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A REDUCTION THEORY OF SECOND ORDER MEROMORPHIC DIFFERENTIAL EQUATIONS, II

BY W. B. JURKAT AND H. J. ZWIESLER

ABSTRACT. — In this article we investigate meromorphic differential equations $X'(z)=A(z)X(z)$ of dimension 2×2 whose exponential behavior near the singularity at infinity is governed by a diagonal matrix $Q(z)$ containing polynomials in $z^{1/2}$ but not in z. We discuss the possible simplifications due to meromorphic equivalence $T^{-1}AT-T^{-1}T'$ and derive normalized standard equations with the least possible number of free parameters in the sense that they cannot be generated by a subcollection with fewer parameters (Theorem 2 and its consequences). We are even able to define "natural" representatives under the above equivalence, which are described by conditions that can easily be checked. To derive these results we determine the possible transformations (mainly Theorem 1) and give a characterization of the minimality of the Poincaré-rank (Proposition 1).

1. Introduction

This article continues our investigations of *meromorphic differential equations* $X'(z)=A(z)X(z)$, abbreviated by [A], which we began in [6]. $A(z)$ is meromorphic at infinity, i. e. holomorphic in a punctured neighborhood of infinity with at most a pole there, and $X(z)$ is a fundamental solution matrix. (All occuring matrices have dimensions 2×2.) If we define $Y(z)=T^{-1}(z)X(z)$ with a *meromorphic transformation* $T(z)$ [i. e. a matrix $T(z)$ which together with its inverse is meromorphic at infinity], then $Y(z)$ solves the equation [B] with $B=T^{-1}AT-T^{-1}T'$ ([4], p. 8). We are interested in finding simple representatives under this equivalence relation which may then be used for closer examinations. This problem as well as many related questions was solved ([6]) in the case when [A] had a formal solution of the form $F(z)z^{\Lambda'}e^{Q(z)}$ ([4], p. 32). Here $Q(z)$ stands for the matrix $\operatorname{diag}(q_1(z), q_2(z))\not\equiv 0$ with polynomials q_1, q_2 without constant term, Λ' denotes a constant, diagonal matrix and $F(z)$ is a formal meromorphic transformation.

In this second part we treat the case when $Q(z)=\operatorname{diag}(q_1(z), q_2(z))\not\equiv 0$ is a polynomial in $z^{1/2}$ without constant term not containing only integral powers of z. Then we know that there exists a formal solution $H(z)=F(z)z^j\operatorname{diag}(1, z^{1/2})Ue^{Q(z)}$ with $U=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $j\in\mathbb{C}$ and a formal meromorphic transformation $F(z)$. Moreover, $q_1(z)$ and $q_2(z)$ are analytic continuations of one another ([4], p. 32) and therefore, even powers of $z^{1/2}$ have the same coefficients whereas the coefficients of odd powers of $z^{1/2}$ differ by a factor -1. This enables us to compute the share of the even powers by using Wronski's identity $(\det \mathrm{H}(z))' = (\operatorname{tr} \mathrm{A}(z)) \det \mathrm{H}(z)$. It shows that $(q_1+q_2)'$ equals the polynomial part $p(z)$ of the trace $\operatorname{tr} \mathrm{A}(z)$, i. e. $\operatorname{tr} \mathrm{A}(z) = p(z) + O(z^{-1})$. [Here the *Landau-symbol* $f(z) = O(g(z))$ is defined for a formal scalar or matrix series by $\deg(f) \leqslant \deg(g)$ where $\deg(f)$ denotes the highest power of z in the formal series of f with non-vanishing coefficient or $\deg(0) = -\infty$.] Hence the even powers of $z^{1/2}$ in q_1, q_2 are given by $\int_0^z p(w)/2\, dw$. If we replace $\mathrm{X}(z)$ by $\mathrm{Y}(z) = \exp\left(-\int_0^z p(w)/2\, dw\right) \mathrm{X}(z)$ the equation [A] is changed to $[\mathrm{A} - \mathrm{I}p(z)/2]$. Although this is not a meromorphic transformation (but still single-valued), it should be executed as a preliminary simplification, because it is a scalar factor which commutes with all other transformations and can be undone in the end. Thus, we require in the following that $\operatorname{tr} \mathrm{A}(z) = O(z^{-1})$. This remains true if we apply a meromorphic transformation $\mathrm{T}(z)$ since $\operatorname{tr}(\mathrm{T}^{-1} \mathrm{A} \mathrm{T}) = \operatorname{tr} \mathrm{A}$ and $\operatorname{tr}(\mathrm{T}^{-1} \mathrm{T}') = O(z^{-1})$. Furthermore it forces $\mathrm{Q}(z) = \operatorname{diag}(q(z), -q(z))$ where $q(z) \not\equiv 0$ is an odd polynomial in $z^{1/2}$. This assures that the leading coefficient of $\mathrm{A}(z)$ is nilpotent ([9], p. 100 ff). Now, it is reasonable to get rid of all superfluous leading coefficients by reducing the *Poincaré-rank* $r = \deg(\mathrm{A}) + 1$ until $r = \deg(\mathrm{Q}) + 1/2 \ (\geqslant 1)$. This is the least possible value since $\mathrm{A} = \mathrm{H}' \mathrm{H}^{-1}$ shows that $\deg(\mathrm{A}) \geqslant \deg(\mathrm{Q}')$.

To this equation we apply Birkhoff's reduction ([4], p. 15). It leads to a meromorphic differential equation [A] with $\mathrm{A}(z) = \sum_{k=-1}^{r-1} \mathrm{A}_k z^k$. We call this the *standard form*. In the case of 2×2-matrices we can always obtain it without increasing the Poincaré-rank ([5], Theorem 1) which therefore is still assumed to be minimal.

These standard equations $(n=2)$ with minimal Poincaré-rank $r > 0$, nilpotent leading coefficient and a trace which is of order $O(z^{-1})$ provide a suitable basis for our further investigations. In section 2 we will show that the minimality of the Poincaré-rank can be nicely characterized in terms of the formal solution. This allows a good description of all possible transformations between our standard equations. They turn out to be mainly polynomial transformations in $1/z$. These are examined in section 3 (mainly Theorem 1) and afterwards we take a closer look at the special case of constant transformations in section 4. They allow the normalization of our standard equations to a form which, in general, cannot be obtained from equations with a smaller number of parameters as will be shown in section 5 (as a consequence of Theorem 2). The same is true for certain subclasses which depend on a smaller, but fixed number of parameters and again cannot be reduced in general. We will even be able to discuss whether a given standard equation can be meromorphically transformed into another one with fewer coefficients different from 0. Finally, section 6 is devoted to explain how natural, unique representatives are chosen within each equivalent class. Examples of such representatives are the generalized Airy equations $y''(z) - z^{2m-1} y(z) = 0$, $m \in \mathbb{N}$ which should be written as a system and transformed by $\mathrm{T}(z) = \operatorname{diag}(z^{-[m/2]}, z^{[m/2]})$ followed, in case of odd m, by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Further examples are the equations with $\mathrm{A}(z) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & -a \end{bmatrix} z^{-1}$, $a, b \in \mathbb{C}$, $b \neq 0$ and $0 \leqq a < 1/2$ (lexicographically) which are discussed in part II of [5].

2. A characterization of the minimality of the Poincaré-rank

Here we will show how the minimality of the Poincaré-rank is reflected in the formal solution. For that, we need the notion of an *analytic transformation* which is a meromorphic transformation whose Laurent-series at infinity starts with a constant invertible term ([4], p. 8). Since everything in this section depends only on the behavior near infinity, we drop the requirement that the equation is in standard form.

PROPOSITION 1. — *Let* [A] *be a formal meromorphic differential equation with an exponential polynomial* $Q(z) \not\equiv 0$ *in its formal solution that contains only odd powers of* $z^{1/2}$. *Then* [A] *has minimal Poincaré-rank if and only if it has a formal solution of the form* $F(z) z^{\lambda'} \operatorname{diag}(1, z^{-1/2}) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} e^{Q(z)}$, *where* $F(z)$ *is an analytic transformation and* $\operatorname{tr} A = ((2\lambda' - 1/2)/z) + O(z^{-2})$.

Proof. — Since a constant transformation does not effect the claim, we may assume w. l. o. g. that $A_{r-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ because it is nilpotent. Then the minimality of the Poincaré-rank is equivalent to $\deg(a_{21}) = r-2$ where we use the notation $A(z) = [a_{ij}(z)]$ $(1 \leqslant i, j \leqslant 2)$.

(i) $\Rightarrow$: Upon introducing $\lambda (\neq 0)$ as one of the two square-roots of the leading coefficient in a_{21} we write $a_{21}(z) = \lambda^2 z^{r-2} + O(z^{r-3})$ (which will be used throughout this paper). If we define the new variable t by $z = t^2$, then $Y(t) = X(t^2)$ satisfies $(d/dt) Y(t) = 2t A(t^2) Y(t)$. We transform it with

$$T(t) = \begin{bmatrix} 1 & -1/(4\lambda) \\ \lambda/t & 1/(4t) \end{bmatrix}$$

into the equation [B] where

$$B(t) = \begin{bmatrix} 2\lambda & 0 \\ 0 & -2\lambda \end{bmatrix} t^{2r-2} + O(t^{2r-3}).$$

This equation has a formal solution of the form $F_B(t) t^{\Lambda'} e^{Q_B(t)}$ with

$$Q_B(t) = \operatorname{diag}(2\lambda, -2\lambda) t^{2r-1}/(2r-1) + O(t^{2r-2}),$$
$$\Lambda' = \operatorname{diag}(\tilde{\lambda}'_1, \tilde{\lambda}'_2) \qquad \text{and} \qquad F_B(t) = I + O(t^{-1}).$$

In the formal solution $F_A(z) z^j \operatorname{diag}(1, z^{1/2}) U e^{Q_A(z)}$ of the original equation [A] we order the polynomials in Q_A accordingly and obtain thus the following equation

$$F_A(t^2) t^{2j} \operatorname{diag}(1, t) U e^{Q_A(t^2)} = T(t) F_B(t) t^{\Lambda'} e^{Q_B(t)} C$$

with a constant, invertible matrix C. Since the exponential terms must cancel, $C = \operatorname{diag}(c_1, c_2)$ with $c_1 c_2 \neq 0$. We use the notation $F_A = [f_{ij}]$, $F_B = [g_{ij}]$ $(1 \leqq i, j \leqq 2)$, and compute the first row of the above equation which yields

$$(f_{11}(t^2)+tf_{12}(t^2))\,t^{2j}=(g_{11}-g_{21}/(4\lambda))\,c_1\,t^{\tilde{\lambda}'_1}=c_1\,t^{\tilde{\lambda}'_1}(1+O(t^{-1})),$$
$$(f_{11}(t^2)-tf_{12}(t^2))\,t^{2j}=(g_{12}-g_{22}/(4\lambda))\,c_2\,t^{\tilde{\lambda}'_2}=-c_2/(4\lambda)\,t^{\tilde{\lambda}'_2}(1+O(t^{-1})).$$

Since the even and odd terms on the left cannot cancel it follows that $\tilde{\lambda}'_1=\tilde{\lambda}'_2=\tilde{\lambda}'$ holds. Then we can compute

$$\mathrm{F_A}(t^2)=\mathrm{T}(t)\,\mathrm{F_B}(t)\,\mathrm{CU}^{-1}\,\mathrm{diag}(1,\,t^{-1})\,t^{\tilde{\lambda}'-2j}$$
$$=\begin{bmatrix} c_1-c_2/(4\lambda)+O(t^{-1}) & (c_1+c_2/(4\lambda))\,t^{-1}+O(t^{-2}) \\ (\lambda c_1+c_2/4)\,t^{-1}+O(t^{-2}) & (\lambda c_1-c_2/4)\,t^{-2}+O(t^{-3}) \end{bmatrix} t^{\tilde{\lambda}'-2j}/2.$$

Now the left-hand-side is an even function of t, forcing either $\lambda c_1+c_2/4=0$ or $\lambda c_1-c_2/4=0$. In the first case $(c_2=-4\lambda c_1)$ we obtain

$$\mathrm{F_A}(t^2)\,t^{2j}=\left(\begin{bmatrix} c_1 & \cdot \\ 0 & \lambda c_1 \end{bmatrix}+O(t^{-1})\right)\mathrm{diag}(1,\,t^{-2})\,t^{\tilde{\lambda}'}=\mathrm{F}(t)\,\mathrm{diag}(1,\,t^{-2})\,t^{\tilde{\lambda}'}$$

where $\mathrm{F}(t)$ must be an even analytic transformation. Inserting this for $\mathrm{F_A}$ we find the claimed formal solution with $\lambda'=\tilde{\lambda}'/2$. In the second case $(c_2=4\lambda c_1)$ we get

$$\mathrm{F_A}(t^2)\,t^{2j}=\left(\begin{bmatrix} c_1 & \cdot \\ 0 & \lambda c_1 \end{bmatrix}+O(t^{-1})\right)t^{\tilde{\lambda}'}\begin{bmatrix} 0 & t^{-1} \\ t^{-1} & 0 \end{bmatrix}.$$

We argue as above and insert it into the formal solution which shows the required form when we multiply it by $\mathrm{diag}(1,\,-1)$ from the right.

