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POSITIVE MEASURE SETS
OF ERGODIC RATIONAL MAPS

MARY REES

ABSTRACT. — In the set of rational maps of degree $d$, and in some other families, a positive measure set of ergodic maps is found.

Introduction

In this paper, we study rational maps of the extended complex plane, and the question of their ergodicity. The set of all rational maps of degree $d$ is a complex manifold of complex dimension $2d+1$, and hence admits a natural Lebesgue measure class, as does any submanifold. We show that, in complex submanifolds of rational maps containing at least one map with the forward orbit of each critical point finite and containing an expanding periodic point, and satisfying a simple non-degeneracy condition, the set of maps, which are ergodic with respect to Lebesgue measure on $C \cup \{\infty\}$, has positive Lebesgue measure. In particular, a positive measure subset of all rational maps of degree $d$ consists of ergodic maps. In contrast, there is a conjecture that the set of maps with zero measure Julia set is open and dense. It will also be clear from the estimates that the positive measure set of ergodic maps which is constructed, consists, in some suitable measure theoretic sense, of expanding maps.

This paper grew, in part, out of an unsubstantiated remark in the first version of the introduction of [R] concerning rational maps with positive exponents. The result obtained in this paper has a resemblance to Jakobson's Theorem [J] for the 1-(real)-parameter family $f_\lambda(x) = \lambda x (1-x) (\lambda \in [0,4])$ of maps of $[0,1]$; that for a positive measure set of $\lambda$, $f_\lambda$ has an absolutely continuous invariant measure, and is ergodic. Fundamentally, the idea of the present proof is the same, although the details are different. (One-dimensional arguments cannot be used.) The proof can be modified to prove Jakobson's Theorem for a larger class of real polynomial maps.

One reason for understanding how maps of an interval—or analytic maps of the plane—can be ergodic, is that it seems to be relevant to the problem of constructing volume-preserving diffeomorphisms with positive exponents. It would possibly be helpful if it turned out that complex analysis theorems would be used to construct
analytic examples. Unfortunately, in the present paper, only Montel's Theorem is used—and only once. However, complex analysis theorems have been used to prove related results. Sullivan used the measurable Riemann Mapping Theorem to prove conservativity of maps with Julia set equal to $S^2$. Also, Herman ([H1], [H2]) has a formula, using the Divergence Theorem, for bounding below exponents in certain examples.

I am heavily indebted to J. C. Yoccoz of this paper who clarified many of the ideas, and entirely reorganized and rewrote most of the paper in an intelligible form. He also introduced ideas such as the use of Schwarz's Lemma, standard univalent function theory, and the method of estimating quantities by realizing them as coefficients in the power series of an analytic function (in 7.1). Many of the proofs—most notably the fundamental estimates of section 5, and 7.1—are his, verbatim.

1.2. Precise formulation of the results. — Let $f$ be a rational map such that the orbit of each critical point is finite but no critical point is periodic. Then any periodic point is expanding. This was proved by Thurston by producing a hyperbolic structure on $S^2$ with ramification points at the critical point forward orbits. Another argument is the following. If $f^n(z_0)=z_0$ and $|(f^n)'(z_0)|=1$, and $\phi$ denotes the local inverse of $f^n$ mapping $f^n(z_0)$ to $z_0$, then $\{\phi^k : k \geq 1\}$ form a normal family on a neighbourhood of $f^n(z_0)$ (because critical point forward orbits are finite). $|(\phi^k)'| \to 0$ because $|(f^n)'(z_0)|=1$. So $\{f^{nk} : k \geq 1\}$ is a normal family in a neighbourhood of $z_0$, which must therefore be the centre of a Siegel disc. This is impossible because the critical point forward orbits are finite. The possibility that $|(f^n)'(z_0)|<1$ is ruled out for the same reason, together with the fact that no critical point is periodic.

Thus, such a map has Julia set equal to $S^2$. This is because there are no wandering domains in the complement of the Julia set—which was proved by Fatou in this case [F] and by Sullivan in general [S2]. It also follows from the fact that the map is expanding with respect to Thurston's hyperbolic metric.

Now let $\{f_\lambda : \lambda \in \Lambda\}$ be an analytic family of rational maps of degree $d$, where for some $\lambda_0 \in \Lambda$, $f_{\lambda_0}$ has all critical points non-periodic, but critical point forward orbits finite. Let $x_1(\lambda)$ be the critical points of $f_\lambda$ counted with multiplicity, $1 \leq i \leq 2d-2=m_\lambda$. Write $f_\lambda(\lambda, z)=f_\lambda^0(z)$, and suppose $f_t(\lambda_0, x_1(\lambda_0))$ is periodic with period $s_t$.

The non-degeneracy condition. — $DF_t(\lambda_0) \neq 0$ for $1 \leq i \leq m_\lambda$, where

$$F_i(\lambda) = f_{t+i}(\lambda, x_i(\lambda)) - f_t(\lambda, x_i(\lambda)).$$

Note that $DF_t(\lambda_0)$ exists even though $x_1$ may not be differentiable at $\lambda_0$. Write $y_i(\lambda)=f_{t+i}(\lambda, x_i(\lambda))$, and let $z_i(\lambda)$ be the periodic point of period $s_t$ with $y_i(\lambda_0)=z_i(\lambda_0)$. Then $y_i, z_i$ are both differentiable functions. Then one sees that:

$$DF_t(\lambda_0) = (f_{t+i}(z_i(\lambda_0))-1) \left( \frac{Dy_i(\lambda_0)}{D\lambda} - \frac{Dz_i(\lambda_0)}{D\lambda} \right),$$

so that $DF_t(\lambda_0) \neq 0$ if and only if $(D/D\lambda)(y_i-z_i)(\lambda_0) \neq 0$.
1.3. The main theorem of this paper is:

**Theorem A.** — Suppose \( \{ f_{\lambda} : \lambda \in \Lambda \} \) is such that \( f_{\lambda_0} \) has all critical point forward orbits finite and critical points non-periodic. Suppose the non-degeneracy condition is satisfied at \( \lambda_0 \). Then for a positive measure set of \( \lambda \), \( f_{\lambda} \) is ergodic with respect to Lebesgue measure, and has an invariant probability measure equivalent to Lebesgue.

1.4. **Corollary.** — A positive measure set of rational maps of degree \( d \) consists of maps which are ergodic with respect to Lebesgue measure, and have invariant probability measures equivalent to Lebesgue.

**Proof.** — Write

\[
f_{\lambda}(z) = f(\lambda, z) = \lambda \frac{z-2}{z^d}, \quad \lambda > 0.
\]

Any rational map near \( \{ f_{\lambda} : |\lambda| > 1/2 \} \) can be written in the form \( (\lambda(z-2)^d + p(z))/(z^d + q(z)) \) for some \( \lambda \), where \( p, q \) are polynomials of degree \( \leq d-1 \). The critical points of \( f_{\lambda} \) are 2, 0, and \( f_{\lambda}(2) = 0, f_{\lambda}(0) = \infty, f_{\lambda}(\infty) = \lambda \). \( f_{\lambda_0}(\lambda_0) = \lambda_0 \) if \( ((\lambda_0 - 2)/(\lambda_0))^{d} = 1 \), that is, \( \lambda_0 = 2/(1-\zeta) \) for \( \zeta \) such that \( \zeta^d = 1, \zeta \neq 1 \). For such a \( \lambda_0 \), \( f_{\lambda_0}(\lambda_0) = 2d/(\lambda_0 - 2) = d(1-\zeta)/\zeta \). So \( \lambda_0 \) is an expanding fixed point of \( f_{\lambda_0} \) if \( d|1-\zeta| > 1 \), which is, in fact, always true if \( \zeta^d = 1 \) and \( \zeta \neq 1 \). To apply Theorem A we need to show:

\[
\left( \frac{\partial f}{\partial z} (\lambda_0, \lambda_0) - 1 \right) \frac{\partial f}{\partial \lambda} (\lambda_0, x_1) + \frac{\partial f}{\partial \lambda} (\lambda_0, \lambda_0) \neq 0
\]

for \( i = 1, 2 \) with \( x_1 = 2, r_1 = 3, x_2 = 0, r_2 = 2, f_2(\lambda, 2) = f_2(\lambda, 0) = \lambda \). So the non-degeneracy conditions become:

\[
\frac{\partial f}{\partial z} (\lambda_0, \lambda_0) - 1 + 1 \neq 0 \quad \text{(twice)}
\]

which is true. So Theorem A can be applied.

2. **Reducions**

2.1. It suffices to prove Theorem A for 1-dimensional families with \( \lambda_0 = 0 \in \mathbb{C} \). For we may write \( \Lambda = \bigcup_{v \in T_{\lambda_0}(\Lambda)} \Lambda_v \) with \( \lambda_0 \in \Lambda_v \) for all \( v \), where \( \Lambda_v \subseteq \Lambda \) is 1-dimensional having tangent \( v \) at \( \lambda_0 \). The non-degeneracy condition at \( \lambda_0 \) is satisfied for the family \( \Lambda_v \) if and only if \( DF_i(\lambda_0) v \neq 0 \). This is true for almost all \( v \) if \( DF_i(\lambda_0) v \neq 0 \). So Theorem A for 1-dimensional families and Fubini's Theorem imply Theorem A. For it is easy to see that the set of \( \lambda \) for which \( f_{\lambda} \) is ergodic is measurable (using the criterion for ergodicity given by Hopf's Theorem, for example.) By changing coordinates we may assume \( \lambda_0 = 0 \in \mathbb{C} \).
2.2. Replacing $f_i$ by $f_{i0}$, we may assume the $z_i(\lambda)$ are fixed expanding points, and that $r_i=1$ or 2 for all $i$. (Of course, this procedure will increase the number $m_0$.) For clearly $f_i$ is ergodic if $f_{i0}$ is, and if $\gamma$ is an invariant measure for $f_{i0}$ which is equivalent to Lebesgue, then so is $(f_i)^*\gamma$, and $(f_i)^*\gamma=\gamma$ by ergodicity of $f_{i0}$. So $\gamma$ is an invariant measure for $f_i$.

3. Elementary Estimates

We now show that, for $\lambda$ near $\lambda_0$, we have certain estimates on $f_{i\nu}$. Throughout, when talking about a rational map $f$ of $S^2$, $|f(x)|$ will denote $\lim_{y \rightarrow x} d(fx, fy)/d(x, y)$, and $d$ will denote the usual spherical metric. The measure used on $S^2$ will be spherical measure.

3.1. Estimates for $f_{0\nu}$. For $1 \leq i \leq m_0$, let $\phi_i$ be the local conjugacy between $f_0$ near $z_i=f_i(0)$ and its linear part; take $U_i=\phi_i^{-1}\{z \in \theta_i\}$ with $\theta_i$ small enough; let $W_i$ be the inverse of $f_0$ defined on $U_i$ such that $W_i(z_i)=z_i$; put $V_i=W_i(U_i)$ (for instance), $V=\bigcup_i V_i$, $U=\bigcup_i U_i$; let $T_i$ be the inverse of $f_{0\nu}'$ sending $z_i$ to $x_i$.

Put $X=S^2-\bigcup_{i=1}^{m_0} T_i V_i$. Then one has, for suitable $a_0$:

1. $|W_i(x)|<e^{-a_0}$, $x \in U_i$;
2. $|(f_{0\nu})^*|X|<C_0/2$ if $n \leq p_0$;
3. $|(f_{0\nu})^*|X|<1/4 C_0$ if $n \geq p_0$;
4. $|(T_i W_0^{*})'(x)|<(1/C_0^2) e^{-an}$ if $x \in U_i-V_i$, $n \geq p_0$.

Here, $f_{0\nu}^n$ is any branch of the inverse of $f_{0\nu}$, and $a_0>0$, $C_0>1$, $p_0 \geq 0$ are fixed. 2,3 follow from Montel's Theorem. 3 also uses the fact that expanding periodic points are dense in $S^2$. For a more detailed explanation of this see [F].

