

ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 20, n° 3 (1987), p. 311-323

http://www.numdam.org/item?id=ASENS_1987_4_20_3_311_0

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BERNSTEIN-GELFAND-GELFAND RECIPROCITY ON PERVERSE SHEAVES ⁽¹⁾

BY R. MIROLLO AND K. VILONEN

0. Introduction

The purpose of this paper is to extend the results of Bernstein-Gelfand-Gelfand on infinite dimensional Lie algebra representations [BGG] to perverse sheaves on a wide class of complex analytic spaces. Our method is to use the inductive construction of perverse sheaves given in [MV1], [MV2]. We give alternative proofs of the theorems in [BGG], and in some sense offer an explanation as to when such results should hold. Our main results are stated in section 1, which follows very closely the introduction to the [BGG] paper.

We thank J. Bernstein and R. MacPherson for bringing these questions to our attention and E. de Shalit for pointing out the reference [Mu] to us.

1. Statement of the main results

Let k be a field which will be fixed throughout this paper. Let A be an associative algebra with identity which is finite dimensional as a vector space over k . We say that a category is of *Artin type* if it is equivalent to the category of finitely generated A -modules for some A .

Such a category \mathcal{A} has several special properties (*see e.g.* [CR]). It satisfies the Krull-Schmidt and Jordan-Holder theorems. Furthermore, it has a finite number of irreducible objects L_1, \dots, L_r and each L_i has a unique projective cover P_i . These modules P_i are precisely all the indecomposable projective modules.

Denote by $[M : L_i]$ the number of times the irreducible module L_i occurs in the Jordan-Holder series of M . The matrix $C_{ij} = [P_i : L_j]$ is called the *Cartan matrix* of \mathcal{A} . As is pointed out in [BGG] the Cartan matrix turns out to be symmetric in many

⁽¹⁾ Partially supported by N.S.F.

important examples. This is the case for modular representations of finite groups (see [CR]) and modules over the restricted universal enveloping algebra of a semi-simple Lie algebra in characteristic p ([H]). A third class of examples is given in [BGG] where the category \mathcal{O} of certain infinite dimensional representations of a complex semi-simple Lie algebra is constructed.

In all these examples a stronger duality principle holds. There is a class of modules M_1, \dots, M_l such that the modules P_i have a decomposition series with factors isomorphic to the M_j . Let $[P_i : M_j]$ denote the number of times M_j occurs in the decomposition series of P_i . We say that the category \mathcal{A} satisfies *BGG reciprocity* if $[P_i : M_j] = [M_j : L_i]$ for all i and j . In this case $C = {}^tDD$, where $D_{ij} = [M_i : L_j]$ is the *decomposition matrix*. In all the above examples we have such a reciprocity. In the case of modular representations $l \neq r$ and the matrix D is not square. In the case of category \mathcal{O} and in our case the matrix D is an upper triangular unimodular square matrix (if we choose a proper ordering for L_1, \dots, L_r). In the category \mathcal{O} the modules M are the Verma modules.

In this paper we want to show that these results are true for the category of perverse sheaves on a wide class of topological spaces. The results in [BBG] can be recovered from ours by applying localization ([BB], [BK]) and the Riemann-Hilbert correspondence ([K], [M]). Here the topological space is the flag manifold of a semi-simple complex group.

Let X be a complex analytic space with a complex analytic Whitney Stratification. Let \mathcal{S}' denote the strata of \mathcal{S} which are not of top dimension. We make the further assumption that $\pi_1(S) = 0$ for all $S \in \mathcal{S}$ and $\pi_2(S) = 0$ for all $S \in \mathcal{S}'$. We will keep this assumption all through this paper. Let $P(X)$ denote the category of perverse sheaves of \mathbf{k} -vector spaces on X which are constructible with respect to the fixed stratification ([BBD], [MV2]). We recall that $P(X)$ is the subcategory of the bounded derived category of \mathbf{k} -sheaves $D^b(X)$ consisting of complexes of \mathbf{k} -sheaves A^* on X satisfying:

(0) $H^k(i^* A^*)$ is a local system of finite rank on S

(1) $H^k(i^* A^*) = 0$ for $k > -\dim_{\mathbb{C}} S$

(2) $H^k(i^! A^*) = 0$ for $k < -\dim_{\mathbb{C}} S$

for all $S \in \mathcal{S}$, where $i : S \rightarrow X$ is the inclusion.

It is shown in [BBD] that the category $P(X)$ is an artinian abelian category.

THEOREM 1.1. — *The category $P(X)$ is of Artin type and its Cartan matrix is symmetric.*

We will prove this theorem in Section 2. We remark here that it suffices to prove that $P(X)$ has enough projectives to conclude that it is of Artin type. This follows from the following well-known

LEMMA 1.2. — *Let \mathcal{A} be an artinian, abelian category with enough projectives, finitely many irreducibles and $\text{Hom}(A, A')$ having a structure of a finite dimensional \mathbf{k} -vector space for all $A, A' \in \mathcal{A}$. Then \mathcal{A} is of Artin type over \mathbf{k} .*

Proof. — Let L_1, \dots, L_m be the irreducible objects and choose projectives $P_i \rightarrow L_i$. Then $P = \bigoplus P_i$ is a projective generator, and \mathcal{A} is equivalent to the category of (right) A -modules where $A = \text{Hom}(P, P)$ (see [B]).