(ii) $\Leftarrow$: We dissect the formal solution $\mathrm{H}(z)=\mathrm{F}(z)\,\mathrm{G}(z)$ with

$$\mathrm{G}(z)=z^{\lambda'}\,\mathrm{diag}\,(1,\,z^{-1/2})\,\mathrm{U}\,e^{Q(z)}.$$

Then $\mathrm{G}(z)$ satisfies the differential equation $[\mathrm{G}'\,\mathrm{G}^{-1}]$ with leading coefficients

$$\begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} z^{r-1}+\begin{bmatrix} \cdot & \cdot \\ \lambda & \cdot \end{bmatrix} z^{r-2}+O(z^{r-3})$$

where $\mathrm{Q}'(z)=\mathrm{diag}(\lambda,\,-\lambda)\,z^{r-3/2}+O(z^{r-5/2})$. Hence it has minimal Poincaré-rank which is the same for $\mathrm{A}=\mathrm{H}'\,\mathrm{H}^{-1}=\mathrm{F}'\,\mathrm{F}^{-1}+\mathrm{F}(\mathrm{G}'\,\mathrm{G}^{-1})\,\mathrm{F}^{-1}$.

(iii) Considering this logarithmic derivative we also obtain the equation between λ' and $\mathrm{tr}\,\mathrm{A}$.

Q.E.D.

Notice that F is uniquely determined by [A] up to an arbitrary scalar constant ($\neq 0$) once Q and λ' have been selected. Furthermore, the proof shows that, if we require $\mathrm{A}_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\mathrm{F}(z)$ has constant term $\mathrm{F}_0=c\begin{bmatrix} 1 & \cdot \\ 0 & \lambda \end{bmatrix}$ $(c\neq 0)$.

Remark 1. — Proposition 1 enables us to discuss all formal transformations between equations of minimal Poincaré-rank which satisfy the stated assumptions. Any such transformation can be preceeded and succeeded by an arbitrary constant similarity; hence we may assume w. l. o. g. that all equations start with $A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Obviously we can change λ' to $\lambda'-k$ with arbitrary $k\in\mathbb{Z}$ using $T(z)=z^k I$. But it is also possible to change λ' to $\lambda'-1/2$ without violating our assumptions. This is achieved by $T(z)=\begin{bmatrix} 0 & z \\ \lambda^2 & 0 \end{bmatrix}$. (Notice that λ^2 is a formal meromorphic invariant.) What other transformations $T(z)$ are there? If $T(z)$ transforms [A] into [B] the corresponding formal solutions are connected by

$$F_A z^{\lambda'_A} \operatorname{diag}(1, z^{-1/2}) U e^{Q_A} C = T F_B z^{\lambda'_B} \operatorname{diag}(1, z^{-1/2}) U e^{Q_B}$$

where we may assume that $Q_A \equiv Q_B$, and hence $C=\operatorname{diag}(c_1, c_2)$, $c_1 c_2 \neq 0$. From this equation we learn that $z^{\lambda'_A-\lambda'_B}$ is a formal series in $z^{1/2}$, i. e. $\lambda'_A \equiv \lambda'_B \pmod{1/2}$. Because we know already how to change λ'_A by arbitrary multiples of 1/2 we can perform this task in a preliminary step and then assume w. l. o. g. $\lambda'_A=\lambda'_B$. Then $\operatorname{diag}(1, z^{-1/2}) UCU^{-1} \operatorname{diag}(1, z^{1/2}) = F_A^{-1} T F_B$ is formally meromorphic which shows that $C=cI$, $c\neq 0$. Since a scalar constant does not change a meromorphic differential equation, we assume w. l. o. g. that $c=1$, i. e. $T=F_A F_B^{-1}$. Hence T *must be formally analytic* with constant term $T_0=\begin{bmatrix} d_1 & d_2 \\ 0 & d_1 \end{bmatrix}$ $(d_1 \neq 0)$. On the other hand, any such T leaves λ' and all other assumed properties $\left(\text{including } A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ untouched. If we furthermore require that all equations are in standard form, we learn that T must be holomorphic in $\mathbb{C} - \{0\}$ (in this article S − T always denotes the difference of the two sets) and has at most a pole at 0. Therefore T must be a polynomial in z^{-1} with constant determinant since Wronski's identity shows that any solution of a standard equation has a determinant of the form $e^{p(z)} z^a$ $(a\in\mathbb{C})$. We may as well restrict ourselves to $\det T \equiv 1$ because a constant, scalar factor, does not change a differential equation.

3. Polynomial transformations

The polynomials in z^{-1} constitute a main part of the transformations we encounter. Therefore, we want to study them more closely in this section. For convenience of notation, we choose the singular point located at infinity rather than at zero. These two points can be easily exchanged replacing z by $1/z$.

Our goal is to clarify how a polynomial with constant determinant can change a differential equation with a singularity of the first kind ([2], p. 111). This turns out to be a purely algebraic problem and thus it makes no difference whether we treat it formally or with convergent series. Therefore, we will not distinguish between these two cases in this section since all statements hold in either one. The stated problem was solved in [6], Theorem 1 for the case of incongruent eigenvalues and we will extend these results. For that purpose we restate the result of section 4 in [6] as

LEMMA 1. — *Let* $F(z)=I+O(z^{-1})=[f_{ij}(z)]\,(1 \leqq i, j \leqq 2)$ *be a power-series in* z^{-1} *and* $K=\operatorname{diag}(k, -k)$, $k \in \mathbb{Z}$. *Then there exists a polynomial matrix* $P(z)$ *such that* $P(z)F(z)z^K=I+O(z^{-1})$ *if and only if there are relatively prime polynomials* $p(z)$, $q(z)$ *satisfying:* $\deg(p)=|k|$, *p is monic*, $\deg(q) \leqq |k|$ *and*

$$qf_{11}+pf_{21}=O(z^{-k-1}) \qquad \text{if} \quad k \geqq 0$$
$$qf_{22}+pf_{12}=O(z^{-|k|-1}) \qquad \text{if} \quad k < 0.$$

Proof. — This is essentially Theorem 1 (with $T^{-1}=P$, $r=0$) and Remark 7 in [6] whose proofs are mainly based on a detailed discussion of the equation $T^{-1}Fz^K=I+O(z^{-1})$. We will only point out the necessary modifications. Actually, $\deg(q) < |k|$ since $\deg(p)=|k|$ and f_{12}, f_{21} are both of order $O(z^{-1})$; but the presented statement is better suited for our later applications. Furthermore, the fact that p and q are relatively prime is equivalent to their uniqueness ([6], proof of Theorem 1) if we assume they possess all the remaining properties. It should be noted that $\det P \equiv 1$ is a consequence of $P=(I+O(z^{-1}))z^{-K}F^{-1}$ since taking determinants on both sides leads to a polynomial on the left side whereas on the right we obtain $1+O(z^{-1})$. Finally, for $k=0$ we find immediately $P=I$.

Q.E.D.

Remark 2. — Notice that P^{-1} is also a polynomial because of $\det P \equiv 1$. We use this as follows to prove the uniqueness of P. Suppose that $P_i F z^K = F_i$ $(i=1, 2)$. Then the polynomial $P_1P_2^{-1}=F_1F_2^{-1}=I+O(z^{-1})$ must equal I. Thus P is unique. It has the form I for $k=0$ and $\begin{bmatrix} s & r \\ q & p \end{bmatrix}$ for $k>0$ resp. $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ for $k<0$ where r, s are polynomials which are uniquely determined by the following conditions: $ps-qr \equiv 1$, $\deg(r) < \deg(p)$, $\deg(s) < \deg(q)$ ([6], proof of Theorem 1).

Remark 3. — Lemma 1 remains true if we allow $f_{21}=O(1)$ for $k>0$ resp. $f_{12}=O(1)$ for $k<0$. To show this in the case $k>0$ where $F(z)=\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}+O(z^{-1})$ we apply Lemma 1 to $\tilde{P}\tilde{F}z^K=I+O(z^{-1})$ with $\tilde{P}=P\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$, $\tilde{F}=\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}^{-1}F=I+O(z^{-1})$. This gives conditions for p and $q+pc$; but these are obviously equivalent to those claimed for p and q. Here we may actually find $\deg(q)=|k|$.

Let us now consider a differential equation [A] with $\deg(A) \leqq -1$. w.l.o.g. we assume that A_{-1} is in lower-triangular Jordan canonical form, which is easily achieved by a constant transformation, with the ordering of its diagonal $\operatorname{diag}(\mu_1, \mu_2)$ as follows. We split (uniquely) $\mu_i=m_i+k_i$ where $0 \leqq m_i < 1$ ["$\geqq$" for complex numbers always indicates the use of the lexicographical ordering (real part first)] and $k_i \in \mathbb{Z}$, thus each m_i belongs to a fixed system of representatives modulo 1 in $\mathbb{C}$ $(i=1, 2)$. Then we require $m_1 \geqq m_2$ and if $m_1=m_2$ we want $k_1 \geqq k_2$. According to [3], p. 192, the equation

[A] has a solution of the form $F(z)z^K z^M$ with $M=\begin{bmatrix} m_1 & 0 \\ m & m_2 \end{bmatrix}$, $m=0$ if $m_1 \neq m_2$, $K=\operatorname{diag}(k_1, k_2)$ and $F(z)=I+O(z^{-1})=[f_{ij}]$. If $M=\begin{bmatrix} m_1 & 0 \\ m & m_1 \end{bmatrix}$ and $k_1=k_2$ we find that $M+K=A_{-1}$ and hence $m=0$ or $m=1$.

On the other hand, every matrix $F(z)z^K z^M$ with these properties and the m_i, k_i chosen as above solves a differential equation [A] of the form described above. This is verified by just computing the logarithmic derivative $(F z^K z^M)'(F z^K z^M)^{-1}$. The matrix $F(z)$ is not unique as will be seen in section 5, thus whenever necessary we choose one of the possible solutions (e. g. the one singled out by Lemma 2). The value $m=1$ can also be obtained when k_1 exceeds k_2 by using the similarity $T=\operatorname{diag}(1, m)$ which has the effect that

$$T^{-1}(I+O(z^{-1}))z^K z^M=(I+O(z^{-1}))z^K z^{\tilde{M}} T^{-1} \qquad \text{with} \quad \tilde{M}=\begin{bmatrix} m_1 & 0 \\ 1 & m_1 \end{bmatrix}.$$

Thus we assume from now on that $m=1$ in this case.