3.2. Estimates for $f_{i\nu}$. Keeping for $W_i$, $T_i$, $U_i$, $V_i$, $X$ the same meaning as before (with $f_i$ instead of $f_0$), take $\eta_0$ sufficiently small to have, for $|\lambda| \leq \eta_0$ (diminishing $a_0$ if necessary):

C1. $|W_i(z)|<e^{-a_0}$ for $z \in U_i$;

\[ W_i^\nu(U_i) \subseteq V_i \subseteq W_i^\nu(U_i), \quad W_i(U_i) \subseteq U_i; \]

C2. $|(f_{i\nu})^*|X| \leq C_0$ for $n \leq p_0$,

\[ \leq e^{-a_0} < \frac{1}{2 C_0} \quad \text{for} \quad p_0 \leq n \leq 3 p_0. \]

Also, given a positive integer $n_0$, if $\eta_0$ is sufficiently small, for $|\lambda| \leq \eta_0$ we have:

C3. critical points stay at least $2 n_0^3$ times in $U$;

C4. let $t_0, \ldots, t_n$ be points such that $t_{i+1}=f_i(t_i)$ and there exist $j \in [0, p_0]$, $k \in [n-p_0, n]$, $L \in [1, m_0]$ such that

\[ p_0 \leq k-j \leq n_0, \quad t_k \in U_L-V_L, \quad t_j=T_L W_L^{k-j-n}(t_k); \]
then for the inverse $S$ of $f_n^r$ which maps $t_n$ to $t_0$, we have $|S'(t_n)| < e^{-a_0 n}$.

C4 is an analogue of 3.1.4 for $f_0$ [using (2)].

It is important that all constants in all future estimates do not depend on $n_0$ (provided it is big enough).

3.3. Recall from the statement of Theorem A that the 1-parameter family $\{f_\lambda : |\lambda| < \eta_0 \}$ may have critical points $x_i(\lambda)$ of $f_\lambda$ emerging from critical points of $f_0$ of higher order. Thus, it may not be possible to bound below the distance between the critical points of $f_\lambda$. The following lemma describes the behaviour of the derivative of $f_\lambda$ and of $f_\lambda^r, f_\lambda^{-1}, f_\lambda^{-r}$ for small $r$, near the critical points.

**Lemma.** Let $F: \{z : |z| \leq B\} \to \mathbb{C}$ have derivative $F'(z) = A(z) \prod_{i=1}^{u} (z - z_i)$ with $|z_i| < B$, $A \neq 0$ in $|z| \leq B$. Then for some $C_1$ depending only $u, A, B$:

$$|F'(z)| \geq \frac{1}{C_1} \text{Min}_i |F(z) - F(z_i)|^{u+1},$$

and if $T$ is a multivalued inverse of $F$ defined on a ball of radius $r$ with connected image, $\text{Im} T$ has diameter $\leq (C_1/\eta_0) r^{1/u+1}$.

**Proof.** Let $|z| < B$ and suppose, for example, that $z_1$ is the critical point of $F$ nearest to $z$. Then, with $z_i = (1-t)z + tz_i$, $0 \leq t \leq 1$, we have, as $|z_i - z_j| \leq 2|z_0 - z_i|$ for $2 \leq i \leq u$:

$$|F'(z_i)| \leq C \prod_{i=1}^{u} |z_i - z_i| \leq C' \prod_{i=2}^{u} |z - z_i|.$$

Hence

$$|F(z) - F(z_i)| \leq \frac{1}{2} C' |z - z_1|^2 \prod_{i=2}^{u} |z - z_i|.$$

So

$$|F(z) - F(z_i)|^u \leq C'' |z - z_1|^{2u} \prod_{i=2}^{u} |z - z_i|^u \leq C' \prod_{i=1}^{u} |z - z_i|^{u+1} \leq C'' |F'(z)|^{u+1},$$

which gives the estimate on $|F'(z)|$. Since $T'(F(z)) = (F'(z))^{-1}$, we have $|T'w| \leq C_1 (\text{Min}_i |w - F(z_i)|)^{-u+1}$ for $w \in \text{Im} T$. If we take any straight line joining points on the boundary of $\text{Im} T$, then its image can be replaced by a path $l$ with the same endpoints, in the same homotopy class relative to the points $F(z_i)$, of length $\leq O(r)$, and with length $\leq O(\eta)$ within $\eta$ of any $F(z_i)$. $T(l)$ then has the same endpoints as
the original straight line in ImT. Then the length of \( T(I) \) is

\[
\int |T'w|d|w| \leq \text{Const.} \int_0^r \eta^{-u/w+1} d\eta \leq \frac{C_1 r^{1/u+1}}{C_0},
\]

enlarging \( C_1 \) if necessary.

Note. — It will be convenient to assume \( C_1 \geq C_0 \).

### 4. Inverses

**4.1. General Definition.** — Given \( \epsilon > 0 \), an inverse is determined by a sequence \( t_0 \ldots t_n \) of points such that \( t_{i+1} = f_\lambda(t_i) \) : it is the multivalued inverse of \( f_\lambda \) with connected image defined on the ball with centre \( t_n \) radius \( \epsilon \), which takes \( t_0 \) as a value at \( t_n \).

**Definition.** — Take \( \epsilon' \) such that any ball \( B \) of radius \( 2\epsilon' \) cutting \( V \) and not contained in \( V \) satisfies \( B \subseteq U \) and \( f_\lambda(B) \subseteq U \setminus V \).

**4.2. Lemma.** — Let \( |\lambda| \leq \eta_0 \), so that \( f_\lambda \) satisfies C1-C4. There exists \( a_1 > 0 \), and, given \( \epsilon' < \eta_0 \), \( q_0 \geq p_0 \) with the following property: let \( \epsilon = (\epsilon'/C_1)^{1} + \eta \) (C1 as in 3.3), \( t_0 \ldots t_n \) be a sequence of points with \( f_\lambda(t_i) = t_{i+1} \) and no \( n_0^2 \) consecutive points \( U, S \) be as above, provided \( B_\epsilon(t_n) \subseteq X \) and radius \( \epsilon \).

Then one has:

(a) \( SB_\epsilon(t_n) \subseteq B_\epsilon(t_0) \);

(b) if \( n \geq q_0 \), \( SB_\epsilon(t_n) \subseteq B_\epsilon(t_0) \) where \( \epsilon_n = \epsilon \exp(-a_1 n) \), \( \epsilon_{q_0} < \epsilon_1/2 \);

(c) if \( B_\epsilon(t_n) \subseteq X \) and \( n \geq p_0 \), \( S \) is univalued and \( |S'| \leq \exp(-a_1 n) \) on \( B_\epsilon(t_n) \).

**Proof.** — If \( B_\epsilon(t_n) \subseteq X \) and \( n \leq 3p_0 \), the lemma follows from C2. Next, suppose \( B_{2\epsilon}(t_n) \subseteq U \) and \( S = S_1 T_L W_L^m \) with length \( S_1 = j \leq 3p_0 \). Then \( |(W_L^m)'| \leq e^{-a_0 m} \) on \( B_\epsilon(t_n) \). So by 3.3, \( T_L W_L^m S_1 \subseteq B_\epsilon(t_n) \) where \( \alpha = (C_1/C_0) e^{1/1+w} e^{-a_0 m/1+w} \) and, also, since \( |S_1'| \leq 0 \) on \( B_\epsilon(t_n) \), \( SB_\epsilon(t_n) \subseteq B_\epsilon(t_0) \).

Also, if \( B_{2\epsilon}(t_n) \subseteq U \setminus V \) and \( m \leq n_0^2 \), then \( f_\lambda(y) \notin B_{2\epsilon}(t_n) \) for \( p \leq n_0^2 \) because \( f_\lambda(y) \in U \) for \( p \leq 2n_0^2 \). So \( S \) is univalued on \( B_\epsilon(t_n) \) and by Schwarz's Lemma

\[
|S'| \leq C_2 \frac{\epsilon^n}{\epsilon} = C_2 C_1 e^{-u/w+1} e^{-a_0 m/1+w}
\]

on \( B_\epsilon(t_n) \). So the lemma is proved if \( B_{2\epsilon}(t_n) \subseteq U \) and \( S \) is as above, provided \( q_0 \), \( a_1 \) are suitably chosen, and we use C4 to bound \( |S'| \) if \( p_0 \leq m \leq n_0 \).

If length \( S \geq 3p_0 \), we use the inductive hypothesis. We can write \( S \) either as \( S_1 S_2 \) where \( j = \text{length } S_1 \), \( p_0 \leq j \leq 3p_0 \) and \( B_\epsilon(t_n) \subseteq X \), or as \( S_1 T_L W_L^m S_2 \) with \( j = \text{length } S_1 \), \( j < p_0 \), \( m \geq p_0 \), length \( S_1 = n - k \) and \( B_{2\epsilon}(t_k) \subseteq U \setminus V \) or \( S_2 = \text{identity} \). This last possibility has already been dealt with. For \( S = S_1 S_2 \) we have \( |S_1'| < e^{-a_0 j} \), completing the proof. For \( S = S_1 T_L W_L^m S_2 \) we have \( |S'| \leq e^{-a_0 j} \) if length \( S_2 \leq p_0 \) and \( m \leq n_0 \) by C4, and otherwise from the earlier estimate we have

\[
|(S_1 T_L W_L^m)'| < e^{-a_1 (k + p_0)} C_0^{-1},
\]
which is \( \leq e^{-a_n} C_0^{-1} \) if length \( S_2 = n - k \leq p_0 \), for suitable \( a_1 \). If length \( S_2 \geq p_0 \) we use the inductive estimates on \( S_2 \) to complete the proof.

4.3. Now let

\[
2 \varepsilon_0 = \left( \frac{\varepsilon_0'}{2 C_1} \right)^{1+u}, \quad 2 \varepsilon_1 = \left( \frac{\varepsilon_0}{2 C_1} \right)^{1+u}
\]

Then

\[
SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_0}(t_0),
\]

\[
SB_{2\varepsilon_1}(t_n) \subseteq B_{\varepsilon_0/2}(t_0)
\]

for any \( n \) and any inverse \( S \). (If length \( S \geq n_0 \), this follows from writing \( S \) as a composition of inverses length \( \leq n_0 \) and \( \geq q_0 \).)

There is \( q_0 \) so that

\[
SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_1/2}(t_0) \quad \text{for } n \geq q_0,
\]

any inverse \( S \), and

\[
SB_{2\varepsilon_0}(t_n) \subseteq B_{\varepsilon_1 \exp(-a_1 n)}(t_0) \quad \text{for } n \geq q_0,
\]

\( S \) as in 4.2.

4.4. Critical Inverses. — Critical inverses are those with \( \varepsilon = \varepsilon_0 \), and where \( t_0 \) is a critical point.

An interval \([k, k+m]\) (with \( k > r_0 \)) is an \( x_i \)-follower if the image of the inverse defined by the sequence \( f_1'(x_i) \ldots f_1^{k+m}(x_i) \) (and \( \varepsilon = \varepsilon_0 \)) contains some critical point \( x_j (1 \leq j \leq m_0) \). Note one necessarily has \( k \geq 2 m_0^2 \).

Definition. — Let \( |\lambda| \leq \eta_0 \). \( \lambda \) is \((N, i, \alpha)\)-good if the number of integers \( L \in [0, N] \) which belong to an \( x_i \)-follower of length \( \geq \eta_0 \) is less than \( \alpha N \).

4.5. From now on, assume that if \( |\lambda| \leq \eta_0 \) then \( f_k \) satisfies C1-C4 and \( f_0 \) has all critical points non-periodic, but with finite forward orbits, and that the non-degeneracy condition is satisfied (although this is only needed for Theorem C). Now Theorem A can be replaced by Theorems B and C.

Theorem B. — There exist \( \alpha_0 > 0, \eta_0 > 0 \) such that if \( f_k \) is \((N, i, \alpha_0)\)-good for all \( N > 0, i \leq m_0 \) and if \( |\lambda| < \eta_0 \) then \( f_k \) is ergodic with respect to Lebesgue measure and there is an \( f_k \)-invariant probability measure equivalent to Lebesgue.

Theorem C. — For any \( \alpha_0 > 0, \eta_0 > 0 \), the set of \( \lambda \) such that \( |\lambda| < \eta_0 \) and \( f_k \) is \((N, i, \alpha_0)\)-good for all \( i \leq m_0, N > 0 \) has positive Lebesgue measure.

4.6. General Inverses. — A general inverse is as in 4.1 with \( \varepsilon = \varepsilon_1 \).

Followers for general inverses. — Let \( S \) be a general inverse determined by a sequence \( t_0 \ldots t_n \). Then \([k, k+L] \subseteq [0, n] \) is a follower (for \( S \)) if some \( x_i \in \text{Im} \mathcal{S} \) where \( \mathcal{S} \) is the
general inverse determined by $t_k, \ldots t_{k+L}$. We shall also call $S$ itself a follower. For any $0 < p < L$, we shall say $[k, k+p]$ is at the left end of the follower and $[k+p, k+L]$ is at the right end.

