ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 19, n° 4 (1986), p. 609-627

[<http://www.numdam.org/item?id=ASENS_1986_4_19_4_609_0>](http://www.numdam.org/item?id=ASENS_1986_4_19_4_609_0)

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ON A GENERALIZATION OF HILBERT'S 21ST PROBLEM

BY RICHARD M. HAIN ⁽¹⁾

1. Introduction

As his 21st problem, Hilbert [12] asked if every linear representation

$$\rho : \pi_1(\mathbb{P}^1 - \{t_1, \dots, t_N\}, t) \rightarrow \mathrm{GL}(n)$$

of the fundamental group of the punctured Riemann sphere arises as the monodromy representation of a system

$$\mathbf{z}'(t) = \mathbf{z}(t) \mathbf{A}(t)$$

of n first order linear ordinary differential equations on $\mathbb{P}^1$ with regular singular points at $\{t_1, \dots, t_N\}$. (i. e., the $n \times n$ matrix $\mathbf{A}(t)dt$ of 1-forms has only simple poles, and these are contained in $\{t_1, \dots, t_N\}$.) Birkhoff [2] and Plemelj [20] showed that the answer is yes when ρ is generic, while Lappo-Danilevsky [16] gave a constructive solution for representations in a neighborhood of the trivial representation. If one allows $\mathbf{A}(t)dt$ to have additional singularities, around which there is no monodromy (so called *apparent singularities*), then all ρ occur as monodromy representations (*cf.* [3], p. 311).

In this paper we consider a generalization of Hilbert's problem (also called the *Riemann-Hilbert* problem) that we now discuss. Suppose that V is a smooth algebraic variety over $\mathbb{C}$ and that X is a smooth compactification of V such that $X-V$ is a divisor D in X with normal crossings. A meromorphic $\mathrm{gl}(n)$ -valued 1-form ω on X , which is holomorphic on V and has logarithmic poles along D , defines a meromorphic connection ∇ on the trivial bundle $\mathbb{C}^n \times X$ by defining

$$\nabla f = df - f \omega,$$

⁽¹⁾ Supported in part by National Science Foundation Grants MCS-8108814(AO4) and DMS-8401175.

where $f: X \rightarrow \mathbb{C}^n$ is a locally defined function. This connection is holomorphic over V and has regular singular points along D in the sense of Deligne [7]. If ω is integrable (i. e., $d\omega + \omega \wedge \omega = 0$), then the connection is flat and we have a monodromy representation

$$\rho: \pi_1(V, x) \rightarrow \mathrm{GL}(n).$$

1.1. GENERALIZED RIEMANN-HILBERT PROBLEM. — Characterize the monodromy representations $\pi_1(V, x) \rightarrow \mathrm{GL}(n)$ of integrable 1-forms on V which have logarithmic singularities along D .

Unlike the conjectured situation for V a Zariski open subset of $\mathbb{P}^1$, not every monodromy representation occurs. To see this, consider the case where $\dim \Omega^1(X \log D) = 1$. Here

$$\Omega^1(X \log D) = \left\{ \begin{array}{l} \text{global meromorphic forms on } X, \text{ holomorphic} \\ \text{on } V \text{ with logarithmic singularities along } D \end{array} \right\}.$$

If $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is a matrix of such 1-forms, then $\omega = \eta A$, where A is a constant matrix and $\eta \in \Omega^1(X \log D)$. The monodromy $\pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is given by $\gamma \mapsto \exp\left(\int_\gamma \eta A\right)$. That is, the monodromy representation factors through the 1-parameter subgroup $\sigma_A: \mathbb{C} \rightarrow \mathrm{GL}(n)$ generated by A :

$$(1.2) \quad \begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & \mathrm{GL}(n), \\ & \searrow \theta & \nearrow \sigma_A \\ & \mathbb{C} & \end{array}$$

where $\theta(\gamma) = \int_\gamma \eta$. The converse is also true; if ρ factors through σ_A , then ρ is the monodromy representation of $\omega = \eta A$. Two interesting examples where $\dim \Omega^1(X \log D) = 1$ are the following.

1.3. If $X = V$ is a complex torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, then $\dim \Omega^1(X) = 1$. Since the subgroup

$$\left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}$$

of $\mathrm{GL}(2)$ is isomorphic to $\mathbb{C}$ and since dz and $\overline{dz}$ are linearly independent in $H^1(X; \mathbb{C})$, the representation $\pi_1(X) \rightarrow \mathrm{GL}(2)$ that takes γ to

$$\begin{pmatrix} 1 & \int_\gamma \overline{dz} \\ 0 & 1 \end{pmatrix}$$

does not factor as in (1.2). Consequently, ρ is not a monodromy representation. Here one should note that ρ fails to be a monodromy representation for *Hodge theoretic*

reasons: the homomorphism θ is the canonical map

$$\pi_1(V) \rightarrow H_1(V; \mathbb{C})/F^0 H_1(V),$$

where $F^0 H_1$ denotes $H^{0,1}(V)^*$.

$$(1.4) \quad \text{If } V = \mathbb{C}^2 - \{(x, y) : x^2 = y^3\}, \quad \text{then } \pi_1(V) \cong \langle a, b : a^2 = b^3 \rangle.$$

Denote the symmetric group on 3 letters by Σ_3 . The representation

$$\rho : \pi_1(V) \rightarrow \Sigma_3 \hookrightarrow GL(3)$$

obtained by taking a to $(1\ 2)$, b to $(1\ 2\ 3)$ and then including Σ_3 into $GL(3)$ as permutation matrices is not a monodromy representation as it has non-abelian image. Here one should note that ρ fails to be a monodromy representation for *group theoretic* reasons: the homomorphism θ is the Hurewicz homomorphism

$$\pi_1(V) \rightarrow H_1(V; \mathbb{C}).$$

In general the image of a monodromy representation is not abelian. For this reason we need to consider the mixed Hodge structure on $\pi_1(V, x)$: The J -adic completion of the complex group ring $\mathbb{C}\pi_1(V, x)$ of the fundamental group is defined to be

$$\mathbb{C}\pi_1(V, x)^\wedge = \varprojlim \mathbb{C}\pi_1(V, x)/J^n,$$

where J denotes the kernel of the algebra homomorphism $\mathbb{C}\pi_1(V) \rightarrow \mathbb{C}$ that takes each element of $\pi_1(V)$ to 1. A theorem, essentially due to Morgan [17] (*cf.* [10], [11]), asserts that $\mathbb{C}\pi_1(V, x)^\wedge$ has a natural mixed Hodge structure and that the Hodge filtration

$$\dots \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq 0$$

is preserved by the multiplication. Consequently, the subspace

$$I = F^0 \cap J + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \dots$$

is a closed ideal. Denote the composite

$$\pi_1(V, x) \rightarrow \mathbb{C}\pi_1(V, x)^\wedge \rightarrow \mathbb{C}\pi_1(V, x)^\wedge / I$$

by θ . Our main result is

THEOREM. — *There exists a topological $\mathbb{C}$ -algebra $\mathcal{A} \subseteq \mathbb{C}\pi_1(V, x)^\wedge / I$ such that*

(a) *$\text{im } \theta \subseteq \mathcal{A}$,*

(b) *$\rho : \pi_1(V, x) \rightarrow GL(n)$ is a monodromy representation of an integrable 1-form on V with logarithmic singularities along D if and only if there exists a continuous $\mathbb{C}$ -algebra*

homomorphism $\varphi : \mathcal{A} \rightarrow \mathrm{GL}(n)$ such that

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & \mathrm{GL}(n) \\ \theta \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{\varphi} & \mathrm{gl}(n) \end{array}$$

commutes.