Next we impose a further condition on the space X . We assume that $\bar{S} - S$ is a Cartier divisor in \mathcal{S} (or empty) for all $S \in \mathcal{S}$. For all $S_i \in \mathcal{S}$ we define perverse sheaves M_i as follows:

$$M_i = j_{!} k_{S_i} [\dim_{\mathbb{C}} S_i]$$

where $j : S_i \rightarrow X$ is the inclusion and k_{S_i} is the constant sheaf on S_i . This makes sense by Lemma 3.1. Let $l(X) = \max \{ \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} S \mid S \in \mathcal{S} \}$.

We say that an object $M \in P(X)$ has a *p-filtration* if it has a filtration whose quotients are M_i 's.

THEOREM 1.3. — *In the category $P(X)$ every projective object has a p-filtration, the BGG reciprocity is satisfied and $P(X)$ has projective dimension $\leq 2l(X)$.*

We will prove this theorem in Section 3.

To recover the results of [BGG] from ours we remark that given a complex semi-simple Lie Algebra \mathfrak{g} the category \mathcal{O}_0 is equivalent to $P(X)$, where $X = G/B$ is the flag manifold with stratification by the Schubert cells.

Remark 1.4. — The condition $\pi_1(S) = 0$ for $S \in \mathcal{S}'$ and $\pi_2(S) = 0$ for all $S \in \mathcal{S}'$ can be replaced by the following weaker condition. Let X be a complex manifold with Whitney stratification \mathcal{S} . We call the stratification \mathcal{S} a *good Whitney stratification* if all the projections $\pi_s : \tilde{\Lambda} \rightarrow S$ are fibre bundles, where $\tilde{\Lambda}_s = T_s^* X - \bigcup_{\substack{s' \neq s \\ s' \in \mathcal{S}}} \overline{T_{s'}^* X}$. For a good Whitney stratification \mathcal{S} the condition we need becomes that the map

$\alpha : \pi_1(\pi_s^{-1}(x)) \rightarrow \pi_1(\tilde{\Lambda}_s)$ is an isomorphism. The crucial point here is that this condition implies that $\pi_1(S) = 0$. By different arguments one can show that for the results of this paper to remain true it suffices to assume that α is injective with $\text{Coker } \alpha = \pi_1(S)$ finite and $\text{char}(\mathbf{k}) = 0$.

2. Construction of projectives and the symmetry of the Cartan matrix

In this section we will prove Theorem 1.1. We first apply the results of [MV1] and [MV2] to reduce it to an algebraic problem. We start by recalling the main construction of these papers.

Let \mathcal{A} and \mathcal{B} be two abelian categories, $F : \mathcal{A} \rightarrow \mathcal{B}$ a right exact functor, $G : \mathcal{A} \rightarrow \mathcal{B}$ a left exact functor and $T : F \rightarrow G$ a natural transformation. Then we define a category $\mathcal{C}(F, G; T)$ as follows. Its objects are pairs $(A, B) \in \text{Ob } \mathcal{A} \times \text{Ob } \mathcal{B}$ together with a commutative diagram

$$\begin{array}{ccc} FA & \xrightarrow{TA} & GA \\ & \searrow m & \nearrow n \\ & B & \end{array}$$

The morphisms are pairs $(f, g) \in \text{Mor } \mathcal{A} \times \text{Mor } \mathcal{B}$ such that the appropriate prism commutes. The category $\mathcal{C}(F, G; T)$ has a natural abelian category structure [MV2].

Recall that we have a complex analytic space X with a fixed stratification \mathcal{S} satisfying $\pi_1(S) = 0$ for all $S \in \mathcal{S}$ and $\pi_2(S) = 0$ for all $S \in \mathcal{S}'$. We denote by \mathcal{V} the category of finite dimensional \mathbf{k} -vector spaces. Under these hypotheses we have

THEOREM 2.1 ([MV1], [MV2]). — *The category $P(X)$ can be constructed by iterating the $\mathcal{C}(F, G; T)$ construction starting with $\mathcal{A} = \mathcal{V}$ and always using $\mathcal{B} = \mathcal{V}$.*

Proof. — This follows from Theorem 3.3 and section 7 of [MV2] using the hypothesis that $\pi_1(S) = 0$ for all $S \in \mathcal{S}$, and $\pi_2(S) = 0$ for all $S \in \mathcal{S}'$.

So in order to prove Theorem 1.1, it suffices to prove theorems about the categories $\mathcal{C}(F, G; T)$ which arise from perverse sheaves. For simplicity, we assume from now on that all abelian categories \mathcal{A} or $\mathcal{C}(F, G; T)$ under discussion come from iterating the $\mathcal{C}(F, G; T)$ construction beginning with \mathcal{V} and always using $\mathcal{B} = \mathcal{V}$. Such categories have a natural \mathbf{k} -vector space structure on their Hom sets.

