)^*; z_2 \in \Gamma(V^2, \mathcal{O}_{V^2}). \]
Taking
\[ z_2 = b + \sum b^{\sigma^{-i}} (h_2)^{\sigma^{-i}} + \ldots + \sigma^{-i} \]
as in the proof of Proposition 3.8, we may assume that \( z_2 \equiv 1 \mod J \). Let \( z_1' \) be the restriction of \( z_1 \) to \( \sigma'(W') \), and similarly define \( z_2' \). Then there are elements \( h_1' \in L^* \), \( h_2' \in (1 + J)^* \) with
\[ \pi^*(h_j)|_{W_j} = z_j' \]
Let \( \omega_i \) and \( \delta_i \) be the symbols
\[ \omega_i = \{ 1 + a_0 + a_1 \sigma'(v), h_1' \} ; \quad \delta_i = \{ h_2', b + c \sigma'(v) \} . \]
Then \( \lambda := \prod \omega_i \delta_i \) has tame symbol
\[ T(\lambda) = (z_1 \text{ on } V_1) - (z_2 \text{ on } V_2) , \]
so
\[ \eta_i (\lambda^*/\lambda)^{-1} = \pi^*(\tau) \text{ in } K_2(L(v), \mathbb{L}(v)) , \]
for some \( \tau \) in \( K_2(S, J) \). Specializing as in step 2 gives (**) , completing the proof of step 3, and the proposition. \( \square \)

4. Main Theorems

4.1. Hilbert's Theorem 90 for Relative \( K_2 \) — We now follow the proof of Suslin in [S] to prove Hilbert's Theorem 90 for relative \( K_2 \). Let \( S \) be a semi-local \( \mathbb{P}IR \) containing the field \( k_0 \). Let \( \alpha \) be a unit in \( S \); we retain the notations \( J, S^a, J^a, T, T^a, \) etc. from part 3. In particular, \( S^a \) is an étale cyclic Galois extension of \( S \), of prime degree \( l \), with Galois group generated by \( \sigma \). For a flat \( S \)-algebra \( W \), with \( W \) a semi-local \( \mathbb{P}IR \), let \( W^a = W \otimes_S S^a, J(W) \subset W \) the Jacobson radical, \( J(W)^a = J(W) W^a \). We have the complex
\[ M(W)_a: \quad K_2(W^a, J(W)^a) \rightarrow K_2(W^a, J(W)^a) \rightarrow K_2(W, J(W)) . \]
Let \( V(W) \) be the homology \( H_1(M(W)_a) \). If \( g: W \rightarrow W' \) is an inclusion of semi-local \( \mathbb{P}IR \)'s, then \( g \) induces \( g^*: V(W) \rightarrow V(W') \); if \( W' \) is finite and étale over \( W \) we have \( g^*: V(W') \rightarrow V(W) \). Since \( V(W^a) = 0 \), and \( g^* \circ g^* = \deg(g) \cdot id \), we get
1. \( V(W) \) is an \( l \)-torsion group for every \( W \).

If \( g: W \rightarrow W' \) is finite and Galois, and of degree \( d \) prime to \( l \), then using the maps
\[ g_*: \quad K_2(W^a, J(W^a)) [1/d] \rightarrow K_2(W^a, J(W^a)) [1/d] \]
and
\[ g_*: \quad K_2(W^a, J(W^a)) [1/d] \rightarrow K_2(W, J(W)) [1/d] \]
defined is paragraph 1.10, we see that
2. \( g^* : V(W) \to V(W') \) is injective.

Let \( x \) be in \( 1 + J(W) \), let \( \mathcal{O} \) be the Azumaya algebra constructed as a crossed product algebra from the Hilbert symbol \( \langle \alpha, x \rangle \) (or the symbol \( [\alpha, x]_p \) if \( l = p \)) as in Serre [Se], let \( g : X \to \text{Spec}(W) \) be the Brauer-Severi variety associated to \( \mathcal{O} \). We let \( W = W/J(W) \), \( \bar{X} = g^{-1}(\text{Spec}(W)) \), and let \( R(X) \) denote the semi-local ring of \( \bar{X} \) in \( X \), with radical \( J(X) \). We note that \( \bar{X} \) is a projective space over \( \overline{W} \) (\( \mathcal{O} \) is split) as \( x \equiv 1 \mod J(W) \). Let \( X^\sigma = X \times_W W^\sigma \), and let \( f : X^\sigma \to X, f : \text{Spec}(W^\sigma) \to \text{Spec}(W) \) be the covering maps. \( X^\sigma \) is also a projective space over \( W^\sigma \).

**Proposition 4.1.** — The map \( g^* : V(W) \to V(R(X)) \) is injective.

**Proof.** — Let \( \eta \) be in \( K_2(W^\sigma, W^\sigma) \) with \( N(1-1) = 1 \), and suppose that \( g^*(\eta) = \lambda^\sigma/\lambda \) for some \( \lambda \) in \( K_2(R(X^\sigma), J(X^\sigma)) \). Let \( z = \partial(\lambda) \), where \( \partial \) is the boundary in the localization sequence

\[
\to K_2(R(X^\sigma), J(X^\sigma)) \to K_1((X^\sigma)^{1/2}, (\overline{X^\sigma})^{1/2}) \to K_1((X^\sigma)^{0/2}, (\overline{X^\sigma})^{0/2}) \to
\]

Then \( z'z = \partial(g^*(\eta)) \) = 0. By our computation of \( K_1(X^{1/2}, \bar{X}^{1/2}) \) and \( K_1((X^\sigma)^{1/2}, (\overline{X^\sigma})^{1/2}) \) in paragraph 1.8, the map

\[
f^* : \ K_1(X^{1/2}, \bar{X}^{1/2}) \to K_1((X^\sigma)^{1/2}, (\overline{X^\sigma})^{1/2})
\]

is injective, so \( z \) can be considered as an element of

\[
K_1(X^{1/2}, \bar{X}^{1/2}) \subset K_1((X^\sigma)^{1/2}, (\overline{X^\sigma})^{1/2}).
\]

As \( \delta(z) = 0 \) in \( K_0((X^\sigma)^{2/3}, (\overline{X^\sigma})^{2/3}) \), and since

\[
f^* : \ K_0(X^{2/3}, \bar{X}^{2/3}) \to K_0((X^\sigma)^{2/3}, (\overline{X^\sigma})^{2/3})
\]

is injective (Corollary 1.12), \( z \) defines a class \( [z] \) in \( E^{1,-2}_X(X, \bar{X}) \). Since \( g^*([z]) \) clearly dies in \( E^{1,-2}_X(X, \bar{X}) \), and \( \mathcal{O} \) is split, Corollary 1.13 implies that \( [z] = 0 \) in \( E^{1,-2}_X(X, \bar{X}) \). Thus \( z = \delta(\tau) \) for some \( \tau \) in \( K_2(R(X), J(X)) \). Modifying \( \lambda \) by \( f^*(\tau) \), we may assume that \( \delta(\lambda) = 0 \). By Corollary 1.6, we have \( \lambda = g^*(\xi) \) for some \( \xi \) in \( K_2(W^\sigma, W^\sigma) \), and thus \( g^*(\eta) = g^*(\xi^\sigma/\xi) \). As \( X^\sigma \) is a projective space over \( W^\sigma \), \( g^* \) is injective, and we get \( \eta = \xi^\sigma/\xi \), completing the proof. □

We now define a direct system \( \{ S_i \mid i \in \mathcal{F} \} \) of \( S \)-algebras with \( S_0 = S \). For each \( x \) in \( 1 + J(S) \), we let \( S_x = R(X) \), where \( X \) is the Brauer-Severi variety over with symbol \( \langle \alpha, x \rangle \) (or \( [\alpha, x]_p \) if \( l = p \)), and for each irreducible separable polynomial \( P \) of degree \( < l \), let \( S_P \) be the normalization of \( S \) in the splitting field of \( P \). For each finite set of \( P \)-s and \( x \)-s, we form the tensor product of the \( S_P \)-s and \( S_x \)-s over \( S \) and normalize, giving an \( S \)-algebra \( T \). We then localize \( T \) with respect to \( JT \), forming \( T' \). Let \( \mathcal{F}_1 \) be the set of such \( T' \). We note that each element of \( \mathcal{F}_1 \) is a PIR, and is flat as an \( S \)-algebra. Repeating this for each \( T \) in \( \mathcal{F}_1 \), and taking localizations of normalizations of all finite tensor products gives \( \mathcal{F}_2 \), etc. We let \( \mathcal{F} \) be the union of all the \( \mathcal{F}_i \). Then \( \{ S_i \mid i \in \mathcal{F} \} \) is a direct, filtering system of PIR's which are flat \( S \)-algebras. Let \( S_\infty \) be direct limit of
the $S_i$ and $J_i$ the direct limit of the $J(S_i), S_i^\infty, J_i^\infty$ defined as in paragraph 3.3. Then $S_i^\infty$ satisfies (I) and (II) of paragraph 3.3. Let $L_\infty$ be the direct limit of the quotient fields $L_i$ of $S_i$. Let $G = K_2(S_i^\infty, J_i^\infty).

