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Annales scientifiques de l'É.N.S. 4e série, tome 19, nº 2 (1986), p. 275-301
<http://www.numdam.org/item?id=ASENS_1986_4_19_2_275_0>

Ann. scient. Éc. Norm. Sup.,
4[e] série, t. 19, 1986, p. 275 à 301.

DEGENERATIONS FOR REPRESENTATIONS OF QUIVERS WITH RELATIONS

BY CHRISTINE RIEDTMANN

1. Introduction

1.1 Let k be an algebraically closed field and Q a finite quiver. The quiver algebra kQ of Q over k has all paths in Q as a k-basis, and the product of two paths $w : i' \to j'$ and $v : i \to j$ is the composed path $wv : i \to j'$ if $i'=j$ and zero otherwise. A twosided ideal I in kQ is called admissible if there exists a natural number N such that $kQ_+^N \subseteq I \subseteq kQ_+^2$, where kQ_+ is the ideal generated by all paths of length 1, the arrows of Q. If I is admissible, kQ/I is a finite-dimensional k-algebra, and conversely any finite-dimensional k-algebra is Morita-equivalent to some kQ/I. We will only consider admissible ideals I.

A representation X of Q over k consists of a finite-dimensional k-vector space X(i) for each vertex i and a k-linear map $X(\alpha) : X(i) \to X(j)$ for each arrow $\alpha : i \to j$. The dimension vector $\underline{\dim}\, X$ is the vector with components $\dim_k X(i)$, and the dimension of X is the natural number $\sum \dim_k X(i)$, where i ranges over the vertices of Q. For a path $v=\alpha_r \ldots \alpha_1 : i \to j$ in Q, the linear map $X(v) : X(i) \to X(j)$ is defined to be the identity map of X(i) if $r=0$ and the composition $X(\alpha_r) \circ \ldots \circ X(\alpha_1)$ otherwise. A representation of (Q, I) is a representation X of Q with the additional property that for each linear combination $\varphi=\sum \lambda_v v$ of paths from i to j in I, the linear map $X(\varphi)=\sum \lambda_v X(v)$ is zero. Note that $I=\bigoplus_{i,j} I(i, j)$, where $I(i, j)$ is the intersection of I with the vector space $kQ(i, j)$ of paths from i to j, since I is twosided.

A morphism $f : X \to Y$ between two representations of (Q, I) is a family (f_i) of linear maps $f_i : X(i) \to Y(i)$ for each vertex i, such that for any arrow $\alpha : i \to j$ the equality $Y(\alpha) \circ f_i = f_j \circ X(\alpha)$ holds. A representation X is indecomposable if any decomposition $X \xrightarrow{\sim} X_1 \oplus X_2$ is trivial. The theorem of Krull-Schmidt says that any representation of (Q, I) can be written as a direct sum of indecomposables, and this decomposition is unique up to isomorphism. The category mod (Q, I) of (finite-dimensional) representations of (Q, I) is equivalent to the category of finite-dimensional kQ/I-modules.

KEY-WORDS. — Degeneration, representations of quivers., Auslander-Reiten quiver.
Classification A. M. S. — 16 A 64, 14 L 30.

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE. — 0012-9593/86/02 275 27/$ 4.70/

1.2 Suppose that Q has n vertices $1, 2, \ldots, n$, and let $\underline{d}=(d_1, \ldots, d_n)$ be in $\mathbb{N}^n$. By $\mathscr{M}(\underline{d})$ we denote the set of representations X of (Q, I) with $X(i)=k^{d_i}$ for all i. A representation X in $\mathscr{M}(\underline{d})$ is given by a $d_j \times d_i$-matrix $X(\alpha)$ for each arrow $\alpha : i \to j$, and these matrices satisfy the equations given by I. Hence $\mathscr{M}(\underline{d})$ can be viewed as an affine variety, which is not necessarily irreducible. The group $G(\underline{d})=\prod_i GL(d_i)$ operates on $\mathscr{M}(\underline{d})$ by base change:

$$(g.X)(\alpha)=g_j \circ X(\alpha) \circ g_i^{-1}$$

for $g=(g_1, \ldots, g_n)\in G(\underline{d})$, $X\in\mathscr{M}(\underline{d})$, and $\alpha : i \to j$. The $G(\underline{d})$-orbits are the isomorphism classes of representations in $\mathscr{M}(\underline{d})$.

If X and Y are in $\mathscr{M}(\underline{d})$, we say that X degenerates to Y and write $X \ggcurly Y$ if Y is contained in the closure of the orbit $G(\underline{d}).X$ of X, with respect to the Zariski-topology. The aim of this paper is to find « algebraic » properties which are equivalent to the existence of a degeneration from X to Y. It is not clear whether such an algebraic description of the geometric behavior always exists.

1.3 Let us consider an example where it does:

$$Q=\circlearrowleft^{\alpha} \qquad \text{and} \qquad I=\langle \alpha^N \rangle, \qquad N\geq 2.$$

A representation X of (Q, I) in $\mathscr{M}(d)$ is given by a nilpotent endomorphism $X(\alpha)$ of k^d, and thus the $G(d)$-orbit of X is uniquely determined by the sizes $N\geq p_1 \geq p_2 \geq \ldots \geq p_d \geq 0$ of the Jordan blocks of $X(\alpha)$, or equivalently by the partition $p(X)=(p_1, p_2, \ldots, p_d)$ of d. Now X degenerates to Y if and only if $p(X) \geq p(Y)$, where the order relation on partitions is given by

$$(p_1, \ldots, p_d)\geq(q_1, \ldots, q_d) \Leftrightarrow p_1+\ldots+p_i\geq q_1+\ldots+q_i \qquad \text{for all} \quad i=1, \ldots, d.$$

The condition $p(X)\geq p(Y)$ is equivalent to rank $\alpha(Y^i)\leq$ rank $\alpha(X)^i$ for all i. A proof for these facts, in a different language, can be found in [6].

1.4 Let us return to an arbitrary (Q, I). For two representations U and Z, we write

$$\langle U, Z \rangle = \dim_k \operatorname{Hom}_{(Q,I)}(U, Z).$$

Using the semicontinuity of the fiber dimension for regular morphisms, it is easy to see that $X \ggcurly Y$ implies

(*) $\langle U, X \rangle \leq \langle U, Y \rangle$ for all representations U (see chapter 2).

We will see under which circumstances this condition is also sufficient to guarantee the existence of a degeneration $X \ggcurly Y$.