GENERAL FACTS ABOUT $\mathcal{C}(F, G; T)$'s. — There are several interesting functors relating \mathcal{A} , \mathcal{V} and the $\mathcal{C}(F, G; T)$ built from $F \xrightarrow{T} G : \mathcal{A} \rightarrow \mathcal{V}$. First, given an object

$$N = (A, B, m, n) \in \mathcal{C}(F, G; T)$$

we have restrictions of N to \mathcal{A} and \mathcal{B} :

$$N|_{\mathcal{A}} = A, \quad N|_{\mathcal{B}} = B.$$

The restriction functors are exact. In fact, a complex $N^* \in \mathcal{C}(F, G; T)$ is exact if and only if $N^*|_{\mathcal{A}}$ and $N^*|_{\mathcal{B}}$ are exact. This follows immediately from the description of kernels and cokernels in $\mathcal{C}(F, G; T)$ in [MV2].

There are three functors \hat{F} , \hat{T} and \hat{G} from \mathcal{A} to $\mathcal{C}(F, G; T)$. If $A \in \mathcal{A}$ we set

$$\begin{array}{ccccc} \hat{F}A = FA & \xrightarrow{TA} & GA, & \hat{T}A = FA & \xrightarrow{TA} & GA, & \hat{G}A = FA & \xrightarrow{TA} & GA. \\ & \searrow \text{Id} & \nearrow TA & & \searrow TA & \nearrow \text{incl.} & & \searrow TA & \nearrow \text{Id} \\ & FA & & & \text{Im } TA & & & GA & \end{array}$$

There are obvious maps $\hat{F} \rightarrow \hat{T} \rightarrow \hat{G}$. Note that these functors \hat{F} , \hat{T} and \hat{G} correspond to the functors ${}^p j_!$, ${}^p j_*$ and ${}^p j_*$ in $P(X)$ (see example 4.6 in [MV2]).

The functor \hat{F} is right exact and the functor \hat{G} is left exact. We also have

LEMMA 2.2. — For $N \in C(F, G; T)$ and $A \in \mathcal{A}$ we have

$$\text{Hom}(\hat{F} A, N) = \text{Hom}(A, N|_{\mathcal{A}})$$

and

$$\text{Hom}(N, \hat{G} A) = \text{Hom}(N|_{\mathcal{A}}, A).$$

This lemma implies that \hat{F} preserves projectives and \hat{G} preserves injectives.

IRREDUCIBLE OBJECTS. — We wish to describe all the irreducible objects in any $\mathcal{C}(F, G; T)$ built on \mathcal{A}, \mathcal{V} .

PROPOSITION 2.3. — The category $\mathcal{C}(F, G; T)$ has the following irreducible objects:

1.

$$\begin{array}{ccc} FO & \longrightarrow & GO \\ & \searrow & \nearrow \\ & k & \end{array}$$

2. $\hat{T}L$, where $L \in \mathcal{A}$ is irreducible.

Hence any $\mathcal{C}(F, G; T)$ has finitely many nonisomorphic irreducible objects.

Proof. — Suppose

$$\begin{array}{ccc} FA & \longrightarrow & GA \\ & \searrow m & \nearrow n \\ & B & \end{array}$$

is irreducible, $A \neq 0$. Then m is surjective and n is injective, because otherwise there would be nontrivial maps of N to or from the irreducible

$$\begin{array}{ccc} FO & \longrightarrow & GO \\ & \searrow & \nearrow \\ & k & \end{array}$$

Hence $N = \hat{T}(N|_{\mathcal{A}}) = \hat{T}A$. We need to show that $A \in \mathcal{A}$ is irreducible. Let $A' \rightarrow A$ be any nonzero map. Then $\hat{T}A' \rightarrow \hat{T}A$ is a nonzero map, and so must be surjective. Hence $A' \rightarrow A$ is surjective, so A is irreducible.

Q.E.D.

REPRESENTABILITY OF FUNCTORS

PROPOSITION 2.4. — Assume \mathcal{A} has enough injectives. Then any left exact functor $G : \mathcal{A} \rightarrow \mathcal{V}$ is representable by an object $R \in \mathcal{A}$.