Elementary properties of followers. — 1. Let $t_0, \ldots t_n$ be a sequence of points with $t_{i+1} = f_i(t_i)$, $S$ the corresponding inverse and $[k, k+m]$ a follower for $S$ (with critical point $x_j$). One has:

$$d(f^L_k(x_j), t_{k+1}) \leq \frac{\epsilon_0}{2}, \quad 0 \leq L \leq m,$$

$$d(f^L_k(x_j), t_{k+L}) \leq \frac{\epsilon_1}{2}, \quad 0 \leq L \leq m - q_0.$$

2. If $S$ is some general inverse, and $[k, k+L+m]$ is a follower of $S$ (with critical point $x_j$) and $[k+L, k+L+p]$ is another follower of $S$ (with $L > r_j$) then $[L, L + \inf (m, p)]$ is an $x_j$-follower. This results from 1.

3. Let $[k, k+m]$ be a follower for some general inverse $S$. Then no other follower for $S$ can start in $[k+2, k+\min(m,n^2)]$. This follows from 2, and the fact that the forward orbit of $x_j$ stays in $U$ for $2n^2$ iterates.

4.7. CANONICAL DECOMPOSITION OF AN INVERSE. — In the canonical decomposition $S = S_0 T_1 \ldots S_n$, the $S_i, T_i$ have the properties explained below.

Let $\lambda$ be $(n, i, \alpha)$-good for $n \leq N$, $i \leq m_0$. Let $S$ be a general inverse of length $n \leq N$ determined by $t_0, \ldots t_n$. Consider all followers $[k, k+L]$ for $S$ such that

1. $t_{k-1} \in X$, if $k > 0$.
2. $[k, k+L]$ is maximal, that is, no $[k, k+p]$ is a follower for $p > L$.
3. $n-k-L < \alpha^{1/2} L$.

If $[k_1, k_1 + L_1], [k_2, k_2 + L_2]$ are two such with $k_1 < k_2$, then $k_2 - k_1 \geq 2 n_0^2$.

Case 1. — If $k_1 + L_1 \leq k_2 + L_2$, then by 4.6.2, $[k_2 - k_1, L_1]$ is an $x_j$-follower for some $i$. So $L_1 - (k_2 - k_1) \leq \alpha L_1$, and:

$$n - k_1 \leq (1 + \alpha^{1/2}) L_1 \leq (1 + \alpha^{1/2}) (k_2 - k_1)/(1 - \alpha),$$

$$n - k_2 \leq (1 - (1 - \alpha)/(1 + \alpha^{1/2})) (n - k_1) = \alpha^{1/2} (n - k_1).$$

Case 2. — If $k_2 + L_2 \leq k_1 + L_1$, then $L_2 \leq \alpha (L_2 + k_2 - k_1)$.

So

$$n - k_2 \leq L_2 (1 + \alpha^{1/2}) \leq \alpha (1 + \alpha^{1/2}) ((n - k_2)/(1 - \alpha),$$

and

$$n - k_2 \leq \alpha (1 + \alpha^{1/2}) (n - k_1)/(1 + \alpha^{3/2}) \leq 2 \alpha (n - k_1).$$

So now, if $[k_i, k_i + L_i] (1 \leq i \leq r)$ are all the followers for $S$ satisfying 1–3 with $k_i < k_{i+1}$, then since $n - k_i \geq 1$,

$$(r-1) \log (2 \alpha^{1/2}) + n \geq 0, \quad \text{and} \quad r \leq 1 + 2 \log(n)/\log(1/\alpha).$$
Let \( T_i \) be the inverse determined by \( t_{k_i} \ldots t_{k_i+r_i} \), mapping a neighbourhood of some \( z_j(\lambda) \in V_j \) to a neighbourhood of \( x_j(\lambda) \in X \). Then we can write \( S = S_0 T_1 \ldots T_i S_i \) (with possibly \( S_0 \) or \( S_i \) = identity). We may have to consider some \( S_{r_i} T_i \) to be defined on balls of radii \( \epsilon_0 \), but with this modification the composition \( S_0 T_1 \ldots S_i \) is well-defined on \( B_{t_n}(t_n) \) by 4.2, 4.3.

### 5. Fundamental estimates for inverses

In this section, the estimates work provided \( \alpha \) is small enough and \( n_0 \) is large enough. Let \( |\lambda| \leq n_0 \) and \( \lambda \) be \((m, i, \alpha)\) good for \( m \leq n \) and \( i \leq m_0 \).

5.1. There is \( a_2 > 0 \) so that if \( S \) is a critical inverse of length \( n \geq q_0 \) determined by \( t_0 \ldots t_n \) and radius \( \epsilon_0 \), then \( \text{Im} \, S \) is contained in the ball radius \( e^{-a_2 n} \epsilon_0 \) round to \( t_0 \).

This follows from 4.2 if \( n < q_0 \).

Now let \( n \geq n_0^2 \). Let \( J \) be the union in \([0, n]\) of followers of length \( \geq n_0 \). Write
\[
J = \bigcup_{i=1}^{k} [c_i, d_i] \quad \text{with} \quad d_i - c_i \geq n_0, \quad c_{i+1} > d_i + 1.
\]

For \( 1 \leq i \leq k \), let \( J'_i \) be the union of \([c_i, d_i]\) and of the followers (of length \( < n_0 \)) starting in \([c_i, d_i]\). Put \( J' = \bigcup J'_i \). Then \#\( J' \) \( \leq 2 \# J \).

Write \( J' = \bigcup_{i=1}^{L} [c'_i, d'_i] \) with \( d'_i - c'_i \geq n_0, \quad c'_{i+1} > d'_i + 1 \).

Let \( J'' \) be the union of \( J' \) and those intervals of \([d'_1 + 1, c'_2], \ldots, [d'_L + 1, n]\) of length \( \leq q_0 \). (Note that \( c'_1 = c'_1 \geq 2n_0^2 \).)

Write \( J'' = \bigcup_{i=1}^{m} [c''_i, d''_i] \) with \( c''_1 = c'_1 = c'_1 \geq 2n_0^2, \quad d''_i - c''_i \geq n_0, \quad c''_{i+1} - d''_i > q_0, \quad d''_m = n \) or \( n - d''_m \geq q_0 \). Then \#\( J'' \) \( \leq 3 \# J \) (if \( q_0 \leq n_0 \)).

By construction (4.6.3) \( d''_i + 1 \) is not contained in any follower (for \( 1 \leq i < m \), and \( i = m \) if \( d''_m \neq n \)).

If \( t_0 \ldots t_n \) is the sequence associated to \( S \) (with \( t_0 \) a critical point) denote by \( S_1 \) the inverse associated to the subsequence \( t_0, \ldots, t_{c'_1}, \) by \( S_1 i_{i-1}, \ldots, t_{c'_i}, \) by \( S_2 i_{i-1} \ldots (1 \leq i \leq m) \) the inverse associated to \( t_{c''_i}, \ldots, t_{d''_i+1}, \) and, if \( d''_m \neq n \), by \( S_{m+1} \) the inverse associated to \( t_{c''_m}, \ldots, t_n \).

So \( S = S_1 \ldots S_{2m+1} \) (perhaps with \( S_{2m+1} = \text{id} \)).

For \( 1 \leq i \leq m \) (and \( i = m \) if \( d''_m \neq n \)) \( S_{2i} \) is a univalued function on the disc of radius \( \epsilon_0 \) centred at \( t_{c''_i+1} \) (because \( d''_i + 1 \) does not belong to any follower), with values in the disc of radius \( \epsilon_0/2 \) centred at \( t_{c''_i} \). By Schwarz's Lemma, a disc of radius \( \epsilon \leq \epsilon_0 \) centred at \( t_{c''_i} \) has its image by \( S_{2i} \) contained in the disc of radius \( \epsilon/2 \) centred at \( t_{c''_i} \).

For \( 1 \leq i \leq m \), if we take the domain of \( S_{2i-1} \) to be \( B_{\epsilon_0}(t_{c''_i}) \), then since \( c''_i \) is the starting point of a follower and \( c''_i - 1 \) is not in a follower, \( t_{c''_i} \) is near a critical point in \( X \) and...
Writing $S_i = T S_i$ with length $T = r_i$ if $t_0 = x_i(\lambda)$, one gets $|S_1| \leq \exp(-a_1 n_1)$ with $n_1 = \text{length } S_1$.

If $d_{r}^n = n$, the image by $S_{2,m}$ of the $\varepsilon_0$-ball centred at $t_n$ is contained in the $\varepsilon_1/2$-ball centred at $t_{n-1}$. If $d_{r}^n \neq n$, the image by $S_{2,m+1}$ of the $\varepsilon_0$-ball centred at $t_n$ is contained in the ball centred at $t_{n+1}$ and radius $\varepsilon_1 \exp(-a_1 n_m)$ where $n_m = \text{length } (S_{2,m+1})$.

One concludes that the image by $S_1 S_2 \ldots S_{2,m+1}$ of $B_{t_0}(t_n)$ is contained in the ball centred at $t_j (j = r_i)$ if $t_0 = x_i(\lambda)$ of radius $\varepsilon_1 \exp\left(-a_1 \sum_{i=1}^{m} n_i \right)$. Since $\sum_{i=1}^{m} n_i \geq (1-3\varepsilon)n$, applying Lemma 1 to $T$, we obtain that $SB_{t_0}(t_n)$ is contained in the ball of radius $\varepsilon_0 \exp(-a_2 n)$ round $t_0$ for suitable $a_2 > 0$.

5.2. There is $a_3 > 0$ such that if $S$ is a general inverse of length $n \geq n_0$ determined by $t_0 \ldots t_n$ and radius $\varepsilon, \varepsilon_0$, then $\text{Im } S$ is contained in the ball radius $\varepsilon_0 \exp(-a_3 n)\varepsilon_1$, round $t_0$.

1. If $n < n_0$ this follows from 4.2.

Now let $n \geq n_0$.

2. If there is $k \in [n_0, n_0^2 - q_0]$ such that $B_{2,k}(t_k) \subseteq X$, write $S = S_1 S_2$ with $S_1$ determined by $t_0 \ldots t_k$. Then $S_2 B_{1}(t_k) \subseteq B_{1}(t_k)$ where $\varepsilon = \exp(-a_3 (n-k))\varepsilon_1$ by the inductive hypothesis, since $n-k \geq q_0$, and $|S_1| \leq \exp(-a_1 k)$ by 4.2.

3. If $S = S_1 S_2 S_3$ where length $S_1 \leq n_0$, length $S_2 \geq (1/2) n$ and $S_2$ is a follower, then the image of $S_3$ is contained in the $\varepsilon_0$-ball which is the domain of the critical inverse associated to $S_2$. So $\text{Im } S_2 S_3$ is contained in the ball radius $\varepsilon$ round $t_k$ with $\varepsilon = \varepsilon_0 \exp(-1/2a_2 n)$, if $k = \text{length } S_1$. We can also have $S_2$ so that $S_1$ = identity or $B_{e_0}(t_k) \subseteq X$, since $t_k$ is near a critical point. So $|S_1| \leq C_0$ and $\text{Im } S$ is contained in the ball radius $C_0 \varepsilon_0 \exp(-1/2a_2 n)$ round $t_0$. Since $n \geq n_0^2$ we can assume this is $\leq \varepsilon_1 \exp(-a_3 n)$ for suitable $a_3$ and $n_0$ large enough.

4. If none of the above is possible then we can write $S = S_1 S_2 S_3$ where length $S_1 \leq n_0$, $S_2$ is a maximal follower and $n_0^2 - q_0 - p_0 \leq \text{length } S_2 < (n/2)$.

Let $m = \text{length } S_2$, $t_k \ldots t_{k+m}$ the sequence corresponding to $S_2$.

Put $m' = [\alpha m] + 1$, and let $S_3$ be the inverse corresponding to $t_{k+m-m'} \ldots t_{n}$. By the inductive hypothesis, $\text{Im } S_3$ is contained in the ball of radius $\varepsilon_1 \exp(-a_3 (n-k-m'))$. As $S_2$ is maximal and $\lambda$ is $(N, i, \varepsilon)$-good, from 4.7 any follower containing $k + m + 1$ has left end point $> k + m - m'$.

