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ABELIAN SURFACES AND KOWALEWSKI'S TOP

BY A. LESFARI (1)

ABSTRACT. — This paper presents a new and systematic method to integrate the problem of Kowalewski's rigid body motion, and leads to a detailed geometric description of the invariant surfaces (tori) on which the motion evolues.

Introduction

This paper deals with a geometric and systematic approach to the integration of Kowalewski's rigid body motion. It is well known that this motion is completely integrable and Kowalewski [16] integrates the problem in terms of hyperelliptic quadratures after a complicated and mysterious change of variables. The classical approach to solving integrable systems was based on solving the Hamilton-Jacobi equation by separation of variables, after an appropriate change of coordinates; for every problem finding this transformation required a great deal of ingenuity. Up to now Kowalewski's method has been neither understood, nor improved nor extended to other cases except for some modest amelioration contributed by Kötter [15] and Kolossoff [14]. This paper presents a new and systematic method to integrate the problem, and leads to a detailed geometric description of the invariant surfaces (tori) on which the motion evolves.

As is well known, a Hamiltonian system

$$\dot{z}=J\frac{\partial H}{\partial z}, \quad J=J(z) \text{ antisymmetric, } z\in\mathbb{R}^{2n}$$

is called completely integrable (in the $\mathscr{C}^\infty$ sense), if it has n constants of the motion $H_1, \ldots, H_n$ in involution with linearly independent gradients. By the Arnold-Liouville theorem, the compact and connected invariant manifolds

$$\bigcap_{j=1}^{n}\{H_j=c_j,\ z\in\mathbb{R}^{2n}\}, \qquad c_j\in\mathbb{R}$$

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are diffeomorphic to a real torus and there is a transformation to so-called action-angle variables, mapping the flow into a straight line motion on that torus. In most examples, the tori (of the Arnold-Liouville theorem) are real parts of complex algebraic tori (called Abelian varieties): they come equipped with an algebraic addition law. Adler and van Moerbeke ([3], [4]) have called such systems algebraically completely integrable. They have developed methods, at first, to recognize such integrable systems among families of Hamiltonian systems and, at second, to integrate such problem in terms of Abelian integrals; this approach is inspired by the work of Kowalewski. For example in [4], the criterion is used to detect the algebraic completely integrable geodesic flows on SO(4) for a left invariant diagonal metric: the only one leading to an integrable flow is Manakov's metric. This problem was integrated using coadjoint orbits on Kac-Moody Lie algebras [2]. Mumford [4] then recognized the nature of its invariant tori and Haine [10] used the Laurent solutions to the differential equations to realize the invariant tori as Prym varieties on which the flow linearizes. Recently, Adler and van Moerbeke ([5], [6], [7]) have classified the algebraically completely integrable geodesic flows on SO(4) for a left invariant metric and developped a general and effective method to integrate such systems.

This paper deals with the Kowalewski case in the dynamic of the rigid body, which will now be explained. With Arnold [8], the differential equations of motion of a rigid body about a fixed point are given by the customary Euler-Poisson equations

$$\text{(Int 1)} \qquad \begin{cases} \dot{\mathrm{M}} = [\mathrm{M}, \Lambda\,\mathrm{M}] + \mu g [\Gamma, \mathrm{L}] \\ \dot{\Gamma} = [\Gamma, \Lambda\,\mathrm{M}] \end{cases}$$

where

$$\mathrm{M} = (\mathrm{M}_{jk})_{1 \leq j, k \leq 3} \equiv \sum m_j e_j \equiv \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \in \mathrm{so}(3)$$

$$\Lambda\,\mathrm{M} = (\Lambda_{jk}\,\mathrm{M}_{jk})_{1 \leq j, k \leq 3} = \sum \mathrm{I}_j^{-1} m_j e_j \in \mathrm{so}(3)$$

$$\Gamma = (\Gamma_{jk})_{1 \leq j, k \leq 3} \equiv \sum \gamma_j e_j \equiv \begin{bmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{bmatrix} \in \mathrm{so}(3)$$

and

$$\mathrm{L} = \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix}.$$

M, Γ, L, $\mathrm{I} = \mathrm{diag}(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$, μ and g denote respectively the angular momentum, the directional cosine of the z-axis (fixed in space), the center of gravity, the principal moment of inertia of the body, the mass of the body and the acceleration of gravity, all expressed in the body coordinates. In the absence of gravity i. e. $\mathrm{L} = 0$, we have the Euler free rigid body motion around a fixed point. The Lagrange top corresponds to the case $I_1=I_2$, $l_1=l_2=0$ i. e. the motion of a body around a fixed point symmetric around one principal axis of inertia and the center of gravity is on the axis of symmetry. The so-called Kowalewski top corresponds to the case $I_1=I_2=2I_3$, $l_3=0$ i. e. the center of gravity belongs to the equatorial plane, passing through the fixed point. Moreover, we may choose $l_2=0$, $\mu g l_1=1$ and $I_3=1$. After the substitution $t \to 2t$ the system (Int 1) becomes

(Int 2)
$$\begin{cases} \dot{m}_1=m_2 m_3 \\ \dot{m}_2=-m_1 m_3+2\gamma_3 \\ \dot{m}_3=-2\gamma_2 \\ \dot{\gamma}_1=2m_3\gamma_2-m_2\gamma_3 \\ \dot{\gamma}_2=m_1\gamma_3-2m_3\gamma_1 \\ \dot{\gamma}_3=m_2\gamma_1-m_1\gamma_2 \end{cases}$$

and possesses the four invariants

(Int 3)
$$\begin{cases} H_1=\frac{1}{2}(m_1^2+m_2^2)+m_3^2+2\gamma_1=C_1 \\ H_2=m_1\gamma_1+m_2\gamma_2+m_3\gamma_3=C_2 \\ H_3=\gamma_1^2+\gamma_2^2+\gamma_3^2=C_3=1 \\ H_4=\left[\left(\frac{m_1+im_2}{2}\right)^2-(\gamma_1+i\gamma_2)\right]\left[\left(\frac{m_1-im_2}{2}\right)^2-(\gamma_1-i\gamma_2)\right]=C_4, \end{cases}$$

where we may choose $C_3=1$ without loss of generality. Let A be the complex affine variety defined by the intersection of the constants of the motion

(Int 4)
$$A=\bigcap_{j=1}^{4}\{H_j=c_j\}\subseteq\mathbb{C}^6.$$

The first section explains how the affine variety A and vector-fields behave after the quotient by some natural involution on A and how these vector-fields become well defined when we take Kowalewski's variables. We show that these variables are naturally related to the so-called Euler's differential equations and can be seen as the addition-formula for the Weierstrassian elliptic function. In the second section, which is the main part of the paper, we show that the Kowalewski top is algebraically completely integrable in the Adler-van Moerbeke sense discussed above. The basic tool for doing this, is to consider the five parameter family of Laurent solutions

$$M(t)=\frac{M^0}{t}+M^1+M^2t+\ldots$$

$$\Gamma(t)=\frac{\Gamma^0}{t^2}+\frac{\Gamma^1}{t}+\Gamma^2+\Gamma^3t+\ldots$$

where the five free parameters α, β, λ, θ, μ appear linearly, whenever they appear for the first time. In fact these expansions contain a lot of information, which can be used to construct the abelian surfaces on which the flow linearizes. For instance, substituting these expansions into the constants of motion, leads to 4 polynomial relations between α, β, λ, θ, μ, hence defining a reducible algebraic curve $\mathscr{D}$ of genus 9, with two components of genus 3, each of which is a double ramified cover of an elliptic curve. Then, we complete A into an abelian surface by adjoining the curve $\mathscr{D}$; the abelian surface obtained this way can be embedded into $\mathbb{CP}^7$ via the sections of the line bundle going with $\mathscr{D}$ and its period matrix has the type

$$\begin{pmatrix}1 & 0 & a & c\\ 0 & 2 & c & b\end{pmatrix}\text{ with Im}\begin{pmatrix}a & c\\ c & b\end{pmatrix}>0.$$

It can also be realized as the dual of the Prym variety of the double cover of the elliptic curve mentioned above.

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This paper is dedicated to Professor A. Sadel.

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References.

Note. Some results were obtained recently by Horozov-van Moerbeke [Abelian surfaces of polarization (1, 2) and Kowalewski's top, *Comm. Pure Applied Math.*, 1987], Adler-van Moerbeke [About Lax pair for the Kowalewski's top (to appear)] and Haine-Horozov in a forthcoming paper.

1. About Kowalewski's procedure

Let

$$f:\quad A \rightarrow A : (m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) \rightsquigarrow (x_1, x_2, m_3, y_1, y_2, \gamma_3)$$

be a birational map on the affine variety A (Int 4) where x_1, x_2, y_1 and y_2 are defined as

$$(1.1)\qquad \begin{cases} 2x_1 = m_1 + i m_2 \\ 2x_2 = m_1 - i m_2 \\ y_1 = x_1^2 - (\gamma_1 + i\gamma_2) \\ y_2 = x_2^2 - (\gamma_1 - i\gamma_2). \end{cases}$$

In term of these new variables, equations (Int. 2) with $t \rightarrow it$ and (Int. 3) take the following form where $C_1 \equiv 6h_1$, $C_2 \equiv 2h_2$ and $C_4 \equiv k^2$

$$(1.2)\qquad \begin{cases} \dot{x}_1 = m_3 x_1 - \gamma_3 \\ \dot{x}_2 = -m_3 x_2 + \gamma_3 \\ \dot{m}_3 = -x_1^2 + y_1 + x_2^2 - y_2 \\ \dot{y}_1 = 2m_3 y_1 \\ \dot{y}_2 = -2m_3 y_2 \\ \dot{\gamma}_3 = x_1(x_2^2 - y_2) - x_2(x_1^2 - y_1) \end{cases}$$

and

$$(1.3)\qquad A: \begin{cases} y_1 y_2 = k^2 \\ m_3^2 = 6h_1 + y_1 + y_2 - (x_1 + x_2)^2 \\ m_3 \gamma_3 = 2h_2 + x_1 y_2 + x_2 y_1 - x_1 x_2 (x_1 + x_2) \\ \gamma_3^2 = 1 - k^2 + x_1^2 y_2 + x_2^2 y_1 - x_1^2 x_2^2. \end{cases}$$

Note that

$$(1.4)\qquad \tau:\quad (x_1, x_2, m_3, y_1, y_2, \gamma_3) \rightsquigarrow (x_1, x_2, -m_3, y_1, y_2, -\gamma_3)$$

is an automorphism of A, of order two. The quotient $\mathbb{B} \equiv A/\tau$ by the involution τ, is a *Kummer surface* defined by

$$(1.5)\qquad \mathbb{B}: \begin{cases} y_1 y_2 = k^2 \\ y_1 R(x_2) + y_2 R(x_1) + R_1(x_1, x_2) + k^2 (x_1 - x_2)^2 = 0 \end{cases}$$

with

$$(1.6)\quad \begin{cases} R(X) \equiv -x^4+6h_1x^2-4h_2x+1-k^2 \\ R_1(x_1, x_2) \equiv -6h_1x_1^2x_2^2+4h_2x_1x_2(x_1+x_2) \\ \qquad -(1-k^2)(x_1+x_2)^2+6h_1(1-k^2)-4h_2^2. \end{cases}$$