After all these preparations we now turn to the question of how to transform such an equation [A] into an equation [B] satisfying the same conditions by a polynomial $T(z)$ with $\det T \equiv 1$. For each equation we choose a solution as above and are led to $F_A z^{K_A} z^{M_A}=TF_B z^{K_B} z^{M_B} C$ with constant, invertible C. Therefore $z^{M_B} C z^{-M_A}$ is single-valued. Since M_A and M_B are Jordan-matrices whose eigenvalues are representatives modulo 1 with the same ordering, we conclude that $M_A=M_B=M$ and C commutes with M. Moreover we learn, by taking determinants on both sides, that $\operatorname{tr} K_A=\operatorname{tr} K_B$ and $\det C=1$. Thus, the diagonal of B_{-1}, denoted by $\operatorname{diag}(\tilde{\mu}_1, \tilde{\mu}_2)$, differs from $\operatorname{diag}(\mu_1, \mu_2)$ only by integers, such that $\operatorname{tr} A_{-1}=\operatorname{tr} B_{-1}$; furthermore if $M_A=\begin{bmatrix} m_1 & 0 \\ m & m_1 \end{bmatrix}$ and $\tilde{\mu}_1=\tilde{\mu}_2=\tilde{\mu}$, then $B_{-1}=\begin{bmatrix} \tilde{\mu} & 0 \\ m & \tilde{\mu} \end{bmatrix}$. To a given [A], we call any B_{-1} *admissable*, if it satisfies the properties just mentioned.

With all these assumptions (normalized A_{-1} etc.) and notations we formulate

THEOREM 1. — *Given* $A(z)=A_{-1}z^{-1}+O(z^{-2})$ *and an admissable* B_{-1}. *Then there exists a polynomial* $T(z)$ *with* $\det T \equiv 1$ *which transforms* [A] *into some* [B] *with* $B(z)=B_{-1}z^{-1}+O(z^{-2})$ *if and only if there are relatively prime polynomials* $p(z)$ *and* $q(z)$ *satisfying one of the following conditions*

$$pf_{21}+qf_{11}=O(z^{-k-1}) \qquad \textit{if} \quad k=\mu_1-\tilde{\mu}_1 \geqq 0$$

or

$$pf_{12}+qf_{22}=O(z^{-k-1}) \qquad \textit{if} \quad k=\tilde{\mu}_1-\mu_1 > 0$$

or

$$p(f_{21}-cf_{22}\,z^{\mu_2-\mu_1})+q(f_{11}-cf_{12}\,z^{\mu_2-\mu_1})=O(z^{-k-1})$$

for some suitable $c\in\mathbb{C}$ *if* $k=\tilde{\mu}_1-\mu_2>0$ *and* $M=m_1 I$ *where* $\deg(p)=k$, *p is monic and* $\deg(q)\leq k$.

Proof. — Our previous discussion shows that the existence of $T(z)$ means that the following equation holds:

$$F_A\,z^{K_A}=T(I+O(z^{-1}))\,z^{K_B}C$$

where F_A, K_A and K_B are given and C is some constant invertible matrix with $\det C=1$ which commutes with M. Because of $\det T\equiv 1$ we know that $P(z)=T^{-1}$ is also a polynomial, i.e. we may as well discuss the existence of P satisfying $PF_A\,z^{K_A}C^{-1}z^{-K_B}=I+O(z^{-1})$.

(i) $m_1\neq m_2$: Then C is diagonal and commutes with K_B, which leads to the (equivalent) equation $C^{-1}PF_A\,z^{K_A-K_B}=C^{-1}(I+O(z^{-1}))C=I+O(z^{-1})$. Since $C^{-1}P$ is also a polynomial we can apply Lemma 1 which immediately proves the claim in this case.

(ii) $M=\begin{bmatrix} m_1 & 0\\ 1 & m_1\end{bmatrix}$: Here $C=\begin{bmatrix} c_1 & 0\\ c_2 & c_1\end{bmatrix}$ where $c_1=\pm 1$ because of $\det C=1$ and therefore $C^{-1}z^{-K_B}=z^{-K_B}\begin{bmatrix} c_1^{-1} & 0\\ -c_2\,z^{\tilde{k}_2-\tilde{k}_1} & c_1^{-1}\end{bmatrix}$ with $K_B=\operatorname{diag}(\tilde{k}_1,\tilde{k}_2)$.

As above our proof is completed when we apply Lemma 1 to $C^{-1}P$ if $\tilde{k}_1=\tilde{k}_2$ (i.e. $\tilde{\mu}_1=\tilde{\mu}_2$) or $c_1 P$ if $\tilde{k}_1>\tilde{k}_2$ since then

$$c_1(I+O(z^{-1}))\begin{bmatrix} c_1^{-1} & 0\\ c_2\,z^{\tilde{k}_2-\tilde{k}_1} & c_1^{-1}\end{bmatrix}=I+O(z^{-1}).$$

(iii) $M=m_1 I$: In this case, C is arbitrary provided $\det C=1$. If C^{-1} and hence C is lower triangular we argue almost as in (ii) the only difference being that $C=\begin{bmatrix} c_{11} & 0\\ c_{21} & c_{22}\end{bmatrix}$ with $c_{11}c_{22}=1$. Therefore Lemma 1 applies to $C^{-1}P$ if $\tilde{\mu}_1=\tilde{\mu}_2$ resp. $\operatorname{diag}(1/c_{11},1/c_{22})P$ if $\tilde{\mu}_1\neq\tilde{\mu}_2$.

Otherwise we decompose $C=LUV$ where L is lower triangular with $\det L=-1$, $U=\begin{bmatrix} 1 & c\\ 0 & 1\end{bmatrix}$ and $V=\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}$ ([1], p. 235). (This is possible since $c_{12}\neq 0$.) Inserting this into the equation for P we obtain the equivalent equation

$$PF_A\,z^{K_A}V^{-1}U^{-1}z^{-K_B}=(I+O(z^{-1}))\,z^{K_B}L\,z^{-K_B}.$$

Now $V^{-1}=V$ and hence

$$z^{K_A}V^{-1}U^{-1}z^{-K_B}=V\begin{bmatrix} 1 & -cz^{k_2-k_1}\\ 0 & 1\end{bmatrix}z^{K}$$

with $K=\operatorname{diag}(k_2-\tilde{k}_1,\,k_1-\tilde{k}_2)=\operatorname{diag}(-k,k)$.

The right-hand-side is treated as in (ii) and we are led to the equivalent equation

$$\tilde{P}\begin{bmatrix} f_{22} & f_{21}-cf_{22}z^{k_2-k_1} \\ f_{12} & f_{11}-cf_{12}z^{k_2-k_1} \end{bmatrix} z^{K}=I+O(z^{-1})$$

with the polynomial $\tilde{P}=L^{-1}PV$ if $\tilde{\mu}_1=\tilde{\mu}_2$ or $\tilde{P}=\operatorname{diag}(l_{11}^{-1}, l_{22}^{-1})PV$ if $\tilde{\mu}_1\neq\tilde{\mu}_2$ $(L=[l_{ij}])$.

For every fixed c we can now apply Lemma 1 which leads to the third of the claimed possibilities since $k_2-k_1=\mu_2-\mu_1$ in this case. We may exclude $\tilde{\mu}_1=\mu_2$, i. e. $k=0$, since then $\mu_1=\mu_2=\tilde{\mu}_1=\tilde{\mu}_2$ because of the prescribed ordering and thus the possibility for $k=\mu_1-\tilde{\mu}_1=0$ always applies with $p\equiv 1$, $q\equiv 0$.

Q.E.D.

Remark 4. — The fact that det T is 1 and the use of Remark 2 enable us to construct one possible T under each of the three conditions in the following way (of course $T=I$ for $k=0$):

$$T=\begin{bmatrix} p & -r \\ -q & s \end{bmatrix} \quad \text{for} \quad k=\mu_1-\tilde{\mu}_1>0,$$

$$T=\begin{bmatrix} s & -q \\ -r & p \end{bmatrix} \quad \text{for} \quad k=\tilde{\mu}_1-\mu_1>0$$

or

$$T=\begin{bmatrix} +r & +p \\ -s & -q \end{bmatrix} \quad \text{for } k=\tilde{\mu}_1-\mu_2>0, \qquad M=m_1 I \text{ and some suitable (fixed) } c\in\mathbb{C},$$

where r, s are polynomials uniquely determined by $ps-rq\equiv 1$ and $\deg(r)<\deg(p)$, $\deg(s)<\deg(q)$. We explain the last case since the other cases are easier. From the proof we learned that $T=\tilde{P}^{-1}=VP^{-1}L$ $(\tilde{\mu}_1=\tilde{\mu}_2)$ resp. $VP^{-1}\operatorname{diag} L$ $(\tilde{\mu}_1\neq\tilde{\mu}_2)$.

Now L resp. diag L commute with B_{-1}. We could also use

$$T=VP^{-1}\operatorname{diag}(-1, 1).$$

The last factor leads to $\det T\equiv 1$. By Remark 2 we know that $P=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and hence obtain the claimed form. (This T may yield a different [B], but still the same B_{-1}.) Notice that always $\deg(T)=k$. The proof of Theorem 1 helps us to discuss how to obtain all possible transformations (not necessarily $\det T\equiv 1$) from a single one.

(i) $m_1\neq m_2$: Here $T(z)$ can be replaced by $T\operatorname{diag}(c_1, c_2)$ with arbitrary $c_1, c_2\in\mathbb{C}-\{0\}$. This matches with the fact that $B(z)$ is determined by B_{-1} up to transformations of the form $\operatorname{diag}(c_1, c_2)$ as can be seen from [4], p. 175.

(ii) $M=\begin{bmatrix} m_1 & 0 \\ 1 & m_1 \end{bmatrix}$: In this case, we can only use $cT(z)$ if $\tilde{k}_1>\tilde{k}_2$ resp. $T(z)\begin{bmatrix} c & 0 \\ d & c \end{bmatrix}$ with arbitrary $d\in\mathbb{C}$ for $\tilde{k}_1=\tilde{k}_2$ $(c\neq 0)$. We should recall that the requirement $m=1$

restricts the class of equations. $\Big($If we dropped it, any transformation $T\,\mathrm{diag}(c_1, c_2)$ resp. $T\begin{bmatrix} c_1 & 0 \\ d & c_2 \end{bmatrix}$ with $c_1 c_2 \neq 0$ would be allowed.$\Big)$ This proves that here for $\tilde{k}_1 > \tilde{k}_2$ (resp. $\tilde{k}_1 = \tilde{k}_2$) meromorphic differential equations with the same coefficient at z^{-1} are meromorphically equivalent if and only if they are identical $\Big($resp. related by the constant transformation $\begin{bmatrix} c & 0 \\ d & c \end{bmatrix}\Big)$. This allows an extension of Theorem 2 and its consequences in [6]. There we considered equations (NSE's) where the eigenvalues of A_{-1} were equal or incongruent modulo 1. We now learn that this assumption can be dropped in the case that $\log z$ occurs in the actual solution. This can be seen by reversing the processes used in [6] in the following way. First one uses shearing-transformations to change the formal monodromy matrix by arbitrary integers. Then it is kept fixed and we can influence the actual monodromy. But this can only be done by polynomial transformations in $1/z$ and then Theorem 1 (with variable $1/z$) saves the validity of the arguments in [6].

(iii) $M = m_1 I$: This case is easy if $\tilde{k}_1 = \tilde{k}_2$, i. e. $B_{-1} = \tilde{\mu}_1 I$. Then $B(z)$ can only be transformed by constant similarities if B_{-1} is to be preserved. Hence T can be replaced by TC for any invertible C.