Let $S_4$, $S_5$ be the general inverses with corresponding sequences $t_k \ldots t_{k+m-m'}$ and $t_{k+m-m'} \ldots t_{k+m+1}$ respectively. There is $b_2$ such that $|f_i| \leq \exp(b_2)$. Then the image by $S_2$ of the $\varepsilon_1$-ball $B_0$ centred at $t_{k+m+1}$ contains the ball $B_1$ centred at $t_{k+m-m'}$ of radius $(\varepsilon_1 \exp(-b_2 (m'+1)))$. So the restriction of $S_4$ to $B_1$ is univalued. By the estimate for critical inverses, the image of $B_0$ by the inverse corresponding to $t_{k+m+1}$ is contained in the ball of centre $t_k$, radius $e_0 \exp(-a_2 m)$, hence this is also the case for the image of $B_1$ by $S_4$. By Schwarz's Lemma, we obtain that the image of the $\varepsilon_1$-ball
centred at \( t_n \) by \( S_2 S_3 = S_4 S_3 \) is contained in the ball centred at \( t_k \) of radius
\[
\frac{\varepsilon_0 \exp(-a_2 m)}{\varepsilon_1 \exp(-b_2 (m' + 1))} \cdot \varepsilon_1 \exp(-a_3 (n - (k + m - m'))
\]

[provided \( \alpha \) is small enough to have \( b_2 (m' + 1) < a_3 (n - (k + m - m')) \)].

If we had \( 0 < a_3 < a_2 - b_2 \alpha \) (and therefore \( \alpha \) small enough) and \( n_0 \) big enough, we get
the desired estimates (as \( |S'_n| < C_0 \)).

5.3. There is a constant \( C > 0 \) such that if \( S = S_0 T_1 S_1 \ldots S_r \) is the canonical decomposition, then
\[
\sum_{i=0}^r \text{Var} \log |S'_i|_0 T_{i+1} \ldots \leq C. \quad C \text{ depends on } \alpha \text{ but not on } n_0.
\]
(The earlier constants \( a_2, a_3 \) did not depend on \( \alpha \).)

Let \( S \) correspond to the sequence \( t_0 \ldots t_n \).

1. Assume first that there exists \( k < n \) such that:
   (i) \([0, k] \) is a maximal follower,
   (ii) \( n - k \geq \alpha^{1/2} k \) (equivalently \( k \leq n/(1 + \alpha^{1/2}) \)).

\( S \) is then of the forms \( T S \) with length \( T \leq 2 \) and we want to estimate the variation of
\( \log |T'| \) on the image \( B \) by \( S \) of \( B_{k_1}(t_k) \).

(a) \( k \leq n_0 \). Let \( S_1 \) be the inverse corresponding to \( t_0 \ldots t_k \), \( S_2 \) the inverse corresponding
to \( t_{k-1} \ldots t_n \) and write \( S_1 = T S_1 = T_j W^j - r_j \). Let \( B = B_{k_1}(t_k) \). Then \( W^j - r_j \) and \( T_j W^j - r_j \) are
univalent functions on \( B \). The image \( B' \) by \( S_2 \) of \( B_{k_2}(t_k) \) is contained in \( B_{(1/2)k_0}(t_k) \) and
is contained in the ball centred at \( t_k \) of radius \( \varepsilon_1 \exp(-a_3 (n - k)) \) if \( n - k \geq q_0 \). So using
the distortion theorem for univalent functions (cf. [D] for instance) one gets
\[
\text{Var}_B \log |(W^j - r_j)'| \leq C', \quad \text{Var}_B \log |(T_j W^j - r_j)'| \leq C'
\]
and if \( n - k \geq q_0 \),
\[
\text{Var}_B \log |(W^j - r_j)'| \leq C' \exp -a_3 (n - k), \quad \text{Var}_B \log |(T_j W^j - r_j)'| \leq C' \exp -a_3 (n - k).
\]

Hence
\[
\text{Var}_B \log |(T_j)'| \leq 2 C', \quad \text{Var}_B \log |(T_j)'| \leq 2 C' \exp \left( -a_3 \frac{\alpha^{1/2} n}{1 + \alpha^{1/2}} \right)
\]
if \( n - k \geq q_0 \), which occurs as soon as \( n \geq ((1 + \alpha^{1/2})/\alpha^{1/2}) q_0 \).

(b) \( k \geq n_0 \). Let \( S_1, S_2 \) have the same meaning as in (a). Put \( k' = \lfloor \alpha/2 \rfloor + 1 \). Let
\( S_4 = T S_4 \) be the inverse corresponding to \( t_0 \ldots t_{k-k'} \). By arguments similar to 5.2.4.,
\( S_4 \) is univalent functions on the disc centred at \( t_{k-k'} \) of radius \( \varepsilon_1 \exp -b_2 (k'+1) \).
On the other hand, if \( S_3 \) is the inverse corresponding to \( t_{k-k'} \ldots t_n \), the image \( B \) by \( S_3 \)
of the \( \varepsilon_1 \) ball centred at \( t_n \) is contained in the ball centred at \( t_{k-k'} \) of radius
\[ \varepsilon_1 \exp (-a_3 (n-k+k')) \] (provided \( n_0 \) is big enough).

Now one has:
\[
\begin{align*}
a_3 (n-k+k') - b_2 (k'+1) & \geq a_3 (n-k) - 2 b_2 (k'-1) \\
& \geq \frac{a_3}{2} (n-k) + k \left( a_3 \frac{\alpha^{1/2}}{2} - 2 b_2 \alpha \right) \geq \frac{a_3}{2} \frac{\alpha^{1/2}}{1+\alpha^{1/2} n}.
\end{align*}
\]
if \( \alpha \) is small enough.

Using the distortion theorem we get:
\[
\begin{align*}
\text{Var}_B (\Log |S'_4|) & \leq C' \exp \left( - \frac{a_3}{2} \frac{\alpha^{1/2}}{1+\alpha^{1/2} n} \right), \\
\text{Var}_B (\Log |\bar{S}_4|) & \leq C' \exp \left( - \frac{a_3}{2} \frac{\alpha^{1/2}}{1+\alpha^{1/2} n} \right),
\end{align*}
\]
hence
\[
\text{Var}_B (\Log |T'|) \leq 2 C' \exp \left( - \frac{a_3}{2} \frac{\alpha^{1/2}}{1+\alpha^{1/2} n} \right).
\]

2. In the general case, if \( S = S_0 \circ T_1 \circ S_1 \ldots \) is the canonical decomposition of \( S \), one obtains from 1 the estimate
\[
\sum_{i=0}^r \text{Var} \Log |S_i| \circ T_{i+1} \circ \ldots \circ S_r \leq C
\]
for suitable \( C \).

For write \( S_i = U_i^1 \ldots U_i^\ell_i \), where \( U_i^j \) has length 1 or 2 and is the end of a follower, and \( U_i^{j+1} \) are univalent inverses of \( f_i \). Then 1 gives an estimate on \( \text{Var} \Log |(U_i^j)'| \), and \( \text{Var} \Log |(U_i^{j+1})'| \leq C' \exp (-a_3 (n-m)) \) by univalent function theory, if \( t_m \) is the centre of domain \( (U_i^{j+1}) \).

6. Proof of Theorem B

6.1. LEMMA. — Let \( F'(z) = A(z) \prod_{i=1}^u (z-z_i) \) for \( |z| \leq r \), where \( |z_i| < r \) and \( |A(z)| > 0 \) for \( |z| \leq r \). Let \( B \) be a ball of radius \( \delta \) in the image of \( F \), and let \( T \) be a multivalued inverse of \( F \) on \( B \) with connected image. Then:
1. If \( w, w' \in B \) with \( \min_i |w-F(z_i)| = C_2 \delta \) and \( \min_i |w'-F(z_i)| = \eta \delta \), then
\[
|T'w'/T'w| \leq C_3 \eta^{-\omega u+1}, \text{ where } C_3 \text{ depends only on } C_2, r, A.
\]
2. \( TB \) has diameter between \( C_3 \min_{z \in B} |T'z| \delta \) and \( (1/C_3) \min_{z \in B} |T'z| \delta \).
Proof. — We may assume all \( F(z) \) are distance \( O(\delta) \) from the boundary of \( B \) [by restricting \( B \) to a smaller ball if necessary, but still of radius \( O(\delta) \)]. We know that \( |T'| \) varies by a bounded proportion on a set of points in \( B \) which are distance \( O(\delta) \) from all \( F(z) \) (since this set is contained in a union of larger balls on each of which \( T \) is univalued), in particular on a neighbourhood of the boundary of \( B \) of width \( O(\delta) \).

Let \( \varepsilon \) be the diameter of \( \text{Im} T \). Let \( F(z_0) = w, F(z_0') = w' \). We first consider \( |F'(z_0)|/|F'(z_0')| \) for \( z_0 \) at a distance \( O(\varepsilon) \) from the boundary of \( \text{Im} T \). In fact, we shall see that this is sufficient.

(A) Let \( z, z' \in \text{Im} T \). Let \( |y - z'| \geq O(|z - z'|) \) for any \( y \in F^{-1}F(z) \cap \text{Im} T \). Then
\[
|F(z) - F(z')| \geq O(|F'(z)| |z - z'|).
\]

For let \( I \) be a connected path with one endpoint at \( z' \) and mapping onto the straight line segment between \( F(z) \) and \( F(z') \). Then the endpoints of \( I \) are at distance \( \geq O(|z - z'|) \) apart, and \( |F(z) - F(z')| = \int_I |F'(y)| dy \). But \( |F'(y)| \geq O(|F'(z)|) \) if \( |y - z_i| \geq O(|z - z_i|) \) for all \( i \), and this is true on a segment of \( I \) of length \( \geq O(|z - z'|) \), giving (A).

Thus in particular:

(B) if \( z' \in \text{Im} T \) is at a distance \( O(\varepsilon) \) from \( \partial \text{Im} T \), then
\[
\delta \geq |F(z') - \partial B| \geq O \left( \max_{w \in \partial B} |F'(w)| \right) \geq O \left( \max_{w \in B} |F'(w)| \right) \geq O(\delta).
\]

This proves 2. If there is no \( z_i \in \text{Im} T \) distance \( O(\varepsilon) \) from the boundary then \( |F'(z_0)|/|F'(z_0')| \) is automatically bounded. If there is at least one \( z_i \) in \( \text{Im} T \), let \( z_1 \) be nearest to \( z_0 \). If some \( z_0' \) in \( \text{Im} T \cap F^{-1}F(z_0) \) has \( |z_0' - z_1| \leq |z_0 - z_1| \) then \( |F'(z_0')| \leq O(|F'(z_0)|) \) and we replace \( z_0 \) by \( z_0' \). We can then still assume \( z_0 \) is at a distance \( O(\varepsilon) \) from \( \partial \text{Im} T \). Now \( |z_0' - z_1| \leq O(|z_0 - z_1|) \) for all \( z_i \) (not just \( z_i \) in \( \text{Im} T \)). We may also assume \( z_0 \) is the nearest element in \( F^{-1}F(z_0) \cap \text{Im} T \) to \( z_1 \), since \( F(z_0) \) is at a distance \( O(\delta) \) from all \( F(z_i) \), and hence \( |F'(z_0)|/|F'(z)| \) is bounded above and below for \( z \in F^{-1}F(z_0) \). This gives:
\[
|F(z_1) - F(z_0)| \geq O \left( \max_{z \in \text{Im} T} |F'(z)| \right) |z_1 - z_0| \quad \text{by (A)}.
\]

Also \( |z_1 - z_0| \geq O(|z_1 - z_0'|) \). For
\[
|F'(z_0')| |z_1 - z_0'| \leq O \left( |F(z_1) - F(z_0')| \right) \quad \text{[by (A)]} \leq O \left( |F(z_1) - F(z_0)| \right) \leq O \left( \max_{z \in \text{Im} T} |F'(z)| \right) \quad \text{[since \( F'(z_0) = O(\max_{z \in \text{Im} T} |F'(z)|) \)].}
\]

So if
\[
|z_1 - z_0| \leq |z_1 - z_0'| \quad \text{then} \quad |z_1 - z_0| \leq |z_1 - z_0'| + |z_0 - z_1| + |z_1 - z_0| \leq O \left( |z_1 - z_0'| \right),
\]
and \( |F'(z_0)| \leq O \left( F'(z_0') \right) \), which implies
\[
|F'(z_0)| |z_1 - z_0| \leq O \left( |F'(z_0)| \right) |z_1 - z_0|.
\]

So \( |z_1 - z_0| \geq O \left( |z_1 - z_0'| \right) \) as required.
Then
\[ \frac{|z_i - z_0|}{|z_i - z'_0|} \leq O\left(\frac{|z_1 - z_0|}{|z_1 - z'_0|}\right) \quad \text{for all} \quad i, \]

since
\[ |z_i - z_0| \leq |z_i - z'_0| + |z'_0 - z_1| + |z_1 - z_0| \leq 2|z'_0 - z_1| + |z_1 - z_0|. \]

Then
\[ \frac{|F'(z_0)|}{|F'(z'_0)|} \leq O\left(\prod_{i=1}^{u} \frac{|z_0 - z_i|}{|z'_0 - z_i|}\right) \]
\[ \leq O\left(\left(\prod_{i=1}^{u} \frac{|z_0 - z_i|}{|z'_0 - z_i|}\right)^{u(u+1)}\right) \]
\[ \leq O\left(\left(\frac{|F'(z_0)|}{|F'(z'_0)|}\right)^{u(u+1)}\right) \leq O\left(\frac{|F(z_0) - F(z_i)|}{|F(z'_0) - F(z_i)|}\right)^{u(u+1)} \]

(using that \(|F'(z_0)|\) is proportional to the maximum of \(|F'(z)|\) on the line segment from \(z'_0\) to \(z_1\) for the denominator)
\[ \leq O\left(\left(\frac{\delta}{\eta \delta}\right)^{u(u+1)}\right) = O\left(\eta^{-u(u+1)}\right). \]

Thus, a set of points at a distance \(O(\delta)\) from \(\partial B\) in \(B\) has image under \(T\) at a distance \(O(\epsilon)\) from \(\partial \text{Im } T\), so the proof is completed.