We say that there is a virtual degeneration from X to Y, and we write $X \succ Y$, if $X\oplus Z$ degenerates to $Y\oplus Z$ for some representation Z. The condition (*) is insensitive to cancelling common direct summands; i. e., for any representation U, $\langle U, X \rangle \leq \langle U, Y \rangle$ holds if and only if $\langle U, X\oplus Z \rangle \leq \langle U, Y\oplus Z \rangle$ is true for some Z. Our first result is that (*) is equivalent to the existence of a virtual degeneration from X to Y, provided that (Q, I) is of finite representation type; i. e., has only finitely many non-isomorphic indecomposable representations. One implication is obvious. As for the other one, we prove more precisely:

THEOREM 1. — *Let* (Q, I) *be of finite representation type, and suppose that* X, $Y \in \mathcal{M}(\underline{d})$ *satisfy* (*). *Then there exists an exact sequence*

$$0 \to A \to B \to C \to 0$$

of representations of (Q, I) *such that* $X \oplus A \oplus C$ *is isomorphic to* $Y \oplus B$.

Since the middle term of an exact sequence always degenerates to the direct sum of the two end terms (2.3), we find a degeneration from $X \oplus B$ to $Y \oplus B$.

1.5 The next question is whether cancellation holds for degenerations, that is, whether the existence of a virtual degeneration implies the existence of a degeneration. In 3.1 we give an example found by J. Carlson of two $k[S, T]/(S^2, T^2)$-modules X and Y such that there is a virtual degeneration $X \succ Y$, but nevertheless X does not degenerate to Y. There seems to be no such example known for pairs (Q, I) of finite representation type.

THEOREM 2. — *Let* (Q, I) *be of finite representation type, and assume there is a virtual degeneration* $X \succ Y$ *in* $\mathcal{M}(\underline{d})$. *Then there exists a degeneration from* X *to* Y, *provided that one of the following conditions is satisfied:*

1) *The underlying (non-oriented) graph* $\overline{Q}$ *of* Q *is a Dynkin diagram* A_n *or* D_n.

2) *The Auslander-Reiten quiver of* (Q, I) *is simply connected and* $\beta(Q, I) \leq 2$.

All the notions used to formulate 2) will be explained in chapter 3. Obviously 2) generalizes 1) for $\overline{Q} = A_n$.

In [1] and [2] S. Abeasis and A. Del Fra study degenerations for quivers with underlying graph A_n and for

$$Q = \begin{smallmatrix} \cdot \searrow \\ \cdot \nearrow \end{smallmatrix} \cdot \to \cdot \cdots \cdot \to \cdot,$$

respectively. Our strategy is the same as theirs; i. e., we find a complete set of obstructions for the existence of a degeneration. The main difference is that we use the theory of Auslander-Reiten sequences. This allows us first to give a handy description of some set of obstructions and to show that it is complete at least for the existence of virtual degenerations (theorem 1). Secondly we can avoid all difficulties arising in [1] from different orientations on the same graph.

1.6 *Remark.* — We could give our definitions for locally bounded k-categories and our results for locally representation-finite categories [4] instead of finite quivers with relations. In particular, theorem 2 is true for the universal cover of a representation-finite selfinjective algebra of class A_n [8]. In order to avoid introducing too many notions, we will just indicate briefly how to reduce from a locally representation-finite category Λ to a category with finitely many objects for the proofs of our theorems. Let $\underline{d}$ be a function from the objects of Λ to $\mathbb{N}$ which takes the value zero except for finitely many objects, and define $\mathcal{M}(\underline{d})$ as before. Let $S(\underline{d})$ be the finite set of objects i of Λ with the property that there exists an indecomposable Λ-module U with $U(i) \neq 0$ and such that $U(j) \neq 0$,

$\underline{d}(j) \neq 0$ for some object j. Then work with the full subcategory $\Lambda(\underline{d})$ of Λ whose objects lie in $S(\underline{d})$.

I would like to thank M. Auslander and H. Kraft for discussions and suggestions, J. Carlson for his example, and the referee for useful remarks. I am grateful also to the University of Bielefeld, where part of this work was done.

2. Virtual degenerations

2.1 Proposition. — *Let* X, Y *be in* $\mathscr{M}(\underline{d})$, *and suppose* X *degenerates to* Y. *Then the following inequalities hold:*

(*) $\langle U, X \rangle \leq \langle U, Y \rangle$ *for all representations* U.

Proof. — There is a quick proof, applying the semi-continuity of the fiber dimension to the projection $p : \mathscr{M}_U \to \mathscr{M}(\underline{d})$ on the first factor, where $\mathscr{M}_U$ is the variety of pairs (Z, f) with Z in $\mathscr{M}(\underline{d})$ and $f : U \to Z$ a morphism of representations. We will give instead an elementary argument, which illustrates the connection of (*) with the behavior of the ranks of linear maps (compare 1.3 and 1.5).

Consider a triple

$$A = ((i_1, \ldots, i_r), (j_1, \ldots, j_s), [\varphi_{j_l i_k}]),$$

where $(i_1, \ldots, i_r)$, $(j_1, \ldots, j_s)$ are sequences of vertices and where $[\varphi_{j_l i_k}]$ is an sxr-matrix with $\varphi_{j_l i_k} \in kQ(i_k, j_l)/I(i_k, j_l)$. With a representation Z of (Q, I) we associate the linear map

$$Z(A) : \bigoplus_{k=1}^{r} Z(i_k) \xrightarrow{[Z(\varphi_{j_l i_k})]} \bigoplus_{l=1}^{s} Z(j_l).$$

The following lemma implies the proposition.

Lemma. — *Let* X, Y *be in* $\mathscr{M}(\underline{d})$.

(*a*) *The inequalities* (*) *are equivalent to*

$$\operatorname{rank} X(A) \geq \operatorname{rank} Y(A)$$

for all triples A.

(*b*) *If* X *degenerates to* Y, rank X(A) ≥ rank Y(A) *for all triples* A.

Proof. — (*a*) Let $\mathscr{C}$ be the path category of Q modulo I; i. e. the k-linear category whose objects are the vertices of Q and whose morphism spaces are

$$\mathscr{C}(i, j) = kQ(i, j)/I(i, j);$$

composition is induced from kQ. A representation Z of (Q, I) is a covariant k-linear functor from $\mathscr{C}$ to the category of finite-dimensional k-vector spaces. In particular, we have for each vertex i the representable function $\mathscr{C}(i, \ )$, the projective cover of the one dimensional representation supported at i. By the Yoneda-lemma, the map

$$\operatorname{Hom}(\mathscr{C}(i, \ ), Z) \to Z(i)$$

given by evaluating at i on the trivial path is an isomorphism which is functorial in i and Z.

