DAVID C. KEYS

$L$-indistinguishability and $R$-groups for quasisplit groups: unitary groups in even dimension

Annales scientifiques de l’É.N.S. 4e série, tome 20, no 1 (1987), p. 31-64

<http://www.numdam.org/item?id=ASENS_1987_4_20_1_31_0>
L-INDISTINGUISHABILITY AND R-GROUPS
FOR QUASI-SPLIT GROUPS:
UNITARY GROUPS IN EVEN DIMENSION

BY C. DAVID KEYS (1)

Let $G$ be a quasi-split group over a local field $F$. In this paper, we determine the structure of L-packets for the minimal unitary principal series representations of $G$. Let $h: G \to \tilde{G}$ be a homomorphism with abelian kernel and cokernel, between quasi-split groups. Restrictions of representations from $\tilde{G}$ to $G$ are studied, and we show that a type of reciprocity holds.

Reducibility of the minimal principal series is determined by the R-group, which we show in this case is isomorphic to the group $S_\varphi$ defined in terms of L-group data arising from a Langlands parameter $\varphi ([13], [17])$.

Information concerning the components of the $\text{Ind} \lambda$ is obtained by analysis on the finite groups $S_\varphi$.

Let $\Pi_\varphi$ be an L-packet consisting of components of a unitary principal series representation. We show that representations $\pi$ in $\Pi_\varphi$ are parametrized by the irreducible representations $\rho(\pi)$ of the group $S_\varphi$. Then we may define a pairing

$$\langle , \rangle: S_\varphi \times \Pi_\varphi \to \mathbb{C}$$

so that

$$\text{trace} \mathcal{A}(r, \lambda) I(g) = \sum_{\pi} \langle r, \pi \rangle \text{trace} \pi(g)$$

for $g \in C_\varphi^\infty(G)$ and $r$ in $R(\lambda) \cong S_\varphi$, where $I = \text{Ind} \lambda$. The number of components in an L-packet, and multiplicities, may be determined by a Möbius inversion formula.

The problem of normalizations of intertwining operators arising in the theory of the R-group defines a 2-cocycle

$$\eta: R \times R \to \mathbb{C}^\times$$

(1) Supported in part by N.S.F. Grant MCS83-01581. Henry Rutgers Research Fellow.
for which the commuting algebra $\text{End}(\text{Ind}\, \lambda)$ is isomorphic to the group algebra $\mathbb{C}[R]$, with multiplication twisted by $\eta$. In section 2, we recall the general theory of the $R$-group, and show that for $\lambda$ one-dimensional, the 2-cocycle $\eta$ is a coboundary. More generally, for $P = MN$ nonminimal and $G$ connected, the cocycle associated to $\text{Ind}(P, G; \sigma)$ has trivial cohomology class if the inducing representation $\sigma$ of $M$ is generic.

Then the commuting algebra is given by the group algebra $\mathbb{C}[R]$. The $L$-packet $\Pi_\phi$ is parametrized by the dual $S_\phi^\vee$, and we define the pairing $\langle , \rangle$ of $S_\phi$ and $\Pi_\phi$.

Define $U(n, n)$, $SU(n, n)$, and $GU(n, n)$ to be the quasi-split unitary groups associated to a separable quadratic extension $E/F$ of local fields. The groups $S_\phi$ which occur for the minimal principal series are explicitly classified. Non-abelian $S_\phi$ more complicated than those known previously, occur for the groups $SU(n, n)$, and multiplicity one fails. The non-abelian $R \cong S_\phi$ are shown to be extensions of a certain reflection group $\bar{R}$ of type $A_1 \times \ldots \times A_1$ by an abelian Galois group $\text{Gal}(L/E)$ with exponent dividing the rank of $\bar{R}$.

An explicit classification of the $R$-groups occurring for the minimal principal series of the quasi-split unitary groups is given in section 3. We may construct a list of characters which have non-trivial $R$-groups, such that any character with non-trivial $R$ is conjugate under the Weyl group to one on the list.

In section 4, we consider the restrictions of representations in an $L$-packet, and show that a type of reciprocity relates restriction of representations to induction of parameters.

More precisely, let $h : G \to \bar{G}$ have abelian kernel and cokernel. Consider representations $\text{Ind}(P, \bar{G}; \tilde{\lambda})$ and $\text{Ind}(P, G; \lambda)$, where $\tilde{\lambda}$ and $\lambda = \lambda \circ h$ have Langlands' parameters $\tilde{\phi}$ and $\phi = h^* \circ \phi$, respectively. Then $S_\phi$ may be considered as a subgroup of $S_\phi$. Let $\pi \in \Pi_\phi$ be parametrized by an irreducible representation $\tilde{\rho}$ of $S_\phi$. Then the components in the restriction of $\tilde{\pi}$ to $G$ are parametrized by the components of the induced representation $\text{Ind}(S_\phi, S_\phi; \tilde{\rho})$ of $S_\phi$. Further, the multiplicity of $\pi \in \Pi_\phi$ in the restriction of $\tilde{\pi}$ to $G$ equals the multiplicity of $\rho = \rho(\pi)$ in the representation $\text{Ind}(S_\phi, S_\phi; \tilde{\rho})$ of $S_\phi$.

In section 5, we give typical examples of the non-abelian groups $S_\phi$ which occur for $SU(n, n)$. The number of irreducible inequivalent components of $\text{Ind} \, \lambda$, and multiplicites, is given by the number of irreducible representations of the groups $S_\phi$, and their degrees. Formulas for these may be found by Möbius inversion.

In section 6, we give an example with $R$ non-abelian, similar in spirit to an example of Vogan, with $\dim \lambda = 2$ and $G$ disconnected (and not quasi-split), for which the cocycle $\eta$ is not a coboundary. We may compute the cocycle in this case, and use the theorem $\text{End}(\text{Ind} \, \lambda) \cong \mathbb{C}[R]$, to determine the number of inequivalent components, their multiplicities, and the action of the intertwining operators.

It is a pleasure to thank Diana Shelstad for her encouragement and for many helpful discussions concerning notions of functoriality.

I would also like to thank the referee for pointing out the simple proof of Theorem 4.1 and for correcting the proof that $S_{\phi} \cong R(\lambda)$.
1. Notation and Definitions

Let $E/F$ be a separable quadratic extension of local fields. Define the groups

$$GU(n,n) = \{ g \in GL(2n,E) \mid g J^t g = xJ \text{ for some } x \in F^* \},$$

$$U(n,n) = \{ g \in GL(2n,E) \mid g J^t g = J \},$$

and

$$SU(n,n) = \{ g \in SL(2n,E) \mid g J^t g = J \}.$$

Here

$$J = \begin{pmatrix} & \ast & 1 \\ \ast & 1 & \\
1 & \ast & \end{pmatrix}$$

and $x \mapsto \bar{x}$ is the Galois automorphism of $E/F$.

These groups are $F$-points of quasi-split algebraic groups defined as $E/F$-forms of $GL(2n) \times GL(1)$, $GL(2n)$, and $SL(2n)$. For an extension $K$ of $E$, define the groups of $K$-points to be $GL(2n,K) \times K^\times$, $GL(2n,K)$, and $SL(2n,K)$.

Let

$$\sigma(g,x) = (\bar{x} J^t g^{-1} J^{-1}, \bar{x})$$

for $(g,x) \in GL(2n,E) \times E^*$, and define the $F$-points of the unitary group of similitudes $GU(n,n)$ to be the fixed points of $\sigma$.

Define the $F$-points of the unitary group $U(n,n)$, and the special unitary group $SU(n,n)$ to be the fixed points of $\sigma(g) = g J^t g^{-1} J^{-1}$, for $g$ in $GL(2n,E)$ or $SL(2n,E)$.

Recall the definition of the $L$-group from [1], [11], or [13]. Realize the Weil group $W_{E/F}$ as

$$\{ x \times \tau \mid x \in E^*, \tau \in \text{Gal}(E/F) \}$$

with multiplication defined by

$$(x \times \tau)(x' \times \tau') = x \cdot \tau(x') \cdot a_{\tau,\tau'} \times \tau \tau'$$

where $a_{\tau,\tau'} = 1$ unless $\tau = \tau' = \sigma$ is the non-trivial element of $\text{Gal}(E/F)$, and $a_{\sigma,\sigma} = a$ is a fixed element of $F^*$ which is not a norm from $E^*$.

The $L$-group is an extension $L^G = L^{G^0} \times W_{E/F}$. Included in $L$-group data are complex groups $(L^{G^0}, L^{B^0}, L^{T^0})$ "dual" to the quasi-split group $G$ with Borel subgroup and maximal torus, as well as root vectors $X_\alpha$ for the simple roots of $L^{T^0}$ in $L^{B^0}$.

For $G = SU(n,n)$, $\widetilde{G} = U(n,n)$, and $\widetilde{G} = GU(n,n)$, we have $L^{G^0} = PGL(2n,\mathbb{C})$, $L^{\widetilde{G}^0} = GL(2n,\mathbb{C})$, and $L^{\widetilde{G}^0} = GL(2n,\mathbb{C}) \times GL(1,\mathbb{C})$. 

ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPERIEURE
The Weil group acts through projection onto $\text{Gal}(E/F)$. The action of $1 \times \sigma$ on $(L^0 G^0, L^0 B^0, L^0 T^0, X_\sigma)$ is the "algebraic dual" of the Galois action on $G$.

The action of $1 \times \sigma$ on $L^0 G^0$ and $L^0 G^0$ is given by $g \mapsto J' g^{-1} J^{-1}$, and the action on $L^0 \tilde{G}^0$ is given by $(g, x) \mapsto (x J' g^{-1} J^{-1}, x)$.

We assemble some explicit formulas which will be useful later.

Take $S$ to be the maximal $F$-split torus

$$S = \{ \text{diag}(a_1, a_2, \ldots, a_n, a_{-n}, \ldots, a_{-1}) \mid a_i = a_i^{-1} \in F^* \}.$$ 

Let $M$, $\tilde{M}$, and $\tilde{\tilde{M}}$ be the centralizers of $S$ in $G = SU(n, n)$, $\tilde{G} = U(n, n)$, and $\tilde{\tilde{G}} = GU(n, n)$, respectively. Then $M$ and $\tilde{M}$ are diagonal with $\tilde{a}_i a_{-i} = 1$, and $\tilde{\tilde{M}}$ is diagonal with $\tilde{a}_i a_{-i} = k \in F^*$, where $a_i \in E^*$. We may take minimal parabolic subgroups

$$P = MN < \tilde{P} = \tilde{M} N < \tilde{\tilde{P}} = \tilde{\tilde{M}} N.$$ 

The root system $\Phi$ of $(G, S)$ is of type $C_n$. Extend the coroots $\alpha^\vee : E^* \to S$ to $\alpha^\vee : E^* \to \tilde{M}$ as follows.

For $\alpha = e_i - e_j$ and $x \in E^*$, let

$$\alpha^\vee (x) = \text{diag}(1, \ldots, x, \ldots, x^{-1}, \ldots, 1, 1, \ldots, x, \ldots, x^{-1}, \ldots, 1)$$

with $x, x^{-1}$ in the $i$-th and $j$-th positions.

For $\alpha = e_i + e_j$ let

$$\alpha^\vee (x) = \text{diag}(1, \ldots, x, \ldots, x, \ldots, 1, 1, \ldots, x^{-1}, \ldots, x^{-1}, \ldots, 1)$$

with $x, x$ in the $i$-th and $j$-th positions.

For $\alpha = 2 e_i$, let

$$\alpha^\vee (x) = \text{diag}(1, \ldots, x, \ldots, 1, 1, \ldots, x^{-1}, \ldots, 1)$$

with $x$ in the $i$-th position.

Take $x \in E^*$ to define an element of $\tilde{M}$, and $x \in F^*$ to define an element of $M$. These formulas will be used in the explicit classification of reducibility.

The coroots corresponding to the simple roots of $\Phi$ determine isomorphisms

$$\tilde{M} \cong (E^*)^n$$

and

$$M \cong (E^*)^{n-1} \times F^*.$$ 

Let $\lambda$ be a quasi-character of $\tilde{M}$ or $M$, and define characters $\lambda_\alpha$ of $E^*$, or $F^*$, by

$$\lambda_\alpha (x) = \lambda(\alpha^\vee (x)).$$

These characters behave under multiplication as the $\alpha^\vee \in \Phi^\vee$.

The Weyl group $W \cong S_n \times \mathbb{Z}_2^2$ acts on $M$ and $\tilde{M}$ as follows. A permutation $(ij)$, i.e., the reflection corresponding to the root $e_i - e_j$, interchanges the $i$-th and $j$-th entries, as
well as the \(-i\)-th and \(-j\)-th entries. A sign change \(c_i\), \(1 \leq i \leq n\), i.e., the reflection corresponding to the root \(2e_i\), interchanges the \(i\)-th and \(-i\)-th entries. The reflection corresponding to the root \(e_i + e_j\) is \((ij) c_i c_j\).

The Weyl group acts on characters of \(M\) and \(M\) by

\[
w^{-1} \lambda(m) = \lambda(wmw^{-1}).
\]

Then \(W\) also acts on the \(\lambda_a\), and \(w^{-1}\lambda_a(x) = \lambda_{wa}(x)\). Note that \(w\lambda = \lambda\) if and only if \(w\lambda_a = \lambda_a\) for all simple roots \(a\).