Consistent with our observations in (1.3) and (1.4), θ factors into a groups theoretic piece and a Hodge theoretic piece:

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\text{group theory}} & \mathbb{C} \pi_1(V, x)^\wedge \\ & \searrow \theta & \downarrow \text{Hodge theory} \\ & & \mathbb{C} \pi_1(V, x)^\wedge / I \end{array}$$

The group theoretic restriction on monodromy representations given by the theorem is that the kernel of each monodromy representation must contain

$$D^\infty := \ker \{ \pi_1(V, x) \rightarrow \mathbb{C} \pi_1(V, x)^\wedge \}.$$

In (1.4), D^∞ is the commutator subgroup of $\pi_1(V, x)$. Even when D^∞ is trivial, the Hodge theoretic component may restrict the possible monodromy representations by imposing rigidity conditions on their images such as in (1.3).

Define the *irregularity* $q(V)$ of V to be $h^{1,0}(X)$. This is independent of the compactification X . When $q(V)=0$ (e.g., $V \subseteq \mathbb{P}^n$) the ideal I is trivial and it appears as though the only restrictions on monodromy representations are group theoretic. In this case we conjecture that if $\pi_1(V, x)$ satisfies a mild group theoretic condition, then $\rho : \pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is a monodromy representation if and only if it factors through $\pi_1(V, x)/D^\infty$:

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & \mathrm{GL}(n) \\ & \searrow & \nearrow \\ & \pi_1(V, x)/D^\infty & \end{array}$$

(A precise statement is given in 7.1.) When $X = \mathbb{P}^1$ the conjecture reduces to the classical Riemann-Hilbert problem as free groups satisfy our technical condition and D^∞ is trivial. In the general case, the techniques of Lappo-Danielevsky [16] and Golubeva [9] can be used to prove the conjecture for representations in a neighborhood of the trivial representation.

As a corollary of our main theorem we are able to characterize unipotent monodromy representations.

THEOREM. — *If $\rho: \pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is unipotent, then ρ is a monodromy representation of a nilpotent 1-form if and only if ρ factors through*

$$\pi_1(V, x) \xrightarrow{0} [\mathbb{C} \pi_1(V, x)/J^n]/F^0 \cap J + F^{-1} \cap J^2 + \dots$$

Since the vanishing of the irregularity $q(V)$ is equivalent to the vanishing of I , and since $\pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is unipotent if and only if it factors through $\pi_1(V, x) \rightarrow \mathbb{C} \pi_1(V, x)/J^n$, we obtain the next result.

COROLLARY. — *For a smooth variety V every unipotent representation $\pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is a monodromy representation of a nilpotent 1-form if and only if $q(V) = 0$.*

A different generalization of Hilbert's 21st problem has been considered by Deligne [7]. He showed that every representation $\rho: \pi_1(V, x) \rightarrow \mathrm{GL}(n)$ is the monodromy representation of some holomorphic vector bundle $E \rightarrow X$ which has an integrable connection with regular singular points along D . Here we are attempting to characterize those representations for which we can choose E to be trivial. In the classical case, allowing non-trivial bundles is equivalent to allowing apparent singularities.

The proof of the main theorem combines K.-T. Chen's de Rham theory for the fundamental group [4] with Deligne's mixed Hodge theory for non-singular varieties [8]. The key ingredient from Chen's theory is the formula (2.5) which gives a formula for the monodromy of a flat connection on a trivial bundle, while the principal ingredient from Hodge theory is the fact that each element of $\Omega^*(X \log D)$ is closed and that the resulting map of $\Omega^*(X \log D)$ into $H^*(V)$ is injective. This implies, amongst other things, that the integrability condition for $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is $\omega \wedge \omega = 0$. This leads us to consider the algebra

$$R = \mathbb{C}\{X_1, \dots, X_l\} / (\sum a_{ij}^k [X_i, X_j], k = 1, \dots, m),$$

where

- $X_1, \dots, X_l$ is a basis of the dual of $\Omega^1(X \log D)$.
- the a_{ij}^k are complex constants given by the cup product $\Omega^1 \otimes \Omega^1 \rightarrow \Omega^2$.
- $\mathbb{C}\{X_1, \dots, X_l\}$ denotes the formal power series in the non-commuting indeterminates $X_1, \dots, X_l$ that are *universally convergent*. That is, the power series that converge absolutely whenever the X_j are specialized to matrices $A_j \in \mathfrak{gl}(n)$.

There is a universal integrable 1-form $\tilde{\omega} \in \Omega^1(X \log D) \otimes R$. Namely

$$\tilde{\omega} = w_1 X_1 + \dots + w_l X_l = \mathrm{id} \in \Omega^1 \otimes (\Omega^1)^*.$$

The relations in R guarantee that $\tilde{\omega} \wedge \tilde{\omega} = 0$. Every integrable form $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is then obtained from $\tilde{\omega}$ by specializing the X_j to matrices. Chen's monodromy formula yields a universal monodromy representation

$$\theta: \pi_1(V, x) \rightarrow R.$$

These, and the topology on R , are described in sections 3 and 4.

Using the Hodge theory for π_1 as developed in [10] and described in [11], we show in section 5 that R is the algebra $\mathcal{A}$ of the theorem and that it is canonically associated to the mixed Hodge structure on $\mathbb{C}\pi_1(V, x)^\wedge$. In section 6 we prove the main theorem and its corollaries while in section 7 we discuss the inverse problem and state a conjecture that generalizes Hilbert's 21st problem to Zariski open subsets of $\mathbb{P}^n$.

The complex of C^∞ forms on a manifold will be denoted by E^*M .