Let $a$ be in $1 + J_\infty$, $b$ in $L_\infty^*$. Then $a = N(x)$ for some $x$ in $1 + J_\infty$. Suppose $a \neq 1$. Then

\[ \{x, 1-a\} = \{x, 1-N(x)\} = 0 \text{ in } G/(1-\sigma)G \]

by Proposition 3.8. If $a = 1$, then $x = z^\alpha/z$, so $\{x, b\} \equiv 0 \text{ mod } (1-\sigma)G$. Thus the map

\[ (1+J_\infty)^* \otimes L_\infty^* \to G/(1-\sigma)G \]

\[ a \otimes b \to \{x, b\} \text{ mod } (1-\sigma)G \]

defines a homomorphism $\Theta: K_2(S_\infty, J_\infty) \to G/(1-\sigma)G$. By Proposition 3.9, $\Theta$ is surjective; clearly $N \circ \Theta = id$. Thus $V(S_\infty) = 0$. Since $V(S) \to V(S_\infty)$ is injective for all $i \in \mathcal{P}$ by (1) and (2), this implies that $V(S) = 0$. Thus we have shown

**Theorem 4.2.** — Let $S$ be a semi-local PIR containing $k_0$. Let $\alpha$ be a unit in $S$, and $S^\alpha$ the extension ring $S[X]/X^l - \alpha$ if $l \neq p = \text{char}(k_0)$, $S[X]/X^p - X - \alpha$ if $l = p$. Let $J$ be the Jacobson radical of $S$, $J^\alpha = JS_\alpha$. Let $\sigma$ be a generator of $\text{Gal}(S^\alpha/S)$. Then

\[ K_2(S^\alpha, J^\alpha) \to K_2(S, J) \]

is exact.

4.2. Torsion in relative $K_2$ and $K_3^{\text{ind}}$. — Using Hilbert's theorem 90 we compute the torsion in $K_2(S, J)$ and in $K_3(E)^{\text{ind}}$, where $E$ is a field.

**Theorem 4.3.** — Let $S$ be a semi-local PIR containing a field $k$ which contains $\mu_n$ for $n$ prime to the characteristics $p$ of $k$. Let $J$ denote the radical of $S$. Then $K_2(S, J)$ is generated by symbols $\{f, \zeta_\alpha\}$, with $f$ in $1 + J$. $K_2(S, J)$ has no $p$-torsion if $p > 0$.

**Proof.** — Suppose $l$ is a prime dividing $n$. Suppose the theorem is true for $n = l$. Let $\eta$ be an $n$-torsion element in $K_2(S, J)$. Then $\eta^l$ is $n/l$ torsion, so by induction we may assume that $\eta^l = \{g, \zeta_\alpha^l\}$ for some $g$ in $1 + J$, so $\eta = \{g, \zeta_\alpha\}^{-1}$ is $l$-torsion, thus is of the form $\{h, \zeta_\alpha\}$. Then $\eta = \{g(h)^{nll}, \zeta_\alpha\}$, as desired.

Consider the generic Kummer extension $S(v)/S(u)$ with $v' = u$; here $S(v)$ is the semi-local ring of $J[v]$ in $S[u]$, and similarly define $S(u)$. Let $f^*: S \to S(v)$, $h^*: S \to S(u)$ be the inclusions. Let $\eta$ be in $K_2(S, J)$. Then

\[ N_{S(v)/S(u)}(f^*(\eta)) = h^*(\eta^l) = 1 \]

so we can write $f^*(\eta)$ as

\[ f^*(\eta) = \tau^\sigma/\tau \text{ for some } \tau \text{ in } K_2(S(v), JS(v)). \]

Let

\[ g: \mathcal{A}_{S}^1 = \text{Spec}(S[v]) \to \mathcal{A}_{S}^1 = \text{Spec}(S[u]) \]

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be the $l$-fold cover. Then $\partial(\tau^*/\tau) = \partial(f^*(\eta)) = 0$ in $K'_1(S[v], JS[v])$, so there is an element $z$ of $K'_1(S[u])$ with $g^*(z) = \partial(\tau)$. The image of $z$ in $K'_1((S/J)[u])$ is 1 at all points of $A^1_S - \{0\}$, since $g$ is étale away from 0. Thus we can add an element $z_0$ of the form

$$z_0 = (a \text{ on } \{u = 0\}), \quad a \in S^*$$

to $z$ so that $z + z_0$ lands in the subgroup $K'_1(S[u], JS[u])$ of $K'_1(S[u])$.

Take $\xi$ in $K_2(S(u), JS(u))$ with

$$\partial(\xi) = z + z_0.$$  

Then $\tau \cdot g^*(\xi)^{-1} = \tau'$ has $\partial(\tau')$ supported on $\{v = 0\}$, and

$$\eta = \tau'^* / \tau'.$$

Since $Z_0 := \{v = 0\}$ is smooth, $\partial(\tau) = (f^*(x) \text{ on } Z_0)$, for some $x$ in $1 + J$. But then $\tau'.\{f^*(x), v\}^{-1}$ has $\partial(\tau'.\{f^*(x), v\}^{-1}) = 0$, so

$$\tau' = \{f^*(x), v\}.f^*(\beta) \text{ for some } \beta \text{ in } K_2(S, J).$$

Thus

$$f^*(\eta) = \tau'^* / \tau'$$

$$= \{(f^*(x), v).\beta\}/\{(f^*(x), v).\beta\}$$

$$= \{f^*(x), \zeta_t\}$$

$$= f^*(\{x, \zeta_t\}),$$

hence $\eta = \{x, \zeta_t\}$, as desired.

If $l = p$, we use the generic Artin-Schreier extension $S(v)/S(u)$ where $v^p - v = u$. Since $S[v]/S[u]$ is étale, the above argument shows that there is no $p$-torsion in $K_2(S, J)$. This completes the proof. □

For a ring $R$, we let $K_3(R)^{dec}$ be the subgroup of $K_3(R)$ generated by products from $K_1(R)$; $K_3(R)^{ind}$ will denote the quotient group

$$K_3(R)^{ind} = K_3(R)/K_3(R)^{dec}.$$  

Let $E$ be a field, and let $E(t)$ be a purely transcendental extension of $E$; then the map

$$K_3(E)^{ind} \to K_3(E(t))^{ind}$$

is an isomorphism. Indeed, the map is clearly injective. We have the exact localization sequence

$$0 \to K_3(E[t]) \to K_3(E(t)) \to \bigoplus_{\delta \text{ prime}} K_2(E[t]/(\delta)) \to 0$$

and the exact sequence of Milnor $K$-theory [Bass-Tate]

$$0 \to K_3^M(E) \to K_3^M(E(t)) \to \bigoplus_{\delta \text{ prime}} K_2^M(E[t]/(\delta)) \to 0$$
compatible with the Quillen localization sequence. As $K_2(F) = K_2^M(F)$, and $K_3(F)^{dec}$ is the image of $K_3^M(F)$ for fields $F$, we see the map $K_3(E)^{ind} \to K_5(E(t))^{ind}$ is surjective. Similarly, if $R$ is a semi-local ring containing $E$, with quotient field $E(t)$, then $K_3(E)^{ind} \to K_3(R)^{ind}$ is an isomorphism.

