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CONSTRUCTIVENESS OF HIRONAKA'S RESOLUTION

BY ORLANDO VILLAMAYOR ⁽¹⁾

Introduction

In [9] Hironaka develops the notion of *local idealistic* presentation for an algebraic scheme X embedded in a regular scheme W . Here we take those results as starting point and we exhibit a *constructive resolution of singularities* (see 2.2)

Roughly speaking, an upper semicontinuous function is defined on a fixed Samuel stratum such that

- (i) the function determines the center of a permissible transformation $\pi_1: X_1 \rightarrow X$.
- (ii) for $\pi_1: X_1 \rightarrow X$ as before, an upper semicontinuous function can be defined at X_1 [as in (i)] such that either there is an improvement of the Hilbert-Samuel functions at X_1 , or there is an improvement on these functions. Repeating (i) and (ii) a finite number of times, say

$$X_r \xrightarrow{\pi_r} X_{r-1} \rightarrow \dots \rightarrow X_1 \xrightarrow{\pi_1} X$$

one can force an improvement (at X_r) of the Hilbert-Samuel function.

In section 1 we introduce the notation and some results (without proofs) required for the *construction*. We refer the reader mainly to [9] for more details and proofs. The definition of constructive resolutions and the development of these are given in section 2.

I thank Prof. Jean Giraud for important suggestions on this work.

§ 1. Throughout this article W will denote a regular algebraic scheme admitting a finite cover by affine sets. Each restriction to these being the spectrum of an algebra of finite type over a fixed field k of characteristic zero. And all patching maps being k -algebra maps.

A map $W_1 \rightarrow W$ will always mean a morphism of finite type.

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We consider pairs of the form (J, b) where b is a positive integer and $J \subset \mathcal{O}_W$ is a coherent sheaf of ideals for which $J_x \neq 0, \forall x \in W$ (J_x denotes the stalk at x).

Given a valuation ring A and a principal ideal $J \subset A$ let $\text{ord}(J)$ denote the value of J with respect to the valuation associated with A .

DEFINITION 1.1. — Assume that (J_1, b_1) and (J_2, b_2) are two pairs as before with the property that for any morphism $h: \text{Spec}(A) \rightarrow W$, where A is a noetherian valuation ring, the following equality holds:

$$\frac{\text{ord}(J_1 A)}{b_1} = \frac{\text{ord}(J_2 A)}{b_2}. \quad (\text{at } \mathbb{Q}).$$

$J_i A$ the ideal induced by J_i via h at A .

This condition defines an equivalence relation among such pairs. We shall say that $(J_1, b_1) \sim (J_2, b_2)$ and the equivalence class of a pair (J, b) , say $\mathcal{A} = ((J, b))$ is called an idealistic exponent at W (see Def. 3, p. 56 [9]).

Assume that $(J_1, b_1) \sim (J_2, b_2)$ and let $\pi: W_1 \rightarrow W$ be any morphism of regular schemes, then $(J_1 \mathcal{O}_{W_1}, b_1) \sim (J_2 \mathcal{O}_{W_1}, b_2)$. So we define for a given idealistic exponent $\mathcal{A} = ((J, b))$ at W , the idealistic exponent $\pi^{-1}(\mathcal{A})$ as:

$$\pi^{-1}(\mathcal{A}) = ((J \mathcal{O}_{W_1}, b)).$$

DEFINITION 1.2. — Let (J_1, b_1) and (J_2, b_2) be two equivalent pairs at W corresponding to the idealistic exponent $\mathcal{A}$. If $x \in W$ then

$$c = \frac{v_x(J_1)}{b_1} = \frac{v_x(J_2)}{b_2},$$

where $v_x(J_i)$ denotes the order of the stalk $J_{i,x}$ at the local regular ring $\mathcal{O}_{W,x}$. We define the order of $\mathcal{A}$ at x to be $v_x(\mathcal{A}) = c$ and the order of $\mathcal{A}$ to be $\text{ord}(\mathcal{A}) = \max_{x \in W} \{v_x(\mathcal{A})\}$.

DEFINITION 1.3. — Given a pair (J, b) at W as in Def. 1.1 we define a reduced subscheme:

$$\text{Sing}^b(J) = \{x \in W \mid v_x(J) \geq b\}$$

A transformation $\pi: W_1 \rightarrow W$ is said to be *permissible* for (J, b) if it is the blowing up with center C , where C is a regular subscheme of W contained in $\text{Sing}^b(J)$.

In this case there is a coherent sheaf of ideals $\bar{J} \subset \mathcal{O}_{W_1}$ such that $J \mathcal{O}_{W_1} = \bar{J} P^b$ where P denotes the sheaf of ideals $\mathcal{O}(-\pi^{-1}(C)) \subset \mathcal{O}_{W_1}$.

We define the transform of (J, b) by π to be the pair $(\bar{J}, b)$ at W_1 .

One can check that if $(J_1, b_1) \sim (J_2, b_2)$ at W then:

(i) $\text{Sing}^{b_1}(J_1) = \text{Sing}^{b_2}(J_2)$ and if $(\bar{J}_i, b_i)$ denotes the transform of (J_i, b_i) , $i=1, 2$ by a permissible map $\pi: W_1 \rightarrow W$, then:

(ii) $(\bar{J}_1, b_1) \sim (\bar{J}_2, b_2)$ at W_1 .

So now let (J, b) be a pair at W , $\pi: W_1 \rightarrow W$ permissible for (J, b) and $\mathcal{A} = ((J, b))$, then we define the subscheme of *singular points*:

$$\text{Sing}(\mathcal{A}) = \text{Sing}^b(J) \subset W$$

A transformation $\pi: W_1 \rightarrow W$ is said to be *permissible for $\mathcal{A}$* if it is permissible for (J, b) and the *transform of $\mathcal{A}$ by the permissible transformation π* to be $\mathcal{A}_1 = ((\bar{J}, b))$ at W_1 where $(\bar{J}, b)$ is the transform of (J, b) . Finally a *sequence of permissible transformation of $\mathcal{A}$ over W* is a sequence

$$\begin{array}{ccccccc} W = W_0 & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\pi_2} & W_2 & \dots & \xleftarrow{\pi_r} W_r \\ \mathcal{A} = \mathcal{A}_0 & & \mathcal{A}_1 & & \mathcal{A}_2 & & \mathcal{A}_r \end{array}$$

where each π_i is permissible for $\mathcal{A}_{i-1}$ and $\mathcal{A}_i$ is the transform of $\mathcal{A}_{i-1}$.

DEFINITION 1.4. — We define on W_1 for some index set Λ

$$E_\Lambda = \{E_\lambda \mid \lambda \in \Lambda\}$$

each E_λ being a smooth hypersurface of W or the empty set. We also assume that these hypersurfaces have only normal crossings i.e. $\bigcup_{\lambda \in \Lambda} E_\lambda (\subset W)$ is a subscheme with only normal crossings.

A monoidal transformation $\pi: W_1 \rightarrow W$ is said to be *permissible for (W, E_Λ)* , if it is the blowing up at a center C which is regular and has only normal crossings with $\bigcup_{\lambda \in \Lambda} E_\lambda$.

In this case the *transform* of (W, E_Λ) is defined as (W_1, E_{Λ_1}) , where $\Lambda_1 = \Lambda \cup \{\beta\}$ and

(i) for each $\lambda \in \Lambda \subset \Lambda_1$, E'_λ is the strict transform of $E_\lambda \subset W$, by this we mean the strict transform of the components of E_λ which are not components of C . $E'_\lambda = \emptyset$ if $E_\lambda = \emptyset$, also if $E_\lambda = C$.

(ii) $E'_\beta = \pi^{-1}(C)$.

It is clear that $\bigcup_{\alpha \in \Lambda_1} E'_\alpha$ consists of hypersurfaces with only normal crossings.

A *permissible tree* is a data of the form:

$$\begin{array}{ccccccc} T: & W = W_0 & \xleftarrow{\pi_1} & W_1 & \xleftarrow{\dots} & W_{r-1} & \xleftarrow{\pi_r} W_r \\ & E_\Lambda = E_{\Lambda_0} & & E_{\Lambda_1} & & E_{\Lambda_{r-1}} & E_{\Lambda_r} \\ & C = C_0 & & C_1 & & C_{r-1} & \end{array}$$

each π_i permissible for $(W_{i-1}, E_{\Lambda_{i-1}})$ and (W_i, E_{Λ_i}) being the corresponding transform.

DEFINITION 1.5. — An isomorphism $\Gamma = (\theta, \gamma): (W, E_\Lambda) \rightarrow (W', E_{\Lambda'})$ consists of:

- (i) A bijection $\gamma: \Lambda \rightarrow \Lambda'$.
- (ii) An isomorphism $\theta: W \rightarrow W'$ inducing by restriction an isomorphism

$$\theta: E_\lambda \rightarrow E_{\gamma(\lambda)}$$

for each $\lambda \in \Lambda$.

Remark 1.6. — Given an isomorphism of pairs $\Gamma: (W, E_\Lambda) \rightarrow (W', E_{\Lambda'})$ as before, and a transformation $\pi_1: W_1 \rightarrow W$ permissible for (W, E_Λ) (Def. 1.4) with center C , then $\theta(C) \subset W'$ has only normal crossings with $\bigcup_{\lambda \in \Lambda'} E_\lambda$ and if π'_1 denotes the corresponding transformation then there is a unique isomorphism $\Gamma_1 = (\theta_1, \gamma_1)$ of the transforms (W_1, E_{Λ_1}) and $(W'_1, E_{\Lambda'_1})$ such that the diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\theta_1} & W'_1 \\ \pi_1 \downarrow & & \downarrow \pi'_1 \\ W & \xrightarrow{\theta} & W' \end{array}$$

is commutative.

Moreover if T is any permissible tree for (W, E_Λ) , then via Γ , T induces a permissible tree over $(W', E_{\Lambda'})$ and the isomorphism Γ can be “lifted” by T .

Remark 1.7. — Let $\mathbb{A} = \text{Spec}(k[X])$ and $P_n: W_n = W \times \mathbb{A}^n \rightarrow W$ the natural projection ($n \geq 0$). Given a pair (W, E_Λ) as in Def. 1.4 we define on each W_n a set $(E_n)_\Lambda$, which consists for each $\lambda \in \Lambda$ of $(E_n)_\lambda = P_n^{-1}(E_\lambda)$.

An isomorphism $\Gamma = (\theta, \gamma): (W, E_\Lambda) \rightarrow (W', E_{\Lambda'})$ (Def. 1.5) induces natural isomorphisms

$$\Gamma_n = (\theta_n, \gamma_n): (W_n, (E_n)_\Lambda) \rightarrow (W'_n, (E_n)_{\Lambda'})$$

for all $n \geq 0$.

DEFINITION 1.8. — Consider now a 3-tuple $(W, \mathcal{A}, E_\Lambda)$ where $\mathcal{A}$ is an idealistic exponent on W and (W, E_Λ) is as in Def. 1.4.

A tree T is said to be *permissible* for $(W, \mathcal{A}, E_\Lambda)$ when the two following conditions hold:

- (a) T is permissible for (W, E_Λ) (Def. 1.4)
- (b) the induced sequence of transformation

$$W = W_0 \xleftarrow{\pi_1} W_1 \leftarrow \dots \leftarrow W_{r-1} \xleftarrow{\pi_r} W_r$$

is permissible for $(W, \mathcal{A})$ in the sense of Def. 1.3.

If $\pi_1: W_1 \rightarrow W$ is permissible for $(W, \mathcal{A}, E_\Lambda)$, let $\mathcal{A}_1$ denote the transform of $\mathcal{A}$ (Def. 1.3) and (W_1, E_{Λ_1}) the transform of (W, E_Λ) (Def. 1.4), then $(W_1, \mathcal{A}_1, E_{\Lambda_1})$ is called the *transform of* $(W, \mathcal{A}, E_\Lambda)$.

The *grove* of $(W, \mathcal{A}, E_\Lambda)$ consists of all possible permissible trees for $(W, \mathcal{A}, E_\Lambda)$.

Let $P_n: W_n = W \times \mathbb{A}^n \rightarrow W$ be as in Remark 1.7 then the *polygrove* of $(W, \mathcal{A}, E_\Lambda)$ consists of the groves of $(W_n, P_n^{-1}(\mathcal{A}), (E_n)_\Lambda)$ for each $n \geq 0$. $P_n^{-1}(\mathcal{A})$ as in Def. 1.1

An *idealistic situation* is a 3-tuple $(W, \mathcal{A}, E_\Lambda)$ as before, together with its polygrove.

DEFINITION 1.9. — An *isomorphism from the idealistic situation* $(W, \mathcal{A}, E_\Lambda)$ to $(W', \mathcal{A}', E_{\Lambda'})$ consists of an isomorphism

$$\Gamma = (\theta: \gamma): (W, E_\Lambda) \rightarrow (W', E_{\Lambda'}) \quad (\text{Def. 1.5})$$

such that the induced isomorphism

$$\Gamma_n = (\theta_n: \gamma_n): (W_n, (E_n)_\Lambda) \rightarrow (W'_n, (E_n)_{\Lambda'}), \quad n \geq 0$$

(Remark 1.7) establish a bijection between those trees of the grove of $(W_n, P_n^{-1}(\mathcal{A}), (E_n)_\Lambda)$ and those of the grove of $(W'_n, P_n^{-1}(\mathcal{A}'), (E_n)_{\Lambda'})$ for all $n \geq 0$. The correspondence of trees via an isomorphism being as in Remark 1.6.

DEFINITION 1.10. — Consider at W an idealistic situation $(W, \mathcal{A}, E_\Lambda)$ and an etale map

$$e: W_1 \rightarrow W$$

then the *restriction by e* of $(W, \mathcal{A}, E_\Lambda)$ is the idealistic situation $(W_1, e^{-1}(\mathcal{A}), (E_1)_\Lambda)$ where:

- (a) for each $\lambda \in \Lambda$, $(E_1)_\lambda = e^{-1}(E_\lambda)$
- (b) if $\mathcal{A}$ is the class of (J, b) , then $e^{-1}(\mathcal{A})$ is the class of (JO_{W_1}, b) (Def. 1.1).

Given a closed point $x \in \text{Sing}(\mathcal{A})$, then an *etale neighbourhood* of $(W, \mathcal{A}, E_\Lambda)$ at x consists of an etale map $e: W_1 \rightarrow W$, an idealistic situation $(W_1, e^{-1}(\mathcal{A}), (E_1)_\Lambda)$ as before, and a point $y \in \text{Sing}(e^{-1}(\mathcal{A}))$ such that $e(y) = x$.