2. Connections on Trivial Bundles

By a *trivialized bundle* over a manifold M we mean a trivial bundle $\mathbb{C}^n \times M \rightarrow M$ with a fixed trivialization. The trivialization, being fixed, gives a 1-1 correspondence between connections ∇ on this bundle and matrix valued 1-forms $\omega \in E^1 M \otimes \text{gl}(n)$ according to the rule

$$(2.1) \quad \nabla f = df - f\omega,$$

where $f: M \rightarrow \mathbb{C}^n$ is smooth. This connection is flat if and only if its curvature vanishes:

$$(2.2) \quad d\omega + \omega \wedge \omega = 0.$$

A 1-form is *integrable* if it satisfies (2.2).

Associated with a connection on a trivialized bundle $\mathbb{C}^n \times M \rightarrow M$ is the *transport function*

$$T: PM \rightarrow GL(n),$$

where PM denotes the space of piecewise smooth paths $\gamma: [0, 1] \rightarrow M$. It is the unique function $PM \rightarrow GL(n)$ such that the parallel transport of a vector $v \in \mathbb{C}^n$ along γ is $vT(\gamma)$. Equivalently, if $\gamma_t \in PM$ is the path defined by $\gamma_t(s) = \gamma(st)$, then $T(\gamma)$ is the solution at $t=1$ of the equation

$$(2.3) \quad dT(\gamma_t) = T(\gamma_t)\gamma_t^*\omega, \quad T(\gamma_0) = I.$$

(cf. [5]). It satisfies $T(\alpha\beta) = T(\alpha)T(\beta)$ whenever $\alpha(1) = \beta(0)$.

An explicit formula for T can be given in terms of ω . The formula is due to Chen. First recall the definition of an iterated integral.

2.4. DEFINITION. — Suppose that R is an associative algebra and that $w_1, \dots, w_r \in E^1 M \otimes R$. For $\gamma \in PM$, define

$$\int_{\gamma} w_1 w_2 \dots w_r = \int_{0 \leq t_1 \leq t_2 \leq \dots \leq t_r \leq 1} A_1(t_1) A_2(t_2) \dots A_r(t_r) dt_1 \dots dt_r,$$

where $\gamma^* w_j = A_j(t) dt$. We regard the iterated integral as a function

$$\int w_1 \dots w_r: PM \rightarrow R.$$

2.5. LEMMA. — Suppose that $\omega \in E^1 M \otimes \mathfrak{gl}(n)$. For each $\gamma \in PM$, there exists $M > 0$ such that

$$\left\| \int_{\gamma} \overbrace{\omega \omega \dots \omega}^r \right\| = O\left(\frac{M^r}{r!}\right)$$

so that the series

$$I + \int_{\gamma} \omega + \int_{\gamma} \omega \omega + \int_{\gamma} \omega \omega \omega + \dots$$

converges absolutely. Further, the transport $T: PM \rightarrow GL(n)$ of the connection on $\mathbb{C}^n \times M \rightarrow M$ given by (2.1) is given by

$$T(\gamma) = I + \int_{\gamma} \omega + \int_{\gamma} \omega \omega + \int_{\gamma} \omega \omega \omega + \dots \quad \square$$

The result follows by solving (2.3) by Picard iteration (cf. [18]) and can be found in [11].

When ω is integrable, the value of T on the path γ depends only on its homotopy class relative to its endpoints. Thus T induces the monodromy representation

$$(2.6) \quad \begin{cases} \rho: \pi_1(M, x) \rightarrow GL(n) \\ \{\gamma\} \mapsto T(\gamma) \end{cases}.$$

3. Universally convergent Power series

Suppose that V is a finite dimensional complex vector space. Denote by $\mathbb{C}\langle V \rangle$ the free associative algebra generated by V . The powers of the maximal ideal J generated by V define a topology on $\mathbb{C}\langle V \rangle$. The completion of $\mathbb{C}\langle V \rangle$ in this topology is a ring of formal power series that we shall denote by $\mathbb{C}\langle\langle V \rangle\rangle$: If $X_1, \dots, X_l$ is a basis of V , then $\mathbb{C}\langle\langle V \rangle\rangle$ is isomorphic to the ring $\mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle$ of formal power series in the non-commuting indeterminates $X_1, \dots, X_l$. A typical element of this ring will be written $\sum a_I X_I$, where $I = (i_1, \dots, i_k)$ is a multi index, $a_I \in \mathbb{C}$ and $X_I = X_{i_1} X_{i_2} \dots X_{i_k}$.

3.1. For the time being fix a basis $\mathcal{X} = \{X_1, \dots, X_l\}$ of V . The power series $\sum a_I X_I$ is *universally convergent* if

$$\sum |a_I| r^{|I|} < \infty$$

for all $r \in \mathbb{R}^+$. Here $|I|$ denotes the length of the multi index I . Thus $\sum a_I X_I$ is universally convergent if and only if $\sum a_I A_I$ converges absolutely whenever the X_j are specialized to elements $A_1, \dots, A_l$ of $\mathfrak{gl}(n)$. It is not immediately clear that this notion is independent of the basis chosen for V . This will be proved in (3.4).

3.2. The set of all universally convergent power series in the indeterminates $\mathcal{X} = \{X_1, \dots, X_l\}$ forms a subalgebra $\mathbb{C}\{\mathcal{X}\}$ of $\mathbb{C}\langle\langle V \rangle\rangle$. Define a topology on $\mathbb{C}\{\mathcal{X}\}$ as follows: For each $\varepsilon > 0$ and $r > 0$, set

$$U_{r,\varepsilon} = \{ \sum a_i X_i \in \mathbb{C}\{\mathcal{X}\} : \sum |a_i| r^{|i|} < \varepsilon \}.$$

The proof of the next proposition is a straightforward exercise.

3.3. PROPOSITION. — *The $\{U_{r,\varepsilon} : r > 0, \varepsilon > 0\}$ form a basis for a topology on $\mathbb{C}\{\mathcal{X}\}$ such that:*

- (a) *the topology induced on $\mathbb{C}$ by the inclusion $\mathbb{C} \rightarrow \mathbb{C}\{\mathcal{X}\}$ is the standard topology;*
- (b) *$\mathbb{C}\{\mathcal{X}\}$ is a topological $\mathbb{C}$ -algebra;*
- (c) *if $s = \sum a_i X_i \in \mathbb{C}\{\mathcal{X}\}$ and $s_n = \sum_{|i| \leq n} a_i X_i$ $n = 0, 1, 2, \dots$ is the sequence of partial sums*

of s , then $s_n \rightarrow s$ in this topology.

Suppose that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_l\}$ is another basis of V .