**Corollary 4.4.** — Let $E$ be a field containing $\mu_n$, $(n, \text{char}(E)) = 1$. Then the $n$-torsion subgroup of $K_3(E)^{ind}$ is a quotient of $\mathbb{Z}/n$. If $E$ has characteristic $p > 0$, then $K_3(E)^{ind}$ has no $p$-torsion.

**Proof.** — Let $R$ be the semi-local ring of $\{0, 1\}$ on $A^1_E$, $J$ the Jacobson radical of $R$. We have the exact sequence

$$\to K_3(R) \to K_3(R) \to K_2(R, J) \to K_2(R) \to$$

which gives the exact sequence

$$\to K_3(R)^{ind} \to K_3(R)^{ind} \to K_2(R, J) \to K_2(R) \to.$$

Since

$$K_3(R)^{ind} = K_3(E)^{ind}, K_3(R)^{ind} = K_3(E(0))^{ind} \oplus K_3(E(1))^{ind}$$

we get the exact sequence

$$0 \to K_3(E)^{ind} \to K_2(R, J) \to K_2(R) \to.$$

From this and Corollary 4.3, it follows that $a(K_3(E)^{ind})$ is generated by symbols of the form $\{f, \zeta\}$, $f \in (1+J)^*/(1+J)^*$, such that the symbol $\{f, \zeta\} = 0$ as an element of $K_2(R)$. In particular, the tame symbol $T(\{f, \zeta\})$ is zero, hence the divisor of $f$ on $A^1_E$ is divisible by $n$.

Thus we can write $f$ as an $n$-th power:

$$f = g^n \text{ some } g \in (E \otimes E R)^*.$$

We normalize $g$ so that $g(0) = 1$. Let $\sigma$ be an element of $\text{Gal}(\bar{E}/E)$. Then

$$g^n = \lambda g$$

for some $\lambda$ in $\mu_n$; evaluating at $0$ shows that $\lambda = 1$. Thus $g$ is in $R^*$. The class of $f \mod ((1+J)^*)^n$ is then determined by the value $g(1) \in \mu_n$, proving the corollary. □

Now we can show

**Theorem 4.5.** — Let $E$ be a number field. The Chern class

$$c_{2,1} : K_3(E)^{ind} \otimes \mathbb{Z}_l \to H^l_\ell(E, \mathbb{Z}_l(2))$$

is an isomorphism, so the $l$-primary torsion in $K_3(E)^{ind}$ is isomorphic to $H^0_\ell(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$.

**Proof.** — We may assume that $E$ contains $\mu_l$. From [Q] and the vanishing of $K_2$ for finite fields, $K_3(E)$ is finitely generated. From the above, the $l$-torsion in $K_3(E)^{ind}$ is
cyclic, hence the $l$-primary torsion is also cyclic. By [B-T], $K_3^M(E)$ is a torsion group; by [Borel] the rank of $K_3(E)$ is $r_2$. Thus $K_3(E)^{ind/l}$ is a $\mathbb{Z}/l$ vector space of dimension between $r_2$ and $1+r_2$. In addition, the Chern class vanishes on the Milnor $K_3$ (this follows from the integral product formula for Chern classes).

Let $\text{symb}: H^1(E, \mu_l^{\otimes 2}) \to K_2(E)$ be the map

$$H^1(E, \mu_l^{\otimes 2}) \to (E^*/(E^*)^l) \otimes \mu_l \to K_2(E),$$

and let $H$ be the kernel of $\text{symb}$. Tate [T] has shown that $H$ is $(\mathbb{Z}/l)^{1+r_2}$. Soule [So] has shown that $c_i$ is surjective for $l > 2$; we give here a proof of surjectivity for all prime $l$: Let $R$ be the semi-local ring of $\{0, 1\}$ on $\mathbb{A}_E^1$. By ([So], Prop. 2) we have the commutative ladder

$$K_3(R; \mathbb{Z}/l) \xrightarrow{\delta} \bigoplus_1 K_2(E(x); \mathbb{Z}/l) \to 0$$

$$c_{2,1} \downarrow \quad \downarrow -c_{1,0}$$

$$H^1(R, \mu_l^{\otimes 2}) \xrightarrow{\delta} \bigoplus_1 H^0(E(x), \mu_l) \to 0;$$

where $\bigoplus_1$ means the sum over codimension one points of $\mathbb{A}_E^1-\{0, 1\}$, and the rows are respectively the localization sequence and Bloch-Ogus sequence for the open subset $\text{Spec}(R)$ of $\mathbb{A}_E^1$. As $c_{1,0}$ induces the isomorphism $(E(x))^* \to H^0(E(x), \mu_l)$, the map

$$K_3(R; \mathbb{Z}/l)^{\ast} K_3(E, \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})/\pi^* H^1(E, \mu_l^{\otimes 2})$$

is surjective. We have the commutative square

$$\begin{array}{ccc}
K_3(R; \mathbb{Z}/l) & \xrightarrow{\delta_K} & K_3(E; \mathbb{Z}/l) \\
c_{2,1} \downarrow & & \downarrow c_{2,1} \\
H^1(R, \mu_l^{\otimes 2}) & \xrightarrow{\delta_H} & H^1(E, \mu_l^{\otimes 2})
\end{array}$$

where $\delta_K$ is the composition

$$K_3(R; \mathbb{Z}/l) \longrightarrow K_3(R/J; \mathbb{Z}/l) = K_3(E(0); \mathbb{Z}/l) \oplus K_3(E(1); \mathbb{Z}/l) \longrightarrow K_3(E; \mathbb{Z}/l)$$

reduce mod $J$

and similarly for $\delta_H$. Since $H^1(R, \mu_l^{\otimes 2}) = R^* \otimes \mu_l$ and $H^1(E, \mu_l^{\otimes 2}) = E^* \otimes \mu_l$, $\delta_H$ is surjective.

Since $\delta_K$ kills $\pi^* K_3(E, \mathbb{Z}/l)$ and $\delta_H$ kills $\pi^* H^1(E, \mu_l^{\otimes 2})$, it follows that $c_{2,1}: K_3(E; \mathbb{Z}/l) \to H^1(E, \mu_l^{\otimes 2})$ is surjective. This incidently shows that $c_{2,1}: K_3(R; \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})$ is also surjective. The surjectivity of $c_{2,1}: K_3(E, \mathbb{Z}/l) \to H^1(E, \mu_l^{\otimes 2})$, together with the computation of $K_3(E)^{ind/l}$ and $H$ implies that the Chern class map

$$c_{2,1}: K_3(E; \mathbb{Z}/l)^{ind} \to H^1(E, \mu_l^{\otimes 2})$$

is an isomorphism. The commutative square ($\star \star$), together with the surjectivity of $c_{2,1}: K_3(R; \mathbb{Z}/l) \to H^1(R, \mu_l^{\otimes 2})$ and $\delta_H$ then implies that $\delta_K$ is surjective ($\delta_K$ is obviously
surjective on the Milnor $K_3$, and hence $K_2(R, J; \mathbb{Z}/l) \to K_2(R, \mathbb{Z}/l)$ is injective, hence $K_2(R, J)/l \to K_2(R)/l$ is injective.

Let $L$ be the quotient field of $R$, $i: \text{Spec}(L) \to \text{Spec}(R)$ the inclusion. Let $i$ be the functor “extension by zero”, from sheaves on $L$ to sheaves of $R$ (for the étale topology). The construction of Chern classes for relative $K$-theory in paragraph 1.12 gives the Chern classes

$$c_{p,q}: K_{2p-2q}(R, J) \to H^q(\text{Spec}(R), i_!(\mu_n^\otimes q)),$$

together with the commutative ladder

$$\begin{array}{cccccccc}
K_3(E)/l^n & \to & K_2(R, J)/l^n & \to & K_2(R)/l^n & \to & (K_2(E)/l^n)^2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
H^1(E, \mu_n^{\otimes 2}) & \to & H^2(R, i_!(\mu_n^{\otimes 2})) & \to & H^2(R, \mu_n^{\otimes 2}) & \to & (H^2(E, \mu_n^{\otimes 2}))^2 & \to 0,
\end{array}$$

the horizontal line coming from the relativization sequence, and the vertical arrows Chern classes. For all $n$, the Chern classes for $K_2(R)/l^n$ and $K_2(E)/l^n$ are isomorphisms. The surjectivity of $\delta_H$ shows that $H^2(R, i_!(\mu_n^{\otimes 2})) \to H^1(R, \mu_n^{\otimes 2})$ is injective, hence the second vertical arrow is an isomorphism for $n=1$.