Given two idealistic situations $(W_1, \mathcal{A}_1, E_{\Lambda_1})$, $(W_2, \mathcal{A}_2, E_{\Lambda_2})$ and closed points $x_1 \in \text{Sing}(\mathcal{A}_1)$, $x_2 \in \text{Sing}(\mathcal{A}_2)$, then x_1 is said to be *equivalent* to x_2 if there are etale neighbourhoods at x_1 and x_2 which are isomorphic i.e. there are etale maps $e_i: W'_i \rightarrow W_i$, $i = 1, 2$, restrictions $(W'_i, e_i^{-1}(\mathcal{A}_i), e_i^{-1}(E)_{\Lambda_i})$, $i = 1, 2$, closed points $y_i \in \text{Sing}(e_i^{-1}(\mathcal{A}_i))$, $i = 1, 2$ and an isomorphism of idealistic situations (Def. 1.9)

$$\Gamma = (\theta, \gamma): (W'_1, e_1^{-1}(\mathcal{A}_1), (e_1^{-1}(E))_{\Lambda_1}) \rightarrow (W'_2, e_2^{-1}(\mathcal{A}_2), e_2^{-1}(E)_{\Lambda_2})$$

such that $\theta(y_1) = y_2$.

Remark 1.10.1. — Let the notation and assumptions be as in Def. 1.9.

Let $e: W'_1 \rightarrow W'$ be an etale map and

$$\begin{array}{ccc} W_1 & \xrightarrow{\theta_1} & W'_1 \\ e_1 \downarrow & & \downarrow e \\ W & \xrightarrow{\theta} & W' \end{array}$$

the commutative diagram arising from the fiber product of $\theta: W \rightarrow W'$ and $e: W'_1 \rightarrow W'$.

Then e_1 is étale and θ_1 induces an isomorphism between the restricted situations (Def. 1.10).

This follows from the definition of excellence.

1.11. — Let $(Z, \bar{E}_\Lambda)$, (W, E_Λ) be as in Def. 1.4 and $i: Z \rightarrow W$ be an immersion of regular schemes. Assume furthermore that the following condition holds:

$$(1.11.1) \quad \forall \lambda \in \Lambda: \bar{E}_\lambda = E_\lambda \cap Z.$$

In this case it is clear that a permissible tree T for $(Z, \bar{E}_\Lambda)$ induces a permissible tree for (W, E_Λ) , say $i(T)$. And the final transform of $(Z, \bar{E}_\Lambda)$ and (W, E_Λ) by T and $i(T)$ still satisfy 1.11.1.

Let $\mathbb{A} (= \text{Spec}(k[X]))$, $W_n = W \times \mathbb{A}^n$, $Z_n = Z \times \mathbb{A}^n$ and $(E_n)_\Lambda$, $(\bar{E}_n)_\Lambda$ be as in Remark 1.7. If $i: Z \rightarrow W$ is such that condition 1.11.1 is satisfied, then the same will hold for

the natural immersions $Z_n \xrightarrow{i_n} W_n$.

DEFINITION 1.11. — Let $(Z, \mathcal{A}, \bar{E}_\Lambda)$, $(W, \mathcal{B}, E_\Lambda)$ be two idealistic situations (Def. 1.8), assume that Z is a subscheme of W , $i: Z \hookrightarrow W$, and that $\bar{E}_\Lambda$ and E_Λ satisfy 1.11.1. Then i is said to be a *strong immersion* if $Z_n \hookrightarrow W_n$ induces a bijection between the grove of $(Z_n, P_n^{-1}(\mathcal{A}), (\bar{E}'_n)_\Lambda)$ and that of $(W_n, P_n^{-1}(\mathcal{B}), (E_n)_\Lambda)$ for all $n \geq 0$.

THEOREM 1.12. — Let

$$(Z_1, \mathcal{A}_1, (\bar{E}_1)_\Lambda) \xrightarrow{i_1} (W, \mathcal{B}, E_\Lambda) \quad \text{and} \quad (Z_2, \mathcal{A}_2, (\bar{E}_2)_\Lambda) \xrightarrow{i_2} (W, \mathcal{B}, E_\Lambda)$$

be two strong immersions (Def. 1.11), and let x_i be a closed point at $\text{Sing}(\mathcal{A}_i) \subset Z_i$ ($i=1, 2$) such that $i_1(x_1) = i_2(x_2)$.

If $\dim(Z_1)_{x_1} = \dim(Z_2)_{x_2}$ then x_1 is equivalent to x_2 (Def. 1.10).

Proof. — Argue as in Theorem 11.1 [8] and construct a retraction from W to Z , locally at some étale neighbourhood of $i_1(x_1) = i_2(x_2)$ which induces an isomorphism of the restricted idealistic situations (Def. 1.10).

THEOREM 1.13.1. — Let x_i be a closed singular point of an idealistic situation $(Z_i, \mathcal{A}_i, E_{\Lambda_i})$ $i=1, 2$ (Def. 1.8). If x_1 and x_2 are equivalent (Def. 1.10) then

$$v_{x_1}(\mathcal{A}_1) = v_{x_2}(\mathcal{A}_2) \quad (\text{Def. 1.2})$$

Proof. — (see Prop. 8, p. 68 [9]).

1.13.2. — We now refer to Definition 1.9, p. 59 [9] for the notion of *tangent vector space* of an idealistic exponent $\mathcal{A}$ at a closed point $x \in \text{Sing}(\mathcal{A}) \subset W$ (say $T_{\mathcal{A}, x}$). This is a subspace of $T_{W, x}$ (the tangent-space of W at x) and we shall denote its codimension by $\tau(\mathcal{A}, x)$.

THEOREM 1.13.2. — Let $(Z_i, \mathcal{A}_i, E_{\Lambda_i})$ $i=1, 2$ and x_i $i=1, 2$ be as in the last theorem. Then

$$\tau(\mathcal{A}_1, x_1) = \tau(\mathcal{A}_2, x_2)$$

and $\tau(\mathcal{A}_1, x_1) \geq 0$ iff $v_{x_1}(\mathcal{A}_1) = 1$ (Def. 1.2).

Proof. — The proof of this fact is similar to that of Theorem 1.13.1.

1.14. Let $Z \hookrightarrow W$ be as before a closed immersion of regular schemes and $Z_n = Z \times \mathbb{A}^n \hookrightarrow W_n = W \times \mathbb{A}^n$ the induced immersions.

Let $(W, \mathcal{A}, E_{\Lambda})$ be an idealistic situation and

$$\begin{array}{c} W \times \mathbb{A}^n = (W_n)_0 \xleftarrow{\pi_1} (W_n)_1 \cdots \leftarrow (W_n)_r \\ (E_n)_{\Lambda} = (E_n)_{\Lambda_0} \quad (E_n)_{\Lambda_1} \quad (E_n)_{\Lambda_r} \\ C_0 \quad C_1 \end{array}$$

a tree over W_n , permissible for $(W_n, P_n^{-1}(\mathcal{A}), (E_n)_{\Lambda})$ (see Def. 1.8). For any such tree let $(Z_n)_i \subset (W_n)_i$ denote the strict transform of $Z_n \subset W_n = (W_n)_0$.

DEFINITION 1.14. — With the notation as before, a regular subscheme $Z \subset W$ is said to have *maximal contact* with the idealistic situation $(W, \mathcal{A}, E_{\Lambda})$ if, for any fix $n \geq 0$ and any tree T of the grove of $(W_n, P_n^{-1}(\mathcal{A}), (E_n)_{\Lambda})$ one has that $C_i \subset (Z_n)_i$ $0 \leq i < r$, or equivalently if $\mathcal{A}_i$ denotes the transform at $(W_n)_i$ of $\mathcal{A}_0 = P_n^{-1}(\mathcal{A})$, then $\text{Sing}(\mathcal{A}_i) \subset (Z_n)_i$, $\forall n \geq 0$.

THEOREM 1.15. — Let $(W, \mathcal{A}, E_{\Lambda})$ be an idealistic situation (Def. 1.8), $Z \overset{i}{\hookrightarrow} W$ a regular subscheme having maximal contact with $\mathcal{A}$, and $(Z, \bar{E}_{\Lambda})$ as in Def. 1.4. If the condition 1.11.1 holds for $(Z, \bar{E}_{\Lambda})$ and (W, E_{Λ}) then, locally at any closed point $x \in \text{Sing}(\mathcal{A})$, either

(a) $\text{Sing} \mathcal{A} = Z$ or

(b) for a convenient restriction of $(Z, \bar{E}_{\Lambda})$ at a Zariski neighbourhood of x (as in Def. 1.10), say $(Z, \bar{E}_{\Lambda})$, there is an idealistic situation $(Z, \mathcal{B}, \bar{E}_{\Lambda})$ such that $i: Z \hookrightarrow W$ is a strong immersion (Def. 1.11).

Proof. — See theorem 5, p. 111 [9].

DEFINITION 1.15. — If (a) ever holds at x , we shall say that x is a *regular point* of $\text{Sing}(\mathcal{A})$.

THEOREM 1.16.1. — Let $(W, \mathcal{A}, E_{\Lambda})$ be an idealistic situation and assume that $\text{ord}(\mathcal{A}) = 1$ (Def. 1.2). Then, locally at any closed point $x \in \text{Sing}(\mathcal{A})$, there is a regular hypersurface H having maximal contact with the restricted idealistic situation (Def. 1.10 and Def. 1.14).

COROLLARY 1.16.1. — Assume that $x \in W$ is not a point at which (locally) $\text{Sing}(\mathcal{A})$ is regular of codimension one (Def. 1.15). And assume also that H is a hypersurface

of maximal contact, $(H, \bar{E}_\Lambda)$ is as in Def. 1.4 and that $(H, \bar{E}_\Lambda)$ and (W, E_Λ) satisfy the condition 1.11.1. Then, after restricting to a convenient Zariski neighbourhood of x , there is an idealistic situation $(H, \mathcal{B}, \bar{E}_\Lambda)$ such that $i: H \hookrightarrow W$ is a strong immersion (Def. 1.11).

THEOREM 1.16.2. — *Let $\pi: W_1 \rightarrow W$ be permissible for an idealistic situation $(W, \mathcal{A}, E_\Lambda)$ (Def. 1.8), assume that $\text{ord}(\mathcal{A})=1$ and let $(W_1, \mathcal{A}_1, E_{\Lambda_1})$ be the transform. Then either $\text{Sing}(\mathcal{A}_1)=\emptyset$ or $\text{ord}(\mathcal{A}_1)=1$. If x is any closed point of $\text{Sing}(\mathcal{A}_1)$:*

$$\tau(\mathcal{A}, \pi(x)) \leq \tau(\mathcal{A}, x)$$

DEFINITION 1.16.3. — Let $(W, \mathcal{A}, E_\Lambda)$ be an idealistic situation, we define

$$\tau(\mathcal{A}) = \inf_{x \in \text{Sing}(\mathcal{A})} \{ \tau(\mathcal{A}, x) \}$$

and

$$R(\tau)(\mathcal{A}) = \{ x \in \text{Sing}(\mathcal{A}) \mid \tau(\mathcal{A}, x) = \tau(\mathcal{A}) \text{ and } x$$

is a regular point of $\text{Sing}(\mathcal{A})$ (Def. 1.15) }.

PROPOSITION 1.16.4 (with the same notation as before). — (a) *The set $R(\tau)(\mathcal{A})$ is a regular subscheme of W , of codimension $\tau(\mathcal{A})$ at any point, and every irreducible component of $R(\tau)(\mathcal{A})$ is a connected component of $\text{Sing}(\mathcal{A})$.*

(b) *Let $\pi: W_1 \rightarrow W$ be permissible for $(W, \mathcal{A}, E_\Lambda)$ (Def. 1.8) and let $(W_1, \mathcal{A}_1, E_{\Lambda_1})$ be its transform, then at a closed point $x \in \text{Sing}(\mathcal{A}_1)$ both conditions:*

(i) *x is regular at $\text{Sing}(\mathcal{A}_1)$ (in the sense of Def. 1.15).*

(ii) $\tau(\mathcal{A}_1, x) = \tau(\mathcal{A})$

will hold if and only if $\pi(x) \in R(\tau)(\mathcal{A})$.

Theorem 1.16.1, 1.16.2 and Prop. 1.16.4 follow from Theorem 1 p. 104 [9].

1.17. WEIGHTED IDEALISTIC SITUATIONS. — Let (W, E_Λ) be as in Def. 1.4 and P_λ the sheaf of ideals ($\subset O_W$) defining E_λ (i. e. $P_\lambda = O(-E_\lambda)$) for each $\lambda \in \Lambda$.

DEFINITION 1.17.1. — A *weighted idealistic situation* is an idealistic situation $(W, \mathcal{A}, E_\Lambda)$ (Def. 1.8) together with:

(i) a set A_λ consisting for each $\lambda \in \Lambda$, of a locally constant function

$\alpha(\lambda): E_\lambda \rightarrow (Q \geq 0)$ (non negative rational numbers) such that if $\mathcal{A} = (J, b)$ and $x \in \text{Sing}^b(J)$, then at $O_{W, x}$:

$$J_x = \prod_{\{\lambda \mid x \in E_\lambda\}} P_{\lambda, x}^{\beta(\lambda)(x)} \cdot \bar{J}_x, \quad \bar{J}_x \not\subset P_{\lambda, x}, \quad \forall \lambda/x \in E_\lambda$$

and $\beta(\lambda)(x) = b \cdot (\alpha(\lambda)(x)) \in (Z \geq 0)$, for some coherent sheaf of ideals $\bar{J} (\subset O_W)$.

(ii) at each closed point $x \in \text{Sing}^b(J)$ define $\Lambda_x = \{ \lambda \in \Lambda \mid x \in E_\lambda \}$. Since these hypersurfaces have only normal crossings at W it follows that $\# \Lambda_x \leq \dim W$. We assume

the existence of a total order at any such Λ_x , say $<$, subject to the following conditions:

(1.17.1.1) Given two closed points $\{x_1, x_2\} \subset E_{\alpha_1} \cap E_{\alpha_2}$ then $\alpha_1 \leq_{x_1} \alpha_2$ if and only if $\alpha_1 \leq_{x_2} \alpha_2$. We denote this weighted idealistic situation by $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$.

We also define the *weighted order of $\mathcal{A}$ at x*

$$w - v_x(\mathcal{A}) = \frac{v_x(\bar{J})}{b} \quad (\text{check consistency}).$$

The *weighted order of $\mathcal{A}$* :

$$w\text{-ord}(\mathcal{A}) = \max_{x \in \text{Sing } \mathcal{A}} \{ w - v_x(\mathcal{A}) \}.$$

And the *weighted singularities of $\mathcal{A}$* :

$$w\text{-Sing}(\mathcal{A}) = \{ x \in \text{Sing}(\mathcal{A}) \mid w - v_x(\mathcal{A}) = w\text{-ord}(\mathcal{A}) \}$$

which is a closed subset of $\text{Sing}(\mathcal{A})$.