The variety A is a double cover of the surface $\mathbb{B}$ branched over the fixed points of the involution τ. To find them, we substitute $m_3=\gamma_3=0$ in the system (1.3), to wit

$$(1.7)\quad \begin{cases} (a) & y_1y_2=k^2 \\ (b) & y_1+y_2=(x_1+x_2)^2-6h_1 \\ (c) & x_2y_1+x_1y_2=x_1x_2(x_1+x_2)-2h_2 \\ (d) & x_2^2y_1+x_1^2y_2=x_1^2x_2^2+k^2-1. \end{cases}$$

Away from the $x_1^2=x_2^2$, we may solve (*b*) and (*d*) in y_1 and y_2 and substitute into the remaining equations; one then finds two curves in x_1 and x_2

$$(1.8)\quad \begin{cases} R(x_1, x_2) \equiv -x_1^2x_2^2+6h_1x_1x_2-2h_2(x_1+x_2)+1-k^2=0 \\ S(x_1, x_2) \equiv (x_1^4+2x_1^3x_2-6h_1x_1^2+1-k^2) \\ \qquad \times (x_2^4+2x_1x_2^3-6h_1x_2^2+1-k^2)+k^2(x_1^2-x_2^2)^2=0 \end{cases}$$

which intersect at the zeroes of the resultant of R, S:

$$(1.9)\qquad \operatorname{Res}(R, S)_{x_2}=x_1^2(x_1^4+6h_1x_1^2+k^2-1)^2P_8(x_1),$$

where $P_8(x_1)$ is a monic polynomial of degree 8. Since the root x_1 must be excluded (it indeed implies that the leading terms of R and S vanish), the possibles intersections of the curve R and S will be

(i) at the roots of $x_1^4+6h_1x_1^2+k^2-1=0$: this is inacceptable, because then one checks that the common roots of R and S would have the property that $x_1^2=x_2^2$, which was excluded.

(ii) at the roots of $P_8(x_1)=0$; there, for generic k and h, $x_1^2 \neq x_2^2$.

Finally, we must analyze the case $x_1^2=x_2^2$ for which one checks that (1.7) has no common roots. Consequently the involution τ has 8 fixed points on the affine variety A. Clearly the vector field (1.2) vanishes at the fixed points of τ.

Now, equations (1.5) imply that

$$y_1=\frac{-1}{2R(x_2)}[R_1(x_1, x_2)+k^2(x_1-x_2)^2+w]$$

$$y_2=\frac{-1}{2R(x_1)}[R_1(x_1, x_2)+k^2(x_1-x_2)^2-w]$$

with a radical w such that

$$w^2=[R_1(x_1, x_2)+k^2(x_1-x_2)^2]^2-4k^2 R(x_1) R(x_2) \equiv \Psi(x_1, x_2).$$

This shows that the surface $\mathbb{B}$ is a double cover of the plane x_1, x_2 ramified along the curve

$$(1.10) \qquad \mathscr{C}: \quad \Psi(x_1, x_2)=0.$$

This equation is reducible and can be written as the product $\Psi_1(x_1, x_2) . \Psi_2(x_1, x_2)$ of two symmetric polynomials (in x_1, x_2) of degree two in each one of the variables x_1, x_2, i. e.,

$$\Psi_1(x_1, x_2)=A(x_1) x_2^2+2 B(x_1) x_2-C(x_1)=A(x_2) x_1^2+2 B(x_2) x_1-C(x_2)$$

where $A(x)$, $B(x)$ and $C(x)$ are three polynomials of degree two in x:

$$\begin{aligned} A(x) &\equiv -2(k+3h_1) x^2+4h_2 x-1 \\ B(x) &\equiv 2h_2 x^2+(2k(k+3h_1)-1) x-2h_2 k \\ C(X) &\equiv x^2+4h_2 kx+2(k^2-1)(k+3h_1)+4h_2^2. \end{aligned}$$

The curve $\mathscr{C}_1$ given by the symmetric equation

$$(1.11) \qquad \mathscr{C}_1: \quad \Psi_1(x_1, x_2)=0$$

is elliptic:

$$(1.12) \qquad \begin{aligned} x_2 &= \frac{-B(x_1) \pm \sqrt{2(k+3h_1)-4h_2^2} \sqrt{R(x_1)}}{A(x_1)} \\ x_1 &= \frac{-B(x_2) \pm \sqrt{2(k+3h_1)-4h_2^2} \sqrt{R(x_2)}}{A(x_2)} \end{aligned}$$

where $R(x)$ is given by (1.6). Let $\mathscr{C}_2$ be the curve defined by (1.11) but after switching the sign of k. The curve $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect in 8 distinct points which happen to be the fixed points of the involution τ, because

$$\operatorname{Res}(\Psi_1, \Psi_2)_{x_2}=16k^2 P_8(x_1)$$

where $P_8(x_1)$ is given by (1.9).

Now, differentiating the symmetric equation $\Psi_1(x_1, x_2)=0$ [or $\Psi_2(x_1, x_2)=0$] with regard to t, one finds

$$\frac{\partial \Psi_1}{\partial x_1} \dot{x}_1+\frac{\Psi_2}{\partial x_2} \dot{x}_2=0$$

where

$$\frac{\partial \Psi_1}{\partial x_1} = 2(A(x_2)x_1 + B(x_2)) = \pm 2\sqrt{2(k+3h_1) - 4h_2^2}\sqrt{R(x_2)} \quad \text{by (1.12)}.$$

Hence

$$\frac{\dot{x}_1}{\sqrt{R(x_1)}} \pm \frac{\dot{x}_2}{\sqrt{R(x_2)}} = 0. \tag{1.13}$$

Since $R(x_1)$ and $R(x_2)$ are two polynomials of the fourth degree in x_1 and x_2 respectively and having the same coefficients, then (1.13) is the so-called Euler's equation. The reader is referred to Halpen [12] and Weil [23] for this theory which we summarize here after.

Let $F(x) = a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$ be a polynomial of the fourth degree. The general integral of Euler's equation

$$\frac{\dot{x}}{\sqrt{F(x)}} \pm \frac{\dot{y}}{\sqrt{F(y)}} = 0$$

can be written in two different ways:

(i) $$F_1(x, y) + 2sF(x, y) - s^2(x-y)^2 = 0$$

where

$$F(x,y) = a_0 x^2 y^2 + 2a_1 xy(x+y) + 3a_2(x^2+y^2) + 2a_3(x+y) + a_4$$

and

$$F_1(x,y) \equiv \frac{F(x)F(y) - F^2(x,y)}{(x-y)^2}.$$

(ii) or in an irrational form

$$\frac{F(x,y) \mp \sqrt{F(x)}\sqrt{F(y)}}{(x-y)^2} = s$$

which can be seen as the addition-formula for the Weierstrassian elliptic function

$$2\wp(u+v) = \frac{(\wp(u)+\wp(v))(2\wp(u).\wp(v) - (1/2)g_2) - g_3 - \wp'(u)\wp'(v)}{(\wp(u)-\wp(v))^2}$$

$$\wp'(u) \equiv \frac{d\wp}{du} = \pm\sqrt{4\wp^3 - g_2\wp - g_3}$$

with $\wp(u) = x$, $\wp(v) = y$, $\wp'^2(u) = F(x)$, $\wp'^2(v) = F(y)$, $F(x) = 4x^3 - g_2 x - g_3$, $2\wp(u+v) = s$ and g_2, g_3 constants.

Let us now apply these facts to Kowalewski's problem with $a_0=-1$, $a_1=0$, $a_2=h_1$, $a_3=-h_2$, $a_4=1-k^2$, $F(x)=R(x)$, $F(x_1, x_2)=R(x_1, x_2)+3h_1(x_1-x_2)^2$ and $s=k+3h_1$. So the polynomial (1.11) which can also be regarded as a solution of (1.13), can also be written as:

$$R_1(x_1, x_2)+2kR(x_1, x_2)-k^2(x_1-x_2)^2=0$$

where $R_1(x_1, x_2)$ is given by (1.6) and has the form

$$R_1(x_1, x_2)=\frac{R(x_1)R(x_2)-R^2(x_1, x_2)}{(x_1-x_2)^2}.$$

Remember that $R(x_1, x_2)$ is given by (1.8). The solution of (1.13) can also be expressed as:

$$\frac{R(x_1, x_2)\mp\sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1-x_2)^2}+3h_1=s. \tag{1.14}$$

Let us carry out the calculations, assuming the polynomial $R(x)$ reduced to the form $4x^3-g_2x-g_3$ and call s_1 (resp. s_2) the relation (1.14) with the sign $-$ (resp. $+$). Now, outside the branch locus of $\mathbb{B}$ (1.5) over $\mathbb{C}^2$, the equation (1.13) is not identically zero and may be written in the form

$$\left\{\begin{aligned}&\frac{\dot{x}_1}{\sqrt{R(x_1)}}+\frac{\dot{x}_2}{\sqrt{R(x_2)}}=\frac{\dot{s}_1}{\sqrt{4s_1^3-g_2s_1-g_3}}\neq 0\\&\frac{\dot{x}_1}{\sqrt{R(x_1)}}-\frac{\dot{x}_2}{\sqrt{R(x_2)}}=\frac{\dot{s}_2}{\sqrt{4s_2^3-g_2s_2-g_3}}\neq 0\end{aligned}\right. \tag{1.15}$$

where $g_2=k^2-1+3h_1^2$ and $g_3=h_1(k^2-1-h_1^2)+h_2^2$, After some algebraic manipulation we deduce from (1.4)

$$(m_3x_1-\gamma_3)^2=R(x_1)+(x_1-x_2)^2y_1$$
$$(m_3x_2-\gamma_3)^2=R(x_2)+(x_1-x_2)^2y_2$$
$$(m_3x_1-\gamma_3)(m_3x_2-\gamma_3)=R(x_1, x_2)$$

and from (1.3)

$$\dot{x}_1^2=R(x_1)+(x_1-x_2)^2y_1$$
$$\dot{x}_2^2=R(x_2)+(x_1-x_2)^2y_2.$$

This together with (1.5) and (1.15) implies that

$$\frac{\dot{s}_1^2}{4s_1^3-g_2s_1-g_3}=\left(\frac{\dot{x}_1}{\sqrt{R(x_1)}}+\frac{\dot{x}_2}{\sqrt{R(x_2)}}\right)^2$$

$$=\frac{(x_1-x_2)^4}{R(x_1)R(x_2)}\left[\left(\frac{R(x_1,x_2)-\sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1-x_2)^2}\right)^2-k^2\right]=4\frac{(s_1-3h_1)^2-k^2}{(s_1-s_2)^2}.$$

In the same way, we find

$$\frac{\dot{s}_2^2}{4s_2^3-g_2s_2-g_3}=\left(\frac{\dot{x}_1}{\sqrt{R(x_1)}}-\frac{\dot{x}_2}{\sqrt{R(x_2)}}\right)^2=4\frac{(s_2-3h_1)^2-k^2}{(s_2-s_1)^2}.$$

Consequently, the system (1.1) can be written as follows

$$(1.16)\qquad \left\{\begin{aligned}&\frac{ds_1}{\sqrt{P_5(s_1)}}+\frac{ds_2}{\sqrt{P_5(s_2)}}=0\\ &\frac{s_1\,ds_1}{\sqrt{P_5(s_1)}}+\frac{s_2\,ds_2}{\sqrt{P_5(s_2)}}=i\,dt\end{aligned}\right.$$

where $P_5(s)\equiv((s-3h_1)^2-k^2)(4s^3-g_2s-g_3)$. As known, such integrals are called *hyperelliptic integrals* and the problem can be integrated in terms of genus two hyperelliptic functions of time. Finally, we have the