6.2. Lemma. — Let \(S = S_0 T_1 S_1 \ldots S_r\) be defined on a ball \(B\). Let \(X\) be any subset of \(B\). For a constant \(D\),
\[ \text{Meas}(S X) \leq D^r \text{Meas}(\text{Im} S) (\text{Meas} X)^{(1 + u)^{-r}}. \]

Proof. — Prove inductively on \(r\) that \(\text{Im}(S_0 T_1 \ldots S_r)\) contains a ball of radius \(\delta\), is contained in a ball of radius \(D_1 \delta\), and that
\[ \text{Meas}(S_0 \ldots S_r X) \leq E_r \delta^2 (\text{Meas}(X))^{(1 + u)^{-r}}, \]
where \(E_r = D_2 E_{r-1}^{1/u+1}\), so that \(E_r \leq E'\) where
\[ \log E \geq \left(1 + \frac{1}{1 + u} + \ldots + \frac{1}{(1 + u)^{r-1}}\right) \log D_2. \]

So assume \(\text{meas}(S_1 \ldots S_r X) \leq E_{r-1} \delta_1^2 (\text{Meas}(X))^{(1 + u)^{-r+1}}\), where \(S_1 \ldots S_r B\) contains a ball of radius \(\delta_1\) and is contained in a ball of radius \(D_1^{-1} \delta_1\). Since \(|S_0|\) varies by a bounded proportion on \(T_1 \ldots S_r B\), it suffices to prove the inductive result for \(T_1 \ldots S_r\). The result will then be true for \(S_0 \ldots S_r\) for \(D_1, D_2\) sufficiently large independent of \(r\).
Write $X_i = S_1 \ldots S_i X$ and let $B_1, B_2$ denote respectively the ball of radius $D_i^{-1} \delta_i$ containing $S_1 \ldots S_i B$ and the ball of radius $\delta_i$ contained in $S_1 \ldots S_i B$.

Take any $z_0 \in B_1$ with $\min_i |z_0 - z_i| = O(D_i^{-1} \delta_i)$, where $z_i (i \leq u)$ are the singularities of $T_i$. We know $|T_i(z_0)|$ is boundedly proportional to $|T_i(z)|$ for any $z \in B_1$ with $\min_i |z - z_i| = O(D_i^{-1} \delta_i)$.

By 6.1 $T_i B_1$ has diameter $\leq \text{Const. } D_i^{-1} \delta_i |T_i(z_0)|$, and $T_i B_2$ contains a ball of radius $\delta = \text{Const. } \delta_i |T_i(z_0)|$.

Then

$$\text{meas}(T_i X_i) = \int x_i (T_i^{-1} z) d |z|^2 = \int x_i (z) |T_i(z)|^2 d |z|^2$$

$$\leq \text{Const. } \int x_i (z) |T_i(z)|^2 \left( \min_i |z - z_i| \right)^{-1} \left( \min_i |z_0 - z_i| \right)^{-1} d |z|^2$$

(by 6.1) where, for some $\rho$, $X_2 = \{ z : \min_i |z - z_i| < \rho \}$ and $\text{meas}(X_2) = \text{meas}(X_i)$.

So

$$\text{meas}(T_i X_i) \leq \text{Const. } \left| T_i(z_0) \right|^2 (D_i^{-1} \delta_i)^2 u(u + 1) \sum_i \int_{|z - z_i| < \rho} |z - z_i|^{-2} u(u + 1) d |z|^2$$

$$= \text{Const. } \left| T_i(z_0) \right|^2 (D_i^{-1} \delta_i)^2 u(u + 1) \rho^u(u + 1).$$

But $\rho^2 = \text{Const. } \text{meas}(X_i) \leq \text{Const. } E_{r-1} \delta_i (\text{meas}(X))^{(1 + u)^{-r+1}}$.

So $\text{meas}(T_i X_i) \leq \text{Const. } \left( \left| T_i(z_0) \right| \right)^2 D_i^{u(u + 1)} E_{r-1}^{1/(u + 1)} \text{meas}(X)^{(1 + u)^{-r}}$.

Then since $\delta = \text{Const. } |T_i(z_0)| \delta_i$ we obtain the result.

6.3. PROPOSITION. Let $|\lambda| \leq \eta_0$ and let $\lambda$ be $(m, i, a)$ — good for $m \leq n$, $i \leq m_0$. Let $X_{n, n}$ be the union of all $\text{Im } S$ with $S$ of length $n$ and such that in the canonical decomposition $S = S_0 T_1 \ldots S_n$, $r > s$. Then $\text{meas}(X_{n, n}) \leq \eta_n$ where $\eta_n$ is independent of $n$ and $\eta_1 \to 0$ as $s \to \infty$.

Proof. Let $S$ be such that there are $> s T's$ in the canonical decomposition of $S$. Write $S = W_1 W_2$, where $W_2$ is determined by $t_{n-m} \ldots t_s$ if $S$ is determined by $t_0 \ldots t_m$, and $n - m$ is the smallest integer such that some $[n - m, n - m + L]$ is a follower with $m - L < 2 \alpha^{1/2} L$. Then $m > (1/3 \alpha)^{1/2}$ by the same argument as in 4.7. Also, if $[p, p + q]$ is a follower in $W_1$ with $n - m - (p + q) < \alpha^{1/2} q$, then by minimality of $n - m$ we know $n - (p + q) > 2 \alpha^{1/2} q$. So $2 \alpha^{1/2} q < \alpha^{1/2} q + m$ and $q < \alpha^{-1/2} m$. In the canonical decomposition of $W_1$ there are $t T's$ where

$$t \leq 1 + \frac{2 \log(\alpha^{-1/2} m)}{\log((1/2) \alpha)}.$$
So if we now take all \( S \) whose union is \( X_{x,a} \) and consider the decomposition \( S = W_1 W_2 \), then we may as well assume \( W_1 \) has radius \( 2\varepsilon_i \) centred on some \( x_i \). If we fix \( i, m, L \) and look at all possible \( W_2 \) with \([0, L]\) an \( x_i \)-follower for \( W_2 \), we see that \( \text{Im} W_2 \leq \text{Im} W_2 \) where \( W_2 \) is the inverse of length \( L \) with domain radius \( \varepsilon_0 \) determined by \( x_i \ldots f^L_1 x_i \).

So \( X_{x,a} \subseteq \bigcup_{i \leq m_0, \varepsilon_i \leq m - L} \{ \text{Im} W_1 W_2 : W_1 \) is inverse length \( n - m \) with domain \( B_{x_{\varepsilon_1}}(x_i) \) and \( W_2 \) is determined by \( x_i \ldots f^L_1 (x_i) \) and radius \( \varepsilon_0 \} \).

Then by 6.2,

\[
\text{meas}(\text{Im} W_1 W_2) \leq \text{meas}(\text{Im} W_1) D \left( e^{-2 \varepsilon_2 L \varepsilon_0^2 (1+u)^{-1}} \leq e^{-m} \text{meas}(\text{Im} W_1) \right) \quad \text{for } \gamma > 0
\]
such that

\[
t \log(1+u) < (1-\gamma') \log L \quad \text{for a } \gamma' > \gamma.
\]

Then summing over \( L, W_1, i, m \), we obtain

\[
\text{meas}(X_{x,a}) \leq \sum \text{meas}(\text{Im} W_1) \sum_{m > (1/3 \alpha)^{1/2}} m^2 e^{-m} \to 0 \text{ as } s \to \infty.
\]

6.4. PROOF OF THEOREM B. — Let \( \lambda \) denote spherical measure, transferred to \( \mathbb{C} \subseteq \), with \( \lambda(\mathbb{C}) = 1 \). Let \( (f^n)_\lambda \) be defined by \( (f^n)_\lambda(A) = \lambda(f^{-n} A) \). Then 6.2, 6.3 show the measures \( (f^n)_\lambda \) are uniformly absolutely continuous with respect to \( \lambda \). For given \( \varepsilon > 0 \), choose \( \eta_\varepsilon < \varepsilon /2 \). Choose a cover \( u \) of \( \mathbb{C} \) of index \( p \), by balls of radius \( \varepsilon_i \). Then \( \{ \text{Im} S : \text{domain } S \in u, \text{length } S = n \} \) is also a cover of index \( p \) for each \( n \). Now choose \( \delta \) so that \( \delta^{(1+u)^{-1}} < \varepsilon /2p \). Then if \( \lambda(A) < \delta \),

\[
(f^n)_\lambda(A) \leq \sum \{ \lambda(SA) : \text{domain } S \in u, \text{length } S = n \}
\]

\[
< \eta_\varepsilon + \sum \{ \lambda(SA) : \text{domain } S \in u, \text{length } S = n, S \text{ has}
\]

\[
\leq s T's \text{ in its canonical decomposition }\}.
\]

Then for \( S \) in the sum, \( \lambda(SA) \leq \lambda(\text{Im } S) \delta^{(1+u)^{-1}} (6.2) < \lambda(\text{Im } S) \varepsilon /2p \). So

\[
(f^n)_\lambda(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2p} \sum \{ \lambda(\text{Im } S) : \text{domain } S \in u, \text{length } S \leq n \} < \varepsilon,
\]

proving uniform absolute continuity.

So then any weak limit of \( (1/(n+1)) \sum_{i=0}^{n} (f^i)_\lambda \) is an invariant measure absolutely continuous with respect to \( \lambda \). (This type of argument comes from [SZ].)

To show ergodicity of \( \lambda \), let \( A \) be any set satisfying \( f_{i_1}(A) \subseteq A, \lambda(A) > 0 \). Since \( \lim \text{diam}(\text{Im } S) \to 0 \) as \( \text{length } S \to \infty \) (uniformly in \( S \)), for sufficiently large \( n \) \( A \) can be approximated to within \( (\varepsilon /p) \lambda(A) \) in \( \lambda \)-measure by a union of sets \( \text{Im } S \) with \( S \) of length
n and domain \( S \in u \). That is, we can have:

\[
\lambda\left(U \{ \text{Im } S : \text{Im } S \cap A \neq \emptyset, \text{ domain } S \in u, \text{ length } S = n \} \right) < \lambda(A) \left(1 + \frac{\varepsilon}{p}\right).
\]

Then \( \sum \lambda(\text{Im } S \setminus A) < \varepsilon \lambda(A) \leq \varepsilon \sum \lambda(\text{Im } S) \), where both summations are taken over \( S \) with length \( S = n \), domain \( S \in u \). So for a set of \( S \) with \( \sum \lambda(\text{Im } S) > \lambda(A)/2 \), \( \lambda(\text{Im } S \setminus A) < 2 \varepsilon \lambda(\text{Im } S) \). Suppose \( \eta_\varepsilon < \lambda(A)/4p \). Then there is an \( S \) with \( \leq s \) 's in its canonical decomposition with \( \lambda(\text{Im } S \setminus A) < 2 \varepsilon \lambda(\text{Im } S) \). Then applying \( f_s \) to \( \text{Im } S \setminus A \) by 5.3, since \( f_s(A) \subseteq A \), \( \lambda(\text{domain } S \setminus A) < \delta(\varepsilon, s) \lambda \) (domain \( S \)) where \( \delta(\varepsilon, s) \to 0 \) as \( \varepsilon \to 0 \). Letting \( \varepsilon \to 0 \), there is at least one ball \( B \) of radius \( \varepsilon \) which is contained a.e. in \( A \). We can also take \( B \) to have non-null intersection with the Julia set of \( f_s \) for the positive measure \( A \) can be reduced if necessary (but keeping the property \( f_s(A) \subseteq A \)) so that it does not contain any ball radius \( \varepsilon \) not intersecting the Julia set. So \( f_s(B) = \mathcal{C} \) for some \( N \), by the expanding property of the Julia set. So \( \lambda(\mathcal{C} - A) = 0 \) and \( \lambda \) is ergodic. By ergodicity, we obtain that any invariant measure absolutely continuous with respect to \( \lambda \) is unique (up to a scalar), and equivalent to \( \lambda \).

