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HIGHER G-INDICES AND APPLICATIONS

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0. Introduction

Suppose a compact Lie group G acts on a closed, connected, smooth manifold M, in such a way that the classifying map $f: M \rightarrow B\pi$, where $\pi = \pi_1(M)$, can be made equivariant (for the trivial G-action on $B\pi$). This is equivalent to assuming $\pi_1(M) \rightarrow \pi_1(M/G)$ splits, and turns out to be automatic if the action is semifree with non-empty fixed set, and if G acts trivially on π. A G-invariant elliptic operator D on M defines a class $[D] \in K_*^G(M)$, and we call $f_*([D]) \in K_*^G(B\pi) = K_*(B\pi) \otimes_{\mathbb{Z}} R(G)$, or sometimes its Chern character in $H_*(\pi, \mathbb{C}) \otimes_{\mathbb{Z}} R(G)$, the *higher G-index* of D. It generalizes the ordinary G-index (which corresponds to the case $\pi = \{1\}$), and in section 1 below, we shall show that it can often be computed by methods arising out of Kasparov's work on the Novikov Conjecture [16]. Then in the rest of this paper, we shall study a number of applications of such higher G-indices, as well as a few closely related topics.

In our applications, D will always be either the Dirac operator on a spin manifold or the signature operator on an oriented manifold, though potentially our theory might also be useful when applied to the Dolbeault operator $(\bar{\partial} + \bar{\partial}^*)$ on a complex manifold. We begin in section 2 with the case of the Dirac operator. Browder and Hsiang [7] have already pointed out the vanishing of the higher G-$\hat{A}$-genus, which is the rational higher G-index of the Dirac operator, for actions of $G = S^1$ on spin manifolds. However, there is a more subtle analogue of this invariant, living in $KO_*^G(B\pi)$, which appears in the problem of trying to determine when a spin G-manifold has a G-invariant Riemannian metric of positive scalar curvature, a problem first studied by Bérard Bergery [5]. In Theorem 2.5, we show that this higher "G-$\hat{\mathscr{A}}$-genus" is sometimes an obstruction to existence of such a metric. On the other hand, in Theorem 2.3 and the examples of (2.7), we show how equivariant surgery can often be used to construct such metrics in the case where the obstruction vanishes. Most interesting is probably Example 2.7 (2), where we construct invariant metrics of positive scalar curvature on n-spheres, for $\mathbb{Z}_p$-actions with a *knotted* $(n-2)$-sphere as fixed set.

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In section 3, we consider the application of our methods to the higher G-signature (*i. e.*, the higher G-index of the signature operator). One general consequence (Theorem 3.8) is a proof of a version of what one can call the Equivariant Novikov Conjecture, which is the equivariant homotopy-invariance of the higher G-signature. When G is cyclic and acts trivially on cohomology (with local coefficients), we obtain a "higher" version of the Atiyah-Singer G-signature formula of [4], which relates the higher signature of M (in the sense of Novikov) to the "twisted" higher signature of the fixed set—*see* Theorem 3.1 below. Similar results, weaker in some respects but stronger in others, were previously found by the second-named author, using different methods ([31], [32]).

The original version of this paper also contained a number of applications of the Teleman signature operator on a Lipschitz manifold (*see* [28], [29], and [12]) and of its higher G-index theory. However, since this section eventually grew to be much longer and more complicated than the whole rest of the paper, we have decided to publish it separately.

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1. The higher G-index theorem

Let M^{2k} be a closed, connected, smooth even-dimensional manifold with fundamental group π on which a compact Lie group G acts by diffeomorphisms. We suppose that $D: \Gamma(E^0) \to \Gamma(E^1)$ is a G-invariant elliptic pseudodifferential operator between sections of two G-vector bundles E^0, E^1 over M, and wish to discuss the *higher G-index* of D, taking both the G-action and the fundamental group into account. For this to make sense, G must act trivially on π, in the sense that there should be a splitting to the map

$$\pi_1(M) \to \pi_1(M/G).$$

This, of course, is equivalent to the existence of a commutative diagram of G-maps

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & B\pi, \\ & M/G \arrow[ur] & \end{tikzcd} \tag{1.1}$$

where $B\pi$ is a $K(\pi, 1)$-space with *trivial* G-action and f is an isomorphism on π_1. This condition already made its appearance in [1], [31], and [32]; a few conditions that guarantee this are given by the following proposition. Note that if G acts freely, it no longer makes sense to say G acts trivially on $\pi_1(M)$ (since M has no G-invariant basepoint), but existence of a splitting (1.1) reduces to a simple algebraic condition on the exact sequence

$$\pi_1(G) \to \pi_1(M) \to \pi_1(M/G) \to \pi_0(G) \to 1.$$

1.2. Proposition. — *Suppose* G, *a compact Lie group, acts semi-freely on a connected, not necessarily compact manifold* M, *with nonempty fixed set* F *and* G *acting trivially on* $\pi_1(M)$. *If either* F *is connected or* $\pi_1(M)$ *has no subgroups that are non-trivial homomorphic images of* $\pi_0(G)$ *(this is no condition at all if* G *is connected), then* $\pi_1(M) \to \pi_1(M/G)$ *is an isomorphism.*

Proof. — Assume $F \neq M$ without loss of generality. It is obvious that $\pi_1(M) \to \pi_1(M/G)$ is surjective (choose a basepoint in F and note that one can lift paths from M/G to M), so it's enough to construct a splitting $\pi_1(M/G) \to \pi_1(M)$. Differentiation of the action at a point of F yields in the direction normal to F a semifree linear representation V of G, which is positive-dimensional. Now it is enough to prove the proposition assuming the codimension of F in M is at least 3, since otherwise we can replace M by $M \times V^2$ (in which the fixed set is $F \times \{(0,0)\}$, which is of codimension at least 3) and use the diagram

$$\begin{array}{ccc} M & \to & M \times V^2 \\ \downarrow & & \downarrow \\ M/G & \to & (M \times V^2)/G \end{array}$$

to obtain a splitting $\pi_1(M/G) \to \pi_1((M \times V^2)/G) \to \pi_1(M \times V^2) \cong \pi_1(M)$.

