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ON MINIMAL IMMERSIONS OF S^{n-1} INTO $S^n(1)$, $n \geqq 4$

BY WU-YI HSIANG AND PER TOMTER

1. Introduction

In the study of basic global objects in high dimensional spherical geometry, one of the outstanding natural problems is the following spherical Bernstein problem formulated by Chern [7]:

The spherical Bernstein problem: Let the $(n-1)$-sphere, S^{n-1}, be imbedded as a minimal hypersurface in $S^n(1)$. Is it (necessarily) an equator?

In the beginning dimension of $n=3$, one has the following strong uniqueness theorems, namely,

THEOREM (Almgren [2], Calabi [3]). — *A minimal immersion of* S^2 *into* $S^3(1)$ *is necessarily the equator.*

THEOREM (Chern [8]). — *A constant mean curvature immersion of* S^2 *into* $S^3(1)$ *is necessarily totally umbilical, i. e.* $O(3)$ *invariant.*

However, in the dimensions $n \geqq 4$, one has a series of recent results on the existence of non-equatorial, imbedded, minimal hyperspheres in $S^n(1)$, which, in our opinion, only begins to expose the profound depth of the above spherical Bernstein problem, or rather, the problem of minimal hyperspheres in $S^n(1)$, $n \geqq 4$. Roughly speaking, up to now, one has the following known examples of non-equatorial minimal hyperspheres:

(i) There exist 18 specific orthogonal transformation groups, $(G, S^n(1))$, each of them accommodates infinitely many, mutually non-congruent, G-invariant examples of (non-equatorial) imbedded minimal hyperspheres (*cf.* [11]).

(ii) Besides the above 18 families of infinite examples with their dimensions n ranging over 4, 5, 6, 7, 8, 10, 12 and 14, there exists at least one more example of non-equatorial, imbedded minimal hypersphere for each isoparametric foliation of $S^n(1)$ with rank$=2$ and $g=3$ or 4 (*cf.* § 2). We refer to [13], [19], [20] for the actual construction of such additional examples.

So far, there are no known examples of non-equatorial, imbedded, minimal hyperspheres in $S^{2m+1}(1)$, $m \geqq 4$. Intuitively speaking, the minimality condition only imposes a single equation among the $n-1$ principal curvatures of a hypersurface in $S^n(1)$, namely,

$\sum k_i = 0$. Such a condition becomes less restrictive for larger n, therefore, the lack of known examples of imbedded minimal hypersurfaces for high dimensional spheres, most likely, merely reflects the limitation of the methods of construction used so far, rather than the actual reality.

From the viewpoint of global analysis, a minimal hypersphere in $S^n(1)$ is exactly a closed integral hypersurface of the simplest possible topological type for the minimal equation of $S^n(1)$. Hence examples of immersed minimal hyperspheres are almost as significant as the imbedded ones. The purpose of this paper is to give a unified, clear-cut proof of the following theorem, which exhibits a great many varieties of examples of immersed minimal hyperspheres in $S^n(1)$ for all $n \geqq 4$.

MAIN THEOREM. — *For each rank two isoparametric foliation of* $S^n(1)$ *with two fixed points,* $n \geqq 4$, *there exist infinitely many mutually non-congruent examples of foliated minimal immersions of* S^{n-1} *into* $S^n(1)$.

The above theorem demonstrates that isoparametric foliations can, in fact, be exploited to produce infinitely many examples of minimal hyperspheres in $S^n(1)$ for all dimension $n \geqq 4$, if one does not insist on the analytically elusive restriction of imbedding. The techniques we use in the proof of the above theorem can also be adapted to produce many other interesting examples of minimal hypersurfaces in compact symmetric spaces.

2. Rank two isoparametric foliations on $S^n(1)$ and foliated minimal hypersurfaces

Recall that an isoparametric hypersurface in a space of constant curvature, $M^{n+1}(c)$, is, by definition, a level surface of an isoparametric function, namely, a function $f : M^{n+1}(c) \to R$ such that Δf and $|\nabla f|^2 \equiv 0 \pmod f$. Geometrically, the level surfaces of such an isoparametric function constitute a parallel foliation of hypersurfaces of constant mean curvatures. Such nice geometric structures of rank one isoparametric foliations were first studied by B. Segre and E. Cartan. In a series of papers of E. Cartan ([4], [5], [6]), he was particularly fascinated by the profound depth of the spherical case and its mysterious connection with the Lie group theory. This subject was somehow forgotten until it was revived by a sequence of recent papers ([9], [14], [15], [16]). High codimensional generalization of isoparametric foliations was proposed in a recent paper of Terng [17], namely, a submanifold N^n in $M^{n+k}(c)$ is isoparametric if its normal bundle, ν, is flat and its second fundamental form has constant principal eigenvalues along any parallel section of ν. An isoparametric foliation of rank k on $M^{n+k}(c)$ is, by definition, a parallel foliation of $M^{n+k}(c)$ by codimension k isoparametric submanifolds and their focal varieties. We refer to [17] for a general theory and some basic theorems on isoparametric foliations. One of its basic features is the existence of a totally geodesic normal section with an induced group generated by reflections, $(W, M^k(c))$, called the associated Coxeter group of the given isoparametric foliation. Let C_0 be an arbitrarily chosen Weyl chamber of $(W, M^k(c))$. Then C_0 intersects each "leave" perpendicularly at exactly one point and the projection map, $p : M^{n+k}(c) \to M^k(c)/W \simeq C_0$, is a Riemannian submersion. A foliated submanifold is, by definition, a submanifold, Σ^{n+d}, which consists of a suitable subcollection of leaves. Hence, it is uniquely determined by the transversal intersection $\Sigma^{n+d} \cap C_0$.