Proof. — This follows from Grothendieck's pro-representability theorem [Mu]. However we give a simple direct proof. G is exact when restricted to injective objects of \mathcal{A} . It suffices to show that there exists $R \in \mathcal{A}$ and a natural isomorphism $\text{Hom}(R, I) = GI$ for I injective in \mathcal{A} . If $I \in \mathcal{A}$, let $h_I = \text{Hom}(I, \cdot)$. Given $v \in GI$ there exists a natural map

$$h_v : h_I \rightarrow G$$

defined by $(h_v N)(f) = (Gf)v$, $f \in \text{Hom}(I, N)$. If $N = I$ then $(h_v I)(\text{Id}_I) = v$. Choose a spanning set v_1, \dots, v_r for GI . Then we get a natural map

$$\phi_I = \bigoplus_i h_{v_i} : h_{I^r} \rightarrow G.$$

ϕ_I has the property that $\phi_I(I) : h_{I^r}(I) \rightarrow GI$ is surjective. Let I_1, \dots, I_m be the indecomposable injectives, corresponding to the irreducibles objects of \mathcal{A} . Consider the sum $I = \bigoplus_k I_k^{r_k}$ and

$$\phi = \bigoplus \phi_{I_k} : h_I = \bigoplus_k h_{I_k^{r_k}} \rightarrow G.$$

Then $\phi : h_I \rightarrow G$ has the property that $\phi(J) : h_I(J) \rightarrow GJ$ is surjective for any injective $J \in \mathcal{A}$. Let $G' = \ker(h_I \xrightarrow{\phi} G)$. Then G' is also exact, so there exists $\phi' : h_{I'} \rightarrow G'$. Hence we have produced a 2-step resolution of G :

$$h_{I'} \rightarrow h_I \rightarrow G \rightarrow 0.$$

By Yoneda's lemma, we get a map $I \xrightarrow{\alpha} I'$. Let $R = \ker \alpha$. Then R represents G because for any injective J ,

$$\text{Hom}(R, J) \cong \text{Hom}(I, J)/\text{Im Hom}(I', I) \cong h_I(J)/\text{Im } h_{I'}(J) \cong GJ.$$

Q.E.D.

Remark. — A similar statement holds for right exact functors $F : \mathcal{A} \rightarrow \mathcal{V}$: If \mathcal{A} has enough projectives, there exists an object $S \in \mathcal{A}$ st

$$FN \cong \text{Hom}(N, S)^*$$

where $*$ is the dual in the sense of \mathbf{k} -vector spaces.

PROJECTIVES AND INJECTIVES IN $\mathcal{C}(F, G; T)$. — The following proposition together with Lemma 1.2 establishes the first part of Theorem 1.1.

PROPOSITION 2.5. — *Suppose \mathcal{A} has enough projectives (injectives). Then any $\mathcal{C}(F, G; T)$ built on \mathcal{A}, \mathcal{V} has enough projectives (injectives).*

Proof. — We must show that any object $N \in \mathcal{C}(F, G; T)$ is covered by a projective. We can assume N is irreducible. If $N = \hat{T}A$, $A \in \mathcal{A}$ irreducible, $A' \rightarrow A$ a projective covering of A , then $\hat{F}A' \rightarrow \hat{T}A$ is a projective cover of N .

So it suffices to cover the new irreducible

$$\begin{array}{ccc} \text{FO} & \longrightarrow & \text{GO} \\ & \searrow \quad \nearrow & \\ & \mathbf{k} & \end{array}$$

By Proposition 2.4 G is representable. Suppose $G \cong \text{Hom}(R, \cdot)$. Form the object P :

$$\begin{array}{ccc} \text{FR} & \longrightarrow & \text{GR} = \text{Hom}(R, R) \\ m \searrow & & \nearrow n \\ & \text{FR} \oplus \mathbf{k} & \end{array}$$

where $m = (\text{Id}, 0)$, $n|_{\text{FR}} = \text{TR}$, $n(0,1) = \text{Id}_R$. For any $N \in \mathcal{C}(F, G; T)$

$$\text{Hom}(P, N) \cong N|_{\mathcal{V}};$$

i.e., a map $P \rightarrow N$ is uniquely determined by the image of the element $(0,1)$ in $N|_{\mathcal{V}}$. Since $|_{\mathcal{V}}$ is exact, P is projective. Clearly, P covers the new irreducible. Hence $\mathcal{C}(F, G; T)$ has enough projectives. A similar proof works for injectives.

Q.E.D.

To complete the proof of Theorem 1.1 we have to prove the symmetry of the Cartan matrix. However this symmetry is not true for $\mathcal{C}(F, G; T)$'s in general. The category of perverse sheaves $P(X)$ has an involution $A \rightarrow A^*$ given by Verdier duality which satisfies

- (a) $\text{Hom}(A_1, A_2^*) \cong \text{Hom}(A_2, A_1^*)$,
- (b) $L^* \cong L$ for L irreducible.

Condition (b) holds because $\pi_1(S) = 0$ for all strata S . If \mathcal{L} is a complex link of S at a point ([MV2], [GM]) then $F(A) = H^{-d-1}(\mathcal{L}, A)$ and $G(A) = H_c^{-d-1}(\mathcal{L}, A)$. By Verdier duality we then have

$$F(A) = G(A^*)^* \quad \text{and} \quad T(A^*) = T(A)^*$$

which means that the representing objects S and R for F and G satisfy $S = R^*$.

Motivated by these considerations we develop a notion of duality for $\mathcal{C}(F, G; T)$'s.

DUALITY. — Let \mathcal{A} be an abelian category. A *duality* on \mathcal{A} is by definition a contravariant functor $A \rightarrow A^*$ st.