Assuming that $\operatorname{codim}(F \subset M) \geqq 3$, we observe that

$$\pi_1((M-F)/G) \cong \pi_1(M) \times \pi_0(G).$$

This can be seen as follows. By a general position argument (or direct calculation with Van Kampen's Theorem), $\pi_1(M-F) \to \pi_1(M)$ is an isomorphism. The action of G on $M-F$, which is assumed to be free, makes $M-F$ into a principal G-bundle over $(M-F)/G$, yielding an exact sequence

$$\pi_1(G) \to \pi_1(M-F) \to \pi_1((M-F)/G) \to \pi_0(G) \to 1.$$

The map $\pi_1(G) \to \pi_1(M-F)$ is trivial, since its composition with the isomorphism $\pi_1(M-F) \to \pi_1(M)$ is given by the inclusion into M of an orbit, and can be computed from the orbit of any point, in particular, from the trivial orbit of a fixed point. Let S be a G-invariant sphere in M linking a component of F, as provided by the equivariant tubular neighborhood theorem ([6], Theorem VI.2.2). There is a similar exact sequence computing $\pi_1(S/G) \cong \pi_0(G)$ (recall dim $S \geqq 2$), so that the inclusion $S/G \hookrightarrow (M-F)/G$ splits the above exact sequence. To deduce the splitting of $\pi_1(M-F) \to \pi_1((M-F)/G)$, it suffices to know that the action of $\pi_0(G)$ on $\pi_1(M-F)$ is trivial, which follows from the assumption that G acts trivially on $\pi_1(M) \cong \pi_1(M-F)$. [The two actions of $\pi_0(G)$ are easily identified using covering-space theory.]

Now we have to compute $\pi_1(M/G)$. Choose disjoint equivariant tubular neighborhoods N_i of the components F_i of F. By repeated application of Van Kampen's theorem,

$$\begin{aligned}\pi_1(M/G) &= \pi_1(((M-F)/G) \cup F_1 \cup F_2 \cup \dots) \\ &= \pi_1((M-F)/G) *_{\pi_1(\partial N_1/G)} \pi_1(F_1) *_{\pi_1(\partial N_2/G)} \pi_1(F_2) * \dots \\ &\cong (\pi_1(M) \times \pi_0(G)) *_{\pi_1(F_1)\times\pi_0(G)} \pi_1(F_1) *_{\pi_1(F_2)\times\pi_0(G)} \pi_1(F_2) * \dots\end{aligned}$$

If F is connected, we have only F_1 to deal with and we get

$$\pi_1(M/G) \cong (\pi_1(M) \times \pi_0(G)) *_{\pi_1(F_1)\times\pi_0(G)} \pi_1(F_1) \cong \pi_1(M).$$

If F is not connected, one must keep track of the way each $\pi_1(\partial N_i/G)$ is identified with $\pi_1(F_i) \times \pi_0(G)$, and unfortunately the various copies of $\pi_0(G)$ may not all coincide with the selected factor of $\pi_0(G)$ in $\pi_1((M-F)/G)$. However, if $\pi_1(M)$ is as in the hypothesis, this cannot cause trouble, and the proof is complete.

1.3. *Remarks.* — 1. All actions of the cyclic groups $\mathbb{Z}_p$ of prime order are semifree.

2. This proposition can be extended to many non-semifree group actions, if one has suitable hypotheses on fixed sets of subgroups. However, even in the semifree case, some hypotheses are needed to eliminate the (projectivized linear) involution on $\mathbb{RP}^2$ with fixed set $\mathbb{RP}^1 \amalg \mathbb{RP}^0$, which satisfies all the hypotheses of the proposition except for connectedness of the fixed set and lack of a homomorphism $\mathbb{Z}_2 \to \pi_1(\mathbb{RP}^2)$, yet has quotient the 2-disk.

3. *See* [7] and [32], Lemma 2, for an analogous statement when $G = S^1$.

1.4. Definition. — Assume D is as above and one has a diagram (1.1). The standard procedure, involving replacing D by an operator D′ of order 0 whose symbol is in the same K-theory class (*e.g.*, $D' = D(1+D^*D)^{-1/2}$) and considering D′ as a bounded G-equivariant Fredholm operator $L^2(E^0) \to L^2(E^1)$, associates to D a class

$$[D] \in K_0^G(M) = KK^G(C(M), \mathbb{C})$$

independent of the choices of D′ and of smooth measures ([14] or [10]). The *higher* G-*index* of D is defined to be $f_*([D]) \in K_0^G(B\pi)$. [Note that since $B\pi$ is not necessarily locally compact and is only defined up to homotopy, $K_0^G(B\pi)$ should be interpreted to mean

$$\varinjlim K_0^G(X) = (\varinjlim K_0(X)) \otimes_{\mathbb{Z}} R(G).$$

where the inductive limit is to be taken over finite subcomplexes X of $B\pi$.]

The name *higher* G-*index* is motivated by the case $G \cong \{1\}$, M oriented but not simply connected, and D the signature operator, in which case the Chern character of the higher index is

$$\mathrm{ch} f_*([D]) = 2^k f_*(\mathscr{L}(M) \cap [M]) \in H_*(B\pi, \mathbb{Q}),$$

which when paired with an element of $H^*(B\pi, \mathbb{Q})$ is one of the *higher signatures* of Novikov. Here $\mathscr{L}$ is the Atiyah-Singer modification of the Hirzebruch $\mathbb{L}$-class, differing from Hirzebruch's polynomial only by certain powers of 2 [4], § 6.

Next we need to recall a number of constructions from [15] (*see* also [15], [10], [11], [13], and [23]). Let $A = C^*(\pi)$ be the group C^*-algebra of π [in fact the reduced C^*-algebra $C_r^*(\pi)$ would work just as well and from some points of view is more satisfactory], and form an A-vector bundle $\mathscr{V}$ over M by

$$\tilde{M} \times_\pi C^*(\pi),$$

where $\tilde{M}$ is the universal cover of M and π acts on $C^*(\pi)$ by left translations. Since [by (1.1)] the classifying map $f: M \to B\pi$ for the principal π-bundle $\tilde{M} \to M$ is G-equivariant, the action of G on M lifts to an action on $\mathscr{V}$. Thus we may form the operator $D_{\mathscr{V}}$, D "with coefficients in $\mathscr{V}$", which is a G-invariant elliptic pseudodifferential A-operator in the sense of [21]; as such, it has an equivariant A-index in $K_0^G(A) = K_0(A) \otimes_{\mathbb{Z}} R(G)$.