If $B_{-1} \neq \tilde{\mu}_1 I$ the situation is much more complicated since now the third possibility in Theorem 1 may produce transformations for different values of c which happens e. g. in the following example:

$$A(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z^{-1} + \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} z^{-2} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z^{-3}$$

is transformed into

$$B(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z^{-1} + \begin{bmatrix} b/2 & 2-3b^2/4 \\ -1 & -b/2 \end{bmatrix} z^{-2} + \begin{bmatrix} 1-b^2/2 & -b+b^3/4 \\ -b & -1+b^2/2 \end{bmatrix} z^{-3}$$

by

$$T(z) = \begin{bmatrix} -(bz/2)+1 & -(bz^2/2)+(b^2 z/4)-b/2 \\ b/2 & (bz/2)-(b^2/4)+1 \end{bmatrix}.$$

Here we see where the problem lies. For $B_{-1} = A_{-1}$ there exist transformations which are not constant due to the third possibility in addition to similarities of the form $\mathrm{diag}(c, 1/c)$ with $c \in \mathbb{C} - \{0\}$. All polynomial transformations which leave B_{-1} unchanged are exactly the factors with which a given T transforming [A] into [B] can be multiplied from the right.

Theorem 1 also enables us to discuss the least possible change of k_1 (a *minimal step*) which at the same time changes k_2 in the opposite direction. For the moment we exclude $M=m_1 I$ and start with trying to increase k_1 to k_1+k ($k \in \mathbb{N}$). This depends on solving $pf_{12}+qf_{22}=O(z^{-k-1})$ with relatively prime polynomials p and q.

Since this prevents q from vanishing identically, we must have $f_{12} \not\equiv 0$ and then $k=\deg(p)=\deg(q)-\deg(f_{12})$ is smallest if $\deg(q)=0$, i. e. q is a constant different from zero. Obviously we can satisfy all conditions for p and q in this case. From the normalization of the differential equation and the chosen solution we see that $\deg(f_{12})=\deg(a_{12})+1$ ([3], p. 179 ff) which shows that k_1 can be increased if and only if $a_{12} \not\equiv 0$ and the minimal step in this case is $-\deg(a_{12})-1$.

In order to decrease k_1 to k_1-k ($k \in \mathbb{N}$) we use an analogous reasoning, but we must keep in mind that for $M=\begin{bmatrix} m_1 & 0 \\ 1 & m_1 \end{bmatrix}$ we require $\tilde{k}_1 \geqq \tilde{k}_2$ which forces $k \leqq (k_1-k_2)/2$. Hence k_1 can be decreased if and only if $a_{21} \not\equiv 0$ for $m_1 \neq m_2$ resp. $0 < -\deg(a_{21})-1 \leqq (k_1-k_2)/2$ for $M=\begin{bmatrix} m_1 & 0 \\ 1 & m_1 \end{bmatrix}$ and the minimal step then is $-\deg(a_{21})-1$.

For $M=m_1 I$ the situation is different because the value k_1+1 could now also be obtained by a transformation of degree k_1-k_2+1 as the third possibility of Theorem 1 shows. We may still discuss the equations for p and q for minimal values of k but because of the three possibilities we can in general for $k>1$ not be sure whether this is the least possible change. As before we can increase k_1 by $-\deg(a_{12})-1$ if $a_{12} \not\equiv 0$ and decrease it by $-\deg(a_{21})-1$ if $-\deg(a_{21})-1 \leqq (k_1-k_2)/2$. In the third possible equation for p and q the value $k=\tilde{\mu}_1-\mu_2$ is bounded below by the inequality $k \geqq (k_1-k_2)/2$ which is equivalent to $k_2+k=\tilde{k}_1 \geqq \tilde{k}_2=k_1-k$. Hence the smallest possible value k produced here is $k=\min(-\deg(a_{21})-1,\, k_1-k_2)$ but only if this is positive and at least $(k_1-k_2)/2$. Now $k=k_1-k_2$ would lead to $\tilde{\mu}_1=\mu_1$, $\tilde{\mu}_2=\mu_2$ which is no change at all. Hence we can definitely decrease k_1 to $k_2-\deg(a_{21})-1$ if $(k_1-k_2)/2 \leqq -\deg(a_{21})-1 < k_1-k_2$.

4. Constant transformations and normalizations

After our careful discussion of all possible meromorphic transformations we will now proceed towards finding simple differential equations in each equivalence class. We recall that so far we have already narrowed our choice to standard equations of minimal Poincaré-rank whose trace is cz^{-1} with arbitrary $c \in \mathbb{C}$ and where $z^{1/2}$ actually occurs in the exponential part $Q(z)$. The simplest transformations are the constant ones, i. e. similarities, which we will discuss now in the context of equations [A] as above with fixed Poincaré-rank $r \in \mathbb{N}$. As we remarked earlier, we can always obtain $A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which will be assumed from now on. Then any allowed similarity must commute with A_{r-1} and hence, w. l. o. g., has the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ since a constant, scalar factor ($\neq 0$) does not change [A]. Such an equation contains $4(r+1)$ complex parameters in the matrices $A_k(-1 \leqq k \leqq r-1)$, and we associate a vector in $\mathbb{C}^{4(r+1)}$ with it. The matrix $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ can now be used to influence A_{-1} which can be brought into one of the two forms: either $\begin{bmatrix} \mu_1 & 0 \\ a & \mu_2 \end{bmatrix}$ or $\begin{bmatrix} \mu & b \\ 0 & \mu \end{bmatrix}$ with $a, b \in \mathbb{C}, b \neq 0$.

These can never be transformed into one another, and we call the new matrix again [A]. If A_{-1} equals μI or $\begin{bmatrix} \mu & b \\ 0 & \mu \end{bmatrix}$ the similarity $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ commutes with A_{-1}, too. Thus we can influcence another coefficient-matrix. (Notice that in this case $r > 1$ since A_{r-2} must not be upper triangular.) For convenience we choose A_{r-2} because of $a_{21}^{(r-2)} \neq 0$ since we required the Poincaré-rank to be minimal. If we demand that in this case the diagonal elements of A_{r-2} are zero, we find a normalization which is always possible. The sole effect, which a similarity can now have, is to exchange μ_1 and μ_2 if $\mu_1 \neq \mu_2$. Since we are only allowed to use $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ this is possible if and only if $a_{21}^{(-1)} \neq 0$, i.e. A_{-1} is not diagonal, and then the transformation has the form $\begin{bmatrix} 1 & (\mu_1-\mu_2)/a_{21}^{(-1)} \\ 0 & 1 \end{bmatrix}$. This shows that we could obtain differential equations which are unique under similarities if we would require $\mu_1 \geqq \mu_2$ in case that A_{-1} is not diagonal. Then cI are the only similarities not destroying any of these properties ($c \in \mathbb{C} - \{0\}$). In order to obtain a more natural approach we will not prescribe this ordering.

In section 2 we saw that it was especially easy to change the formal monodromy λ'. Is this still true in the presence of our normalizations? Obviously $z^k I$ still changes λ' to $\lambda' - k$ by an arbitrary integer k. Thus we only have to examine the transformation $T(z) = \begin{bmatrix} 0 & z \\ a_{21}^{(r-2)} & 0 \end{bmatrix}$ which yields $\lambda' - 1/2$. It leads from $A(z) = [a_{ij}(z)]$ to the matrix $\begin{bmatrix} a_{22} & z a_{21}/a_{21}^{(r-2)} \\ a_{21}^{(r-2)} a_{12}/z & a_{11} - 1/z \end{bmatrix}$ which has all the required properties unless $A_{-1} = \begin{bmatrix} \mu & b \\ 0 & \mu \end{bmatrix}$ with $b \neq 0$. This last case can be avoided if we undertake a preliminary treatment as proposed by Theorem 1. To do this we first replace z by $1/z$ to exchange the points zero and infinity which changes $A(z)$ to $-A(1/z)/z^2 = B(z)$ with $B_{-1} = \begin{bmatrix} -\mu & -b \\ 0 & -\mu \end{bmatrix}$. Then we use $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and a minimal step to obtain $\operatorname{diag}(-\mu+k, -\mu-k)$ as coefficient of z^{-1} where $k = -\deg(b_{21}) - 1 > 0$ [recall the permutation applied to $B(z)$]. The value k can also be directly found from $a_{21}(z) = \sum_{j=k-1}^{r-2} a_{21}^{(j)} z^j$ where $a_{21}^{(k-1)} \neq 0$. This shows that $k \leqq r-1$ because of $a_{21}^{(r-2)} \neq 0$. If we undo the replacement of z by $1/z$ we see that an analytic transformation exists changing A_{-1} to $\operatorname{diag}(\mu-k, \mu+k)$ $(k \in \mathbb{N})$ which enables us to restore all the required properties by using a final similarity. These preparations are analytic at infinity and hence do not change λ' but they show how we can avoid the case $A_{-1}=\begin{bmatrix}\mu & b\\ 0 & \mu\end{bmatrix}$. Therefore we can always change λ' by an arbitrary, integral multiple of 1/2. Because this is done so easily we may assume that we fix one of these values of λ' and leave it untouched during the remaining transformation which then must be a polynomial in z^{-1}. The only assumption we will make about this choice is that if $\mu_1-\mu_2\in\mathbb{Z}$ then actually $\mu_1-\mu_2\equiv 0 \pmod 2$. Otherwise we are not able to make congruent eigenvalues of A_{-1} equal by means of polynomial transformations in z^{-1}. If $\mu_1-\mu_2$ is odd, we change λ' to $\lambda'-1/2$ and thereby obtain an even difference. Then we can demand w. l. o. g. that $\operatorname{tr} A\equiv 0$ since this is guaranteed by the use of $e^{p(z)}z^{\lambda'/2-1/4}$ instead of $e^{p(z)}$ alone as a preliminary scalar factor. We are now able to define when we call [A] a *normalized standard equation* (NSE), viz.

(i) $A(z)$ is in standard form with minimal Poincaré-rank $r\in\mathbb{N}$.

(ii) $\operatorname{tr} A\equiv 0$.

(iii) $A_{r-1}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$.

(iv) A_{-1} is either lower triangular or $\begin{bmatrix}\mu & b\\ 0 & \mu\end{bmatrix}$ with $b\neq 0$.

(v) If A_{-1} equals $\begin{bmatrix}\mu & b\\ 0 & \mu\end{bmatrix}$ or μI, then $\operatorname{diag} A_{r-2}=\operatorname{diag}(0, 0)$.

(vi) If $\mu_1\equiv\mu_2 \pmod 1$ holds then already $\mu_1\equiv\mu_2 \pmod 2$.

[Notice that $\operatorname{tr} A\equiv 0$ leads in condition (iv) and (v) immediately to $\mu=0$.]

These NSE's will turn out to contain the least number of parameters because they cannot be generated by a subcollection with less parameters. It is in this sense that they are the simplest representatives. Of course, the above normalizations cause some dependencies among the $4(r+1)$ complex parameters of such an equation. The four coefficients of A_{r-1} are fixed by (iii). Because of condition (ii) the r parameters in the (2,2)-position depend on the remaining ones and condition (iv) shows that one further parameter (in A_{-1}) is zero. Hence an NSE contains $3r-1$ free parameters, i. e. parameters whose domain contains an open set. If we consider only those NSE's where A_{-1} has eigenvalues μ_1, μ_2 that are congruent modulo 1, then $\operatorname{tr} A\equiv 0$ and condition (vi) force $\mu_1, \mu_2\in\mathbb{Z}$. Hence they assume only countably many, discrete values which reduces the number of free parameters to $3r-2$. Furthermore we may restrict our attention to NSE's with $A_{-1}=\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}$ which have $3r-3$ free parameters for $b\neq 0$ and $3r-4$ free parameters for $b=0$ as condition (v) shows. This last case is only possible if $r\geqq 2$.

Any meromorphic transformation between NSE's is a polynomial in z^{-1} which can have the following two effects: exchange of μ_1 and μ_2 or leading to μ_1+k, μ_2-k $(k\in\mathbb{Z})$. This suggests to describe its impact by grouping the NSE's in types. To do that we recall the splitting $\mu_i=m_i+k_i$ with $0\leqq m_i<1$ and $k_i\in\mathbb{Z}$ $(i=1, 2)$. Besides the integers k_1, k_2 we also introduce two criteria:

(1) $m_1 > m_2$, $m_1 < m_2$ or $m_1 = m_2$.