In this paper we shall only consider those rank 2 isoparametric foliations on $S^{n+2}(1)$ which have exactly two point-leaves. The Weyl chamber of $(W, S^2(1))$ of such an isoparametric foliation is a spherical lune of angle π/g, $g=2$, 3, 4, or 6, where the two vertices are exactly the pair of point-leaves and the two boundary arcs parametrize the two types of focal varieties. We shall denote the multiplicities of focal directions of the two types of focal varieties by m_1 and m_2 respectively. Then one has the following (theoretical) possibilities [1]:

(i) $g=2$: m_1, m_2 can be an arbitrary pair of positive integers,

(ii) $g=3$: $m_1=m_2=1$, 2, 4 or 8,

(iii) $g=6$: $m_1=m_2=1$ or 2,

(iv) $g=4$: then $m_1 \leqq m_2$ must satisfy one of the following three restrictions, namely

(4 A) m_1+m_2+1 is divisible by $2^k := \min\{2^\alpha \mid 2^\alpha > m_1\}$,

(4 B 1) m_1 is a power of 2 and $2\,m_1$ divides (m_2+1),

(4 B 2) m_1 is a power of 2 and $3\,m_1$ divides $2(m_2+1)$.

It is convenient to parametrize the spherical lune, C_1, by the polar coordinate system $\{(r, \theta);\ 0 \leqq r \leqq \pi,\ 0 \leqq \theta \leqq \pi/g\}$, $ds^2 = dr^2 + \sin^2 r\, d\theta^2$. Since the n-dimensional leaf determined by (r, θ) is foliated by g pairwise orthogonal round spheres whose radii depend on the spherical distance to the focal submanifolds, one finds the following volume functions:

(i) $$g=2: \quad v(r, \theta) = K.(\sin r)^{m_1+m_2}.(\sin\theta)^{m_1}.(\cos\theta)^{m_2}$$

(ii) $$g=3: \quad v(r, \theta) = K.(\sin r)^{3m}.\left[\sin\theta.\cos\left(\theta+\frac{\pi}{6}\right)\cos\left(\theta-\frac{\pi}{6}\right)\right]^m$$

(iii) $$g=6: \quad v(r, \theta) = K.(\sin r)^{6m}.\left[\sin\theta.\cos\theta.\cos\left(\theta+\frac{\pi}{6}\right)\right.$$
$$\left..\cos\left(\theta-\frac{\pi}{6}\right)\cos\left(\theta+\frac{\pi}{3}\right)\cos\left(\theta-\frac{\pi}{3}\right)\right]^m$$

(iv) $$g=4: \quad v(r, \theta) = K.(\sin r)^{2m_1+2m_2}.[\sin\theta.\cos\theta]^{m_1}$$
$$.\left[\cos\left(\theta+\frac{\pi}{4}\right)\cos\left(\theta-\frac{\pi}{4}\right)\right]^{m_2}.$$

LEMMA 1. — *For a given rank 2 isoparametric foliation on* $S^{n+2}(1)$ *of the above type, the generating curve of a foliated minimal hypersurface,* Σ^{n+1}, *is characterized by the following ODE:*

$$(*) \qquad \frac{d\alpha}{ds} + \cos r\frac{d\theta}{ds} + \sin r\frac{d\theta}{ds}.\frac{\partial}{\partial r}\ln v - \frac{1}{\sin r}\frac{dr}{ds}.\frac{\partial}{\partial\theta}\ln v = 0$$

where α is the angle between the generating curve, $\gamma=\Sigma^{n+1}\cap C_0$, and the radial direction, and v is the above volume function.

Proof. — Straightforward computation will show that the mean curvature of Σ^{n+1} at an interior point, say $p=(r(s), \theta(s))$, of γ is equal to the mean curvature of γ in C_0 minus the normal derivative of ln v, namely,

$$H(\Sigma^{n+1}, p)=H(\gamma, p)-\frac{d}{dn}\ln v(r, \theta)\big|_p$$

$$=\left(\frac{d\alpha}{ds}+\cos r\frac{d\theta}{ds}\right)-\left(-\sin r\frac{d\theta}{ds}\frac{\partial}{\partial r}\ln v+\frac{1}{\sin r}\frac{dr}{ds}\frac{\partial}{\partial\theta}\ln v\right).$$

Hence, Σ^{n+1} is minimal if and only if its generating curve γ satisfies the ODE $(*)$.

Q. E. D.

3. The reduced minimal equation and its singularities

From now on, we shall simply call the ODE $(*)$ the reduced minimal equation. Observe that the volume function $v(r, \theta)$ is of the form $K.(\sin r)^n.g(\theta)$; the reduced minimal equation may be reformulated as the following dynamical system:

$$(*') \qquad \begin{aligned} \frac{dr}{ds}&=\cos\alpha \\ \frac{d\theta}{ds}&=\frac{\sin\alpha}{\sin r} \\ \frac{d\alpha}{ds}&=-(n+1)\frac{\sin\alpha}{\sin r}.\cos r+\frac{\cos\alpha}{\sin r}G(\theta) \end{aligned}$$

where $G(\theta)=(d/d\theta)\ln g(\theta)=g'(\theta)/g(\theta)$.

LEMMA 2. — $(d/d\theta)\,G(\theta)+n+1<0$, *and* $G(\theta)$ *decreases from* $+\infty$ *to* $-\infty$ *as* θ *varies from* 0 *to* π/g.