DEFINITION 1.17.2 (notation as in Definition 1.9). — Two weighted idealistic situations $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ and $(W', \mathcal{A}', E_{\Lambda'}, A_{\Lambda'})$ are said to be *isomorphic* if there is an isomorphism of the underlying idealistic situation $(W, \mathcal{A}, E_\Lambda)$ and $(W', \mathcal{A}', E_{\Lambda'})$, induced by an isomorphism

$$\Gamma: (\theta, \gamma): (W, E_\Lambda) \rightarrow (W', E_{\Lambda'}) \quad (\text{Def. 1.9})$$

such that:

(i) for each $\lambda \in \Lambda$ let $\alpha(\lambda) \in A_\Lambda$ and $\alpha'(\gamma(\lambda)) \in A_{\Lambda'}$ be the corresponding functions, then

$$\alpha(\lambda) = \alpha'(\gamma(\lambda)) \circ (\theta|_{E_\lambda}): E_\lambda \rightarrow (Q \geq 0)$$

(ii) at any closed point $x \in \text{Sing}(\mathcal{A})$, $\lambda_1 < \lambda_2$ (at Λ_x) if and only if $\gamma(\lambda_1) < \gamma(\lambda_2)$ (at $\Lambda'_{\theta(x)}$).

(From Theorem 1.13.1 it follows that only (ii) must be checked)

DEFINITION 1.17.3 (notation as in Def. 1.10). — Consider a weighted idealistic situation $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ and an etale map $e: W_1 \rightarrow W$ then the *restriction by e* consists of:

(i) the restriction of the idealistic situation

$$(W_1, e^{-1}(\mathcal{A}), (E_1)_\Lambda) \quad (\text{Def. 1.10})$$

(ii) $(e^{-1}(A))_\Lambda = \{ \alpha'(\lambda) \mid \lambda \in \Lambda \}$ where

$$\alpha'(\lambda) = \alpha(\lambda) \circ e|_{e^{-1}(E_\lambda)}, \quad \forall \lambda \in \Lambda$$

(iii) At a closed point $x \in \text{Sing}(e^{-1}(\mathcal{A}))$, given $\lambda_1, \lambda_2 \in \Lambda_x$, define $\lambda_1 \leq_x \lambda_2$ if and only if $\lambda_1 < \lambda_2$. The restriction by e of $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ is again a weighted idealistic situation.

Given two weighted idealistic situations $(W_i, \mathcal{A}_i, E_{\Lambda_i}, A_{\Lambda_i})$ $i=1,2$ and closed points $x_i \in \text{Sing}(\mathcal{A}_i)$, then x_1 and x_2 are said to be *equivalent* (as singular points of *weighted* idealistic situations) if there are restrictions at etale neighbourhoods of x_i ($i=1,2$) and an isomorphism as in Def. 1.10 which is also isomorphism of weighted idealistic situations (Def. 1.17.2).

Remark. — So far we have not defined a notion of transform of weighted idealistic situations, at least not as *weighted* idealistic situations.

DEFINITION 1.17.4. — Let $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ be as before. A transformation $\pi: W_1 \rightarrow W$ is said to be *w-permissible* if:

- (i) π is permissible for the idealistic situation $(W, \mathcal{A}, E_\Lambda)$ (Def. 1.8).
- (ii) In the case that $w\text{-ord}(\mathcal{A}) > 0$ (Def. 1.17.1), and if π is the blowing up at center $C \subset W$ then $C \subset w\text{-Sing}(\mathcal{A})$.

If $\pi: W_1 \rightarrow W$ is a *w-permissible* transformation as before and (W_1, E_{Λ_1}) is the transform of (W, E_Λ) (see Def. 1.4), then $\Lambda_1 = \Lambda \cup \{\beta\}$ and we define now A_{Λ_1} as follows:

- (i) for each $\lambda \in \Lambda \subset \Lambda_1$, let $\alpha'(\lambda) = \alpha(\lambda) \circ \pi|_{E'_\lambda}$ where E'_λ is the strict transform of E_λ (Def. 1.4).

- (ii) $\alpha'(\beta)|_{\pi^{-1}(c_i)} = \sum_{\{\lambda \mid c_i \subset E_\lambda\}} \alpha'(\lambda) \circ \pi + w\text{-ord}(\mathcal{A})$

where the c_i are the connected components of C , so $\alpha'(\beta): \pi^{-1}(C) \rightarrow Q$ is a locally constant function. Now we define at each closed point $x \in \text{Sing}(\mathcal{A}_1)$ $[\mathcal{A}_1]$ the transform of $\mathcal{A}$ (Def. 1.3)] a total order at $(\Lambda_1)_x$:

- (i) If $\beta \in (\Lambda_1)_x$ [i. e. if $x \in \pi^{-1}(C)$] and $\beta \neq \alpha \in (\Lambda_1)_x$ then $\beta <_x \alpha$.
- (ii) Given $\alpha_1 \neq \beta \neq \alpha_2$, then $\alpha_1 <_x \alpha_2$ if and only if $\alpha_1 <_{\pi(x)} \alpha_2$.

$(W, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ is now a weighted idealistic situation called the *transform* of $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ by π , which we also denoted by $(W_1, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) \xrightarrow{\pi} (W, \mathcal{A}, E_\Lambda, A_\Lambda)$.

Remark 1.17.5. — Let $\Gamma: (\theta, \gamma): (W, \Lambda) \rightarrow (W', \Lambda')$ define an isomorphism of the weighted idealistic situations $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ and $(W', \mathcal{A}', E_{\Lambda'}, A_{\Lambda'})$ (Def. 1.17.2). Let $\pi: W_1 \rightarrow W$ be a *w-permissible* transformation for $(W, \mathcal{A}, E_\Lambda, A_\Lambda)$ (Def. 1.17.4). Then there exists a unique isomorphism of weighted idealistic situations Γ' such that the diagram

$$\begin{array}{ccc} (W, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) & \xrightarrow{\Gamma'} & (W_1, \mathcal{A}'_1, E_{\Lambda'_1}, A_{\Lambda'_1}) \\ \pi \downarrow & & \downarrow \pi' \\ (W, \mathcal{A}, E_\Lambda, A_\Lambda) & \xrightarrow{\Gamma} & (W', \mathcal{A}', E_{\Lambda'}, A_{\Lambda'}) \end{array}$$

commuts, where π' corresponds to π via Γ and $(W'_1, \mathcal{A}'_1, E_{\Lambda'_1}, A_{\Lambda'_1})$ is the transform of $(W', \mathcal{A}', E_{\Lambda'}, A_{\Lambda'})$.

Remark 1.17.6. — With the notion as in Def. 1.17.1.

Let $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ be a weighted idealistic situation and $t = w\text{-ord}(\mathcal{A})$. If $\mathcal{A} = ((J, b))$ then:

$$(a) \quad t_1 = b \cdot t = \max_{x \in W} \{v_x(\bar{J})\} \text{ and}$$

$$(b) \quad w\text{-Sing}(\mathcal{A}) = \{x \in \text{Sing}(\mathcal{A}) \mid v_x(\bar{J}) = t_1\}.$$

When $t > 0$ we attach to (J, b) a new idealistic pair $w(J, b)$ as follows:

If $t \geq 1$, then: $w(J, b) = (\bar{J}, t_1)$.

If $0 < t < 1$, then: $w(J, b) = (\langle \prod P_{\lambda}^{(b)} t_1, \bar{J}^{b-t_1} \rangle, t_1(b-t_1))$ where $t_1 = tb$, and $\bar{J}$ and $P_{\lambda}^{(b)}$ are as in Def. 1.17.1. Now we can check:

(i) If $(J, b) \sim (J', b') \Rightarrow w(J, b) \sim w(J', b')$ (check first that $(\bar{J}, b) \sim (\bar{J}', b')$, notation as before).

(ii) If $w(\mathcal{A})$ denotes $(w(J, b))$, then $\text{Sing}(w(\mathcal{A})) = w\text{-Sing}(\mathcal{A})$. So $\pi: W_1 \rightarrow W$ is w -permissible for $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ if and only if it is permissible for $(W, w(\mathcal{A}), E_{\Lambda})$ (Def. 1.17.4 and Def. 1.8).

(iii) Let $\pi: W_1 \rightarrow W$ be as in (ii) and let $(W_1, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ be the transform of $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ (Def. 1.17.4). Then:

$$w\text{-ord}(\mathcal{A}_1) \leq w\text{-ord}(\mathcal{A})$$

and if the equality holds, then $w(\mathcal{A}_1)$ is the transform (simply as idealistic situation) of $w(\mathcal{A})$ (Def. 1.8).

Remark 1.17.7. — Given a weighted idealistic situation $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$, assume $w\text{-ord}(\mathcal{A}) > 0$, and let $w(\mathcal{A})$ be as before, then: $\text{ord}(w(\mathcal{A})) = 1$.

Remark 1.17.8. — If $(W, \mathcal{A}, E_{\Lambda})$ is an idealistic situation (Def. 1.8) and $\text{ord}(\mathcal{A}) = 1$ (Def. 1.2) then it can be given a structure of weighted idealistic situation, taking A_{Λ} to consists of the functions $\alpha(\lambda)$ which are constantly equal to zero along E_{λ} for each $\lambda \in \Lambda$ (Def. 1.17.2).

Note also that in this case $w\text{-Sing}(\mathcal{A}) = \text{Sing}(\mathcal{A})$. So the notions of w -permissibility and of permissibility coincide (Def. 1.17.4 and Def. 1.8).

If $\pi: W_1 \rightarrow W$ is permissible for $(W, \mathcal{A}, E_{\Lambda})$ [w -permissible for $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$] and $(W, \mathcal{A}_1, E_{\Lambda_1})$ $((W_1, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1}))$ denotes the transform. Then again A_{Λ_1} consists of functions $\alpha(\lambda): E_{\lambda} \rightarrow Q$ such that $\alpha(\lambda)(x) = 0 \quad \forall x \in E_{\lambda}, \forall \lambda \in \Lambda_1$.

1.18. IDEALISTIC SPACES

DEFINITION 1.18.1. — By $(C(m), \Lambda)$ we denote a category, where the objects are those weighted idealistic situations $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ where $\dim W = m$ (Def. 1.17.1) and a map $(W_1, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1}) \rightarrow (W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ is an etale map $e: W_1 \rightarrow W$ such that id_{W_1} induces an isomorphism (Def. 1.17.2) between $(W_1, \mathcal{A}_1, E_{\Lambda_1}, A_{\Lambda_1})$ and the restriction of $(W, \mathcal{A}, E_{\Lambda}, A_{\Lambda})$ by e (Def. 1.17.3).

To simplify the notation, given an object $\alpha \in C(m, \Lambda)$ we denote

$$\alpha = (W(\alpha), \mathcal{A}(\alpha), E_{\Lambda_\alpha}, A_{\Lambda_\alpha}).$$

A subset C of $C(m, \Lambda)$ consists, for each $\alpha \in C(m, \Lambda)$ of a locally closed subset $C(\alpha) \subset \text{Sing}(Q(\alpha)) \subset W(\alpha)$ subject to the following conditions:

1. Given $\alpha \xrightarrow{j} \beta$ in $C(m, \Lambda)$, then $e(j)^{-1}(C(\beta)) = C(\alpha)$ where $e(j): W(\alpha) \rightarrow W(\beta)$ is the associated etale map.
2. Given $\alpha_1, \alpha_2 \in C(m, \Lambda)$ and closed points $x_i \in W(\alpha_i)$, if x_1 and x_2 are equivalent (Def. 1.17.3), then $x_1 \in C(\alpha_1) \Leftrightarrow x_2 \in C(\alpha_2)$.

DEFINITION 1.18.2. — An *idealistic space of dimension m* is a map χ from a set I to $C(m, \Lambda)$ ($\dim \chi = m$).

A *closed subset C* of χ consists of a subset C of $C(m, \Lambda)$ such that for each $\alpha \in I$ $C(\chi(\alpha)) (\subset W(\chi(\alpha)))$ is a closed subset. A closed subset C of χ is said to be *permissible* for χ if $C(\chi(\alpha))$ is w -permissible for $\chi(\alpha)$ in the sense of Def. 1.17.4. In such case the *transform of χ by C* is defined by $\chi': I \rightarrow C(m, \Lambda)$ where $\chi'(\alpha)$ is the transform of $\chi(\alpha)$ by $C(\alpha)$ (Def. 1.17.4). This we denote by $\chi' \rightarrow \chi$ and π is said to be a *permissible transformation* with center C .

A point $x \in \chi$ consists of a closed point $x_\alpha \in \text{Sing}(\mathcal{A}(\chi(\alpha)) \subset W(\chi(\alpha))$ (for some $\alpha \in I$) together with all those $x_\beta \in \text{Sing}(\mathcal{A}(\chi(\beta)) \subset W(\chi(\beta))$ ($\beta \in I$) such that x_α and x_β are equivalent (Def. 1.17.3).

DEFINITION 1.18.3. — A m -dimensional idealistic space $\chi: I \rightarrow C(m, \Lambda)$ is said to be *restrictive to an n -dimensional idealistic space* if $n \leq m$ and there are idealistic spaces $\chi_n: \bar{I} \rightarrow C(n, \Lambda)$ and $\chi_m: \bar{I} \rightarrow C(m, \Lambda)$ such that:

1. Points of χ are locally equivalent to points of χ_m and the converse also holds (local equivalence always as in Def. 1.17.3).

2. For each $\alpha \in \bar{I}$ there is a strong immersion (Def. 1.11), disregarding the weighted structure, induced by $W(\chi_n(\alpha)) \xrightarrow{i(\alpha)} W(\chi_m(\alpha))$ such that two points

$$x_i \in \text{Sing}(\mathcal{A}(\chi_n(\alpha_i)) \subset W(\chi_n(\alpha_i)),$$

$i=1,2$ are equivalent points at $C(n, \Lambda)$ (Def. 1.18.2) if and only if $i(\alpha_i)(x_i)$ are equivalent as points of χ_m [at $C(m, \Lambda)$].

Remark 1.18.4. — Given χ_n and χ_m as before, permissible center for χ_n and χ_m coincide (via i) and if $\chi'_m \rightarrow \chi_m$ and $\chi'_n \rightarrow \chi_n$ are the permissible transforms at an identified center, then (1) and (2) hold for χ'_n and χ'_m .

Remark 1.18.5. — Suppose that for each $\alpha \in I$,

$$\chi_m(\alpha) = (W(\chi_m(\alpha)), \mathcal{A}(\chi_m(\alpha)), E_{\Lambda_\alpha}, A_{\Lambda_\alpha})$$

is such that all functions $\alpha(\lambda)$ (Def. 1.17.1) [for all $\lambda \in \Lambda(\alpha)$] are constant functions equal to zero *i. e.*

$\alpha(\lambda): E_\lambda \rightarrow \mathbb{Q}$ is such that $\alpha(\lambda)(x)=0, \forall x \in E_\lambda, \forall \lambda \in \Lambda(\alpha)$. Assume that this also holds for any $\alpha \in \bar{I}$ at $\chi_n(\alpha)$, then (2) of Def. 1.18.3 can be replaced by:

(2') For each $\alpha \in \bar{I}$ there is a strong immersion, disregarding the weighted structure, induce by:

$$W(\chi_n(\alpha)) \hookrightarrow_{i(\alpha)} W(\chi_m(\alpha))$$

1.19. When we consider a fixed idealistic space $\chi: I \rightarrow C(m, \Lambda)$, and $\alpha \in I$ we denote $\chi(\alpha) = (W(\chi(\alpha)), \mathcal{A}(\chi(\alpha)), E_{\Lambda_{\chi(\alpha)}}, A_{\Lambda_{\chi(\alpha)}})$ by $(W(\alpha), \mathcal{A}(\alpha), E_{\Lambda\alpha}, A_{\Lambda\alpha})$.