(2) If $m_1 = m_2$ then either the monodromy M is $m_1 \mathrm{I}$ (no logarithm) or $\begin{bmatrix} m_1 & 0 \\ m & m_1 \end{bmatrix}$ ($m \neq 0$, with logarithm).

There exist exactly four possibilities according to (1) and (2). Now a *type* is defined as the set of all NSE's corresponding to a particular choice of k_1, k_2 and one of the four possibilities. From this definition one deduces immediately that there are only countably many types and each NSE belongs to exactly one of them.

Remark 5. — The condition that $\operatorname{tr} \mathrm{A}$ vanishes identically restricts the possible values of m_i, k_i ($i=1, 2$). First we see that $m_1 + m_2 \in \mathbb{Z}$, and $0 \leqq m_i < 1$ leads to the following possibilities:

$$m_1 + m_2 = 0 \quad \Leftrightarrow \quad m_1 = m_2 = 0 \quad \Leftrightarrow \quad k_1 + k_2 = 0$$

or

$$m_1 + m_2 = 1 \quad \Leftrightarrow \quad m_1 \neq m_2 \quad \Leftrightarrow \quad k_1 + k_2 = -1,$$

since condition (vi) excludes the case $m_1 = m_2 = 1/2$.

Remark 6. — The discussion in Remark 4 proves that two NSE's within the same type [where we exclude $\mathrm{M}=0$ in (2)] can only be meromorphically equivalent if they are similar. But then the normalizations in this section guarantee that they are identical. Hence *any NSE is unique within its type under meromorphic equivalence* (if $\mathrm{M} \neq 0$). If $\mathrm{M}=0$, the situation is much more intricate. We know already that besides similarities also transformations of degree $\mu_1 - \mu_2$ may relate NSE's within one type. Therefore uniqueness is only true if we restrict our transformations to similarities or our NSE's to $\mathrm{A}_{-1} = 0$ since in the latter case the arguments of [4], p. 175 apply.

For the rest of this section and the next two sections we exclude the case $\mathrm{M}=0$ unless something different is stated. We want to discuss how the NSE's of one type can be transfromed into NSE's of another type. For that purpose, we fix the two types, say t_1 and t_2. Then Remark 6 guarantees that to any NSE in t_1 there exists at most one meromorphically equivalent NSE in t_2. We call the totality of all these equivalent pairs the *complete mapping* between t_1 and t_2. These complete mappings are the main objects of our further investigations because they describe exactly the possible effects of meromorphic transformations, namely exchanging μ_1 and μ_2 or changing them by an integer (in opposite directions).

5. On the number of parameters

The key to our results lies in the observation that the complete mappings between types can also be characterized by the way in which they transform the original parameters. The occuring functions are not just arbitrary, but show a specifically piecewise rational behavior. This makes them accessible for considerations which involve algebraic independence or measure theory. It turns out that the best suited measure for our purposes is the *p-dimensional outer Hausdorff-measure* on $\mathbb{R}^n$ ($n \in \mathbb{N}$, $0 \leqq p \leqq n$) which is defined for a set $S \subseteqq \mathbb{R}^n$ as $H_p(S) = \sup_{\varepsilon > 0} \inf \sum_{i=1}^{\infty} d^p(A_i)$ where $S \subseteqq \bigcup_{i=1}^{\infty} A_i$ and the diameters $d(A_i) < \varepsilon$ for all i ([7], p. 53). Moreover S is called *σ_p-finite* if $S = \bigcup_{i=1}^{\infty} S_i$ with $H_p(S_i) < \infty$ (for all i). The sets S which we encounter are not just arbitrary ones, but rather possess a *precise dimension p*, by which we mean that S is σ_p-finite with $H_p(S) > 0$. We want to apply these notions to functions between complex parameters. Therefore we think of these parameters as split into real and imaginary part, which yields twice as many real parameters, without actually mentioning it.

In order to define the essential properties of the occuring functions we fix subsets X, Y, Z ($\neq \varnothing$) of $\mathbb{C}^N$ (with fixed $N \in \mathbb{N}$). Moreover in the sequel, all polynomials and rational functions belong to $\mathbb{Q}(x)$. Then a set $S \subseteqq X$ is called a P-*set* in X if there exist polynomials $p_1, \ldots, p_r, \ldots, p_s$ ($r, s \in \mathbb{N}_0$, $r \leqq s$) such that

$$S = \{x \in X : p_j(x) = 0 \text{ for } 1 \leqq j \leqq r \text{ and } p_k(x) \neq 0 \text{ for } r < k \leqq s\}.$$

A function f mapping a subset of X into Y is called a PR-*function* ($f: X \to Y$) if there is a disjoint partition $X = \bigcup_{i=0}^{n} X_i$ ($n \in \mathbb{N}_0$) of X such that $X_1, \ldots, X_n$ are non-empty P-sets in X, f is defined exactly on $\bigcup_{i=1}^{n} X_i = X - X_0$ and the restrictions $f|_{X_i}: X_i \to Y$ have (componentwise) rational representations (for $1 \leqq i \leqq n$). Two PR-functions $f: X \to Y$ and $g: Y \to Z$ can be composed to $h = g \circ f$ which is defined on the maximal set $X' \subseteqq X - X_0$ such that $f(X') \subseteqq Y - Y_0$ and maps X' into Z. A PR-function $f: X \to Y$ is called PR-*invertible* or a PR-*transformation* if f possesses an inverse g which is also a PR-function ($g: Y \to X$). We use the notation $g = f^{-1}$ and assume w. l. o. g. that the partitions $X = \bigcup_{i=0}^{n} X_i$ for f and $Y = \bigcup_{j=0}^{n'} Y_j$ for g are always such that $n = n'$ and $f(X_i) = Y_i$ ($1 \leqq i \leqq n$) which can be obtained by refining the initial partitions ([6], section 3). Now we list the properties of these functions which we will use later:

(i) The composition of two PR-transformations is a PR-transformation.

(ii) Suppose f is a PR-function (on X) and $M \subseteqq X$ is σ_p-finite resp. satisfies $H_p(M) = 0$ for some p ($0 \leqq p \leqq 2N$). Then $f(M)$ is σ_p-finite resp. $H_p(f(M)) = 0$. Thus, a PR-transformation preserves the precise dimension of a set.

(iii) If $x_1, \ldots, x_N$ ($(x_1, \ldots, x_N) \in X - X_0$) have transcendency degree t ([8], p. 229) over a field F with $\mathbb{Q} \subseteqq F \subseteqq \mathbb{C}$ and f is a PR-transformation on X, then the components of $f(x_1, \ldots, x_N)$ have the same transcendency degree over F.

The proofs of these properties as well as a detailed discussion of these concepts can be found in [6], section 3 and Lemma 3 and Proposition 4. In the discussion of transformations between NSE's we make extensive use of Theorem 1. It relates the transformation to the coefficient-functions of a solution. Thus, in a first step, we examine the relation between the differential equations and their solutions $F(z) z^K z^M$ which were introduced in section 3.

LEMMA 2. — *Let a meromorphic differential equation* [A], *normalized as in section* 3 (*but without requiring that* $m=1$ *for* $\mu_1 \neq \mu_2$) *be given. It has a unique solution of the form* $F(z) z^K z^M$ *with* $F(z)=I+O(z^{-1})$ *if, in case of* $\mu_1-\mu_2 \in \mathbb{N}$, *we further demand that in* f_{21} *the coefficient of* $z^{\mu_2-\mu_1}$ *is zero.*

Proof. — According to [3], p. 191 f we know that [A] has a solution $F(z) z^K z^M$ with $F(z)=I+O(z^{-1})$. Then any solution is of the form $F(z) z^K z^M C$ with constant, invertible C. Hence the question is: How can we use C to obtain another solution $\tilde{F}(z) z^{\tilde{K}} z^{\tilde{M}}$ with $\tilde{F}(z)=I+O(z^{-1})$. From [3], p. 179 ff we learn that our K and M are uniquely determined by [A]. Hence $z^M C z^{-M}=z^{-K} F^{-1} \tilde{F} z^K$ must be single-valued which shows that C must commute with M ([4], p. 38). We distinguish three cases:

(i) $m_1 \neq m_2$: Here $C=\operatorname{diag}(c_1, c_2)$ and therefore $\tilde{F}=FC=C+O(z^{-1})$. Hence $C=I$, i. e. the solution is unique.

(ii) $m_1=m_2$, $M=\begin{bmatrix} m_1 & 0 \\ m & m_1 \end{bmatrix}$, $m \neq 0$: In this case $C=\begin{bmatrix} c_1 & 0 \\ c_2 & c_1 \end{bmatrix}$ and thus $\tilde{F}=F\begin{bmatrix} c_1 & 0 \\ c_2 z^{k_2-k_1} & c_1 \end{bmatrix}$. From that we deduce $c_1=1$ and the coefficient of $z^{k_2-k_1}$ in the (2,1)-position f_{21} can be annihilated. Recall that in this case $k_2 \leqq k_1$. Hence our extra-assumption for $\mu_1-\mu_2 \in \mathbb{N}$ leads to $c_2=0$, and again this solution is unique.

(iii) $m_1=m_2$, $M=m_1 I$: Here C is arbitrary with $\det C \neq 0$. But

$$\begin{bmatrix} c_{11} & c_{12} z^{k_1-k_2} \\ c_{21} z^{k_2-k_1} & c_{22} \end{bmatrix}=z^K C z^{-K}=F^{-1} \tilde{F}=I+O(z^{-1})$$

shows that $C=I$ for $k_1=k_2$ resp. $\tilde{F}=F\begin{bmatrix} 1 & 0 \\ c_{21} z^{k_2-k_1} & 1 \end{bmatrix}$ for $k_1>k_2$. Therefore we draw the same conclusions as in (ii) to see that the prescribed solution is unique.

Q.E.D.

Remark 7. — From [3], p. 179 ff we also see that in this unique solution with $F=[f_{ij}]$ and $f_{ij}(z)=\sum_{k=0}^{\infty} f_{ij}^{(k)} z^{-k}$ the coefficients $f_{ij}^{(k)}$ are rational functions of the coefficients $a_{uv}^{(w)}$ $(1 \leqq w \leqq k+1)$ where we use the notation $A=[a_{uv}]$, $a_{uv}(z)=\sum_{w=1}^{\infty} a_{uv}^{(w)} z^{-w}$. Furthermore the (2,1)-element in M is, in case of $\mu_1-\mu_2 \in \mathbb{N}_0$, a rational function of the $a_{uv}^{(w)}$ $(1 \leqq w \leqq \mu_1-\mu_2+1)$. Moreover the denominators of these rational functions must be polynomials in μ_1, μ_2. In the case that $A(z)$ has only a fixed, finite number of coefficient-matrices which may not vanish, we see that m and the $f_{ij}^{(k)}$ are rational functions of the parameters in $A(z)$. This applies e. g. to our standard equations if we replace z by $1/z$. With this information we will now describe more carefully the parameter-sets we encounter. The NSE's are defined in terms of equations $\left(\operatorname{tr} A \equiv 0,\ A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ or conditions that certain expressions do not vanish ($a_{21}^{(r-2)} \neq 0$ because of the minimal Poincaré-rank); furthermore if $\mu_1 \neq \mu_2$ then $a_{12}^{(-1)}$ is 0 and $\mu_1-\mu_2$ is not an odd integer and if $\mu_1-\mu_2=0$ we have the choice between $a_{21}^{(-1)} \neq 0$, $a_{12}^{(-1)}=0$ resp. $a_{21}^{(-1)}=0$, $a_{11}^{(r-2)}=0$ corresponding to the different normalizations of A_{-1}. Thus the parameter-set of all NSE's is the finite union of linear manifolds from which countably many lower-dimensional submanifolds are excluded. The main-part is determined by:

$$\operatorname{tr} A \equiv 0, \qquad A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$a_{21}^{(r-2)} \neq 0, \qquad a_{21}^{(-1)} \neq 0, \qquad a_{12}^{(-1)}=0, \qquad \mu_1-\mu_2 \notin \mathbb{Z}$$

and has precise dimension $6r-2$. All other submanifolds have smaller precise dimensions. A type is a subset of all NSE's and includes inequalities of the type $k_1 \leqq \mu_1 < k_1+1$ for some given $k_1 \in \mathbb{Z}$ as well as e. g. $m_1=\mu_1-k_1 > \mu_2-k_2=m_2$. Furthermore for $m_1=m_2=0$ we distinguish between the cases in which $\log z$ does resp. does not occur; this is decided by checking whether $m \neq 0$ or $m=0$ and by Remark 7 this results in checking if a certain polynomial vanishes. This shows that our types are Borel-sets and hence always measurable. Furthermore we deduce that in many cases our types have precise dimensions, viz.