THEOREM 1.1. — (*i*) *The complex affine variety* A (Int 4) *is a double ramified cover on the Kummer surface* $\mathbb{B}$ (1.5), *with eight branch points* (*=zeroes of the polynome* $P_8(x_1)$ (1.9)). (ii) *The surface* $\mathbb{B}$ *is a double cover of plane* (x_1, x_2) *ramified along two elliptic curves intersecting each other at the* 8 *points above.* (iii) [16] *The system of differential equations* (Int 2) *is reduced to the system* (1.16) *which can be integrated in terms of genus* 2 *hyperelliptic functions.*

2. A geometric approach to study Kowalewski's top

The system (Int 2) can be written as a Hamiltonian vector field

$$(2.1)\qquad \dot{z}=J\frac{\partial H_1}{\partial z};\qquad z=\begin{bmatrix}m_1\\m_2\\m_3\\\gamma_1\\\gamma_2\\\gamma_3\end{bmatrix},\quad J=\begin{pmatrix}M & \Gamma\\ \Gamma & 0\end{pmatrix}\quad \text{with } M,\Gamma\in so(3).$$

The second Hamiltonian vector field

$$(2.2)\qquad \dot{z}=J\frac{\partial H_4}{\partial z}$$

is quartic and is written explicitly as

$$\dot{m}_1=A\,m_2m_3+B(-m_1m_3+2\gamma_3)$$

$$\dot{m}_2 = \mathrm{B}\, m_2 m_3 - \mathrm{A}\,(-m_1 m_3 + 2\gamma_3)$$
$$\dot{m}_3 = 2\,\mathrm{A}\,(-m_1 m_2 + \gamma_2) + \mathrm{B}\,(m_1^2 - m_2^2 - 2\gamma_1)$$
$$\dot{\gamma}_1 = (\mathrm{A}\, m_2 - \mathrm{B}\, m_1)\,\gamma_3$$
$$\dot{\gamma}_2 = (\mathrm{A}\, m_1 + \mathrm{B}\, m_2)\,\gamma_3$$
$$\dot{\gamma}_3 = (\mathrm{B}\, m_1 - \mathrm{A}\, m_2)\,\gamma_1 - (\mathrm{B}\, m_2 + \mathrm{A}\, m_1)\,\gamma_2$$

where $\mathrm{A} \equiv m_1^2 - m_2^2 - 4\gamma_1$ and $\mathrm{B} \equiv 2(m_1 m_2 - 2\gamma_2)$. These vector fields are in *involution*, i. e.

$$\{\mathrm{H}_1, \mathrm{H}_4\} = \left\langle \frac{\partial \mathrm{H}_1}{\partial z}, \mathrm{J}\frac{\partial \mathrm{H}_4}{\partial z} \right\rangle = 0$$

and the remaining ones are *Casimir functions*, i. e.

$$\mathrm{J}\frac{\partial \mathrm{H}_2}{\partial z} = \mathrm{J}\frac{\partial \mathrm{H}_3}{\partial z} = 0.$$

To illustrate the method (announced in the introduction) in a simple example, let us first examine the *Euler rigid body motion* around a fixed point. The system (Int 1) reduces in this case to the following equation

(*a*)
$$\dot{\mathrm{M}} = [\mathrm{M}, \Lambda\,\mathrm{M}]$$

which is explicitely given by

(*b*)
$$\begin{cases} \dot{m}_1 = (\lambda_3 - \lambda_2)\, m_2 m_3 \\ \dot{m}_2 = (\lambda_1 - \lambda_3)\, m_{1\ 3} \\ \dot{m}_3 = (\lambda_2 - \lambda_1)\, m_1 m_2 \end{cases}$$

where $\lambda_j = 1/\mathrm{I}_j\,(1 \leqq j \leqq 3)$ and has the two first integrals

(*c*)
$$\begin{cases} \mathrm{H}_1 = m_1^2 + m_2^2 + m_3^2 = c_1 \\ \mathrm{H}_2 = \lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 = c_2 \end{cases}$$

where c_1 and c_2 are constants. The system (*b*) with Hamiltonian $(1/2)\,\mathrm{H}_1$ is completely integrable on the phase space which is the sphere $\mathrm{H}_1 = c_1$. For appropriate values of the constants, this sphere intersects the ellipsoid $\mathrm{H}_2 = c_2$ in two circles and any representative point of (*b*) moves on a circle of the sphere (uniform motion). It is well known that the system (*b*) can be integrated in terms of elliptic functions. From system (*b*) together with conditions (*c*) we obtain the following expression

(*d*)
$$\int_{m_3(t_0)}^{m_3(t)} \frac{dm}{[(a_1^2 - m^2)\,(a_2^2 - m^2)]^{1/2}} = a_3\,(t - t_0)$$

where

$$a_3^2=(\lambda_2-\lambda_3)(\lambda_3-\lambda_1), \qquad a_1^2=\frac{\lambda_2 c_1-c_2}{\lambda_2-\lambda_3}$$

and

$$a_2^2=\frac{c_2-\lambda_1 c_1}{\lambda_3-\lambda_1}, \qquad \lambda_1\neq\lambda_2\neq\lambda_3.$$

If we replace $t\in\mathbb{R}$ by $t\in\mathbb{C}$ then the function $m_3(t)$ is an elliptic function on a complex torus and must have a Laurent expansion around an arbitrary complex valued constant t_0. Therefore since a_1, a_2 and a_3 only depend on c_1 and c_2, these two free parameters must enter somewhere in the expansion of m_3 around the blow up point. Indeed, it is easy to see that the Laurent series solution of the system (a) has the form

$$\mathrm{M}(t)=\sum_{k=0}^{\infty}\mathrm{M}^k(t-t_0)^{k-1}.$$

Substituting this series into (a), one finds at the 0-th step a non-linear equation

$$(e) \qquad \mathrm{M}^0+[\mathrm{M}^0, \Lambda\,\mathrm{M}^0]=0,$$

and at the k-th step, a linear equation

$$(\mathscr{L}-k\,\mathrm{I})(\mathrm{M}^k)=\text{terms containing } \mathrm{M}^j \text{ for } 1\leqq j<k,$$

with $\mathscr{L}$ being the Jacobian of (e). The matrix $(\mathscr{L}-k\,\mathrm{I})$ is always invertible, unless $k=2$ and then its rank equals 1. Consequently the coefficient M^2 contains two free parameters, which account for c_1 and c_2. In fact, there is a much richer structure involved in this example. Namely the circle of the sphere $(\mathrm{H}_1=c_1)$ extends to the complex torus mentioned above, the flow is mapped by the integral (d) into a straight line motion on that torus and the functions $\mathrm{M}(t)$ is meromorphic. The complex intersection

$$\bigcap_{i=1}^{2}\{\mathrm{H}_i(\mathrm{M})=c_i\}\subseteqq\mathbb{C}^3$$

is the affine part of an elliptic curve $\subseteqq\mathbb{CP}^3$ which is the above torus. This torus has an algebraic addition law connecting $p(t_1+t_2)$ to $p(t_1)$ and $p(t_2)$ where $p(t)\equiv(m_1(t), m_2(t), m_3(t))$ is a solution of equations (b).

This example shows that the classical way of solving Euler's equations in terms of elliptic functions can be understood as a characterization of this complex torus. So the main question is how to complete the affine variety (defined by the intersection of the constants of the motion) and the differential equations on it into a non-singular compact complex algebraic variety? In the above example, we have completed the affine variety by the points of blow up, which are captured automatically by projectivizing the equations of this variety in $\mathbb{CP}^3$. But this procedure can never work in general; indeed, an abelian variety of dimension bigger or equal than two is never a complete intersection. Therefore if the affine part of an abelian surface is defined by 4 equations in $\mathbb{C}^6$, then the obvious embedding into $\mathbb{CP}^6$, by making the equations projective, must have one or more singularities at infinity. In fact, we shall show that the Laurent expansions can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down.

A. Asymptotic expansions. — Let M and Γ have the following asymptotic expansions

$$M=\sum_{k=0}^{\infty} M^k t^{k-1} \quad \text{and} \quad \Gamma=\sum_{k=0}^{\infty} \Gamma^k t^{k-2}. \tag{2.3}$$

Substituting (2.3) in the differential equations (Int 1), at the 0-th step, the coefficients of t^{-2} (for M) and t^{-3} (for Γ), yield a non-linear system

$$\begin{cases} M^0+[M^0, \Lambda M^0]+[\Gamma^0, L]=0 \\ 2\Gamma^0+[\Gamma^0, \Lambda M^0]=0 \end{cases} \tag{2.4}$$

and at the k-th step ($k \geqq 1$), the coefficients of t^{k-2} (for M) and t^{k-3} (for Γ) lead to a system of linear equations in M^k and Γ^k

$$(\mathscr{L}-k\,I)\begin{pmatrix} M^k \\ \Gamma^k \end{pmatrix}=0 \quad \text{for} \quad k=1 \tag{2.5}$$

$$=\begin{bmatrix} -\sum_{j=1}^{k-1}[M^j, \Lambda M^{k-j}] \\ -\sum_{j=1}^{k-1}[\Gamma^j, \Lambda M^{k-j}] \end{bmatrix} \quad \text{for} \quad k>1.$$

where $\mathscr{L}$ denotes the linear operator

$$\mathscr{L}\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} [M^0, \Lambda X]+[X, \Lambda M^0]+[Y, L]+X \\ [\Gamma^0, \Lambda X]+[Y, \Lambda M^0]+2Y \end{pmatrix}.$$

In the basis $(e_j)_{1 \leqq j \leqq 6}$, $\mathscr{L}$ is given by the matrix

$$\mathscr{L}=\begin{bmatrix} 1 & m_3^0 & m_2^0 & 0 & 0 & 0 \\ -m_3^0 & 1 & -m_1^0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & -\gamma_3^0 & 2\gamma_2^0 & 2 & 2m_3^0 & -m_2^0 \\ \gamma_3^0 & 0 & -2\gamma_1^0 & -2m_3^0 & 2 & m_1^0 \\ -\gamma_2^0 & \gamma_1^0 & 0 & m_2^0 & -m_1^0 & 2 \end{bmatrix}$$

which is the Jacobian of (2.4).

Lemma 2.1. — *The non linear system* (2.4) *defines two lines and two points.*

Proof. — The system (2.4) has the following explicit form

$$(2.6)\quad \begin{cases} (a) & m_1^0+m_2^0 m_3^0=0 \\ (b) & m_2^0-m_1^0 m_3^0+2\gamma_3^0=0 \\ (c) & m_3^0-2\gamma_2^0=0 \\ (d) & 2\gamma_1^0+2m_3^0\gamma_2^0-m_2^0\gamma_3^0= \\ (e) & 2\gamma_2^0+m_1^0\gamma_2^0-2m_3^0\gamma_1^0= \\ (f) & 2\gamma_3^0+m_2^0\gamma_1^0-m_1^0\gamma_2^0=0. \end{cases}$$

Equations (a), (b) and (c) imply that

$$(g)\qquad m_1^0+2m_2^0\gamma_3^0=0$$

$$(h)\qquad 2\gamma_3^0+(1+(2\gamma_2^0)^2)m_2^0=0.$$

From equations (c), (d) and (h), it follows that

$$(j)\qquad 4\gamma_1^0+2(2\gamma_2^0)^2+(1+(2\gamma_2^0)^2)(m_2^0)^2=0$$

and by (c), (e), (f), (g), (h) and (j), we obtain

$$m_2^0(1-\gamma_1^0+2(\gamma_2^0)^2)=0$$
$$\gamma_2^0(1-4\gamma_1^0-(2\gamma_2^0)^2)=0.$$

We now distinguish several cases:

Case I. — If $m_2^0=\gamma_2^0=0$ then the solution of (2.6) is identically zero.