We define the map $\text{symb}: H^1(R, \mu_n^{\otimes 2}) \to K_2(R, J)$ by

$$f \otimes \zeta \mapsto \{f, \zeta\}: \quad f \in \mathbb{R}^\times,$$

where we identify $H^1(R, \mu_n^{\otimes 2})$ with $\mathbb{R}^\times \otimes \mu_n$ via the Chern class $c_{1,1}$. From the product formula for Chern classes, we have

$$c_{2,2}(a, b) = -c_{1,1}(a) \cup c_{1,1}(b)$$

for $a$ in $K_1(R, J) = (1+J)^\times$, $b$ in $K_1(R) = \mathbb{R}^\times$. This gives the commutative ladder

$$\begin{array}{cccccccc}
K_2(R, J) & \to & K_2(R, J)/l^n & \to & K_2(R, J)/l^{n+1} & \to & K_2(R, J)/l & \to 0 \\
\uparrow \text{symb} & & \uparrow & & \uparrow & & \uparrow & \\
H^1(R, i_!(\mu_n^{\otimes 2})) & \to & H^2(R, i_!(\mu_n^{\otimes 2})) & \to & H^2(R, i_!(\mu_n^{\otimes 2})) & \to & H^2(R, i_!(\mu_n^{\otimes 2})) & \to 0,
\end{array}$$

with the second row exact, and the first row exact, except possibly at $K_2(R, J)/l^n$. This and induction shows that the Chern class for $K_2(R, J)/l^n$ is an isomorphism for all $n$.

From the localization sequence on $\mathbb{A}^1_k$, together with a knowledge of $K_2(E)$, and $K_1$ of number fields, it follows that $K_2(R)$ has no divisible subgroups. As $K_3(E)$ is finitely generated, $K_2(R, J)$ has no divisible subgroups as well. Thus for $n$ sufficiently large, the $l$-primary torsion in $K_3(E)^{\text{ind}}$ injects into $K_2(R, J)/l^n$. From the ladder ($\star$), it follows that the Chern class $c_{2,1}: K_3(E)^{\text{ind}} \to H^1(E, \mu_n^{\otimes 2})$ is injective on the $l$-primary torsion for large $n$. From this, the surjectivity of $c_{2,1}$, and the computation of the ranks of $K_3(E)^{\text{ind}}$ and $H^1(E, \mathbb{Z}_l(2))$ (the latter due to Tate [T]) it follows that the Chern class gives an isomorphism on the limits

$$c_{2,1}: K_3(E)^{\text{ind}} \otimes \mathbb{Z}_l \to H^1(E, \mathbb{Z}_l(2))$$

proving the theorem. □
Using this result, we can refine the statement of Corollary 4.4.

**Corollary 4.6.** — Let $E$ be a field, $l$ a prime with $(l, \text{char}(E)) = 1$. Then the $l$-primary torsion in $K_3(E)^{\text{ind}}$ is isomorphic to $H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2))$. If $F$ is an extension field of $E$, then the map

$$K_3(E)^{\text{ind}} \to K_3(F)^{\text{ind}}$$

is injective.

**Proof.** — The second statement follows from the first. If $E$ is a finite field, the computation of the torsion is due to Quillen [Q2]; for $E$ a number field this is part of Theorem 4.5. In particular, if $E \to F$ is a map of fields which are finite over the prime field, the induced map

$$K_3(E)^{\text{ind}} \to K_3(F)^{\text{ind}}$$

is injective.

In the general case, since $K$-theory commutes with direct limits, we may assume that $E$ is finitely generated over the prime field $F_0$.

Let $k$ be the field of constants in $E$. Let $\eta$ be an $l$-primary torsion element of $K_3(k)^{\text{ind}}$. Let $g^*: k \to E$ denote the inclusion, and suppose that $g^*(\eta) = 0$ in $K_3(E)^{\text{ind}}$. Then there is a regular $k$-algebra $A$ of finite type, $A$ a domain with quotient field $E_0 \subset E$, such that $h^*(\eta) = 0$ in $K_3(A)^{\text{ind}}$. Here $h^*: k \to A$ is the inclusion. Taking an $F$-valued point $j^*: A \to F$ of $A$, with $F$ a finite extension of $k$, we see that $j^* h^*(\eta) = 0$, contradicting the injectivity of $K_3(k)^{\text{ind}} \to K_3(F)^{\text{ind}}$. Thus there is a natural inclusion

$$\Phi: H^0(E, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to K_3(E)^{\text{ind}} \{l\}.$$

To show that $\Phi$ is surjective, we may assume that $E$ contains $\mu_l$. Then $K_3(E)^{\text{ind}}$ is cyclic by Corollary 4.4, hence $\Phi$ is surjective.

4.3. **Co-torsion in $K_3^{\text{ind}}$.** — We now compute $K_3^{\text{ind}}/\mu$ for fields. Let $E$ be a field, $R$ the semi-local ring of $\{0, 1\}$ in $A_b^1$, $J$ the Jacobson radical of $R$, $R$ the quotient $R/J$. For an $R$-scheme $T$ let $T = T \times_{\text{Spec}(R)} \text{Spec}(R)$. Let $R(T)$ denote the semi-local ring of $T$ in $T$. We consider a chain of $R$-schemes

$$X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \text{Spec}(R)$$

such that $X_{i+1}$ is the Brauer-Severi scheme over $\text{Spec}(R(X_i))$ associated to a central simple algebra $D_{i+1}$ over $R(X_i)$, with $D_{i+1}$ split.

**Lemma 4.7.** — For each $i = 1, \ldots, n$ there is field $E_i \supset E$, a smooth $E_i$-scheme $Y_i$, with

$$Y_i \cong \mathbb{P}^n_{E_i} \times_{E_i} \text{Spec}(E_i[t](0, 1)).$$
and finite maps

\[ X_i \leftarrow Y_i; R(X_{i-1}) \rightarrow E_i[t][0,1] \text{ such that the diagram} \]

\[
\begin{array}{ccc}
X_i & \leftarrow & Y_i \\
\downarrow & & \downarrow \pi_i \\
X_{i-1} & \leftarrow & \text{Spec}(E_i[t][0,1]) \mathbb{P}^{n_i} \\
& & \downarrow \\
& & \text{Spec}(E_i)
\end{array}
\]

commutes.

Proof. — By Tsen’s theorem, \( \mathcal{O}_1 \otimes_E E_0 \) is split for some finite extension \( E_0 \) of \( E \). Let \( R \rightarrow E_0[t][0,1] \) the natural inclusion and \( Y_i \) the fiber product \( X_i \times_E E_0 \). In general, suppose we have the diagram (\( \star \)). Let \( F_i \) be the function field \( E_i(\mathbb{P}^{n_i}) \). Then the semi-local ring \( R(Y_i) \) of \( Y_i \) in \( Y_i \) is \( F_i[t][0,1] \) and \( R(Y_i) \) is finite over \( R(X_i) \). Take the fiber product \( X_{i+1}' \)

\[ X_{i+1}' = X_{i+1} \times_{R(X_i)} R(Y_i) \rightarrow \text{Spec}(R(Y_i)) \]

Then \( X_{i+1}' \) is split by a finite extension \( E_{i+1} \) of \( F_i \). Letting \( Y_{i+1} \) be the fiber product

\[ Y_{i+1} = X_{i+1}' \times_{F_i} E_{i+1} \rightarrow E_{i+1}[t][0,1] \]

continues the induction. \( \square \)