7. Proof of theorem C

The following proposition is the key idea in the proof of Theorem C. As mentioned before, the idea of using the power series of an analytic function is due to the referee.

7.1. Proposition. — Let \( N \geq n_{01/2} \), let \( |\lambda| \leq \eta_0 \) and \( \lambda \) be \((n, i, \alpha)\)-good for all \( n \leq N \), \( i \leq m_0 \). The union of images of inverses \( S \) of length \( N \) such that more than \( a_n \) points in \([0, N]\) are in a follower for \( S \) of length \( \geq n_0 \) has measure \( \leq \exp(-a_s N) \), where \( a_s \) depends on \( \alpha \) but not on \( n_0 \).

Note. — Later we shall need to apply this with follower length \( \geq (1/4)n_0 \), which is easily deduced by changing variable names. We use \( n_0 \) here for easier writing.

Proof. — (a) Let \( S \) be an inverse of length \( N \). Let \( c'_1 \ldots c'_s \) be the starting points of followers of \( S \) of length \( \geq n_0 \). Put \( c_i = c'_i - 1 \) or \( c'_i \) depending on whether there exists a follower starting at \( c'_i - 1 \) or not. Reindex to have no repetitions amongst the \( c_i \) so that \( c_1 < \ldots < c_s \). Let \( d'_i \) be the biggest integer contained in a follower starting at \( c_i \) or \( c_i + 1 \): if \( d'_i \geq c_i + 1 \) (respectively \( d'_i = N \)) put \( d_i = c_i + 1 \) (respectively \( d_i = N \)); if \( d'_i < c_i + 1 \) and there is a maximal follower (of length \( < n_0 \)) starting in \([c_i, d'_i]\) with right endpoint bigger than \( d'_i \), then call this right endpoint \( d_i \). In all cases one has \( d_i - c_i \geq n_0 \), \( 1 \leq i \leq s \).

Define

\[
\begin{align*}
u_i = d_i - c_i & \quad \text{for} \quad 1 \leq i \leq s, \\
u_i = c_{i+1} - d_i & \quad \text{for} \quad 1 \leq i < s, \\
u_0 = c_1, \\
u_s = N - d_s.
\end{align*}
\]
Then \( \sum u_i + \sum v_i = N \) and \( u_i \geq n_0 \) for \( 1 \leq i \leq s \).

(b) Estimate the union of images of \( S \) which have fixed \( s, u, v \). Write \( S = V_0 U_1 V_1 \ldots V_s \) where \( V_i \) has length \( v_i \) and \( U_i \) has length \( u_i \).

We may assume \( U_i \) is determined by one of the following:

(i) a sequence \( x_j \ldots f_{n}^{i} x_{j} \) and \( U_i \) has domain \( B_{\varepsilon_0}(f_{n}^{i}(x_{j})) \);

(ii) a sequence \( a, x_j \ldots f_{n}^{i-1}(x_{j}) \), where \( f_{n}(a) = x_{j} \) and the domain has radius \( \varepsilon_0 \) centre \( f_{n}^{i-1}(x_{j}) \);

(iii) \( U_i = W_i X_i \) where \( W_i \) is determined by a sequence as in (a) or (b) [but with \( u_i \) replaced by length \( (W_i) \)] and \( X_i \) is determined by \( x_k \ldots f_{n}^{m}(x_{k}) \) where \( m = \text{length} X_i < n_0 \) and the domain has radius \( \varepsilon_0 \), centre \( f_{n}^{m}(x_{k}) \).

Then in all cases, if \( V_i \neq \text{identity and} \ V_i \) is determined by \( t_0 \ldots t_m \), \( U_i \) is univalued on \( B_{2 \varepsilon_1}(t_i) \). If \( V_i \neq \text{identity but} \ V_k \) is univalued for \( j \leq k \leq i \) then \( U_j \ldots U_i \) is univalued on \( B_{2 \varepsilon_1}(t_i) \). Then the distortion theorem for univalent functions [D] gives \( |(U_j \ldots U_i)|^2 \leq C^2 \text{ meas } (\text{Im } (U_j \ldots U_i)) \) on \( \text{Im } V_i \ldots V_s \) if \( i < s \) or if length \( (V_i \ldots V_s) \geq s_0 \).

By (i)–(iii) there are \( \leq m_0^2 \) possibilities for each \( U_i \). We can also assume domain \( (V_{i-1}) \) is the ball radius \( \varepsilon_1 \) round the starting point of \( U_i \), \( i \leq s \), so there are \( m_0(d + 1) \) possibilities for domain \( (V_{i-1}) \). \( V_{i-1} \) is univalued on this ball, so we have, by the distortion theorem

\[
| (V_{i-1}) |^2 \leq C \text{ Meas } (\text{Im } (V_{i-1})) \quad \text{on } \text{Im } U_i \ldots V_{s}, \quad i \leq s.
\]

We can also assume domain \( (V_s) \) is one of a finite number of balls \( B(w_p, 2\varepsilon_1) \) by taking a finite cover of \( S \) by balls \( B(w_p, \varepsilon_1) \): every ball of radius \( \varepsilon_1 \) will then be in some \( B(w_p, 2\varepsilon_1) \). So for fixed \( s, u, v \), we obtain a bound for the measure of the union of all possible \( \text{Im } V_0 U_1 \ldots U_s \) of the form \( (K n_0)^S \exp(-2a_3 \sum u_i) \) for some constant \( K \).

(c) So to prove the proposition, it is sufficient to show that

\[
\sum_{s \geq 1} \sum_{n \leq k \leq N} (K n_0)^S \exp\left(-2a_3 k\right) \sum_{u_1 + \ldots + u_k = k} 1 \leq \exp(-a_4 N).
\]

The last sum is less than the coefficient of \( X^k Y^{N-k} \) in \( (X^{n_0} \frac{1}{1-X})^2 \left( \frac{1}{1-Y} \right)^s \).

We have (as a formal series)

\[
\sum_{s \geq 0} (K n_0)^S \left( \frac{X^{n_0}}{1-X} \right)^s \left( \frac{1}{1-Y} \right)^2 = \frac{1}{1-(n_0 KX^{n_0}/(1-X)(1-Y)))}.
\]

For \( n_0 \) big enough, this function is holomorphic and bounded for \( |X| \leq \exp(-a_3 \alpha), |Y| \leq \exp(-a_3 \alpha) \); hence the coefficient of \( X^k Y^{N-k} \) is less than \( K' \exp(a_3 \alpha N) \). So the
triple summation is bounded by
\[ \sum_{2N \leq k \leq N} K' \exp(a_3 \alpha k) \exp(-2a_3 k) \leq K' N \exp(-a_3 \alpha N). \]

This gives the result.

7.2. THE FUNCTION \( g_n^i(\lambda) \): FIRST ESTIMATE. — Now define \( g_n^i(\lambda) = f_n(\lambda, y_i(\lambda)) \). Let \( N_i \) be the largest integer such that \( g_n^i \) maps \( \{ \lambda : |\lambda| \leq \eta_0 \} \) into \( U_i \) for all \( n \leq N_i \) so that \( N_i \geq 2n_0^2 \).

PROPOSITION. — There is a constant \( K_1 \) such that
\[ \frac{1}{K_1} |(f_n^i)'(z_i(0))| \leq \left| \frac{d g_n^i(\lambda)}{d \lambda} \right| \leq K_1 |(f_n^i)'(z_i(0))| \quad \text{for all } n \leq N_i, \quad |\lambda| \leq \eta_0, \]
and such that the image of \( \{ \lambda : |\lambda| \leq \eta_0 \} \) under \( g_{N_i}^i \) contains the ball radius 2\( \varepsilon_0 \) round \( z_i(0) \), assuming \( \eta_0, \varepsilon_0 \) and \( U_i \) are small enough (independently of \( n_0 \)).

Proof. — Denote by \( \varphi_\lambda \) the conjugacy with image \( U_\lambda \), \( \varphi_\lambda(0) = z_\lambda \), \( \varphi_\lambda(0) = 1 \), such that \( f_n(\varphi_\lambda(z)) = \varphi_\lambda(\mu(\lambda) z) \) for \( z \in \varphi_\lambda^{-1}(U_\lambda) \), where \( \mu(\lambda) = f_n(z_\lambda) \). Put \( t(\lambda) = \varphi_\lambda^{-1}(y_i(\lambda)) \), and write \( \varphi_\lambda(z) = \varphi(\lambda, z) \). Then \[ |\partial \varphi/\partial \lambda(z, z)|, |\partial \varphi/\partial z(z, z)| \pm 1 \] are bounded for \( |\lambda| \leq \eta_0 \), \( z \in U_\lambda \) and \( \varphi_\lambda^{-1}(U_\lambda) \) is bounded. Then for \( n \leq N_i \), \( g_n^i(\lambda) = \varphi(\lambda, (\mu(\lambda))^n t(\lambda)) \), and \( (\mu(\lambda))^n t(\lambda) \in \varphi_\lambda^{-1}(U_\lambda) \). Then:
\[ \frac{d g_n^i(\lambda)}{d \lambda} = \frac{\partial \varphi}{\partial \lambda}(\lambda, (\mu(\lambda))^n t(\lambda)) + \frac{\partial \varphi}{\partial z}(\lambda, (\mu(\lambda))^n t(\lambda)) B, \]
where \( B = (\mu(\lambda))^{-1} n \mu'(\lambda) t(\lambda) + (\mu(\lambda))^n t'(\lambda) \).

So \( |d g_n^i/d \lambda| \cdot ||B|^{-1} \) is bounded above and below. \( |\mu'(\lambda)| \) is bounded for \( |\lambda| \leq \eta_0 \), and we have seen that \( (\mu(\lambda))^n t(\lambda) \) is too, for \( |\lambda| \leq \eta_0, \quad n \leq N_i \). Then \( t'(\lambda) \) is near \( t'(0) \), and since \( y_i(\lambda) - z_i(\lambda) = \varphi(\lambda, t(\lambda)) - \varphi(\lambda, 0) \), and \( t(0) = 0, \quad t'(0) = y_i'(0) - z_i'(0) \neq 0 \).

So \( |B|^{-1} |\mu(\lambda)|^{-n} \) is bounded above and below. So the proposition follows, provided we can show that, for \( |\lambda| \leq \eta_0 \) and \( n \leq N_i \), \( |\mu(\lambda)|^{-n} |\mu(0)|^{-n} \) is bounded above and below. Suppose \( |\lambda| \leq |\mu(0)|^{-N_i/2} \). Then, since \( |\mu'| \) is bounded,
\[ \log |\mu(\lambda)|^{-n} - \log |\mu(0)|^{-n} \leq O(n |\mu(0)|^{-N_i/2}) \leq 1 \quad \text{for } n \leq N_i, \]
So for such \( \lambda \), \( |\mu(\lambda)|^{-n} |\mu(0)|^{-n} \) is bounded above and below. We deduce that \( \eta_0 < |\mu(0)|^{-N_i/2} \). For otherwise, \( |d g_n^i/d \lambda| \cdot |\mu(0)|^{-n} \) is bounded for \( |\lambda| \leq |\mu(0)|^{-N_i/2}, \quad n \leq N_i, \) and \( g_n^i \) maps the set \( \{ |\lambda| \leq |\mu(0)|^{-N_i/2} \} \) to a set of diameter \( O(|\mu(0)|^{N_i/2}) \). So \( \eta_0 < |\mu(0)|^{-N_i/2} \), and \( |d g_n^i/d \lambda(\lambda)| \cdot |\mu(0)|^{-n} \) is bounded above and below for \( |\lambda| \leq \eta_0, \quad n \leq N_i, \) as required.

7.3. THE FUNCTION \( g_n^i(\lambda) \): FULL ESTIMATE.

DEFINITION. — Let \( C_{N_i, i}(\lambda) \) be the connected component of \( (g_n^i)^{-1}(B(g_n^i(\lambda), \varepsilon/4)) \) which contains \( \lambda \). From now on, choose \( \alpha_0 \) such that with \( \alpha = \alpha_0 \), the estimates of paragraph 5 work.
PROPOSITION. - Let \( N \geq N_1 + q_0 \), let \( \lambda \) be \((n, j, \alpha_0)\) good for all \( n \leq N, j \leq m_0 \). Suppose \( N \) is not in any \( x_f \)-follower for \( \lambda \). Then \( g_N^1 \) maps \( C_{N, i}(\lambda) \to 1 \) to \( B(g_N^1(\lambda), \varepsilon_1/4) \) and

\[
\frac{1}{K_2} |(f_N^1)'(y_i(\lambda))| \leq |(g_N^1)'(\mu)| \leq K_2 |(f_N^1)'(y_i(\lambda))|
\]

for all \( \mu \in C_{N, i}(\lambda) \), and suitable \( K_2 \).