With a triple A as above, we associate the morphism of functors

$$\mu(\mathrm{A}): \bigoplus_{l=1}^{s} \mathscr{C}(j_l, \ ) \xrightarrow{[\mathscr{C}(\varphi_{j_l i_k}, \ )]} \bigoplus_{k=1}^{r} \mathscr{C}(i_k, \ )$$

and its cokernel $\mathrm{U}=\zeta(\mathrm{A})$. For any representation Z, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Hom}(\mathrm{U}, \mathrm{Z}) & \to & \bigoplus_{k=1}^{r} \mathrm{Hom}(\mathscr{C}(i_k, \ ), \mathrm{Z}) & \xrightarrow{\mu(\mathrm{A})(\mathrm{Z})} & \bigoplus_{l=1}^{s} \mathrm{Hom}(\mathscr{C}(j_l, \ ), \mathrm{Z}) \\ & & \downarrow \wr & & \downarrow \wr \\ 0 \to \mathrm{Ker}\, \mathrm{Z}(\mathrm{A}) & \longrightarrow & \bigoplus_{k=1}^{r} \mathrm{Z}(i_k) & \xrightarrow{\mathrm{Z}(\mathrm{A})} & \bigoplus_{l=1}^{s} \mathrm{Z}(j_l) \end{array}$$

We conclude that

$$\operatorname{rank} \mathrm{Z}(\mathrm{A}) + \langle \mathrm{U}, \mathrm{Z} \rangle = \sum_{k=1}^{r} \dim \mathrm{Z}(i_k) = \sum_{k=1}^{r} d_{i_k} \qquad \text{for} \qquad \mathrm{Z} \ \text{in} \ \mathscr{M}(\underline{d}).$$

This implies *a*), because any representation U of (Q, I) has a projective presentation and is therefore of the form $\zeta(\mathrm{A})$ for some A.

(*b*) For a triple A, we consider the map

$$f: \mathscr{M}(\underline{d}) \to \mathrm{Hom}_k\left(\bigoplus_{k=1}^{r} k^{d_{i_k}}, \bigoplus_{l=1}^{s} k^{d_{j_l}}\right) = \mathrm{H}_{\mathrm{A}}$$

given by sending Z to the matrix Z(A).

This is a $\mathrm{G}(\underline{d})$-equivariant polynomial map, where the operation of $\mathrm{G}(\underline{d})$ on H_{A} is defined by

$$g.[\psi_{j_l i_k}] = [g_{j_l} \circ \psi_{j_l i_k} \circ g_{i_k}^{-1}]$$

for $g=(g_1, \ldots, g_n) \in \mathrm{G}(\underline{d})$ and $[\psi_{j_l i_k}] \in \mathrm{H}_{\mathrm{A}}$.

For any natural number r, the set

$$\{ [\psi_{j_l i_k}] \in \mathrm{H}_{\mathrm{A}} : \operatorname{rank} [\psi_{j_l i_k}] \leq r \}$$

is closed, since a matrix lies in this set if and only if all its $(r+1) \times (r+1)$-minors vanish. Therefore the set

$$\{ \mathrm{Z} \in \mathscr{M}(\underline{d}) : \operatorname{rank} \mathrm{Z}(\mathrm{A}) \leq r \}$$

is closed as well. In particular, any Y in $\overline{\mathrm{G}(\underline{d}).\mathrm{X}}$ satisfies $\operatorname{rank} \mathrm{Y}(\mathrm{A}) \leq \operatorname{rank} \mathrm{X}(\mathrm{A})$.

Remark. — With a triple A we can also associate the morphism

$$[\mathrm{D}\mathscr{C}(\ , \varphi_{j_l i_k})]: \bigoplus_{l=1}^{s} \mathrm{D}\mathscr{C}(\ , j_l) \to \bigoplus_{k=1}^{r} \mathrm{D}\mathscr{C}(\ , i_k)$$

between injective representations, and its kernel $\sigma(\mathrm{A}) = \mathrm{U}$. Here D denotes the usual duality for vector spaces. The dual arguments of the above yield, for $\mathrm{Z} \in \mathscr{M}(\underline{d})$,

$$\langle \mathrm{Z}, \mathrm{U} \rangle + \operatorname{rank} \mathrm{Z}(\mathrm{A})^t = \sum_{l=1}^{s} d_{j_l}.$$

Since each representation admits an injective presentation, the inequalities (*) for X, $Y \in \mathcal{M}(\underline{d})$ are equivalent to

$$\langle X, U \rangle \leq \langle Y, U \rangle \quad \text{for all representations } U.$$

2.2 In order to prove theorem 1, we need the notion of Auslander-Reiten sequences, which are the almost split sequences of [3].

For each indecomposable non-projective representation V, there exists a non-split exact sequence

$$\Sigma_V : 0 \to \tau V \xrightarrow{f} E_V \xrightarrow{g} V \to 0$$

with the following properties:

(i) τV is indecomposable.

(ii) Any morphism $h : X \to V$ which is not a retraction factors through g.

Such a sequence Σ_V is called an Auslander-Reiten sequence stopping at V. The representations τV and E_V are uniquely determined by V, up to isomorphism. Moreover, any morphism $h : \tau V \to X$ which is not a section factors through f.

Dually, for each non-injective indecomposable V′, there exists an Auslander-Reiten sequence

$$0 \to V' \to E'_{V'} \to \tau^{-1}V' \to 0$$

starting at V′; it is at the same time an Auslander-Reiten sequence stopping at $\tau^{-1}V'$.

THEOREM 1. — *Let* (Q, I) *be of finite representation type, and suppose that* X, $Y \in \mathcal{M}(\underline{d})$ *satisfy* (*). *Then there exists an exact sequence*

$$0 \to A \to B \to C \to 0$$

of representations of (Q, I) *such that* $X \oplus A \oplus C$ *is isomorphic to* $Y \oplus B$.

Proof. — Define

$$\delta(V) = \langle V, Y \rangle - \langle V, X \rangle$$

for V indecomposable. By our hypothesis, we have $\delta(V) \geq 0$, and since X, Y both lie in $\mathcal{M}(\underline{d})$, $\delta(V) = 0$ if V is projective. We set

$$0 \to A \to B \to C \to 0 = \bigoplus_V \Sigma_V^{\delta(V)};$$

i. e., we take the direct sum of $\delta(V)$ copies of the Auslander-Reiten sequence stopping at V for each indecomposable non-projective V.

For any indecomposable U, we obtain an exact sequence

$$0 \to \mathrm{Hom}(U, A) \to \mathrm{Hom}(U, B) \to \mathrm{Hom}(U, C) \to k^{\delta(U)} \to 0,$$

using the property (ii) of Auslander-Reiten sequences. This yields

$$\langle U, A \rangle - \langle U, B \rangle + \langle U, C \rangle = \delta(U) = \langle U, Y \rangle - \langle U, X \rangle,$$

and thus

$$\langle U, A\oplus C\oplus X\rangle = \langle U, B\oplus Y\rangle.$$

By an unpublished result of Auslander, which follows quite easily from the existence of Auslander-Reiten sequences [3], the numbers $\langle U, Z\rangle$, for U indecomposable, characterize Z up to isomorphism, and therefore $X\oplus A\oplus C \simeq Y\oplus B$.