1.5. Theorem (higher G-index theorem). — *Under the above hypotheses,*

$$\operatorname{ind}_A D_{\mathscr{V}} = [\mathscr{V}] \otimes_{C(M)} [D] = \beta(f_*([D])),$$

where $\otimes_{C(M)}$ *is the Kasparov pairing between* $[D] \in K_0^G(M)$ *and*

$$[\mathscr{V}] \in K_G^0(M; A) \cong K_0^G(C(M, A)),$$

with values in $K_0^G(A) = K_0(A) \otimes_{\mathbb{Z}} R(G)$, *and* β *is the Kasparov map* $K_*(B\pi) \to K_*(A)$, *extended to the equivariant case by tensoring with the identity map on* $R(G)$. (*This makes sense since* G *acts trivially on* $B\pi$ *and on* A.)

Proof. — The first identity is almost exactly the same as Theorem 3.1 of [23], the only difference being that everything is G-equivariant. The second identity follows as in the proof of Theorem 3.3 of [23], using functoriality of the Kasparov product and the fact that by construction, the bundle $\mathscr{V}$ is pulled back from the universal bundle $E\pi \times_\pi C^*(\pi)$ over $B\pi$, via the G-map f.

For purposes of applications to spin manifolds, we shall also need the analogue of Theorem 1.5 in real equivariant K-theory. In the only case of interest, M^n will be a manifold with a G-invariant Riemannian metric and spinor bundle, and D will be the Dirac operator. There is no longer any point in assuming n is even, since KO-index theory is interesting even in certain odd dimensions.

1.6. Theorem (higher G-$\mathscr{A}$ index theorem). — *Let* M^n *be a closed Riemannian spin* G-*manifold* (*with* G *preserving both the Riemannian metric and the spinor bundle*) *with* G-*equivariant classifying map* $f: M \to B\pi$ *as in* (1.1). *Let*

$$[D] \in K_n KO^G(C^{\mathbb{R}}(M), \mathbb{R}) = KO_n^G(M)$$

be the KO-*fundamental class defined by the real Dirac operator* D, *and* $\mathscr{V}$ *as above* (*now* *with* $A=C^*_{\mathbb{R}}(\pi)$ *as in* [39]). *Then*

$$\mathrm{ind}_A(D_{\mathscr{V}})=[\mathscr{V}]\otimes_{C^{\mathbb{R}}(M)}[D]=\beta(f_*([D]))\in KO^G_n(A),$$

where $\beta: KO_*(B\pi)\to KO_*(A)$ *is the Kasparov map in* KO-*theory as defined in* [25], *extended in the obvious way to the equivariant case* (*for trivial* G-*action*).

Caution. — For A a real C*-algebra with trivial G-action, it is not necessarily true that $KO^G_*(A)\cong KO_*(A)\otimes_{\mathbb{Z}} RO(G)$. *See* [27] or [3], § 8, for the correct substitutes and for calculations of $KO^G_n(pt)$.

Proof. — Again, except for insertions of the G's, one can carry over *verbatim* the discussion in [25], Theorems 3.3 and 3.4.

For the applications to higher G-indices, we really want an index theoretic interpretation for $f_*([D])$ and not for $\beta(f_*([D]))$. Thus it is important to know when β is injective (either integrally or after tensoring with $\mathbb{Q}$). As explained in [15], [16], [13], [10], [11], [23], and [25], the injectivity of β is intimately linked to the Novikov Conjecture and certain L-theoretic refinements. At least at present, it seems that for most groups for which the Novikov Conjecture can be proved by any method, one can also prove something about injectivity of β, which is what we want to exploit. We recall in particular the following two results. The proofs obviously carry over to K^G_* and KO^G_* (with G acting trivially on both sides).

1.7. THEOREM (Kasparov—[15] and [16]; *see* also [10], [11], and [25]). — *If* π *is the fundamental group of some complete Riemannian manifold of nonpositive sectional curvature* (*not necessarily compact or of finite volume*), *then* β *is a split injection*

$$K^G_*(B\pi)\to K^G_*(C^*(\pi)),\qquad KO^G_*(B\pi)\to KO^G_*(C^*_{\mathbb{R}}(\pi)).$$

1.8. THEOREM ([23], Theorem 2.6 and [25], Theorem 2.8). — *If* π *is a solvable group having a composition series for which the composition factors are torsion-free abelian, then* β *is an isomorphism* $K^G_*(B\pi)\to K^G_*(C^*(\pi))$ *and* $KO^G_*(B\pi)\to KO^G_*(C^*_{\mathbb{R}}(\pi))$.

Using Theorems 1.7 and 1.8, one may view Theorems 1.5 and 1.6 as giving explicit interpretations of the higher G-index $f_*([D])$ in terms of the A-index of a certain elliptic A-operator $[A=C^*(\pi)$ or $C^*_{\mathbb{R}}(\pi)]$, assuming π is reasonable and torsion-free. For a somewhat larger class of groups π (*see* [23], Proposition 2.7), we get such an interpretation for $f_*([D])$ viewed *rationally* in $H_*(B\pi, R(G)\otimes_{\mathbb{Z}}\mathbb{Q})=H_*(\pi,\mathbb{Q})\otimes_{\mathbb{Z}} R(G)$. In the remainder of this paper, we shall give some sample geometric applications of the index theorems. Undoubtedly there are others involving, say, the Dolbeault operator applied to holomorphic actions on complex manifolds.

Before we get to the applications, it is useful to give a homological formula for the (rational) higher G-index, which reduces to the formula of [4] in case π is trivial.