**Lemma 4.8.** — *The map*

\[ K_2(R(X_i), J(X_i)) \rightarrow K_2(R(X_{i+1}), J(X_{i+1})) \]

*is injective.*

Proof. — Let \( X = X_0, X' = X_{i+1}, Y = Y_0, F = F_i \). We have the commutative ladder with exact rows

\[
\begin{array}{ccc}
\rightarrow & K_3(R(Y)) & K_3(\overline{R}(Y)) \\
& \uparrow & \uparrow \\
& K_3(R(X)) & K_3(\overline{R}(X)) \\
& \uparrow & \uparrow \\
& K_3(F(0))^{\text{ind}} \oplus K_3(F(1))^{\text{ind}} \cong K_3(\overline{R})^{\text{ind}} & K_3(\overline{R}(X))^{\text{ind}}
\end{array}
\]

Since each \( \mathcal{O}_j \) is split, \( \overline{R}(X) \) is a pure transcendental extension of \( \overline{R} \), hence the map

\[ K_3(E(0))^{\text{ind}} \oplus K_3(E(1))^{\text{ind}} \cong K_3(\overline{R})^{\text{ind}} \rightarrow K_3(\overline{R}(X))^{\text{ind}} \]

is an isomorphism. Similarly, the map

\[ K_3(F(0))^{\text{ind}} \oplus K_3(F(1))^{\text{ind}} \rightarrow K_3(\overline{R}(Y))^{\text{ind}} \]

is an isomorphism. Let \( X_0 \) and \( X_1 \) denote the two irreducible components of \( X \), and similarly define \( Y_0 \) and \( Y_1 \). \( Y_0 \) and \( Y_1 \) are both projective spaces over \( F \); and \( X_0 \) and \( X_1 \) are projective spaces over a subfield \( k \) of \( F \), so we can identify \( X_0 \) and \( X_1 \), \( Y_0 \) and
Y₁. Then the image of
\[ K₃(R(X)) \to K₃(R(X)) = K₃(k(X₀)) \oplus K₃(k(X₁)) \]
and
\[ K₃(F(Y)) \to K₃(F(Y)) = K₃(F(Y₀)) \oplus K₃(F(Y₁)) \]
contain the respective diagonals. By taking the difference maps
\[ K₃(k(X₀)) \oplus K₃(k(X₁)) \to K₃(k(X₀)) \]
and
\[ K₃(F(Y₀)) \oplus K₃(F(Y₁)) \to K₃(F(Y₀)) \]
and noting that the maps
\[ K₃(R(X)) \to K₃(R(X)) \]
are surjective, we can rewrite the ladder above as
\[ \to K₃(R(Y))^{\text{ind}} \to K₃(F(Y₀))^{\text{ind}} \to K₂(R(Y), J(Y)) \to K₂(R(Y)) \]
\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]
\[ \to K₃(R(X))^{\text{ind}} \to K₂(k(X₀))^{\text{ind}} \to K₂(R(X), J(X)) \to K₂(R(X)) \]

Let L(X), L(Y) be the quotient fields of R(X), R(Y). Then by Proposition 2.3, the maps
\[ K₃(R(X))^{\text{ind}} \to K₃(L(X))^{\text{ind}}, \quad K₃(R(Y))^{\text{ind}} \to K₃(L(Y))^{\text{ind}} \]
are injective. Thus from Corollary 4.6, the vertical arrows in left-hand the commutative square of (⋆) are injective. On the other hand, since R(Y) = F[t_{0}, t_{1}], the image of K₃(R(Y)) in K₃(F(Y₀)) is exactly K₃(F(Y₀))^{\text{dec}}, i.e. the map K₃(R(Y))^{\text{ind}} \to K₃(F(Y₀))^{\text{ind}} is the zero map. Thus K₃(R(X))^{\text{ind}} \to K₃(k(X₀))^{\text{ind}} is the zero map as well, and we have the exact sequence
\[ 0 \to K₃(k(X₀))^{\text{ind}} \to K₂(R(X), J(X)) \to K₂(R(X)) \]

By a similar argument, we have the exact sequence
\[ 0 \to K₃(k(X₀))^{\text{ind}} \to K₂(R(X'), J(X')) \to K₂(R(X')) \]

By Suslin (Theorem 3.6 [S]) the map K₂(L(X)) → K₂(L(X')) is injective. This implies that K₂(R(X)) → K₂(R(X')) is injective; the map K₃(k(X₀))^{\text{ind}} → K₃(k(X₀))^{\text{ind}} is also injective by Corollary 4.6, hence
\[ K₂(R(X), J(X)) \to K₂(R(X'), J(X')) \]
is injective, completing the proof. □
THEOREM 4.9. — Let $X = X_0$, $X' = X_{i+1}$, $\pi: X' \to \text{Spec}(R(X))$ the projection. The map

$$\pi^*: K_2(R(X), J(X)) \to E_2^0(-2)(X', \mathcal{R}') \subset K_2(R(X'), J(X'))$$

is an isomorphism.

Proof. — We recall from paragraph 1.6 that $E_2^0(-2)$ is the kernel of

$$K_2(R(X'), J(X')) \to K_1(X^{1/2}, X'^{1/2}) \subset K_1(X)^{1/2}.$$ 

The injectivity of $\pi^*$ follows from the previous lemma. We have the commutative diagram with exact rows:

$$
\begin{array}{cccccc}
K_2(\mathcal{M}_{X'}^{1/2}, \mathcal{R}) & \to & K_2(\mathcal{M}_{X'}^{1/2}, \mathcal{R}) & \to & K_1(X'^{1/2}, \mathcal{R}^{1/2}) & \to & K_1(\mathcal{M}_{X'}^{1/2}, \mathcal{R}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K_3(R(X')) & \to & K_3(\mathcal{R}(X')) & \to & K_2(R(X'), J(X')) & \to & K_2(R(X')) \\
\pi^* & \uparrow & \pi^* & \uparrow & \pi^* & \uparrow & \pi^* \\
K_3(R(\mathcal{R}(X))) & \to & K_2(\mathcal{R}(X), J(X)) & \to & K_2(R(\mathcal{R}(X))) & \to & K_2(\mathcal{R}(X)) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

The columns are all complexes. The second and fifth columns are exact since $\mathcal{R}'$ is a projective space over $\mathcal{R}(X)$; the fourth column is exact since Suslin has shown that

$$K_2(\mathcal{R}(X)) \to H^0(X_{k(\mathcal{R}(X)), X_2})$$

is an isomorphism, and $R(X)$ is semi-local. In addition, the image of the tame symbol $\alpha$ is the same as the image of $\alpha$ restricted to $K_2^0(\mathcal{R}(X'))$. Since we can lift $K_2^0(\mathcal{R}(X'))$ to $K_2(\mathcal{R}(X'))$, the surjectivity of $\pi^*: K_2(R(X), J(X)) \to E_2^0(-2)(X', \mathcal{R}')$ follow from a diagram chase. □

Let $k_2(S, \mathcal{S})$ denote $K_2(S, \mathcal{S})/l$, where $l$ is a prime different from $\text{char}(E)$, $S$ an $E$-scheme with closed subscheme $\mathcal{S}$.

COROLLARY 4.10. — Suppose that $E$ contains $\mu_n$ and that the division algebra $\mathcal{D} = \mathcal{D}_{i+1}$ is the crossed product algebra coming from the symbol $(a, b)_n$, $a \in (1 + J(X))^*$, $b \in L(X)^*$. The kernel of

$$\pi^*: k_2(R(X), J(X)) \to k_2(R(X'), J(X'))$$

is the subgroup generated by $\{a, b\}$. If $\mathcal{D}$ is split, then $\pi^*$ is injective.

Proof. — This follows from Theorem 5.9 and a diagram chase as in [M-S]. □

THEOREM 4.11. — The Chern class map

$$c_{2, 2}: K_2(R, J)/l^n \to H^2(R(X), i_1(\mu_n)^{\otimes 2})$$

is an isomorphism.