2.3 COROLLARY. — *Let* (Q, I) *be of finite representation type, and let* X, $Y\in\mathcal{M}(\underline{d})$. *Then* X *and* Y *satisfy* (*) *if and only if there exists a virtual degeneration from* X *to* Y.

Proof. — If $X\oplus Z$ degenerates to $Y\oplus Z$, the inequalities (*) hold for $X\oplus Z$ and $Y\oplus Z$ and thus for X and Y (proposition 2.1). To show the converse, we have to prove that an exact sequence

$$0\to A\to B\to C\to 0$$

gives rise to a degeneration $B\gg A\oplus C$, which is a consequence of the following lemma.

Let Z be a representation in $\mathcal{M}(\underline{d})$, filtered by subrepresentations

$$Z=Z_0\supseteq Z_1\supseteq\ldots\supseteq Z_r\supseteq Z_{r+1}=0,$$

set $\overline{Z_k}=Z_k/Z_{k+1}$, and denote the associated graded representation by

$$\operatorname{gr} Z=\bigoplus_{k=0}^{r}\overline{Z_k}.$$

LEMMA. — *There exists a one parameter subgroup* $\lambda: k^*\to G(\underline{d})$ *such that* $\lambda(t).Z$ *is isomorphic to* Z, $t\in k^*$, *and* $\lim_{t\to 0}\lambda(t).Z=\operatorname{gr} Z$. *In particular,* Z *degenerates to* gr Z.

Compare [7], where it is shown in addition that, conversely, any degeneration $\lambda(1).Z\gg\lambda(0).Z$ for a 1 PSG λ is of the form $Z\gg\operatorname{gr} Z$ for some filtration.

Proof. — For each vertex i, we choose a basis of $Z(i)$ which is adapted to the filtration. Then the matrix of $Z(\alpha)$ is of the form

$$\begin{bmatrix} \overline{Z_0}(\alpha) & 0 & 0 & \cdots & 0 \\ A_{10} & \overline{Z_1}(\alpha) & 0 & & \vdots \\ A_{20} & A_{21} & \overline{Z_2}(\alpha) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{r0} & A_{r1} & \cdots & A_{r,r-1} & \overline{Z_r}(\alpha) \end{bmatrix}$$

for any arrow α. We choose for $\lambda(t)$ the base change that replaces each basis vector x in $Z_k(i)\backslash Z_{k+1}(i)$ by $t^k x$. The matrix of $(\lambda(t).Z)(\alpha)$ is

$$\begin{bmatrix} \overline{Z_0}(\alpha) & 0 & 0 & \cdots & & 0 \\ tA_{10} & \overline{Z_1}(\alpha) & 0 & & & \vdots \\ t^2A_{20} & tA_{21} & \overline{Z_2}(\alpha) & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ & & & & \ddots & 0 \\ t^rA_{r0} & t^{r-1}A_{r1} & \cdots & & tA_{r,r-1} & \overline{Z_r}(\alpha) \end{bmatrix}.$$

We see that

$$\lim_{t\to 0} \lambda(t).Z = \operatorname{gr} Z.$$

2.4 REMARK. — Let (Q, I) be an arbitrary quiver with relations, and let $\mathscr{F}$ be the category of k-linear contravariant finitely presented functors from the category of representations of (Q, I) to the category of vector spaces. Suppose that F_1, $F_2 \in \mathscr{F}$ have the property that

$$\begin{cases} \dim_k F_1(U) \leq \dim_k F_2(U) & \text{for all } U, \\ \dim_k F_1(P) = \dim_k F_2(P) & \text{for all projectives } P. \end{cases}$$

As suggested by H. Lenzing, it would be interesting to know if there exists an $F_3 \in \mathscr{F}$ such that

$$\dim_k F_3(U) = \dim_k F_2(U) - \dim_k F_1(U) \quad \text{for all } U.$$

Without loss of generality, one may assume that F_1, F_2 are representable. If such an F_3 always exists for some (Q, I), the same arguments as above imply that (*) is equivalent to the existence of a virtual degeneration $X \succ Y$.

3. Cancellation for degenerations

3.1 First we give an example, due to J. Carlson, where cancellation does not hold; i. e., we find two representations X and Y such that there is a virtual degeneration $X \succ Y$ although X does not degenerate to Y. Let Q be the quiver

$$Q = \alpha \circlearrowleft \cdot \circlearrowright \beta$$

and $I = \langle \alpha\beta - \beta\alpha, \alpha^2, \beta^2 \rangle$. Note that kQ/I is the algebra $k[\alpha, \beta]/(\alpha^2, \beta^2)$. For $\lambda \in k$, let M_λ be the two-dimensional representation of (Q, I) given by

$$M_\lambda(\alpha) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M_\lambda(\beta) = \begin{bmatrix} 0 & 0 \\ \lambda & 0 \end{bmatrix}.$$

Denote by P the only indecomposable projective representation, which is at the same time injective.

The Auslander-Reiten sequence containing P as a direct summand of the middle term has the form

$$0 \to \operatorname{rad} P \xrightarrow{\begin{bmatrix} j \\ p \end{bmatrix}} P \oplus \operatorname{rad} P/\operatorname{soc} P \xrightarrow{[-p\, j]} P/\operatorname{soc} P \to 0,$$

where j and p are inclusion and canonical projection [3]. For each $\lambda \in k$, there are exact sequences

$$0 \to M_\lambda \to \operatorname{rad} P \to k \to 0$$
$$0 \to k \to P/\operatorname{soc} P \to M_\lambda \to 0,$$

where k is the unique one-dimensional representation. So we obtain degenerations

$$P \oplus \operatorname{rad} P/\operatorname{soc} P \succcurlyeq \operatorname{rad} P \oplus \operatorname{soc} P \succcurlyeq M_\lambda \oplus M_\mu \oplus k^2,$$

for any $\lambda, \mu \in k$, and since $\operatorname{rad} P/\operatorname{soc} P \simeq k^2$, this yields a virtual degeneration

$$P \succ M_\lambda \oplus M_\mu.$$

On the other hand, the endomorphism ring of P is 4-dimensional, so that the orbit $G(\underline{d}).P$ has dimension 12, where $\underline{d} = (4)$. For $\lambda \neq \mu$, the orbit of $M_\lambda \oplus M_\mu$ has dimension 10, and hence the closure of the union of the orbits of all $M_\lambda \oplus M_\mu$, $\lambda \neq \mu$, is 12-dimensional as well and cannot be contained in the irreducible variety $\overline{G(\underline{d}).P}$.