Hence (since the relevant local inverse of \( f_N^1 \) is univalued on \( B(\varepsilon_1, (f_N^1)'(y_i(\lambda))) \)),

\[
\frac{1}{K} e^{A_3 N} \leq |(g_N^1)'(\mu)|
\]

for some \( K \).

Proof. — We shall make use of 7.2, which, in particular, proves the proposition for \( N \), with \( K_2 \) replaced by \( K_1 \). To prove the proposition for \( N \), choose \( n < N \) such that:

(a) \( N - 2bN < n < N - bN \), where \( b \) is small, depending on \( \alpha_3 \);

(b) \( n \) is not in any \( x_f \)-follower for \( \lambda \).

Then either by an inductive hypothesis of by 7.2, whenever \( \mu \in C_{n, i}(\lambda) \) (or \( |\mu| \leq \eta_0 \) if \( n \leq N \)) we can assume for suitable \( K_2 \) that:

(i) \( K_2^{-1/2} \leq |(g_N^1)'(\lambda)| |(g_N^1)'(\mu)| \leq K_1^{1/2} \), and,

(ii) \( A_n^{-1} |(f_N^1)'(y_i(\lambda))| \leq |(g_N^1)'(\lambda)| \leq A_n |(f_N^1)'(y_i(\lambda))| \)

for \( A_n \) such that \( \prod_{m \geq n} (1 + e^{-A_3 m/2}) A_n < K_1^{1/2} \).

From (i), there is a map \( \tilde{f} \) defined on \( B(g_n^1(\lambda), \varepsilon_1/4) \) which is a local inverse of \( g_N^1 \).

On the other hand, as \( N \) is not in any follower for \( \lambda \), there is a univalent inverse \( G \) of \( f_N^1 \) defined on \( \tilde{B} = B(g_N^1(\lambda), \varepsilon_1) \) by 5.2 \( G(\tilde{B}) \) is contained in \( B(g_n^1(\lambda), \varepsilon_1, \exp(-A_3 (N - n))) \). Let \( C \) be the boundary of \( G(\tilde{B}) \). By the inductive hypothesis on \( g_n^1 \), \( |\lambda - \mu| < K \varepsilon_1 \exp(-(A_3 N) \).

So

\[
d(f_N^1 - n(g_n^1(\mu)), g_N^1(\mu)) < e^{b_2 (N - n)} K \varepsilon_1 (N - n) < K \varepsilon_1 e^{(2b_2 - A_3) N (N - n)} < e_1 \frac{\varepsilon_1}{4}
\]

if \( b \) is small enough, where \( b_2 \) is such that

\[
|f_n^1|, \quad \frac{\partial^2 f}{\partial \mu \partial z}(\mu, z) \leq e^{b_2}.
\]

So each point in the ball radius \( 3\varepsilon_1/4 \) round \( g_N^1(\lambda) \) must be \( g_N^1(\mu) \) for exactly one \( \mu \) with \( g_n^1(\mu) \in G(\tilde{B}) \). So \( g_N^1 \) has a univalued inverse on \( B(g_n^1(\lambda), \varepsilon_1/4) \) and

\[
K_2^{-1/2} \leq \left| \frac{(g_N^1)'(\lambda)}{(g_N^1)'(\mu)} \right| \leq K_1^{1/2}
\]

for all \( \mu \in C_{N, i}(\lambda) \), for suitable \( K_2 \).
Also since
\[(g_N^i)'(\lambda) = (f_N^{1,0})'(g_N^i(\lambda)) (g_N^i)'(\lambda) + \frac{\partial}{\partial \lambda} f_{N-i+1}(\lambda, g_N^i(\lambda)) = (f_N^{1,0})'(g_N^i(\lambda)) (g_N^i)'(\lambda) (1 + O(e^{-a_3 N/2}))\]
for \(b\) sufficiently small, we have:
\[A_N^{-1} |(f^N_N)'(y_i(\lambda))| \leq |(g_N^i)'(\lambda)| \leq A_N |(f^N_N)'(y_i(\lambda))| \]
using the inductive hypothesis (ii).

**Good Inverses and Good Parameter Values.** — The following lemmas 7.4 — 7.7 are for showing that good inverses (which were shown in Proposition 7.1 to be a large proportion of the whole) give rise to good parameter values.

Let \(\lambda\) be \((n, i, \alpha_0)\) — good for \(n \leq N\) and \(i \leq m_0\). The constant \(b_2\) as in § 5 is such that \(|f_\mu'| \leq e^{b_2}\) for all \(|\mu| \leq \eta_0\), and also such that second derivatives of \(f(\mu, z)\) are bounded by \(e^{b_2}\).

**7.4. Lemma.** — If \(\mu \in C_{N,i}(\lambda), r + k \leq b N\) and \(x_j(\mu) \in \text{Im} S\), where \(S\) is the inverse of \(f_\mu^k\) with domain radius \(e/2\) \((e_1/2 \leq e \leq e_0)\) defined by \(g_{N+r}^i(\mu) \ldots g_{N+r+k}^i(\mu)\), then \(x_j(\lambda) \in \text{Im} S'\) where \(S' = \text{the inverse of } f_\mu^k\) with domain radius \(e\) and defined by \(f_\mu^i(g_{N}^i(\mu)) \ldots f_\mu^{i+k}(g_{N}^i(\mu))\).

**Proof.** — By 7.3, \(|\lambda - \mu| \leq K e e^{-a_3 N}\).

So \(d(x_j(\mu), x_j(\lambda)) < e^{-a_12 N} e_1\) for \(a_12 > 0\). So it suffices to show the \(e^{-a_12 N} e_1\) — neighbourhood of \(\text{Im} S\) is contained in \(\text{Im} S'\).

\[d(g_{N+r}^i(\mu), f_\mu^i(g_{N}^i(\mu)) < re^{b_2 r} |\lambda - \mu| < e^{-(1/2) a_3 N} e_1/4 < e^{b_2 k} e_{1/4}\] if \(b\) is small enough.

So \(\text{Im} S \cap \text{Im} S' \neq \emptyset\), because \(\text{Im} S'\) contains a ball radius \(e^{b_2 k} e^{(e_1/2)}\) round \(f_\mu^i(g_{N}^i(\mu))\).

So it suffices to show, if \(Y\) denotes the \(e^{-a_12 N} e_1\) — neighbourhood of the boundary of \(\text{Im} S\), \(Y \cap \partial(\text{Im} S') = \emptyset\). Now \(f_\mu^i(Y)\) is contained in the \(ke^{b_2 k} e^{-a_12 N} e_1\) neighbourhood of the boundary of \(B(\mu, g_{N+r+k}^i(\mu), e), \) and \(f_\mu^i(\partial(\text{Im} S'))\) is contained in the \(ke^{b_2 k} |\lambda - \mu|\) neighbourhood of the boundary of \(B(g_{N+r+k}^i(\mu), e),\) and \(ke^{b_2 k} |\lambda - \mu| < e^{-(a_3/2) N} e_1\) for \(b\) sufficiently small. So these neighbourhoods are clearly disjoint for \(b\) small enough, and \(Y \cap \partial(\text{Im} S') = \emptyset\) as required.

**7.5.** Continue to assume \(\lambda\) is \((n, i, \alpha_0)\) — good for \(n \leq N, i \leq m_0\). Now fix \(i,\) and (unless \(N \leq N_\mu\) when there are no conditions) assume that \(\mu \in C_{N,i}(\lambda)\) satisfies:

I. \(g_{N}^i(\mu) \in \text{Im} S\) for \(S\) an inverse of length \([b N]\) for \(\lambda\), where \(S\) is \((n, i, \alpha_0/4, n_0/2)\) — good for \((\alpha_0/4) N \leq n \leq b N\), that is, at most \(\alpha_0/4) n\) integers are contained is followers for \(S_i\) of length \(\geq n_0/2\) if \(S_i\) is determined by \(t_{00} \ldots t_n\) and \(S\) by \(t_0 \ldots t_{[b N]}\).

II. \(B(g_{N}^i(\mu), e^{a_6 N} e_1) \subseteq B(g_{N}^i(\lambda), e_1/4),\) where \(a_6\) is a fixed positive number < \(a_1 b/10 m_0\).

(Actually, condition II is not needed yet, but it seems best to give both conditions together.)

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Lemma. — For \( \mu \) as above, \([N + r, N + r + L] \subseteq [N, N + n]\) is an \( x_{r} \)-follower for \( \mu \) with \( L \geq r_{0} \) only if \([r, r + L - k]\) is a follower for \( S \) for all \( k \geq r_{0} \), where \( S \) is as in I and \( r_{0} > q_{0} \) is such that \( e^{-r_{0}} < 1/4 \). Thus, if \( \alpha_{0} N/4 \leq n \leq b N \), The number of points in \([N, N + n]\) which are in \( x_{r} \)-followers for \( \mu \) of length \( \geq n_{0} \) is \( < (\alpha_{0}/2) n \).

Proof. — If \([N + r, N + r + L] \) is an \( x_{r} \)-follower for \( \mu \), and \( L \geq r_{0} \), then by the definition of \( a_{1} \) (and 4.2) there is \( x_{j}(\mu) \in \text{Im} \( \mathcal{S} \) \), where \( \mathcal{S} \) is the inverse of \( f_{\lambda}^{--k} \) defined by \( f_{\lambda}^{--k}(g_{N}(\mu)) \ldots f_{\lambda}^{--k-1}(g_{N}(\mu)) \) and radius \( \varepsilon_{1}/2 \) for all \( k \geq r_{0} \).

Then by 7.4, \( x_{j}(\lambda) \in \text{Im} \( \mathcal{S}' \) \), where \( \mathcal{S}' \) is the inverse of \( f_{\lambda}^{--k} \) defined by \( f_{\lambda}^{--k}(g_{N}(\mu)) \ldots f_{\lambda}^{--k-1}(g_{N}(\mu)) \) with domain radius \( \varepsilon_{1}/2 \). Now by I, the union of \([r, r + L - r_{0}]\) in \([0, n]\) with \( L - r_{0} \geq n_{0}/2 \) contains \( < (\alpha_{0}/4) n \) points if \((\alpha_{0}/4) N \leq n \leq b N\). So the number of points in followers \([N + r, N + r + L] \subseteq [N, N + n]\) with \( L \geq n_{0} \) and \((\alpha_{0}/4) N \leq n \leq b N\) is \( < (\alpha_{0}/4) n \) if \( n_{0} \) is large enough.

7.6. Suppose \( \mu, \lambda \) are as in 7.5 and suppose that for \( N' \in (N, N(1 + b)] \), \( C_{N', i}(\mu) \subseteq C_{N', i}(\lambda) \). Let \( v \in C_{N', i}(\mu) \).

Lemma. — If \( \mathcal{S} \) is the inverse determined by \( f_{\lambda}^{--k}(g_{N}(\mu)) \ldots f_{\lambda}^{--k-1}(g_{N}(\mu)) \) with radius \( \varepsilon_{1}/2 \), and \( \mathcal{S} \) is determined by \( f_{\lambda}^{--k}(g_{N}(\mu)) \ldots f_{\lambda}^{--k-1}(g_{N}(\mu)) \) with radius \( \varepsilon \) and either \( N' - (r + k) - N > q_{0} \) with \( \varepsilon = \varepsilon_{1} \), or \( \varepsilon \geq \varepsilon_{0} \), then \( \text{Im} \( \mathcal{S} \) \subseteq \text{Im} \( \mathcal{S}' \) \).