$6r-2$ if $m_1 \neq m_2$,

$6r-4$ if $\mu_1=\mu_2=0$ and $\log z$ occurs,

$6r-8$ if $\mu_1-\mu_2=0$ and $\log z$ does not occur (recall Remark 5).

In the following the sets X, Y contain exactly the parameter vectors of all NSE's belonging to some given type. We assume again that types with $M=0$ are excluded so that in the case of criterion (2) we are always in the situation that $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$. In this situation we can now formulate the main result about the structure of the transformations between types. It will enable us to explain that our NSE's have the least number of parameters, but it is also the key to a discussion of possible simplifications of individual equations.

THEOREM 2. — *The complete mapping between two given types induces a PR-transformation between the coefficients of their occuring NSE's. This PR-transformation can be explicitly determined.*

Proof: First we should remark that there are many pairs of types between whose NSE's no transformations at all are possible, namely $m_1 \neq m_2$ can be never be transformed into $m_1=m_2$ or vice versa. In these cases the claim is trivial since $X=X_0$ where X is the parameter-set of NSE's of the initial type. Another trivial case is the identity map of one type onto itself which is a PR-transformation. From now on we exclude these cases. The exchange of μ_1 and μ_2 for $\mu_1 \neq \mu_2$ is one of the complete mappings since its occurrence is competely determined by the type because of opposite orderings of m_1 and m_2 in criterion (1) and reversed values of k_1 and k_2; the corresponding transformation was computed in section 4 and leads to a PR-function. Since every meromorphic transformation possesses an inverse we can analogously construct the transformation leading from the final type back to the initial one. But then the uniqueness of an NSE within its type as explained in Remark 5 shows that the two transformations are inverses of one another thus proving the claim.

The last conclusion is true in all of our cases, hence in the following it suffices to show that a complete mapping leads to a PR-function. Let us first consider the case $m_1 \neq m_2$. We want to apply Theorem 1 of which only the first two possibilities can occur since $M=0$ is excluded.

To do this we must replace z by z^{-1}, i.e. consider $-A(z^{-1})/z^2$ instead of $A(z)$. We call this matrix again $A(z)$. First we have to normalize A_{-1} so that it matches the assumptions of Theorem 1. For that purpose we diagonalize it by $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$. The new parameters are rational functions of the old ones. If the ordering due to criterion (1) of the type does not match the requirements of Theorem 1 we permute using $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We call the resulting matrix again $A(z)$. The integers k_1, k_2 of the second type as well as the ordering of m_1, m_2 prescribed there enable us to determine the admissable matrix B_{-1} from which we compute the value $\tilde{\mu}_1 - \mu_1 = k \in \mathbb{Z}$ by which the eigenvalues of A_{-1} are changed. Notice that k depends only on the two types. Since $k=0$ was treated before we assume w.l.o.g. that $k > 0$. We compute the coefficients $f_{12}^{(j)}$, $f_{22}^{(j)}$ of the series $f_{12}(z)$, $f_{22}(z)$ in the unique solution of Lemma 2 $(0 \leqq j \leqq 2k)$ from $A(z)$ by means of rational functions as explained in Remark 7. Hence these are also rational functions of the original parameters. Now we must solve $p(z) f_{12}(z) + q(z) f_{22}(z) = O(z^{-k-1})$ with relatively prime polynomials p, q satisfying: $\deg(p) = k, p$ is monic and $\deg(q) \leqq k$. This is equivalent to solving a system of $2k+1$ equations in the $2k+1$ unknown coefficients of p and q. We recall from the proof of Lemma 1 that relatively prime here is equivalent to uniqueness. But this uniqueness can be checked by determining whether the determinant of the system vanishes or not, which is a polynomial condition leading to a P-set. In the first case there are no transformations for the corresponding NSE's whereas in the second case the coefficients of p and q are determined by Cramer's rule as rational functions of the parameters. Then the transformation $T(z)$ is completed by computing polynomials r and s as explained in Remark 4 which is again done by solving a system of $2k$ linear equations for the $2k$ coefficients of r and s. Since we already know that r and s are unique we can use Cramer's rule immediately. Thus $T(z)$ is found and $T(z^{-1})$ can be applied to the original $A(z)$ yielding a standard equation $B(z)$ of minimal Poincaré-rank r with the prescribed coefficient-matrix B_{-1}. Thus the parameters of $B(z)$ are rational functions of the original parameters. The last step is to find an NSE in the second type which is similar to $B(z)$. For that purpose we first permute $B(z)$ by

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if this is required by the ordering of m_1 and m_2 in the second type. Then we use $\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$ to transform the coefficient-matrix of z^{-1} to the form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Obviously this last step also leads to a PR-function. More precisely, we have seen that the complete mapping is factored into three steps, of which the first is always possible while the other two involve a polynomial restriction in each case and the complete mapping requires both of these restrictions. Moreover we see that the complete mapping is given by a single rational transformation on the P-set where it is defined. Summarizing the construction we learn from Remark 2 and Theorem 1 that the above procedure yields the complete mapping between the two types, and we see that it in fact induces a PR-function which proves the claim in this case. The case $k < 0$ is treated analogously. Now we continue with $m_1 = m_2 = 0$, i. e. $M = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$. The proof proceeds along the same line as for $m_1 \neq m_2$, and we will only point out the necessary adjustments. For $k_1 \neq k_2$ we normalize A_{-1} as before. Whereas in case $k_1 = k_2 = 0$ we face two situations. If $a_{12}^{(-1)} \neq 0$ and hence $a_{21}^{(-1)} = a_{11}^{(r-2)} = 0$ we permute A by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, otherwise ($a_{12}^{(-1)} = 0$) we leave A. In all the cases we now have to guarantee that m is 1 before Theorem 1 applies. Hence we compute m which is a rational function of the parameters and use $T = \operatorname{diag}(1, m)$ which leads to $m = 1$. These steps can always be performed. Then Theorem 1 is applied as before and followed by a similarity which leads to an NSE of the given type. In this last step we face again the exceptional situation corresponding to $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ as coefficient-matrix of z^{-1} if $\tilde{k}_1 = \tilde{k}_2 = 0$ in the image type. Here it is characterized by $b_{12}^{(r-1)} = 0$ since $B_{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Thus we can use the transformation $\begin{bmatrix} 0 & 1 \\ b_{21}^{(r-1)} & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ in this case where c is used to adjust B_{r-2}.

With these supplements the previous arguments carry over to this case if we notice that $\tilde{k}_1 = \tilde{k}_2 = 0$ in the image type only if $k_1 \neq k_2$ in the original type since $k = 0$ was treated before and is thus excluded. We see again that the complete mapping is given by a rational transformation on a single P-set unless $k_1 = k_2 = 0$ either in the preimage type or in the image type. To complete the proof we merely observe that each of the steps above allows the explicit computations of all necessary quantities.

Q.E.D.

Theorem 2 enables us to decide whether a given NSE of one type can be meromorphically transformed into another type by checking certain polynomial equations. If it is possible the new equation is unique and can be explicitly computed.

Furthermore the proof shows that the coefficients of the transformation-matrix T are given as PR-functions of the original parameters. [We remind the reader of the fact that the degree of the polynomial $T(z^{-1})$ is given by the two types as Remark 4 tells.]

COROLLARY 1. — *Let a collection of NSE's be given whose parameters form a set which is σ_p-finite (resp. with $H_p=0$ resp. with precise dimension p) for some $p\,(0 \leqq p < 6r-2)$. Then the parameters of all meromorphically equivalent NSE's form a set which is still σ_p-finite (resp. with $H_p=0$ resp. with precise dimension p.).*

Proof. — All the properties under consideration are σ-additive. Hence we can restrict our attention to only those initial NSE's which belong to a fixed type and consider all NSE's which are meromorphically equivalent to them and lie in another fixed type. By Theorem 2 these two types are connected by a PR-transformation whose properties assure exactly what we want to prove.

Q.E.D.

This Corollary is the exact formulation of the statement that our NSE's are simplest possible. Their parameters form a set of precise dimension $6r-2$ which cannot be obtained by meromorphic equivalence from a subcollection with a parameter-set S that satisfies $H_{6r-2}(S)=0$. The same is true if we restrict ourselves to equations with $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$, whose parameter-set has precise dimension $6r-4$ and which can also not be generated from NSE's with a smaller set of parameters.

Although Theorem 2 clarifies the structure of the complete mapping between two types as a PR-transformation of the corresponding coefficients this may still lead to the apprehension that the involved functions are very complicated. In the following we will demonstrate that actually one rational function suffices to transform almost all NSE's of one type into the other if such a transformation is possible at all. It is only on the remaining nullset that other functions occur.

COROLLARY 2. — *Let two types be given such that both satisfy either $m_1 \neq m_2$ or $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} (m \neq 0)$. Then there exists a polynomial p over $\mathbb{Z}$ such that:*

(i) *Any NSE of the first type can certainly be meromorphically transformed into one of the second type if p does not vanish when the coefficients of the original NSE are inserted. And then the new coefficients are rational functions of the original ones.*

(ii) *All NSE's of the first type for which p vanishes yield a parameter-set which is σ_{6r-4}-finite if $m_1 \neq m_2$ resp. σ_{6r-6}-finite if $m_1=m_2=0$.*

Proof. — We want to take a closer look at the proof of Theorem 2. First we recall that the assumptions for the two types do not immediately make a transformation impossible since they assure that k_1+k_2 stays constant (*see* Remark 5). The cases of the identity ($p \equiv 1$) or the exchange of μ_1 and μ_2 if they are different ($p=a_{21}^{(-1)}$) obviously satisfy the claim. Hence they are excluded from now on. In the proof of Theorem 2 the transformation between the two types was constructed in four steps. In the first A_{-1} was normalized according to the requirements of Theorem 1. Then the coefficients $f_{ij}^{(l)}$ and m were computed. Afterwards the polynomials p, q, r and s were obtained from Theorem 1 and Remark 4. And finally the resulting equation was normalized corresponding to the second type. We show that each of these steps is executed by applying one (componentwise) rational function to the coefficients of the differential equation if one polynomial does not vanish. The normalization of A_{-1} is always possible and can be carried out by one rational function if $A_{-1}=\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ is avoided. This is guaranteed when we require $a_{21}^{(-1)} \neq 0$.