Proof. — Straightforward case-by-case verification. For example, in the case $g=2$, $G(\theta)=m_1\cot\theta-m_2\tan\theta$,

$$\frac{d}{d\theta}G(\theta)=-m_1(1+\cot^2\theta)-m_2(1+\tan^2\theta)<-(m_1+m_2+2)=-(n+1).$$

Notice that $G(\theta)=g'(\theta)/g(\theta)$ and $g(0)=g(\pi/g)=0$.

Q. E. D.

Set θ_0 to be the unique value of $\theta\in[0, \pi/g]$ such that $G(\theta_0)=0$. It is easy to see that $(*)$ has the following two obvious solutions, namely,

(i) The "equator solution": $r\equiv\pi/2$ whose inverse image is an equator in $S^{n+2}(1)$.

(ii) The "meridian solution": $\theta \equiv \theta_0$ whose inverse image is the suspension of the isoparametric submanifold passing through $(\pi/2, \theta_0)$ which is itself a minimal submanifold in the above equator $S^{n+1}(1)$.

Observe that the inverse image of a solution curve, γ, of $(*)$ which starts at one of the boundary arcs of C_0 and terminates at the other boundary arc of C_0 is necessarily an immersed minimal hypersphere in $S^{n+2}(1)$. Therefore, the basic approach of this paper is exactly to establish the existence of infinitely many, geometrically distinct, solution curves of the ODE $(*)$ of the above type for each given case of isoparametric foliation. However, points of the two boundary arcs of C_0, namely, $B_1 = \{(r, 0); 0 < r < \pi\}$ and $B_2 = \{r, \pi/g); 0 < r < \pi\}$, are singularities of the ODE $(*)$. Therefore, one needs the following lemmas concerning the behavior of solution curves which start or terminate at points of B_1 or B_2.

LEMMA 3. — *To each boundary point $(b, 0) \in B_1$ [resp. $(b, \pi/g) \in B_2$], there exists a unique solution curve, $\gamma_b^{(1)}$ (resp. $\gamma_b^{(2)}$) which starts at $(b, 0)$ [resp. $(b, \pi/g)$], namely*

$$\gamma_b^{(1)}(s) = (r_1(b, s), \theta_1(b, s)) \quad \text{and} \quad \lim_{s \to 0+} \gamma_b^{(1)}(s) = (b, 0)$$

$$\gamma_b^{(2)}(s) = (r_2(b, s), \theta_2(b, s)) \quad \text{and} \quad \lim_{s \to 0+} \gamma_b^{(2)}(s) = (b, \pi/g).$$

Moreover, both $r_i(b, s)$ and $\theta_i(b, s)$, $i = 1$ or 2, are automatically analytic functions of b and s.

Proof. — Let $\gamma_b^{(1)}$ (resp. $\gamma_b^{(2)}$) be a given solution curve of the ODE $(*)$ starting at $(b, 0)$ [resp. $(b, \pi/g)$]. Then it follows easily from $(*)$ that γ_b^i must be automatically perpendicular to B_i, $i = 1$ or 2. Therefore, its inverse image, $p^{-1}(\gamma_b^i)$, is a smooth minimal hypersurface in $S^{n+2}(1)$ and hence, it follows from the standard regularity theorem of elliptic PDE that $p^{-1}(\gamma_b^i)$ is analytic. This, in turn, implies that γ_b^i is also analytic. The rest of the proof of Lemma 3 is by the method of formal power series substitution and majorization.

Changing variable from s to θ, $(*)$ becomes:

$$\theta \frac{d^2 r}{d\theta^2} + \theta G(\theta) \frac{dr}{d\theta} = (n+2)\theta \cot r \left(\frac{dr}{d\theta}\right)^2 - \frac{1}{\sin^2 r} \theta G(\theta) \left(\frac{dr}{d\theta}\right)^3,$$

where $\theta G(\theta)$ is analytic near $\theta = 0$.

We refer to Proposition 1 of [10], or, in greater generality, to [21], for such a proof of uniqueness, existence, and analytical dependence of power series solutions of this type of singular differential equations.

Q. E. D.

Equation $(*)$ is symmetric with respect to reflection of parameter; hence any solution curve which hits the boundary can be continued back along the same trajectory with a discontinuous jump in α at the boundary; hence all solution curves may be considered as defined for all s. Close by solution curves will generically avoid the boundary, but one has the phenomenon of "sharp turning" close to the boundary.

Let $B_i[a, c]$ be a compact segment of B_i with $a \leqq r \leqq c$. One can always choose a sufficiently small $\delta > 0$ such that the δ-collar neighbourhood of $B_i[a, c]$ is foliated by the small segments of γ_b^i, $a \leqq b \leqq c$. In other words, one may introduce a new coordinate system (t, θ) for the above δ-collar neighbourhood $N^{(i)}[a, c]$, $i = 1$ or 2, such that γ_b^i is given by $t \equiv b$. Since the behavior of solution curves of $(*)$ in $N_\delta^{(2)}$ is essentially the same as that of $N_\delta^{(1)}$, we shall, for simplicity of notation, only state the latter case.

Let $\Gamma_{r, \theta, \alpha}(s) = (r(s), \theta(s), \alpha(s))$ be the unique solution curve of $(*')$ with initial conditions $r(0) = r$, $\theta(0) = \theta$, $\alpha(0) = \alpha$, where $r \in (0, \pi)$, $\theta \in (0, \pi/g)$, $\alpha \in \mathbf{R} \pmod{2\pi}$, and let $\gamma_{r, \theta, \alpha}(s) = (r(s), \theta(s))$ be its projection to the orbit space.