The group \(\tilde{M}\) should be considered as the group \(T_F\) of \(F\)-rational points of an algebraic torus \(T\), which splits over \(E\). A character \(\tilde{\lambda}\) of \(\tilde{M}\) corresponds [13] to an admissible homorphism

\[
\tilde{\phi} : W_{E/F} \rightarrow L\tilde{T},
\]

where

\[
\tilde{\phi}(x \times \tau) = \tilde{\phi}_0(x \times \tau) \times x \times \tau.
\]

Recall that \(\text{Hom}(E^x, \tilde{M}) \cong \text{Hom}(L\tilde{T}, \mathbb{C}^\times)\). The map

\[
\tilde{\phi}_0 : W_{E/F} \rightarrow L\tilde{T}
\]

is determined on \(E^x\) by the requirement that

\[
\alpha^x \circ \tilde{\phi}_0 = \tilde{\lambda} \circ \alpha^x = \tilde{\lambda}_a,
\]

i.e., that for each (simple) root \(\alpha\), the following diagram commutes:

\[
\begin{array}{ccc}
E^x & \subset & W_{E/F} \\
\downarrow & & \downarrow \\
\tilde{M} & \rightarrow & L\tilde{T}
\end{array}
\]

Let the non-trivial element \(\sigma\) of \(\text{Gal}(E/F)\) satisfy \(\sigma^2 = a\) in the Weil group. Then we also require

\[
\tilde{\phi}_0(1 \times \sigma) : \tilde{\phi}_0(1 \times \sigma) = \tilde{\phi}_0(a \times 1).
\]

The Langlands parameter

\[
\tilde{\phi} : W_F \rightarrow L\tilde{G}
\]

corresponding to \(\text{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda})\) is given by (the equivalence class of) the map

\[
W_F \rightarrow W_{E/F} \rightarrow L\tilde{T} \rightarrow L\tilde{G}.
\]

More generally, we should use the "thickened" Weil group \(W_F \times SL(2, \mathbb{C})\) to account for special representations [13]. \(W_{E/F}\) will suffice for the purposes of this paper.
Define the parameter \( \varphi: W_{E/F} \to {}^LT \subset {}^LG \) corresponding to an Ind \((P, SU(n,n); \lambda)\) by the same method. If \( \lambda \in M^\wedge \) is the restriction of \( \tilde{\lambda} \in \tilde{M}^\wedge \), then the diagrams immediately define a lift

\[
\tilde{\varphi}_0: \quad W_{E/F} \to {}^LG^0 = GL(2n, \mathbb{C})
\]
of the map

\[
\varphi_0: \quad W_{E/F} \to {}^LG^0 = PGL(2n, \mathbb{C}).
\]

Then defining the lift \( \tilde{\varphi} = \tilde{\varphi}_0 \times id \), we easily use the formulas for the \( \alpha^\vee \) given above to check that

\[
\tilde{\varphi}_0(x) \tau^{-1} = \tilde{\varphi}_0(\tau x)
\]
for \( x \in E^\times \) and \( \tau \in \text{Gal}(E/F) \). Then \( \tilde{\varphi} \) is in fact a homomorphism, hence is a lift of \( \varphi \).

In general, for the unitary principal series induced from a minimal parabolic subgroup of a quasi-split group, one uses Theorem 2(a) of Langlands' "Representations of Abelian Algebraic Groups" to define the parameter \( \varphi_0: W_F \to {}^L M \) corresponding to a character \( \lambda \) of \( M \), since \( M \) is the group of \( F \)-rational points of an algebraic torus \( T \) which splits over some Galois extension \( E/F \). See [1]. Then the Langlands' parameter corresponding to \( \text{Ind} (P, G; \lambda) \) is given by the composition \( \varphi: W_F \to {}^L M \subset {}^L G \).

For a unitary character \( \lambda \) with parameter \( \varphi = \varphi(\lambda) \), define the L-packet \( \Pi_\varphi \) to be the set of irreducible components of \( \text{Ind} \lambda \).

**2. R-group and L-indistinguishability**

We first recall the general theory of the R-group, which determines reducibility of the unitary principal series. The commuting algebra \( \text{End} (\text{Ind} \lambda) \) is given as a group algebra \( C[R]^n \) with multiplication twisted by a 2-cocycle \( \eta \). For \( \lambda \) one-dimensional, we show that \( \eta \) has trivial cohomology class. This allows us to parametrize the L-packet \( \Pi_\varphi \) by the dual \( S^\wedge_\varphi \cong R^\wedge \), and to define a pairing of \( \Pi_\varphi \) with \( S^\wedge_\varphi \).

Standard intertwining operators

\[
A (\tilde{w}, \lambda): \quad \text{Ind} (P, G; \lambda) \to \text{Ind} (P, G; w \lambda)
\]
are defined initially for certain \( s \) by an integral formula

\[
(A (\tilde{w}, \lambda) f)(g) = \int_{N \cap w \tilde{N} w^{-1}} f(g \tilde{w}) dn,
\]
where \( \tilde{w} \) is a fixed representative for the element \( w \) of the Weyl group \( W = N(S)/M \), and \( \tilde{P} = \tilde{M} \tilde{N} \) is the parabolic opposed to \( P \). The operators are then defined as meromorphic functions of \( s \) by analytic continuation [20], and satisfy the cocycle relation

\[
A (\tilde{w}_1 \tilde{w}_2, \lambda) = A (\tilde{w}_1, w_2 \lambda) A (\tilde{w}_2, \lambda),
\]
provided \( \text{length} (w_1 w_2) = \text{length} (w_1) + \text{length} (w_2) \).

Note that if the coset representative \( \bar{w} \) is changed by an element \( m \) of \( M \), the operator changes by \( \delta^{-1/2} \lambda^{-1} (m) \). Let

\[
w = w_{a_1} w_{a_2} \cdots w_{a_l}
\]

be a reduced expression for \( w \) as a product of simple reflections, where \( l = \text{length} (w) \). The expression is not unique. However, we may fix representative \( \bar{w}_a \) for each simple reflection so that the representative

\[
\bar{w} = \bar{w}_{a_1} \bar{w}_{a_2} \cdots \bar{w}_{a_l}
\]

for \( \bar{w} \) is independent of the reduced decomposition chosen.

Fix such representatives \( \bar{w} \) and write \( A (w, \lambda) \) for \( A (\bar{w}, \lambda) \). We may define normalized intertwining operators

\[
\mathcal{A} (w, \lambda) = \gamma (w, \lambda, s)^{-1} A (w, \lambda)
\]

for certain meromorphic functions \( \gamma \) of \( s \), the \( c \)-functions of Harish-Chandra [20]. The normalization may be chosen so that

\[
\mathcal{A} (w_{\alpha}, \lambda_{\alpha}) \mathcal{A} (w_{\alpha}, \lambda_{\alpha}) = I
\]

for all simple roots \( \alpha \).

Let \( W (\lambda) = \{ w \in W : w \lambda = \lambda \} \).

\textbf{Theorem 2.1} (Harish-Chandra, [20]). — Suppose \( \lambda \) is unitary. Then the commuting algebra of \( \text{Ind} (P, G; \lambda) \) is spanned by the operators

\[
\{ \mathcal{A} (w, \lambda) : w \in W (\lambda) \}.
\]

Plancherel factors \( \mu_\alpha \) are defined by

\[
A (w_{\alpha}, w_{\lambda}) A (w_{\alpha}, \lambda) = c^2 \mu_\alpha (\lambda, s)^{-1} I,
\]

where \( c^2 \) is a certain positive constant. Let

\( \Delta' = \{ \text{roots } \alpha : \mu_\alpha (\lambda, s) \text{ has a zero at } s = 0 \} \)

and let \( W' \) be the subgroup of \( W \) generated by the reflections \( w_{\alpha} \) for \( \alpha \in \Delta' \). Then [21]

\[
W' = \{ w : \mathcal{A} (w, \lambda) \text{ is scalar} \}.
\]

\textbf{Theorem 2.2} (Silberger [21]). — Suppose \( \lambda \) is unitary. Then the dimension of the commuting algebra of \( \text{Ind} (P, G; \lambda) \) is \( | W (\lambda) / W' | \).

Define

\[
R = R (\lambda) = \{ w \in W (\lambda) : \alpha > 0 \text{ and } \alpha \in \Delta' \text{ imply that } w \alpha > 0 \}.
\]

Then \( R \) is a complement for \( W' \) in \( W (\lambda) \).
Harish-Chandra’s Theorem and the decomposition

\[ W(\lambda) = R \times \lambda W' \]

imply that the normalized operators

\[ \{ \mathcal{A}(w, \lambda) \mid w \in R \} \]

span the commuting algebra of \( \text{Ind} (P, G; \lambda) \), since \( W' \) corresponds to scalar operators. By Silberger’s Theorem, the operators corresponding to \( R \) form a linear basis for the commuting algebra.

By Schur’s Lemma, the intertwining operators satisfy the cocycle relation with no condition on the lengths of the Weyl group elements, up to a scalar. Thus, \( a \text{ priori} \), the map \( w \rightarrow \mathcal{A}(w, \lambda) \) is only a projective homomorphism. We define a 2-cocycle \( \eta \) of the Weyl group by

\[ \mathcal{A}(w_1 w_2, \lambda) = \eta(w_1, w_2) \mathcal{A}(w_1, w_2 \lambda) \mathcal{A}(w_2, \lambda). \]

Then the commuting algebra of \( \text{Ind} (P, G; \lambda) \) is isomorphic to the group algebra \( \mathbb{C}[R] \), with multiplication twisted by the 2-cocycle \( \eta \).

We show that the cocycle

\[ \eta : W \times W \rightarrow \mathbb{C}^* \]

is a coboundary if \( \lambda \) is one-dimensional. Fix the choices of coset representatives for Weyl group elements as above. The following is in [5].

**Theorem 2.3.** — Suppose \( \lambda \) is one-dimensional and that the intertwining operators corresponding to the simple reflections \( w_\alpha \) are normalized so that

\[ \mathcal{A}(w_\alpha w_\beta, \lambda) = 1. \]

Then \( \eta \equiv 1 \), i.e., the cocycle relation holds with no condition on the lengths of the Weyl group elements.

Since the restriction of the 2-cocycle \( \eta \) to \( R \times R \) is trivial, we have the following theorem for the unitary minimal principal series representations of connected reductive \( \rho \)-adic groups.

**Theorem 2.4.** — Suppose \( \lambda \) is a one-dimensional unitary character with \( \rho = R(\lambda) \). Then

(i) The commuting algebra of \( \text{Ind} (P, G; \lambda) \) is isomorphic to the group algebra \( \mathbb{C}[R] \),

(ii) \( \text{Ind} \lambda \) decomposes with multiplicity one if and only if \( R \) is abelian.

(iii) The inequivalent irreducible components \( \pi_\alpha \) of the representation \( \text{Ind} \lambda \) of \( G \) are parametrized by the irreducible representations \( \rho_\alpha = \rho(\pi_\alpha) \) of \( R \), and

(iv) The multiplicity with which a component \( \pi_\alpha = \pi(\rho_\beta) \) occurs in \( \text{Ind} \lambda \) is equal to the dimension of the representation \( \rho_\beta \in R^* \) which parametrizes it.

**Proof.** — The result \( \eta \equiv 1 \) implies (i) and (ii). The parametrization of (iii) is determined as follows. Let \( \rho \) be an irreducible representation of \( R \), with dimension \( \dim \rho \) and
character $\theta_p$. Then, summing over $r \in R$, the operators

$$P_p = \left| R \right|^{-1} \dim \sum_{r} \overline{\theta_p(r)} \mathcal{A}(r, \lambda)$$

form $\left| R^\wedge \right|$ orthogonal projections onto invariant subspaces of $\text{Ind} \lambda$, by the orthogonality relations for the characters $\theta_p$. Each is non-zero by Silberger's Theorem.

Further, each $P_p$ is central in the commuting algebra, since the characters $\theta_p$ are class functions on $R$. Thus, if we let $V_p = P_p V$, where $V$ is the space of $\text{Ind} \lambda$, then the $V_p$ are pairwise disjoint.

Thus

$$V = V_{p(1)} \oplus V_{p(2)} \oplus \ldots \oplus V_{p(k)}$$

as an orthogonal direct sum, with the $V_{p(i)}$ disjoint, $1 \leq i \leq k = \left| R^\wedge \right|$. But since $\text{End}(\text{Ind} \lambda) \cong \mathbb{C}[R]$, the number of inequivalent components of $\text{Ind} \lambda$ equals the number of conjugacy classes in $R$, which equals $k$. Thus the subspaces $V_p$ are isotypic. Note that the elementary idempotents of $\mathbb{C}[R]$ further decompose each $V_p$ into its irreducible components.

Then the $P_p$ are the projections onto the isotypic components of $\text{Ind} \lambda$, and the component $\pi$ corresponding to $\rho$ occurs with multiplicity $d(\rho)$, the degree of $\rho$.

If $\pi$ corresponds to $\rho$ under the identification $\Pi_{\pi} \cong R^\wedge$, we write $\rho = \rho(\pi)$ and $\pi = \pi(\rho)$. Note that $\pi(\rho) \mapsto \pi(\rho \otimes \rho')$ determines an action of the group of one-dimensional characters $\rho'$ of $R$ on the set of components $\Pi_{\pi}$. The action is faithful and simply transitive on the subset of the $L$-packet $\Pi_{\pi}$ consisting of the components which occur with multiplicity one.

The inverse map to $\rho \mapsto \pi$ is explicitly constructed as follows. Given an irreducible component $(\pi, V_0)$ of $\text{Ind} \lambda$, fix a non-zero vector $v_0$ in $V_0$. We consider the translates of $v_0$ under $R$. Let $W$ be the span of the vectors

$$\{ \mathcal{A}(r, \lambda) v_0 \mid r \in R \}.$$ 

Since $\text{End}(\text{Ind} \lambda) \cong \mathbb{C}[R]$, $W$ intersects each irreducible component $V_1$ of the $\pi$-isotypic subspace of $\text{Ind} \lambda$. In fact, $W$ intersects each irreducible component in a one-dimensional subspace, by Schur's Lemma. Thus the dimension of $W$ equals the multiplicity of $\pi$ in $\text{Ind} \lambda$. The representation $\rho$ of $R$ is defined by extending

$$\rho(r)[\mathcal{A}(r', \lambda) v_0] = \mathcal{A}(rr', \lambda) v_0$$

to $W$ by linearity.

Suppose $V_1$ is an irreducible component of $\text{Ind} \lambda$ equivalent to $V_0$, and let $v_1$ be a non-zero vector in $W \cap V_1$. By Schur's Lemma.

$$\mathcal{A}(r, \lambda) V_1 \subseteq V_1 \text{ if and only if } \mathcal{A}(r, \lambda) \text{ acts on } V_1 \text{ as a scalar } c.$$ 

This happens if and only if $\mathcal{A}(r, \lambda) v_1 = cv_1$, i.e., $\rho(r)v_1 = cv_1$. 