1.9. THEOREM. — *Let* G *be a compact Lie group acting smoothly on a closed, connected, smooth* G-*manifold* M^{2k}, *let* D *be a* G-*invariant elliptic pseudodifferential operator over* M, *and* $f: M\to X$ *any continuous* G-*map with* X *a trivial* G-*space* [*i.e., any continuous map factoring through* M/G; *in particular, one may take* $X=B\pi$ *in the situation of*

(1.1)]. *View the Chern character of* $f_*([\mathrm{D}])$ *as a map* $\mathrm{G} \to \mathrm{H}_*(\mathrm{X}, \mathbb{C})$, *by identifying elements of* $\mathrm{R}(\mathrm{G})$ *with their characters on* G. *Then for* $g \in \mathrm{G}$,

$$\mathrm{ch} f_*([\mathrm{D}])(g) = \sum_i (-1)^{n_i} f_* \left(\left\{ \frac{\mathrm{ch}\, \iota_i^*(u)(g)}{\mathrm{ch}\, \lambda_{-1}(\mathrm{N}_i^g \otimes_{\mathbb{R}} \mathbb{C})(g)} \mathscr{T}(\mathrm{M}_i^g) \right\} \cap [\mathrm{TM}_i^g] \right) \in \mathrm{H}_*(\mathrm{X}, \mathbb{C}),$$

where the sum is taken over the various components M_i^g *of the fixed set* M^g. *Here* $n_i = \dim(\mathrm{M}_i^g)$, N_i *denotes the normal bundle of* M_i^g *in* M, $\iota_i^*(u)$ *is the restriction of the symbol class of* D *to* TM_i^g, *and* $\mathscr{T}$ *denotes the Todd class of the complexified tangent bundle.*

Proof. — Without loss of generality, we may take X compact. Let E be any complex vector bundle over χ (with trivial G-action) and consider $\mathrm{D}_{f^*(\mathrm{E})}$, the operator D "with coefficients in $f^*(\mathrm{E})$," defined using a suitable connection on $f^*(\mathrm{E})$. Since $f^*(\mathrm{E})$ has trivial G-action ([4], Theorem 3.9) gives

$$(\mathrm{ind}\, \mathrm{D}_{f^*(\mathrm{E})})(g) = \sum_i (-1)^{n_i} \left\langle \frac{\mathrm{ch}\, \iota_i^*(u)\, \iota_i^*\, \mathrm{ch} f^*(\mathrm{E})}{\mathrm{ch}\, \lambda_{-1}(\mathrm{N}_i^g \otimes_{\mathbb{R}} \mathbb{C})(g)} \mathscr{T}(\mathrm{M}_i^g), [\mathrm{TM}_i^g] \right\rangle,$$

which says exactly that the two sides of our desired equality agree when paired with $(\mathrm{ch}\, \mathrm{E}) \in \mathrm{H}^*(\mathrm{X}, \mathbb{C})$. Since $\mathrm{ch}: \mathrm{K}^0(\mathrm{X}) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{H}^{\mathrm{even}}(\mathrm{X}, \mathbb{C})$ is an isomorphism, this proves the result.

2. The higher G-$\tilde{\mathscr{A}}$-genus and invariant metrics of positive scalar curvature

In this section, we shall apply the results of paragraph 1 in the case of a compact Lie group G acting on a closed spin manifold M. We assume the action of G lifts to an action on the spinor bundle, so that the theory of paragraph 1 applies to the Dirac operator. As explained in [2], Proposition 2.1, this is not much of a restriction if G is connected; it is also no restriction if G is finite of odd order.

We begin by showing how Theorem 1.9 gives a result of Browder and Hsiang [7]. We emphasize that this is not intended to be a particularly convincing application of our theory, since none of the deeper aspects of paragraph 1 (these include the Miščenko-Fomenko index theorem and Kasparov's results on the Novikov Conjecture) are involved here, and furthermore, we don't see any way to simplify the hardest aspect of [7], which is to get rid of the assumption of a splitting of the map on fundamental groups. Therefore, we only state the elementary case to point out the relation between our methods and their result.

2.1. Theorem (Browder-Hsiang [7]). — *Let* M *be a closed, connected spin manifold admitting a non-trivial action of the circle group* $\mathrm{G} = \mathrm{S}^1$, *and suppose a splitting* (1.1) *exists for the classifying map* $f: \mathrm{M} \to \mathrm{B}\pi$ *of the universal cover of* M. *Then the higher* $\hat{\mathrm{A}}$-*genus* $f_*(\hat{\mathbb{A}}(\mathrm{M}) \cap [\mathrm{M}]) \in \mathrm{H}_*(\mathrm{B}\pi, \mathbb{Q})$ *vanishes.* (*Here* $\hat{\mathbb{A}}$ *denotes the total* $\hat{\mathrm{A}}$-*class of* M.)

Proof. — Because of [2], Proposition 2.1, it is no loss of generality to suppose the action is of even type (*i.e.*, preserves the spin structure). Then specializing Theorem 1.9 in the case of a generator z_0 of S^1 with positive, very small imaginary part gives a formula for $\mathrm{ch}\, f_*([D])(z_0)$ (D the Dirac operator for an S^1-invariant metric on M) which is essentially identical to the one appearing in [2], § 1.5, except that it takes values in $H_*(B\pi, \mathbb{C})$. Hence the Atiyah-Hirzebruch proof goes through unchanged.

We turn now to a more serious application of our theory, to the following problem first studied in [5] by L. Bérard Bergery: Suppose M^n is a closed, connected, smooth manifold admitting both a metric of positive scalar curvature and an action of a compact Lie group G. When does M admit a *G-invariant* metric of positive scalar curvature?

In the case where G is finite and acts freely, this is equivalent to asking when M/G admits a metric of positive scalar curvature, a problem studied in detail in [24], § 3, and in [25], Theorem 1.3. On the other hand, when G contains a non-trivial connected semi-simple compact Lie group, this is always possible (a result of [18], quoted in [5]) and when $G = S^1$ acting freely, M admits a G-invariant metric of positive scalar curvature if and only if M/G admits a metric of positive scalar curvature [5], Theorem C. Hence we concentrate here on the case where G is finite or $G = S^1$, but where the action isn't free. (The case $G = S^1$, M 3-dimensional is completely settled in [5].)

Fortunately, the problem is not completely hopeless because of the following two positive results.

2.2. THEOREM (Bérard Bergery [5, Theorem 11.1]). — *If* G *acts on* M *preserving a metric of positive scalar curvature, and if* M′ *can be obtained from* M *by an equivariant surgery of codimension at least three, then* M′ *has a metric of positive scalar curvature invariant under its* G*-action.*

This enables one to reduce, for group actions "without codimension-two complications," the problem of invariant positive scalar curvature to the consideration of cobordism classes. To make matters simple we shall consider only $G = \mathbb{Z}_p$ (p a prime) and simply connected manifolds, although the general result (*see* Remark 2.4 below) can be proven in virtually the same way.