Case II. — If $\gamma_2^0=0$ and $\gamma_1^0=1$ it follows that the set of solutions of (2.6) is

$$(2.7)\qquad (m_1^0,\, m_2^0,\, m_3^0,\, \gamma_1^0,\, \gamma_2^0,\, \gamma_3^0)=(0,\, 2\varepsilon,\, 0,\, 1,\, 0,\, -\varepsilon),\ \varepsilon\equiv\pm i.$$

Case III. — If $1-\gamma_1^0+2(\gamma_2^0)^2=1-4\gamma_1^0-(2\gamma_2^0)^2=0$, this immediately implies $\gamma_1^0=1/2$ and $\gamma_2^0=\pm i/2$. From equations (c), (g) and (h), it follows that $m_3^0=\pm i$, $m_1^0\pm i m_2^0=0$ and $\gamma_3^0=0$. Consequently, we find the two lines

$$(2.8)\qquad (m_1^0,\, m_2^0,\, m_3^0,\, \gamma_1^0,\, \gamma_2^0,\, \gamma_3^0)=(\alpha,\, \varepsilon\alpha,\, \varepsilon,\, 1/2,\, \varepsilon/2,\, 0),\qquad \varepsilon\equiv\pm i$$

where α is a free parameter. This concludes the proof of lemma 2.1.

LEMMA 2.2. — *The system* (2.5) *has in*

(i) *case II* (2.7), 2 *degrees of freedom for* $k=2$ *and* 1 *degree of freedom for* $k=3$ *and* 4.

(ii) *case III* (2.8), 1 *degree of freedom for* $k=1$, 2, 3 *and* 4.

The *proof* of this lemma, is a linear algebra problem which is a straight forward computation. Since we are interested in 5-parameter Laurent solution, we only consider (ii). The eigenvectors V_1, V_2 V_3 and V_4 corresponding to the eigenvalues $k=1$, 2, 3 and 4 of the matrix

$$\mathscr{L}=\begin{bmatrix} 1 & \varepsilon & \varepsilon\alpha & 0 & 0 & 0 \\ -\varepsilon & 1 & -\alpha & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & \varepsilon & 2 & 2\varepsilon & -\varepsilon\alpha \\ 0 & 0 & -1 & -2\varepsilon & 2 & \alpha \\ -\varepsilon/2 & 1/2 & 0 & \varepsilon\alpha & -\alpha & 2 \end{bmatrix}$$

are

$$V_1=(\varepsilon(\alpha^2-2),\ -\alpha^2,\ \alpha,\ 0,\ 0,\ 1)$$
$$V_2=(\alpha,\ \varepsilon\alpha,\ -2\varepsilon,\ 1,\ \varepsilon,\ 0)$$
$$V_3=(-\varepsilon(\alpha^2-4),\ \alpha^2+8,\ -3\alpha,\ -3\varepsilon\alpha,\ 3\alpha,\ 6)$$
$$V_4=(\alpha(\alpha^2-8),\ \varepsilon\alpha(\alpha^2+12),\ -4\varepsilon(\alpha^2-3),\ 6(\alpha^2+2),\ 6\varepsilon(\alpha^2-3),\ 20\varepsilon\alpha).$$

We denote by β, λ, θ and μ, the free parameters obtained respectively at $k=1$, 2, 3 and 4. If we put

$$X^j\equiv(m_1^j,\ m_2^j,\ m_3^j,\ \gamma_1^j,\ \gamma_2^j,\ \gamma_3^j),\qquad 1\leqq j\leqq 4$$

the solutions of (2.5) are given by

$$X^1=\beta V_1$$
$$X^2=-\beta^2 A_1+3\lambda V_2$$
$$X^3=-\frac{\alpha\beta^3}{4}A_2+\frac{\beta\lambda}{2}A_3+\theta V_3$$
$$X^4=\frac{\beta^4}{80}A_4-\frac{\beta^2\lambda}{40}A_5+\frac{\lambda^2}{10}A_6+\frac{\beta\theta}{20}A_7-\mu V_4$$

where

$$A_1\equiv(\alpha(\alpha^2+2)/2,\ \varepsilon\alpha(\alpha^2-2)/2,\ 0,\ 1,\ 0,\ \varepsilon\alpha)$$
$$A_2\equiv(\varepsilon\alpha(\alpha^2-2),\ -\alpha(\alpha^2+6),\ \alpha^2+6,\ \varepsilon(\alpha^2-2),\ -\alpha^2-6,\ 0)$$
$$A_3\equiv(\varepsilon(7\alpha^2+8),\ -7\alpha^2+16,\ 3\alpha,\ 3\varepsilon\alpha,\ -3\alpha,\ 0)$$
$$A_4\equiv(\alpha(13\alpha^4+74\alpha^2+40),\ \varepsilon\alpha(13\alpha^4-26\alpha^2-80),\ -4\varepsilon(3\alpha^4-11\alpha^2+10),\ 2(9\alpha^4+32\alpha^2-20),\ 2\varepsilon(9\alpha^4-33\alpha^2+30),\ 0)$$
$$A_5\equiv(\alpha(85\alpha^2-208),\ \varepsilon\alpha(85\alpha^2+212),\ -4\varepsilon(5\alpha^2+37),\ -6(5\alpha^2+2),\ 2\varepsilon(15\alpha^2+111),\ 0)$$
$$A_6\equiv(63\alpha,\ 63\varepsilon\alpha,\ -72\varepsilon,\ 108,\ 108\varepsilon,\ 0)$$
$$A_7\equiv(\alpha(31\alpha^2+80),\ \varepsilon(31\alpha^2-100),\ -4\varepsilon(11\alpha^2-5),\ 6(11\alpha^2-10),\ 6\varepsilon(11\alpha^2-5),\ 0).$$

To conclude the generic solution blows up after a finite time according to a Laurent series within a 5 parameters family of Laurent solutions. By the majorant method, any formal Laurent series solution of a system of differential equations with quadratic right hand side automatically converges. Now it is easily checked that

$$(\mathscr{L}+\mathrm{I})\begin{pmatrix}\mathrm{M}^0\\ \Gamma^0\end{pmatrix}=0$$

and it follows from Lemma 2.2 that

$$\det(\mathscr{L}-k\,\mathrm{I})=(k+1)\,k\,(k-1)\,(k-2)\,(k-3)\,(k-4).$$

Consequently, we have

THEOREM 2.1. — *The system of differential equations* (Int 2) *possesses Laurent series solution*

$$(2.9)\qquad m_j=\sum_{k=0}^{\infty} m_j^k\, t^{k-1} \quad \textit{and} \quad \gamma_j=\sum_{k=0}^{\infty}\gamma_j^k\, t^{k-2},\qquad 1\leqq j\leqq 3$$

which depend on 5 *free parameters* (= dim (*phase space*) − 1), *with leading terms given by* (2.8).

B. DIVISORS OF POLES. — We now search for the set of Laurent solution which remain confined to a fixed affine invariant surface, related to specific values of c_1, c_2, c_4, i. e.

$$\mathscr{D}_\varepsilon=\left\{\begin{array}{c}\text{The Laurent solutions } m_j(t),\ \gamma_j(t),\ 1\leqq j\leqq 3\\ \text{such that } \mathrm{H}_k(m_j(t),\gamma_j(t))=c_k,\ 1\leqq k\leqq 4\end{array}\right\}$$

= 4 polynomial relations between α, β, λ, θ and μ;

$$\begin{aligned} c_1&=(\alpha^2-4)\,\beta^2+18\,\lambda\\ -\varepsilon\,c_2&=(\alpha^2+2)\,\beta^3-6\,\beta\lambda+12\,\theta\\ 8&=(5\,\alpha^2-2)\,\beta^4-6\,\beta^2\,\lambda+84\,\beta\theta-240\,\mu\\ 8\,c_4&=(\alpha^2-1)\,((11\,\alpha^2+10)\,\beta^4+54\,\beta^2\,\lambda+108\,\beta\theta+240\,\mu)\end{aligned}$$

= algebraic curve

$$=\left\{\begin{array}{c}(\alpha,\beta,\varepsilon)\text{ such that } \mathrm{P}(\alpha,\beta,\varepsilon)\equiv(\alpha^2-1)\,(\alpha^2-1)\,\beta^4-\mathrm{P}(\beta))+c_4=0\\ \mathrm{P}(\beta)\equiv c_1\,\beta^2-2\,\varepsilon\,c_2\,\beta-1\\ \varepsilon\equiv\pm i\end{array}\right\}.$$

The map

$$(2.10)\qquad \sigma_\varepsilon:\quad \mathscr{D}_\varepsilon\to\mathscr{D}_\varepsilon:(\alpha,\beta,\varepsilon)\mapsto(-\alpha,\beta,\varepsilon)$$

is an involution on $\mathscr{D}_\varepsilon$. The quotient $\mathscr{D}_\varepsilon^0=\mathscr{D}_\varepsilon/\sigma_\varepsilon$ by the involution σ_ε is an elliptic curve defined by

$$(2.11)\qquad \mathscr{D}_\varepsilon^0:\quad u^2=\mathrm{P}^2(\beta)-4\,c_4\,\beta^4.$$

The curve $\mathscr{D}_\varepsilon$ is a 2-sheeted ramified covering of $\mathscr{D}_\varepsilon^0$

$$\varphi_\varepsilon: \quad \mathscr{D}_\varepsilon \to \mathscr{D}_\varepsilon^0 : (\alpha, u, \beta,) \mapsto (u, \beta, \varepsilon) \tag{2.12}$$

$$\mathscr{D}_\varepsilon : \begin{cases} \alpha^2 = \dfrac{2\beta^4 + P() + u}{2\beta^4} \\ u^2 = P^2(\beta) - 4c_4\beta^4. \end{cases} \tag{2.13}$$

Let us now look more closely at certain points of interest on the non-singular version of the curve $\mathscr{D}_\varepsilon$. For β sufficiently small,

$$\alpha^2 = \frac{2\beta^4 + P(\beta) + \sqrt{P^2(\beta) - 4c_4\beta^4}}{2\beta^4} = 1 - c_4 + O(\beta)$$

and

$$\alpha^2 = \frac{2\beta^4 + P(\beta) - \sqrt{P^2(\beta) - 4c_4\beta^4}}{2\beta^4} = \frac{1}{\beta^4}(-1 + O(\beta)).$$