DEFINITION 1.19.1. — An idealistic space $\chi: I \rightarrow C(m, \Lambda)$ is said to be *quasi-compact* if there is a finite subset $\{\alpha_1, \dots, \alpha_n\} \subset I$ such that for any $\alpha \in I$ and any closed point $x \in \text{Sing}(\mathcal{A}(\alpha)) \subset W(\alpha)$ there is an index $i, 1 \leq i \leq n$ and a point $y \in \text{Sing}(\mathcal{A}(\alpha_i))$ such x and y are locally equivalent (Def. 1.17.3).

If x is a point of χ (Def. 1.18.2), say that $x_1 \in W(\alpha_1)$ belongs to the class of x , then we define *the order of χ at x*

$$\text{ord}_x(\chi) = v_{x_1}(\mathcal{A}(\alpha_1)) \quad (\text{Def. 1.2})$$

and

$$\tau(\chi, x) = \tau(\mathcal{A}(\alpha_1), x_1), \quad (\text{Def. 1.13.2})$$

the consistency of these definitions are given by Theorems 1.13.1 and 1.13.2.

The *order of χ* is:

$$\text{ord } \chi = \max_{\alpha \in I} \{ \text{ord } \mathcal{A}(\alpha) \} \quad (\text{Def. 1.2})$$

The *weighted order of χ* is:

$$w\text{-ord}(\chi) = \max_{\alpha \in I} \{ w\text{-ord}(\mathcal{A}(\alpha)) \} \quad (\text{Def. 1.17.1})$$

and

$$\tau(\chi) = \inf_{\alpha \in I} \{ \tau(\mathcal{A}(\alpha), x) \mid x \in \text{Sing}(\mathcal{A}(\alpha)) \}.$$

1.19.2. One can check that the following are closed subsets of χ in the sense of Definition 1.18.2.

1. $\text{Sing } \chi: (\text{Sing } \chi)(\alpha) = \text{Sing}(\chi(\alpha)) = \text{Sing}(\mathcal{A}(\alpha)) \subset W(\alpha), \forall \alpha \in I$.
2. $w\text{-Sing } \chi: (w\text{-Sing } \chi)(\alpha) = w\text{-Sing}(\mathcal{A}(\alpha)) \subset W(\alpha), \forall \alpha \in I$.
3. If $\tau = \tau(\chi)$ then $F(\tau)(\chi)$:

$$F(\tau)(\chi)(\alpha) = \{ x \in \text{Sing } \mathcal{A}(\alpha) \mid \tau(\mathcal{A}(\alpha), x) = \tau \}$$

4. If $\tau = \tau(\chi)$ then $R(\tau)(\chi)$:

$$R(\tau)(\chi)(\alpha) = \{x \in \text{Sing } \mathcal{A}(\alpha) \mid \tau(\mathcal{A}(\alpha), x) = \tau\}$$

and

x is regular at $\text{Sing } \mathcal{A}(\alpha)$ (Def. 1.15) }.

Remark 1.19.2. — $R(\tau)(\chi)$ is a component of $\text{Sing } \chi$ in the sense that $\forall \alpha \in I$, $R(\tau)(\chi)(\alpha)$ is a union of connected components of $(\text{Sing } \chi)(\alpha) = \text{Sing } (\mathcal{A}(\alpha))$ (see Proposition 1.16.4).

DEFINITION 1.19.3. — Given $\chi: I \rightarrow C(m, \Lambda)$ such that $w\text{-ord}(\chi) > 0$ (Def. 1.19.1), define $w(\chi): I \rightarrow C(m, \Lambda)$ by:

$$w(\chi)(\alpha) = (W(\alpha), w(\mathcal{A}(\alpha)), E_{\Lambda\alpha}, A'_{\Lambda\alpha})$$

$w(\mathcal{A}(\alpha))$ as in 1.17.6 and all functions of $A'_{\Lambda\alpha}$ being constantly equal to zero (see Remark 1.17.8).

Now one can check that $w(\chi)$ is an idealistic space for which:

- (i) $\text{ord}(w(\chi)) = 1$ (Def. 1.19.1).
- (ii) $\text{Sing}(w(\chi)) = w\text{-Sing}(\chi)$.
- (iii) If $\pi: \chi_1 \rightarrow \chi$ is a permissible transformation (Def. 1.18.2) then $w\text{-ord } \chi_1 \leq w\text{-ord } \chi$.
- (iv) If the equality holds at (iii) then naturally $\pi: w(\chi_1) \rightarrow w(\chi)$ is a permissible transformation.

THEOREM 1.20. — Let $\chi: I \rightarrow C(m, \Lambda)$ be a quasi-compact m -dimensional idealistic space of order 1 (Def. 1.19.1). If $E_{\Lambda\alpha} = \emptyset \forall \alpha \in I$, then $\tau(\chi) > 1$, and χ is restrictive to a quasi-compact idealistic space of dimension $m-1$ (Def. 1.18.3).

Proof. — Follows from theorems 1.16.1 and 1.12.

§ 2. Constructive Resolutions

2.1. Recall from 1.19.3 that if $\pi: \chi_1 \rightarrow \chi$ is a permissible transformation of idealistic spaces, then

$$w\text{-ord}(\chi_1) \leq w\text{-ord}(\chi).$$

DEFINITION 2.1. — Fix a sequence of idealistic spaces and permissible transformations (1.18.2):

$$(2.1.1) \quad \chi_0 \xleftarrow{\pi_1} \chi_1 \xleftarrow{\pi_2} \chi_2 \xleftarrow{\dots} \chi_r$$

and assume that $w\text{-ord}(\chi_0) = w\text{-ord}(\chi_r) > 0$, we shall say that χ_0 is a *new* space and χ_0 is the *birth* of χ_r .

In this case [(2.1.1) being fixed], we define $\tau(w\chi_r)$ to be $\tau(w(\chi_0))[\tau(\chi_0)$ as in Def. 1.19.1 and $w(\chi_i)$ as in 1.19.3].

Let $\chi_0: I \rightarrow C(m, \Lambda)$, then (2.1.1) induces for each $\alpha \in I$ a sequence of w -permissible transformations of weighted idealistic situations

$$(W^{(0)}(\alpha), \mathcal{A}^{(0)}(\alpha), E_{\Lambda(\alpha)}^{(0)}, A_{\Lambda(\alpha)}^{(0)}) \xleftarrow{\pi_1} (W^{(1)}(\alpha), \mathcal{A}^{(1)}(\alpha), E_{\Lambda(\alpha)}^{(1)}, A_{\Lambda(\alpha)}^{(1)}) \dots \\ \xleftarrow{\pi_r} (W^{(r)}(\alpha), \mathcal{A}^{(r)}(\alpha), E_{\Lambda(\alpha)}^{(r)}, A_{\Lambda(\alpha)}^{(r)})$$

For each $\alpha \in I$ we define $(E_{\Lambda(\alpha)}^{(r)})^+, (E_{\Lambda(\alpha)}^{(r)})^-$ such that

$$E_{\Lambda(\alpha)}^{(r)} = (E_{\Lambda(\alpha)}^{(r)})^+ \cup (E_{\Lambda(\alpha)}^{(r)})^-.$$

(i) $(E_{\Lambda(\alpha)}^{(r)})^-$ consists of the strict transform at $W^{(r)}(\alpha)$ of elements of $E_{\Lambda(\alpha)}^{(0)}$ [as in (i) of Def. 1.4].

(ii) $(E_{\Lambda(\alpha)}^{(r)})^+$ consists of the strict transforms at $W^{(r)}(\alpha)$ of the exceptional locus of π_j , $j = 1, 2, \dots, r$ [as in (ii) Def. 1.4].

A *partial resolution* of χ consists of a sequence of permissible transformations

$$\chi = \chi_0 \xleftarrow{\pi_1} \chi \xleftarrow{\pi_2} \chi_2 \dots \xleftarrow{\pi_r} \chi_r \xleftarrow{\pi_{r+1}} \chi_{r+1}$$

such that $w\text{-ord}(\chi) = w\text{-ord}(\chi_r) > w\text{-ord}(\chi_{r+1})$. And a *resolution* is a sequence

$$\chi_0 \leftarrow \dots \leftarrow \chi_s$$

of permissible transformations, and $\text{Sing } \chi_s = \emptyset$.

2.2. At this point we want to establish the meaning of a *constructive resolution of quasi compact idealistic spaces of dimension m* .

On any partially ordered set $(D, <)$ consider the discrete topology, then a constructive resolution of χ consists of:

(i) An upper semicontinuous function $\varphi: \text{Sing } \chi \rightarrow D$ such that

$$\underline{\text{Max}} \varphi = \{x \in \text{Sing } \chi \mid \varphi(x) \text{ is maximum}\}$$

is the center of a permissible transformation

$$\pi_1: \chi_1 \rightarrow \chi.$$

(ii) If $\pi_1: \chi_1 \rightarrow \chi$ [as in (i)] is not a resolution of χ (Def. 2.1), then there is an upper semicontinuous function $\varphi_1: \text{Sing } \chi_1 \rightarrow D$, such that:

(a) $\varphi(\pi_1(x)) \geq \varphi_1(x)$, $\forall x \in \text{Sing } \chi_1$

(b) If $\pi(x) \notin \underline{\text{Max}} \varphi$ then $\varphi_1(x) = \varphi(\pi(x))$

(c) $\underline{\text{Max}} \varphi_1$ is permissible at χ_1

(iii) Assume that a sequence

$$\chi = \chi_0 \leftarrow \chi_1 \leftarrow \chi_2 \cdots \leftarrow \chi_r$$

has been defined, that $\text{Sing } \chi_r \neq \emptyset$, and also that the functions $\varphi_i: \chi_i \rightarrow D$ are given $i=0, \dots, r$. Then $\underline{\text{Max}}(\varphi_r)$ is the center of a permissible transformation say π_{r+1} :

$$\chi_r \xleftarrow{\pi_{r+1}} \chi_{r+1}$$

such that either χ_{r+1} is a resolution of χ_r or there is an upper semicontinuous function $\varphi_{r+1}: \chi_{r+1} \rightarrow D$ and conditions (a), (b) and (c) of (ii) (with the obvious adjustment of subindices) hold.

(iv) For some r , $\text{Sing } \chi_r = \emptyset$ i.e.

$$\chi = \chi_0 \xleftarrow{\pi_1} \chi_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} \chi_r$$

is a resolution (Def. 2.1).

(v) Suppose that $\text{ord}(\chi) = 1$, that $\text{Sing}(\chi) = R(\tau)(\chi)$ (1.19.2) and $\chi \xleftarrow{\pi_1} \chi_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} \chi_r$ have been constructed, and assume that only hypersurfaces arising as exceptional locus from this sequence of permissible transformations intersect $\text{Sing}(\chi_r)$ [which is also regular (Prop. 1.16.4)], then

$$\underline{\text{Max}} \varphi_r = \text{Sing } \chi_r$$

i.e. φ_r is constant at $\text{Sing } \chi_r$.

Remark 2.2.1. — Let χ_r be as in (v) then φ_r is constantly equal to some $c \in D$. If

$$\chi_r \xleftarrow{\pi_r} \chi_{r+1}$$

is any permissible transformation and $\text{Sing } \chi_{r+1} \neq \emptyset$ then all conditions on χ_r hold also on χ_{r+1} , and if we define $\varphi_{r+1}: \text{Sing } \chi_{r+1} \rightarrow D$ by $\varphi_{r+1} = c$ (the constant function), then condition (iii) still holds.

Remark 2.2.2. — On a ordered set $(D, \leq)$ we may assume the existence of an element $\infty_D \in D$ such that $\lambda < \infty_D, \forall \lambda \in D, \lambda \neq \infty_D$. If not we can “add” such an element to D .

Given D_1 and D_2 as before we consider on $D_1 \times D_2$ the lexicographic order, then $\infty_{D_1 \times D_2} = (\infty_{D_1}, \infty_{D_2})$.

$\mathbb{Z}$ (or $\mathbb{Z} \cup \{\infty\}$) will be considered with the usual order.

2.3. We begin by constructing an upper semicontinuous function T from which φ will derive.

First we consider the case of an idealistic space of dimension m , say $\chi: I \rightarrow C(m, \Lambda)$ and weighted order zero (Def. 1.19.1).

2.3.1. Case $\dim \chi = m$ and $w - \text{ord } \chi = 0$.

At each closed point $x \in \text{Sing } \chi$ define $\Lambda_x = \{\alpha \in \Lambda \mid x \in E_\alpha\}$ [see Def. 1.17.1 (ii)] and recall that $\# \Lambda_x \leq m$.

Let now $T: \text{Sing } \chi \rightarrow \mathbb{Z}^3 \times \Lambda^m$ be defined as follows

$$\begin{aligned} T(1)(x) &= 0 \\ T(2)(x) &= -\mathcal{B}(x) \quad \text{where} \quad \mathcal{B}(x) = \min \{k \mid \exists i_1 < i_2 < \dots < i_k \\ i_j \in \Lambda_x, j=1, 2, \dots, k \quad \text{and} \quad \alpha(i_1)(x) + \dots + \alpha(i_k)(x) \geq 1\}. \end{aligned}$$

If $\mathcal{B} = \mathcal{B}(x)$ then

$$T(3)(x) = \max \{ \alpha(i_1)(x) + \dots + \alpha(i_{\mathcal{B}})(x) \mid i_1 < \dots < i_{\mathcal{B}} \}$$

and

$$E_{i_j} \in \Lambda_x, i=1, 2, \dots, \mathcal{B} \}.$$

Now consider $\Lambda_x^{\mathcal{B}} = \Lambda_x \times \dots \times \Lambda_x$ ($\mathcal{B}$ -times) with the lexicographic ordering, and:

$$\begin{aligned} \beta &= (\beta_1, \dots, \beta_{\mathcal{B}}) = \max \{ (\beta_1 \dots \beta_{\mathcal{B}}) \mid \beta_1 > \beta_2 > \dots > \beta_{\mathcal{B}}, \beta_i \in \Lambda_x \\ i=1, 2, \dots, \mathcal{B} \quad \text{and} \quad \alpha(\beta_1)(x) + \dots + \alpha(\beta_{\mathcal{B}})(x) &= T(3)(x) \}. \end{aligned}$$

Define:

$$T(4)(x) = (\beta, \infty) \in \Lambda^m (\beta \in \Lambda_x^{\mathcal{B}} \subset \Lambda^{\mathcal{B}} \text{ and } \infty = \infty_{\Lambda^{m-\mathcal{B}}} \in \Lambda^{m-\mathcal{B}})$$

We shall now define at $\text{Img } T \subset \mathbb{Z}_3 \times \Lambda^m$ a partial order, without a notion of order at Λ , but extending the lexicographic order at $\mathbb{Z}^3$.