2.3. THEOREM. — *Assume* $\mathbb{Z}_p$ *acts smoothly on a simply connected manifold* M^n, *where* $n \geqq 5$, *preserving a spin structure, and such that no component of the fixed set* F *has codimension* 2. *If* M *is equivariantly spin cobordant to another* (*not necessarily connected*) *spin* $\mathbb{Z}_p$*-manifold* M′, *and if* M′ *has an invariant metric of positive scalar curvature, then so does* M.

Proof. — Consider an equivariant cobordism W in two stages (see *Fig.* 1). To begin with, W restricts to a cobordism of the fixed sets F and F′ along with their "fixed-point data" (determined by the equivariant normal bundles). In this way (*see* [9], §§ 40-43) the equivariant spin cobordism group $\Omega_n^{\mathrm{Spin},\, \mathbb{Z}_p}$ maps to a direct sum of groups of the form $\Omega^{\mathrm{Spin}}_{n-2(k_1+\ldots+k_r)}(BU(k_1)\times\ldots\times BU(k_r))$, where k_i are the complex dimensions of the eigenbundles of the $\mathbb{Z}_p$-action on the normal bundles. Any such cobordism between F and F′ is the result of spin surgeries on F′ preserving maps to the appropriate classifying space. Thus, they can be thickened to $\mathbf{Z}_p$-equivariant surgeries on all of M′. Any surgery in this step is on a sphere of dimension less than that of the fixed set, and hence is of codimension at least three in M′, so that this produces a new manifold M″ with an invariant metric of positive scalar curvature and F as fixed set. There is an equivariant spin cobordism of this manifold M″ to M. Now all surgeries are taking place in the free part, which by the codimensionality hypothesis is simply connected. Actually, we work on the quotient and observe that $\pi_1 = \mathbf{Z}_p$ and we have a spin cobordism over $\mathbf{Z}_p$ of the free parts, so that all remaining surgeries can be taken of codimension at least three, as in the proof of [24], Theorem 2.2.

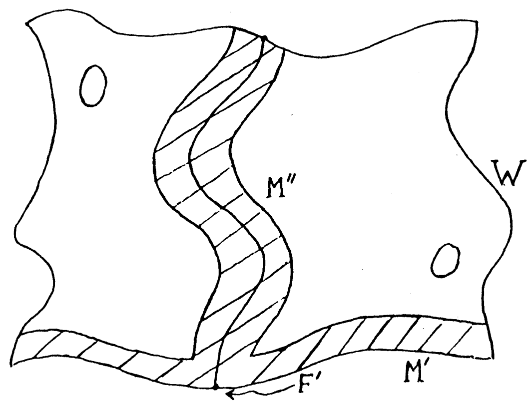

Fig. 1. — An equivariant spin cobordism. The first step puts an invariant metric on the "upper boundary" M″ of the shaded region.

2.4. *Remark.* — For the case of general compact G, one should assume that whenever $H \subset H' \subset G$ are closed subgroups, the codimension of each component of $M^{H'}$ in the relevant components of M^H is either zero or at least three. For nonsimply connected manifolds, one must, of course, also take the fundamental group into account in the bordism group. We shall see some examples later.

The following theorem gives our main necessary condition for existence of an invariant metric of positive scalar curvature. It follows from the higher $G-\tilde{\mathscr{A}}$ index theorem (Theorem 1.6 above) precisely as in the proof of [25, Theorem 3.4].

2.5. THEOREM. — *Suppose* M^n *is a closed spin manifold and* G *acts smoothly on* M, *preserving the spin structure. Assume that* M *admits a G-invariant metric of positive scalar curvature. Also assume* $\pi = \pi_1(M) \to \pi_1(M/G)$ *splits* (see *Proposition* 1.2). *If* [D] *is the fundamental class in* $KO_n^G(M)$ *defined by the real Dirac operator, the higher* G-$\tilde{\mathscr{A}}$*-class* $\beta(f_*([D]))$ *vanishes in* $KO_n^G(C^*_{\mathbb{R}}(\pi))$. *If* $\beta : KO_*(B\pi) \to KO_*(C^*_{\mathbb{R}}(\pi))$ *is injective,* $f_*([D])$ *vanishes in* $KO_n^G(B\pi)$.

2.6. *Remark.* — For $\pi_1(M)$ as in 1.7 or 1.8, this gives vanishing of the higher G-$\tilde{\mathscr{A}}$ index. Even if β is only rationally injective, Theorem 2.5 gives vanishing of the higher G-Â-class in $H_*(\pi, \mathbb{Q}) \otimes_{\mathbb{Z}} R(G)$ (just as in the Browder-Hsiang situation), and one can do many explicit rational calculations by restricting to fixed sets as in Theorem 1.9.

2.7. *Examples.* — We now give some examples of group actions on manifolds of positive scalar curvature, for some of which invariant metrics of positive scalar curvature exist, and for some of which no such invariant metric exists.

1. Although every connected sum of any number of copies of $S^2 \times S^2$ clearly has a metric of positive scalar curvature there are $\mathbb{Z}_p$ actions on such manifolds that have no such invariant metrics. The construction goes as follows: Let ρ be any 2-dimensional complex representation space not containing a trivial 1-dimensional subrepresentation. The sphere $S(\rho)$ modulo the action of $\mathbb{Z}_p$ is some 3-dimensional lens space L. Suppose p is an odd prime. Now pL bounds a spin manifold (even taking the fundamental group into account) since $\Omega_3^{spin}(B\mathbb{Z}_p) = \mathbb{Z}_p$, and after some surgeries on (0- and) 1-spheres one can assume that π_1 of this manifold is $\mathbb{Z}_p$. Consider the universal cover and glue in p copies of ρ. If necessary, take p copies of this manifold (and again do some surgeries) to guarantee that its signature is a multiple of $16p$. (Rohlin's theorem implies that it is already a multiple of 16.) One can now equivariantly pipe on some number of p times the Kummer surface, permuted freely by the $\mathbb{Z}_p$ to obtain a simply connected spin manifold of zero signature with a $\mathbb{Z}_p$ action with isolated fixed points all with local representation ρ. By Wall's theorem [30] the connected sum with a sufficiently large number of $S^2 \times S^2$'s (which we do equivariantly) will be diffeomorphic to a connected sum of $S^2 \times S^2$'s. On the other hand the local formula for the $\mathbb{Z}_p$-$\tilde{A}$-genus in $KO_4^{\mathbb{Z}_p}$ shows that it is nonzero, so that there is no invariant metric of positive scalar curvature.