At $\beta = \infty$, the curve $\mathscr{D}_\varepsilon$ behaves as follows

$$2(\alpha^2 - 1)\beta^2 = c_1 \pm \sqrt{c_1^2 - 4c_4} + O(\beta).$$

Now, the curve $\mathscr{D}_\varepsilon$ has 4 points at infinity $p_j (1 \leqq j \leqq 4)$ and 4 branch points $q_j \equiv (\alpha = 0, u = -2\beta^4 - P(\beta), \beta^4 + P(\beta) + c_4 = 0)$ $(1 \leqq j \leqq 4)$ on the elliptic curve $\mathscr{D}_\varepsilon^0$. The divisor structure of α, β on $\mathscr{D}_\varepsilon$ is

$$(\alpha) = \sum_{1 \leqq j \leqq 4} q_j - \sum_{1 \leqq j \leqq 4} p_j$$

$$(\beta) = 4 \text{ zeroes} - \sum_{1 \leqq j \leqq 4} p_j.$$

Let $g(\mathscr{D}_\varepsilon^0) =$ genus of $\mathscr{D}_\varepsilon^0$, $g(\mathscr{D}_\varepsilon) =$ genus of $\mathscr{D}_\varepsilon$, $n = \#$ of sheets and $v = \#$ of branch points. Then by the Riemann-Hurwitz's formula

$$g(\mathscr{D}_\varepsilon) = n(g(\mathscr{D}_\varepsilon^0) - 1) + 1 + \frac{v}{2} = 3.$$

The map

$$(\alpha, u, \beta, \varepsilon) \mapsto (\alpha, u, -\beta, -\varepsilon)$$

is an isomorphism between $\mathscr{D}_{\varepsilon = i}$ and $\mathscr{D}_{\varepsilon = -i}$ and so we have the following commutative diagram

$$\begin{array}{ccc} \mathscr{D}_{\varepsilon = i} & \xrightarrow{\ \sim\ } & \mathscr{D}_{\varepsilon = -i} \\ {\scriptstyle \varphi_{\varepsilon = i}} \downarrow & & \downarrow {\scriptstyle \varphi_{\varepsilon = -i}} \\ \mathscr{D}_{\varepsilon = i}^0 & \xrightarrow{\ \sim\ } & \mathscr{D}_{\varepsilon = -i}^0 \end{array}$$

Thus, we have proved

THEOREM 2.2. — *The divisors of poles* $\mathcal{D}_{\varepsilon=\pm i}$ (2.13) *of the functions* m_j, $\gamma_j (1 \leqq j \leqq 3)$ *are two isomorphic irreducible Riemann surfaces of genus* 3. *They are 2-sheeted ramified coverings of two elliptic curves* $\mathcal{D}^0_{\varepsilon=\pm i}$ (2.11).

Remark. — From the Poincaré residue formula, we know that the 3 holomorphic differentials on $\mathcal{D}_\varepsilon$ are of the form

$$\frac{g(\alpha, \beta, \varepsilon)\, d\beta}{(\partial P/\partial\alpha)(\alpha, \beta, \varepsilon)} = \frac{g(\alpha, \beta, \varepsilon)\, d\beta}{\alpha u}$$

where $g(\alpha, \beta, \varepsilon)$ is a polynomial of at most degree five in α and β. It is easy to verify that

$$(2.14) \qquad \omega_0 = \frac{d\beta}{u}, \qquad \omega_1 = \frac{(\alpha^2 - 1)\,\beta^2\, d\beta}{\alpha u}, \qquad \omega_2 = \frac{d\beta}{\alpha u}$$

form effectively a basis of holomorphic differentials on $\mathcal{D}_\varepsilon$. Observe that $\sigma^*_\varepsilon \omega_0 = \omega_0$ and $\sigma^*_\varepsilon \omega_j = -\omega_j (j = 1, 2)$ for the involution σ_ε (2.10).

C. ABELIAN SURFACE. — Let T be a smooth surface compactifying A (Int 4) and let

$$\mathcal{D} \equiv \mathcal{D}_{\varepsilon=i} + \mathcal{D}_{\varepsilon=-i} \subseteqq T$$

be a divisor (to be shown later). Consider a basis $1, f_1, \ldots, f_N$ of the vector space

$$\mathcal{L}(\mathcal{D}) = \{f,\, f \text{ meromorphic on } T,\, (f) \geqq -\mathcal{D}\}$$

and the holomorphic map

$$T \to \mathbb{CP}^N : \quad p \mapsto (1, f_1(p), \ldots, f_N(p)),$$

considered projectively, because if at p some $f_j(p) = \infty$, we divide by f_j having the highest order pole near p, which makes every element finite. This defines a map of T into $\mathbb{CP}^N$. The *Kodaira embedding theorem* tells us that if the line bundle associated with the divisor is positive, then for some $k \geqq 0$, $\in \mathbb{Z}$, the functions of $\mathcal{L}(k\mathcal{D})$ embed smoothly T into $\mathbb{CP}^N$ and then by Chow's theorem, T can be realized as an algebraic variety, i. e.

$$T = \bigcap_j \{P_j(z) = 0,\, z \in \mathbb{CP}^N\}$$

where $P_j(z)$ are homogeneous polynomials. In fact we shall show that in our case, $k = 1$ suffices i. e. the divisor $\mathcal{D}$ provides a smooth embedding into $\mathbb{CP}^7$, via the meromorphic section of $\mathcal{L}(\mathcal{D})$. The Riemann-Roch theorem and the adjunction formula (on abelian surfaces) together imply that

$$\dim \mathcal{L}(\mathcal{D}) = N + 1 = g(\mathcal{D}) - 1.$$

Based on this motivation, we wish to find a set of polynomial functions $\{f_0=1, f_1, \ldots, f_N\}$ having a simple pole along $\mathscr{D}$ such that the embedding of $\mathscr{D}$ with those functions into $\mathbb{CP}^N$ yields a curve of genus $N+2$. Let

$$\mathscr{L}^r = \left\{ \begin{array}{l} \text{polynomials } f \text{ of degree} \leqq r \text{ behaving like} \\ \dfrac{f^0}{t} + f^1 + O(t) \text{ mod. the constants of motion} \end{array} \right\}$$

and let $\{f_0=1, f_1, \ldots, f_{N_r}\}$ be a basis of $\mathscr{L}^r$. The map

$$\mathscr{D} \to \mathbb{CP}^{N_r}: \quad p \mapsto \lim_{t \to 0} t(f_0(p), \ldots, f_{N_r}(p)) = (0, f_1^0(p), \ldots, f_{N_r}^0(p))$$

maps the curve $\mathscr{D}$ into $\tilde{\mathscr{D}}^r \subseteqq \mathbb{CP}^{N_r}$. We look for r such that

$$g(\tilde{\mathscr{D}}^r) = N_r + 2, \qquad \tilde{\mathscr{D}}^r \subseteqq \mathbb{CP}^{N_r}.$$

LEMMA 2.3:

$$\mathscr{L}^0 = \{f_0 = 1\}$$

$$\mathscr{L}^1 = \mathscr{L}^0 \oplus \{f_1 = m_1, f_2 = m_2, f_3 = m_3\}$$

$$f_1 = \frac{\alpha}{t} + \varepsilon(\alpha^2 - 2)\beta + O(t)$$

$$f_2 = \frac{\varepsilon\alpha}{t} - \alpha^2\beta + O(t)$$

$$f_3 = \frac{\varepsilon}{t} + \alpha\beta + O(t)$$

$$\mathscr{L}^2 = \mathscr{L}^1 \oplus \left\{ f_4 = \gamma_3, f_5 = -\frac{1}{4}(f_1^2 + f_2^2) \right\}$$

$$f_4 = \frac{\beta}{t} - \varepsilon\alpha\beta^2 + O(t)$$

$$f_5 = \frac{\varepsilon\alpha\beta}{t} + \beta^2 + O(t)$$

$$\mathscr{L}^3 = \mathscr{L}^2 \oplus \{f_6 = f_3 f_5 + f_1 f_4\}$$

$$f_6 = \frac{\varepsilon(\alpha^2-1)\beta^2}{t} - \alpha(\varepsilon c_2 - c_1\beta + (\alpha^2-1)\beta^3) + O(t)$$

$$\mathscr{L}^4 = \mathscr{L}^3 \oplus \{f_7 = (f_2\gamma_1 - f_1\gamma_2)f_3 + 2f_4\gamma_2\}$$

$$f_7 = \frac{\varepsilon}{t}(-\varepsilon c_2 + c_1\beta - 2(\alpha^2-1)\beta^3) - \alpha(2 + 3\varepsilon c_2\beta - c_1\beta^2 + 2(\alpha^2-1)\beta^4) + O(t).$$

The *proof* of the previous lemma can easily done by inspection of the expansions (2.9).

PROPOSITION 2.1. — *$\mathscr{L}^4$ provides an embedding of $\tilde{\mathscr{D}}^4$ into projective space such that*

$$(\alpha, u, \beta) \in \mathscr{D} \mapsto \lim t(f_0, f_1, \ldots, f_7) = (0, f_1^0, \ldots, f_7^0) \in \mathbb{CP}^7$$

and the genus of $\tilde{\mathscr{D}}^4$ is

$$g(\tilde{\mathscr{D}}^4) = 9.$$

Proof. — It turns out that neither $\mathscr{L}^1$, nor $\mathscr{L}^2$, nor $\mathscr{L}^3$ yields a curve of the right genus; in fact

$$g(\tilde{\mathscr{D}}^r) \neq \dim \mathscr{L}^r + 1, \qquad r = 1, 2, 3$$

For instance, the embedding into $\mathbb{CP}^3$ via $\mathscr{L}^1$ does not separate the sheets, so we proceed to $\mathscr{L}^2$ and we show that

$$g(\tilde{\mathscr{D}}^2 \text{ as embedded into } \mathbb{CP}^5) - 2 > 5$$

which contradicts the fact that $N + 1 = g(\tilde{\mathscr{D}}) - 1$, so we look at $\mathscr{L}^3$ and we find that

$$g(\tilde{\mathscr{D}}^3 \text{ as embedded into } \mathbb{CP}^6) - 2 > 6$$

and the contradiction persists. We proceed now to $\mathscr{L}^4$ and if we denote by

$$\left(\frac{f_0}{f_4}, \ldots, \frac{f_7}{f_4}\right) \equiv F_k = F_k^0 + F_k^1 t + \ldots, \quad 0 \leqq k \leqq 7$$

and

$$p_j \equiv (\alpha = \pm 1,\ u = \pm \beta^2 \sqrt{c_1^2 - 4c_4},\ \beta = \infty) \qquad (1 \leqq j \leqq 4)$$

the 4 points at infinity of $\mathscr{D}_\varepsilon$, we obtain

$$F_k^0(p_j) = \left(0, \frac{\alpha}{\beta}, \frac{\varepsilon\alpha}{\beta}, \frac{\varepsilon}{\beta}, 1, \varepsilon\alpha, \varepsilon(\alpha^2 - 1)\beta, -\frac{c_2}{\beta} + \frac{\varepsilon(1-u)}{\beta^2}\right)(p_j)$$
$$= (0, 0, 0, 0, 1, \pm\varepsilon, 0, \mp\varepsilon\sqrt{c_1^2 - 4c_4}) = 4 \text{ distinct points}$$

and using the transformation

$$\mathscr{D}_{\varepsilon = i} \to \mathscr{D}_{\varepsilon = -i}: \quad (\alpha, u, \beta) \rightsquigarrow (-\alpha, -u, \beta),$$

we have that

$$F_k^0(p_j)\big|_{\mathscr{D}_{\varepsilon=i}} = F_k^0(p_j)\big|_{\mathscr{D}_{\varepsilon=-i}},$$

implying that the 4 points p_j on one curve are identified pairwise with the 4 corresponding points on the other curve. Let $s \equiv 1/\beta$ be a local parameter for p_j, we have

$$\frac{\partial F_k^0}{\partial s}(p_j) = (0, \pm 1, \pm\varepsilon, \varepsilon, 0, \pm\varepsilon, 0, -c_2)$$

which shows that

$$\left.\frac{\partial \mathrm{F}_k^0}{\partial s}(p_j)\right|_{\mathscr{D}_{\varepsilon=i}} \neq \left.\frac{\partial \mathrm{F}_k^0}{\partial s}(p_j)\right|_{\mathscr{D}_{\varepsilon=-i}}$$

and consequently the curve $\mathscr{D}_{\varepsilon=i}$ intersects transversely the curve $\mathscr{D}_{\varepsilon=-i}$ in 4 points at infinity (*fig.* 1).