It suffices to define a notion of $T(x) < T(y)$ at closed points $x, y \in \text{Sing } \chi$ for which $T(j)(x) = T(j)(y) = a_j, j=1, 2$ and 3 ($a_1 = 0$ by assumption).

Let $J = \{x \in \text{Sing } \chi \mid T(j)(x) = a_j, j=1, 2, 3\}$. One can check (at each $\alpha \in I$) that the irreducible components of J are open subset of irreducible components of $\text{Sing } \chi$ of dimension $m + a_2$ [at $W(\alpha)$]. Now we say that $T(4)(x) < T(4)(y)$ if there are closed points $\{x_0 = x_1, \dots, x_2 = y\} \subset J$ such that:

- (a) $T(4)(x_i) \in \Lambda_{x_{i+1}}^{-a_2}, i=0, \dots, l-1$
- (b) for some i as before $T(4)(x_i) < T(4)(x_{i+1})$ at $\Lambda_{x_{i+1}}^{-a_2}$.

The consistency of this definition follows from (1.17.1.1) and Def. 1.17.2 (ii).

This order is not a total order at $\text{Img } T$, and the existence of maximal elements follows from the hypothesis of quasi-compactness on χ .

The maximal elements might not be unique as shown in the following examples:

Examples. — Consider at $W = \text{Spec}(C[x, y, z])$ hypersurfaces

$$E_1 = \{x=0\}, \quad E_2 = \{x=1\}, \quad E_3 = \{y=0\}, \quad E_4 = \{z=0\},$$

and given $\{i, j\} \in \Lambda_x$ let $i < j$ iff $i < j$ (at $\mathbb{Z}$).

Define also $T_{ij} = E_i \cap E_j$.

Example 1. — Let (J, b) be defined at W by $J = \langle x(x-1)z \rangle$ and $b=2$. Then $\text{Sing}^{(b)}(J) = T_{14} \cup T_{24}$, T is maximal along $\text{Sing}^b(J)$ and

$$\max T = \{ (0, -2, 1, (1, 4, \infty)); (0, -2, 1, (2, 4, \infty)) \}.$$

Example 2:

$$J = \langle x(x-1) \cdot y \cdot z \rangle, \quad b=2.$$

$$\text{Sing}^b J = T_{14} \cup T_{24} \cup T_{34} \cup T_{13} \cup T_{23}$$

in this case $\max T = \{ (0, -2, 1, (3, 4, \infty)) \}$ is reached exactly along T_{34} .

Remark 2.3.1. — One can check that T is upper semicontinuous, moreover for a fixed $d \in \mathbb{Z}^3 \times \Lambda^m$ the condition $T > d$ is closed at $\text{Sing} \chi$.

Recall now from Def. 1.17.4 the notion of total order at Λ_x after a permissible transformation and check that $T = \varphi$ satisfies all conditions of 2.2.

2.3.2. Case of $\dim \chi = m$ and $w\text{-ord}(\chi) > 0$. Consider χ together with a fixed sequence

$$\chi^{(-r)} \xleftarrow{\pi-r} \chi^{(-r+1)} \leftarrow \dots \chi^{(-1)} \xleftarrow{\pi-1} \chi^{(0)} = \chi$$

in the conditions of the sequence (2.1.1) of Def. 2.1, so that $\chi^{(-r)}$ is the birth of χ and $E_\Lambda = E_\Lambda^+ \cup E_\Lambda^-$ ($E_\Lambda(\alpha) = E_\Lambda^+(\alpha) + E_\Lambda^-(\alpha)$, $\forall \alpha \in I$) are defined.

Now let $T: w\text{-Sing} \chi \rightarrow \mathbb{Z}^3 \times \Lambda^m$ be defined for each $x \in w\text{-Sing} \chi$ by:

$$T(1)(x) = w\text{-ord}(\chi) \quad (\text{Def. 1.9.1})$$

$$T(2)(x) = \begin{cases} 0 & \text{if } x \in R(\tau)(w(\chi)) \quad (1.19.2 \text{ and } 1.19.4) \\ 1 & \text{if } x \notin R(\tau)(w(\chi)) \end{cases}$$

OBSERVATION 2.3.2. — $R(\tau)(w\chi)$ is a “component” of $w\text{-sing} \chi$ (Remark 1.19.2), this fact can be checked at any $\text{Sing}(w(\mathcal{A} \alpha)) \subset W(\alpha)$ ($\alpha \in I$). Moreover the definitions of $\tau(\chi)$ (Def. 2.1) together with Proposition 1.16.4 and 1.19.3 assert that a point $x \in R(\tau)(w(\chi))$ if and only if the final imagen of such point at $\chi^{(-r)}$ is a point of $R(\tau)(w(\chi^{(-r)}))$.

Now define:

$$n(x) = \# \{ \alpha \in \Lambda_x \mid E_\alpha \in E_\Lambda^- \}$$

$$m(x) = \# \{ \alpha \in \Lambda_x \mid E_\alpha \in E_\Lambda^- \text{ and } w\text{-Sing}(\chi) \notin E_\alpha \text{ locally at } x \}$$

and finally

$$T(3)(x) = \begin{cases} n(x) & \text{if } x \notin R(\tau) \\ m(x) & \text{if } x \in R(\tau) \end{cases}$$

And $T(4)(x) = \infty \in \Lambda^m$.

The function T_1 takes values at $\mathbb{Q}$, but since we assume that χ is quasi-compact there is $n \in \mathbb{Z}$ such that $\text{Img } T_1 \subset 1/n \mathbb{Z} \subset \mathbb{Q}$, and $1/n \mathbb{Z} \simeq \mathbb{Z}$ as ordered sets.

Remark 2.3.2. — The fact T is well defined follows from the notion of equivalence of points at weighted idealistic situations (Def. 1.17.3) and Theorems 1.13.1 and 1.13.2.

OBSERVATION 2.3.3. — If $\dim \chi = m = 1$ (Def. 1.18.2) then $T = \varphi$ satisfies all conditions of 2.2.

Remark 2.3.4. — If $w\text{-ord } \chi > 0$ then T reaches a *unique* maximal value along $w\text{-Sing}(\chi)$. And for a fixed element $d \in \mathbb{Z} \times \Lambda^m$ both $\{x \in w\text{-Sing } \chi \mid T(X) \geq d\}$ and $\{x \in w\text{-Sing } \chi \mid T(X) > d\}$ are closed subsets (Def. 1.18.2) included in $w\text{-Sing } \chi$. In fact the values of T are taken in the totally ordered discrete subset $\mathbb{Z}^3 \times \infty (\subset \mathbb{Z}^3 \times \Lambda^m)$.

DEFINITION 2.4. — A *preparation procedure* of an idealistic space χ of weighted order bigger than zero, consists of a sequence of permissible transformation

$$\chi \xleftarrow{\pi_1} \chi_1 \dots \xleftarrow{\pi_s} \chi_s \xleftarrow{\pi_{s+1}} \chi_{s+1}$$

such that $w\text{-ord } \chi = w\text{-ord } \chi_s$ and either $w\text{-ord } \chi_{s+1} < w\text{-ord } \chi_s$ or, if $w\text{-ord } \chi_{s+1} = w\text{-ord } \chi_s$ then $T(3)(x) = 0, \forall x \in w\text{-Sing}(\chi_{s+1})$.

DEFINITION 2.5. — Let

$$\beta: \chi^{(-r)} \xleftarrow{\pi_{-r}} \chi^{(-r+1)} \leftarrow \dots \leftarrow \chi^{(0)} = \chi$$

be as in 2.3.2, i. e. $\chi^{(-r)}$ is the birth of χ (Def. 2.1), and let $\pi: \chi \rightarrow \chi^{(-r)}$ denote the composition of the intermediate transformation. Then given $x \in w\text{-Sing}(\chi)$ we define the *birth of x* to be the point $\pi(x) \in w\text{-Sing}(\chi^{(-r)})$.

2.6. Here we define a notion of an *inductive procedure*. Let the assumptions and notation be as in Def. 2.5. Assume also that $T(3)(x) = 0, \forall x \in w\text{-Sing}(\chi)$, and that this condition does not hold at $\chi^{(-1)}$.

Now fix $x \in w\text{-Sing}(\chi)$ and let $y \in w\text{-Sing}(\chi^{(-r)})$ denote the birth of x . $\chi^{(-r)}: I \rightarrow C(m, \Lambda)$. Choose $\alpha \in I$ such that

$$y \in w\text{-Sing}(\mathcal{A}^{(-r)}(\alpha)) \subset W^{(-r)}(\alpha).$$

Now $w\text{-Sing}(\mathcal{A}^{(-r)}(\alpha)) = \text{Sing}(w(\mathcal{A}^{(-r)}(\alpha)))$ (Remark 1.17.6), and $\text{ord}(w(\mathcal{A}^{(-r)}(\alpha))) = 1$ (Remark 1.17.7).

So Theorem 1.16.1 asserts that there is a regular hypersurface H , such that $y \in H \subset W^{(-r)}(\alpha)$, having maximal contact with $W(\mathcal{A}^{(-r)}(\alpha))$ locally at y .

After a convenient restriction assume that H has maximal contact with $W(\mathcal{A}^{(-r)}(\alpha))$.

The sequence of permissible transformations $\beta : \chi^{(-r)} \leftarrow \dots \leftarrow \chi^{(0)}$ gives rise to:

(1) a sequence of w -permissible transformations over

$(W^{(-r)}(\alpha), \mathcal{A}^{(-r)}(\alpha), E_\Lambda(-r), A_\Lambda(-r))$ (Def. 1.17.4):

$$\begin{aligned} (W^{(-r)}(\alpha), \mathcal{A}^{(-r)}(\alpha), E_{\Lambda^{(-r)}(\alpha)}, A_{\Lambda^{(-r)}(\alpha)}) &\leftarrow \dots \\ &\leftarrow (W^{(0)}(\alpha), \mathcal{A}^{(0)}(\alpha), E_{\Lambda^{(0)}(\alpha)}(0), A_{\Lambda^{(0)}(\alpha)}(0)). \end{aligned}$$

(2) a sequence of permissible transformations over

$$(W^{(-r)}(\alpha), w(\mathcal{A}^{(-r)}(\alpha)), E_{\Lambda^{(-r)}(\alpha)}) \quad (\text{Def. 1.8}).$$

Since $\text{ord}(w(\mathcal{A}^{(-r)}(\alpha))) = 1$ (Remark 1.17.7), it can be interpreted as a sequence of w -permissible transformations (see Remark 1.17.8).

$(W^{(-r)}(\alpha), w(\mathcal{A}^{(-r)}(\alpha)), E_{\Lambda^{(-r)}(\alpha)}, \bar{A}_{\Lambda^{(-r)}(\alpha)}) \leftarrow \dots$

$$\leftarrow (W^{(0)}(\alpha), w(\mathcal{A}^{(0)}(\alpha)), E_{\Lambda^{(0)}(\alpha)}, \bar{A}_{\Lambda^{(0)}(\alpha)}).$$

Let H_1 denote the final strict transform of $H (\subset W^{(-r)}(\alpha))$ at $W^{(0)}(\alpha)$, and let $E_{\Lambda^{(0)}(\alpha)} = E_{\Lambda^{(0)}(\alpha)}^+ \cup E_{\Lambda^{(0)}(\alpha)}^-$ be as in 2.3.2.

Now we consider two cases

2.6 (a) Case $T(2)(y) = 1$. In this case, $y \notin R(\tau(w(\mathcal{A}^{(-r)})))$. Since $R(\tau(w(\mathcal{A}^{(-r)})))$ is a connected component of $w\text{-Sing}(\mathcal{A}^{(-r)}) = \text{Sing}(w(\mathcal{A}^{(-r)}))$ (Proposition 1.16.4), we may assume after shrinking that $R(\tau(w(\mathcal{A}^{(-r)}))) = \emptyset$ (at $W^{(-r)}(\alpha)$).

Now one can check at $W^{(0)}(\alpha)$ that $\bar{E}_\lambda = E_\lambda \cap H_1$ is empty or a smooth hypersurface for $E_\lambda \in E_{\Lambda^{(0)}(\alpha)}^+$, and $\bar{E}_\lambda = \emptyset$ if $E_\lambda \in E_{\Lambda^{(0)}(\alpha)}^-$ [at least locally at $w\text{-Sing}(\chi)$].

Let $\bar{E}_\Lambda = \{\bar{E}_\lambda \mid \lambda \in \Lambda\}$, then the inclusion $H \subset W^{(0)}(\alpha)$ and $(H_1, \bar{E}_\Lambda), (W^{(0)}(\alpha), E_\Lambda)$ satisfy the condition 1.11.1.

On the other hand H_1 has maximal contact with $w(\mathcal{A}^{(0)}(\alpha))$ at $W^{(0)}(\alpha)$. One can check that the conditions are given for Theorem 1.15, (b) to hold, so that there is an idealistic situation (Def. 1.8) $(H_1, \mathcal{B}, \bar{E}_\Lambda)$ such that $i : H_1 \hookrightarrow W^{(0)}(\alpha)$ is a strong immersion from $(H_1, \mathcal{B}, \bar{E}_\Lambda)$ to $(W^{(0)}(\alpha), w(\mathcal{A}^{(0)}(\alpha)), E_\Lambda)$ (Def. 1.11).

$\mathcal{B}$ might have order bigger than $1 = \text{ord}(w(\mathcal{A}^{(0)}(\alpha)))$ (Remark 1.17.7). We define the weighted idealistic situation $(H_1, \mathcal{B}, \bar{E}_\Lambda, \bar{A}_\Lambda)$ where $\bar{A}_\Lambda = \{\alpha(\lambda) \mid \lambda \in \Lambda\}$ such that $\alpha(\lambda)(x) = 0, \forall x \in \bar{E}_\lambda (\forall \bar{E}_\lambda \in \bar{E}_\Lambda)$.

Arguing as before at each point y , we construct a restriction of $w(\chi)$ to an $m-1$ dimensional idealistic space $\bar{\chi}^{(0)}$ (Def. 1.18.3). Theorem 1.12 asserts that $\bar{\chi}^{(0)}$ is quasi-compact (Def. 1.19.1). And $\text{Sing } \bar{\chi}^{(0)} = (\text{Sing } w(\chi^{(0)}) - R(\tau(w(\chi^{(0)})))$ which consists of "connected components" of $\text{Sing } w(\chi^{(0)})$ (Remark 1.19.2).

In this case we define the restriction of $w(\chi^0)$ to be $\bar{\chi}^{(0)}$.

2.6 (b) Case $T(2)(y) = 0$ i.e. $y \in R(\tau(w(\mathcal{A}^{(-r)})))$.

After a convenient restriction we may assume that $R(\tau(w(\mathcal{A}^{(-r)}))) = \text{Sing}(w(\mathcal{A}^{(-r)}))$ (Def. 1.19.3).