3.4. PROPOSITION. — (a) *The set of universally convergent power series in $\mathbb{C}\langle\langle V \rangle\rangle$ does not depend upon the choice of a basis of V . That is, $\mathbb{C}\{\mathcal{X}\} = \mathbb{C}\{\mathcal{Y}\}$.*

(b) *The topology on the set of universally convergent power series does not depend upon the choice of basis.*

Proof. — We can write $X_i = \sum c_{ij} Y_j$. Let $C = \max |c_{ij}|$. First observe that if $s = \sum a_i X_i \in \mathbb{C}\{\mathcal{X}\}$ and

$$a_k = \max_{|i|=k} |a_i|,$$

then

$$\sum |a_i| r^{|i|} \leq \sum a_k (rl)^k,$$

where $l = \dim V$. Now, rewriting s in terms of the Y_j 's we have

$$s = \sum b_j Y_j$$

where $|b_j| \leq a_k l^k C^k$ and $k = |j|$. Therefore

$$\sum_j |b_j| r^{|j|} \leq \sum_k a_k (lCr)^k < \infty.$$

It follows that $\mathbb{C}\{\mathcal{X}\} \subseteq \mathbb{C}\{\mathcal{Y}\}$ and, by symmetry, that $\mathbb{C}\{\mathcal{X}\} = \mathbb{C}\{\mathcal{Y}\}$. This proves (a).

Denote by

$$V_{u,\varepsilon} = \{ \sum b_j Y_j : \sum |b_j| u^{|j|} < \varepsilon \}$$

the basic open sets of the topology on $\mathbb{C}\{\mathcal{Y}\}$ defined by $\mathcal{Y}$. If $s = \sum a_i X_i = \sum b_j Y_j \in U_{r,\varepsilon}$, then, as above,

$$\sum_j |b_j| u^{|j|} \leq \sum_k a_k (lCu)^k \leq \sum_i |a_i| (lCu)^{|i|} < \varepsilon$$

when $u \leq r/lC$. Thus $U_{r,\varepsilon} \subseteq V_{u,\varepsilon}$ when $u = r/lC$. It follows that the identity $\mathbb{C}\{X\} \rightarrow \mathbb{C}\{Y\}$ is continuous. By symmetry, $\mathbb{C}\{X\} \rightarrow \mathbb{C}\{Y\}$ is a homeomorphism. $\square$

We shall denote the ring of universally convergent power series in $\mathbb{C}\langle\langle V \rangle\rangle$ with the topology defined in (3.2) by $\mathbb{C}\{V\}$. It has a nice universal mapping property.

3.5. PROPOSITION. — *There is a 1-1 correspondence between continuous $\mathbb{C}$ -algebra homomorphisms $\mathbb{C}\{V\} \rightarrow \mathfrak{gl}(n)$ and $\mathbb{C}$ linear maps $V \rightarrow \mathfrak{gl}(n)$ such that*

$$\begin{array}{ccc} & V & \\ & \searrow & \\ \downarrow & & \nearrow \mathfrak{gl}(n) \\ \mathbb{C}\{V\} & & \end{array}$$

commutes.

Proof. — Given a $\mathbb{C}$ -algebra homomorphism $\mathbb{C}\{V\} \rightarrow \mathfrak{gl}(n)$, one obtains a $\mathbb{C}$ -linear map by restriction. Conversely, a $\mathbb{C}$ -linear map $\varphi: V \rightarrow \mathfrak{gl}(n)$ extends to a $\mathbb{C}$ -algebra homomorphism $\varphi: \mathbb{C}\langle V \rangle \rightarrow \mathfrak{gl}(n)$. According to (3.3c), $\mathbb{C}\langle V \rangle$ is dense in $\mathbb{C}\{V\}$. Thus it suffices to show that $\varphi: \mathbb{C}\langle V \rangle \rightarrow \mathfrak{gl}(n)$ is continuous.

Choose a basis $X_1, \dots, X_l$ of V . Let $A_j = \varphi X_j$ and $r = \max \|A_j\|$. If $\{p_m(X_1, \dots, X_l)\}_{m=1}^\infty$ is a sequence of polynomials in $\mathbb{C}\langle V \rangle$ converging to 0, then, for each $\varepsilon > 0$ there exists N such that $p_m(X_1, \dots, X_l) \in U_{r,\varepsilon}$ whenever $m \geq N$. But this implies that

$$\|\varphi p_m(X_1, \dots, X_l)\| = \|p_m(A_1, \dots, A_l)\| < \varepsilon,$$

which implies that φ is continuous. $\square$

4. The basic construction

Now suppose that V is a smooth complex algebraic variety and that X is a smooth completion of V such that $X - V$ is a divisor D in X with normal crossings. Denote the algebra of global meromorphic differentials on X that are holomorphic on V and have, at worst, logarithmic poles along D by $\Omega^*(X \log D)$. By a theorem of Deligne ([8], (3.2.14))

$$(4.1) \quad \Omega^p(X \log D) = F^p H^p(V; \mathbb{C}) \subseteq H^p(V; \mathbb{C}),$$

where F^* denotes the Hodge filtration associated with the mixed Hodge structure on the cohomology of V . Implicit in (4.1) are the facts:

4.1 (a) each element of $\Omega^*(X \log D)$ is closed,

4.1 (b) no non-zero element of $\Omega^*(X \log D)$ is exact.

Denote the dual of $\Omega^p(X \log D)$ by W_p . The dual of the cup product

$$\Omega^1(X \log D) \otimes \Omega^1(X \log D) \rightarrow \Omega^2(X \log D)$$

is a map

$$\Delta: W_2 \rightarrow W_1 \otimes W_1.$$

Let R be the closed ideal of $\mathbb{C}\langle\langle W_1 \rangle\rangle$ generated by the image of Δ . Set

$$A = \mathbb{C}\langle\langle W_1 \rangle\rangle / R.$$

4.2. *Remarks.* — (a) Choose bases $w_1, \dots, w_l$ of $\Omega^1(X \log D)$, $z_1, \dots, z_m$ of $\Omega^2(X \log D)$ and a dual basis $X_1, \dots, X_l$ of W_1 . Then

$$\mathbb{C}\langle\langle W_1 \rangle\rangle = \mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle$$

and R is the closed ideal generated by

$$\sum a_{ij}^k [X_i, X_j], \quad k = 1, \dots, m,$$

where the complex constants a_{ij}^k are defined by

$$w_i \wedge w_j = \sum a_{ij}^k z_k$$

and $[A, B] = AB - BA$.