Proof. — We prove the stronger result that

$$c_{2, 2}: K_2(R(X), J(X))/l^n \to H^2(R(X), i_1(\mu_n)^{\otimes 2})$$
is an isomorphism. An argument as in the case of \(K_2\) of fields reduces to the case \(n=1\); we may also assume that \(E\) contains \(\mu_n\). In this case, for \(X=X_0\), \(H^2(R(X), i_! (\mu_n)_{\otimes 2})\) just the kernel of the restriction map

\[
H^2(R(X), (\mu_n)_{\otimes 2}) \rightarrow H^2(R(X), (\mu_2)_{\otimes 2})
\]

so \(\{a, b\} \in K_2(R(X), J(X))\) goes to zero under \(c_{2, 2}\) if and only if the crossed product algebra \((a, b)_l\) is split. By Corollary 4.10, \(\{a, b\} = 0\) in \(k_2(R(X), J(X))\). We now prove that

\[
c_{2, 2}: K_2(R(X), J(X))/l \rightarrow H^2(R(X), i_! (\mu_n)_{\otimes 2})
\]

is an isomorphism by induction on the length of an element \(\eta\) in the kernel

\[
\eta = \sum \{a, b\}.
\]

This is done by going up to the Brauer-Severi scheme associated to \(\{a, b\}_l\) and using the corollary above. \(\Box\)

**Theorem 4.12.** — The Chern class

\[
c_{2, 1}: K_3(E, \mathbb{Z} / p) \rightarrow K_3(E, \mathbb{Z} / 2)
\]

is an isomorphism.

**Proof.** — We reduce as in the proof of Theorem 4.5 to the case \(n=1\), and may assume that \(E\) contains \(\mu_n\). We have the commutative ladder

\[
\begin{array}{cccccc}
K_3(R, J; \mathbb{Z} / l) & \rightarrow & K_3(R; \mathbb{Z} / l) & \rightarrow & K_3(E; \mathbb{Z} / l) & \rightarrow & K_3(R, J)/l \\
\downarrow \varepsilon & & \downarrow \gamma & & \downarrow \delta & & \downarrow \iota \\
H^1(R, i_! (\mu_{n^2})) & \rightarrow & H^1(R, \mu_{n^2}) & \rightarrow & H^1(E, \mu_{n^2}) & \rightarrow & H^2(R, i_! (\mu_{n^2})) \rightarrow H^2(R, \mu_{n^2})
\end{array}
\]

We have already shown that \(\delta\) is surjective. Since \(\beta\) is surjective, \(\alpha\) is also surjective. As in the proof of Theorem 4.5, \(\alpha\) and \(\beta\) factor through \(K_3(R; \mathbb{Z} / l)/K_3(E; \mathbb{Z} / l)\) and \(H^1(R, \mu_{n^2})/H^1(E, \mu_{n^2})\) respectively. We claim that \(\varepsilon\) maps \(K_3(R, J; \mathbb{Z} / l)\) onto \(\mathbb{1}(H^1(R, i_! (\mu_{n^2})))\).

Indeed, we have the commutative triangle

\[
\begin{array}{ccc}
K_3(R; \mathbb{Z} / l) & \rightarrow & iK_2(R) \\
\gamma \downarrow & & \uparrow \text{symb}_R \\
H^1(R, \mu_l) \otimes \mu_l & = & R \otimes \mu_l
\end{array}
\]

The image \(\mathbb{1}(H^1(R, i_! (\mu_{n^2})))\) is \((1+J)^x / l; \) let \(f\) be in \((1+J)^x\), and let \(\eta\) be a lifting of the element \(\{f, \zeta_d\}\) of \(iK_2(R, J)\) to \(K_3(R, J; \mathbb{Z} / l)\). Then

\[
\text{symb}_R \circ \gamma \circ \kappa(\eta) = \text{symb}_R (f \otimes \zeta_d).
\]

On the other hand, the kernel of \(\text{symb}_R\) injects into \(H^1(E, \mu_{n^2})^2\), hence \(\gamma \circ \kappa(K_3(R, J; \mathbb{Z} / l))\) maps isomorphically onto \(iK_2(R)\) via \(\text{symb}_R\). Thus
\[ \gamma \circ \kappa(\eta) = (f \otimes \zeta_i), \] proving our claim. Since
\[ \overline{\gamma}: K_3(R; Z/l)/K_3(E; Z/l) \to H^1(R, \mu_3^\otimes)/H^1(E, \mu_3^\otimes) \]
is an isomorphism, \( \delta \) is an isomorphism, as claimed. \( \square \)

Let \( E/F \) be a finite Galois extension of fields which are finitely generated over the prime field. Since \( H^1_{et}(E, \mathbb{Z}/l) = 0 \), the Hochschild-Serre spectral sequence shows that
\[ H^1(F, Z/I(2)) = H^1(E, Z/I(2))^{Gal(E/F)}. \]

In addition, using the Bloch-Ogus sequence relating \( H^1_{et}(E(t), \mu_r^\otimes) \) and \( H^1_{et}(A_{et}^1, \mu_r^\otimes) \), we find that \( H^1(\cdot, Z/I(2)) \) is invariant under pure transcendental extensions.

**Theorem 4.13.** — Let \( E \) be a field. Then the map
\[ c_{2,1}: \lim_{\to} K_3(E)^{ind}/l^m \to H^1_2(E, Z/I(2)) \]
is an isomorphism, so the kernel of \( c_{2,1}: K_3(E)^{ind} \to H^1_2(E, Z/I(2)) \) is the maximal \( l \)-divisible subgroup of \( K_3(E)^{ind} \). If \( E_0 \) is the field of constants in \( E \), then
\[ K_3(E_0)^{ind}/l^m \to K_3(E)^{ind}/l^m \]
is an isomorphism. If \( E \to F \) is an algebraic Galois extension with Group \( G \), such that every finite quotient of \( G \) has order prime to the characteristic, then
\[ K_3(E)^{ind} = (K_3(F)^{ind})^G. \]

**Proof.** — Suslin has shown that
\[ \ker(H^1(E_0, (\mu_r)^\otimes) \to \mu_r K_2(E_0)) \to \ker(H^1(E, (\mu_r)^\otimes) \to \mu_r K_2(E)) \]
is an isomorphism, and that these kernels are the image under \( c_{2, 1} \) of \( K_3(E_0)^{ind} \) and \( K_3(E)^{ind} \) respectively. In addition, he has shown that the map
\[ H^1(E_0, Z/I(2)) \to H^1(E, Z/I(2)) \]
is an isomorphism. The first two results follow from this, Theorem 4.5 and Theorem 4.12. To prove the third, we may assume that \( F \) is finite over \( E \), of degree say \( d \). Since \( K_3(E)^{ind} \to K_3(F)^{ind} \) is injective we have the inclusions
\[ d \cdot K_3(E)^{ind} \subseteq d \cdot (K_3(F)^{ind})^G \subseteq K_3(E)^{ind} \subseteq (K_3(F)^{ind})^G \]
Thus we need only show that
\[ K_3(E)^{ind}/l = (K_3(F)^{ind}/l)^G. \]
for all \( l \mid d \). The result now follows from the isomorphism
\[ H^1(E, Z/I(2)) \to H^1(F, Z/I(2))^G \]
Let $F$ be a field. We recall the definition of Bloch's group $B(F)$. Let $D(F)$ be the free abelian group on $F^* - \{1\}$; $P(F)$ the quotient of $D(F)$ by the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})].$$

The map $D(F) \to F^* \otimes F^*/\langle a \otimes b + b \otimes a \rangle$ gotten by sending $[x]$ to $x \otimes (1-x)$ descends to $P(F)$. $B(F)$ is defined to be the kernel of

$$T(F) = F^* \otimes F^*/\langle a \otimes b + b \otimes a \rangle.$$

**Corollary 4.14.** Let $E$ be a field containing an algebraically closed field. Then Bloch's group $B(E)$ is uniquely $l$-divisible for $l$ prime to the characteristic.

**Proof.** We may assume that $E$ is finitely generated over the algebraic closure of the prime field. Suslin has shown that $B(E)$ is just $K_3(E)^{ind}$ modulo the image of $Q_l/Z_l(2)$. By Corollary 4.6 $B(E)$ is torsion free. Since $H^1(E, Z_l(2)) = 0$ by Suslin's computation (Cor. 2.7 [S]), it follows from the previous theorem that $B(E)$ is $l$-divisible.