2. (Construction of Metrics of Positive Scalar Curvature Invariant Under $\mathbb{Z}_p$ Actions on the Sphere with a Knot as Fixed Set.) Now we consider the first interesting codimension-two case, that of high-dimensional counterexamples to the P. A. Smith Conjecture that a $\mathbb{Z}_p$ action on the n-sphere with a codimension two subsphere as fixed set is in fact a linear action (or at least, that the subsphere is unknotted). We shall show that all of these actions for p an odd prime preserve some metric of positive scalar curvature, and in so doing, perhaps partially elucidate the nature of the difficulties unique to codimension two. We let K denote the fixed set, X its closed exterior in the sphere (*i. e.*, the complement of an open tubular neighborhood of K), and Σ denote a Seifert surface for K (*i. e.*, an oriented submanifold of the sphere with boundary K). We note that Σ can always be "closed" in a canonical way by gluing on a disk along K. We shall call the result of this operation Σ'. One can show that Σ' is always a spin manifold (it is a codimension-one oriented submanifold in the spin manifold obtained by surgery on K). The proof is somewhat simpler in case Σ' is spin-nullcobordant (which certainly implies $\tilde{\mathscr{A}} = 0$), so we do this case first.

First we show that the class of $X/\mathbb{Z}_p$ in $\Omega_n^{spin}(B\pi_1(X/\mathbb{Z}_p), B\mathbb{Z})$ is zero. This is true because it clearly lies in the image of $\Omega_n^{spin}(X/\mathbb{Z}_p, B\mathbb{Z})$ under the natural map $X/\mathbb{Z}_p \to B\pi_1(X/\mathbb{Z}_p)$. Alexander duality applied to X shows that the inclusion of the S^1 in the boundary of X induces an isomorphism on homology, so that the spectral sequence of the covering shows that this is true for $X/\mathbb{Z}_p$ as well, so that the Atiyah-Hirzebruch spectral sequence applies to deduce the result for Ω^{spin}, which leads to the vanishing of the relative group. (On the other hand, $\Omega^{spin}(B\pi_1(X/\mathbb{Z}_p), B\mathbb{Z})$ can be very large.) As a result, $X/\mathbb{Z}_p$ is spin-cobordant, keeping track of fundamental group of interior and boundary, to $S^1 \times D^{n-1}$. As a result, it can be obtained by a sequence of surgeries with codimension at least three from the complement of the unknot. This is unfortunately not of much help since the surgeries might touch the boundary (in such a way that we could not extend them inward toward the knot). The trouble is that the cobordism restricted to the boundary might not be the product cobordism $S^1 \times S^{n-2} \times I$. The obstruction to coboarding the given cobordism on the boundary to this one lies in $\Omega_n^{spin}(B\,\mathbb{Z}) = \Omega_n^{spin} \oplus \Omega_{n-1}^{spin}$. If this obstruction vanished, one could then attach this cobordism to the one on the boundary and get the desired cobordism between $X/\mathbb{Z}_p$ and $S^1 \times D^{n-1}$. The element that we have does have vansihing first component since p times that component can be viewed as the result of surgery along K, which is spin cobordant to the sphere, and Ω_*^{spin} has no p-torsion for p odd. The second component is exactly the class of Σ'. If this vanishes we are done since the original action can be obtained from a linear one (which preserves the standard metric of constant positive scalar curvature) by a sequence of equivariant codimension-at-least-three surgeries, so the result follows from the theorem of Bérard Bergery [5], Theorem 11.1. If not, then it cannot be so obtained. However, one can instead arrange for the boundary to be the cobordism $(\Sigma - D^{n-1}) \times S^1$. Then one can take covers and glue in a copy of $(\Sigma - D^{n-1}) \times D^2$ equivariantly. This gives an equivariant spin cobordism to the linear action *which respects the fundamental group of the complement*, so that the method of proof of the reduction to spin cobordism for the case of noncodimension-two fixed set applies, and one sees that in any case one gets the invariant metric.

3. All of the above examples were simply connected. Now we shall study the effect of crossing, say Example 1, with S^1. If S^1 has trivial action then 2.5 (and 2.6) apply and there is no invariant metric of positive scalar curvature. On the other hand, if we give S^1 an action by rotation, the product action is free. Hence there is an invariant metric of positive scalar curvature if and only if the quotient can be given positive scalar curvature. The quotient is the mapping torus of a generator of the $\mathbb{Z}_p$-action on $\#_k S^2 \times S^2$. This is a 5-manifold with $\pi_1 = \mathbb{Z}$, and hence by [25], Theorem 3.6, has a positive-scalar-curvature metric if and only if the higher $\hat{A}$-genus vanishes. Since that is clearly the case here, for this product action there is an invariant metric of positive scalar curvature.

4. For S^1-actions, Theorem 2.5 does not give as much information. But according to [5], Theorem C, a free S^1-action has an invariant metric of positive scalar curvature if and only if there is a metric of positive scalar curvature on the quotient. This excludes the case of a principal S^1-bundle M^5 over a K3-surface K^4. Then $(M - (S^3 \times D^2)) \cup_{S^3 \times S^1} (D^4 \times S^1)$ can be made into a semi-free S^1-manifold, which is simply connected if M is, and thus has a metric of positive scalar curvature (since $\Omega_5^{spin} = 0$). (Here take the $S^3 \subset M$ to be S^1-invariant, with S^3/S^1 a 2-sphere in K, let S^1 act on $D^4 = $ (unit ball of $\mathbb{C}^2$) by rotation, and let S^1 act trivially on D^2.) However, this manifold does not have an invariant metric of positive scalar curvature, since S^1-equivariant surgery in codimension 4 gives back M, which has no such metric. (This example answered a question asked to us by Bérard Bergery, of whether surgery theory could be used to find an example of a manifold with a metric of positive scalar curvature, but with no invariant such metric for a semi-free S^1-action.)