(b) For future reference we record the following fact. The graded vectorspace W is a connected coalgebra with diagonal $\Delta: W \rightarrow W \otimes W$ dual to the cup product. We can apply Adam's cobar construction (cf. [5]) to get a differential graded algebra $\mathcal{F}(W)$. It follows from the definition of the cobar construction that A is the J -adic completion of $H_0(\mathcal{F}(W))$, where J denotes the augmentation ideal. Since W is commutative, $H_0(\mathcal{F}(W))$ is a Hopf algebra and A has a natural complete Hopf algebra structure. $\square$

Let $\mathcal{A}$ be the image of $\mathbb{C}\{W_1\}$ in A under the canonical projection $\mathbb{C}\langle\langle W_1 \rangle\rangle \rightarrow A$. Give $\mathcal{A}$ the topology induced by the surjection $\mathbb{C}\{W_1\} \rightarrow \mathcal{A}$.

4.3. PROPOSITION: (a) $\mathcal{A}$ is a topological $\mathbb{C}$ -algebra.

(b) There is a 1-1 correspondence between continuous $\mathbb{C}$ -algebra homomorphisms $\mathcal{A} \rightarrow \mathfrak{gl}(n)$ and $\mathbb{C}$ -linear functions $\varphi: W_1 \rightarrow \mathfrak{gl}(n)$ for which the composite

$$W_2 \xrightarrow{\Delta} W_1 \otimes W_1 \xrightarrow{\varphi \otimes \varphi} \mathfrak{gl}(n) \otimes \mathfrak{gl}(n) \xrightarrow{\text{mult}} \mathfrak{gl}(n)$$

is zero.

Proof. — A homomorphism $\mathcal{A} \rightarrow \mathfrak{gl}(n)$ determines a function $W_1 \rightarrow \mathfrak{gl}(n)$ with the required property by restriction. Conversely, a $\mathbb{C}$ -linear map $\varphi: W_1 \rightarrow \mathfrak{gl}(n)$ determines a continuous function $\hat{\varphi}: \mathbb{C}\{W_1\} \rightarrow \mathfrak{gl}(n)$ by (3.5). Let $\mathcal{R} \subseteq \mathbb{C}\langle\langle W_1 \rangle\rangle$ be the ideal generated by the image of Δ and $\bar{\mathcal{R}}$ its closure in $\mathbb{C}\{W_1\}$. One can easily check, using 3.3(c), that

$$\bar{\mathcal{R}} \supseteq \mathbb{C}\{W_1\} \cap R.$$

If φ satisfies the condition, then $\hat{\varphi}$ vanishes on $\bar{\mathcal{R}}$ and thus induces an algebra homomorphism $\mathcal{A} \rightarrow \mathfrak{gl}(n)$ which is continuous, as $\mathcal{A}$ has the quotient topology. $\square$

Now suppose that $\mathbb{C}^n \times X \rightarrow X$ is a trivial holomorphic vector bundle over X . Since two trivializations differ by a morphism $g: X \rightarrow GL(n)$ and since $GL(n)$ is affine, g is constant and each such bundle has an essentially unique trivialization. Connections on this bundle, holomorphic over V and with regular singular points along D , correspond to $\mathfrak{gl}(n)$ -valued 1-forms $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ by (2.1). Fix such a 1-form ω . We can express ω in terms of a basis $w_1, \dots, w_l$ of $\Omega^1(X \log D)$:

$$\omega = \sum w_j A_j,$$

where each A_j is a constant matrix. Since each w_j is closed (4.1(a)), ω is integrable if and only if

$$0 = \omega \wedge \omega = \sum w_i \wedge w_j A_i A_j = \frac{1}{2} \sum a_{ij}^k z_k [A_i, A_j].$$

(Here we are using the notation of 4.2(a).) Since the z_k 's are linearly independent, we have proved:

4.4. PROPOSITION. — *The 1-form $\omega = \sum w_j A_j \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is integrable if and only if*

$$\sum a_{ij}^k [A_i, A_j] = 0$$

for each k . $\square$

Combining this with (4.3(b)) we obtain:

4.5. PROPOSITION. — *There is a 1-1 correspondence between integrable 1-forms $\omega \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ and continuous $\mathbb{C}$ -algebra homomorphisms $\mathcal{A} \rightarrow \mathfrak{gl}(n)$.* $\square$

4.6. We now define the universal integrable connection. As in 4.2(a), choose a basis $w_1, \dots, w_l$ of $\Omega^1(X \log D)$ and a dual basis $X_1, \dots, X_l$ of W_1 . Set

$$\tilde{\omega} = w_1 X_1 + \dots + w_l X_l \in \Omega^1(X \log D) \otimes \mathcal{A}$$

[This corresponds to $\text{id} \in \text{Hom}(\Omega^1, \Omega^1) \approx \Omega^1 \otimes W_1$ and is thus independent of the choice of basis.] As in (4.4), one checks that $\tilde{\omega}$ is integrable. It follows (cf. [4]) that the group homomorphism

$$\begin{aligned} \theta: \pi_1(V, x) &\rightarrow \mathcal{A} \\ \{\gamma\} &\mapsto 1 + \int_{\gamma} \tilde{\omega} + \int_{\gamma} \tilde{\omega} \tilde{\omega} + \int_{\gamma} \tilde{\omega} \tilde{\omega} \tilde{\omega} + \dots \end{aligned}$$

is well defined. Furthermore, (2.5) implies that $\text{im } \theta \subseteq \mathcal{A}$. One should think of

$$\theta: \pi_1(V, x) \rightarrow \mathcal{A}$$

as the universal monodromy representation.

Now suppose that $\omega = \sum w_j A_j \in \Omega^1(X \log D) \otimes \mathfrak{gl}(n)$ is an integrable 1-form. By (4.5) this determines a continuous $\mathbb{C}$ -algebra homomorphism $\varphi: \mathcal{A} \rightarrow \mathfrak{gl}(n)$ such that

$\varphi(X_j) = A_j$. Let $\rho: \pi_1(V, x) \rightarrow GL(n)$ be the associated monodromy representation. By (2.6) and (3.3(c)) the diagram

$$(4.7) \quad \begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n) \\ \theta \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{\varphi} & gl(n) \end{array}$$

commutes. Conversely, if we are given a continuous $\mathbb{C}$ -linear homomorphism $\varphi: \mathcal{A} \rightarrow gl(n)$ and a representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ such that (4.7) commutes, then, by (2.6) again, ρ is the monodromy representation of $\sum w_j \varphi(X_j)$. This completes the proof of the following result.