5. Relative $K_2$ and $l$-adic cohomology

We now proceed to prove an analogue of the theorem of Merkurjev and Suslin for relative $K_2$ of semi-local PIR's. Since the receptor cohomology group for the relevant Galois symbols are the étale cohomology groups of $\text{Spec}(R)$, $R$ a semi-local PIR, we need a good cohomology theory with $Z_l(i)$ coefficients. Uwe Jannsen [J] has constructed such a theory by viewing $Z_l(i)$ as an object in the category of inverse systems of étale sheaves. A similar theory has been constructed by Dwyer and Friedlander [D-F], using étale homotopy theory.

5.1. Continuous cohomology. Let $\mathscr{S}_{et}(X)$ denote the category of sheaves in the small étale site over $X$, $\mathscr{A}$ the category of abelian groups. If $\mathscr{A}$ is an abelian category, let $\mathscr{A}^N$ denote the category of inverse systems in $\mathscr{A}$ indexed by the natural numbers. Jannsen defines the continuous cohomology on $X$ of the limit $F$ of an inverse system $(F_n) \in \mathscr{S}_{et}(X)^N$, $H^*_{cont}(X, \lim(F_n))$, to be the derived functors of the composition

$$(F_n) \to (H^0_{et}(X, F_n)) \to \lim(H^0_{et}(X, F_n))$$

from $\mathscr{S}_{et}(X)^N$ to $\mathscr{A}$. The functor $H^*_{cont}$ satisfies many of the properties of continuous Galois cohomology; in particular if $X$ is the spectrum of a field, and the $(F_n)$ satisfies the Mittag-Leffler condition (e.g. all $F_n$ sheaves of finite groups) then $H^*_{cont}(X, F)$ is the usual continuous Galois cohomology. There is a Hochschild-Serre spectral sequence if
X is over a field, and short exact sequences
\[ 0 \rightarrow \limlim (H^p(X, F_n)) \rightarrow H^p_{\text{cont}}(X, \limlim (F_n)) \rightarrow \limlim (H^p_{\text{cont}}(X, F_n)) \rightarrow 0. \]

In particular, if X is of finite type over \( Z[1/l] \), the cohomology groups \( H^p(X, F_n) \) are finite if the \( F_n \) are sheaves of finite groups, hence
\[ H^p_{\text{cont}}(X, \limlim (F_n)) = \limlim H^p(X, F_n). \]

Let X be a scheme essentially of finite type over a field \( k \). Let \( \text{Fin}(X/) \) be the category of pairs \((Y, f)\), where \( Y \) is a scheme of finite type over \( Z[1/l] \) and \( f: X \rightarrow Y \) is a morphism. A morphism from \((Y, f)\) to \((Z, g)\) is a commutative diagram
\[
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow & \searrow \\
Y & \rightarrow & \\
\end{array}
\]

Then X is the inverse limit
\[ \limlim Y \]
\[
\text{Fin}(X/)
\]
hence \( K_*(X) \) is the direct limit
\[ K_*(X) = \limlim K_*(Y). \]
\[
\text{Fin}(X/^{op})
\]

We have the Chern classes ([Gillet] or [So]):
\[ c_{p, q} : K_{2p-q}(Y) \rightarrow \limlim H^q_{\text{et}}(Y, (\mathbb{Z}/l)^{\oplus p}) = H^q_{\text{cont}}(Y, Z_l(p)). \]

This defines the Chern classes \( c_{p, q} : K_{2p-q}(X) \rightarrow H^q_{\text{cont}}(X, Z_l(p)) \) via the composition
\[
K_{2p-q}(X) \xrightarrow{\sim} \limlim K_{2p-q}(Y) \xrightarrow{c_{p, q}} \limlim H^q_{\text{cont}}(Y, Z_l(p)) \rightarrow H^q_{\text{cont}}(X, Z_l(p)).
\]

If \( X = \text{Spec}(R) \), where \( R \) is a semi-local PIR with Jacobson radical \( J \), and \( i : x \rightarrow X \) the inclusion of the generic point, we similarly get Chern classes
\[ c_{p, q} : K_{2p-q}(R, J) \rightarrow H^q_{\text{cont}}(X, i_!(Z_l(p))), \]
compatible with the relativization sequences for K-theory and continuous cohomology. This is done using the relative Chern classes of paragraph 1.12 and a limit argument as above.
5.2. Merkurjev-Suslin for relative $K_2$.

**Theorem 5.1.** — Let $R$ be a semi-local PIR containing a field $k_0$, $J$ the Jacobson radical. Let $l$ be a prime distinct from $\text{char}(k_0)$. Then the Chern class

$$c_{2,2} : K_2(R, J)/l^n \to H^2(R, i_1(\mu_l)^{\otimes 2})$$

is an isomorphism. The map

$$c_{2,2} : K_2(R, J) \to H^2_{\text{cont}}(R, i_1(\mathbb{Z}_l(2)))$$

is injective mod the maximal $l$-divisible subgroup of $K_2(R, J)$. If all the residue fields of $R$ are finite extensions of the prime field, then $c_{2,2}$ induces a natural isomorphism

$$K_2(R, J) \{l\} \to H^2(R, i_1(\mathbb{Z}_l(2)) \{l\}).$$

**Proof.** — To prove the first statement, we reduce to the case $n=1$, and may assume that $R$ contains $\mu_l$. Let $\bar{R} = R/J$. By Theorem 4.13, the map

$$c_{2,1} : K_3(\bar{R}; \mathbb{Z}/l)^{\text{ind}} \to H^1(\bar{R}, \mathbb{Z}/l(2))$$

is an isomorphism. Arguing as in Theorem 4.5, the map

$$c_{2,1} : K_3(R; \mathbb{Z}/l)^{\text{ind}} \to H^1(R, \mathbb{Z}/l(2))$$

is surjective. Since $R$ contains $\mu_l$, the map

$$H^1(R, \mathbb{Z}/l(2)) \to H^1(\bar{R}, \mathbb{Z}/l(2))$$

is surjective, hence

$$K_3(R; \mathbb{Z}/l)^{\text{ind}} \to K_3(\bar{R}; \mathbb{Z}/l)^{\text{ind}}$$

is surjective. Thus

$$K_2(R, J)/l \to K_2(R)$$

and

$$H^2(R, i_1(\mu_l)^{\otimes 2}) \to H^2(R, (\mu_l)^{\otimes 2})$$

are injective. Since $\iota K_1(R, J) = 0$ and $\iota K_1(R) \to \iota K_1(\bar{R})$ is injective, the relativization sequence

$$\to K_2(R, J; \mathbb{Z}/l) \to K_2(R; \mathbb{Z}/l) \to K_2(\bar{R}; \mathbb{Z}/l) \to$$

yields the commutative ladder

$$0 \to K_2(R, J)/l \to K_2(R)/l \to K_2(\bar{R})/l \to$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$0 \to H^2(R, i_1(\mu_l)^{\otimes 2}) \to H^2(R, (\mu_l)^{\otimes 2}) \to H^2(\bar{R}, (\mu_l)^{\otimes 2})$$

Thus

$$c_{2,2} : K_2(R, J)/l \to H^2(R, i_1(\mu_l)^{\otimes 2})$$
is an isomorphism as claimed. Passing to the limit, we see that
\[ c_{2,2} : \lim K_2(R, J)/\mathfrak{p} \to \lim H^2(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \]
is an isomorphism. We have the commutative diagram
\[
\begin{array}{ccc}
K_2(R, J) & \to & H^2_{\text{cont}}(R, i_1(\mathbb{Z}_l(2))) \\
\downarrow & & \downarrow \\
\lim K_2(R, J)/\mathfrak{p} & \to & \lim H^2(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \\
\end{array}
\]
proving the second statement. To prove the third, we note that our assumptions on R, together with Quillen’s finiteness theorem [Q3] for the K-theory of number rings, implies that the maximal \( l \)-divisible subgroup of \( K_2(R, J) \) is just the prime to \( l \) torsion. We may assume that \( k_0 \) contains \( \mu_\mathfrak{p} \). Then the sequence
\[ H^1(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \to H^2_{\text{cont}}(R, i_1(\mathbb{Z}_l(2))) \to H^2_{\text{cont}}(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \]
together with the symbol map
\[ \text{symb} : H^1(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \to iK_2(R, J) \]
\[ f \otimes \zeta \to \{f, \zeta\} \]
shows that \( c_{2,2} \) maps \( iK_2(R, J) \) onto \( iH^2_{\text{cont}}(R, i_1(\mathbb{Z}_l(2))) \), completing the proof. \( \square \)