- (a) If $A, B \in \mathcal{A}$, $\text{Hom}(A, B^*) \cong \text{Hom}(B, A^*)$ naturally;
- (b) $*$ is fully faithful.

Condition (b) is equivalent to

- (b') The natural map $A \rightarrow A^{**}$ is an equivalence of categories.

Suppose \mathcal{A} has a duality $*$. We wish to extend $*$ to $\mathcal{C}(F, G; T)$. However, we need some conditions on the representing objects for F and G .

We assume that $S=R^*$ and that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(R, A)^* & \xrightarrow{(TA)^*} & \mathrm{Hom}(A, R^*) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A^*, R^*)^* & \xrightarrow{T(A^*)} & \mathrm{Hom}(R, A^*) \end{array}$$

commutes: i. e., $T(A^*)=(TA)^*$ under the above identifications.

If

$$N = \begin{array}{ccc} & FA & \xrightarrow{TA} GA \\ & \searrow & \nearrow \\ & B & \end{array}$$

we can let N^* be the object

$$\begin{array}{ccc} (GA)^* & \xrightarrow{(TA)^*} & (FA)^* \\ & \searrow & \nearrow \\ & B^* & \end{array} = \begin{array}{ccc} \mathrm{Hom}(R, A)^* & \xrightarrow{(TA)^*} & \mathrm{Hom}(A, R^*) \\ & \searrow & \nearrow \\ & B^* & \end{array}$$

Then $*$ is a duality on $\mathcal{C}(F, G; T)$ extending the duality on \mathcal{A} . Note that $*\hat{F}=\hat{G}^*$, $*\hat{T}=\hat{T}^*$ and $*$ fixes the new irreducible object. Hence if $*$ on \mathcal{A} fixes irreducible objects in \mathcal{A} , $*$ on $\mathcal{C}(F, G; T)$ will also fix irreducible objects in $\mathcal{C}(F, G; T)$.

SYMMETRY OF THE CARTAN MATRIX. — Let L_1, \dots, L_r be the distinct irreducibles in \mathcal{A} , and $P_i \rightarrow L_i$ the projective covers of L_i . Note that the L_i 's have the property that $\dim_k \mathrm{Hom}(L_i, L_j) = \delta_{ij}$.

Consider the Grothendieck group $K(\mathcal{A})$. This is a free abelian group with basis $[L_1], \dots, [L_r]$. We have by definition

$$[N] = \sum_{j=1}^r [N : L_j] [L_j], \quad N \in \mathcal{A}.$$

Note that $[N : L_j] = \dim_k \mathrm{Hom}(P_j, N)$, because both sides are additive functions on $K(\mathcal{A})$ which agree when $N=L_i$.

Recall that the Cartan matrix of \mathcal{A} is $C_{ij}=[P_i : L_j]$. We are interested in the *symmetry* of C_{ij} .

The category of perverse sheaves $P(X)$ is constructed by iterating the $\mathcal{C}(F, G; T)$ construction where $F=\mathrm{Hom}(\cdot, S)^*$ and $G=\mathrm{Hom}(R, \cdot)$ are represented by dual objects $S=R^*$. Since $*$ fixes irreducibles, $[R^*]=[R]$ in the Grothendieck group. Therefore the following proposition shows that the Cartan matrix for $P(X)$ is symmetric, and completes the proof of Theorem 1.1.

PROPOSITION 2.6. — *The Cartan matrix of $\mathcal{C}(F, G; T)$ is symmetric precisely when the Cartan matrix of \mathcal{A} is symmetric and $[R]=[S]$ in $K(\mathcal{A})$.*

Proof. — In $\mathcal{C}(F, G; T)$ let $\hat{L}_i = \hat{T}L_i$, $\hat{P}_i = \hat{F}P_i$, $1 \leq i \leq r$,

$$\begin{array}{ccccc} \hat{L}_{r+1} = FO & \longrightarrow & GO, & \hat{P}_{r+1} & \longrightarrow & \hat{L}_{r+1} \\ & \searrow & & \nearrow & & \\ & & \mathbf{k} & & & \end{array}$$

its projective cover.

Let \hat{C}_{ij} be the Cartan matrix of $\mathcal{C}(F, G; T)$. If $i, j \leq r$ then by adjunction $\hat{C}_{ij} = \dim \operatorname{Hom}(\hat{F}P_j, \hat{F}P_i) = \dim \operatorname{Hom}(P_j, P_i) = C_{ij}$. So we need only check that that $\hat{C}_{i, r+1} = \hat{C}_{r+1, i}$, $1 \leq i \leq r$. Write the new projective

$$\begin{array}{ccc} \hat{P} = FR & \longrightarrow & GR \\ & \searrow & \nearrow \\ & FR \oplus \mathbf{k} & \end{array}$$

in terms of \hat{P}_i 's:

$$\hat{P} = \hat{P}_{r+1} \oplus \bigoplus_{i=1}^r \hat{P}_i^{\alpha_i}$$

Then $\dim \operatorname{Hom}(\hat{P}_{r+1}, \hat{P}_i) = \dim \operatorname{Hom}(\hat{P}, \hat{P}_i) - \sum_{j=1}^r \alpha_j C_{ij}$, and $\dim \operatorname{Hom}(\hat{P}_i, \hat{P}_{r+1}) = \dim \operatorname{Hom}(\hat{P}_i, \hat{P}) - \sum_{j=1}^r \alpha_j C_{ji}$. So we need to compare $\operatorname{Hom}(\hat{P}_i, \hat{P})$ and $\operatorname{Hom}(\hat{P}, \hat{P}_i)$.