Let α and $H \subset W^{(-r)}(\alpha)$ be as before. Since H has maximal contact with $w(\mathcal{A}^{(-r)}(\alpha))$, apply Theorem 1.15 case (b) if possible (see Remark I below) and let $(H, \mathcal{B}, E_\emptyset, A_\emptyset)$ induce a strong immersion with $(W^{(-r)}(\alpha), w(\mathcal{A}^{(-r)}(\alpha)), E_\emptyset, A_\emptyset)$ (we do not assume that $E_\Lambda^{(-r)} = \emptyset$ at $\chi^{(-r)}(\alpha)$).

One can check that, by this procedure an $m-1$ dimensional idealistic space $\bar{\chi}^{(-r)}$ has been defined such that:

- (i) $\bar{\chi}^{(-r)}$ is quasi-compact
- (ii) $\text{Sing } \bar{\chi}^{(-r)} = \text{Sing } w(\chi^{(-r)}) = w\text{-Sing}(\chi^{(-r)})$
- (iii) The permissible sequence $\beta : \chi^{(-r)} \leftarrow \dots \leftarrow \chi$ induces a permissible sequence

$$\bar{\beta} : \bar{\chi}^{(-r)} \leftarrow \dots \leftarrow \bar{\chi}^{(0)}.$$

(iv) $\text{Sing } \bar{\chi}^{(j)} = \text{Sing } w(\chi^{(j)})$, $j = -r, \dots, 0$.

(v) $w(\chi^{(0)})$ is restrictive to $\bar{\chi}^{(0)}$ (Def. 1.18.3).

In this case we define the restriction of $w(\chi^{(0)})$ to be $\bar{\chi}^{(0)}$ (with birth $\bar{\chi}^{(-r)}$).

Remark 2.6.1. — Let $\bar{\chi}^{(0)}$ be the restriction of $w(\chi^{(0)})$ as in 2.6 (a) or 2.6 (b), then:

(i) $\text{Sing}(\bar{\chi}^{(0)}) = w\text{-Sing}(\chi)$ (disregarding eventually connected components of the second term).

(ii) the function $T : w\text{-Sing}(\chi) \rightarrow \mathbb{Z}^3 \times \Lambda^m$; is constant along $\text{Sing}(\bar{\chi}^{(0)})$

Remark I. — The procedure of 2.6 is not defined at x if and only if

- (i) $\tau(\chi^{(-r)}) = 1$
- (ii) $T(2)(y) (= T(2)(x)) = 0$

since, in that case and only in that case Theorem 1.15 b) does not apply.

2.7

2.7.1. Before going into the development of this section we sketch the strategy to follow in a simplified form.

So we start with a pair (J, b) and $E = \{E_1, \dots, E_n\}$ hypersurfaces with only normal crossings in a regular scheme W of dimension m (as in § 1). Recall that if χ is the induced idealistic space, then permissible transformations over χ correspond to w -permissible transformations over $(J, b), E$ (Def. 1.18.2). Say

$$\begin{array}{ccccccc} \chi & & \chi_1 & \dots & \dots & \dots & \chi_r \\ (J, b) \leftarrow (J_1, b) & \dots & \dots & \dots & \dots & \dots & \leftarrow (J_r, b) \\ E & & E_1 & & & & E_r \end{array}$$

where: (i) (J_i, b) is the transform of (J_{i-1}, b) (Def. 1.3).

(ii) $J_i = MJ^{(i)}$, M a monomial (Def. 1.17.1).

(iii) $w\text{-ord}(J) \geq \dots \geq w\text{-ord}(J_r)$ (Remark 1.17.6 (iii)).

(iv) $w\text{-Sing } \chi_i = \text{Sing}(w - \chi_i) = \text{Sing } w(J_i, b)$ [$w(J_i, b)$ as in 1.17.6].

The notion of birth of χ_r (and of $E_r = E_r^- \cup E_r^+$) of Def. 2.1 corresponding to the smallest index k for which $w\text{-ord}((J_k, b)) = w\text{-ord}((J_r, b))$.

If the weighted order of (J_r, b) is zero *i. e.* if J_r is locally a monomial, the resolution of (J_r, b) will follow easily. So assume that $w\text{-ord}(J_r, b) > 0$ (as in 2.3.2).

For further simplification we restrict our attention to the functions on $w\text{-Sing } \chi_r$ defined by $T(1)$ [constantly equal to $w\text{-ord}(J_r, b)$] and $T(3)$, $T(3)(x) = n(x)$ (as in 2.3.2).

These two functions turn out to be substantial for this procedure of resolution.

In 2.7.2 we study the maximal value of this function (in a lexicographic sense) along $w\text{-Sing}(\chi_r)$, say $\text{Max } T_r = (\omega, n)$. We set

$$\text{Max } T_r = \{x \in w\text{-Sing}(\chi_r) / T(x) = (\omega, n)\}.$$

Fix $x \in \text{Max}(T_r)$, then $n(x) = n$, and say $\{E_1, \dots, E_n\} = \{E_i \in E_r^- / x \in E_i\}$, E_i locally defined by $x_i = 0$.

Then $\text{Max } T_r$ is the singular locus of a new pair of order 1 (Def. 1.2), say $T_r(J_r, b)$, where:

$$T_r(J_r, b) \sim w(J_r, b) \cap (\langle x_1 \rangle, 1) \cap \dots \cap (\langle x_n \rangle, 1)$$

or equivalently, if $w(J_r, b) = (\mathcal{A}, d)$

$$T_r(J_r, b) \sim (\mathcal{A} + (x_1^d) + \dots + (x_n^d), d)$$

[$\sim$: isomorphic in the sense of idealistic situations (Def. 1.9)].

If $n=0$, in 2.6 we showed that the problem of resolution of $\omega(J_r, b)$ (the problem of "lowering" the weighted order), is a problem of resolution of an idealistic space of dimension smaller than m .

n is to be thought of as an obstruction in this sense.

The main results in this section are: [see conditions (1), (2), (3) and (4) of 2.7.3 for precise statements].

(a) The lowering of n [or of $\omega = w\text{-ord}$ of (J_r, b)], is equivalent to the resolution of the pair $T_r(J_r, b)$.

(b) The problem of resolution of $T_r(J_r, b)$ is a problem of resolution of idealistic spaces of dimension smaller than m .

Of course the number n , or any $n(x)$ is bounded by m . There cannot be more than m -hypersurfaces with normal crossings at $x \in W$.

2.7.2. Consider a sequence

$$\beta : \chi^{(-r)} \xleftarrow{\pi_{-r}} \chi^{(-r+1)} \leftarrow \dots \chi^{(-1)} \xleftarrow{\pi_{-1}} \chi^0 = \chi$$

of permissible transformations over an m -dimensional idealistic space $\chi^{(-r)} : I \rightarrow C(m, \Lambda)$ such that

$$w\text{-ord}(\chi^{(-r)}) = w\text{-ord}(\chi) > 0.$$

We assume, inductively on r , that each π_j is a permissible transformation with center C_j , uniquely determined by an upper semicontinuous function on the “closed” sets $w\text{-Sing}(\chi^j)$.

In 2.3.2 we have constructed a function T on each $w\text{-Sing}(\chi^{(j)})$ which is upper semicontinuous. Now define for each such $T : \text{Max}(T(\chi^{(j)}))$ or simply.

$\text{Max}(T) = \text{maximum value of } T \text{ at } w\text{-Sing}(\chi^{(j)}), \text{ and}$

$\underline{\text{Max}}(T) = \{x \in w\text{-Sing}(\chi^{(j)}) \mid T(x) = \text{Max } T\}$

(see Remark 2.3.4).

Assume that the following conditions hold:

- (i) $C_j \subset \underline{\text{Max}} T \subset w\text{-Sing} \chi^{(j)}$
- (ii) for any $x \in w\text{-Sing}(\chi^{(j+1)})$; $T(\pi_j(x)) \geq T(x)$.

DEFINITION 2.7.2. — When these conditions hold then for each $x \in \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi)$ we define:

1. $m\text{-Sing}(x) = T(x) (= \text{Max}(T))$.
2. the m -birth of x as the image y of x by the natural map $\pi : \chi \rightarrow \chi^{(-j)}$ where $-j$ is the smallest index for which $T(x) = \text{Max}(T(\chi^{(-j)}))$.

Remark. — Given x as before, let y be the m -Sing birth of x and z the birth of x (Def. 2.5). Then z is also the birth of y .

2.7.3. In 2.6 we studied a sequence β (as before) such that $w\text{-ord}(\chi^{(-r)}) = w\text{-ord}(\chi) > 0$ and the additional hypothesis that $T(3)(x) = 0, \forall x \in w\text{-Sing}(\chi)$. In this section we consider the case that $\text{Max } T = (d_1, d_2, d_3, \infty)$ ($T : w\text{-Sing}(\chi) \rightarrow \mathbb{Z}^3 \times \Lambda^m$) where $d_3 > 0$ and we want to construct now a preparation procedure (Def. 2.4).

Let $-j$ and y be as before and $F^{(-j)} = \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi^{(-j)})$, let z denote the birth of y and let $\alpha \in I$ be such that $z \in w\text{-Sing}(\mathcal{A}^{(-r)}(\alpha)) \subset W^{(-r)}(\alpha)$ where $\chi^{(-r)}(\alpha) = (W^{(-r)}(\alpha), \mathcal{A}^{(-r)}(\alpha), E_{\Lambda^{(-r)}(\alpha)}, A_{\Lambda^{(-r)}(\alpha)})$.

Now $w\text{-Sing}(\mathcal{A}^{(-r)}(\alpha)) = \text{Sing}(w(\mathcal{A}^{(-r)}(\alpha)))$ and $\text{ord}(w(\mathcal{A}^{(-r)}(\alpha))) = 1$ (Remark 1.17.1). Again by theorem 1.16.1 there is a smooth hypersurface $H^{(-r)} \subset W^{(-r)}(\alpha)$ such that $z \in H^{(-r)}$ and $H^{(-r)}$ has maximal contact with $w(\mathcal{A}^{(-r)}(\alpha))$ [after shrinking $W^{(-r)}(\alpha)$].

If $H^{(-j)}$ denotes the strict transform of $H^{(-r)}$ at $W^{(-j)}(\alpha)$ by the maps induced over $W^{(-r)}(\alpha)$, then $y \in H^{(-j)}$ and $H^{(-j)}$ has maximal contact with $w(\mathcal{A}^{(-j)}(\alpha))$ (which is the transform of the idealistic exponent $w(\mathcal{A}^{(-r)}(\alpha))$ at $W^{(-j)}(\alpha)$ [Remark 1.17.6 (iii)]. Recall (as in 2.6) that $H^{(-j)}$ has normal crossings with $E_{\Lambda^{(j)}(\alpha)}^+$ (2.1). If $w(\mathcal{A}^{(-j)}(\alpha))$ is defined locally at y by a pair (J, b) , then consider the idealistic exponent

$$K = ((J + \sum_{y \in E_s \in \Gamma} P_s^b, b)), \quad \Gamma = (E_{\Lambda^{(j)}(\alpha)})^-$$

(2.1) where $P_s \subset \mathcal{O}_{W^{(j)}(\alpha)}$ is the sheaf of ideals $\mathcal{O}(-E_s)$.

One can check that:

- (a) $\text{Sing } K = F^{(-j)}$ (locally at y).
- (b) K is well defined independently of the election of (J, b) .

Remark. — Assume that $T(2)(y) = (T(2)(z)) = 0$ then

$$(J + \sum_{y \in E_s \in \Gamma} P_s^b, b) \sim (J + \sum_{y \in E_t \in \Gamma'} P_t^b, b) \quad (\text{Def. 1.1})$$

where $\Gamma' = \{E_t \in (E_{\Lambda^{(j)}})^- \mid w\text{-Sing}(\chi^{(-j)}) \notin E_t\}$ (locally at y).

Since $H^{(j)}$ has maximal contact with $w(\mathcal{A}^{(-j)}(\alpha)) = ((J, b))$, then it also has maximal contact with K .

Now consider at $W^{(-j)}(\alpha)$ the weighted idealistic situation $(W^{(-j)}(\alpha), K, (E_{\Lambda^{(-j)}(\alpha)})^+, \bar{A}_{\Lambda^{(-j)}(\alpha)})$ where $(E_{\Lambda^{(-j)}(\alpha)})^+$ is as before and $\bar{A}_{\Lambda^{(-j)}(\alpha)}$ consists of functions $\alpha(\lambda) : E_\lambda \rightarrow \mathbb{Q}$, for each $E_\lambda \in (E_{\Lambda^{(-j)}(\alpha)})^+$ where $\alpha(\lambda)(x) = 0, \forall x \in E_\lambda$.

Now for each $E_\lambda \in (E_{\Lambda^{(j)}(\alpha)})^+$ let $\bar{E}_\lambda = E_\lambda \cap H^{(-j)}$ and define $E_{\bar{\Lambda}} = \{\bar{E}_\lambda \text{ (as before)}\}$ and $A_{\bar{\Lambda}} = \{\alpha(\lambda) : \bar{E}_\lambda \rightarrow \mathbb{Q} \text{ (}\bar{E}_\lambda \text{ as before)} \text{ such that } \alpha(\lambda)(x) = 0, \forall x \in \bar{E}_\lambda\}$.

$E_{\bar{\Lambda}}$ consists of hypersurfaces (at $H^{(-j)}$) with only normal crossings.

We claim that the conditions of Theorem 1.15 (b) are given (see Remark II below), so that there is an idealistic exponent $\mathcal{B}$ at $H^{(-j)}$ and a strong immersion

$$(H^{(-j)}, \mathcal{B}, E_{\bar{\Lambda}}) \hookrightarrow (W^{(-j)}(\alpha), K, (E_{\Lambda^{(-j)}(\alpha)})^+).$$

Arguing in the same way for all points $x \in \underline{\text{Max}}(T) \subset \chi^0 = \chi$ and all election of hypersurfaces $H^{(-r)}$, we construct an $m-1$ dimensional idealistic space $\bar{\chi}^{(-j)}$ which is quasi-compact and satisfies the following conditions:

- (1) $\text{Sing } \bar{\chi}^{(-j)} = \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi^{(-j)})$.
- (2) The permissible sequence

$$\chi^{(-j)} \xleftarrow{\pi_{-j}} \chi^{(-j+1)} \leftarrow \dots \xleftarrow{\pi_{-1}} \chi^{(0)} = \chi$$

induces a permissible sequence

$$(A) : \bar{\chi}^{(-j)} \leftarrow \bar{\chi}^{(-j+1)} \leftarrow \dots \leftarrow \bar{\chi}^{(0)}$$

over $\bar{\chi}^{(-j)}$ such that $\text{Sing}(\bar{\chi}^{(l)}) = \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi^{(l)})$ for all $l = -j, -j+1, \dots, 0$.