3.2 We now turn to our positive results. First we show that we may replace I be a smaller ideal.

Let (Q, I) be arbitrary, and let J be a twosided ideal containing I. Even though J is not necessary admissible, we denote by mod (Q, J) the full subcategory of mod (Q, I) of representations Z of Q satisfying $Z(\varphi) = 0$ for all $\varphi \in J$. By $\mathcal{M}_I(\underline{d})$ and $\mathcal{M}_J(\underline{d})$ we denote the algebraic varieties of representations Z of (Q, I) and (Q, J), respectively, with $Z(i) = k^{d_i}$ for all i. Clearly $\mathcal{M}_J(\underline{d})$ is a closed $G(\underline{d})$-stable subset of $\mathcal{M}_I(\underline{d})$, and therefore the closure in $\mathcal{M}_I(\underline{d})$ of the orbit of some $Z \in \mathcal{M}_J(\underline{d})$ is contained in $\mathcal{M}_J(\underline{d})$. As an immediate consequence we have:

LEMMA. — *If* $X \succ Y$ *implies* $X \succcurlyeq Y$ *for* X *and* Y *in* $\mathcal{M}_I(\underline{d})$, *then* $X \succ Y$ *implies* $X \succcurlyeq Y$ *for* X *and* Y *in* $\mathcal{M}_J(\underline{d})$.

3.3 THEOREM 2. — *Let* (Q, I) *be of finite representation type, and assume there is a virtual degeneration* $X \succ Y$ *in* $\mathcal{M}(\underline{d})$. *Then* X *degenerates to* Y *in the following cases:*

1) *If the underlying graph* $\overline{Q}$ *of* Q *is a Dynkin diagram* A_n *or* D_n.

2) *If the Auslander-Reiten quiver of* (Q, I) *is simply connected and* $\beta(Q, I) \leq 2$.

Note that we may assume $I = 0$ in the first case by the preceding lemma.

We will prove the theorem by examining the « building blocks » of degenerations. Let X, Y be in $\mathcal{M}(\underline{d})$ for some $\underline{d}$. We call a degeneration $X \geqslant Y$ or a virtual degeneration $X \succ Y$ irreducible if $X \not\simeq Y$ and if, for any Z in $\mathcal{M}(\underline{d})$, $X \geqslant Z \geqslant Y$ or $X \succ Z \succ Y$, respectively, implies that Z is isomorphic to X or to Y.

If X degenerates to Y and $X \not\simeq Y$, we have

$$\dim G(\underline{d}).X > \dim G(\underline{d}).Y \qquad \text{or equivalently} \quad \dim_k \operatorname{End} X < \dim_k \operatorname{End} Y.$$

Therefore there is a finite sequence $X \geqslant Z_1 \geqslant \ldots \geqslant Z_r \geqslant Y$ of irreducible degenerations whenever $X \geqslant Y$ and $X \not\simeq Y$. In general, there seems to be no reason why such a finite sequence of irreducible virtual degenerations should exist for a virtual degeneration $X \succ Y$. By 2.1 we still have

$$\dim_k \operatorname{End} X = \langle X, X \rangle \leq \langle X, Y \rangle \leq \langle Y, Y \rangle = \dim_k \operatorname{End} Y,$$

but the inequality might not be strict for $X \not\simeq Y$. It would be interesting to have an example of a virtual degeneration $X \succ Y$ with $X \not\simeq Y$ and $\dim_k \operatorname{End} X = \dim_k \operatorname{End} Y$. For such an example, (Q, I) has to be of infinite type: If (Q, I) is of finite type, $X \succ Y$ and $X \not\simeq Y$ implies that the sum

$$\Sigma(\langle U, Y \rangle - \langle U, X \rangle)$$

over all indecomposables U is positive and finite, and hence there is a finite sequence of $X \succ Z_1 \succ \ldots \succ Z_r \succ Y$ of irreducible virtual degenerations.

Thus the following proposition implies theorem 2.

PROPOSITION. — *Let* (Q, I) *be of finite representation type, satisfying* 1) *or* 2) *of theorem 2, and let* X *and* Y *be in* $\mathcal{M}(\underline{d})$ *with no nontrivial direct summand in common. If there is an irreducible virtual degeneration* $X \succ Y$, *then there exists an exact sequence*

$$0 \to Y_1 \to X \to Y_2 \to 0$$

with

$$Y \simeq Y_1 \oplus Y_2.$$

It follows easily from 3.2 that in case 1) we only have to prove this for $I = 0$.

3.4 Proposition 3.3 tells us more precisely that in some cases an irreducible degeneration is given by a short exact sequence. This is not true for an arbitrary (Q, I) of finite representation type:

Let $$Q = 1 \xrightarrow{\alpha} 2 \circlearrowleft \beta \qquad \text{and} \qquad I = \langle \beta^2 \rangle,$$

and set

$$X = k \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad Y = k \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

For $t \in k^*$, consider the element

$$g_t = \left([1], \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix} \right) \in G(\underline{d}), \qquad \underline{d} = (1, 2),$$

and let

$$X_t = g_t . X = k \xrightarrow{\left[\begin{smallmatrix} t \\ 1 \end{smallmatrix}\right]} k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Obviously, X degenerates to Y, and the degeneration is irreducible, since

$$\dim G(\underline{d}).X - \dim G(\underline{d}).Y = \dim_k \operatorname{End} Y - \dim_k \operatorname{End} X = 1 .$$

Since Y is indecomposable, this degeneration is not obtained from a short exact sequence.

This example leads us to a new type of degenerations, for arbitrary (Q, I).

PROPOSITION. — *If*

$$0 \to A \to A \oplus X \to Y \to 0$$

is exact, X *degenerates to* Y.

The same conclusion holds for an exact sequence

$$0 \to Y \to X \oplus A \to A \to 0 .$$

The example above is of this type : Take the sequence

$$0 \longrightarrow \left(0 \to k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \xrightarrow{\left(0,\left[\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right]\right),\ \left(0,\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right)} \begin{matrix} 0 \to k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \oplus \\ k \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{matrix} \xrightarrow{\left(0,\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right),\ \left(1,\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\right)} k \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} k^2 \circlearrowleft \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longrightarrow 0$$

For this (Q, I), any irreducible virtual degeneration is either the one above or is given by a short exact sequence. It would be interesting to find other types of irreducible degenerations.