Then the $f_{ij}^{(l)}$ and m are rational functions of the old coefficients. To find p, q, r and s we had to solve two systems of linear equations which was possible if the determinant D of the first was different from zero. Then Cramer's rule gave the needed rational functions. Finally we permuted by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if required by the second type and tried to normalize the obtained matrix $B(z)$ which is done by applying $\begin{bmatrix} 1 & 0 \\ -b_{11}^{(r-1)}/b_{12}^{(r-1)} & 1/b_{12}^{(r-1)} \end{bmatrix}$ to come up with the leading coefficient $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ if $b_{12}^{(r-1)} \neq 0$. Hence we consider $p=a_{21}^{(-1)} D\, b_{12}^{(r-1)}$ as the desired polynomial. Part (i) is thus proved, and it remains to show that also (ii) holds. First we discuss this for types with $m_1 \neq m_2$. To any type with $\operatorname{diag}(A_{-1})=\operatorname{diag}(\mu_1, -\mu_1)$ we add all equations of the form $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^{r-1}+\ldots+\begin{bmatrix} \mu_1 & 0 \\ 0 & -\mu_1 \end{bmatrix} z^{-1}$ with $\deg(a_{12})=r-2$ and vanishing trace. Their parameter vectors form the sets X, Y. Notice that the additional equations occur only if $r \geqq 2$ and are not equivalent to any of the NSE's in the given type. With the equations of an initial set X we want to perform a minimal step leading to μ_1+1 resp. μ_1-1. This is done by diagonalizing A_{-1} using $\begin{bmatrix} 1 & 0 \\ a_{21}^{(-1)}/(2\mu_1) & 1 \end{bmatrix}$. Then the possibility to change μ_1 as required depends on either $a_{12}^{(0)} \neq 0$ or $a_{21}^{(0)}-a_{21}^{(-1)} a_{11}^{(0)}/\mu_1-a_{12}^{(0)}(a_{21}^{(-1)})^2/(4\mu_1^2) \neq 0$ (see section 3). These two polynomials vanish on σ_{6r-4}-finite P-sets. If we avoid the corresponding set we can change μ_1 as desired and normalize the resulting equation to obtain parameter-vectors in a set Y. The inverse operation is again a minimal step and thus we find that we deal with a PR-transformation $(f: X \to Y)$ such that f resp. f^{-1} are defined everywhere but on σ_{6r-4}-finite sets $X_0 \subseteqq X$ resp. $Y_0 \subseteqq Y$. In $X-X_0$ and $Y-Y_0$ we consider the sets formed by the additional equations which are not NSE's. Together with their images under f resp. f^{-1} they form σ_{6r-4}-finite sets. Hence if we restrict f to the NSE's in X and Y, we see that the domain of f and its image-set fill up X and Y up to σ_{6r-4}-finite sets. The same is true if we simply want to exchange μ_1 and μ_2 which depends on $a_{21}^{(-1)} \neq 0$. This shows that the complete mapping between two types (with $m_1 \neq m_2$) maps almost all NSE's of the first type onto almost all equations of the second type (the exceptional sets beeing σ_{6r-4} finite) if $\tilde{k}_1=k_1 \pm 1$ or $\tilde{k}_2=k_1 \pm 1$ (minimal steps). But then it holds in general since every complete mapping includes a multiple application of minimal steps. What must be changed if $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$? First we consider $k_1=k_2=0$. Here we include the equations with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^{r-1}+\ldots+\begin{bmatrix} 0 & 0 \\ a_{21}^{(-1)} & 0 \end{bmatrix}$ and $a_{12}^{(r-2)} \neq 0$, $a_{21}^{(r-1)} \neq 0$ and vanishing trace into X. Then the minimal step depends on $a_{12}^{(0)} \neq 0$. Since the original set has precise dimension $6r-4$, the same now holds for its image. But then we can use the above reasoning to deduce the same consequences for the complete mapping between two types with $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$, where only $6r-4$ is replaced by $6r-6$. Furthermore we find inductively that every type has a parameter-set of precise dimension $6r-4$.

But then p satisfies also part (ii) since $a_{21}^{(-1)} \neq 0$ excludes only a σ_{6r-4}- (resp. σ_{6r-6}-) finite set, $D \neq 0$ is in this case necessary according to Theorem 1 and $b_{12}^{(r-1)} \neq 0$ excludes the possibility of a final equation where the coefficient of z^{-1} is upper triangular, again belonging to a set which is σ_{6r-4}-(resp. σ_{6r-6}-) finite.

Q.E.D.

Remark 8. — This proof also shows that to every type there belongs a parameter-set of precise dimension $6r-2$ for $m_1 \neq m_2$ resp. $6r-4$ for $M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ $(m \neq 0)$; especially the rational expression for m in the coefficients of the differential equation is not only not identically zero on this set, but its zeros can constitute at most a set which is σ_{6r-6}-finite. It follows:

The complete mapping between two types is either empty or takes almost all equations of the first type onto almost all equations of the second type (in the sense that for $r=1$ there are no exceptions and for $r>1$ the exceptional sets are σ_{6r-4}- resp. σ_{6r-6}-finite as can be seen in the proof).

We already pointed out that Corollary 1 can be applied to various situations. One of these will be discussed in the following. In the theory of meromorphic differential equations the equivalence classes under meromorphic equivalence are characterized by invariants, i. e. quantities which are associated to the differential equations but remain unchanged if the equations are transformed ([4], p. 124). Some of these are easily identified in the equation whereas others (e. g. connection matrices) have a rather difficult relation to the equation. Therefore it is reasonable to fix those invariants which are easy to compute and investigate the restricted PR-transformations. This view is supported by the fact that the fixed invariants give rise to invariant relations between our parameters which introduce a certain structure into these PR-transformations. These invariants are the equivalence-classes modulo 1 of the actual monodromy (i. e. m_1, m_2) as well as the formal invariant $Q(z)$ ([4], p. 124). Notice that $\lambda'=1/4$ because of the condition $\operatorname{tr} A \equiv 0$; but even if we relax this to $\operatorname{tr} A = O(z^{-1})$ we can obtain $\lambda' \equiv m_1+m_2$ (mod 1). How is $Q(z)$ related to $A(z)$? We want to explain that under the conditions, that $\operatorname{tr} A \equiv 0$ and [A] has minimal Poincaré-rank $r \geqq 1$ with $A_{r-1}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we can compute $Q'(z)=\operatorname{diag}(q'(z), -q'(z))$ exactly as the leading part of the eigenvalues of $A(z)$. Because of $\operatorname{tr} A \equiv 0$ these eigenvalues are $\lambda(z)$ resp. $-\lambda(z)$, and we know that $\det A = -\lambda^2(z)$. On the other hand $A(z)$ can be computed as the logarithmic derivative of the formal solution $F(z)z^{1/4}\operatorname{diag}(1, z^{-1/2})\, U e^{Q(z)}$ of Lemma 1 which yields

$$\mathrm{A}(z)=\mathrm{F}\left(\begin{bmatrix} 1/(4z) & z^{1/2}q'(z) \\ z^{-1/2}q'(z) & -1/(4z) \end{bmatrix}+\mathrm{F}^{-1}\mathrm{F}'\right)\mathrm{F}^{-1}.$$

We recall that $\deg(q')=r-3/2$ and therefore obtain

$$\det \mathrm{A}=\det\left(\begin{bmatrix} 1/(4z) & z^{1/2}q' \\ z^{-1/2}q' & -1/(4z) \end{bmatrix}+O(z^{-2})\right)=-(q')^2+O(z^{r-3}).$$

Hence we find $\lambda^2=(q')^2(1+O(z^{-r}))$ and, if we choose λ with the same leading coefficient as q',

$$\lambda=q'(1+O(z^{-r}))=q'+O(z^{-3/2})$$

or

$$(q'+O(z^{-3/2}))^2=a_{11}^2+a_{12}a_{21}.$$

The last equation suggests to replace $a_{21}^{(r-2)}$ by q_{r-1}^2 where we use the notation $q'(z)=\sum_{k=0}^{r-1} q_k z^{k-1/2}$. We require $q_{r-1}>0$ (lexicographically) and notice that all results so far remain correct since q_{r-1} is an invariant and therefore unchanged during our transformation. Then the equation between q' and the a_{ij} shows that if we replace the parameters $a_{21}^{(k)}$ by q_{k+1} $(-1\leqq k\leqq r-3)$ the two parameter-sets are related by a PR-transformation (actually rational transformation) where the denominators are powers of q_{r-1}. If we fix $m_1\neq m_2$ and $q_0, \ldots, q_{r-1}$ the permitted NSE's are determined from a_{11} (which contains exactly $r-1$ free complex parameters and one free integer) and a_{12} (which contains exactly $r-1$ further free complex parameters).

If $\mathrm{M}=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m\neq 0$ we first consider only NSE's with $\mathrm{A}_{-1}=\begin{bmatrix} 0 & 0 \\ . & 0 \end{bmatrix}$.

If $q_0, \ldots, q_{r-1}$ are fixed these NSE's are determined from a_{11} (which contains exactly $r-1$ free complex parameters) and a_{12} whose coefficients are restricted by the requirement that $a_{21}^{(-1)}\neq 0$. When we investigate the equation $(q'+O(z^{r-3/2}))^2=a_{11}^2+a_{12}a_{21}$ we find that $a_{21}^{(-1)}$ occurs there for the first time in the coefficient of z^{r-2} which is also true for $a_{12}^{(0)}$. Thus $a_{21}^{(-1)}\neq 0$ is equivalent to the fact that $a_{12}^{(0)}$ avoids a certain value which is determined by a polynomial in q_k $(0\leqq k\leqq r-1)$, $a_{11}^{(k)}$ $(0\leqq k\leqq r-2)$ and $a_{12}^{(k)}$ $(1\leqq k\leqq r-2)$ divided by q_{r-1}^2. Thus $a_{12}^{(k)}$ are also free complex parameters for $0\leqq k\leqq r-2$ besides this condition for $a_{12}^{(0)}$. Thus we face a parameter set which does not satisfy $\mathrm{H}_{4r-4}=0$. On the other hand for given q' and $k_1\in\mathbb{Z}$, a_{21} is completely determined by a_{11}, a_{12} which contain only $2r-2$ complex parameters $\left(\text{even if } \mathrm{A}_{-1}=\begin{bmatrix} 0 & . \\ 0 & 0 \end{bmatrix} \text{ since then } \deg a_{11}<r-2\right)$. Thus we find:

The NSE's with fixed values of m_1, m_2, $q_0, \ldots, q_{r-1}$ *and* $\mathrm{M}\neq m_1\mathrm{I}$ *have a parameter-set in* $\mathbb{C}^{4(r+1)}$ *of precise dimension* $4r-4$ *regardless of whether* $m_1=m_2$ *or* $m_1\neq m_2$. These numbers can therefore not be reduced by meromorphic transformations.

After these measure-theoretic considerations we also want to look at a single NSE and ask whether we can simplify it, e. g. by transforming it into an NSE with the maximal number of zero-coefficients. In general this can only be found out by checking whether countably many polynomials vanish or not when we insert the coefficients of our NSE. Nevertheless we can replace this in many cases by computing the transcendency degree ([8], p. 229) of the coefficients.

COROLLARY 3. — *Suppose that* [A] *and* [B] *are meromorphically equivalent NSE's with* $M \neq m_1 I$. *If the coefficients of* $A(z)$ *have transcendency degree* t *over a field* F $(\mathbb{Q} \subseteqq F \subseteqq \mathbb{C})$, *then the coefficients of* $B(z)$ *have the same transcendency degree over* F.

This follows immediately from the general properties of PR-transformation (rational coefficients!).

This Corollary can be applied in many concrete situations with $F=\mathbb{Q}$ or $F=\mathbb{Q}(m_1, m_2)$ or $F=\mathbb{Q}(m_1, m_2, q_{r-1}, \ldots, q_0)$.

Remark 9. — At this point we can eliminate the requirement that in the case of eigenvalues μ, $-\mu$ of A_{-1} which are congruent modulo 1 their difference should in fact be even. The case of an odd difference can be adjusted by $\begin{bmatrix} 0 & z^{1/2} \\ \lambda^2 z^{-1/2} & 0 \end{bmatrix}$. $\left(\text{Notice that } A_{-1}=\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ cannot occur in this case.}\right)$ Of course, this transformation is not meromorphic but it induces a rational transformation between the coefficients which enables us to extend all measure-theoretic results to this case as well as the considerations concerning the transcendency degree. Thus Theorem 2 and its Corollaries remain true if we replace the assumption "$M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$" by the two possibilities "$M=\begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ or $M=\begin{bmatrix} 1/2 & 0 \\ m & 1/2 \end{bmatrix}$" with $m \neq 0$.