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPERIEURE
Finally, we note that $\rho$ is an irreducible representation of $R$ and $P_\rho$ is the projection onto the $\pi$-isotypic subspace of $\text{Ind} \, \lambda$. By the Theorem, there exists an irreducible representation $\tau$ of $R$ such that $P_\tau$ is the projection onto the $\pi$-isotypic subspace. In particular,

$$v_0 = P_\tau v_0 = |R|^{-1} \dim \tau \sum \overline{\theta_t(r)} \mathcal{A}(r, \lambda) v_0 = |R|^{-1} \dim \tau \sum \overline{\theta_t(r)} \rho(r) v_0$$

is non-zero. Thus $\tau$ is contained in $\rho$. Then a counting argument as in the proof of the Theorem implies that each $\rho$ arising this way is irreducible, and $\rho = \tau$.

The correspondence between the set of components and the dual $R^\vee$ depends on the choice of normalizations of intertwining operators giving $\text{End} \, (\text{Ind} \, \lambda) \cong \mathbb{C}[R]$. But any two such normalizations must differ by a one-dimensional character $\rho'$ of $R$, i.e.,

$$\mathcal{A}'(r, \lambda) = \rho'(r) \mathcal{A}(r, \lambda).$$

Then the parametrization of the $L$-packet $\Pi_\rho$ by $R^\vee$ will be shifted by the character $\rho'$. If $\pi$ corresponds to $\rho(\pi)$ with the first normalization, it corresponds to $\rho(\pi) \otimes \rho'$ with the second. We write $\pi(\rho) = \pi' (\rho \otimes \rho')$.

For quasi-split $G$, $M = T(F)$ is the group of $F$-rational points of a torus, and $L$-packets of representations of $M$ are singletons. A quasi-character $\lambda$ of $M$ corresponds to an admissible homomorphism $\varphi_\tau = \varphi_\tau(\lambda)$ of the Weil group $W_F$ into $^L T$. Assume that $\lambda$ is unitary. Then the composition

$$\varphi : W_F \to ^L T \subset ^L G$$

parametrizes an $L$-packet $\Pi_\rho$ of representations of $G$, consisting of the components of $\text{Ind} \, \lambda$, and all $L$-indistinguishability in this case is accounted for by reducibility.

We recall the $L$-group description ([12], [13]) of the reducibility group and the decomposition $W(\lambda) = R(\lambda) \times_s W'$, and check that $R(\lambda) \cong S_\rho$ in our situation. The proof is the same as for real groups, modulo knowledge of Plancherel measure. The key point concerns the location of zeros of the Plancherel measure. See Lemma 5.3.15 of [18].

Langlands introduces several groups attached to a parameter $\varphi$. Let $S_\varphi$ be the centralizer of $\varphi(W_F) \subseteq ^L G$ in $^L G^0$, $S^0_\varphi$ the connected component of the identity in $S_\varphi$, and $Z^{W_F}$ the Weil group invariants in the center of $^L G^0$. Then define

$$S_\varphi = S_\varphi / Z^{W_F} S^0_\varphi.$$

Let $U$ be the connected component of the identity in the Weil group invariants $(^L T^0)^W_F$ of $^L T^0$. Let $N$ be the normalizer of $^L T^0$ in $^L G^0$.

**Lemma 2.5.** — Let $P = MN$ be a minimal parabolic subgroup of a quasi-split group $G$. Suppose $\lambda \in M^\circ$ corresponds to the parameter $\varphi$. Then

(i) The centralizer of $U$ in $^L G^0$ is $^L T^0$.

(ii) $U$ is a maximal torus of $S^0_\varphi$.

(iii) $^L T^0 \cap S_\varphi = (^L T^0)^W_F = U. Z^{W_F}$.

(iv) $^L T^0 \cap S^0_\varphi = U$. 

4° SÉRIE — TOME 20 — 1987 — N° 1
(v) \( N \cap S_{\varphi}^0 \) is equal to the normalizer of \( U \) in \( S_{\varphi}^0 \).

**Proof.** — The centralizer of \( U \) in \( ^1G^0 \) equals \( ^1T^0 \), since \( U \) contains regular elements of \( ^1T^0 \).

Note that \( U \) is contained in \( S_{\varphi}^0 \). Then (ii) follows from (i). Statement (iv) also follows easily.

For (iii), it is enough to note that \( (^1T^0)^{W_F} \) is connected if \( ^1G^0 \) has trivial center.

For (v), first note that an element \( x \) of \( S_{\varphi}^0 \) which normalizes \( U \) also normalizes the centralizer \( ^1T^0 \) of \( U \). To prove the reverse inclusion, start with an element \( x \) in \( N \cap S_{\varphi}^0 \). The inner automorphism \( \text{Int}(x) \) carries \( ^1T^0 \) into \( ^1T^0 \) and commutes with the image \( \phi(W_F) \). This implies that \( \text{Int}(x) \) commutes with the usual action of the Weil group on \( ^1T^0 \), and hence preserves both the invariants \( (^1T^0)^{W_F} \) and the identity component \( U \).

**Proposition 2.6.** — The following sequence is exact.

\[
1 \to (N \cap S_{\varphi}^0)/U \to (N \cap S_{\varphi}^0)/U. Z^{W_F} \to S_{\varphi}^0/Z_{\varphi}^0. Z^{W_F} \to 1.
\]

Moreover, the middle term in the exact sequence can be identified with the stabilizer \( W(\lambda) \) of \( \lambda \) in the Weyl group of \( G \).

With this identification, \( W(S_{\varphi}^0, U) \) is the subgroup \( W'(\lambda) \) of \( W(\lambda) \). Thus \( S_{\varphi} \cong R(\lambda) \).

**Proof.** — The exactness of the sequence is easy. For the surjectivity at the right end one uses the conjugacy of maximal tori in \( S_{\varphi}^0 \). Let \( s \in S_{\varphi} \). Then \( s \) normalizes \( S_{\varphi}^0 \), and \( sUs^{-1} \) is a maximal torus of \( S_{\varphi}^0 \). Hence there exists an element \( t \in S_{\varphi}^0 \) such that \( sUs^{-1} = tUt^{-1} \). Then \( t^{-1}s \) normalizes \( U \), hence normalizes \( ^1T^0 \). Thus \( t^{-1}s \in N \cap S_{\varphi}^0 \) maps to the same class as \( s \).

The identification of \( (N \cap S_{\varphi})/U. Z^{W_F} \) with \( W(\lambda) \) requires some care. For each \( \omega \) in the Weyl group \( W \), we get \( ^1\omega : ^1T \to ^1T \), and by Langlands' correspondence for tori

\[
W(\lambda) = \{ \omega \in W | ^1\omega \circ \varphi \text{ is equivalent to } \varphi \}.
\]

Let \( \omega \in W \) and let \( n \in N \) be a representative for \( \omega \), where \( \omega \) is viewed as a \( W_F \)-invariant element of the Weyl group of \( ^1T^0 \). The \( W_F \)-invariance of \( \omega \) implies that \( n \) normalizes \( ^1T \), not just \( ^1T^0 \).

We claim first that \( ^1\omega \circ \varphi_T \) is equivalent to \( \text{Int}(n) \circ \varphi_T \). The truth of the claim is independent of the choice of representative \( n \) for \( \omega \) and is clear if \( n \) is invariant under \( W_F \). Therefore the claim follows from

**Lemma.** — Any \( \omega \in W \) has a \( W_F \)-invariant representative \( n \in N \).

This is Lemma 6.2 in the survey article of Borel in the Corvallis proceedings if \( G \) has a cyclic splitting field. The same proof works in general.

Using the claim, we see that \( \omega \) fixes \( \lambda \) if and only if \( \text{Int}(n) \circ \varphi_T \) is equivalent to \( \varphi_T \), that is, if and only if there is a \( t \in ^1T^0 \) such that \( tn \in S_{\varphi} \). Therefore

\[
W(\lambda) = ((N \cap S_{\varphi}, ^1T^0)/^1T^0 = (N \cap S_{\varphi})/(U. Z^{W_F}).
\]
By the exact sequence,

\[ S_\alpha = W(\lambda)/W(S^0_\psi, U), \quad \text{since} \quad W(S^0_\psi, U) = (N \cap S^0_\psi)/U. \]

We want to identify \( S^\alpha \) with the \( R \)-group \( R(\lambda) \). All that remains is to show that \( W(S^0_\psi, U) \) is the subgroup \( W'(\lambda) \) of \( W(\lambda) \) generated by reflections \( w_\alpha \) with respect to reduced roots \( \alpha \) for which the Plancherel factor \( \mu_\alpha(\lambda, s) \) has a zero at \( s=0 \).

Let \( S \) be a maximal \( F \)-split torus in \( T \) and let \( \Sigma_\alpha \) be the set of reduced positive roots of \( S \) in \( G \). A root \( \alpha \in \Sigma_\alpha \) determines a Levi subgroup \( M_\alpha \) of \( G \) containing \( T \). The root spaces of \( T \) in the Lie algebra \( \text{Lie}(M_\alpha) \) are precisely the root spaces of \( T \) in \( \text{Lie}(G) \) corresponding to roots whose restrictions to \( S \) is some multiple of \( \alpha \).

The group \( ^t(M_\alpha)^0 \) occurs naturally as a subgroup of \( ^tG^0 \) containing \( ^tT^0 \). The root spaces of \( ^tT^0 \) in \( \text{Lie}(^t(M_\alpha)^0) \) are the root spaces of \( ^tT^0 \) in \( \text{Lie}(^tG^0) \) corresponding to the coroots of \( T \) in \( M_\alpha \).

We now claim that the action of \( W_F \) on \( ^tG^0 \) preserves the subgroup \( ^t(M_\alpha)^0 \), on which it acts in the usual way (from the definition of the \( L \)-group). First, there is an \( \omega \) in \( W \) such that \( \omega \alpha \) is simple. From the Lemma, there is a \( W_F \)-invariant representative \( n \in N \) of \( \omega \). Then \( ^t(M_\alpha)^0 = \text{Int}(n^{-1})(^t(M_{\omega\alpha})^0) \). Since \( \text{Int}(n^{-1}) \) commutes with the action of the Weil group, we are reduced to the case in which \( \alpha \) is simple. Then the claim is obvious.

Recall that \( U \) is a maximal torus of \( S^0_\psi \). Any root of \( U \) in \( S^0_\psi \) is the restriction of some root of \( ^tT^0 \) in \( ^tG^0 \). The group \( \varphi(W_F) \) acts on \( \text{Lie}(^tG^0) \) via the adjoint action \( ^tG \rightarrow \text{Aut}(\text{Lie}(^tG^0)) \) and preserves the subspaces \( \text{Lie}(^t(M_\alpha)^0) \). We get one positive root of \( U \) in \( S^0_\psi \) for every \( \alpha \in \Sigma_\alpha \) such that \( \varphi(W_F) \) fixes a non-zero vector in \( \text{Lie}(^t(M_\alpha)^0) \cap \text{Lie}(\text{unipotent radical of } ^tB^0) \).

We need to show that the following are equivalent:

(A) The image \( \varphi(W_F) \) fixes a non-zero vector in \( V_\alpha \).

(B) The Plancherel factor \( \mu_\alpha(\lambda, s) \) has a zero at \( s=0 \).

Since \( \mu_\alpha(\lambda, s) \) is the same for \( M_\alpha \) as it is for \( G \), we may assume that the \( F \)-rank of \( G \) is 1 and that \( G = M_\alpha \). Further, the truth of (A) and (B) is unchanged if \( G \) is replaced by the simply connected cover of its derived group. The two statements are also compatible with restriction of scalars. Thus we have only two cases to consider.

Case 1. \(- G = \text{SL}(2, F) \) and \( \lambda \) is a character of \( F^\times \).

Then

\[ V_\alpha = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \]

with \( W_F \) acting on the entry * by the character \( \lambda \).

Therefore (A) is equivalent to \( \lambda = 1 \).

Case 2. \(- G \) is a quasi-split \( \text{SU}(2,1) \) associated to a separable quadratic extension \( E/F \). Then \( \lambda \) is a character of \( E^\times \).
Then

\[ V_x = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} = V_1 \oplus V_2 \]

with

\[ V_1 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \]

Then \( W_F \) acts on \( V_1 \) by the representation of \( W_F \) induced from the character \( \lambda \) of \( W_E \), and \( W_F \) acts on \( V_2 \) by the character \( \text{sgn}_{E/F} \cdot \lambda|_{F^\times} \).

The fixed points of \( W_F \) on \( V_x \) are the direct sum of the fixed points on \( V_1 \) and \( V_2 \).

Therefore (A) holds if and only if \( \lambda \equiv 1 \) or \( \text{sgn}_{E/F} \cdot \lambda|_{F^\times} \equiv 1 \).

Then from the explicit formulas for \( \mu_\psi(\lambda, s) \) in [15] and [7], we see that (A) is equivalent to (B) in both of the cases.

Remarks. – From the explicit normalizations of the operators \( \mathcal{A}(w_\psi, \lambda) \) for the groups \( \text{SL}(2) \) and \( \text{SU}(2, 1) \) over real and non-archimedean fields, one knows more than just the location of the zeros of Plancherel measure. The normalizing factors may be shown to be given in terms of Euler factors, as conjectured by Langlands. Further details and applications will appear in a joint paper with F. Shahidi.

Now that we know \( S_\phi \cong \mathbb{R} \), and thus \( \Pi_\psi \cong S_\phi^\circ \) (Theorem 2.4), we define a pairing of \( S_\psi \times \Pi_\psi \) by

\[ \langle r, \pi \rangle = \langle r, \rho \rangle = \text{trace } \rho(r) \]

for \( \rho = \rho(\pi) \in \mathbb{R}^\circ \).

For \( g \in C_c^\infty(G) \), the operators

\[ I(g) = \int_G g(x) \text{Ind}_\psi(\lambda)(x) \, dx \]

and

\[ \pi(g) = \int_G g(x) \pi(x) \, dx \]

have finite rank, \( \pi = \pi(\rho) \in \Pi_\psi \), and we may take traces.