5. If G is semisimple and nonabelian, then according to [18] there is always an invariant positive-scalar-curvature metric for any effective G-action. As a result, (and for π_1-split actions, higher) G-$\hat{\mathscr{A}}$ vanishes. We note that this fails for actions not preserving a spin structure.

3. The higher G-signature theorem and some of its applications

In this section, we apply the higher G-index formula, very much in the style of the original paper of Atiyah and Singer, to show that for appropriately "homologically trivial" G-actions on manifolds with a suitably restricted fundamental group, there is a formula that relates the higher signature of the manifold to "twisted" higher signatures of fixed sets. (For earlier results, see [31] and [42].) The idea is simply this: the G-index of the twisted signature operator in the equivariant K-group $K_*^G(C^*(\pi))$ can be identified (by the method of Kaminker and Miller [13], for instance) with the image of an equivariant Miščenko-Ranicki higher signature of the simplicial chain complex. When the action is homologically trivial, the latter degenerates to something essentially unequivariant. On the other hand, assuming a suitable form of the Novikov Conjecture, the G-index of the signature operator can be given a homological interpretation, yielding the formula.

3.1. Theorem. — *Let* M^{2k} *be a connected, closed, oriented manifold and* π *a group with the property that* $\beta: K_*(B\pi) \to K_*(C^*(\pi))$ *is rationally injective. (In the notation of* [23], *this is* SNC2—*by* [16], §6, *and by* [23], *Propositions* 2.7 *and* 2.8, *it suffices to assume that* π *has a subgroup of finite index which, modulo a finite normal subgroup, has a discrete embedding in a connected Lie group.) Suppose* $G=\mathbb{Z}_n$ *acts smoothly on* M, *preserving the orientation, that* $f: M \to B\pi$ *factors through* M/G, *and that* G *acts trivially on* $H^*(M; \mathbb{R}[\pi])$ *(the local coefficients being determined by f). Then*

$$2^k f_*(\mathscr{L}(M) \cap [M]) = \sum_j 2^{t-r} f_* \{ [\mathscr{L}(\nu(-1))^{-1} e(\nu(-1)) \mathscr{L}(F_j)]$$

$$\times \prod_{0<\theta<\pi} \left[\left(i \tan \frac{\theta}{2} \right)^{-s(\theta)} \mathscr{M}^\theta(\nu(\theta)) \right] \cap [F_j] \Big\} \in H_*(B\pi, \mathbb{C}),$$

where (F_j) *are the components of the fixed set* $F=M^g$ *of a generator g of* $G=\mathbb{Z}_n$, $2t=\dim F_j$, $\nu(-1)=(-1)$*-eigenspace of the normal bundle of* F_j, *e its Euler class,* $2r=\dim \nu(-1)$, $s(\theta)=$ *complex dimension of the subbundle* $\nu(\theta)$ *of the normal bundle of* F_j *where g acts by* $e^{i\theta}$, *and* $\mathscr{M}^\theta$ *is as in* [4], p. 581. *(For simplicity of notation, we suppress the subscript j on t, r, etc. As in* [4], *Theorem* 6.12, *one needs to use local coefficients in case* F_j *isn't orientable.)*

We note some special cases.

3.2. COROLLARY (cf. [26]). — *If the fixed set* $F = M^G$ *is empty, then the higher signature of* M *in* $H_*(B\pi, \mathbb{Q})$ *is zero.*

This was proven for free actions in [31], *where it was shown to hold for all homologically trivial actions on manifolds with fundamental group* π *if* and only if *the Novikov Conjecture holds for* π.

3.3. COROLLARY. — *If* $n=2$ (i. e., *if we are dealing with an orientation-preserving involution*), *then the higher signature of the self-intersection of* $F = M^G$ *is equal to that of* M.

This follows from the proof of Proposition 6.15 of [4].

3.4. COROLLARY. — *If* $n=2$ *and* $\dim F < (1/2) \dim M$, *then the higher signature of* M *is zero* [*in* $H_*(B\pi, \mathbb{Q})$].

Proof. – We can push F away from itself by general position.

Corollary 3.4 can be seen to be equivalent to the Novikov Conjecture. Though there is a (less precise) version of Theorem 3.1 for non-smooth actions, Corollary 3.4 fails for PL actions, even in the simply connected case.

3.5. *Remark.* – Changing the generator of $\mathbb{Z}_n$ in (3.1) does not change F, nor the left-hand side of the equation, but it does change the θ's and their characteristic classes. The formula obtained by averaging (3.1) over all choices of generators was shown in [32] to be equivalent to the Novikov Conjecture, at least for semifree actions. However, our present formula appears to be somewhat stronger. One could prove our formula for groups in Cappell's class [8] by combining the arguments below with [32] and with well-known connections between K-theory and L-theory.

3.6. *Remark.* – As in Theorem 2.1, one can partially reprove for S^1-actions the result of [32] that the higher signatures of M and F always agree. This is also true (by a trick of [17] attributed to one of us) for $\mathbb{Z}_n$-actions that extend to S^1-actions.

Now for the *proof of* (3.1). Using the factorization

$$\begin{tikzcd} \pi_1(M) \arrow[rr, "f_*"] \arrow[dr] & & \pi, \\ & \pi_1(M/G) \arrow[ur] & \end{tikzcd}$$

we can define an element $\sigma_\pi(G, M) \in L^{2k}(\mathbb{Z}[1/n][G \times \pi])$, extending the symmetric signature of Miščenko [19] and Ranicki [22]. To define σ_π, note that f defines a covering of M with covering group π, and the algebraic chain complex of this cover (for some G-equivariant triangulation or G-CW decomposition), with the obvious $G \times \pi$ action, defines an element of the group of algebraic Poincaré complexes. (L-groups are decorated by indicating the type of modules allowed and the amount of control of bases demanded. Our L-groups are constructed from projective modules over $\mathbb{Z}[1/n][G \times \pi]$ that are free – even based if you like – over $\mathbb{Z}[1/n][\pi]$. All of this only affects 2-torsion which will not be of interest to us here. Once we invert the order n of G, permutation modules become projective. Actually, we can merely invert the order of the isotropy.)