PROPOSITION 1. — *For any positive* ε *there exists a positive* δ *such that any solution curve* $\gamma_{r, \theta, \alpha}(s)$ *which intersects* $N_\delta^{(1)}[a, c]$ *at* $s = 0$, *has* $|(\pi/2) - \alpha(s_0)| < \varepsilon$ *for some* $s_0 \in (0, \varepsilon)$.

The details of this argument involve delicate estimates, [19]. Let $b \in [a, c]$, by reflectional symmetry we may assume $b \leqq \pi/2$. Consider $\Gamma_{b, \theta, \alpha}(s)$, and let $\alpha \in [-\pi/2, \pi/2 - \varepsilon]$. From $(*')$ it follows that $\alpha(s) \geqq 0$ as long as $\alpha(s) \in [-\pi/2, 0)$ and $r(s) \leqq \pi/2$, [δ is chosen less than θ_0, so $G(\theta(s))$ remains positive]. From $(*')$ and Lemma 2 it follows that for a given $\varepsilon > 0$, one can determine a $\delta > 0$ such that in $N_\delta^{(1)}[a, c]$, $d\alpha/ds$ will be dominated by the term $(\cos\alpha/\sin r)\, G(\theta)$ as long as $|\cos\alpha| > \varepsilon$, $\theta < \delta$. It follows that for sufficiently small δ, $\alpha(s_1) \geqq 0$ for some $s_1 \in (0, \varepsilon/10)$. When $\alpha(s) \geqq 0$, θ is increasing, and $G(\theta)$ is decreasing; hence it is more delicate to estimate the increase in $\alpha(s)$.

From $(*')$ and the form of $g(\theta)$ it is clear that we can choose a θ_1 such that $\dot{\alpha} > k(\cos\alpha/\sin r)\cot\theta$ for $\alpha \in [0, \pi/2 - \varepsilon]$, $\theta \in (0, \theta_1)$, and k a positive constant depending on n and $g(\theta)$.

Then $d\alpha/d\theta > k \cot\alpha \cot\theta$. The solution of the equation $d\alpha/d\theta = k \cot\alpha\cot\theta$ with initial condition $\alpha(\theta_2) = 0$ is $\cos\alpha = (\sin\theta_2)^k/(\sin\theta)^k$ for $0 < \theta_2 < \theta < \theta_1$. Comparing with this we obtain for our orbit: $\cos\alpha(\theta) \leqq (\sin\theta_2)^k/(\sin\theta)^k$. Hence

$$\begin{aligned}\cos(\alpha(2^{p+1}\theta_2)) &\leqq \frac{(\sin\theta_2)^k}{(\sin 2^{p+1}\theta_2)^k} \\ &= \left(\frac{\sin\theta_2}{2^{p+1}(\cos 2^p\theta_2)(\cos 2^{p-1}\theta_2)\ldots(\cos\theta_2)(\sin\theta_2)}\right)^k \\ &\leqq 2^{-k(p+1)}(\cos 2^p\theta_2)^{-k(p+1)}.\end{aligned}$$

Choose p so large that $2^{-k(p+1)} < \varepsilon/8$ and then θ_2 so small that

$$2^{p+1}\theta_2 < \max\left(\theta_1, \frac{k\varepsilon^2}{8}\right)$$

and

$$\cos^{k(p+1)}(2^p\theta_2) > 1/2, \qquad \cos(2^{p+1}\theta_2) > 1/2.$$

Then $\cos(\alpha(2^{p+1}\theta_2)) < \varepsilon/4$. Choosing δ small enough to satisfy the above conditions for θ_2 and setting $\theta_2 = \theta(s_1)$, we now observe that $\alpha(\theta)$ reaches $\pi/2 - \varepsilon$ for a $\theta = \theta_3 < 2^{p+1}$

$\theta_2 < k\,\varepsilon^2/8$. Let $\theta(s_0)=\theta_3$, then

$$\dot{\alpha} > k\cos\alpha\cot\theta > k\sin\varepsilon\cot\theta_3 > \frac{\varepsilon k}{1,2}\cot(2^{p+1}\theta_2)$$

$$> \frac{\varepsilon k}{2,4}\sin^{-1}\left(\frac{k\,\varepsilon^2}{8}\right) > \frac{2}{\varepsilon} \qquad \text{for} \quad s_1 < s < s_0 \quad (\varepsilon \text{ assumed small}).$$

Since

$$\alpha(s_0)-\alpha(s_1) < \frac{\pi}{2} \qquad \text{and} \qquad s_0 - s_1 > s_0 - \frac{\varepsilon}{10},$$

it follows that $s_0 < \varepsilon$ [otherwise $\alpha(s_0)-\alpha(s_1) > 2/\varepsilon . 9/10\,\varepsilon = 1.8 > \pi/2$]. This concludes the argument when $\alpha(0)\in[-\pi/2, \pi/2-\varepsilon]$. [Case (a)].

Case (b): $\alpha(0)\in[\pi/2+\varepsilon, 3\pi/2)$. From Lemma 5 (c) $\alpha(s)$ has no relative minimum when $\alpha(s)\in(\pi, 3\pi/2)$. If $\dot{\alpha}(0)<0$, it follows that $\dot{\alpha}(s)<0$ as long as $\alpha(s)\in(\pi, 3\pi/2)$ and $\theta(s)<\theta_0$. By the argument in case (a), the conclusion follows. If $\dot{\alpha}(0)>0$, there are the following possibilities:

(i) $\alpha(s)$ increases past $3\pi/2$, this reduces to case (a).

(ii) $\alpha(s)$ increases to $3\pi/2$ as the solution enters B_1.