4.8. LEMMA. — *A representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ is the monodromy representation of an integrable 1-form $\omega \in \Omega^1(X \log D) \otimes gl(n)$ if and only if there exists a continuous $\mathbb{C}$ -linear algebra homomorphism $\varphi: \mathcal{A} \rightarrow gl(n)$ such that (4.7) commutes. $\square$*

5. The mixed Hodge structure on π_1

Let $V = X - D$ be as in section 4. Denote the complex group ring of $\pi_1(V, x)$ by $\mathbb{C}\pi_1(V, x)$ and its augmentation ideal (i.e., the kernel of the algebra homomorphism $\mathbb{C}\pi_1 \rightarrow \mathbb{C}$ that takes each $g \in \pi_1$ to 1) by J . The J -adic completion

$$\mathbb{C}\pi_1(V, x)^\wedge = \varprojlim \mathbb{C}\pi_1(V, x)/J^s$$

has a natural complete Hopf algebra structure (cf. [21], Appendix A). The following theorem, essentially due to Morgan [17], is proved in [10] (see also [11]).

5.1. THEOREM. — *There are filtrations $W_\bullet, F^\bullet$ on $\mathbb{C}\pi_1(V, x)^\wedge$ by closed subspaces such that*

- (a) *$W_\bullet$ is the complexification of an increasing filtration of $\mathbb{Q}\pi_1(V, x)^\wedge$;*
- (b) *on each truncation $\mathbb{C}\pi_1(V, x)/J^{s+1}$ of $\mathbb{C}\pi_1(V, x)^\wedge$, the filtrations induce a mixed Hodge structure;*
- (c) *the filtrations are preserved by the product and coproduct of $\mathbb{C}\pi_1(V, x)^\wedge$;*
- (d) $J = F^{-1} \cap J + F^{-2} \cap J^2 + F^{-3} \cap J^3 + \dots$ $\square$

From (d) above it follows that the closed subspace

$$I = F^0 \cap J + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \dots$$

is an ideal (in fact, a Hopf ideal) of $\mathbb{C}\pi_1(V, x)^\wedge$.

The next result relates the algebra A of section 4 to the mixed Hodge theory of $\pi_1(V)$.

5.2. LEMMA. — *There is a natural isomorphism of complete Hopf algebras*

$$\mathbb{C}\pi_1(V, x)^\wedge / I \rightarrow A$$

that is natural with respect to base point preserving morphisms of smooth algebraic varieties.

Proof. — Denote the complex of C^∞ forms on X with logarithmic singularities along D by $L^\bullet$ and let $F^\bullet$ be the usual Hodge filtration of $L^\bullet$ obtained by counting dz 's. The base point $x \in V$ induces an augmentation $L^\bullet \rightarrow \mathbb{C}$ so that we can apply the reduced bar construction (see [5]) to obtain a d.g. Hopf algebra $\bar{B}(L^\bullet)$. This is naturally isomorphic to the complex of iterated integrals on the space $P_{x,x}V$ of piecewise smooth loops in V based at x : The element $[w_1 | \dots | w_r]$ of $\bar{B}(L^\bullet)$ corresponds to the iterated integral $\int w_1 \dots w_r$ restricted to $P_{x,x}V$.

Let $\mathbf{B}_s$ be the increasing filtration of $\bar{B}(L^\bullet)$ by length. That is, $\mathbf{B}_s$ is the linear span of the iterated integrals $[w_1 | \dots | w_r]$ where $r \leq s$. Chen's π_1 de Rham theorem ([5], cf. [11]) asserts that, for each $s \geq 0$, integration induces a natural isomorphism

$$\mathbf{B}_s H^0(\bar{B}(L^\bullet)) \xrightarrow{\sim} \text{Hom}(\mathbb{C} \pi_1(V, x)/J^{s+1}, \mathbb{C}).$$

Since $\pi_1(V, x)$ is finitely generated, each $\mathbb{C} \pi_1(V, x)/J^{s+1}$ is finite dimensional. Therefore $\text{Hom}(H^0(\bar{B}(L^\bullet)), \mathbb{C}) \cong \varprojlim \text{Hom}(\mathbf{B}_s H^0(\bar{B}(L^\bullet)), \mathbb{C})$

$$\cong \varprojlim \mathbb{C} \pi_1(V, x)/J^{s+1} \cong \mathbb{C} \pi_1(V, x)^\wedge.$$

This is easily seen to be an isomorphism of complete Hopf algebras.

The Hodge filtration $F^\bullet$ of $L^\bullet$ extends to one of $\bar{B}(L^\bullet)$ by defining $F^p \bar{B}(L^\bullet)$ to be the linear span of the $[w_1 | \dots | w_r]$ such that $w_j \in F^{p_j} L^\bullet$ and $p_1 + \dots + p_r \geq p$. It follows from the proof of (5.1) given in [10] (see also [11]) that

$$(5.3) \quad F^p \text{Hom}(\mathbb{C} \pi_1(V, x)/J^{s+1}, \mathbb{C}) \cong \mathbf{B}_s H^0(F^p \bar{B}(L^\bullet)).$$

The holomorphic log complex $\Omega^\bullet = \Omega^\bullet(X \log D)$ is a sub d.g. algebra of $L^\bullet$. We therefore have a Hopf algebra homomorphism

$$H^0(\bar{B}(\Omega^\bullet)) \rightarrow H^0(\bar{B}(L^\bullet)).$$

It follows from (4.1), by examining the E_1 term of the Eilenberg-Moore spectral sequence (i.e., the spectral sequence associated with the filtration $\mathbf{B}_s$), that this map is an inclusion and that

$$H^0(\bar{B}(\Omega^\bullet)) \cong \mathbb{C} + F^1 \cap \mathbf{B}_1 + F^2 \cap \mathbf{B}_2 + \dots$$

Dualizing, we see from (5.3) that the kernel of the surjection

$$\mathbb{C} \pi_1(V, x)^\wedge \rightarrow \text{Hom}(H^0(\bar{B}(\Omega^\bullet)), \mathbb{C})$$

is $I = F^0 \cap J + F^{-1} \cap J^2 + \dots$

Finally, the duality between bar and cobar and 4.2(b) yield the following isomorphisms of complete Hopf algebras:

$$\begin{aligned}
 (5.4) \quad \text{Hom}(H^0(\bar{B}(\Omega')), \mathbb{C}) &\cong \varprojlim \text{Hom}(\mathbf{B}_s H^0(\mathbf{B}(\Omega')), \mathbb{C}) \\
 &\cong \varprojlim H_0(\mathcal{F}(W.))/J^{s+1} \\
 &\cong H_0(\mathcal{F}(W.))^\wedge \cong A. \quad \square
 \end{aligned}$$

Our final task is to show that the natural map

$$\pi_1(V, x) \rightarrow \pi_1(V, x)^\wedge / I$$

and the map $\theta: \pi_1(V, x) \rightarrow A$ correspond under the isomorphism of (5.2).