The parametrization of \( \Pi_\psi \) by the dual \( \mathbb{R}^\circ = S_\phi^\circ \) is defined so that the operators \( \mathcal{A}(r, \lambda) \) act on and permute the \( \text{dim } \rho \) irreducible subspaces of the \( \pi(\rho) \)-isotopic component as the representation \( \rho \). Thus we get
THEOREM 2.7:
\[
\text{trace} \mathcal{A} (r, \lambda) I (g) = \sum_{\rho} \langle r, \rho \rangle \text{trace} \pi (\rho) (g)
\]
for \( g \) in \( C^\infty (G) \), where the sum is over an L-packet.

Suppose we change the normalization of intertwining operators by a character \( \rho' \) of \( R \). Recall that the parametrization \( \Pi_{\varphi} \cong R^\wedge \) is shifted by \( \rho' \), i.e., \( \pi' (\rho) = \pi (\rho^{-1} \otimes \rho) \).

Then, if \( \mathcal{A}' (r, \lambda) = \rho' (r) \mathcal{A} (r, \lambda) \), the formula (2.7) for the new normalization and parametrization gives
\[
\rho' (r) \text{trace} \mathcal{A} (r, \lambda) I (g) = \text{trace} \mathcal{A}' (r, \lambda) I (g)
\]
\[
= \sum_{\rho} \theta_\rho (r) \text{trace} \pi' (\rho) (g)
\]
\[
= \sum_{\rho} \theta_\rho (r) \text{trace} \pi (\rho^{-1} \otimes \rho) (g)
\]
\[
= \sum_{\rho} \theta_{\rho' \otimes \rho} (r) \text{trace} \pi (\rho) (g)
\]
\[
= \rho' (r) \sum_{\rho} \theta_\rho (r) \text{trace} \pi (\rho) (g).
\]

Thus a change in normalization amounts to multiplying both sides of (2.7) by the scalar \( \rho' (r) \). Note that the possible choices for a “base representation” of \( \Pi_{\varphi} \) to be indexed by the trivial character of \( R \), correspond to the one-dimensional characters \( \rho' \) of \( R \).

3. Classification of R-groups for SU (n, n), U (n, n), and GU (n, n)

We give an explicit classification of the reducible (unitary) principal series by constructing a list of characters with non-trivial R-groups, such that any \( \lambda \) with non-trivial R (\( \lambda \)) is conjugate under the Weyl group to one on the list. Multiplicity one fails for SU (n, n).

A less explicit classification is easy to obtain, and may be generalized. The R-groups \( \tilde{R} \) for U (n, n) are contained in the group \( \mathbb{Z}_2^* \) of sign changes in the Weyl group. Further, the \( \tilde{R} \) are the reflection groups of type \( A_1 \times \ldots \times A_1 \) generated by the reflections \( w_\alpha \) corresponding to the roots \( \alpha = 2e_i \) for which the restriction of \( \lambda_\alpha \) to \( F^* \) is the character \( \text{sgn}_{E/F} \) of order 2 attached to \( E/F \) by class field theory. Then the R-groups for the unitary group of similitudes GU (n, n) consist of the subgroups (of index 2) of even sign changes in these \( \tilde{R} \).

The R-groups for SU (n, n) are “generalized dihedral” groups, given as extensions
\[
1 \rightarrow \tilde{R} \rightarrow R \rightarrow \text{Gal} (L/E) \rightarrow 1
\]
of the $\mathcal{R}$ by the Galois group of an abelian extension $L/E$. The quotient $R/\mathcal{R}$ embeds naturally into the group of characters $(E^\times)^\wedge$, and determines the extension $L/E$ by class field theory.

For $x \in E^\times$, let $X_i = (2e_i)^\wedge (x) \in \tilde{M}$. Fix a character $\lambda \in M^\wedge$. Then define a map

$$\Theta : W(\lambda) \to (E^\times)^\wedge,$$

by

$$\Theta(w) = \lambda_w, \quad \text{where} \quad \lambda_w(x) = \lambda(X_i w X_i^{-1} w^{-1}).$$

This expression is independent of $i$, since $w$ fixes $\lambda$.

More generally, let $h : G \to \tilde{G}$ be a homomorphism between quasi-split groups with abelian kernel and cokernel. Fix a character $\lambda$ of $M$ and let $\tilde{\lambda}$ be a character of $\tilde{M}$ for which $\lambda = \tilde{\lambda} \circ h$. Suppose $w$ fixes $\lambda$. Then the map

$$\tilde{m} \mapsto w \tilde{\lambda}(\tilde{m}), \quad (w_1) = \tilde{\lambda}(w^{-1} \tilde{m} w \tilde{m}^{-1})$$

defines a character $\Theta(w) = \lambda_w$ of the abelian group $\tilde{M}$. Note that $\lambda_w$ is trivial on $h(M)$, so $\Theta$ factors through a map

$$\Theta : W(\lambda) \to (\tilde{M}/h(M))^\wedge.$$

Note that $\lambda_w \equiv 1$ if and only if $w \in W(\tilde{\lambda})$, by definition.

**Lemma 3.1.** — The map $\Theta$ is a homomorphism, and the kernel of $\Theta$ contains $W'(\lambda)$. Further, if $\lambda = \tilde{\lambda} \circ h$, then $\ker \Theta = W(\tilde{\lambda})$, $W'(\lambda) = W'(\tilde{\lambda})$, and $R(\lambda) \cap \ker \Theta = R(\tilde{\lambda})$.

**Proof.** — Let $w_1$ and $w_2$ be in $W(\lambda)$. Then

$$\Theta(w_1)(\tilde{m}) \Theta(w_2)(\tilde{m}) = \lambda_{w_1}(\tilde{m}) \lambda_{w_2}(\tilde{m}) = \tilde{\lambda}(w^{-1}_1 \tilde{m} w_1 \tilde{m}^{-1}) \lambda(w_2^{-1} \tilde{m} w_2 \tilde{m}^{-1})$$

We use the facts that a commutator $w_2^{-1} \tilde{m} w_2 \tilde{m}^{-1}$ is in the image $h(M)$ and that $w$ fixes the character $\lambda = \tilde{\lambda} \circ h$.

Since the operator $\mathcal{A}(w_2, \tilde{\lambda})$ is identified with $\mathcal{A}(w_2, \lambda)$ on the space $V(\tilde{\lambda}) \cong V(\lambda)$, one operator is scalar if and only if the other is. It follows that $W'(\lambda) = W'(\tilde{\lambda})$, and the rest of the Lemma follows.

**Corollary 3.2.** — Let $\lambda$ be the restriction of $\tilde{\lambda}$ to $M$. Then the quotient $R(\lambda)/R(\tilde{\lambda})$ is abelian.

**Proof.** — The map

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE
\[ \mathbb{R}(\lambda)/\mathbb{R}(\bar{\lambda}) = \mathbb{R}(\lambda)/\mathbb{R}(\lambda) \cap \ker \Theta \subset (\bar{\mathbb{M}}/\mathbb{M})^* \]

induced by \( \Theta \) is injective, so we may consider the quotient as a subgroup of \((\bar{\mathbb{M}}/\mathbb{M})^*\), which is abelian.

We now consider the case of the inclusion of groups \( \text{SU}(n, n) \subset \text{U}(n, n) \). Then the quotient \( \mathbb{R}(\lambda)/\mathbb{R}(\bar{\lambda}) \) may be considered as a subgroup of \( E^\times \). The group of characters given as the image of \( \Theta \) arises in an interesting fashion, as in [9]. See also [3] and [16].

Let
\[ L(\bar{\lambda}) = \{ w \in W \mid w \bar{\lambda} = \bar{\lambda} \otimes (\omega \circ \text{det}) \text{ for some one-dimensional } \omega \}, \]

and
\[ \bar{L}(\bar{\lambda}) = \{ \omega \mid w \bar{\lambda} = \bar{\lambda} \otimes (\omega \circ \text{det}) \text{ for some } w \in W \}. \]

Here \( \omega \) is a character of \( E^1 = \{ x \in E^\times \mid \text{Norm}_{E/F}(x) = 1 \} \). By Hilbert’s Theorem 90, \( E^1 = \{ x \bar{x}^{-1} \mid x \in E^\times \} \). Thus we may identify the group of characters \( \omega \) of \( E^1 \) with the characters \( \nu \) of \( E^\times \) which are trivial on \( F^\times \), via the isomorphism determined by \( \omega(x/x^\alpha) = \nu(x) \).

First, we show that \( L(\bar{\lambda}) = W(\lambda) \). Restricting \( w \bar{\lambda} = \bar{\lambda} \otimes (\omega \circ \text{det}) \) to \( M \) gives \( L(\bar{\lambda}) \subseteq W(\lambda) \). If \( w \in W(\lambda) \), we may define \( \omega \) satisfying \( w \bar{\lambda} = \bar{\lambda} \otimes (\omega \circ \text{det}) \) by
\[ \omega(x/x^\alpha) = w \bar{\lambda} \otimes \bar{\lambda}^{-1} (x^\alpha(x)), \]

where \( \alpha = 2e_n \). Thus \( w \in L(\bar{\lambda}) \), and \( W(\lambda) \subseteq L(\bar{\lambda}) \). Note that the expression defining \( \omega \) is just \( \Theta(w) \).

There is a natural homomorphism \( L(\bar{\lambda}) \rightarrow \bar{L}(\bar{\lambda}) \) with kernel \( W(\bar{\lambda}) \). With the above identifications, this homomorphism is just \( \Theta \).

We first classify the groups \( R = R(\bar{\lambda}) \) which occur for the unitary group \( \text{U}(n, n) \). Let \( \lambda = \bar{\lambda}|_M \), so that
\[ R(\bar{\lambda}) = R(\lambda) \cap W(\bar{\lambda}) = R(\lambda) \cap \ker \Theta. \]

**Proposition 3.3.** — For any \( R \)-group \( R(\lambda) \) for \( \text{SU}(n, n) \), \( R(\lambda) \cap \ker \Theta \) is contained in the group \( Z_{2}^n \) of sign changes in the Weyl group. In particular, any \( R(\bar{\lambda}) \) for \( \text{U}(n, n) \) is contained in \( Z_{2}^n \).

**Proof.** — Suppose that \( w = sc \in R(\bar{\lambda}) = R(\lambda) \cap \ker \Theta \), with \( s \) in \( S_n \) and \( c \in Z_{2}^n \). We must show that \( s = 1 \).

If \( s \neq 1 \), we may as well conjugate by an element of \( S_n \) to assume that the orbit of \( n \) under \( s \) is
\[ n \mapsto k \mapsto k + 1 \mapsto \ldots \mapsto n - 1 \mapsto n. \]

Then by conjugation by elements of \( Z_{2}^n \), we may assume that \( c \) changes the sign of at most one root \( 2e_i \), say \( 2e_n \), in the orbit.

If \( c(2e_i) = +2e_i \) for all \( k \leq i \leq n \), then \( w \in R(\lambda) \subseteq \ker \Theta \) implies
\[ \begin{align*}
1 \equiv \lambda_w(x^{-1}) &= \lambda((2e_n)^\nu(x^{-1})w(2e_n)^\nu(x)w^{-1}) \\
 &= \lambda((2e_n)^\nu(2e_k)^\nu(x)) = \lambda((e_k-e_n)^\nu(x))
\end{align*} \]
for all \( x \in E^* \).

Thus \( e_k-e_n \in \Delta' \). But then \( e_k-e_n > 0 \) and \( w(e_k-e_n) = e_{k+1} - e_k < 0 \) contradict \( w \in R \).

If \( c(2e_n) = -2e_n \) and \( c(2e_i) = +2e_i \) for \( k \leq i \leq n - 1 \). then \( w \in R(\tilde{\lambda}) \subseteq \ker \Theta \) implies
\[ \begin{align*}
1 \equiv \lambda_w(x) &= \lambda((2e_n)^\nu(x)sc(2e_n)^\nu(x^{-1})(sc)^{-1}) \\
 &= \lambda((2e_n)^\nu(x)s(2e_n)^\nu(x)sc^{-1}) \\
 &= \lambda((2e_n)^\nu(x)(2e_k)^\nu(x)) = \lambda((e_k+e_n)^\nu(x))
\end{align*} \]
for all \( x \in E^* \).

Thus \( e_k + e_n \in \Delta' \). But \( e_k + e_n > 0 \) and
\[ w(e_k + e_n) = sc(e_k + e_n) = s(e_k - e_n) = e_{k+1} - e_k < 0 \]
contradict \( w \in R \).

**Corollary 3.4.** — Suppose \( w = sc \) and \( w' = s'c' \) are in \( R(\lambda) \), with \( s \) and \( s' \) in \( S_n \), and \( c \) and \( c' \) in \( Z_n^2 \). Then the permutations \( s \) and \( s' \) commute.

**Proof.** — Since the commutator \([w, w']\) is in the coset \([s, s']\). \( Z_n^2 \) in the Weyl group, and \([R, R] \leq R \cap \ker \Theta \leq Z_n^2 \), the commutator \([s, s'] \) must be trivial.

Thus, the \( R \)-groups which occur for \( \tilde{G} = U(n, n) \) are products of the 2-element group \( Z_2 \), with the number of factors bounded by \( n = \text{rank } \tilde{G} \). The following gives a more precise description of the groups \( R(\tilde{\lambda}) \).

**Lemma 3.5.** — (i) Suppose a product of sign changes \( c = c_k c_{k+1} \ldots c_n \) is in \( R(\tilde{\lambda}) \), where \( 1 \leq k \leq n \). Then each \( c_i \) is in \( R(\tilde{\lambda}) \), for \( k \leq i \leq n \).

(ii) If \( c = c_k c_{k+1} \ldots c_n \) is in \( \mathcal{R}(\lambda) \), and \( 2 \leq k \leq n \), then each \( c_i \) is in \( \mathcal{R}(\lambda) \), for \( k \leq i \leq n \). Further, \( c_i \) is in \( R(\tilde{\lambda}) \), where \( \lambda = \tilde{\lambda}|_{\mathcal{M}} \).