(iii) $\alpha(s)$ reaches a relative maximum $\alpha_m=\alpha(s_m)\leqq 3\pi/2$. If $\alpha_m = 3\pi/2$, we get $r(s_m)=\pi/2$ from $(*')$, hence this would be the equator solution, which is a contradiction. Hence $\alpha(s_m)<3\pi/2$, and $\dot{\alpha}(s)<0$ for $s>s_m$; this reduces to the previous case $\dot{\alpha}(0)<0$. The estimate on s_0 is obtained as above.

Q. E. D.

Proposition 1 demonstrates that small perturbations of initial data around $s=-s'$ for $\gamma_b^{(1)}(s)$, $s'>0$, will give a solution curve which follows $\gamma_b^{(1)}$ closely towards the boundary B_1, turns sharply, and leaves $N_\delta^{(1)}[a, c]$ with initial data close to $\gamma_b^{(1)}$. However, since δ is small, this is in the region where the Lipschitz constant of $(*')$ is not under control, and we cannot conclude from Proposition 1 that the solution continues to follow $\gamma_b^{(1)}$ closely. For this conclusion we need the following:

LEMMA 4. — *Let* $N_\delta^{(1)}[a, c]$ *be the above* δ*-collar neighbourhood with the new coordinate system* (t, θ). *Let* $\{\gamma_n;\, n\in N\}$ *be a sequence of solution curves of* $(*)$ *such that* γ_n *enters* N_δ *at* $(t_n^{(1)}, \delta)$, *reaches its unique* θ*-minimal at* $(r_n^{(2)}, \theta_n^{(2)})$ *and exits* N_δ *at* $(t_n^{(3)}, \delta)$. *If* $\theta_n^{(2)}>0$ *for all n*, $\lim t_n^{(2)}$ *exists, and* $\lim\theta_n^{(2)}=0$, *then* $\lim t_n^{(1)}=\lim t_n^{(2)}=\lim t_n^{(3)}$.

Proof. — Let $\alpha_n^{(1)}$ and $\alpha_n^{(3)}$ be respectively the entrance and exit directions of γ_n. Obviously, one may assume without loss of generality that $\lim t_n^{(j)}$ and $\lim \alpha_n^{(j)}$ both exist for $j=1$ or 3. Then it follows from the assumption $\lim\theta_n^{(2)}=0$ that

$$\lim\alpha_n^{(1)} = -\frac{\pi}{2}, \qquad \lim\alpha_n^{(3)} = \frac{\pi}{2}$$

because $\lim \alpha_n^{(1)} \neq -\pi/2$ or $\lim \alpha_n^{(3)} \neq \pi/2$ clearly implies that $\lim \theta_n^{(2)}$ exists and is non-zero. Moreover, one must have

$$\lim t_n^{(1)} = \lim t_n^{(2)} = \lim t_n^{(3)},$$

for otherwise $\lim t_n^{(1)} = b_1 \neq b_3 = \lim t_n^{(3)}$, and the segment of γ_n before (resp. after) its θ-minimal point would approach the curve $t = b_1$ (resp. $t = b_3$) as the limiting curve, which is clearly impossible for a sequence of solution curves of $(*)$.

Q. E. D.

PROPOSITION 2. — *Compact segments of solution curves of the ODE* $(*)$ *depend continuously on their initial conditions, also when they contain some bouncing back points on the singular boundary arcs* B_1 *and* B_2.

Proof. — Let $\gamma_b^{(1)}$ be as in Lemma 3, and let $B_1[a, c]$ and $N_\delta^{(1)}[a, c]$ be as above with $a < b < c$. It is sufficient to consider a sequence of solution curves to $(*')$: $\Gamma_n = (t_n, \theta_n, \alpha_n)$ which enter $N_\delta^{(1)}[a, c]$ at $(t_n^{(1)}, \delta)$ with entry direction $\alpha_n^{(1)}$; here $t_n^{(1)} \to b$ and $\alpha_n^{(1)} \to \alpha^1$, the entry direction of $\gamma_b^{(1)}$. By the standard theorem on continuous dependence on initial conditions Γ_n contains points $(t_n^4, \theta_n^4, \alpha_n^4)$ converging to $(b, 0, -\pi/2)$. From Proposition 1 it follows that for sufficiently large n, α_n gets close to $\pi/2$ while γ_n is still in $N_\delta^{(1)}[a, c]$. Hence, for a fixed $b \in (a, c)$, it follows that for sufficiently large n, the solution curves (t_n, θ_n) enter $N_\delta^{(1)}$ et $(t_n^{(1)}, \delta)$, reach their unique $\theta_{\min}$, and then leave $N_\delta^{(1)}$ at points $(t_n^{(3)}, \delta)$. By Lemma 4 the initial data of these curves at the exit points converge to the corresponding initial data for $\gamma_b^{(1)}$: $(b, \delta, -\alpha^1)$. This is a fixed point where we have control over the Lipschitz constant; hence we can apply the standard theorem on continuous dependence on initial conditions beyond this point.

Q. E. D.

4. Some qualitative features of solution curves under deformations

LEMMA 5. — *Let* $(r(s), \theta(s), \alpha(s))$ *be any solution of* $(*')$. *Then*

(*a*) *Any relative maximum (minimum) of* $r(s)$ *occurs with* $r > \pi/2$ $(r < \pi/2)$.

(*b*) *Any relative maximum (minimum) of* $\theta(s)$ *occurs with* $\theta > \theta_0$ $(\theta < \theta_0)$.

(*c*) *Any relative maximum (minimum) of* $\alpha(s)$ *occurs with* α *in the first or third (second or fourth) quadrant.*

Proof. — (*a*) and (*b*) are quite obvious. We shall only prove (*c*) as follows.