$$b_1, \ldots, b_4 \equiv (\alpha=\infty, \beta=0)$$
$$p_1, \ldots, p_4 \equiv (\alpha=\pm 1, u=\pm \beta^2 \sqrt{c_1^2-4c_4}, \beta=\infty)$$

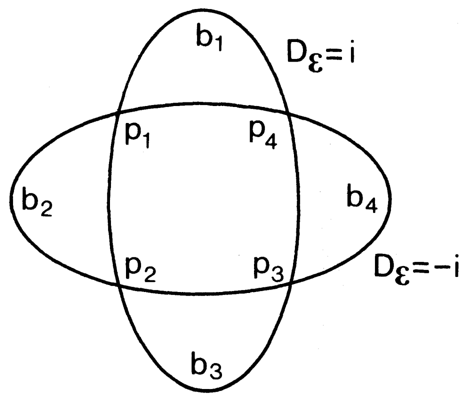

Fig. 1

In a neighbourhood of the points $b_j \equiv (\alpha=\infty, \beta=0)$ $(1 \leqq j \leqq 4)$ it is more convenient to divide the functions $f_0, \ldots, f_7$ by f_1 and one can see that if we put

$$\left(\frac{f_0}{f_1}, \ldots, \frac{f_7}{f_1}\right) \equiv \mathscr{F}_k = \mathscr{F}_k^0 + \mathscr{F}_k^1 t + \ldots, \quad 0 \leqq k \leqq 7$$

that

$$\mathscr{F}_k^0(b_j) = (0, 1, \varepsilon, 0, 0, 0, \mp 1, 0) = 4 \text{ distinct points}$$

and consequently

$$g(\tilde{\mathscr{D}}^4 \text{ as embedded into } \mathbb{CP}^7) - 2 = 7$$

i. e. $g(\tilde{\mathscr{D}}^4) = 9$. This completes the proof of proposition 2.1.

Let $\mathscr{L} = \mathscr{L}^4$ and $\tilde{\mathscr{D}} = \tilde{\mathscr{D}}^4 \subseteq \mathbb{CP}^7$. Next we wish to construct a surface strip around $\tilde{\mathscr{D}}$ which will support the commuting vector fields.

(i) At all points where α, $\beta \neq 0$, ∞, the Laurent solutions are nicely convergent (by the majorant method). Therefore, at most points of $\tilde{\mathscr{D}}$ there is a transversal fiber to the curve. Hence this defines a smooth surface strip around $\tilde{\mathscr{D}}$ except at the bad points.

(ii) Now we need to construct a surface strip around $\tilde{\mathcal{D}}$ at the bad points as well. For doing that, we must introduce the concept of projective normality. Ultimately, we wish to prove that in the various charts

$$\left(\frac{f_j}{f_k}\right)^{\cdot} = \text{polynomial}\left(\frac{f_j}{f_k}\right), \qquad 0 \leqq j \leqq 7, \quad k \text{ fixed}. \tag{2.15}$$

This enables one to show that f_j/f_k is a bona fide Taylor series starting from every point in a neighbourhood of the point in question $\subseteqq \mathbb{CP}^7$.

PROPOSITION 2.2. — *The orbits of the vector field* (2.1) *going through the curve $\mathcal{D}$ form a smooth surface Σ near $\mathcal{D}$ such that*

$$\Sigma \setminus \mathcal{D} \subseteqq \mathrm{A}.$$

Moreover, the variety

$$\mathrm{T} = \mathrm{A} \cup \Sigma$$

is smooth, compact and connected.

Proof. — Let $\mathrm{I} = \{t \in \mathbb{C},\ -\delta < t < \delta\}$ be an interval, let

$$(t, p) \rightsquigarrow \varphi(t, p) = \{(\mathrm{M}(t, p), \Gamma(t, p)),\ t \in \mathrm{I}, p \in \mathcal{D}\},$$

be the orbit of the vector field (2.1) going through the point $p \in \mathcal{D}$, let $\Sigma_p \subset \mathbb{CP}^7$ be the surface element formed by the divisor $\mathcal{D}$ and the orbits going through p, and set $\Sigma \equiv \bigcup_{p \in \mathcal{D}} \Sigma_p$. Consider the curve $\mathcal{D}' = \pi \cap \Sigma$ where $\pi \subset \mathbb{CP}^7$ is a hyperplane transversal to the direction of the flow. If $\mathcal{D}'$ is smooth, then using the implicit function theorem the surface Σ is smooth. But if $\mathcal{D}'$ is singular at 0, then Σ would be singular along the trajectory (t-axis) which go immeidately into the affine part A. Hence, A would be singular which is a contradiction because A is the fibre of a morphism from $\mathbb{C}^6$ to $\mathbb{C}^4$ and so smooth for almost all the four constants of the motion c_j. Next, let $\bar{\mathrm{A}}$ be the projective closure of A into $\mathbb{CP}^6$, let $z = (z_0, z_1 = m_1/z_0, \ldots, \gamma_3/z_0) \in \mathbb{CP}^6$, let $\mathscr{C} = \bar{\mathrm{A}} \cap (z_0 = 0)$ be the locus at infinity. (In fact, we have $\mathscr{C} = d_1 \cup d_2 \cup \mathrm{S}'$ where d_1, d_2 are straight lines and S′ a circle with $d_1 \cap d_2 = \varnothing$, $d_j \cap \mathrm{S}' = 1$ point, $j = 1, 2$) and let $f = (f_0, f_1, \ldots, f_7) \in \mathscr{L}(\mathcal{D})$. Consider the map

$$\bar{\mathrm{A}} \subseteqq \mathbb{CP}^6 \to \mathbb{CP}^7 : z \rightsquigarrow f(z),$$

and let $\mathrm{T} = f(\bar{\mathrm{A}})$. In a neighbourhood $\mathscr{V}(p) \subseteqq \mathbb{CP}^7$ of p, we have

$$\Sigma_p = \mathrm{T}$$
$$\Sigma_p \setminus \mathcal{D} \subseteqq \mathrm{A}.$$

Oherwise there would an element of surface $\Sigma'_p \subseteqq \mathrm{T}$ such that

$$\Sigma_p \cap \Sigma'_p = t\text{-axis}$$

$$\text{orbit } \varphi(t, p) = t\text{-axis} \setminus p \subseteqq \mathrm{A},$$

and hence A would be singular along the t-axis which is impossible. Since the variety $\bar{\mathrm{A}} \cap (z_0 \neq 0)$ is irreducible and since the generic hyperplane section $\pi_{\text{gen.}}$ of $\bar{\mathrm{A}}$ is also irreducible, all hyperplane sections are connected and hence $\mathscr{C}$ is also connected. Now, consider the graph $(f) \subseteqq \mathbb{CP}^6 \times \mathbb{CP}^7$ of the map f, which is irreducible together with $\bar{\mathrm{A}}$. It follows from the irreducibility of $\mathscr{C}$ that a generic hyperplane section graph $(f) \cap (\pi_{\text{gen.}} \times \mathbb{CP}^7)$ is irreducible, hence the special hyperplane section

$$\pi_{\text{sp.}} \equiv \text{graph } (f) \cap ((z_0 = 0) \times \mathbb{CP}^7)$$

is connected and therefore the projection map

$$pr_{\mathbb{CP}^7}(\pi_{\text{sp.}}) = f(\mathscr{C}) \equiv \mathscr{D}$$

is connected. Hence, the variety

$$\mathrm{A} \cup \Sigma = \mathrm{T}$$

is compact, connected and embeds smoothly into $\mathbb{CP}^7$ via f. This concludes the proof of proposition 2.2.

In fact, we shall prove a somewhat stronger statement than (2.15), namely that (2.15) is namely that (2.15) is satisfied with quadratic polynomials. By inspection one sees that the functions $f_0, \ldots f_7$ do not satisfy that property. For example

$$\left(\frac{f_0}{f_4}\right)^{\cdot} = \frac{f_1 \gamma_2 - f_2 \gamma_1}{(f_4)^2} \neq \text{polynomial} \left(\frac{f_j}{f_4}\right), \qquad 0 \leqq j \leqq 7.$$

Hence we must take functions with higher order poles. Let us consider for instance $\mathscr{L}(2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i})$.

Proposition 2.3. — *We have* $\dim \mathscr{L}(2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i}) = 18$ *and* $2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i}$ *is projectively normal and embeddable into* $\mathbb{CP}^{17}$.

Proof. — For ease of manipulation, we are going to use x_1, x_2, y_1, y_2 given by (1.1) instead of the functions $m_1, m_2, \gamma_1, \gamma_2$. Let $g_0 = 1, g_1, \ldots, g_{17}$ be a basis of $\mathscr{L}(2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i})$ where $g_1 = x_1$, $g_2 = x_2$, $g_3 = f_3$, $g_4 = f_4$, $g_5 = f_5$, $g_6 = f_6$, $g_7 = f_7$, $g_8 = g_2^2$, $g_9 = g_2 g_3$, $g_{10} = g_2 g_4$, $g_{11} = y_2$, $g_{12} = g_1 g_{11}$, $g_{13} = g_1^2 g_{11}$, $g_{14} = g_1^3 g_{11}$, $g_{15} = g_2 g_5$, $g_{16} = g_2 g_6$ and $g_{17} = g_2 g_7$. These meromorphic functions have the properties

$$(g_1) = -\mathscr{D}_{\varepsilon=-i}$$

$$(g_2) = -\mathscr{D}_{\varepsilon=i}$$

$$(g_k) = -\mathscr{D}_{\varepsilon=i} - \mathscr{D}_{\varepsilon=-i}, \qquad 3 \leqq k \leqq 7$$

$$(g_8) = -2\mathscr{D}_{\varepsilon=i}$$

$$(g_k) = -2\mathscr{D}_{\varepsilon=i} - \mathscr{D}_{\varepsilon=-i}, \qquad k = 9, 10, 15, 16, 17$$

$$(g_{11}) = -2\mathscr{D}_{\varepsilon=i} + 2\mathscr{D}_{\varepsilon=-i}$$

$$(g_{12}) = -2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i}$$

$$(g_k) = -2\mathscr{D}_{\varepsilon=i}, \qquad k=13,\ 14.$$

Hence, the map

$$2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i} \to \mathbb{CP}^{17} : p \rightsquigarrow \lim_{t\to 0} t\,(g_0(p), \ldots, g_{17}(p)) = (0,\ g_1^0(p), \ldots, g_{17}^0(p))$$

maps the curve $2\mathscr{D}_{\varepsilon=i} + \mathscr{D}_{\varepsilon=-i}$ into $\mathbb{CP}^{17}$. In fact, in a neighbourhood of the points at infinity $p_j = (\alpha = \pm 1, u = \pm \beta^2 \sqrt{c_1^2 - 4c_4}, \beta = \infty)$ $(1 \leqq j \leqq 4)$ it is simpler to divide the functions $g_0, \ldots, g_{17}$ by g_{10} which makes $(g_k/g_{10})(p_j)$ $(0 \leqq k \leqq 17)$ finite. Whereas in the neighbourhood of the points $b_j = (\alpha = \infty, \beta = 0)$, it is more convenient to divide by g_8 which makes $(g_k/g_8)(p_j)$ $(0 \leqq k \leqq 17)$ finite. Next, using the vector field (2.1), we show that in a neighbourhood of the points p_j, and modulo linear combination of the constants of motion

$$\left(\frac{g_k}{g_{10}}\right)^{\cdot} = \frac{\dot{g}_k g_{10} - g_k \dot{g}_{10}}{(g_{10})^2}, \qquad 0 \leqq k \leqq 17$$

$$= \frac{\text{quadratic polynomial in } (g_0, \ldots, g_{17})}{(g_{10})^2}$$

$$= \text{quadratic polynomial} \left(\frac{g_0}{g_{10}}, \ldots, \frac{g_{17}}{g_{10}}\right).$$

Also, in a neighbourhood of the points b_j, we have

$$\left(\frac{g_k}{g_8}\right)^{\cdot} = \text{quadratic polynomial} \left(\frac{g_0}{g_8}, \ldots, \frac{g_{17}}{g_8}\right).$$

This finishes the proof of proposition 2.3.