**Proof.** — Let \( \alpha = 2e_i \). Recall that \( w = c_i \) fixes a character \( \tilde{\lambda} \) of \( \tilde{\mathcal{M}} \) if and only if the restriction of \( \lambda_a \) to the norm subgroup \( N_{E^f}(E^+) \) is trivial. Then \( c_i \in R(\tilde{\lambda}) \) if and only if the restriction \( \lambda_a = \tilde{\lambda}_a|_{E^f} \) is the character \( \text{sgn}_{E^f} \) of \( F^* \) of order 2 which is trivial on norms from \( E^\ast \).

To show (i), let \( c \) be any sign change in \( R(\tilde{\lambda}) \). For any \( i \) with \( c(2e_i) = -2e_i \),
\[ \tilde{\lambda}((2e_j)^\nu(x)c_i(2e_j)^\nu(x^{-1})c_i^{-1}) \equiv 1 \quad \text{for} \quad j \neq i, \]
while
\[ \lambda((2e_j)^\nu(x)c_i(2e_j)^\nu(x^{-1})c_i^{-1}) = \tilde{\lambda}((2e_j)^\nu(x)c_i(2e_j)^\nu(x^{-1})c_i^{-1}) \equiv 1. \]

So \( c \in W(\tilde{\lambda}) \) and \( c(2e_j) = -2e_j \) imply that \( c_i \in W(\tilde{\lambda}) \). But \( \beta > 0 \) and \( c \beta > 0 \) imply that \( c_i \beta > 0 \) also. Thus \( c \in R(\tilde{\lambda}) \) implies that \( c_i \in R(\tilde{\lambda}) \), for all \( i \) with \( c(2e_i) = -2e_i \).

For part (ii), take \( n \geq 2 \), and suppose \( c \in \mathcal{R}(\lambda) \) with \( c(2e_i) = 2e_i \) and \( c(2e_j) = -2e_j \). We check that \( c_i \in W(\lambda) \) by noting that

\[ \text{ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPERIEURE} \]
for all $y \in F^*$,

$$c_i \lambda((2e_i)\gamma(y)) = \lambda((2e_i)\gamma(y)),$$

for all $x \in E^*$, and $j \neq i$, and

$$c_i \lambda((e_i - e_j)\gamma(x)) = \lambda((e_i - e_j)\gamma(x))$$

for all $x \in E^*$.

Then $c_i \in R(\lambda)$ as before. Further,

$$\lambda((2e_i)\gamma(x)) c_i((2e_i)\gamma(x^{-1})c_i^{-1}) \equiv 1$$

implies that $c_i \in \ker \Theta$, so $c_i \in R(\lambda) \cap \ker \Theta = R(\tilde{\lambda})$.

**Remark.** — The hypothesis $2 \leq k \leq n$ in part (ii) is necessary, since it is possible for the longest Weyl element $w_0 = c_1 c_2 \ldots c_n$ to be in an $R(\lambda)$ for $SU(n, n)$ with no other sign changes in $R(\lambda)$. This happens if and only if $w_0$ fixes $\lambda$ and $\Theta$ sends $w_0$ to a character of $F^*$ of order 2, other than $\text{sgn}_{E/F}$.

We may now explicitly classify the reducible Ind $\tilde{\lambda}$ which occur for $U(n, n)$. By Proposition 3.3, $R(\tilde{\lambda})$ is contained in the group of sign changes in the Weyl group.

Suppose $R(\tilde{\lambda})$ is non-trivial. Pick an element $c$ in $R$ for which the set

$$I = \{ i \mid c(2e_i) = -2e_i \}$$

is maximal. Then by Lemma 3.5 and the maximality of $I$, $R(\tilde{\lambda})$ is the reflection group $\langle c_i \mid i \in I \rangle$ of type $A_1 \times A_1 \times \ldots \times A_1$ generated by the $c_i$, for $i \in I$.

Without loss of generality, we may replace $\tilde{\lambda}$ by a conjugate under $W$ to assume that $I = \{ i \mid k \leq i \leq n \}$ for some $k$, $(1 \leq k \leq n)$, and then

$$R(\tilde{\lambda}) = \langle c_k, c_{k+1}, \ldots, c_n \rangle.$$

Recall that a character $\tilde{\lambda}$ of $\tilde{M}$ may be defined by the $n$ characters $\tilde{\lambda}_\alpha$ of $E^*$, for $\alpha = e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$, and $2e_n$. Equivalently, we may specify the $n$ characters of $E^*$ corresponding to the roots $\alpha = 2e_1, 2e_2, \ldots, 2e_n$.

Finally, the condition $R(\Delta') \subseteq \Delta'$ in the definition of $R$ requires that the characters $\tilde{\lambda}_\alpha$ are distinct, for $\alpha = 2e_k, 2e_{k+1}, \ldots, 2e_n$, if

$$R = R(\tilde{\lambda}) = \langle c_k, c_{k+1}, \ldots, c_n \rangle.$$

**Theorem 3.6.** — Let $G = U(n, n)$. Any character $\tilde{\lambda}$ of $\tilde{M}$ with non-trivial $R$-group is conjugate under the Weyl group to one of the following.

For the roots $\alpha = 2e_k, \ldots, 2e_n$ let $\tilde{\lambda}_\alpha$ be distinct characters of $E^*$ with $\tilde{\lambda}_\alpha|_{F^*} = \text{sgn}_{E/F}$.

The characters corresponding to $2e_1, \ldots, 2e_{k-1}$ are arbitrary, subject to the condition that $k$ is minimal with respect to this property.

The corresponding $R$-group $R(\tilde{\lambda}) \cong Z_2 \times \ldots \times Z_2$ is generated by the reflections with respect to the roots $2e_i$, where $k \leq i \leq n$. 

---

4° SERIE - TOME 20 - 1987 - N° 1
Thus Ind \( \lambda \) decomposes with multiplicity one, since \( R(\lambda) \) is abelian, and components \( \tilde{\pi} \) of the corresponding \( L \)-packet are parametrized by characters \( \rho \) of \( R(\lambda) \).

Then the operator \( \mathcal{A}(\tilde{r}, \tilde{\lambda}) \) acts on the space of \( \tilde{\pi}(\tilde{\rho}) \) as multiplication by the scalar

\[
\tilde{\rho}(\tilde{r}) = \langle \tilde{r}, \tilde{\rho} \rangle,
\]

for \( \tilde{r} \) in \( R(\lambda) = S_{\tilde{\rho}} \).

The integrated representations corresponding to \( \text{Ind} \lambda \) and \( \tilde{\pi}(\tilde{\rho}) \) satisfy

\[
\text{trace} \mathcal{A}(\tilde{r}, \tilde{\lambda}) I(g) = \sum_{\tilde{\rho} \in R} \langle \tilde{r}, \tilde{\rho} \rangle \text{trace} \pi(\tilde{\rho})(g)
\]

and

\[
\text{trace} \pi(\tilde{\rho})(g) = |R|^{-1} \sum_{\tilde{\rho} \in R} \langle \tilde{r}, \tilde{\rho} \rangle \text{trace} \mathcal{A}(\tilde{r}, \tilde{\lambda}) I(g)
\]

for \( g \in C_{\iota}^\infty (\tilde{G}) \).

The same situation appears in [6] for the unramified unitary principal series of simply-connected, semi-simple \( p \)-adic groups. We may define a pairing as above in this case, fixing the \( K_0 \)-class-I component as the base representation to be indexed by the trivial character of \( R \). Note that in the case \( \Phi \) is of type \( C_n \) and \( q_{a/2} \neq 1 \), then \( R \) is trivial, and that \( G \) has only one conjugacy class of \textit{hyperspecial} maximal compact subgroups in this case.

We now classify the \( R \)-groups which occur for \( SU(n, n) \). A character \( \lambda \) of \( M \) is defined by \( n-1 \) characters \( \lambda_\alpha \) of \( E^\times \), for the short roots \( \alpha = e_1 - e_2, \ldots, e_{n-1} - e_n \), and a character \( \lambda_\beta \) of \( F^\times \), for \( \beta = 2e_n \).

Recall that \( \Theta \) induces an injection

\[
R/R \cap \ker \Theta \cong (E^\times)^\wedge.
\]

If \( R \cap \ker \Theta \) is trivial, then \( R \) is abelian, and by \((b)\) of Lemma 3.5, \( R \cap Z_2^\times \) is either the group \( \langle w_0 \rangle \), or is trivial. Further, \( R \cap \ker \Theta \) is trivial implies that \( R \) acts on \( \{ \pm 2e_i | 1 \leq i \leq n \} \) without fixed points. Thus \( |R| \) divides \( 2n \). Suppose \( m \) divides \( 2n \). Then if \( R \) is to have exponent \( m \), \( R \) must inject into \( (E^\times/F^\times (E^\times)^m)^\wedge \). Hence the order of \( R \) divides both \( 2n \) and the index \([E^\times : F^\times (E^\times)^m]\). In particular, \( |R| \) divides both \( 2n \) and \([E^\times : F^\times (E^\times)^m]\).

Note that the longest Weyl element \( w_0 \) is in \( R(\lambda) \) if and only if \( \lambda_\alpha \equiv 1 \) for \( \alpha = e_1 - e_2, \ldots, e_{n-1} - e_n \), \( \lambda_\beta \) has order 2 for \( \beta = 2e_n \), and \( \Lambda = \emptyset \). Suppose \( w_0 \in R \). Then, \( R \cap Z_2^\times = \langle w_0 \rangle \) if and only if \( \lambda_\beta \) has order 2 and \( \lambda_\beta \neq sgn_{E/F} \).

Now consider the case \( R \cap \ker \Theta \) is non-trivial. Since \( R \cap \ker \Theta \leq Z_2^\times \), \( R \) will contain a non-trivial product of sign changes. Then if

\[
I = \{ i \mid \exists c \in R \text{ with } c(2e_i) = -2e_i \},
\]

\text{ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPERIEURE}
R contains each reflection $c_i$ by Lemma 3.5. Thus $\bar{R} = R \cap \ker \Theta$ is the reflection group of type $A_1 \times A_1 \times \ldots \times A_1$ corresponding to the root system

$$\tilde{\Lambda} = \{ \pm 2e_i \mid i \text{ in } I \}.$$

If $\tilde{\lambda} \in \tilde{\Lambda}$ restricts to $\lambda$, then

$$\bar{R} = R(\tilde{\lambda}) \subseteq R(\lambda).$$

Let $R^* = \{ r \in R \mid \alpha > 0 \text{ and } \alpha \in \tilde{\Lambda} \text{ imply that } r\alpha > 0 \}.

Then standard arguments concerning reflection groups imply that $R^*$ is a complement of $\bar{R}$ in $R$.

Since $R^*$ acts on $\tilde{\Lambda}$, and $R^* \cap \ker \Theta$ is trivial, any non-identity element of $R^*$ acts on $\{ 2e_i \mid i \in I \}$ without fixed points. Thus, the order of $R^*$ divides $|I| = \text{rank}(\tilde{\Lambda}) \leq n$, and the order of any $R$-group for $SU(n, n)$ is bounded by $n 2^n$. The action of $R^*$ on $\mathbb{C}[\tilde{\Lambda}]$ is a multiple of the regular representation.

Note that $\Theta$ induces an inclusion $R^* \subseteq (E^*/F^*)^\vee \subseteq (E^*)^\vee$. Since $|R^*|$ divides $|I|$, $R^*$ may be considered as a subgroup of the (finite) group of characters of $E^*/F^*$ of order dividing $|I|$. By class field theory, this subgroup determines an abelian extension $L$ of $E$ with $\text{Gal}(L/E) \cong R^*$. This provides an arithmetic condition limiting the complexity of the $R = R^* \times R$ which can occur. Suppose $m$ is a divisor of the rank of $\bar{R}$. Then if $R^*$ is to have exponent $m$, its order must divide the index $[E^*/F^*(E^*)^m]$, since then $R^*$ embeds into $(E^*/F^*(E^*)^m)^\vee$.

The exact sequence $1 \to \bar{R} \to R \to R^* \to 1$ describes $R$ as an extension of the reflection group $\bar{R} = Z_2 \times \ldots \times Z_2$ by an abelian group $R^* \cong \text{Gal}(L/E)$. Further, any non-trivial element of $R^*$ permutes the generators $\{ w_{\alpha} \mid \alpha = 2e_i \in \tilde{\Lambda} \}$ of $\bar{R}$ with no fixed points. Thus the order of $R^*$ divides the rank of $\bar{R}$, and $R$ is non-abelian if and only if both $\bar{R}$ and $R^*$ are non-trivial.

Non-abelian $R$ of order $m 2^k$ will occur, where $k \leq n$ and $m$ divides $k = \text{rank } \bar{R}$, subject to the arithmetic conditions above.

We examine the conjugacy classes in the Weyl group to determine which ones may contain elements of an $R$-group. We then find elements which may be used to "build" the $R$-groups for $SU(n, n)$. We may then construct an explicit list of possible $R$-groups for $SU(n, n)$ and use the arithmetic conditions on the extension $E/F$ to determine the existence of characters $\lambda$ for which these groups occur. Every finite abelian group will occur as an $R^*$ for some $G$.

Conjugacy classes in Weyl groups are parametrized by certain admissible graphs [2]. An element $w$ in a Weyl group $S_n \times_s Z_2^n$ of type $C_n$ acts on $\{ \pm 2e_i \mid 1 \leq i \leq n \}$ to produce cycles

$$2e_{j(1)} \mapsto \pm 2e_{j(2)} \mapsto \ldots \mapsto \pm 2e_{j(r)} \mapsto \pm 2e_{j(1)}.$$

If $w' (2e_{j(1)}) = +2e_{j(1)}$, say that the cycle is positive.

If $w' (2e_{j(1)}) = -2e_{j(1)}$, say that the cycle is negative.
Then two elements of $W$ are conjugate if and only if they have the same signed cycle type.

Note that a positive cycle is conjugate in $W$ to one involving no sign changes, and that a negative cycle is conjugate to one involving a single sign change.

Thus conjugacy classes in $W$ are parametrized by graphs of the form

$$A_i(1) + A_i(2) + \ldots + C_j(1) + C_j(2) + \ldots,$$

where

$$\sum (i(k) + 1) + \sum j(k) = n.$$ 

Here, $A_i$ is the class of a positive cycle of length $i + 1$, and $C_j$ is the class of a negative cycle of length $j$.