At an α-extremal point, one has $\dot{\alpha} = 0$ and it follows from $(*')$ by differentiation and substitution that

$$\ddot{\alpha} = \sin^{-2} r . \left[\frac{d}{d\theta} G(\theta) + n + 1\right] . \sin\alpha \cos\alpha.$$

Hence (c) follows from Lemma 2, which proves that

$$\frac{d}{d\theta}G(\theta)+(n+1)$$

is always negative.

Q. E. D.

DEFINITION 1. — Set the regions I-IV in the Weyl chamber C_0 to be

$$\mathrm{I}: \quad \left(\frac{\pi}{2}, \pi\right)\times\left(\theta_0, \frac{\pi}{g}\right),$$

$$\mathrm{II}: \quad \left(0, \frac{\pi}{2}\right)\times\left(\theta_0, \frac{\pi}{g}\right),$$

$$\mathrm{III}: \quad \left(0, \frac{\pi}{2}\right)\times(0, \theta_0),$$

$$\mathrm{IV}: \quad \left(\frac{\pi}{2}, \pi\right)\times(0, \theta_0).$$

DEFINITION 2. — Let $\Gamma_b(s)=(r_b(s), \theta_b(s), \alpha_b(s))$ be the solution of $(*')$ with $\Gamma_b(0)=(b, 0, \pi/2)$, and let $\gamma_b(s)$ be the corresponding solution curve of $(*)$ in C_0. Let $r^i_M(b)$ and $r^i_m(b)$ be respectively the i-th maximum and i-th minimum of $r_b(s)$, $s>0$, and define their corresponding argument to be s^i_{Mb} and s^i_{mb}, namely, $r^i_M(b)=r_b(s^i_{Mb})$ and $r^i_m(b)=r_b(s^i_{mb})$. Similarly, let $\alpha^i_M(b)=\alpha_b(t^i_{Mb})$ and $\alpha^i_m(b)=\alpha_b(t^i_{mb})$ be respectively the i-th maximum and i-th minimum of $\alpha_b(s)$, $s>0$.

DEFINITION 3. — Let γ^+_b be the largest segment of $\gamma_b(s)$, $s>0$, which does not touch the singular boundary.

LEMMA 6. — *Let $b\in(0, \pi/2)$. Then all critical points for $r_b(s)$ and $\alpha_b(s)$ along γ^+_b are non-degenerate and vary continuously with b. None of them can coincide.*

Proof. — By the uniqueness theorem of ODE applied to $(*)$, $r^i_M(b)$, $r^i_m(b)\neq\pi/2$ for all i. It is easy to see that $\ddot{r}=(n+1).\cot r$ at a critical point of $r(s)$. Hence all of them are non-degenerate and stable and vary continuously with b.

It follows from Lemma 5 that a critical point of α is non-degenerate provided that $\sin \alpha.\cos\alpha\neq 0$. Suppose the contrary, that there is a place with $\dot{\alpha}_b(s_0)=0$ and $\cos\alpha_b(s_0)=0$. Then it follows from $(*')$ that $\cos r_b(s_0)=0$, i. e., $r_b(s_0)=\pi/2$, which implies, again by the uniqueness theorem, that γ_b is the equator solution and hence a contradiction. Similarly, if there is a place with $\dot{\alpha}_b(s_1)=0$ and $\sin\alpha_b(s_1)=0$, then it follows from $(*')$ that $G(\theta_b(s_1))=0$, namely, $\theta_b(s_1)=\theta_0$ and hence γ_b must be identical with the meridian solution which is clearly impossible. Therefore, all critical points of α along γ^+_b must be non-degenerate and stable and, moreover, the critical points of $r_b(s)$ and $\alpha_b(s)$ can never coincide.

Q. E. D.

DEFINITION 4. — The N-th (r, α)-pattern of γ_b, $P^N(b)$, is the finite sequence of symbols constructed as follows: we assign symbols r^i_M, r^i_m, α^i_M, α^i_m in the same order as the critical points $r^i_M(b)$, $r^i_m(b)$, $\alpha^i_M(b)$, $\alpha^i_m(b)$ occurring along γ^+_b, $s>0$, until we have passed the N-th critical point or $\gamma_b(s)$ has reached the boundary.

Remark. – By Lemma 6, the above $P^N(b)$ is well defined.

COROLLARY. — *Assume that γ^+_b has at least N critical points among the $r^i_M(b)$, $r^i_m(b)$, $\alpha^i_M(b)$ and $\alpha^i_m(b)$ for $b=b_1$. Then the N-th (r, α)-pattern, $P^N(b)$, is locally constant around $b=b_1$.*

Observe that any change of the (r, α)-pattern, $P^N(b)$, must result from γ_b hitting the boundary. This provides a scheme for establishing the existence of "closed" solution curves γ_b which start and terminate at boundary points, or rather, oscillating back and forth between two boundary points.

LEMMA 7. — *Let $\gamma_b(s)$ intersect B_1 or B_2 at $s=s_1$ for $b=b_1$. Variation of b around b_1 then allows a continuous crossing of critical points of $\alpha_b(s)$ and $r_b(s)$ at the intersection point according to the following scheme:*

Intersection point in:

I: An $r_M(b)$ crosses an $\alpha_M(b)$.

II: An $r_m(b)$ crosses an $\alpha_m(b)$.

III: An $r_m(b)$ crosses an $\alpha_M(b)$.

IV: An $r_M(b)$ crosses an $\alpha_m(b)$.