5.5. PROPOSITION. — *The diagram*

$$\begin{array}{ccc}
 \pi_1(V, x) & \rightarrow & \mathbb{C} \pi_1(V, x)^\wedge \\
 \theta \downarrow & & \downarrow \\
 A & \cong & \mathbb{C} \pi_1(V, x)^\wedge / I
 \end{array}$$

commutes, where θ is the map constructed in (4.6).

Proof. — Choose a basis $w_1, \dots, w_l$ of $\Omega^1(X \log D)$ and a dual basis $X_1, \dots, X_l$ of W_1 . As in 4.2(a), we can identify A with

$$\mathbb{C} \langle\langle X_1, \dots, X_l \rangle\rangle / (\sum a_{ij}^k [X_i, X_j]),$$

where the a_{ij}^k are complex constants. The universal connection form is

$$\tilde{\omega} = w_1 X_1 + \dots + w_l X_l$$

and θ takes a loop γ to

$$1 + \int_\gamma \tilde{\omega} + \int_\gamma \tilde{\omega} \tilde{\omega} + \dots = 1 + \sum \int_\gamma w_i X_i + \sum \int_\gamma w_i w_j X_i X_j + \sum \int_\gamma w_i w_j w_k X_i X_j X_k + \dots$$

On the other hand, the isomorphism (5.4) takes $X_{i_1} \dots X_{i_r}$ to the linear functional on $H^0(\bar{B}(\Omega'))$ induced by the functional on $\bar{B}(\Omega')$ that takes $[w_{i_1} | \dots | w_{i_r}]$ to 1 and all other $[w_{j_1} | \dots | w_{j_s}]$ to 0.

Consequently, the composite

$$\pi_1(V, x) \xrightarrow{\theta} A \cong \text{Hom}(H^0(\bar{B}(\Omega')), \mathbb{C})$$

takes γ to the functional induced by the functional on $\bar{B}(\Omega')$ defined by

$$[w_{i_1} | \dots | w_{i_r}] \rightarrow \int_\gamma w_{i_1} \dots w_{i_r}.$$

It follows that the diagram commutes. $\square$

6. Main results

Let $V = X - D$ be as in section 3. Combining (4.8), (5.2) and (5.5), we obtain our main theorem.

6.1. THEOREM. — *There is a topological $\mathbb{C}$ -algebra $\mathcal{A}$ contained in*

$$\mathbb{C} \pi_1(V, x)^\wedge / F^0 \cap J + F^{-1} \cap J^2 + \dots$$

such that

(a) *the image of the natural homomorphism*

$$\theta: \pi_1(V, x) \rightarrow \mathbb{C} \pi_1(V, x) / F^0 \cap J + F^{-1} \cap J^2 + \dots$$

is contained in $\mathcal{A}$;

(b) *a representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ is the monodromy representation of an integrable 1-form $\omega \in \Omega^1(X \log D) \otimes gl(n)$ if and only if there is a continuous $\mathbb{C}$ -algebra homomorphism $\varphi: \mathcal{A} \rightarrow gl(n)$ such that the diagram*

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n) \\ \theta \downarrow & & \downarrow \\ \mathcal{A} & \rightarrow & gl(n) \end{array}$$

commutes. $\square$

The theorem imposes obvious conditions on monodromy representations. Let $R = \ker \theta$ and

$$D^\infty = \ker \{ \pi_1(V, x) \rightarrow \mathbb{C} \pi_1(V, x)^\wedge / I \}.$$

6.2. COROLLARY. — *If $\rho: \pi_1(V, x) \rightarrow GL(n)$ is a monodromy representation, then*

$$\ker \rho \supseteq R \supseteq D^\infty. \quad \square$$

6.3. Remark. — Using the Hopf algebra structure of $A = \mathbb{C} \pi_1(V)^\wedge / I$, we can get slightly more information. Let

$$g = \{ X \in A : \Delta X = 1 \otimes X + X \otimes 1 \}, \quad \mathcal{G} = \{ X \in 1 + J : \Delta X = X \otimes X \}$$

be the set of primitive elements of A and group-like elements of A , respectively. Here $\Delta: A \rightarrow A \hat{\otimes} A$ denotes the coproduct. If g is finite dimensional, then $g \subseteq \mathcal{A}$. Since $\mathcal{G} = \exp g$, $\mathcal{G}$ is also in $\mathcal{A}$. Finally, the fact that $\text{im } \theta \subseteq \mathcal{G}$ implies that if g is finite dimensional, then $\rho: \pi_1(V, x) \rightarrow GL(n)$ is a monodromy representation if and only if it factors through a homomorphism $\mathcal{G} \rightarrow GL(n)$ of complex Lie groups:

$$\begin{array}{ccc} \pi_1(V, x) & \rightarrow & \mathcal{G} \\ & \searrow \rho & \swarrow \\ & GL(n) & \end{array}$$

For example, if $\dim \Omega^1(X \log D) = 1$, then $\mathcal{G} = \mathbb{C}$. In this way we obtain the characterization of monodromy representations given in (1.2). Unfortunately, $\mathfrak{g}$ is seldom finite dimensional.

We now consider the unipotent case. By the Kolchin-Engel theorem [13], a representation $\rho: G \rightarrow GL(n)$ is unipotent if and only if the induced map $\mathbb{C}G \rightarrow \mathfrak{gl}(n)$ induces a map

$$\bar{\rho}: \mathbb{C}G/J^n \rightarrow \mathfrak{gl}(n).$$

A matrix valued 1-form ω is said to be *nilpotent* if there exists a nilpotent Lie subalgebra $\mathfrak{n}$ of $\mathfrak{gl}(n)$ such that $\omega \in \Omega^1(X \log D) \otimes \mathfrak{n}$. Monodromy representations of integrable nilpotent 1-forms are always unipotent.

6.4. THEOREM. — *For a unipotent representation, the following are equivalent:*

(a) *There exists a $\mathbb{C}$ -linear algebra homomorphism*

$$\mathbb{C}\pi_1(V, x)^\wedge/J^n + I \rightarrow \mathfrak{gl}(n)$$

such that

$$\begin{array}{ccc} \pi_1(V, x) & \xrightarrow{\rho} & GL(n) \\ \downarrow & & \downarrow \\ \pi_1(V, x)/J^n + I & \xrightarrow{\varphi} & \mathfrak{gl}(n) \end{array}$$

commutes.

(b) ρ is the monodromy representation of an integrable, nilpotent form.

Proof. — As in (5.2), we have an algebra isomorphism

$$\mathbb{C}\pi_1(V, x)^\wedge/J^n + I \cong \mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle / (\sum a_{ij}^k [X_i, X_j]) + J^n.$$

From (2.5) and (4.6) we conclude that if we are given φ as in 6.4(a), then ρ is the monodromy representation of $\omega = \sum w_j \varphi(X_j)$. Since φ is an algebra homomorphism and each X_j is nilpotent in $\mathbb{C}\pi_1(V, x)/J^n + I$, ω is a nilpotent connection. Thus (a) implies (b).