It follows from the previous proposition that at the bad points p_j, b_j the series $(g_k/g_{10})(p_j)$, $(g_k/g_8)(b_j)$ $(0 \leqq k \leqq 17)$ converges as a consequence of Picard's theorem applied to the system of ordinary differential equations $(g_k/g_l)^{\cdot}$ $(l = 8$ or $10)$.

PROPOSITION 2.4. — *The two commuting vector fields* (2.1) *and* (2.2) *extend holomorphically and remain independent on* T.

Proof. — Let φ^{t_1} and φ^{t_2} be the flows generated respectively by vector fields (2.1) and (2.2) and consider a point $p \in T \setminus A = \mathscr{D}$. For δ sufficiently small.

$$\varphi^{\tau_1}(p), \quad \forall \tau_1, \quad -\delta < \delta < \tau_1 < \delta$$

is well defined and $\varphi^{\tau_1}(p) \in A$. Then we may define φ^{t_2} on T by

$$\varphi^{t_2}(q) = \varphi^{-t_1} \varphi^{t_2} \varphi^{t_1}(q), \qquad q \in \mathscr{V}(p) = \varphi^{-\tau_1}(\mathscr{V}(\varphi^{\tau_1}(p))),$$

where $\mathscr{V}(p)$ is a neighbourhood of p. By commutativity one can see that φ^{t_2} is independent of t_1,

$$\varphi^{-t_1-\delta_1}\varphi^{t_2}\varphi^{t_1+\delta_1}(q)=\varphi^{-t_1}\varphi^{-\varphi_1}\varphi^{t_2}\varphi^{t_1}\varphi^{\delta_1}=\varphi^{-t_1}\varphi^{t_2}\varphi^{t_1}(q)$$

We affirm that $\varphi^{t_2}(q)$ is holomorphic away from $\mathscr{D}$. This because $\varphi^{t_2}\varphi^{t_1}(q)$ is holomorphic away from $\mathscr{D}$ and that φ is holomorphic in $\mathscr{V}(p)$ and maps bi-holomorphically $\mathscr{V}(p)$ onto $\mathscr{V}(\varphi^{\tau_1}(p))$. This completes the proof of proposition 2.4.

Since the flows φ^{t_1} and φ^{t_2} are holomorphic and independent on $\mathscr{D}$, we can show along the same lines as in the Arnold-Liouville theorem that T is a torus. And that will be done, by considering the holomorphic map

$$\Psi:\quad \mathbb{C}^2\to \mathrm{T}:(t_1, t_2)\rightsquigarrow \Psi(t_1, t_2)=\varphi^{t_1}\varphi^{t_2}(p)$$

for a base point $p\in \mathrm{A}$. Then

$$\mathrm{L}=\{(t_1, t_2)\in\mathbb{C}^2 : \Psi(t_1, t_2)=p\}$$

is a lattice of $\mathbb{C}^2$, hence

$$\Psi:\quad \mathbb{C}^2/\mathrm{L}\to \mathrm{T}$$

is a biholomorphic diffeomorphism. Therefore $\mathrm{T}\subseteqq\mathbb{CP}^7$ is conformal to a complex torus $\mathbb{C}^2/\mathrm{L}$ and an abelian surface as a consequence of Chow. Finally, we have the

THEOREM 2.3. — *T is an abelian surface on which the Hamiltonian flows* (2.1) *and* (2.2) *are straight lines motions.*

PROPOSITION 2.5. — *There are on* T *two holomorphic differentials dt_1 and dt_2 such that*

$$dt_1|_{\mathscr{D}_\varepsilon}=\omega_1 \qquad \textit{and} \qquad dt_2|_{\mathscr{D}_\varepsilon}=\omega_2$$

where ω_1 and ω_2 and the two holomorphic differentials (2.14) *on $\mathscr{D}_\varepsilon$.*

Proof. — Let $p\in\mathscr{D}_\varepsilon\cap\{\alpha u\neq 0\}$ where u is given by (2.11). Around the point p, we consider two coordinates on T,

$$\tau=\frac{1}{m_3}=-\varepsilon t+O(t^2)$$

$$x=\begin{cases} x_1=-i\beta+O(t) \text{ along } \mathscr{D}_{\varepsilon=i}\\ x_2=i\beta+O(t) \text{ along } \mathscr{D}_{\varepsilon=-i}.\end{cases}$$

We denote by $\partial/\partial t_1$ (resp. $\partial/\partial t_2$) the derivate according to the vector field (2.1) [resp. (2.2)]. Obviously we then have

$$dt_1=\frac{1}{\Delta(\tau, x)}\left(\frac{\partial x}{\partial t_2}d\tau-\frac{\partial \tau}{\partial t_2}dx\right)$$

$$dt_2=\frac{1}{\Delta(\tau, x)}\left(-\frac{\partial x}{\partial t_1}dt+\frac{\partial \tau}{\partial t_1}dx\right)$$

with

$$\Delta(\tau, x)\equiv\frac{\partial \tau}{\partial t_1}\cdot\frac{\partial x}{\partial t_2}-\frac{\partial \tau}{\partial t_2}\cdot\frac{\partial x}{\partial t_1}.$$

By direct computation using the asymptotic expansions, we find that

$$\frac{\partial \tau}{\partial t_1}=-\varepsilon+O(t), \qquad \frac{\partial \tau}{\partial t_2}=-4\varepsilon(\alpha^2-1)\beta^2+O(t)$$

$$\frac{\partial x}{\partial t_1}=-2\alpha\beta^2+O(t), \qquad \frac{\partial x}{\partial t_2}=8\alpha((\alpha^2-1)\beta^4-\mathrm{P}(\beta))+O(t)$$

where $\mathrm{P}(\beta)\equiv c_1\beta^2-2\varepsilon c_2\beta-1$. From which one can deduce the two differentials dt_1 and dt_2. The restrictions of dt_1 and dt_2 to the curve $\mathscr{D}_\varepsilon$ are given by

$$\left.\begin{aligned} dt_1|_{\mathscr{D}_\varepsilon}&=k_1\frac{(\alpha^2-1)\beta^2\,d\beta}{\alpha u}\\ dt_2|_{\mathscr{D}_\varepsilon}&=k_2\frac{d\beta}{\alpha u}\end{aligned}\right\}\ k_1, k_2\in\mathbb{C}$$

and are the two holomorphic differentials ω_1, ω_2 (2.14) on $\mathscr{D}_\varepsilon$. This completes the proof of proposition 2.5.

PROPOSITION 2.6. — *The vector field* (2.1) [*resp.* (2.2)] *is regular along* $\mathscr{D}$, *transversal to* $\mathscr{D}$ *at every point* $\beta\neq 0$ (*resp.* $\beta\neq\infty$) *and* (*doubly*) *tangent at* $\beta=0$ (*resp.* $\beta=\infty$).

Proof. — Using the same notation as in proposition 2.1, one can see that

$$\exists\, \mathrm{F}_k^0, \mathrm{F}_l^1:\quad \det\begin{bmatrix}\frac{\partial}{\partial s}\mathrm{F}_k^0(p_j) & \frac{\partial}{\partial s}\mathrm{F}_l^0(p_j)\\ \mathrm{F}_k^1(p_j) & \mathrm{F}_l^1(p_j)\end{bmatrix}\neq 0,$$

and consequently the vector field (2.1) is transversal to $\mathscr{D}$ at the 4 points p_j of $\mathscr{D}_{\varepsilon=i}\cap\mathscr{D}_{\varepsilon=-i}$. From proposition 2.5, the function

$$\frac{\omega_1}{\omega_2}=\frac{k_1}{k_2}(\alpha^2-1)\beta^2\sim\frac{1}{\beta^2}$$

is meromorphic along a neighbourhood of $b_j=(\alpha=\infty, \beta=0)$ $(1\leqq j\leqq 4)$ and provides the tangent to the curve $\mathscr{D}$ in the coordinates t_1 and t_2. The function ω_1/ω_2 vanishes, whenever the vector field (2.2) is tangent to $\mathscr{D}$ and has a pole whenever (2.1) is tangent to $\mathscr{D}$. Hence the zeroes b_j of ω_2 provide the 4 points of tangency of the vector field

(2.1) to $\mathscr{D}$. We find that

$$\forall\ \mathscr{F}_{\varkappa}^{0},\ \mathscr{F}_{l}^{1},\ \det\begin{bmatrix}\dfrac{\partial}{\partial\beta}\mathscr{F}_{\varkappa}^{0}(b_j) & \dfrac{\partial}{\partial\beta}\mathscr{F}_{l}^{0}(b_j)\\ \mathscr{F}_{\varkappa}^{1}(b_j) & \mathscr{F}_{l}^{1}(b_j)\end{bmatrix}=0,$$

and consequently (2.1) is (doubly) tangent to $\mathscr{D}$ at 4 points b_j, which concludes the proof of proposition 2.6.

PROPOSITION 2.7. — *The differentials* ω_1, ω_2, $f_j^0\,\omega_2\ (1 \leqq j \leqq 7)$ *form a basis for the space* $\Omega(\mathscr{D})$ *of holomorphic differentials on* $\mathscr{D}$.