Proof. – The proofs for the above four separate cases are essentially identical. Therefore, we shall only give the proof of case IV as follows:

Assume that $r_b(s_1)>\pi/2$, $\theta_b(s_1)=0$ for $b=b_1$. By the continuous dependence of $\gamma_b(s)$ on b and Lemma 5, there are only the following possibilities for γ_b with $|b-b_1|$ sufficiently small:

(i) $\alpha_b(s)$ reaches a relative minimum $\alpha^j_m(b)>-\pi/2$ and then turns sharply up to $\pi/2-\varepsilon$. Here $\alpha_b(s)$ actually reaches $\pi/2$, and hence an $r^i_M(b)$, after a short s-interval (which tends to zero as $b\to b_1$). This follows since both terms of $\dot{\alpha}$ in $(*')$ are positive in the given region, and the first term can be estimated by a positive constant. Hence we need only choose ε small.

(ii) $\alpha_b(s)$ decreases to $-\pi/2$ as $\gamma_b(s)$ approaches an intersection with the boundary.

(iii) $\alpha_b(s)$ decreases beyond $-\pi/2$, i. e., an $r^i_M(b)$, and then turns sharply to $-3\pi/2+\varepsilon$. By choosing b sufficiently close to b_1, one may assume that γ_b remains in IV up to that stage. Now, we claim that $\alpha_b(s)$ must reach a relative minimum within a very short s-interval. For otherwise, the curve γ_b must continue almost vertically with $\dot{\alpha}\approx 0$ over this stretch. The first term (positive) of $\dot{\alpha}$ in $(*')$ is approximately constant, while the second (negative) term decreases in absolute value at a steady rate, a contradiction.

Hence, in case (i) an $\alpha_m^j(b)$ precedes an $r_M^i(b)$, in case (ii) they coincide, and in case (iii) an $\alpha_m^j(b)$ succeeds an $r_M^i(n)$. This proves Lemma 7 for region IV.

Q. E. D.

5. Evolution of solution curves and the proof of the main theorem

In this section we shall provide a uniform proof of the main theorem stated in the Introduction which covers all cases of rank two isoparametric foliations on $S^{n+2}(1)$ with exactly two point-leaves. Let us begin with a brief outline of the basic ideas involved in such a proof. As was already pointed out in paragraph 2, the construction of infinitely many foliated minimal hyperspheres in $S^{n+2}(1)$ can be reduced to the proof of existence of infinitely many global solution curves of the ODE $(*)$ which start at B_1 and terminate at B_2. It follows from Lemmas 3 and 4 that compact segments of the following family of solution curves

$$S(B_1)=\{\gamma_b;\ 0<b<\pi,\ \gamma_b(0)=(b,\ 0)\}$$

varies continuously with respect to the parameter b. Observe that $\gamma_{\pi/2}$ is exactly the "equator solution" which bounces back and forth between B_1 and B_2 along the line $r\equiv\pi/2$. Therefore, small perturbations of $\gamma_{\pi/2}$, namely, γ_b with $|b-\pi/2|$ sufficiently small, consist of solution curves which "wrap around" the line $r\equiv\pi/2$ many times before they wander away (*cf.* Lemmas 8 and 9). The key point of the proof of the main theorem is to study the change of the $(r,\ \alpha)$-pattern of γ_b as b varies from $\pi/2-\varepsilon$ to ε.

LEMMA 8. — *Let* $\Gamma(s)=(r(s),\ \theta(s),\ \alpha(s))$ *be a solution of* $(*')$ *with* $0<\alpha(0)<\pi/2$, $\dot\alpha(0)<0$, $(r(0),\ \theta(0))\in\mathrm{III}$. *Then* $\gamma(s)=(r(s),\ \theta(s))$ *escapes the region* III *by crossing* $\theta=\theta_0$ *for an* $s=s_0>0$; *furthermore,* $\dot\alpha(s)<0$ *for* $s\in(0,\ s_0]$. *In particular this holds for* γ_b, $b\in(0,\ \pi/2)$.

Proof. — In III we have $\dot\alpha>0$ at $\alpha=0$ and $\dot\alpha<0$ at $\alpha=\pi/2$, hence $\alpha(s)\in(0,\ \pi/2)$ until $\gamma(s)$ escapes III. By Lemma 5 (c), $\alpha(s)$ has no relative minimum, hence $\dot\alpha(s)$ remains negative until the escape. By $(*)$ wehave $\dot\alpha>0$ at $r=\pi/2$, hence the escape must be across $\theta=\theta_0$. Since $\gamma_b(s)$ starts in III, and $\dot\alpha(s)<0$ for $\alpha\in(\pi/2,\ \pi)$, we must have $\alpha_b(s)<\pi/2$ to begin with, and the result follows.

Q. E. D.

LEMMA 9. — *For any* N, *there exists an* $\varepsilon>0$ *such that*

$$P^{2N}(b)=(r_M^1,\ r_m^1,\ r_M^2,\ r_m^2,\ \ldots,\ r_M^N,\ r_m^N)$$

for $b\in(\pi/2-\varepsilon,\ \pi/2)$.