Conversely, given an integrable, nilpotent 1-form $\omega = \sum w_i A_i$ on V , define

$$\hat{\varphi}: \mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle \rightarrow \mathfrak{gl}(n)$$

by $\hat{\varphi}(X_j) = A_j$. Since the A_j lie in a nilpotent sub Lie algebra of $\mathfrak{gl}(n)$, it follows from Engel's theorem that $\hat{\varphi}$ induces a homomorphism

$$\bar{\varphi}: \mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle/J^n \rightarrow \mathfrak{gl}(n).$$

The integrability of ω implies that $\bar{\varphi}$ induces a homomorphism

$$\varphi: \mathbb{C}\pi_1(V, x)^\wedge/J^n + I \cong \mathbb{C}\langle\langle X_1, \dots, X_l \rangle\rangle/J^n + (\sum a_{ij}^k [X_i, X_j]) \rightarrow \mathfrak{gl}(n)$$

That the diagram in 6.4(a) commutes follows from (2.6) and (4.6). $\square$

As in the introduction, we define the *irregularity* $q(V)$ of $V=X-D$ to be $h^{1,0}(X)$. This is independent of the choice of a smooth completion X of V .

6.5. COROLLARY. — *The following statements are equivalent for a smooth variety V :*

(a) *Every unipotent representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ is the monodromy of a nilpotent integrable 1-form $\omega \in \Omega^1(X \log D) \otimes gl(n)$.*

(b) $q(V)=0$.

(c) $W_1 H^1(V; \mathbb{C})=0$.

Proof. — The equivalence of (b) and (c) follows from [8]: (3.2.14). If $W_1 H^1(V; \mathbb{C})=0$, then one can easily check that $I=0$. Applying (6.4), we see that (c) implies (a).

Suppose that $W_1 H^1(V) \neq 0$. Consider the unipotent representation

$$\begin{aligned} \pi_1(V, x) &\rightarrow GL(\mathbb{C} \pi_1(V, x)/J^2) \\ g &\mapsto \{ U \mapsto Ug \} \end{aligned}$$

given by right multiplication. This representation is unipotent and factors through the Hurewicz homomorphism

$$\pi_1(V, x) \rightarrow H_1(V) \hookrightarrow GL(\mathbb{C} \pi_1(V, x)/J^2)$$

as there is a ring isomorphism

$$\begin{aligned} \mathbb{C} \pi_1(V, x)/J^2 &\cong \mathbb{C} \oplus H_1(V; \mathbb{C}) \\ g &\mapsto (1, [g]) \end{aligned}$$

where $(\lambda, a) \cdot (\mu, b) = (\lambda\mu, \mu a + \lambda b)$ is the multiplication on the right hand side. Since $W_1 H^1(V) \neq 0$, $F^0 H_1(V) \neq 0$. Consequently this representation does not factor through

$$\mathbb{C} \pi_1(V, x)/F^0 \cap J + J^2 \cong \mathbb{C} \oplus H_1(V)/F^0.$$

That is, (c) implies (a). $\square$

In the case when $X = \mathbb{P}^n$, we recover a result of Aomoto [1].

6.6. THEOREM (Aomoto). — *If V is a Zariski open subset of $\mathbb{P}^n$, then every unipotent representation of $\pi_1(V, x)$ is the monodromy representation of an integrable 1-form ω on V with logarithmic singularities at infinity.* $\square$

7. The inverse problem

When V is a Zariski open subset of $\mathbb{P}^m$, the ideal

$$I = F^0 \cap J + F^{-1} \cap J^2 + \dots = 0.$$

Thus, in some sense, only group theory and not Hodge theory imposes conditions on

monodromy representations. Let D^∞ be the kernel of the natural map $\pi_1(V, x) \rightarrow \mathbb{C} \pi_1(V, x)$. According to (6.2), the kernel of every monodromy representation has to contain D^∞ .

7.1. CONJECTURE. — Assume that V is a Zariski open subset of $\mathbb{P}^m$ [or, more generally, that $W_1 H^1(V) = 0$.] Suppose that there exist $x_1, \dots, x_l \in \pi_1(V, x)$ such that

(a) $[x_1], \dots, [x_l]$ are linearly independent in $H_1(V; \mathbb{Z})$,

(b) $x_1, \dots, x_l$ generate $\pi_1(V, x)/D^\infty$.

Then a representation $\rho: \pi_1(V, x) \rightarrow GL(n)$ is the monodromy representation of an integrable 1-form $\omega \in \Omega^1(X \log D) \otimes gl(n)$ if and only if $\ker \rho \supseteq D^\infty$.

When $X = \mathbb{P}^1$, then $\pi_1(V, x)$ is free and $D^\infty = 1$. In this case the conjecture reduces to the classical Riemann-Hilbert problem.

The fundamental groups of many Zariski open subsets of $\mathbb{P}^m$ satisfy the conditions in (7.1). For example, it holds when V is the complement of a union of hyperplanes. It would be interesting to know a larger class of examples of open subsets of $\mathbb{P}^n$ for which it holds as well as an example where it fails.

The condition arises as follows. Suppose that V satisfies the hypotheses of (7.1) and that $\rho: \pi_1(V, x) \rightarrow GL(n)$ satisfies $\ker \rho \supseteq D^\infty$. Set

$$W_j = \log \rho(x_j).$$

If ρ is the monodromy representation of $\omega = w_1 A_1 + \dots + w_l A_l$, then by taking logarithms of (2.6) we have formal power series expansions

$$(7.2) \quad W_j = A_j + \sum_{|I| \geq 2} a_I A_{i_1} \dots A_{i_s}, \quad j = 1, \dots, l.$$

One can formally invert the power series (7.2) to find power series

$$(7.3) \quad A_j = W_j + \sum b_I W_{i_1} \dots W_{i_s}.$$

Golubeva [9] has shown that, when each $\|\rho(x_j) - I\|$ is small enough, the series (7.3) converge absolutely and that ρ is the monodromy representation of the connection $\omega = \sum w_j A_j$. Thus we have:

7.4. THEOREM. — Suppose that the Zariski open subset V of $\mathbb{P}^m$ satisfies the hypotheses of (7.1). If $\rho: \pi_1(V, x) \rightarrow GL(n)$ satisfies $\ker \rho \supseteq D^\infty$ and if each $\|\rho(x_j) - I\|$ is sufficiently small, then ρ is the monodromy representation of an integrable 1-form with logarithmic singularities at infinity. $\square$

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(Manuscrit reçu le 1^{er} avril 1986.)

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