- (3) If $\bar{\chi}^{(-j)} \leftarrow \bar{\chi}^{(-j+1)} \leftarrow \dots \leftarrow \bar{\chi}^{(0)} \leftarrow \dots \leftarrow \bar{\chi}^{(k)}$ is a permissible sequence [extending that of (2)] then it induces a permissible sequence

$$(\chi^{(-r)} \dots \leftarrow) \chi^{(-j)} \leftarrow \dots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \dots \leftarrow \chi^{(k)}$$

at permissible centers $C_l (-r \leq l \leq k)$ such that (i) and (ii) of 2.7 hold. Moreover $\text{Sing } \bar{\chi}^{(l)} = \underline{\text{Max}}(T) \subset w\text{-Sing}(\chi^{(l)})$ $0 \leq l \leq k$ and

$$\text{Max}(T : w\text{-Sing}(\chi) \rightarrow \mathbb{Z}^3 \times \Lambda^m) > \text{Max}(T : w\text{-Sing}(\chi^k) \rightarrow \mathbb{Z}^3 \times \Lambda^m)$$

if and only if $\text{Sing } \bar{\chi}^{(k)} = \emptyset$.

- (4) Conversely, if $\chi^{(-r)} \leftarrow \dots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \dots \leftarrow \chi^{(k)}$ is an extension of $\chi^{(-r)} \leftarrow \dots \leftarrow \chi^0 = \chi$ by permissible transformations at centers

$$C_j \subset \underline{\text{Max}} T \subset w\text{-Sing}(\chi^{(j)}), \quad 0 \leq j \leq k$$

such that (i) and (ii) of 2.7 hold, and if

$$\text{Max}(\text{T} : w\text{-Sing}(\chi^{(k)}) \rightarrow \mathbb{Z}^3 \times \Lambda^m) = \text{Max}(\text{T} : w\text{-Sing}(\chi) \rightarrow \mathbb{Z}^3 \times \Lambda^m)$$

then it induces a sequence of permissible transformations

$$\bar{\chi}^{(-j)} \leftarrow \dots \leftarrow \bar{\chi}^{(0)} \leftarrow \bar{\chi}^{(1)} \leftarrow \dots \leftarrow \bar{\chi}^{(k)}$$

and $\text{Sing}(\bar{\chi}^{(l)}) = \underline{\text{Max}} \text{T} \subset w\text{-Sing} \chi^{(l)} \quad l=0, \dots, k$.

Remark II. — The construction of the restricted situation at y would not be possible if and only if:

- (1) $\tau(\chi^{(-r)}) = 1$
- (2) $\text{T}(2)(y) = 0$
- (3) $\text{T}(3)(y) = 0$

(see Remark I) but we assumed in the construction of 1.7.2 that $\text{T}(3)(y) \neq 0$.

2.8. Now let $D_m = \mathbb{Z}^3 \times \Lambda^m$, $J_m = D_m \times D_{m-1} \times \dots \times D_1$ and suppose that the theorem of constructive resolutions (2.2) holds in dimension smaller than m .

We assume that the sequence (A) is a constructive sequence, i.e. that there is a resolution

$$\chi^{(-j)} \leftarrow \chi^{(-j+1)} \leftarrow \dots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \dots \leftarrow \chi^{(l)}, \chi^{(0)} = \chi$$

together with functions $\psi_{m-1}^{(k)} : \text{Sing} \bar{\chi}^{(k)} \rightarrow J_{m-1}$, $-j \leq k < l$ satisfying the conditions at 2.2 (see observation 2.3.3). Recall that $\text{Sing}(\bar{\chi}^{(k)}) = \underline{\text{Max}}(\text{T}) \subset w\text{-Sing}(\chi^{(k)})$ where now:

$$\chi^{(-k)} \leftarrow \dots \leftarrow \chi^{(0)} \leftarrow \chi^{(1)} \leftarrow \dots \leftarrow \chi^{(l)}, \chi^{(0)} = \chi$$

is the permissible sequence constructed with these centers.

Moreover this maximum value of T along $w\text{-Sing}(\chi^{(-s)})$ is the same, say c , for all $-j \leq s \leq l$.

So if c_1 is the maximum of T along $w\text{-Sing}(\chi^{(l)})$ (assuming that the birth of $\chi^{(l)}$ is still $\chi^{(-k)}$), then $c_1 < c$. But this simply means that

$$\text{Max} \{ \text{T}(3)(x) \mid x \in w\text{-Sing}(\chi^{(l)}) \} < \text{Max} \{ \text{T}(3)(x) \mid x \in w\text{-Sing}(\chi^{(-k)}) \}$$

But $\text{T}(3)(x) \leq m = \dim \chi^{(l)}$ (Def. 1.18.2). So repeating this argument we are left in the situation at which either $w\text{-ord}(\chi^{(l)}) < w\text{-ord}(\chi^{(-k)})$ or $w\text{-ord}(\chi^{(l)}) = w\text{-ord}(\chi^{(-k)})$ and $\text{T}(3)(x) = 0, \forall x \in w\text{-Sing}(\chi^{(l)})$. In this way we have constructed a preparation procedure (Def. 2.4) and now the inductive procedure of 2.6 can be applied.

In either case at $F^{(s)} = \{ x \in w\text{-Sing}(\chi^{(s)}) \mid F(x) \text{ is maximum} \} = \underline{\text{Max}}(\text{T})$ define $\psi_m^{(k)}(x) = (\text{T}(x), \psi_{m-1}^{(k)}(x))$; this defines a map:

$$\psi_m^{(k)} : F^{(k)} \rightarrow D_m \times J_{m-1} (= J_m)$$

We are still left with the case (within $w\text{-ord}(\chi) > 0$) where:

$$\begin{aligned} T(2)(x) &= 0, & \forall x \in w\text{-Sing}(\chi) \\ T(3)(x) &= 0, & \forall x \in w\text{-Sing}(\chi) \end{aligned}$$

and $\tau(w(\chi)) = 1$.

In this case and only in this case, the procedure introduced before are of no use. But then $w\text{-Sing} \chi$ is regular at each point and $w\text{-Sing} \chi$ itself is a center of a permissible transformation and such transformation defines a resolution of $w(\chi)$. On the other hand the function $T : w\text{-Sing}(\chi) \rightarrow D_m$ is constant. So we define

$$\psi_m(x) = (T(x), \infty) \in D_m \times J_{m-1} = J_m$$

Finally, if $w\text{-ord}(\chi) = 0$ define

$$\psi_m : \text{Sing} \chi \rightarrow J_m$$

by

$$\psi_m(x) = (T(x), \infty)$$

(Remark 2.3.1 asserts that a resolution of χ can be “constructed”).

2.9. With the assumption of constructive resolutions of singularities for idealistic spaces of dimension smaller than m , we have produced in 2.8, for any m -dimensional idealistic space χ a unique resolution:

$$(A) \quad \begin{array}{ccccccc} & & & & \Pi_r & & \\ \chi_0 & \leftarrow & \chi_1 & \cdots & \leftarrow & \chi_r & \leftarrow \cdots \leftarrow \chi_n \\ Y_0 & & Y_1 & & & Y_r & \end{array}$$

where each $\chi_r \xleftarrow{\Pi_r} \chi_{r+1}$ is a permissible transformation with center $Y_r \subset \text{Sing} \chi_r$.

DEFINITION 2.9.1. — Given a point $x \in \text{Sing} \chi_r$, if $x \notin Y_r$ we identified it with a point $x \in \text{Sing} \chi_{r+1}$ in such a way that $\Pi_r : \text{Sing} \chi_{r+1} \rightarrow \text{Sing} \chi_r$ is locally an isomorphism (at x). Since (A) is finite there is a well defined number $r' \geq r$ which is maximal with the condition that $\Pi_{r'} : \text{Sing} \chi_{r'} \rightarrow \text{Sing} \chi_r$ (the composition of all intermediate maps) is an isomorphism locally at x . We say that “ $x \in \text{Sing} \chi_{r'}$ ”. In this case $x \in Y_{r'} \subset \text{Sing}(\chi_{r'})$, because of the maximality of r' , r' is called the *level* of x .

DEFINITION 2.9.2. — Given an upper semicontinuous function $h : F \rightarrow (D, \leq)$, if $(D, \leq)$ is totally ordered then set $\text{Max } h = \{\text{maximal value of } h\}$ (a unique element) and $\underline{\text{Max}} h = \{x \mid h(x) = \text{Max}(h)\}$. If D is not totally ordered, then $\text{Max } h$ might consist of more than one element. We will assume moreover that for each $x \in F$, there is a totally ordered subset $(D_x, <) \subset (D, <)$ and that $\text{Im } g(h) \subset D_x$ locally at x .

Examples of these maps are given by

$T : \text{Sing } w(\chi) \rightarrow D$ as pointed out in 2.3.1 and 2.3.2.

Now $\underline{\text{Max}} h$ becomes a disjoint union of closed sets

$$\underline{\text{Max}} h = \bigcup_{d \in \underline{\text{Max}} h} \underline{\text{Max}} (h)(d), \quad \underline{\text{Max}} (h)(d) = \{x \mid h(x) = d\}$$

LEMMA 2.9.3. — Suppose we are given the following data:

$$(B) \quad \begin{array}{ccccccc} & & \Pi_0 & & \pi_1 & & \Pi_j \\ \chi & \leftarrow & \chi_1 & \leftarrow & \chi_j & \leftarrow & \chi_n \\ Y_0 \subset F_0 & Y_1 \subset F_1 & Y_j \subset F_j & & & & \end{array}$$

and upper semicontinuous functions $h_r: F_r \rightarrow (D, \leq)$ such that:

- (i) the data (B) is a resolution of χ .
- (ii) $F_r \subset \text{Sing } \chi_r$ is closed, Y_r is the center of Π_r and $Y_r \subset \underline{\text{Max}} (h_r)$.
- (iii) if $x \in F_{r+1}$ and $\Pi(x) \in F_r$, then $h_{r+1}(x) \leq h_r(\Pi_r(x))$ and the equality holds if moreover $\Pi(x) \notin Y_r$.
- (iv) $\text{ST}(F_r) \subset F_{r+1}$ [$\text{ST}(F_r)$ strict transform of F_r], ($\text{ST}(F_r) = \emptyset$ if $Y_r = F_r$).
- (v) If $x \in Y_s$ ($s > r$) and $\Pi_r^s(x) \in Y_r$, then $h_s(x) \leq h_r(\Pi_r^s(x))$.
- (vi) If $s > r$, $\forall x \in F_s \exists d \in \underline{\text{Max}} h_r$ such that $h_s(x) \leq d$ and if equality holds then $\Pi_r^s(x) \in \underline{\text{Max}} h_r$ (Π_r^s : the composition of all intermediate maps).

Define now $H_r: \text{Sing } \chi_r \rightarrow (D, \leq)$ as follows: given $x \in \text{Sing } \chi_r$ let r' be the level of x (Def. 2.9.1) then $x \in Y_{r'}$ and we define $H_r(x) = h_{r'}(x)$. We claim that

- (a) If $x \in F_r$, $H_r(x) = h_r(x)$ i. e. H_r extends h_r .
- (b) $H_r(x) \leq H_{r-1}(\Pi(x))$ and equality holds if $\Pi(x) \notin Y_{r-1}$.
- (c) H_r is upper semicontinuous, $\text{Max } H_r = \text{Max } h_r$ and $\underline{\text{Max}} H_r = \underline{\text{Max}} h_r$.

Remark 2.9.3.1. — In the conditions of (vi), if $h_s(x) = d$ then $x \in \underline{\text{Max}} h_s$.

Proof (of the Lemma). — (a) Let $x \in F_r$ and r' be the level of x . We must prove that $h_r(x) = h_{r'}(x)$, this follows from (iv) and (iii).

(b) If $\Pi(x) \notin Y_{r-1}$, then level of x and $\Pi(x)$ coincide, so $H_{r-1}(\Pi(x)) = H_r(x)$. If $\Pi(x) \in Y_{r-1}$ then the level of $\Pi(x)$ is $r-1$ and $H_{r-1}(\Pi(x)) = h_{r-1}(\Pi(x))$. Let r' be the level of x , then $x \in Y_{r'}$ and clearly $\Pi_{r-1}^{r'}(x) = \Pi(x)$ so

$$H_r(x) = h_{r'}(x) \leq h_{r-1}(\Pi(x)) = H_{r-1}(\Pi(x))$$

[inequality due to (v)].

(c) Given $d \in D$, we define

$$U = \{x \in \text{Sing } \chi_r / H_r(x) \geq d\}$$

$$V = \bigcup_{(s, d') \in \Gamma} \Pi_r^s(F(s, d'))$$

$$\Gamma = \{(s, d') / d' \in \underline{\text{Max}} (h_s) \text{ and } d' \geq d \text{ and } s \geq r\},$$

$$F(s, d) = \underline{\text{Max}} (h_s)(d') = \{x \in \underline{\text{Max}} (h_s) / h_s(x) = d'\}.$$

We claim that $U=V$. In which case, since each Π_r^s is proper and the $F(s, d')$ are closed, U is a finite union of closed sets.

Fix $x \in U$, $H_r(x) = d' \geq d$ and let r' be the level of x . Then $x \in Y_{r'} (\subseteq \text{Max } h_{r'})$ so $d' \in \text{Max } h_{r'}$ and $d' \geq d$ i. e. $(r', d') \in \Gamma$, so $x \in \Pi_{r'}^s(F(r', d'))$ i. e. $x \in V$.

If $x \in V$ there is $y \in \text{Max}(h_s)(d') ((s, d') \in \Gamma)$ such that $\Pi_r^s(y) = x$, so $h_s(y) = d' \in \text{Max}(h_s)$ and $d' \geq d$.

Let s' be the level of y and r' the level of x . Clearly $s' \geq r'$, $\Pi_{r'}^{s'}(y) = x \in Y_{r'}$ and $y \in Y_{s'}$, so

$$H_r(x) = h_{r'}(x) \geq h_s(y) = h_{s'}(y) = d' \geq d$$

[inequality do to (v)] i. e. $x \in U$.

Let us show that $\text{Max } h_r = \text{Max } H_r$. First we prove that: $\forall d \in \text{Max } H_r, \exists d' \in \text{Max } h_r$ such that $d \leq d'$. In fact if $H_r(x) = d$ for some point $x \in \text{Sing } \chi_r$ of level r' , then $x \in Y_{r'} \subset F_{r'}$ and $h_{r'}(x) = d$. By (vi) there is $d' \in \text{Max}(h_{r'})$ such that $d \leq d'$. Since (a) is proved it follows that $\text{Max } h_r = \text{Max } H_r$. Again because of (a), $\text{Max } h_r \subseteq \text{Max } H_r$ and the equality is clear from (vi).

Remark 2.9.4. — Suppose that the sets F_r are replaced by $F^{(r)}$ satisfying:

(a) $\text{Max}(h_r) \subset F^{(r)} \subset F_r$ and $F^{(r)}$ is closed

(b) Condition (iv) of Lemma 2.9.3.

and (c) $h_r': F^{(r)} \rightarrow D$ are defined by restricting h_r to $F^{(r)}$.

With this conditions we assert that:

(1) the statement of the Lemma still holds.

(2) If H_r' is defined as in the Lemma then $H_r' = H_r$.

Proof of (1) is straightforward [see Remark 2.9.3.1 for (vi)] and (2) is due to the fact that the construction of H_r depends only on $h_s|_{Y_s}, \forall s \geq r$, and $Y_s \subset \text{Max } h_s \subset F^{(s)}$.