Proof. — Geometrically speaking, the above lemma asserts that γ_b wraps around the line $r\equiv\pi/2$ at least N times before it encounters its first α-critical point if b is sufficiently close to $\pi/2$. Its analytical proof is as follows:

Choose a sufficiently small $\delta>0$ and set

$$R_1 \text{ (resp. } R_2 \text{ or } R_3) = \left\{ (r, \theta);\ \left| r-\frac{\pi}{2} \right| < \delta,\ \theta \leqq \delta \left(\text{resp. } \delta < \theta < \frac{\pi}{g} - \delta \text{ or } \theta \geqq \frac{\pi}{g} - \delta \right) \right\}.$$

Suppose that γ is a segment of a solution curve of $(*)$ which goes from R_1 (resp. R_3) to R_3 (resp. R_1) through R_2. The linear approximation of the ODE $(*')$ in the region R_2 is the following ODE

$$(*_0) \qquad \begin{cases} \dot{\bar{r}} = \bar{\alpha}, & \dot{\theta} = 1 \quad (\text{resp. } -1), \\ \dot{\bar{\alpha}} = \ddot{\bar{r}} = -(n+1)\bar{r} - \bar{\alpha}\,.\,G(\theta) \end{cases}$$

where $\bar{r}=r-(\pi/2)$ and $\bar{\alpha}=(\pi/2)-\alpha$. Along an integral curve of $(*_0)$, one has

$$\frac{d}{d\theta} ln \left(\frac{d\bar{r}}{d\theta}\,.\,g(\theta) \right) = \left(\frac{d\bar{r}}{d\theta} \right)^{-1} . \left[\frac{d^2\bar{r}}{d\theta^2} + \frac{d\bar{r}}{d\theta}\,.\,G(\theta) \right] = \left(\frac{d\bar{r}}{d\theta} \right)^{-1} . (-(n+1)\bar{r}) \geqq 0$$

as long as $r<\pi/2$ (by Lemma 5 (a), $d\bar{r}/d\theta=\bar{\alpha}$ stays >0 for $r<\pi/2$). Then

$$\frac{d\bar{r}}{d\theta} \geqq \frac{d\bar{r}}{d\theta}(\theta_1)\frac{g(\theta_1)}{g(\theta)} \qquad \text{for} \quad \theta \geqq \theta_1.$$

From the form of $g(\theta)$ it follows that there then is a positive constant C such that

$$\frac{d\bar{r}}{d\theta} \geqq \frac{C}{\sin(\pi/g-\theta)}$$

in this region. The integral of the right side is unbounded as $\theta \to \pi/g$; hence the solution curve γ must cross the line $r=\pi/2$ on the way up.

Now, by choosing ε sufficiently small, $b\in(\pi/2-\varepsilon, \pi/2)$, it follows from Lemma 3 and Proposition 1 that γ_b goes through R_2 back and forth for at least N round trips. Furthermore, in this region the coefficients of $(*')$ can in this way be approximated by the coefficients of $(*_0)$ with arbitrary accuracy. Since compact segments of solution curves also depend continuously on coefficients, it follows by repeated applications of the above approximation and Lemma 8, that the curve γ_b must go through at least N cycles of

$$\text{III} \to \text{II} \to \text{I} \to \text{IV} \to \text{III},$$

and make sharp turns in R_1 and R_3. Moreover, it follows from the argument in (i) of the proof of Lemma 7, that such a solution curve γ_b must have an r-minimum (resp. r-maximum) after it makes a sharp turn in R_1 (resp. R_3). We observe that $\dot{\alpha}_b(s)$ remains negative throughout the above N cycles of wrapping around the line $r\equiv\pi/2$ as follows:

By Lemma 8 $\dot{\alpha}_b<0$ in III. By $(*')$ both terms of $\dot{\alpha}_b$ are negative in II until the crossing in R_2. After crossing into I, it follows as in Lemma 8 that α has no minimum before $\alpha=0$; by Proposition 1 α must turn sharply in I near B_2. By $(*')$ $\dot{\alpha}$ is now negative in I until $\alpha = -\pi/2$. It follows as in Lemma 8 that γ_b must cross from I into IV, and the argument continues in the same way.

Q. E. D.

Finally, after the preparation of all the above nine lemmas, the proof of the main theorem can be given as follows:

Proof of the Main Theorem. — Let us consider the geometric feature of the (r, α)-pattern of the subfamily of $S(B_1)$ with $b < \pi/2$. *For* b sufficiently close to $\pi/2$, it follows from Lemma 9 that γ_b has an arbitrarily long sequence of r-extremals before it encounters its first α-extremal. On the other hand, it is quite easy to show (*cf.* [13], [19]) that γ_b will have an α-minimum before it encounters any r-extremals if b is sufficiently close to 0. Set

$$b_i = \inf\left\{ b \mid P^{2i-1}(c) = (r_M^1, r_m^1, \ldots, r_M^i) \text{ for } c \in \left(b, \frac{\pi}{2}\right)\right\},$$

$$\bar{b}_i = \inf\left\{ b \mid P^{2i}(c) = (r_M^1, \ldots, r_M^i, r_m^i) \text{ for } c \in \left(b, \frac{\pi}{2}\right)\right\}.$$

It follows from the above discussion that b_i, $\bar{b}_i$, $i \in N$, are well defined values in $(0, \pi/2)$. Moreover, by Lemma 7, γ_{b_i}(resp. $\gamma_{\bar{b}_i}$) intersects the boundary arc of IV (resp. II) at a point when r_M^i (resp. r_m^i) and α_m^1 coincides. This proves the existence of infinitely many solution curves with distinct geometric characteristics, $\{\gamma_{\bar{b}_i};\ i \in N\}$ (resp. $\{\gamma_{b_i};\ i \in N\}$), which start at B_1 and terminate at B_2 (resp. B_1). The inverse images of such solution curves provide infinitely many foliated minimal immersions of S^{n+1} (resp. generalized Clifford torus) into $S^{n+2}(1)$ which are, clearly, mutually non-congruent. This completes the proof of the main theorem.

Remark 1. — The above method, in fact, produces many more varieties of minimal hyperspheres than is exhibited in the above proof.

Remark 2. — In higher dimensions, most examples of minimal hyperspheres we constructed in the above process actually have stable cones in R^{n+3}.

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