Suppose again that $\tilde{R} = R \cap \ker \Theta$ is non-trivial. Then $\tilde{R}$ is the reflection group associated to

$$\tilde{\Lambda} = \{ \pm 2e_i | \exists r \in R \text{ with } r(2e_i) = -2e_i \}.$$ 

If $R = \tilde{R}$, then we are done, since $R \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and the classification is like that for $U(n, n)$. Otherwise, $R^*$ is non-trivial.

Suppose that $R^*$ is non-trivial. We want to determine the conjugacy classes of $W$ which may contain elements of an $R^*$. Suppose $R$ contains an element $w = sc$, where $s \in S_n$ and $c \in \mathbb{Z}^*_n$, with $s \neq 1$. By Proposition 3.3, $w$ has no fixed points in $\{ \pm 2e_i | 1 \leq i \leq n \}$. Suppose the shortest cycle in $s$ has length $k$. Then $w^k = s^j c'$, with $c' \in \mathbb{Z}^*_n$.

If $s^k = 1$, then all cycles of $s$ have length $k$, so $w$ has type

$$A_{k-1} + A_{k-1} + \ldots + C_k + C_k + \ldots,$$

and further, $c' \in R$. By Lemma 3.5 (if $c' \neq w_0$ or if we assume $\tilde{R}$ non-trivial), $R$ will contain each sign change occurring in $c'$. Thus, to determine generators of $R$, we may multiply $w$ (if necessary) by some of these sign changes to assume that each cycle is positive. So if $s^k = 1$, we may assume without loss of generality that $w$ is of type $A_{k-1} + \ldots + A_{k-1}$. In the case $\tilde{R}$ is trivial, $w$ may be a product of negative cycles, of type $C_k + \ldots + C_k$, with $w^k = w_0 = \text{the longest Weyl element}$. In this case $R \cap \mathbb{Z}^*_n = \langle w_0 \rangle$.

If instead $s^k \neq 1$, then $w^k = s^k c'$ again has no fixed points. Since $s$ contains a cycle of length $k$, $s^k$ does have fixed points, so there exists an index $i$ such that

$$w^k(2e_i) = c'(2e_i) = -2e_i.$$ 

In fact, $c'(2e_i) = -2e_i$ for each $i$ fixed by $s^k$, by Proposition 3.3. This implies that each cycle of length $k$ of $s$ must be negative, since the action of $c'$ on the roots $2e_i$ in an orbit associated to such a cycle is the product of the orbits of all the $c_i$ associated with the cycle.
Finally, \( w^{2k} = s^{2k} c^k \) does fix some root \( 2e_i \), so \( s^{2k} = 1 \) by Proposition 3.3. Thus \( s \) is a product of cycles of lengths \( k \) and \( 2k \). Further, any sign change occurring in \( w^{2k} = c^k \) will be in \( R \) by Lemma 3.5, so to determine generators of \( R \), we may assume without loss of generality that each cycle of \( w \) of length \( 2k \) is positive. Then \( w \) has conjugacy type \( A_{2k-1} + \ldots + A_{2k-1} + C_k + \ldots + C_k \), and \( w^A \) has type \( A_1 + \ldots + A_1 + C_1 + \ldots + C_1 \).

The action of \( R^* \) on \( \hat{R} \) is given as \( k \) copies of the regular representation of \( R^* \), where \( k = \text{rank} \, \hat{R} / R^* \).

**Theorem 3.7.** — Any \( R \)-group for \( SU(n, n) \) fits into a split sequence

\[
1 \to \hat{R} \to R \to R^* \to 1,
\]

with \( R \) non-abelian if and only if both \( \hat{R} \) and \( R^* \) are non-trivial.

\[
\hat{R} = R \cap \ker \theta \cong \mathbb{Z}_2^t
\]

is a reflection group of type \( A_1 \times \ldots \times A_1 \) associated to a root system

\[
\hat{\Delta} = \{ \pm 2e_i \mid \exists c \in R \text{ with } c(2e_i) = -2e_i \}
\]

and is an \( R \)-group for \( U(n, n) \).

Since \( \hat{R} \) is a reflection group, the group

\[
R^* = \{ r \in R \mid \alpha \in \hat{\Delta} \text{ and } \alpha > 0 \implies r \alpha > 0 \}
\]

is a complement of \( \hat{R} \) in \( R \), with the order of \( R^* \) dividing the rank \( d \) of \( \hat{\Delta} \). Further, \( R^* \) injects naturally into the finite group of characters \( (E^*/F^*)^\hat{\Delta} \). Thus \( R^* \) may be identified with an abelian Galois group \( \text{Gal}(L/E) \).

If \( \hat{R} \) is non-trivial, it is generated by sign changes of type \( A_0 + \ldots + A_0 + C_1 \), and \( R^* \) may be built from elements of types \( A_{m-1} + \ldots + A_{m-1} + C_k + \ldots + C_k \) and \( A_{2k-1} + A_{2k-1} + C_k + \ldots + C_k \) (or \( w_0 \)), in case \( R \cap \mathbb{Z}_2^t = \langle w_0 \rangle \), or from elements of types \( A_{m-1} + \ldots + A_{m-1} + C_k + \ldots + C_k \).

### 4. Restriction of Representations and Reciprocity

We will study restrictions of representations between

\[
G = SU(n, n) \leq \tilde{G} = U(n, n) \leq \bar{G} = GU(n, n).
\]

More generally, we consider restrictions determined by a homomorphism

\[
h : G \to \tilde{G}
\]

with abelian kernel and cokernel, between quasi-split groups. We may assume that \( h(\mathcal{P}) \leq \mathcal{P} \). If \( \tilde{\lambda} |_{\mathcal{M}} = \lambda \), i.e., \( \tilde{\lambda} \circ h = \lambda \), then \( \text{Ind} (\mathcal{P}, \tilde{G}; \tilde{\lambda}) |_G \cong \text{Ind} (\mathcal{P}, G; \lambda) \). Further,
R(\tilde{\lambda}) \subseteq R(\lambda). The homomorphism \( h \) induces a map \( h^*: \bar{G} \rightarrow \bar{G}, \) and \( S_\varphi \subseteq S_\psi \) if \( \varphi = h^* \circ \tilde{\varphi}. \)

Recall from section 2 that the components of an \( \text{Ind}(\tilde{\lambda}) \) are parametrized by the dual of the group \( R(\tilde{\lambda}) \cong S_\varphi. \)

We show that the following type of reciprocity holds. If a component \( \tilde{\pi} \) of \( \text{Ind} \tilde{\lambda} \) is parametrized by an irreducible representation \( \tilde{\rho} \) of \( R(\tilde{\lambda}) \), then the components of the restriction of \( \pi \) to \( G \) are parametrized by the irreducible components of the induced representation \( \text{Ind}(R(\tilde{\lambda}), R(\lambda); \tilde{\rho}) \) of \( R(\lambda). \)

The reciprocity result follows from a simple algebraic lemma. Let \( A \rightarrow B \) be a ring homomorphism. Let \( W_A \) be an \( A \)-module, and let \( W_B \) and \( V \) be \( B \)-modules. Let \( E = \text{End}_B(V) \). Then \( \text{Hom}_B(W_B, V) \) and \( \text{Hom}_A(W_A, V) \) have natural \( E \)-module structures, using composition of homomorphisms to define the multiplication. There is a natural homomorphism.

\[
\text{Hom}_A(W_A, W_B) \rightarrow \text{Hom}_E(\text{Hom}_B(W_B, V), \text{Hom}_A(W_A, V)).
\]

**Lemma.** Suppose that \( W_B \) is a direct summand of \( V \). Then the above homomorphism is an isomorphism.

**Proof.** Choose \( B \)-module maps \( i : W_B \rightarrow V \) and \( p : V \rightarrow W_B \) with \( p \circ i = id \) on \( W_B. \) Then the inverse map is given by

\[
\psi \mapsto (g_\psi : w_A \mapsto p(\psi(i)(w_A)))
\]

where \( \psi \in \text{Hom}_E(\text{Hom}_B(W_B, V), \text{Hom}_A(W_A, V)) \) and \( w_A \in W_A. \)

Fix Langlands parameters \( \varphi \) and \( \tilde{\varphi} \) corresponding to (unitary) principal series \( \text{Ind}(P, G; \lambda) \) and \( \text{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda}) \), with \( \tilde{\varphi} \) a lift of \( \varphi. \) Then we have \( \lambda = \tilde{\lambda}|_\mathfrak{m}. \) \( R(\tilde{\lambda}) \) is normal in \( R(\lambda) \), and the quotient \( R(\lambda)/R(\tilde{\lambda}) \) is abelian. The commuting algebras of \( \text{Ind} \lambda \) and \( \text{Ind} \tilde{\lambda} \) are given by the group algebras \( C[R(\lambda)] \) and \( C[R(\tilde{\lambda})] \), respectively. Further, the restriction of a standard intertwining operator for \( \tilde{G} \) is a standard intertwining operator for \( G, \) i.e.,

\[
\mathcal{A}(w, \tilde{\lambda})|_G = \mathcal{A}(w, \tilde{\lambda}|_\mathfrak{m}) = \mathcal{A}(w, \lambda).
\]

**Theorem 4.1.** Let \( \tilde{\pi} \) be an irreducible component of \( \text{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda}), \) corresponding to an irreducible representation \( \tilde{\rho} \) of \( S_\tilde{\varphi} \). Let \( \pi \) be an irreducible component of \( \text{Ind}(P, G; \lambda), \) corresponding to an irreducible representation \( \rho \) of \( S_\varphi \). Then the multiplicity of \( \pi \) in the restriction of \( \tilde{\pi} \) from \( \tilde{G} \) to \( G \) is equal to the multiplicity of \( \rho \) in the restriction of \( \tilde{\rho} \) to \( S_\varphi, \) which equals the multiplicity of the representation \( \rho \) in \( \text{Ind}(S_\varphi, S_\varphi; \tilde{\rho}). \)

In particular, the irreducible components of the restriction of \( \tilde{\pi} \) to \( G \) are parametrized by the irreducible components \( \rho \) of the induced representation \( \text{Ind}(S_\varphi, S_\varphi; \tilde{\rho}) \) of \( S_\varphi. \)

**Proof.** It is enough to show that \( \text{Hom}_G(\pi, \tilde{\pi}) \cong \text{Hom}_{S_\varphi}(\tilde{\rho}, \rho). \) Apply the Lemma with \( A = C[G], B = C[\tilde{G}], V = \text{Ind}(\tilde{P}, \tilde{G}; \tilde{\lambda}), W_A = \pi \) and \( W_B = \tilde{\pi}. \)
Note that $E = \text{End}_d(V) = C[S_d]$ and that
\[ \bar{\rho} \cong \text{Hom}_R(W, V) \quad \text{and} \quad \rho \cong \text{Hom}_\Lambda(W, V). \]

Consider now the inclusion $SU(n, n) \subseteq U(n, n)$. Recall that $R(\bar{\lambda})$ is then a reflection group, and thus has a complement $R^*$ in $R(\lambda)$.

Let $\bar{\rho}$ be an irreducible representation of $R(\bar{\lambda})$ and let $Z(\bar{\rho})$ be the centralizer of $\bar{\rho}$ in $R(\lambda)$. The subgroup $Z^* = Z(\bar{\rho}) \cap R^* \leq R^* \cong \text{Gal}(L/E)$ coming from the inclusion $R(\bar{\lambda}) \leq Z(\bar{\rho}) \leq R(\lambda)$ corresponds to an intermediate field extension
\[ E \subseteq K \subseteq L, \]
with $Z^* \cong \text{Gal}(L/K)$ and
\[ R(\lambda)/Z(\bar{\rho}) \cong R^*/Z^* \cong \text{Gal}(K/E). \]

The intersections of the kernels of the characters for each of the subgroups
\[ Z^* \subseteq (E^*/F^*)^* \subseteq (E^*)^* \]
give a lattice of norm subgroups $N(K^*)$ of $E^*$.

Define a lattice of intermediate subgroups $G^K$ between $SU(n, n)$ and $U(n, n)$ by
\[ G^K = \{ g \in U(n, n) \mid \det(g) \in N(K^*) \}, \]
for $E \subseteq K \subseteq L$. Then the restriction of $\text{Ind}(\bar{\rho}, U(n, n); \bar{\lambda})$ to $G^K$ has commuting algebra $C[Z(\bar{\rho})] = C[\text{Gal}(L/K) \times \bar{R}]$, and restrictions among the various $G^K$ satisfy the reciprocity result.

In the next section we count the number of characters fixed by subgroups $Z$ between $R$ and $\bar{R}$ and use Möbius inversion to calculate the number of components, and multiplicities, in an $L$-packet.

5. Structure of $L$-packets

The number of components and multiplicities. — We give examples of typical non-abelian groups $S^\varphi$ which occur for $SU(n, n)$, and derive formulas for the number of components in a general $L$-packet, and multiplicities, using Möbius inversion. The examples illustrate the reciprocity describing restriction of representations between the groups $U(n, n)$ and $SU(n, n)$, and also the parametrization $\Pi_\varphi \cong S^\varphi$. The multiplicities are just the dimensions of the corresponding irreducible representations of the group $S^\varphi$.