Another way to describe this construction, using complex coefficients, is as follows. $\mathbb{C}[G \times \pi] = \mathbb{C}[G][\pi]$, which breaks up according to the representations of

G. Now (for the chosen decoration, or inverting 2), the L-theory of a product ring is the product of the L-theories, so that our obstruction lies in $L^{2k}(\mathbb{C}[\pi]) \otimes R(G)$. For $g \in G$ one can now define $\sigma_\pi(g, M)$, the higher signature of M, as a weighted sum of the components of this element in $L^{2k}(\mathbb{C}[\pi]) \otimes R(G)$ according to their characters on g. Explicitly:

$$\sigma_\pi(g, M) = \sum_{\rho \in \widehat{G}} \chi_\rho(g) [\sigma_\pi(G, M)]_\rho,$$

where $[\sigma_\pi(G, M)]_\rho$ is the ρ-primary part of $\sigma_\pi(G, M)$. When $\pi = \{e\}$, this is the G-signature defined as the trace of a representation by Atiyah and Singer [4], pp. 578-580.

Now recall that if A is a C*-algebra, there is a map $L^*(A) \to K_*(A)$ that arises from spectral theory. Composing this map [for $A = C^*(\pi)$] with the map in L-theory induced by the inclusion $\mathbb{C}[\pi] \hookrightarrow C^*(\pi)$ gives a map (discussed in [20] or [13])

$$m: \quad L^{2k}(\mathbb{C}[\pi]) \to K_0(C^*(\pi)) \qquad \text{or} \qquad L^{2k}(\mathbb{C}[\pi]) \otimes R(G) \to K_0^G(C^*(\pi)),$$

in terms of which the higher G-signature theorem computes the image $m(\sigma_\pi(G, M))$. The connection is provided by the following result.

3.7. Proposition. — *In the above situation, the* $G-C^*(\pi)$*-index of the signature operator of* M *with coefficients in the flat* $C^*(\pi)$*-bundle defined by* $f: M \to B\pi$ *coincides with* $m(\sigma_\pi(G, M))$, *the image of the equivariant Miščenko-Ranicki symmetric signature.*

Proof. — There is no difficulty in carrying through the proof in [13] once one gets used to dealing with projective, rather than free, modules. The only significant change is that complexes of $C^*(\pi)$-modules and chain homotopy equivalences between them must be chosen G-equivariant.

Now we can complete the proof of Theorem 3.1. Using the higher G-index theorem (1.5), we obtain from Proposition 3.7 the equality

$$\beta(f_*([D])) = m(\sigma_\pi(G, M)) \in K_0(C^*(\pi)) \otimes_{\mathbb{Z}} R(G).$$

Now $\operatorname{ch} f_*([D])(1) = 2^k f_*(\mathscr{L}(M) \cap [M])$, the usual higher signature. On the other hand, $\operatorname{ch} f_*([D])(g)$ may be computed using the localization theorem of [27] as in Theorem 1.9, yielding (by the calculation in [4], p. 581), the right-hand side of the formula in the theorem. Since we assumed that β is rationally injective, we only need to show that

$$m(\sigma_\pi(g, M)) = m(\sigma_\pi(1, M)) = m(\sigma_\pi(M))$$

(after tensoring with $\mathbb{Q}$ or $\mathbb{R}$).

But assuming that G acts trivially on $H^*(M; \mathbb{R}[\pi])$, we even show that $\sigma_\pi(G, M) = \sigma_\pi(M)$. In fact, this is equivalent to showing that the image of $\sigma_\pi(G, M) \in L^{2k}(\mathbb{C}[G][\pi])$ in $L^{2k}(\mathscr{I}_G[\pi])$ vanishes, where $\mathscr{I}_G$ is the augmentation ideal of $\mathbb{C}[G]$. But when the action of G is homologically trivial, the cochain complex $C^*(\tilde{M}) \otimes_{\mathbb{C}[G]} \mathscr{I}_G$ is acyclic, hence chain contractible, and this is obvious.

In fact the hypothesis of the theorem is perhaps stronger than necessary. It does not seem to follow that $\sigma_\pi(G, M) = \sigma_\pi(M)$ if G only acts "homologically trivially in the middle dimension," but the definition of m is such that $m(\sigma_\pi(G, M))$ only depends on the hermitian form on middle dimensional cochains, exactly as in [4], pp. 574-576 (*see* for instance [13, Proposition 3.8] or the discussion in [20] in the analogous case of higher signatures defined by Fredholm representations). Thus in some cases it might suffice only to assume G acts trivially on $H^k(M; \mathbb{R}[\pi])$.

Let us note that in the course of proving Theorem 3.1, we have also established an equivariant version of Kasparov's theorem in [16]. For this we no longer require any homological triviality.

3.8. THEOREM (Equivariant Novikov Conjecture). — *Suppose* M^{2k} *is a connected, closed, oriented smooth manifold, and suppose* π *is a discrete group such that* $\beta : K_*(B\pi) \to K_*(C^*(\pi))$ *is rationally injective. Suppose a finite group* G *acts smoothly on* M, *preserving the orientation, and that* $f : M \to B\pi$ *factors through* M/G. *Then the higher* G-*signature of* M *in* $H_*(B\pi, \mathbb{Q}) \otimes_{\mathbb{Z}} R(G)$ *is an (oriented) equivariant homotopy invariant.*

Proof. — This follows immediately from Proposition 3.7 and Theorem 1.5, once we make the obvious observation that $\sigma_\pi(G, M)$ is an oriented equivariant homotopy invariant.

3.9. *Remark.* — One can also formulate versions of the Equivariant Novikov Conjecture for cases where the map $f : M \to B\pi$ can be made equivariant for a *non-trivial* action of the finite group G on $B\pi$. For instance, one may consider the case where M has a G-fixed base-point x_0 but G acts non-trivially on $\pi = \pi_1(M, x_0)$. In this case, the higher G-signature will live in $K_*^G(B\pi) \otimes_{\mathbb{Z}} \mathbb{Q}$, which may be computed using the localization theorem, and $\sigma_\pi(G, M)$ will live in the L-theory of $\mathbb{C}[\pi \times G]$. One case where one can get reasonable results is when $B\pi$ can be chosen to be a complete manifold of non-positive curvature on which G acts by *isometries*, in which case the results of Kasparov (as presented in [11]) can be made G-equivariant. We shall defer a more complete discussion to another paper.

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