By adjunction

$$\dim \operatorname{Hom}(\hat{P}_i, \hat{P}) = \dim \operatorname{Hom}(P_i, R) = [R : L_i]$$

and

$$\dim \operatorname{Hom}(\hat{P}, \hat{P}_i) = \dim FP_i = \dim \operatorname{Hom}(P_i, S)^* = [S : L_i].$$

So \hat{C}_{ij} is symmetric $\Leftrightarrow [R]=[S]$ in $K(\mathcal{A})$.

Q.E.D.

3. The BGG reciprocity

In this section we will give a proof of theorem 1.3. We start with some topological considerations. As in the previous section, after this the rest of the proof is purely algebraic.

We recall that we have a complex analytic space X with an analytic stratification \mathcal{S} satisfying the Whitney conditions. As before we assume that $\pi_1(S) = 0$ for all $S \in \mathcal{S}$ and

$\pi_2(S)=0$ for all $S \in \mathcal{S}'$. From now on we assume furthermore that $\bar{S}-S$ is a Cartier divisor in \bar{S} (or empty) for all $S \in \mathcal{S}$. (If X is algebraic this means that $S \rightarrow X$ is affine.)

LEMMA 3.1. — *Let $S \in \mathcal{S}$ and $j: S \rightarrow X$ be the inclusion. Then $j_! k_S[\dim_C S]$ is perverse. (See [BBD] 4.1.3 for the algebraic case).*

Proof. — Clearly $j_! k_S[\dim_C S]$ satisfies the first perversity condition. It remains to check the second condition or equivalently the first perversity condition for the dual $Rj_* k_S[\dim_C S]$. Let $d = \dim_C S$.

Cutting by a normal slice reduces the problem to the case of a point stratum $S' = \{x\} \subset \bar{S}$. Let $i: \{x\} \rightarrow X$. Then if B is a small neighborhood of x in X ,

$$H^k(i^* Rj_* k_S[d]) = H^{k+d}(Rj_* k_S)_x \simeq H^{k+d}(B, Rj_* k_S) \simeq H^{k+d}(B \cap S, \mathbf{k}) \simeq 0 \quad \text{for } k > 0$$

because $B \cap S$ is a Stein manifold of dimension d .

Q.E.D.

For any stratum $S_k \in \mathcal{S}$ we define the object M_k by $M_k = j_! k_{S_k}[\dim_C S_k]$, where $j: S_k \rightarrow X$ is the inclusion.

The construction of the objects M_k can be done inductively as follows. Let $X \subset \hat{X}$ such that X is stratified by S_1, \dots, S_{r-1} and let $\hat{X} - X = S_r$. We assume that $\dim S_k \geq \dim S_h$ if $k \leq h$. Let $\hat{j}: X \rightarrow \hat{X}$, $j_k: S_k \rightarrow X$ and $\hat{j}_k: S_k \rightarrow \hat{X}$ be the inclusions. Then if we denote $\hat{M}_k = \hat{j}_! k_{S_k}[\dim_C S_k]$ we have $\hat{M}_k = \hat{j}_! M_k$ for $k \leq r-1$. Or because \hat{M}_k is perverse we can phrase this as $\hat{M}_k = {}^p\hat{j}_! M_k$ for $k \leq r-1$.

If we interpret this in terms of the $\mathcal{C}(F, G; T)$ via theorem 2.1 we get that $\hat{M}_k = \hat{F}(M_k)$ for $k \leq r-1$ and $\hat{M}_r = \hat{L}_r$.

LEMMA 3.2. — *We have $L^1 \hat{F}(M_k) = 0$.*

Proof. — It suffices to show that given any exact sequence $0 \rightarrow N' \rightarrow N \rightarrow M_k \rightarrow 0$ the sequence $0 \rightarrow \hat{F}N' \rightarrow \hat{F}N \rightarrow \hat{F}M_k \rightarrow 0$ is exact. Because ${}^p j_! M_k = j_! M_k$ we have an exact sequence $0 \rightarrow {}^p j_! N' \rightarrow {}^p j_! N \rightarrow {}^p j_! M_k \rightarrow 0$ in $P(X)$, but this is just the exact sequence $0 \rightarrow \hat{F}N' \rightarrow \hat{F}N \rightarrow \hat{F}M_k \rightarrow 0$.

Remark. — Because we are using a *fixed* stratification in our definition of $P(X)$ it is *not* true that $\text{Ext}^k(A, B)$ is the same in $P(X)$ and in $D^b(X)$. It is however, clearly true for $k=0,1$.