The simplest non-abelian $S^\varphi \cong R(\lambda)$ occurs for the rank 2 group $SU(2, 2)$. Define $\lambda \in M^\varphi$ by requiring $\lambda = \lambda \circ \alpha^\varphi = \alpha^\varphi \circ \varphi$ to have order 2 in $(E^*/F^*)^*$ for $\alpha = e_1 - e_2$, and...
\( \lambda_\beta = \text{sgn}_{E/F}, \) the character of \( F^* / N(E^*) \) of order 2, for \( \beta = 2e_2. \) Then \( \Delta' = \emptyset \) and

\[
R(\lambda) = W(\lambda) = W.
\]

The isomorphism \( R(\lambda) \cong S_\nu \) is transparent. The subgroup

\[
R^* = \langle w_\eta \rangle = \langle (12) \rangle
\]
maps onto the subgroup \( \langle \lambda_\alpha \rangle \) of \( (E^*)^n \). The subgroup

\[
\bar{R} = \langle w_\eta w_\sigma w_\eta, w_\eta \rangle = \langle c_1, c_2 \rangle
\]
of sign changes is the R-group for a representation \( \text{Ind} \tilde{\lambda} \) of \( U(2, 2) \), where \( \lambda = \tilde{\lambda} |_{\text{M}}. \)

Then, \( \text{Ind} \lambda \) decomposes in this case into 4 components of multiplicity 1, and 1 component of multiplicity 2. The group \( \bar{R} \) determines reducibility under \( U(2, 2) \), and \( \text{Ind} \tilde{\lambda} \) decomposes into 4 components of multiplicity 1. These 4 components are indexed by the characters \( \tilde{\rho} \) of \( \bar{R} \), and the intertwining operator \( \mathcal{A}(\tilde{r}, \lambda) \) acts on the space \( V(\tilde{\rho}) \) by the scalar \( \tilde{\rho}(\tilde{r}) \), for \( \tilde{r} \) in \( \bar{R} \).

The group \( R^* \) governs behavior of restriction to \( SU(2, 2) \). The two components of \( \text{Ind} \tilde{\lambda} \) indexed by the trivial character \( \tilde{\rho}_1 \), and the character \( \tilde{\rho}_2 \) defined by

\[
\tilde{\rho}_2(w_\eta) = \tilde{\rho}_2(w_\eta w_\sigma w_\eta) = -1,
\]
each decompose into two irreducible subspaces on restriction to \( SU(2, 2) \). These are the intersections of \( V(\tilde{\rho}_1) \) and \( V(\tilde{\rho}_2) \) with the +1 and -1 eigenspaces of the operator \( \mathcal{A}(w_\sigma, \lambda) \), since \( R^* = \langle w_\eta \rangle \) stabilizes \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \). This accounts for the 4 components of multiplicity 1 in \( \text{Ind} \tilde{\lambda} \), indexed by the one-dimensional representations \( \rho \) of \( R \). For \( w \) in \( R \), the operator \( \mathcal{A}(w, \lambda) \) acts on these \( V(\rho) \) by the scalar \( \rho(w) \).

The other two components of \( \text{Ind} \tilde{\lambda} \) in this example, indexed by the characters defined by

\[
\tilde{\rho}_3(w_\eta) = \tilde{\rho}_4(w_\eta w_\sigma w_\eta) = 1
\]
and

\[
\tilde{\rho}_3(w_\eta w_\sigma w_\eta) = \tilde{\rho}_4(w_\eta) = -1,
\]
remain irreducible, but become equivalent, upon restriction to \( SU(2, 2) \). The characters \( \tilde{\rho}_3 \) and \( \tilde{\rho}_4 \) are conjugate by \( w_\sigma \), and the operator \( \mathcal{A}(w_\sigma, \lambda) \) gives the equivalence under \( SU(2, 2) \). This accounts for the component \( V(\rho_0) \) of \( \text{Ind} \tilde{\lambda} \) occurring with multiplicity 2, where

\[
\rho_0 = \text{Ind}(\tilde{R}, R; \tilde{\rho}_3) \cong \text{Ind}(\tilde{R}, R; \tilde{\rho}_4).
\]

Note that

\[
\rho_0(c_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
give the action of the operators \( \mathcal{A}(c_1, \lambda) \), \( \mathcal{A}(c_2, \lambda) \), and \( \mathcal{A}(w, \lambda) \), respectively, on the subspaces \( V(\tilde{\rho}_3) \oplus V(\tilde{\rho}_4) \).

Thus we know how each intertwining operator in the commuting algebra acts on the \( \text{SU}(2,2) \)-irreducible subspaces. In particular, if \( \mathcal{A}(w, \lambda) \) acts as a scalar on an irreducible component, we know the scalar. It is the value \( \rho(w) \) if the component is indexed by a one-dimensional character \( \rho \). It is the (non-zero) diagonal entry corresponding to the space \( V(\tilde{\rho}_3) \) or \( V(\tilde{\rho}_4) \) in the matrices defining the two-dimensional \( \rho_0 \) above.

It follows that the pairing of \( S_\phi \times \Pi_\phi \) given by
\[
\langle r, \rho \rangle = \theta_\phi(r) = \text{trace } \rho(r)
\]
for \( r \) in \( R \cong S_\phi \) and \( \rho \) in \( R^* \cong \Pi_\phi \) satisfies
\[
(\star) \quad \text{trace } \mathcal{A}(r, \lambda) I(g) = \sum_{\pi(\rho)} \langle r, \rho \rangle \text{trace } \pi(\rho)(g)
\]
for \( g \in C^c_\phi(G) \), where \( I = \text{Ind}(\lambda) \), and the sum is over \( \pi(\rho) \in \Pi_\phi \cong S_\phi^* \).

If a different normalization of intertwining operators is used, the parametrization \( \Pi_\phi \cong S_\phi^* \) will be shifted by a one-dimensional \( \rho' \in S_\phi^* \) and each side of (\( \star \)) will be multiplied by \( \rho'(r) \). We may consider the component parametrized by the trivial character as a base representation in \( \Pi_\phi \). Then such a shift corresponds to the choice of the component \( V(\rho') \) as the base.

To study the general non-abelian \( S_\phi \) which occur for \( \text{SU}(n, n) \), recall from section 4 that the component parametrized by
\[
\rho = \text{Ind}(Z(\tilde{\rho}), R; \xi)
\]
occurs in the L-packet with multiplicity \( |R/Z(\tilde{\rho})| \). Here, \( Z(\tilde{\rho}) \) is the centralizer of an irreducible representation \( \tilde{\rho} \) of \( S_\phi^* \), and \( \xi \) is an irreducible component of \( \rho = \text{Ind}(S_\phi^*, Z(\tilde{\rho}); \tilde{\rho}) \). Consider the formula
\[
|R| = \sum_{\tilde{\rho}} |Z^*(\tilde{\rho})| \cdot |R/Z(\tilde{\rho})|^2
\]
with a different point of view. Instead of summing over orbits \( \{\tilde{\rho}\} \) of \( R \) in \( \tilde{R} \), sum over subgroups \( Z \) with \( \tilde{R} \leq Z \leq R \) and count the number of representations of \( R \) induced irreducibly from \( Z \).

Setting \( Z = Z^* \times \tilde{R} \), let \( c(Z^*, R^*) \) be the number of characters of \( \tilde{R} \) with stabilizer \( Z^* \) in \( R^* \). Then there are \( c(Z^*, R^*) \cdot |Z^*| \) characters \( \xi \) of \( Z \) to consider, which

\[4^e \text{ SÉRIE — TOME 20 — 1987 — N° 1}\]
form \( c(Z^*, R^*)|Z^*|.|R^*/Z^*|^{-1} \) orbits. For each orbit there is a component which occurs with multiplicity \( |R^*/Z^*| \).

Thus, in general, there are

\[
\sum c(Z^*, R^*)|Z^*|.|R^*/Z^*|^{-1} = m^{-2} |R^*| \sum c(Z^*, R^*)
\]

components of multiplicity \( m = |R^*/Z^*| \), where the sums are over all subgroups \( Z^* \) of index \( m \) in \( R^* \).

Since \( R^* = \text{Gal}(L/E) \), \( Z^* = \text{Gal}(L/K) \), and \( R^*/Z^* = \text{Gal}(K/E) \), the sum may be considered as over all field extensions \( K/E \) of degree \( m \) lying in \( L \).

Note that \( |\bar{R}| = \sum c(Z^*, R^*) \), summing over all \( Z^* \leq R^* \). The correct formula for the dimension of the commuting algebra

\[
|R| = |R^*|. |\bar{R}| = \sum (c(Z^*, R^*)|Z^*|.|R^*/Z^*|^{-1}) |R^*/Z^*|^2
\]

follows immediately.

Then the number of inequivalent components in the \( L \)-packet is given by the sum, over \( Z^* \leq R^* \),

\[
\sum c(Z^*, R^*)|Z^*|. |R^*/Z^*|^{-1} = \sum |R^*|^{-1} c(Z^*, R^*) . |Z^*|^2.
\]

Next, the structure theorem describing the extension

\[
1 \to \bar{R} \to R \to R^* \to 1
\]

gives an inductive method to define the coefficients \( c(Z^*, R^*) \). This will allow us to give explicit formulas for the \( c(Z^*, R^*) \) in terms of Möbius inversion.

If \( N = \text{rank } \bar{R} = |\bar{A}| \), then \( |R^*| \) divides \( N \). The \( N \)-dimensional representation of \( R^* = \text{Gal}(L/E) \) on the space \( C[\bar{A}] \) is the multiple \( k \) times the regular representation of \( R^* \), where \( k = N/|R^*| \). The action of \( R^*/Z^* \) on \( C[\bar{A}(\text{mod } Z^*)] \) is the same multiple \( k \) of the regular representation of \( R^*/Z^* \).

It follows that \( c(R^*/Z^*) = c(Z^*, R^*) \) depends only on the quotient \( R^*/Z^* \).

We give formulas for the number of characters

\[
c(Z^*, R^*) = c_1(Z^*, R^*)
\]

with stabilizer \( Z^* \) in the case that \( k = 1 \), i.e., \( N = |R^*| \). Then, if \( N = k . |R^*| \), with \( k > 1 \), the number of characters \( c_k(Z^*, R^*) \) with stabilizer \( Z^* \) may be found from the \( c_1(Z^*, R^*) \) by similar combinatorial arguments.

We remark that

\[
c_k(Z, R^*) = \sum c_1(Z_1, R^*) c_1(Z_2, R^*) \ldots c_1(Z_k, R^*),
\]

where the sum is over \( k \)-tuples \( (Z_1, Z_2, \ldots, Z_k) \) of subgroups with

\[
Z_1 \cap Z_2 \cap \ldots \cap Z_k = Z.
\]
Define the number $c(H) = c_1(H)$ inductively for any finite abelian group $H$ by

$$2^{|H|} = \sum_{Z \leq H} c(Z, H) = \sum_{Z \leq H} c(H/Z).$$

The sum is over the lattice of subgroups of $H$, partially ordered by inclusion. Then by the Möbius inversion formula [14],

$$c(H) = \sum_{Z \leq H} \mu(Z, H) 2^{|Z|},$$

where $\mu$ is the generalized Möbius function defined inductively by

$$\mu(H, H) = 1$$

and

$$\mu(Z, H) = - \sum_{Z \leq X \leq H} \mu(Z, X).$$

For this lattice, $\mu(Z, H)$ depends only on the quotient $H/Z$. The quotients of $R^* \cong \text{Gal}(L/E)$ correspond to the lattice of fields lying between $L$ and $E$.

If $R^*$ is cyclic, then so are the quotients $R^*/Z^*$, and we write $c(m)$ for $c(R^*/Z^*)$ if $|R^*/Z^*| = m$. In this case, we have the inversion formula

$$2^n = \sum_{m | n} c(m) \iff c(n) = \sum_{m | n} \mu(n/m) 2^m$$

where $\mu$ is the usual Möbius function defined by

$$\mu(1) = 1,$$

$$\mu(n) = (-1)^k$$ if $n$ is a product of $k$ distinct primes, and

$$\mu(n) = 0$$ otherwise.

**Example.** — Define the character $\lambda$ of $M$ for $\text{SU}(n, n)$ by

$$\lambda_\gamma = \lambda \circ \gamma = \alpha^\gamma \circ \psi = \psi$$

of order $n$, for the roots $\alpha = e_1 - e_n$, $e_2 - e_3, \ldots, e_{n-1} - e_n$, and $\lambda_\beta = sgn_{E/F}$ for $\beta = 2 e_n$. Then $\Delta' = \emptyset$ and $R(\lambda) = R^* \times \hat{R}$, where $R^* = \langle (12\ldots n) \rangle \cong Z_n$ and $\hat{R} \cong Z_2^n$ is the group of all sign changes in the Weyl group. Since $R^*$ is cyclic in this example,

$$c(m) = c(Z_m) = \sum_{d \mid m} \mu(m/d) 2^d.$$

There are

$$c(m) m^{-2} n = nm^{-2} \sum_{d \mid m} \mu(m/d) 2^d$$

components in the $L$-packet which occur with multiplicity $m$, for each $m$ dividing $n$.

There are a total of

$$\sum_{m \mid n} c(m) m^{-2} n = n \sum_{m \mid n} m^{-2} \sum_{d \mid m} \mu(m/d) 2^d$$
inequivalent components in the L-packet.

Note that the number of components of multiplicity one is 2n.

For example, if \( n = 12 \), \( R^* \cong \mathbb{Z}_{12} \), and \( \bar{R} \cong \mathbb{Z}_2^{12} \), then \( \Pi_\varphi \) contains 24 components of multiplicity 1, 6 components of multiplicity 2, 8 components of multiplicity 3, 9 components of multiplicity 4, 18 components of multiplicity 6, and 335 components of multiplicity 12.

As another example, let \( n = 77 \), \( R^* \cong \mathbb{Z}_{77} \), and \( \bar{R} \cong \mathbb{Z}_7^{17} \). Then \( \Pi_\varphi \) contains 154 components of multiplicity 1, 198 components of multiplicity 7, 1,302 components of multiplicity 11, and 1,962,541,914,958,813,595,274 components of multiplicity 77.

Now, we give some examples to illustrate the case of a non-cyclic \( R^* \).

Consider the group \( SU(4,4) \), with rank \( n = 4 \). Suppose the characters \( \lambda_2 = \psi \) and \( \lambda_4 \) all have order 2 in \( (E^* / F^*)^\times \), for the roots \( \alpha = e_1 - e_2 \) and \( e_3 - e_4 \), and the root \( \beta = e_2 - e_3 \). Let the character \( \lambda_2 = \text{sgn}_{E/F} \) for \( \beta = 2e_4 \). Then \( R^* = \langle (12) \rangle \), \( (13) \), \( (24) \) \( \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle \lambda_2, \lambda_4 \rangle \), and \( \bar{R} \cong \mathbb{Z}_2^4 \) is the group of sign changes in the Weyl group.