Proof. — Let $f = \begin{bmatrix} \xi \\ \sigma \end{bmatrix} : A \to A \oplus X$ be the morphism given in the exact sequence and set $f_t = \begin{bmatrix} \xi + t . \mathbb{1} \\ \sigma \end{bmatrix}$, $t \in k$. Choose a k-supplement Z for $f(A)$ in $A \oplus X$. There exists an open neighborhood U of 0 such that for $t \in U$, f_t is injective and Z is a supplement for $f_t(A)$. The representation $Z_t = A \oplus X / \operatorname{im} f_t$ thus defined is isomorphic to X whenever $\xi + t . \mathbb{1}$ is an isomorphism, which it is for almost all $t \in U$, and Z_0 is isomorphic to Y.

3.5 We now introduce the notions we need for the proof of proposition 3.3.

A pair (Γ, τ) consisting of a quiver Γ and a bijection $\tau : \mathscr{P} \to \mathscr{I}$ between two subsets of vertices of Γ is a translation quiver if:

(*a*) Γ contains no loops nor multiple arrows and

(*b*) for each $x \in \mathscr{P}$, the set x^- of tails of arrows with head x coincides with the set $(\tau x)^+$ of heads of arrows with tail τx.

The full subquiver of Γ given by the vertices x, τx, and x^- is called the mesh of Γ which starts at τx and stops at x, for $x \in \mathscr{P}$. If $\alpha: y \to x$ an arrow of Γ, $x \in \mathscr{P}$, we set $\sigma\alpha: \tau x \to y$ (see Fig. 1).

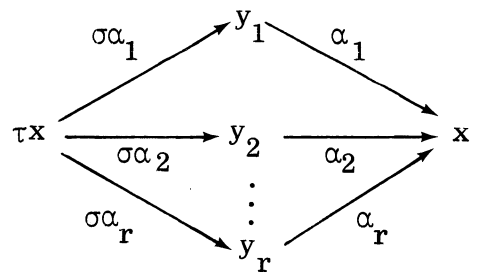

FIG. 1.

If Γ is locally finite (x^+ and x^- are finite for all vertices x), we define the mesh category $k(\Gamma)$: Its objects are the vertices of Γ, and

$$k(\Gamma)(x, y) = k\Gamma(x, y)/\mathrm{J}(x, y),$$

where $k\Gamma(x, y)$ is the vector space spanned by all paths from x to y in Γ and $\mathrm{J}(x, y)$ is the intersection of this vector space with the ideal generated by the « mesh relations »

$$\vartheta_x = \sum_{\alpha: y \to x} \alpha . \sigma\alpha$$

with $y \in x^-$, for all $x \in \mathscr{P}$. For a path $v: x \to y$ in Γ, we denote by $\bar{v}$ its residue class in $k(\Gamma)(x, y)$.

Let (Q, I) be arbitrary. The Auslander-Reiten quiver $\Gamma_{Q,I}$ of mod (Q, I) has as vertices representatives of the indecomposable representations. There is an arrow $\alpha: x \to y$ in $\Gamma_{Q,I}$ if there exists an irreducible morphism $f: x \to y$; i. e., a non-isomorphism with the property that for each factorization $f = h \circ g$ either g is a section or h is a retraction. The connection between irreducible morphisms and Auslander-Reiten sequences gives $\Gamma_{Q,I}$ the structure of a translation quiver (see [3]): For each non-projective vertex x there is a unique non-injective vertex τx and an Auslander-Reiten sequence

$$0 \to \tau x \to \bigoplus_{i=1}^{r} y_i \to x \to 0.$$

Moreover, $(\tau x)^+ = \{ y_1, \ldots, y_r \} = x^-$. So the translation τ is the Auslander-Reiten translation D*Tr*, which maps the non-projective vertices of $\Gamma_{Q,I}$ bijectively onto the non-injective ones.

If (Q, I) is of finite representation type and if $\Gamma_{Q,I}$ contains no oriented cycles, the full subcategory ind (Q, I) of mod (Q, I) whose objects are the vertices of $\Gamma_{Q,I}$ is equivalent to $k(\Gamma_{Q,I})$. So we may work in $k(\Gamma_{Q,I})$ in order to prove proposition 3.3.

A map $\pi:\Gamma_1 \to \Gamma_2$ between connected translation quivers is called a covering map if π induces bijections from x^+ to $(\pi x)^+$ and x^- to $(\pi x)^-$ for all vertices x. A connected translation quiver Γ is simply connected if any covering $\pi:\Gamma' \to \Gamma$ is an isomorphism. Let Γ be a translation quiver. For a vertex x of Γ, we let $\beta^+(x)$ and $\beta^-(x)$ be the number of vertices in x^+ and x^-, respectively, which are not projective-injective; i. e., for which either τ or τ^{-1} is defined. We let $\beta(\Gamma)$ be the bigger one of the two numbers

$$\sup_x \beta^+(x) \quad \text{and} \quad \sup_x \beta^-(x),$$

and we set $\beta(Q, I)=\beta(\Gamma_{Q,I})$ for a pair (Q, I).

3.6 We now explain our strategy for the proof of proposition 3.3, which will be carried out separately in all the cases we consider in the following chapters.

Let X and Y be in $\mathcal{M}(\underline{d})$ such that there exists a virtual degeneration $X \succ Y$, and set

$$\delta_{X,Y}(U)=\langle U, Y \rangle-\langle U, X \rangle$$

for any representation U. Given an exact sequence

$$\Sigma: 0 \to A \to B \to C \to 0.$$

define

$$\delta_\Sigma(U)=\langle U, A \rangle-\langle U, B \rangle+\langle U, C \rangle=\delta_{B,A\oplus C}(U).$$

We say that Σ is admissible for (X, Y) if $\delta_\Sigma(U) \leq \delta_{X,Y}(U)$ for all U.

In each case we consider, we will define a set $\mathcal{S}$ of non-split exact sequences with indecomposable end terms such that the following is true:

Given $X \not\simeq Y$ with $X \succ Y$, $\mathcal{S}$ contains a sequence

$$\Sigma: 0 \to A \to B \to C \to 0$$

which is admissible for (X, Y) and such that $Y \simeq A \oplus C \oplus Y'$ for some Y'.

Then we can prove proposition 3.3: Let X and Y be in $\mathcal{M}(\underline{d})$, without a direct summand in common, and such that there is an irreducible virtual degeneration $X \succ Y$. Choosing Σ as above, we have

$$\langle U, X \rangle=\langle U, Y \rangle-\delta_{X,Y}(U) \leq \langle U, Y \rangle-\delta_\Sigma(U)=\langle U, Y' \oplus B \rangle$$

for all U, and, by theorem 1, $X \succ Y' \oplus B$. By construction, $Y' \oplus B$ degenerates to Y, and $Y' \oplus B \not\simeq Y$, since Σ does not split. The irreducibility of $X \succ Y$ implies $X \simeq Y' \oplus B$, and since X and Y have no direct summand in common, we conclude that $X \simeq B$ and $Y \simeq A \oplus C$.