We now turn to algebra. We make the additional hypothesis that $L^1 \hat{F}(M_k) = 0$ at every stage of the construction of our $\mathcal{C}(F, G; T)$. We also assume duality at every stage.

Recall [BBG] that we say that N has a *p-filtration* if there is a filtration $N_1 \subset N_2 \subset \dots$ such that $N_k/N_{k+1} \cong M_i$ for some i . We will start by proving a lemma about the existence of *p-filtrations* which in particular shows that every projective object has a *p-filtration*.

LEMMA 3.3. — *Let \mathcal{C} be a category which is constructed by iteration with the above hypotheses. Then N has a *p-filtration* if and only if $\text{Ext}^1(M_i, N^*) = 0$ for all i .*

Proof. — We proceed by induction. Assume that it is true for \mathcal{A} and construct a $\mathcal{C}(F, G; T)$ from \mathcal{A} . Suppose $\text{Ext}^1(\hat{M}_i, N^*) = 0$ for $i = 1, \dots, r+1$.

To calculate $\text{Ext}^1(\hat{M}_{r+1}, \hat{N}^*)$ we use the resolution

$$0 \rightarrow \hat{F}R \rightarrow \hat{P} \rightarrow \hat{M}_{r+1} \rightarrow 0.$$

This gives

$$\text{Hom}(\hat{P}, \hat{N}) \rightarrow \text{Hom}(\hat{F}R, \hat{N}^*) \rightarrow \text{Ext}^1(\hat{M}_{r+1}, \hat{N}^*) \rightarrow 0.$$

$$\text{Let } \hat{N} = FA \xrightarrow{m} B \xrightarrow{n} GA.$$

Then $\text{Hom}(\hat{P}, \hat{N}^*) = B^*$, and

$$\text{Hom}(\hat{F}R, \hat{N}^*) = \text{Hom}(R, \hat{N}^*|_{\mathcal{A}}) = G(A^*) = (FA)^*.$$

Therefore $\text{Ext}^1(\hat{M}_{r+1}, \hat{N}^*) \cong \text{Coker}(m^*) = \text{Ker}(m)^*$.

Hence m is an injection. It follows from this that we have a short exact sequence

$$0 \rightarrow \hat{F}(\hat{N}|_{\mathcal{A}}) \rightarrow \hat{N} \rightarrow \hat{M}_{r+1}^{\oplus q} \rightarrow 0, \quad q \geq 0.$$

So it is enough to show that $F(\hat{N}|_{\mathcal{A}})$ has a p -filtration or since $L^1 \hat{F}(M_i) = 0$ that $\hat{N}|_{\mathcal{A}}$ has a p -filtration. But $L^1 \hat{F}M_i = 0$ means we have

$$\text{Ext}^1(M_i, \hat{N}^*|_{\mathcal{A}}) = \text{Ext}^1(\hat{M}_i, \hat{N}^*) = 0.$$

and therefore $\hat{N}^*|_{\mathcal{A}}$ has a p -filtration.

For the converse it suffices to check that $\text{Ext}^1(\hat{M}_i, \hat{M}_j^*) = 0$. By duality $\text{Ext}^1(\hat{M}_i, \hat{M}_j^*) = \text{Ext}^1(\hat{M}_j, \hat{M}_i^*)$. If either i or j is $\leq r$, then $\text{Ext}^1(\hat{M}_i, \hat{M}_j^*) = 0$ by adjunction and the vanishing of $L^1 \hat{F}R$. And $\text{Ext}^1(\hat{M}_{r+1}, \hat{M}_{r+1}^*) = 0$ as before.

Q.E.D.

Next we give a proof of the BGG reciprocity. Assume that we have constructed a category \mathcal{C} by iteration, where $F(A) = G(A^*)^*$, $(TA)^* = TA^*$, and $L^1 \hat{F}(M_k) = 0$ for all k at each stage of the iteration.

Note that the decomposition matrix $D = [M_i : L_j]$ is unipotent upper triangular and therefore the M_i form a basis for $K(\mathcal{C})$. Let $E = [P_i : M_j]$, where $[P_i] = \sum [P_i : M_j][M_j]$ in $K(\mathcal{C})$. Since the P_i have a p -filtration the matrix E has positive entries.