6. Representatives

All of our results on the irreducibility of NSE's were obtained when $M=0$ was excluded. The reason for this can be found in the observation that NSE's with $M=0$ have $3r-4$ free, complex parameters if $k_1=k_2$ but seem to have $3r-3$ complex parameters for $k_1 \neq k_2$ if we take the polynomial equation $m=0$ into account. This suggests that these NSE's can be reduced if we can attain $k_1=k_2$ by meromorphic transformations. Unfortunately this is not always possible as we will explain, but it gives the idea at least to minimize $|\mu_1-\mu_2|$. This is another moment where our discussion of the minimal steps is fruitful. They enable us to decide whether or not $|\mu_1-\mu_2|$ can be decreased. For that purpose we recall that $\mu_1=-\mu_2 \in \mathbb{Z}$ holds and we only deal with the case $\mu_1 \neq 0$. To clarify our strategy we start with the easiest case $\mu_1 < \mu_2$, i. e. $\mu_1 < 0$, and A_{-1} diagonal. Then we can apply Theorem 1 immediately after we replaced z by z^{-1}. There the first and third possibility match our situation since $|\mu_1-\mu_2|$ should become smaller. The first condition leads to a minimal value of $k=-\deg(a_{21}(z^{-1}))+1$

but this is only possible if $\tilde{\mu}_1 = \mu_1 + k \leqq 0$ due to the imposed ordering $\tilde{\mu}_1 \leqq \tilde{\mu}_2$, i. e. if $-\deg(a_{21}(z^{-1})) + 1 \leqq -\mu_1$. Otherwise we can reduce $|\mu_1 - \mu_2| = -2\mu_1$ only by using the third possibility. Again we pay attention to the fact that we want $|\tilde{\mu}_1 - \tilde{\mu}_2| < |\mu_1 - \mu_2|$. Since in the present case $k = -\deg(a_{21}(z^{-1})) + 1 > -\mu_1$, this is possible if and only if $k < |\mu_1 - \mu_2|$ (see our discussion at the end of section 3).

Altogether this proves that $|\mu_1 - \mu_2|$ can be decreased if and only if $-\deg(a_{21}(z^{-1})) + 1 < |\mu_1 - \mu_2|$. The other cases are now easily reduced to the one which we discussed. If A_{-1} is not diagonal, but still lower triangular because of our normalizations, we first diagonalize it using $\begin{bmatrix} 1 & 0 \\ a_{21}^{(-1)}/2\mu_1 & 1 \end{bmatrix}$ and then apply the above result with $a_{21}(z)$ replaced by

$$a_{21}(z) - a_{11}(z)\, a_{21}^{(-1)}/\mu_1 - a_{12}(z)\,(a_{21}^{(-1)})^2/(4\mu_1^2).$$

In the case $\mu_1 > \mu_2$ we diagonalize A_{-1} and then permute A by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ which leads to the above situation with a_{21} replaced by a_{12}. To summarize these results we introduce the function

$$h(z) = \begin{cases} a_{21}(z) - a_{11}(z)\, a_{21}^{(-1)}/\mu_1 - a_{12}(z)\,(a_{21}^{(-1)})^2/(4\mu_1^2) & \text{if} \quad \mu_1 < \mu_2 \\ a_{12}(z) \quad \text{if} \quad \mu_1 \geqq \mu_2 \end{cases}$$

Then $|\mu_1 - \mu_2|$ *is minimal if and only if*

$$-\deg(h(z^{-1})) + 1 \geqq |\mu_1 - \mu_2|.$$

The assumption that [A] is an NSE and hence not diagonal enables us to find a bound for the minimal $|\mu_1 - \mu_2|$, namely $|\mu_1 - \mu_2| \leqq r$; if $\mu_1 < \mu_2$ holds and A_{-1} is diagonal we even know that $|\mu_1 - \mu_2| < r$. Since $\mu_1, \mu_2 \in \mathbb{Z}$ we learn that $r \geqq 2$ is necessary since this follows immediately from our inequalities if $\mu_2 = -\mu_1 \neq \mu_1$; and for $\mu_1 = 0$ we know already that $\log z$ must occur if $r = 1$.

Let us now discuss this minimal situation. We know that a non-trivial similarity can only be applied to it if A_{-1} is not diagonal and then results in an exchange of μ_1 and μ_2. In view of the definition of $h(z)$ this suggests that *we require* $\mu_1 > \mu_2$ *whenever* A_{-1} *is not diagonal* which simplifies $h(z)$ to $a_{21}(z)$ in the case of $\mu_1 < \mu_2$. Then $A(z)$ is unique under similarities, and we can only influence it using the third possibility in Theorem 1 with $0 < k = |\mu_1 - \mu_2|$. Up to constant factors this is done by the transformation $T(z) = \begin{bmatrix} 1 & p(z) \\ 0 & 1 \end{bmatrix}$ with $p(z) = \sum_{l=0}^{k} p_l z^{-l} (p_k \neq 0)$. We apply it to $A(z) = [a_{ij}]$ if $\mu_1 < \mu_2$ and obtain the matrix $\begin{bmatrix} a_{11} - a_{21}p & a_{12} + 2a_{11}p - p' - a_{21}p^2 \\ a_{21} & -a_{11} + a_{21}p \end{bmatrix}$ (recall $a_{22} = -a_{11}$). We already know that $h(z) = a_{21}(z)$ satisfies $-\deg(h(z^{-1})) + 1 \geqq |\mu_1 - \mu_2| = k$; but it can even be seen that this must be a strict inequality because otherwise we apply the shearing $\operatorname{diag}(1, z^k)$ and come up with a coefficient of z^{-1} consisting of one single Jordan-block. This would be a case where $\log z$ occurs in contradiction to our assumption that $M = m_1 I$. Hence $-\deg(a_{21}(z^{-1}))+1 > k$. From that we learn that we can choose p_k arbitrary and always compute the other coefficients from the (1,2)-position recursively such that the resulting equation is an NSE satisfying all of our assumptions ($p_k = 0$ can be included because it leads to $p \equiv 0$). This suggests to use p_k to annihilate the coefficient of z^s in $a_{11}(z)$ where $s = -k - \deg(a_{21}(z^{-1})) \geqq 0$. Thus *the* NSE *with minimal value* $|\mu_1 - \mu_2|$ *is unique if we furthermore require that* $a_{11}^{(s)} = 0$ *with* $s = -|\mu_1 - \mu_2| - \deg(a_{21}(z^{-1}))$. This could even be done if $\mu_1 - \mu_2 = 0$ or if $A_{-1} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $b \neq 0$. The case $\mu_1 > \mu_2$ is treated analogously by $T = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}$ where p has no constant term and shows that [A] *is unique if* $a_{11}^{(s)} = 0$ *with* $s = -|\mu_1 - \mu_2| - \deg(a_{12}(z^{-1}))$.

These normalizations do not only lead to a unique representative in each equivalence class but also to an NSE without an unnecessarily large number of free parameters. This suggests a similar procedure for the other cases. There we also want a prescribed ordering of μ_1 and μ_2 as well as minimality of $|\mu_1 - \mu_2|$. If $M = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$, $m \neq 0$, we treat the equation as before, but then we can only use the second condition of Theorem 1 which allows to influence $|\mu_1 - \mu_2|$ by $k = -\deg(h(z^{-1}))+1$ if $k \leqq |\mu_1| = |\mu_1 - \mu_2|/2$. Thus in this case $|\mu_1 - \mu_2|$ *is minimal if and only if* $-\deg(h(z^{-1}))+1 > |\mu_1 - \mu_2|/2$, with an exception if $A_{-1} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $b \neq 0$. To include this case also we simply permute A by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Finally, consider $m_1 \neq m_2$. Here minimizing $|\mu_1 - \mu_2|$ is equivalent to minimizing $|\mathrm{Re}(\mu_1 - \mu_2)|$.

Again we want to apply Theorem 1 and prepare (only for convenience) the equation as we did above with $\mathrm{Re}(\mu_1) \leqq \mathrm{Re}(\mu_2)$ [replace μ_i be $\mathrm{Re}(\mu_i)$]. Then the second condition of Theorem 1 can be used and yields a minimal change of μ_1 to $\mu_1 + k$ with $k = -\deg(h(z^{-1}))+1$. This is only a progress if $\mathrm{Re}(\mu_1)+k < \mathrm{Re}(\mu_2)$. Hence $|\mu_1 - \mu_2|$ *is minimal if and only if* $-\deg(h(z^{-1}))+1 > |\mathrm{Re}(\mu_1 - \mu_2)|$. If equality holds we have two different NSE's with minimal $|\mathrm{Re}(\mu_1 - \mu_2)| \geqq 1$ which can be meromorphically transformed into one another with a minimal step. An application of a minimal step $\begin{bmatrix} 1 & p(z) \\ 0 & 1 \end{bmatrix}$ [if $\mathrm{Re}(\mu_1) < \mathrm{Re}(\mu_2)$] resp. $\begin{bmatrix} 1 & 0 \\ p(z) & 1 \end{bmatrix}$ [if $\mathrm{Re}(\mu_1) > \mathrm{Re}(\mu_2)$] shows that the values μ_1 of the two equations differ by the sign of $\mathrm{Im}(\mu_1)$ (even if we prescribe $\mu_1 \geqq \mu_2$ when A_{-1} is not diagonal). Hence we can guarantee uniqueness if we require $\mathrm{Im}(\mu_1) < 0$. Furthermore, note that in this case $\mathrm{Re}(\mu_1 - \mu_2) \in \mathbb{Z}$ holds and therefore $\mathrm{Re}(\mu_1)$ is not a free parameter. Hence these equations belong to a subset of precise dimension $6r-3$. We can summarize these observations in a definition of "meromorphic representatives" as follows (compare with the definition of NSE's in section 4)

(i)-(ii) remain unchanged;

(iii′) A_{-1} is lower triangular;

(iv′) $A_{r-1}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ or $A_{r-1}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ $\Bigg($but the second case may occur only if $A_{-1}=\begin{bmatrix}0&0\\b&0\end{bmatrix}$, $b\neq 0\Bigg)$;

(v′) $\mu_1 \geqq \mu_2$ when A_{-1} is not diagonal; and $\mu_1-\mu_2$ is not an odd integer;

(vi′) $|\mathrm{Re}(\mu_1-\mu_2)| \leqq -\deg(h(z^{-1}))+1$ unless $M=\begin{bmatrix}0&0\\m&0\end{bmatrix}$, $m\neq 0$, in which case $|\mathrm{Re}(\mu_1-\mu_2)| < 2(-\deg(h(z^{-1}))+1)$ with

$$h=\begin{cases} a_{12} & \text{if } \mathrm{Re}(\mu_1) \geqq \mathrm{Re}(\mu_2) \\ a_{21} & \text{otherwise} \end{cases}$$

(vii′) If $M=0$ or $A_{r-1}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$, then the coefficient $a_{11}^{(s)}$ is zero where $s=-|(\mu_1-\mu_2)|-\deg(h(z^{-1}))$;

(viii′) If $\mu_1 \not\equiv \mu_2 \pmod 1$ and $|\mathrm{Re}(\mu_1-\mu_2)|=-\deg(h(z^{-1}))+1$ then $\mathrm{Im}(\mu_1)<0$.

We then know that *these "representatives" are unique under meromorphic equivalence.*

Thus we have found "natural" representatives which can easily be identified and such that every NSE can be transformed by an explicit meromorphic transformation into one of them. We remark that the number of possible types corresponding to these representatives is $O(r)$.

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