We have the Chern classes
\[ c_{2,1} : K_3(R, J; \mathbb{Z}/\mathfrak{p}) \to H^1(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \]
and
\[ c_{2,2} : K_2(R, J) \to H^2(R, i_1(\mu_\mathfrak{p})^{\otimes 2}). \]
As in the absolute case, these are compatible with the Bockstein homomorphisms, i.e., we have the commutative square
\[
\begin{array}{ccc}
K_3(R, J; \mathbb{Z}/\mathfrak{p}) & \to & \rho K_2(R, J) \subset K_2(R, J) \\
\downarrow & & \downarrow \\
H^1(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) & \to & H^2(R, i_1(\mu_\mathfrak{p})^{\otimes 2}) \\
\end{array}
\]
In addition, if \( R \) contains \( \mu_\mathfrak{p} \), the map
\[ \overline{c}_{2,1} : \rho K_2(R, J) \to H^1(R, i_1(\mu_\mathfrak{p})^{\otimes 2})/c_{2,1}(K_3(R, J)) \]
satisfies
\[ \overline{c}_{2,1} \left( \{f, \zeta_\mathfrak{p}\} \right) = f \otimes \zeta_\mathfrak{p} \mod c_{2,1}(K_3(R, J)); \quad f = f \mod (1 + J)^{\mathfrak{p} \mathfrak{p}}. \]
We have the relativization sequence:

\[ H^0_{\text{cont}}(\mathcal{R}, Z_i(2)) \to H^1_{\text{cont}}(\mathcal{R}, i_i Z_i(2)) \to H^1_{\text{cont}}(\mathcal{R}, Z_i(2)) \to H^0_{\text{cont}}(\mathcal{R}, Z_i(2)). \]

Suppose that \( \mathcal{R} \) is essentially of finite type over \( Z \). Then the \( H^0 \) terms both vanish, and the \( \lim^1 \) terms for the \( H^1 \)'s also vanish so we get

\[ H^1_{\text{cont}}(\mathcal{R}, Z_i(2)) = \lim H^1(\mathcal{R}, (\mu_r)^{\otimes 2}). \]

By the Bloch-Ogus sequence, we get

\[ \lim H^1(\mathcal{R}, (\mu_r)^{\otimes 2}) = \lim H^1(L, (\mu_r)^{\otimes 2}) \]
\[ = H^1(L, Z_i(2)); \]

by Suslin's theorem (Cor. 2.7 [S]), the restriction map

\[ H^1_{\text{cont}}(\mathcal{R}, Z_i(2)) \to H^1_{\text{cont}}(\mathcal{R}, Z_i(2)); \]

is injective, hence

\[ H^1_{\text{cont}}(\mathcal{R}, i_i Z_i(2)) = 0. \]

**Lemma 5.2.** — Let \( \mathcal{R} \) be a semi-local PIR. Then

\[ c_{2,1}(K_3(\mathcal{R}, J)) = 0. \]

**Proof.** — We may suppose the \( \mathcal{R} \) is essentially of finite type over \( Z \). We have the commutative diagram

\[ \begin{array}{ccc}
K_3(\mathcal{R}, J) & \to & K_3(\mathcal{R}, J, Z/\mathfrak{p}) \\
\downarrow^{c_{2,1}} & & \downarrow^{c_{2,1}} \\
0 = H^1_{\text{cont}}(\mathcal{R}, i_i Z_i(2)) & \to & H^1(\mathcal{R}, i_i (\mu_r)^{\otimes 2})
\end{array} \]

which proves the lemma. \( \Box \)

Let \( L \) be the quotient field of \( \mathcal{R} \), \( L_0 \) the field of constants in \( L \), and \( R_0 = L_0 \cap \mathcal{R} \). \( R_0 \) is a semi-local PIR and \( \mathcal{R} \) is a smooth, faithfully flat extension of \( R_0 \). We call \( R_0 \) the ring of constants in \( \mathcal{R} \).

**Theorem 5.3.** — Suppose \( \mathcal{R} \) contains \( \mu_r \). Let \( R_0 \) be the ring of constants in \( \mathcal{R} \). Then the following are equivalent

(a) \( \{f, \xi_r\} = 0 \) in \( K_2(\mathcal{R}, J) \)

(b) \( f = f_0 \xi_r^r \), with \( g \) in \( (1 + J)^* \), \( f_0 \) in \( (1 + J_0)^* \), and \( \{f_0, \xi_r\} = 0 \) in \( K_2(\mathcal{R}_0, J_0) \).

**Proof.** — This is the same as the proof of Theorem 3.5 in [S]. \( \Box \)
THEOREM 5.4. — Let $R \to S$ be a smooth faithfully flat extension of semi-local PIR's with $R$ algebraically closed in $S$. Suppose that $JS \cap R = J$. Then

$$K_2(R, J) \to K_2(S, JS)$$

is injective.

Proof. — The same as the proof of Theorem 3.9 of [S].

Since $c_{2,1}: K_2(R, J) \to \text{H}^1_{\text{cont}}(R, i_1(\mu_2)^{\otimes 2})$ is the zero map, we get a well-defined map

$$\Phi: K_2(R, J) \{i\} \to \text{H}^1_{\text{cont}}(R, i_1 \mathbb{Q}_l/\mathbb{Z}_l(2)).$$

COROLLARY 5.5. — There is a natural surjection

$$\text{H}^1(R, i_1(\mu_2)^{\otimes 2}) \to \pi K_2(R, J).$$

If $R$ has characteristic zero, then

$$\Phi: K_2(R, J) \{i\} \to \text{H}^1_{\text{cont}}(R, i_1 \mathbb{Q}_l/\mathbb{Z}_l(2))/\text{Im}(\text{H}^1_{\text{cont}}(R, i_1 \mathbb{Q}_l(2)))$$

is an isomorphism. If $R$ has characteristic $p > 0$, $p \neq 1$ then

$$\Phi: K_2(R, J) \{i\} \to \text{H}^1_{\text{cont}}(R, i_1 \mathbb{Q}_l/\mathbb{Z}_l(2))$$

is an isomorphism.

Proof. — Make the obvious modifications in the argument Suslin uses to prove Theorems 3.9, 3.10, and Corollary 3.13 in [S].

COROLLARY 5.4. — Let $\mathcal{D}$ be an Azumaya algebra over $R$ with $\mathcal{D}$ split, $\pi: X \to \text{Spec}(R)$ the associated Brauer-Severi scheme. Then the map

$$\pi^*: K_2(R, J) \to E_2^{-2}(X, X)$$

is an isomorphism.

Proof. — Same as Theorem 4.9.

COROLLARY 5.5. — Let $\mathcal{D}$ be an Azumaya algebra over $R$ with $\mathcal{D}$ split. Then there is a unique homomorphism

$$\text{Nrd}: K_2(\mathcal{D}, \mathcal{D}) \to K_2(R, J)$$

such that for every smooth extension $R \to S$ splitting $\mathcal{D}$, the diagram

$$\text{Nrd}: K_2(\mathcal{D}, \mathcal{D}) \to K_2(R, J)$$

commutes.
Proof. - Let X/R be the associated Brauer-Severi scheme. Define Nrd to be the composition

\[ \xymatrix{ K_2(\mathcal{O}, \mathcal{O}) \ar[r] & K_2(X, X) = K_2(R, J) \oplus K_2(\mathcal{O}, \mathcal{O}) \oplus \cdots \ar[r] & E_2^{-2}(X, X) \ar[d] \ar[r] & K_2(R, J). \quad \Box } \]

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