PROPOSITION 2.9.5. — Given the resolution (A) of 2.9, let F_r be defined as:

(A) $F_r = \text{Sing } w(\chi_r)$ if $w\text{-ord}(\chi_r) > 0$.

(B) $F_r = \text{Sing } \chi_r$ if $w\text{-ord}(\chi_r) = 0$

and set $T_r: F_r \rightarrow D$ as in 2.3.1 and 2.3.2, then all the conditions of Lemma 2.9.3 are satisfied.

Proof. — (i) and (ii) follow by construction.

(iv): If $w\text{-ord}(\chi_r) > 0$ and the strict transform of $F_r = \text{Sing } w(\chi_r)$ is non-empty, then the $w\text{-ord}(\chi_{r+1}) = w\text{-ord}(\chi_r)$ and $w(\chi_{r+1})$ is the transform of $w(\chi_r)$ (2.7). Now (iv) is clear in this case.

If $w\text{-ord}(\chi_r) = 0$ then $F_r = \text{Sing } \chi_r$, $w\text{-ord } \chi_{r+1} = 0$ and $F_{r+1} = \text{Sing } \chi_{r+1}$, so also in this case (iv) is clear.

(iii) We prove it by considering different cases:

(a) $w\text{-ord}(\chi_{r+1}) < w\text{-ord}(\chi_r)$. In this case it is clear that $w\text{-ord}(\chi_r) > 0$ and as discussed above [in the prove of (iv)], $F_r = \text{Sing } w(\chi_r)$ must be Y_r , (iii) is now obvious from these remarks.

(b) $w\text{-ord}(\chi_{r+1}) = w\text{-ord}(\chi_r) = \omega > 0$. The first coordinate of T_r is constant along F_r (equal to ω) and the same holds at F_{r+1} . The second coordinate is $T(2)$, the good behavior of this function is given by Prop. 1.16.4 which states that $T(2)(x) = T(2)(\Pi(x))$, $\forall x \in \text{Sing}(\chi_{r+1})$. So that we are left with proving (iii) by looking at the function $T(3)$, now the statement follows from the fact that E_{r+1}^- is the strict transform of E_r^- and by the construction of (A) in terms of T [condition (1) (2) (3) and (4) of 2.7.2].

(c) If $w\text{-ord}(\chi_{r+1}) = w\text{-ord}(\chi_r) = 0$ we refer to Remark 2.3.1.

(v) (a) $w\text{-ord}(\chi_s) < w\text{-ord}(\chi_r)$ there is nothing to prove. We must consider the cases.

(b) $w\text{-ord}(\chi_s) = w\text{-ord}(\chi_r) > 0$ and (c) $w\text{-ord}(\chi_s) = w\text{-ord}(\chi_r) = 0$ both undergo essentially the same proofs as those given above for (b) and (c) of (iii).

(vi): is clear from the construction of (A) in terms of T .

PROPOSITION 2.9.6. — *Let (A), F_r , Y_r be as in Prop. 2.9.5, if each F_r is replaced by $F^{(r)} = \underline{\text{Max}} T_r$, then the conditions of Remark 2.9.4 hold.*

Proof. — the non trivial point is to show that condition (iv) of Lemma 2.9.3 still holds i. e. $\text{ST}(F_r') \subset F_{r+1}'$.

If $w\text{-ord}(\chi_r) > 0$, there is an $n-1$ dimensional idealistic space $\bar{\chi}^{(l)}$ such that $\text{Sing}(\bar{\chi}^{(l)}) = \underline{\text{Max}}(T_r) (= F^{(r)})$, and if $\text{Max}(T_r) = d$ then the lowering of d is equivalent to

the resolution of $\bar{\chi}^{(l)}$ [conditions (1), (2), (3) and (4) of 2.7.2], so we look at $\chi_r \xleftarrow{\Pi} \chi_{r+1}$.

If $\text{Max} T_{r+1} < d$, Y_r must be $\text{Sing} \bar{\chi}^{(l)} (= F_r)$ and there is nothing to prove. If $\text{Max} T_{r+1} = d$ then $\underline{\text{Max}} T_{r+1}$ is the singular locus of $\bar{\chi}^{l+1}$ which is the transform of $\bar{\chi}^l$ by a permissible map $\bar{\chi}^l \leftarrow \bar{\chi}^{l+1}$, but then the $\text{ST}(\text{Sing} \bar{\chi}^l) \subset \text{Sing} \bar{\chi}^{l+1}$ as was to be shown.

If $w\text{-ord}(\chi_r) = 0$ then $F_r^{(r)}$ is the center i. e. $F^{(r)} = Y_r$ and there is nothing to prove.

2.9.7. In 2.8 we defined at $F^{(s)} = \underline{\text{Max}} T_s$ a function

$$\psi_m^s = F^{(s)} \rightarrow D = D_m \times J_m$$

in such a way that $p_1' \circ \psi_m^s = T_s$ (p_1' projection on D_m).

THEOREM 2.9.7. — *The data*

$$(A) \quad \begin{array}{ccccccc} \chi_0 & \leftarrow & \chi_1 & \leftarrow & \chi_j & \leftarrow & \chi_n \\ Y_0 \subset F^{(0)} & & Y_1 \subset F^{(1)} & & Y_j \subset F^{(j)} & & \end{array}$$

together with the functions $\psi_m^r: F^{(r)} \rightarrow D$ satisfies the conditions of Lemma 2.9.3. In particular there are, for each s , functions $\psi_m^s: \text{Sing} \chi_s \rightarrow D_m \times J_m$ making of (A) a constructive resolution in the sense of 2.2.

Proof. — After Prop 2.9.6, (i), (ii) and (iv) deserve no proof (vi) is clear from the construction of (A) [recall that $Y_s = \underline{\text{Max}} \psi_m^s$, and for $s > r$, x and d as in (vi) then $h_s(x) < d$].

(iii) (a) If $w\text{-ord}(\chi_r) = 0$, then ψ_m^r is basically T_r and again this case is in Prop. 2.9.6.

(b) If $w\text{-ord}(\chi_r) > 0$ and $\text{Max} T_r > \text{Max} T_{r+1}$, then $Y_r = \underline{\text{Max}} T_r (= F^{(r)})$ and the assertion is clear.

(c) If $\text{Max } T_r = \text{Max } T_{r+1}$, there is $\bar{\chi}^l$ (as in the proof of Prop 2.9.6) such that $F^{(r)} = \text{Sing}(\bar{\chi}^l)$, $F^{(r+1)} = \text{Sing}(\bar{\chi}^{l+1})$.

Now $T(x) = T(\Pi(x))$ so one must prove (iii) for ψ_{m-1} and now x and $\Pi(x)$ are singular points of an $m-1$ dimensional resolution.

But ψ_{m-1} is constructive and (iii) follows from (ii), of 2.2.

(v) Reduces immediatly to the case $T_s(x) = T_r(\Pi_r^s(x))$ and undergoes essentially the same argument of the proof of (c) given just above.

2.10

Remark 2.10.1. — Why $T(2)$?

As pointed out in 2.7, the role of $T(2)$ is not essential for our constructions *i.e.* we can define $T(2)(x) = 1$ whenever $T(1)(x) > 0$ without affecting the general strategy. However if we consider $(J, 1)$, E , $J = \langle x, y \rangle \subset \mathbb{C} \mid x, y, z \mid$, $E = \{E_1\}$, $E_1 = \{z=0\} \subset \mathbb{C}^3$, then one can check that the number of unnecessary quadratics transformations applied before solving the pair, will diminish if we do consider this function.

2.10.1. — At this point we give a punctual construction of the functions ψ_m defined at 2.8.

Let χ an idealistic space of dimension m , if $w\text{-ord } \chi = 0$ *i.e.* if χ is locally a monomial, ψ_m reduces to $T(2.3.1)$.

We consider therefore the case $w\text{-ord } \chi > 0$. In order to simplify set (J, b) as in paragraph 1 and (J_r, b) arising from $(J, b) \leftarrow (J_1, b) \dots \leftarrow (J_r, b) \dots \leftarrow (J_n, b)$ with the notations and assumptions of 2.7.1, where only the functions $T(1)$ and $T(3)$ were considered [*i.e.* $T(2)(x) = 1$ if $T(1)(x) > 0$].

So let (ω, n) be $\text{Max } T_r$, and $k \leq r$ be the smallest number for which $\text{Max } T_k = (\omega, n_0)$. Recall from 2.7.1 that $T_r(J_r, b)$ was an “ $m-1$ -dimension” idealistic pair such that $\text{Max } T_r = \text{Sing } T_r(J_r, b)$ and that

$$(J_k, b) \xleftarrow{\Pi_k} \dots \leftarrow (J_r, b)$$

induces a sequence of permissible maps:

$$T_k(J_k, b) \xleftarrow{\Pi_k} \dots \leftarrow T_r(J_r, b),$$

each $T_i(J_i, b)$ being the transform of $T_{i-1}(J_{i-1}, b)$ (Def. 1.3), for $i > k$.

Given $x \in \text{Sing}(J_p, b)$ we express $\psi_m^p(x)$ by three coordinates, the first two corresponding to T_p , the third to ψ_{m-1}^p . We begin by defining, inductively on p , sets $E_{x,p}^-$ as follows:

(i) if $\omega - v_x(J_p, b) < \omega - v_{\pi(x)}(J_{p-1}, b)$ ($\Pi = \Pi_{p-1}$) (Def 1.17.1), or if $p=0$:

$$E_{x,p}^- = \{E_i \in E_p / x \in E_i\}$$

(ii) if $\omega - v_x(J_p, b) = \omega - v_{\pi(x)}(J_{p-1}, b)$

$$E_{x,p}^- = \{ST(E_i) / E_i \in E_{p-1, \Pi(x)}^- \text{ and } x \in ST(E_i)\}$$

(as usual ST denotes the strict transform).

Now we claim that:

$$(a) T_p(1)(x) = \omega - v_x(J_p, b)$$

$$(b) T_p(3)(x) = E_{p,x}^-$$

(c) If $q (\leq p)$ is the smallest index for which $T_q(\Pi_q^p(x)) = T_p(x)$. Consider at a neighbourhood of $y = \Pi_q^p(x)$ the pair:

$$(\mathcal{A}, d) = w(J_{q,y}, b) \cap (x_1, 1) \cap (x_2, 1) \cap \dots \cap (x_h, 1)$$

[notation as in 2.7.1, where $h = T_q(3)(y)$ and $x_i = 0$ defines $E_i \in E_{q,y}^-$ locally at y]. Then the third coordinate is $\psi_{m-1}^t(x)$, $t = p - q$ and ψ_{m-1}^t arises from the constructive resolution of the $m - 1$ dimensional pair $(\mathcal{A}, d)$.

Let r denote the level of $x (r \geq p)$ (Def. 2.9.1) and recall the definition of $\psi_m^p(x)$ in terms of the level of x (2.9.3 and 2.9.7).

Point (a) is clear and (b) will follow by proving inductively on p , that:

$$(d) E_{x,p}^- = \{E_i \in E_r^- / x \in E_i\}.$$

In the case (i), either $p = 0$ or the weighted order of (J_r, b) is smaller than that of (J_{p-1}, b) and (d) follows in this case from the definition of E_r^- in terms of the weighted orders of the pairs (2.1).

In the case (ii), if s is the level of $\Pi(x)$, clearly $s \leq r$ and (with the identifications of Def. 2.9.1)

$$w\text{-ord}(J_s) = \omega - v_{\Pi(x)}(J_s, b) = \omega - v_x(J_r, b) = w\text{-ord}(J_r)$$

since $\Pi(x) \in Y_s \subset \underline{\text{Max}} \psi_m$ and $x \in Y_r \subset \underline{\text{Max}} \psi_m$. So (d) follows now from the relations between E_s^- and E_r^- given in 2.1.

Now that (d) is settled (for any p) we prove (c). So let $s (\geq q)$ be the level of y and r as before that of x . Clearly $s \leq r$. On the other hand $y \in Y_s \subset \underline{\text{Max}} T_s$ and $x \in Y_r \subset \underline{\text{Max}} T_r$, so:

$$\text{Max } T_s = T_s(y) = T_r(x) = \text{Max } T_r = (w, n_0).$$

In particular $k \leq s$ (k defined as above).

Consider the composition of the intermediate maps: Π_k^s and the point $z = \Pi_k^s(y)$. If the level of z is the level of y , Π_k^s is the identity map locally at y and (c) follows from (d) and the construction of $T_k(J_k, b)$ (2.7.1).

If Π_k^s would not be an isomorphism at y , since $\Pi_q^s = \text{id}$, then $k < q$ contradicting the minimality of q .

So if x is considered as a point of $\text{Sing}(J_r, b)$, the point $\Pi_k^r(x) \in \text{Sing}(J_k, b)$ (which is the m -birth of x Def. 2.7.2) has the same level as y .

Suppose now that the function T_p is replaced by $T_p(1)$ and q by $q_1 (\leq p)$: the smallest index for which $y_1 = T_p(1)(\Pi_{q_1}^p(x)) = T_p(1)(x)$. Then the same argument as above will show that the birth of $x \in \text{Sing}(J_r, b)$ (Def. 2.5) has the same level as y_1 . Therefore in

the construction of 2.7.3 the election of the hypersurface of maximal contact can be done locally at y_1 .

REFERENCES

- [1] S. S. ABHYANKAR and T. T. MOH, *Newton-Puiseux Expansion and Generalized Tschirnhausen Transformation* (*Crell Journal*, Vol. 260, 1973, pp. 47-83 and Vol. 261, 1973, pp. 29-54).
- [2] J. M. AROCA, H. HIRONAKA and J. L. VICENTE, *The Theory of Maximal Contact* (*Memo. Mat. del Inst. Jorge Juan*, Vol. 29, Madrid, 1975).
- [3] J. M. AROCA, H. HIRONAKA and J. L. VICENTE, *Desingularization Theorems* (*Memo. Mat. del Inst. Jorge Juan*, Vol. 30, Madrid, 1977).
- [4] B. M. BENNETT, *On the Characteristic Function of a Local Ring* (*Ann. Math.*, Vol. 91, 1970, pp. 25-87).
- [5] J. GIRAUD, *Sur la théorie du contact maximal* (*Math. Zeit.*, Vol. 137, 1974, pp. 285-310).
- [6] J. GIRAUD, *Remarks on Desingularization Problems* (*Nova acta Leopoldina*, NF 52, Nr. 240, 1981, pp. 103-107).
- [7] H. HIRONAKA, *Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero, I-II* (*Ann. Math.*, Vol. 79, 1964).
- [8] H. HIRONAKA, *Introduction to the Theory of Infinitely Near Singular Points* (*Memo Math. des Inst. Jorge Juan*, Vol. 28, Madrid, 1974).
- [9] H. HIRONAKA, *Idealistic Exponents of Singularity* (*Alg. Geom.*, *J. J. Sylvester Symp.*, John Hopkins Univ. Baltimore, Maryland 1976, John Hopkins Univ. Press 1977).

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