The following lemma will give us the left end of the desired exact sequence Σ.

LEMMA. — *Let* (Q, I) *be of finite representation type and suppose that* $\Gamma=\Gamma_{Q,I}$ *contains no oriented cycle. Choose* $X \succ Y$ *in* $\mathcal{M}(\underline{d})$, *set* $\delta=\delta_{X,Y}$, *and let* a *be a vertex of* Γ *such that*

$\delta(a)>0$ *and* $\delta(z)=0$ *whenever there exists a non-trivial path from* z *to* a *in* Γ. *Then* τa *is a direct summand of* Y.

Proof. — Since $\delta(a)>0$, a is not projective. Consider an Auslander-Reiten sequence

$$0 \to \tau a \xrightarrow{f} \oplus z \to a \to 0.$$

Using that every morphism starting at τa which is not a section factors through f, we obtain two exact sequences:

$$0 \to \operatorname{Hom}(a, X) \to \oplus \operatorname{Hom}(z, X) \to \operatorname{Hom}(\tau a, X) \to k^{\lambda} \to 0$$
$$0 \to \operatorname{Hom}(a, Y) \to \oplus \operatorname{Hom}(z, Y) \to \operatorname{Hom}(\tau a, Y) \to k^{\mu} \to 0,$$

where λ and μ are the multiplicities of τa in X and Y, respectively. Hence

$$\delta(a)=\delta(a)-\sum\delta(z)+\delta(\tau a)=\mu-\lambda>0,$$

and we conclude that $\mu>0$.

4. The case $\overline{Q}=A_n$

Let Q be a quiver with underlying graph A_n, $I=0$, and set $\Gamma=\Gamma_{Q,I}$. We view Γ as a subtranslation quiver of $\mathbb{Z}A_n$, which we now describe (see [8]).

4.1 Let K be the quiver

$$K=1 \to 2 \to 3 \ldots n-1 \to n.$$

Then $\mathbb{Z}A_n$ is obtained from $\mathbb{Z}\times K$ by adding an arrow $(i,j) \to (i+1,j-1)$ for $i\in\mathbb{Z}$ and $2\leq j\leq n$. The translation is given by $\tau(i,j)=(i-1,j)$. We say that the vertices (i, n) and $(i, 1)$ lie on the upper and the lower border, respectively.

Embed the opposite quiver Q^{op} of Q into $\mathbb{Z}A_n$ in such a way that all τ-orbits of vertices in $\mathbb{Z}A_n$ are hit. Then the Auslander-Reiten quiver Γ can be identified with the full subtranslation quiver of $\mathbb{Z}A_n$ given by the vertices lying on or between Q^{op} and $\nu(Q^{op})$, where ν is the Nakayama permutation $\nu(i,j)=(i+j-1,n+1-j)$ (see [5]). We write τ and $\tau_{\mathbb{Z}A_n}$ for the translation on Γ and $\mathbb{Z}A_n$, respectively, if we have to distinguish. We have to work with Q^{op} instead of Q since an arrow $i \to j$ of Q yields an irreducible map $\mathscr{C}(j, \ ) \to \mathscr{C}(i, \ )$ between projective indecomposables (see (2.1).

4.2 Let (i,j) be such that the vertex $(i+1,j)$ of $\mathbb{Z}A_n$ belongs to Γ and such that $2\leq j\leq n-1$. Then we say the two paths

$$(i,j) \to (i,j+1) \to (i+1,j) \qquad \text{and} \qquad (i,j) \to (i+1,j-1) \to (i+1,j)$$

are homotopic. We call two paths v, w: $x \to y$ in Γ homotopic if they are equivalent under the equivalence relation generated by the « mesh homotopies » just defined. A path in Γ is called essential if it is not homotopic to a path factoring through $(i, n) \to (i+1,n-1) \to (i+1,n)$ or $(i,1) \to (i,2) \to (i+1,1)$.

Obviously any two paths v, $w: x \to y$ in Γ are homotopic, and therefore they yield the same morphism $\bar{v}, \bar{w} \in k(\Gamma)(x, y)$, up to the sign, which depends on the number of meshes of Γ « lying between v and w »; i. e., the number of mesh homotopies used to get from v to w, modulo 2. So we have that $k(\Gamma)(x, y)$ equals k or 0 depending on the existence of an essential path from x to y.

4.3 For two vertices a and c of Γ with $k(\Gamma)(a, c) \neq 0$, we define the rectangle

$$R_{a,c} = \{ d : k(\Gamma)(a, d) \neq 0 \neq k(\Gamma)(d, c) \}$$

(see Fig. 2). Note that $R_{a,c}$ intersects the upper and the lower border of $\mathbb{Z}A_n$ at the same time if and only if $c = \nu(a)$, and then a is projective and c injective.

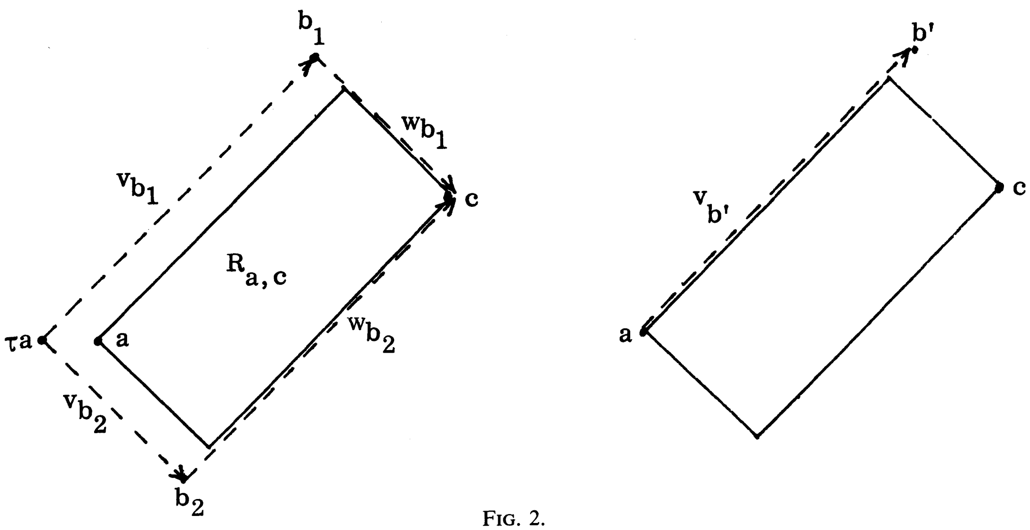

FIG. 2.