THEOREM 3.4 (BGG Reciprocity). — *We have $E = {}^tD$ and therefore $C = {}^tDD$.*

Proof. — We proceed by induction. Let E and D be the decomposition matrices of \mathcal{A} and \hat{E} and \hat{D} the corresponding matrices in $\mathcal{C}(F, G; T)$. Because the P_i have p -filtrations and $L^1 \hat{F}(M_j) = 0$ we have

$$\hat{E}_{ij} = E_{ij} \quad \text{if } 1 \leq i, j \leq r$$

and

$$E_{i, r+1} = 0 \quad \text{if } 1 \leq i \leq r.$$

So to prove the proposition we must only check that

$$\hat{D}_{r+1, i} = \hat{E}_{i, r+1} \quad \text{if } 1 \leq i \leq r+1,$$

i. e.

$$[\hat{P}_{r+1} : \hat{M}_i] = [\hat{M}_i : \hat{L}_{r+1}].$$

We have the short exact sequence $0 \rightarrow \hat{F}R \rightarrow \hat{P} \rightarrow \hat{L}_{r+1} \rightarrow 0$ where R represents G and \hat{P} is the new projective constructed in paragraph 2. Write

$$\hat{P} = \hat{P}_{r+1} \oplus \bigoplus_{i=1}^r \hat{P}_i^{\alpha_i}$$

as before. Then if $1 \leq i \leq r$

$$[\hat{P}_{r+1} : \hat{M}_i] = [\hat{P} : \hat{M}_i] - \sum_{j=1}^r \alpha_j [\hat{P}_j : \hat{M}_i] = [R : M_i] - \sum_{j=1}^r \alpha_j E_{ji};$$

$$\begin{aligned} [\hat{M}_i : \hat{L}_{r+1}] &= \dim \operatorname{Hom}(\hat{P}_{r+1}, \hat{M}_i) \\ &= \dim \operatorname{Hom}(\hat{P}, \hat{M}_i) - \sum_{j=1}^r \alpha_j \dim \operatorname{Hom}(\hat{P}_j, \hat{M}_i) \\ &= \dim \hat{F}M_i - \sum_{j=1}^r \alpha_j D_{ij} \\ &= \dim \operatorname{Hom}(M_i, R^*) - \sum_{j=1}^r \alpha_j D_{ij}. \end{aligned}$$

So we must show that $[R : M_i] = \dim \operatorname{Hom}(M_i, R^*)$. We have $\operatorname{Ext}^1(M_i, M_j^*) = 0$ and $\dim \operatorname{Hom}(M_i, M_j^*) = \delta_{ij}$ (this can easily be established by induction). Using this and the fact that R has a p -filtration we see that $[R : M_i] = \dim \operatorname{Hom}(M_i, R^*)$.

Q.E.D.

We will conclude by proving that the projective dimension of $P(X) \leq 2l(X)$, where

$$l(X) = \dim_{\mathbb{C}} X - \min \{ \dim_{\mathbb{C}} S \mid S \in \mathcal{S} \}.$$

We define another length function $l(k)$ by induction as follows. $l(1) = 0$. Suppose \mathcal{A} has objects M_1, \dots, M_r and $\mathcal{C} = \mathcal{C}(F, G; T)$ is constructed with representing object R . Let $l(r+1) = l(r)$ if R has a decomposition series with M_k such that $l(k) < l(r)$, $l(r+1) = l(r) + 1$ otherwise. Note that if X has strata S_1, S_2, \dots then $l(k) \leq \operatorname{codim}_{\mathbb{C}} S_k$. Let $l(\mathcal{C}) = \max_{k \geq 1} l(k)$. Then $l(\mathcal{C}) \leq l(X)$.

LEMMA 3.5. — We have $p.d. M_i \leq l(i)$.

Proof. — We proceed by induction. Construct $\mathcal{C}(F, G, T)$ from \mathcal{A} . Since \mathcal{A} has finite projective dimension by induction, $L^q \hat{F}(M_i) = 0$ for all $q > 0$, i. e. the modules M_i are \hat{F} -acyclic. Hence $p.d. \hat{F}M_i = p.d. M_i$. Therefore it is enough to prove the result for

the new $\hat{M}_{r+1} = \hat{L}_{r+1}$. But we have a short exact sequence

$$0 \rightarrow \hat{F}R \rightarrow \hat{P} \rightarrow \hat{L}_{r+1} \rightarrow 0$$

so $\text{p. d. } \hat{M}_{r+1} \leq \text{p. d. } R + 1 \leq l(r+1)$, because R has a p -filtration.

Q.E.D.

PROPOSITION 3.6. — *We have $\text{p. d. } L_i \leq 2l(\mathcal{C}) - l(i)$ and hence $\text{p. d. } P(X) \leq 2l(X)$.*

Proof. — If $i = r+1$ this follows from Lemma 3.5. Consider the short exact sequence

$$0 \rightarrow \hat{K}_i \rightarrow \hat{M}_i \rightarrow \hat{L}_i \rightarrow 0.$$

The module \hat{K}_i has a decomposition series involving only \hat{L}_j , where $l(j) > l(i)$. We proceed by descending induction on i . Hence assume that $\text{p. d. } \hat{L}_j \leq 2l(\mathcal{C}) - l(j)$ for $j > i$. Then

$$\text{p. d. } \hat{L}_i \leq \max(\text{p. d. } \hat{M}_i, 1 + \text{p. d. } \hat{K}_i) \leq \max(l(i), 1 + 2l(\mathcal{C}) - (l(i) + 1)) = 2l(\mathcal{C}) - l(i).$$

Q.E.D.

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(Manuscrit reçu le 6 octobre 1986,
révisé le 4 mai 1987).

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