Proof. — The adjunction formula gives us a map, the Poincaré residue map, between meromorphic 2-forms on T with a pole along $\mathscr{D}$ and holomorphic 1-forms on $\mathscr{D}$. Applied to the 2-form $\omega = f_j\,dt_1 \wedge dt_2$ with $f_j \in \mathscr{L}(\mathscr{D})$,

$$\begin{aligned}\omega=\frac{dt_1\wedge dt_2}{(1/f_j)} &\to \operatorname{Res}\omega\Big|_{\mathscr{D}} = -\frac{dt_1}{(\partial/\partial t_2)(1/f_j)}\Big|_{\mathscr{D}}\\ &= \frac{dt_2}{(\partial/\partial t_1)(1/f_j)}\Big|_{\mathscr{D}} = \frac{dt_2}{(\partial/\partial t_1)(t_1/f_j^0)+O(t_1^2))}\Big|_{\mathscr{D}}\\ &= f_j^0\,dt_2\,|_{\mathscr{D}} = f_j^0\,\omega_2.\end{aligned}$$

Hence

$$\Omega(\mathscr{D}) = \{\omega_1,\ \omega_2\} \oplus \{f_j^0\,\omega_2,\ 1 \leqq j \leqq 7\}$$

and this finishes the proof of proposition 2.7.

Remark. — It is interesting to observe that the embedding of $\mathscr{D}$ into $\mathbb{CP}^7$ is the canonical embedding,

$$\mathscr{D} \to \mathbb{CP}^7 : p \rightsquigarrow (\omega_2,\ f_1^0\,\omega_2,\ \ldots,\ f_7^0\,\omega_2).$$

As we have seen, the involution τ (1.4) has 8 fixed points on the affine variety A. In fact, it has 8 other fixed points at infinity given by the branch points of $\mathscr{D}_\varepsilon$ on $\mathscr{D}_\varepsilon^0$. This is the object of the

PROPOSITION 2.8. — *The involution* τ (1.4) *on the Abelian surface* T *coming from the one defined on the affine variety* A *has* 8 *fixed points at infinity.*

Proof. — This is easily proved as follows. From the asymptotic expansions (2.9), one can see that the functions m_1, m_2, γ_1, γ_2 remain invariable by the transformation

$$(t,\ \alpha,\ \beta) \rightsquigarrow (-t,\ -\alpha,\ \beta)$$

whereas m_3, γ_3 change into $-m_3$, $-\gamma_3$. Then the involution τ (1.4) is transformed at infinity into an involution σ_ε (2.10) on $\mathscr{D}_\varepsilon$. Now the fixed points of σ_ε are given by the branch points of $\mathscr{D}_\varepsilon$ on $\mathscr{D}_\varepsilon^0$. This concludes the proof of proposition 2.8.

D. Prym variety. — As is well known, if the period matrix of an abelian variety A is (Δ, Z) with

$$\Delta=\begin{bmatrix}\delta_1 & & \\ & \ddots & \\ & & \delta_n\end{bmatrix},\qquad \delta_1\,|\,\delta_2\,|\ldots|\,\delta_n\in\mathbb{N},\quad Z \text{ symmetric, } \operatorname{Im} Z>0$$

then the period matrix of the dual abelian variety of A [i.e. the group $\operatorname{Pic}^0(A)$ of holomorphic line bundles on A with chern class zero] is $(\delta_n\Delta^{-1}, \delta_n\Delta^{-1}Z\Delta^{-1})$.

Theorem 2.3. — *The abelian variety* T *is characterized as the dual* Prym *variety* Prym $(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)^*$ *of the genus* 3 *curve* $\mathscr{D}_\varepsilon$ (2.13) *which is a double cover of the elliptic curve* $\mathscr{D}_\varepsilon^0$ (2.11).

Proof. — Let A_j, B_j $(1\leqq j\leqq 3)$ a basis of cycles of $\mathscr{D}_\varepsilon$ (*Fig.* 2) such that: $(A_j, A_k)=(B_j, B_k)=0$, $(A_j, B_k)=\delta_{jk}$, $\sigma_\varepsilon(A_1)=A_3$, $\sigma_\varepsilon(B_1)=B_3$, $\sigma_\varepsilon(A_l)=-A_l$, $\sigma_\varepsilon(B_l)=-B_l$, $(1\leqq j,\ k\leqq 3,\ l=2,\ 3)$ for the involution σ_ε (2.10).

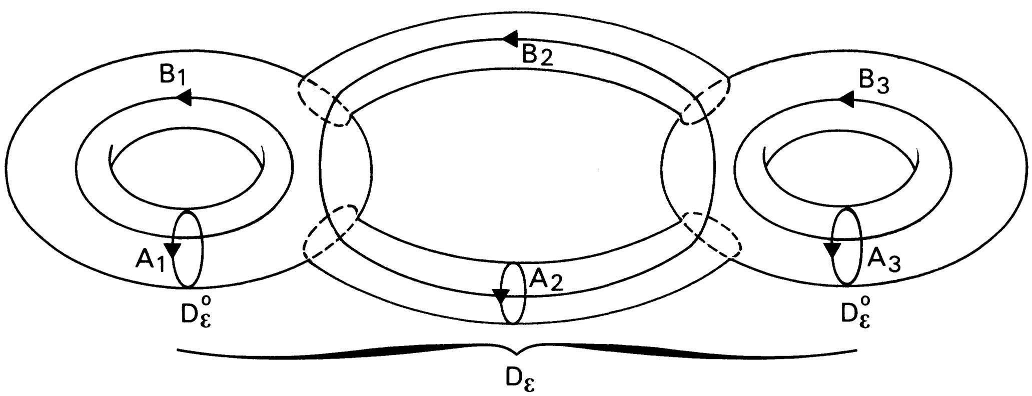

Fig. 2

From the double cover φ_ε (2.12), we can construct a subabelian variety of the Jacobi variety $\operatorname{Jac}(\mathscr{D}_\varepsilon)$ of $\mathscr{D}_\varepsilon$, called a Prym variety $\operatorname{Prym}(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)$: the involution σ_ε on $\mathscr{D}_\varepsilon$, extends by linearity to a map $\sigma_\varepsilon : \operatorname{Jac}(\mathscr{D}_\varepsilon)\to\operatorname{Jac}(\mathscr{D}_\varepsilon)$. Up to points of order two, $\operatorname{Jac}(\mathscr{D}_\varepsilon)$ splits up into an even part $\operatorname{Jac}(\mathscr{D}_\varepsilon^0)$ and an odd part $\operatorname{Prym}(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)$. The period matrix of this Prym variety is explicitelty given by

$$\Omega=\begin{matrix} 2\int_{A_1}\omega_1 & \int_{A_2}\omega_1 & 2\int_{B_1}\omega_1 & \int_{B_2}\omega_1 \\ 2\int_{A_1}\omega_2 & \int_{A_2}\omega_2 & 2\int_{B_1}\omega_2 & \int_{B_2}\omega_2 \end{matrix}$$

where ω_1 and ω_2 are two holomorphic differentials on $\mathcal{D}_\varepsilon$ (2.14). Let us call

$$U=\begin{bmatrix} 2\int_{A_1}\omega_1 & \int_{A_2}\omega_1 \\ 2\int_{A_1}\omega_2 & \int_{A_2}\omega_2 \end{bmatrix},\qquad V=\begin{bmatrix} 2\int_{B_1}\omega_1 & \int_{B_2}\omega_1 \\ 2\int_{B_1}\omega_2 & \int_{B_2}\omega_2 \end{bmatrix},$$

$$e'_1=2\begin{bmatrix} \int_{A_1}\omega_1 \\ \int_{A_1}\omega_2 \end{bmatrix}\quad\text{and}\quad e'_2=\begin{bmatrix} \int_{A_2}\omega_1 \\ \int_{A_2}\omega_2 \end{bmatrix}.$$

Observe that in the new basis $e_1=e'_1/2$, $e_2=e'_2$, the period matrix Ω takes the form

$$\Omega=(2\Delta^{-1}, 2\Delta^{-1}Z\Delta^{-1}) \tag{2.16}$$

where

$$\Delta=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix},\qquad Z=U^{-1}V\Delta \text{ symmetric and } \operatorname{Im} Z>0.$$

Consider now a basis dt_1, dt_2 of holomorphic differentials on T, the map $T\to\mathbb{C}^2/L_{\Omega^*}: p \rightsquigarrow \int_{p_0}^{p}\begin{pmatrix}dt_1\\ dt_2\end{pmatrix}$, a basis a_j, b_j $(j=1, 2)$ of $H_1(T, \mathbb{Z})$, the period matrix

$$\Omega^*=\begin{bmatrix} \int_{a_1}dt_1 & \int_{a_2}dt_1 & \int_{b_1}dt_1 & \int_{b_2}dt_1 \\ \int_{a_1}dt_2 & \int_{a_2}dt_2 & \int_{b_1}dt_2 & \int_{b_2}dt_2 \end{bmatrix}$$

and the lattice

$$L_{\Omega^*}=\left\{\sum_1^2 l_j\int_{a_j}\begin{pmatrix}dt_1\\ dt_2\end{pmatrix}+n_j\int_{b_j}\begin{pmatrix}dt_1\\ dt_2\end{pmatrix} : l_j, n_j\in\mathbb{Z}\right\}$$

associated to Ω^*. By the Lefschetz hyperplane theorem, the map $H_1(\mathcal{D}_\varepsilon, \mathbb{Z})\to H_1(T, \mathbb{Z})$ induced by the inclusion $\mathcal{D}_\varepsilon \subsetneq T$ is surjective and consequently there are 4 cycles a_j, b_j $(j=1, 2)$ on the curve $\mathcal{D}_\varepsilon$ such that

$$\Omega^*=\begin{bmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{bmatrix}$$

and

$$L_{\Omega^*}=\left\{\sum_1^2 l_j \int_{a_j}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}+n_j\int_{b_j}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} : l_j,\ n_j \in \mathbb{Z}\right\}$$

where $\omega_1 = dt_1|_{\mathscr{D}_\varepsilon}$, $\omega_2 = dt_2|_{\mathscr{D}_\varepsilon}$ (proposition 2.5); hence the 4 cycles a_j, b_j $(j=1, 2)$ in $\mathscr{D}_\varepsilon$ which we look for are A_j, B_j $(j=1, 2)$ and they generate $H_1(T, \mathbb{Z})$ such that

$$\Omega^*=\begin{bmatrix}\int_{A_1}\omega_1 & \int_{A_2}\omega_1 & \int_{B_1}\omega_1 & \int_{B_2}\omega_1\\ \int_{A_1}\omega_2 & \int_{A_2}\omega_2 & \int_{B_1}\omega_2 & \int_{B_2}\omega_2\end{bmatrix}$$

is a Riemann matrix ([1]). Since $U=2U^*\Delta^{-1}$ and $V=2V^*\Delta^{-1}$ we have $2\Delta^{-1}U^{-1}V\Delta\Delta^{-1}=(U^*)^{-1}2V^*\Delta^{-1}$ and from (2.16), we deduce that $\Omega^*=(\Delta, Z)$. Consequently T and $\text{Prym}(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)^*$ i.e. dual of $\text{Prym}(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)$, are two abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_{\Omega^*}$. By Chow's theorem, T and $\text{Prym}(\mathscr{D}_\varepsilon/\mathscr{D}_\varepsilon^0)^*$ are then algebraically isomorphic. This completes the proof of theorem 2.3.

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([1]) Otherwise there would exist a non trivial linear combination $\sum^2 l_j A_j + n_j B_j = \partial\rho$ with ρ a 2-chain on T. But dt_1 and dt_2 are closed differentials on T, so the Stokes' formulà implies that

$$\sum_1^2 l_j \int_{A_j}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}+n_j\int_{B_j}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\int_{\partial\rho}\begin{pmatrix}dt_1\\ dt_2\end{pmatrix}=0.$$

This is a contradiction because the columns of Ω^* must be linearly independent.

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