Proof. — It suffices to show \( \text{Im} \( \mathcal{S} \) \subseteq f_{\lambda}^{--k} \) (domain \( \mathcal{S} \)) for all \( v \in C_{N', i}(\mu) \). For then the set of \( v \) with \( \text{Im} \( \mathcal{S} \) \subseteq \text{Im} \( \mathcal{S}' \) \) is open and closed and contains \( \mu \), and must be all of \( C_{N', i}(\mu) \) since this is connected. Thus it suffices to show

\[ d(f_{\lambda}^{--k+1}(g_{N}(\mu)), f_{\lambda}^{--k-1}(g_{N}(\mu))) < \frac{\varepsilon}{2} \quad \text{for} \quad v \in C_{N', i}(\mu). \]

Now \( \text{Im} \( \mathcal{S}' \) \subseteq \mathcal{B}(f_{\lambda}^{--k}(g_{N}(\mu)), \varepsilon/2) \) if \( \mathcal{S}' \) is the inverse of \( f_{\lambda}^{--k}(g_{N}(\mu)) \ldots f_{\lambda}^{--k+1}(g_{N}(\mu)) \) with radius \( \varepsilon_{1}, \) and either \( N' - (r + k) - N > q_{0} \), \( \varepsilon = \varepsilon_{1} \), or \( \varepsilon \geq \varepsilon_{0} \).

But since \( C_{N', i}(\mu) \) is connected, \( f_{\lambda}^{--k+1}g_{N}(C_{N', i}(\mu)) \subseteq \text{Im} \( \mathcal{S}' \) if

\[ f_{\lambda}^{--k+1}(g_{N}(C_{N', i}(\mu))) \subseteq f_{\lambda}^{--k-1}(g_{N}(C_{N', i}(\mu))) \] \( \subseteq \mathcal{B}(f_{\lambda}^{--k-1}(g_{N}(\mu)), \varepsilon_{1}) \),

that is, if \( d(f_{\lambda}^{--k-1}(g_{N}(\mu)), g_{N}(\mu)) < \varepsilon_{1} \). But this true because \( |\lambda - \mu|, |\lambda - \nu| < K e^{-d_{N} N} \varepsilon_{1} \) by assumption and \( N' - N < b N \), so \( d(f_{\lambda}^{--k-N}(g_{N}(\mu)), g_{N}(\mu)) < \varepsilon_{1} \) and similarly for \( v \), and \( d(g_{N}(\mu), g_{N}(\nu)) < \varepsilon_{1}/4 \).

7.7. We now extend the result of 7.5 to \( v \in C_{N', i}(\mu) \).

Lemma. — Under the same conditions as in 7.6, \([N + r, N + r + L] \subseteq [N, N + n]\) is an \( x_{r} \)-follower for \( v \) with \( L \geq r_{0} \) only if \([r, r + L - k]\) is a follower for \( S \) for all \( k \geq r_{0} \), where \( S \) is as in I and \( g_{N}^{k}(\mu) \in \text{Im} \( \mathcal{S} \) \). Then if \( \alpha_{0} N/4 \leq n \leq b N \), the number of points in \([N, N + n]\) which are in \( x_{r} \)-followers for \( v \) of length \( \geq n_{0} \) is \( < (\alpha_{0}/2) n \).

Proof. — This is very similar to 7.5. If \([N + r, N + r + L] \subseteq [N, N + n]\) is a follower for \( v \), then \([r, r + L - r_{0}]\) is such that the inverse \( \mathcal{S} \) of \( f_{\lambda}^{--r_{0}} \) determined by \( f_{\lambda}^{--r_{0}}(g_{N}(\mu)) \ldots f_{\lambda}^{--r_{0}-1}(g_{N}(\mu)) \) with domain radius \( \varepsilon_{1}/2 \) has some \( x_{j}(\lambda) \in \text{Im} \( \mathcal{S} \) \). Then by
7.6, the inverse $S$ of $f_{x_0}^{L-r_0}$ determined by $f_{x_0}(g_{x_0}^r(\mu)) \cdot \cdots \cdot f_{x_0}^{L-r_0}(g_{x_0}^r(\mu))$ with domain radius $\varepsilon_1$ has $x_j(\lambda) \in \text{Im} \ S$, since $\text{Im} \ S \subseteq \text{Im} \ S$. Then the proof is completed as in 7.5.

7.8. We now consider extra conditions on $N' \in [N, N(1+b)]$ where $N'$ is as 7.6-7.7. We continue with the assumptions of 7.4-7.7 on $\mu$, $v$.

III. $N' = N + n$ is such that $[n - r_0, n]$ does not intersect any follower for $S$ of length $\geq (1/2) n_0$ and radius $\varepsilon_4$, where $S$ is as in I.

By 7.5, 7.7, III implies $N'$ does not intersect any follower of length $\geq n_0$ for $\mu$, or $v$.

IV. $x_j(\lambda) \notin \text{Im} \ S$, where $S$ is determined by $f_{x_0}^{N - N - r}(g_{x_0}^r(\mu)) \cdot \cdots \cdot f_{x_0}^{N - N}(g_{x_0}^r(\mu))$ for $r \leq n_0$ and radius $4 \varepsilon_0$.

Then by 7.4, $N'$ is not in a follower of length $r \leq n_0$ for $\mu$. Also, by 7.6, $x_j(\lambda) \notin \text{Im} \ S'$ where $S'$ is determined by $f_{x_0}^{N - N - r}(g_{x_0}^r(\nu)) \cdot \cdots \cdot f_{x_0}^{N - N}(g_{x_0}^r(\nu))$ and radius $2 \varepsilon_0$. Then by 7.4, $N'$ is not in a follower of length $\leq n_0$ for $\nu$.

7.9. A SET OF INTERVALS. — Recall from 7.2 that $N_i$ is the largest integer such that $g_i^r$ maps $\{ \lambda : |\lambda| \leq \eta_0 \}$ into $U_i$ for all $n \leq N_i$.

Define $R_0 = 2n_0^2$. We can find disjoint intervals $I_k$ of integers, for $k \geq 1$, such that if $R_k$ is the least element of $I_k$, then:

(a) $(1 + (b/2)) R_k \leq N \leq (1 + (3b/4)) R_k$ for all $N \in I_{k+1}$.

(b) there is no $N_i \in I_k$, and if $N_i < R_k$, then

$$|N_i - R_k| > \frac{b}{10 \ m_0} \ R_k.$$  

(c) $I_k$ has width $>(b/10 \ m_0) \ R_k$ and $<(b/4) \ R_k$.

Proof of Theorem C. — We shall define sets $g_k = \{ \lambda : |\lambda| \leq \eta_0 \}$ such that if $\lambda \in \limsup g_k$, then $\lambda$ is $(n, i, \alpha_0)$ — good for all $n$, all $i \leq m_0$, thus proving Theorem C.

Let

$$g_0 = \{ \lambda : |\lambda| \leq \frac{1}{2} \eta_0 \}.$$  

As in 7.2, let $N_i$ be the largest integer such that $g_i^r$ maps $\{ \lambda : |\lambda| \leq \eta_0 \}$ into $U_i$ for all $n \leq N_i$.

Suppose inductively that $g_k$ has been defined so that:

V for each $\lambda \in g_k$, there exists an integer $N(k, \lambda) \in I_k$ such that if $D_k' (\lambda) = C_{n_i} (\lambda)$ for $N = N(k, \lambda)$, then for all $\mu \in D_k' (\lambda)$, $\mu$ is $(n, i, \alpha_0)$ — good for $n \leq N(k, \lambda)$ and $(n, i, \alpha_0/2)$ — good for $n = N(k, \lambda)$, and $N(k, \lambda)$ is not in a follower for $\mu$. In particular, if $\lambda = \mu$, these things are true for all $i \leq m_0$.

Then define $g_{k+1} = \bigcup \{ \bigcap_{i \leq m_0} E_k(\lambda) : \lambda_i \in g_k \}$ where $E_k(\lambda)$ is

(a) $\{ \mu \in D_k'(\lambda) : \mu$ satisfies I and II for $N = N(k, \lambda) \}$ if $N_i < N(k, \lambda)$;

(b) $\{ \mu \in D_k'(\lambda) : \mu$ satisfies I and II for $N = N_i \}$ if $N(k, \lambda) < N_i < R_{k+1}$;

(c) $E_k(\lambda) = D_k'(\lambda) = \{ |\lambda| \leq (1/2) \eta_0 \}$ if $R_{k+1} < N_i$. 

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Then if $\mu \in \mathcal{E}_k(\lambda)$, $\lambda_i \in g_k$ and $N = N(k, \lambda_i)$, $\mu$ is $(n, i, \alpha_0)$—good for $n \leq N$, and $(N, i, \alpha_0/2)$—good, hence $(n, i, \alpha_0)$—good for $n \leq N(1 + (\alpha_0/2))$. Then by 7.5, $\mu$ is $(n, i, \alpha_0/2)$—good for $N(1 + (\alpha_0/4)) \leq n \leq N(1 + b)$.

Now (unless $R_{k+1} < N$, in which case there is no condition for $i$) choose $N' = N(k + 1, \mu) \in I_{k+1}$ such that III and IV are satisfied for all $i$ (where for $i, N$ in III and IV is taken to be $N(k, \lambda_i)$ if $\mu \in \cap E^i_k(\lambda_i)$, $\lambda_i \in g_k$). This is possible since the proportion of integers which must be avoided for each $i$ is $< n_0/2 n_0^2$ (for IV) and for III: the proportion of integers which must be avoided for followers of length $> n_0/2$ is $< (\alpha_0/4) (1 + (2 r_0/n_0))$.

By 7.8, $N'$ is not in an $x_i$-follower for $\mu$ (for any $i$), nor for $v$, provided that the condition $C_{N'}(\mu) \subseteq C_{N-i}(\lambda)$ is satisfied [if $\lambda = \lambda_i$ and $N = N(k, \lambda_i)$, where $\mu \in E^i_k(\lambda_i)$].

But since $\mu$ is $(n, i, \alpha_0)$—good for $n \leq N'$ and all $i$, and $N'$ is not in an $x_i$-follower for $\mu$,

$$\frac{|dg_{N'/dv}|}{|dg_{N/dv}|}$$

is minorized on $C_{N', i}(\mu)$ by $O(e^{-\alpha_3 (N'-N)})$, and thus, II and $N' - N > \alpha_3 b/10 m_0$ (from the conditions on the intervals $I_k$ in 7.9) imply $C_{N', i}(\mu) \subseteq C_{N-i}(\lambda)$ as required.

We can then deduce from 7.7 that $v$ is $(n, i, \alpha_0)$—good for $n \leq N'$, and $(N', i, \alpha_0/2)$—good, if $v \in C_{N', i}(\mu)$, and from 7.8 that $N'$ is in an $x_i$-follower for $v$. Thus $g_k+1$ satisfies condition V as required.

Finally, we have to bound $\operatorname{meas} (g_k \setminus g_{k+1})$. Now

$$g_k \setminus g_{k+1} \subseteq \bigcup_{i} \bigcup_{\lambda \in I_i} (F^i_k(\lambda)) \setminus E^i_k(\lambda))$$

Here, $F^i_k(\lambda)$ is the largest ball centred on $\lambda$ and contained in $D^i_k(\lambda)$ (which is a ball up to bounded distortion) and $I_i$ is a countable subset of $g_k$ with $g_k \subseteq \bigcup_{\lambda \in I_i} F^i_k(\lambda)$, chosen so that no point in $\{|\lambda| \leq \eta_0\}$ lies in more than 20 of the balls $F^i_k(\lambda)$.

Now, by 7.3 the following inequalities are equivalent.

$$\operatorname{meas} (D^i_k(\lambda) \setminus E^i_k(\lambda)) < \operatorname{Const.} e^{-\alpha_8 R_k} \operatorname{meas} (D^i_k(\lambda)),$$

$$\operatorname{meas} (\{g^i_k(\mu) : \mu \in D^i_k(\lambda) \setminus E^i_k(\lambda)) \} < \operatorname{Const.} e^{-\alpha_8 R_k} \operatorname{meas} \{g^i_k(\mu) : \mu \in D^i_k(\lambda) \}.$$

But the latter is true by conditions I, II and 7.1.

So

$$\operatorname{meas} (F^i_k(\lambda) \setminus E^i_k(\lambda)) < \operatorname{const.} e^{-\alpha_8 R_k} \operatorname{meas} (F^i_k(\lambda)).$$

So

$$\operatorname{meas} (g_k \setminus g_{k+1}) < \operatorname{Const.} e^{-\alpha_8 R_k} \sum_{i \leq m_0} \sum_{\lambda \in I_i} \operatorname{meas} (F^i_k(\lambda)) < \operatorname{Const.} e^{-\alpha_8 R_k} \times \operatorname{meas} \{|\lambda| \leq \eta_0\}).$$
So

\[ \sum_k \text{meas}(g_k \setminus g_{k+1}) < \text{meas}(g_0) \]

for \( n_0 \) sufficiently large such that \( \sum_k \text{const. } e^{-\alpha g} R_k < 1 \), and thus \( \text{meas}(\limsup g_k) > 0 \). The proof is completed.

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