We may then use the values \( c(1) = 2 \), \( c(\mathbb{Z}_2) = 2 \), \( c(\mathbb{Z}_2 \times \mathbb{Z}_2) = 8 \) in the formulas derived above to find that the L-packet for this example contains 8 components of multiplicity 1, 6 components of multiplicity 2, and 2 components of multiplicity 4.

For \( n = 4 \), the example above with \( R^* \cong \mathbb{Z}_4 \) cyclic would give instead 8 components of multiplicity 1, 2 components of multiplicity 2, and 3 components of multiplicity 4.

Next, to illustrate the case \( k = 2 \), consider a similar example for \( SU(8,8) \), with \( R^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \bar{R} \cong \mathbb{Z}_2^2 \) with rank 8, instead of rank 4 as above. Considering the lattice of subgroups of \( R^* \), and using the values for the \( c_1 \) above, we count \( c_2 \)\( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{Z}_2 = 216 \) characters of \( \bar{R} \) with trivial stabilizer in \( R^* \), \( c_2 \)\( (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) = 12 \) for each of the three subgroups of \( R^* \) isomorphic to \( \mathbb{Z}_2 \), and \( c_2 \)\( (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2) = 4 \). Then our formulas above give 16 components of multiplicity 1, 36 components of multiplicity 2, and 54 components of multiplicity 4.

Finally, return to the case \( SU(12,12) \). Given a subgroup \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \) of \( (E^* / F^*)^\times \), we may define \( \lambda \) with \( R^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \) and \( \bar{R} \cong \mathbb{Z}_2^{12} \) as follows. Let \( \lambda_\beta = \text{sgn}_{E/F} \) for \( \beta = 2e_{12} \). Take as generators for \( R^* \) the (commuting) permutations

\[
(1 2 3) (4 5 6) (7 8 9) (10 11 12),
\]

\[
(1 4) (2 5) (3 6) (7 10) (8 11) (9 12),
\]

and

\[
(1 7) (2 8) (3 9) (4 10) (5 11) (6 12).
\]

Then use the definition of the embedding

\[
R^* \subseteq (E^* / F^*)^\times
\]

to determine the characters \( \lambda_\alpha \). The situation is reminiscent of the explicit classification of \( R \)-groups for \( SL(n) \) [5].

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPERIEURE
We may then use the values \( c(1) = 2, c \left( Z_2 \right) = 2, c \left( Z_2 \times Z_2 \right) = 8, c \left( Z_3 \right) = 6, c \left( Z_2 \times Z_3 \right) = c \left( Z_6 \right) = 54, \) and \( c \left( Z_2 \times Z_2 \times Z_3 \right) = 3912 \) to find that the L-packet in this case contains 24 components of multiplicity 1, 18 of multiplicity 2, 8 of multiplicity 3, 6 of multiplicity 4, 54 of multiplicity 6, and 326 of multiplicity 12.

The reader may check as an exercise that the extension
\[
1 \rightarrow R \sim \cong Z_2^8 \rightarrow R \rightarrow R^* \cong Z_2 \times Z_2 \times Z_2 \rightarrow 1
\]
gives an L-packet for \( SU(8,8) \) containing 16 components of multiplicity 1, 28 of multiplicity 2, 28 of multiplicity 4, and 23 of multiplicity 8. Note first that \( c \left( Z_2 \times Z_2 \times Z_2 \right) = 184. \)

Then a similar example for \( SU(16,16) \) given by an extension
\[
1 \rightarrow \bar{R} \cong Z_2^{16} \rightarrow R \rightarrow R^* \cong Z_2 \times Z_2 \times Z_2 \rightarrow 1,
\]
which has \( k = \text{rank } \bar{R}/|R^*| = 2, \) would give 32 components of multiplicity 1, 168 of multiplicity 2, 756 of multiplicity 4, and 7,992 of multiplicity 8.

6. An example with non-trivial cocycle

We present an example of a minimal principal series representation with non-abelian \( R \)-group, for which the 2-cocycle \( \eta \) arising in the problem of normalization of intertwining operators in the theory of the \( R \)-group, has non-trivial cohomology class. Then the commuting algebra is not the group algebra \( C[R], \) but instead the twisted group algebra \( C[R]^\eta. \) The example is similar to an example of Vogan’s.

Let \( G \) be an extension of \( SU(2,2) \times SU(2,2) \) by the dihedral group \( D \) of order 8. Let \( D \) be generated by \( X \) and \( Y, \) with relations \( X \) and \( Y \) have order 2, and \( XY \) has order 4.

Let \( X \) act on the first factor of \( SU(2,2) \times SU(2,2) \) as conjugation by the diagonal element \( \text{diag} \left( 1, 1, x_0, x_0 \right), \) where \( x_0 \in F^* \) is not a norm from \( E^*=, \) and act trivially on the second factor.

Let \( Y \) act trivially on the first factor and act on the second factor as conjugation by \( \text{diag} \left( a_0, 1, \tilde{a}_0^{-1} \right), \) where \( a_0 \) is fixed below.

Take \( P_0 = M_0 N_0 \) a minimal parabolic in \( SU(2,2). \) Then
\[
P = (P_0 \times P_0) \times D = MN
\]
is a minimal parabolic subgroup of \( G, \) with
\[
M = (M_0 \times M_0) \times D.
\]

Let \( \sigma_0 \) be the irreducible 2-dimensional representation of the dihedral group \( D, \) and let \( \lambda \) be the character of \( M_0 \cong E^* \times F^* \) defined by the conditions that \( \lambda_\alpha \) is a character of \( E^* \) of order 2, for \( \alpha = e_1 - e_2, \) such that the restriction \( \lambda_\alpha |_{F^*} \) is trivial, and that \( \lambda_\beta = \text{sgn}_{E/F} \) is the character of \( F^* \) of order 2 attached to \( E/F \) by local class field theory, for \( \beta = 2e_2. \)
We choose the element $a_0$ above and the character $\lambda$ so that

$$\lambda(a_0) = -1.$$ 

Note that we have $\lambda_\sigma(x_0) = -1$. We will consider the unitarily induced representation $\text{Ind}(P, G; \sigma)$, where $\sigma = (\lambda, \lambda, \sigma_0)$ is irreducible and 2-dimensional. The stabilizer $W(\sigma)$ of $\sigma$ is the entire Weyl group $W = W_0 \times W_0$ of $G$, where $W_0$ is the Weyl group for $SU(2,2)$.

In general, if $w \sigma \cong \sigma$, consider the normalized integral intertwining operator

$$\mathcal{A}(w, \sigma) : \text{Ind}(P, G; a) \rightarrow \text{Ind}(P, G; w \sigma).$$

If

$$T : V_{w \sigma} \rightarrow V_{\sigma}$$

gives the equivalence $w \sigma \cong \sigma$, we may define

$$\sigma(w) = T$$

to extend the representation $\sigma$ of $M$ on $V_\sigma = V_{w \sigma}$ to a representation $\sigma$ of the group $\langle w \rangle \times M$.

Then $\sigma(w)$ will be unique up to a scalar, which must be an $m$-th root of unity, where $m$ is the order of $w$, since we want

$$\sigma(w)^m = \sigma(w^m) = I.$$ 

Thus

$$\sigma(w) \mathcal{A}(w, \sigma) : \text{Ind}(P, G; \sigma) \rightarrow \text{Ind}(P, G; \sigma)$$

is a self-intertwining operator. The operators

$$\{ \sigma(w) \mathcal{A}(w, \sigma) | w \in R = R(\sigma) \}$$

form a linear basis for the commuting algebra of $\text{Ind}(P, G; \sigma)$.

Define a 2-cocycle $\eta : R \times R \rightarrow \mathbb{C}^\times$ by

$$\sigma(w_1, w_2) \mathcal{A}(w_1, w_2, \sigma) = \eta(w_1, w_2) \mathcal{A}(w_1, \sigma) \mathcal{A}(w_2, \sigma).$$

Then the commuting algebra is given as the group algebra $\mathbb{C}[R]_\eta$, with multiplication twisted by $\eta$.

The coset representatives for elements of $W$ and the normalizations of the integral intertwining operators $\mathcal{A}(w, \sigma)$ are chosen as in section 2, so that the cocycle relation

$$\mathcal{A}(w_1, w_2, \sigma) = \mathcal{A}(w_1, w_2 \sigma) \mathcal{A}(w_2, \sigma).$$
holds with no condition on the lengths of the Weyl group elements. Then the 2-cocycle is determined by the conditions

$$\sigma(w_1 w_2) = \eta(w_1, w_2) \sigma(w_1) \sigma(w_2).$$

Thus the possible obstruction arises from a problem involving finite groups.

For the case of 1-dimensional $\sigma$, the condition $w \sigma \equiv \sigma$ means $w \sigma = \sigma$, and we can always take $\sigma(w) = 1$ to extend $\sigma$. Thus $\eta \equiv 1$, and $\text{End}(\text{Ind} \sigma) \cong \mathbb{C}[R]$, as in section 2. We hope that these remarks help to clarify the nature of the results of [5].

One general result for arbitrary parabolics of $p$-adic groups may be obtained as in the case of groups over finite fields. See § 6 of [4]. Suppose $G$ connected. If there is a component of $\text{Ind} \sigma$ occurring with multiplicity one, then $\mathbb{C}[R]_H$ has a one-dimensional representation, which implies that $\eta$ is a coboundary. We may then absorb it in our normalizations, so in fact $\text{End}(\text{Ind} \sigma) \cong \mathbb{C}[R]$. In particular, this is the case if the supercuspidal representation $\sigma$ of $M$ has a Whittaker model.

We show that in the present example, the 2-cocycle $\eta$ is not a coboundary, i.e., the projective homomorphism

$$w \mapsto \sigma(w) \cdot \mathfrak{A}(w, \sigma)$$

or equivalently,

$$w \mapsto \sigma(w),$$

can not be made into a homomorphism.

Write $(w_\alpha, 1), (w_\beta, 1), (1, w_\alpha), \text{ and } (1, w_\beta)$, for $\alpha = e_1 - e_2$ and $\beta = 2e_2$, for the simple reflections generating $W = W_0 \times W_0$. Since the standard intertwining operators are essentially $SL_2$ operators, we check immediately that $A' = \emptyset$ in our example, so $R = W(\sigma) = W$.

Try to extend the representation $\sigma$ consistently to a representation of $R$.

Since $(w_\alpha, 1)X(w_\alpha, 1)^{-1} = (w_\alpha, 1)X(w_\alpha, 1)^{-1}X^{-1}X = X$, we must define $\sigma(w_\alpha, 1)$ to satisfy

$$(1) \quad \sigma(w_\alpha, 1) \sigma(X) \sigma(w_\alpha, 1)^{-1} = \sigma(X).$$

Since

$$(w_\beta, 1)X(w_\beta, 1)^{-1} = (w_\beta, 1)(Xw_\beta^{-1}X^{-1}, 1)X = (\text{diag}(1, x_0, x_0^{-1}, 1), 1)X,$$

we must define $\sigma(w_\beta, 1)$ so that

$$(2) \quad \sigma(w_\beta, 1) \sigma(X) \sigma(w_\beta, 1)^{-1} = \text{sgn}(x_0) \sigma(X) = -\sigma(X).$$

Since $Y$ acts trivially on the first factor of $SU(2,2) \times SU(2,2)$, we get

$$(3) \quad \sigma(w_\alpha, 1) \sigma(Y) \sigma(w_\alpha, 1)^{-1} = \sigma(Y),$$

and

$$(4) \quad \sigma(w_\beta, 1) \sigma(Y) \sigma(w_\beta, 1)^{-1} = \sigma(Y).$$
Similarly, considering the actions of $X$ and $Y$ on the second factor, we require the extension of $\sigma$ to satisfy

\[
\sigma(1, w_a) \sigma(X) \sigma(1, w_a)^{-1} = \sigma(X),
\]

(5)

\[
\sigma(1, w_q) \sigma(X) \sigma(1, w_q)^{-1} = \sigma(X)
\]

(6)

\[
\sigma(1, w_a) \sigma(Y) \sigma(1, w_a)^{-1} = -\sigma(Y),
\]

(7)

and

\[
\sigma(1, w_p) \sigma(Y) \sigma(1, w_p)^{-1} = \sigma(Y).
\]

(8)

Since $\sigma(w_a, 1)$ must commute with $\sigma(X)$ and $\sigma(Y)$, it must be a scalar, and $(w_a, 1)^2 = 1$ implies

\[
\sigma(w_a, 1) = \pm I.
\]

Similarly,

\[
\sigma(1, w_p) = \pm I.
\]

Further, since the operators $\sigma(X)$ and $\sigma(Y)$ anticommute, the operator $\sigma(1, w_a) \sigma(X)^{-1}$ must commute with both $\sigma(X)$ and $\sigma(Y)$, and then

\[
\sigma(1, w_a) \sigma(X)^{-1} = \pm I = \varepsilon_a I.
\]

Similarly,

\[
\sigma(w_p, 1) \sigma(Y)^{-1} = \pm I = \varepsilon_p I.
\]

But then

\[
\sigma(1, w_a) = \varepsilon_a \sigma(X)
\]

and

\[
\sigma(w_p, 1) = \varepsilon_p \sigma(Y)
\]

must also anticommute;

\[
\sigma(1, w_a) \sigma(w_h, 1) = -\sigma(w_p, 1) \sigma(1, w_a),
\]

although $(1, w_a)$ and $(w_p, 1)$ commute in $W$. Thus $\sigma$ can only be extended to a projective representation of $R$, and the 2-cocycle $\eta$ gives non-trivial cohomology.

Finally, we can compute the 2-cocycle $\eta$ and examine

\[
\text{End(Ind } \sigma) \cong C[W]_\eta
\]

to show that Ind $\sigma$ has 8 components occurring with multiplicity 2, and 2 components with multiplicity 4.

If $\eta$ were trivial, we would instead get 16 components of multiplicity 1, 8 components with multiplicity 2, and 1 component with multiplicity 4.
REFERENCES


(Manuscrit reçu le 30 juillet 1984, révisé le 3 novembre 1986).

David KEYS,
Department of Mathematics,
Rutgers, N.C.A.S.,
Newark, N.J. 07102.