From now on we suppose that a is not projective. We denote by $\mathscr{B}_{a,c}$ the set of vertices of Γ which lie in a mesh stopping in $R_{a,c}$ but neither in $R_{a,c}$ nor in $\tau(R_{a,c})$. If $R_{a,c}$ does not hit either border, we have

$$\mathscr{B}_{a,c} = R_{\tau a,c} \setminus (R_{a,c} \cup \tau(R_{a,c})) = \{ b_1, b_2 \}$$

as in Fig. 2. Otherwise $\mathscr{B}_{a,c}$ consists of one vertex only. For $b \in \mathscr{B}_{a,c}$, we let

$$v_b : \tau a \to b \quad \text{and} \quad w_b : b \to c$$

be the unique paths in Γ between these vertices (Fig. 2). We define a sequence

$$\Sigma_{a,c} : 0 \to \tau a \overset{[\bar{v}_b]}{\to} \oplus b \xrightarrow{[\varepsilon_b \bar{w}_b]} c \to 0 .$$

where b ranges over $\mathscr{B}_{a,c}$ and where $\varepsilon_b = \pm 1$ with the only condition that

$$\varepsilon_{b_1} \varepsilon_{b_2} = (-1)^{\operatorname{card} R_{a,c} - 1}$$

for $\mathscr{B} = \{ b_1, b_2 \}$.

LEMMA. — $\Sigma_{a,c}$ *is exact, and*

$$\delta_{\Sigma_{a,c}}(z)=\begin{cases} 1 & \textit{for } z\in \mathrm{R}_{a,c} \\ 0 & \textit{otherwise.}\end{cases}$$

Proof. — There exists an essential path $v: z \to c$ in Γ which is not homotopic to one factoring through a vertex $b\in\mathscr{B}_{a,c}$ if and only if z lies in $\mathrm{R}_{a,c}$. So we have:

$$\operatorname{coker}(\bigoplus_b k(\Gamma)(z, b) \to k(\Gamma)(z, c))=\begin{cases} k & \text{for } \ z\in \mathrm{R}_{a,c}, \\ 0 & \text{otherwise},\end{cases}$$

and similarly

$$\operatorname{coker}(\bigoplus_b k(\Gamma)(b, z) \to k(\Gamma)(\tau a, z))=\begin{cases} k & \text{for } \ z\in \tau(\mathrm{R}_{a,c}), \\ 0 & \text{otherwise}.\end{cases}$$

In particular, the statement about $\delta_{\Sigma_{a,c}}$ is a consequence of the exactness of $\Sigma_{a,c}$.

We show that for any vertex z the induced sequence

$$0 \to k(\Gamma)(z, \tau a) \to \bigoplus_b k(\Gamma)(z, b) \to k(\Gamma)(z, c)$$

is exact, leaving the dual assertion to the reader. Since an injective vertex j cannot belong to $\tau(\mathrm{R}_{a,c})$, the map

$$\bigoplus_b k(\Gamma)(b, j) \to k(\Gamma)(\tau a, j)$$

is surjective. Choose maps $i_k : \tau a \to j_k$ from τa to injective vertices j_k such that $[i_k] : \tau a \to \oplus j_k$ is an injective envelope. There is a commutative triangle

$$\begin{array}{ccc} k(\Gamma)(z, \tau a) & \hookrightarrow & \bigoplus_b k(\Gamma)(z, b) \\ & \searrow \quad \swarrow & \\ & \bigoplus_k k(\Gamma)(z, j_k) & \end{array}$$

which guarantees that $k(\Gamma)(z, [\overline{v}_b])$ is injective.

By the definition of ε_b, we have

$$\sum_b \varepsilon_b \overline{w}_b \overline{v}_b = 0 .$$

On the other hand, let $f_b\in k(\Gamma)(z, b)$ be such that

$$\sum_b \varepsilon_b \overline{w}_b f_b = 0 .$$

We have to find $g\in k(\Gamma)(z, \tau a)$ with $f_b=\overline{v}_b g$. We may assume that for some $b_1\in\mathscr{B}_{a,c}$ there is an essential path $v'_1 : z \to b_1$ with $f_{b_1}=\overline{v}'_1$, and we will suppose that the second coordinate of b_1 is greater than the second coordinate of τa (as in Fig. 2).

If $w_{b_1}v'_1$ is not essential, it is homotopic to a path factoring through $(i, 1) \to (i, 2) \to (i+1, 1)$ for some i, because v'_1 is essential. Therefore v'_1 is homotopic to $v_{b_1}u$ for some path $u: z \to \tau a$. Taking $g=\overline{u}$ certainly works if $\mathscr{B}_{a,c}=\{ b_1 \}$. In case $\mathscr{B}_{a,c}=\{ b_1, b_2 \}$, the path $v_{b_2}u: z \to b_2$ is not essential, hence $k(\Gamma)(z, b_2)=0$ and $f_{b_2}=0=\overline{v}_{b_2}\overline{u}$.

If $w_{b_1}v'_1$ is essential, the condition

$$\sum_b \varepsilon_b \overline{w}_b f_b = 0$$

implies that $\mathscr{B}_{a,c} = \{ b_1, b_2 \}$ and that $w_{b_1}v'_1$ is homotopic to $w_{b_2}v'_2$ for some essential path $v'_2 : z \to b_2$ with $f_{b_2} = \pm \overline{v}'_2$. But then v'_i is homotopic to $v_{b_i}u$ for some path $u : z \to \tau a$, $i = 1, 2$, and we can take $g = \overline{u}$.

4.4 Let $\mathscr{B}'_{a,c}$ be the set of vertices of Γ that belong to a mesh starting in $R_{a,c}$ but neither to $R_{a,c}$ nor to $\tau^{-1}(R_{a,c})$ (see Fig. 2). Note that this situation is not exactly dual to the one studied in the preceding paragraph as c might be injective. In particular, $\mathscr{B}'_{a,c}$ can be empty, whereas $\mathscr{B}_{a,c}$ cannot. For $b' \in \mathscr{B}'_{a,c}$, let $v_{b'} : a \to b'$ be the only path between these vertices. Consider the map

$$f_{a,c} = [\overline{v}_{b'}] : a \to \oplus b',$$

where b' ranges over $\mathscr{B}'_{a,c}$. If c is not injective, this is just the left half of $\Sigma_{\tau^{-1}a, \tau^{-1}c}$. However, if c is injective, $f_{a,c}$ is not a monomorphism. For any vertex z of Γ, we set

$$\delta'_{a,c}(z) = \dim_k \operatorname{coker} (\bigoplus_{b'} k(\Gamma)(b', z) \to k(\Gamma)(a, z)).$$

The following lemma is easy to prove.

LEMMA.

$$\delta'_{a,c}(z) = \begin{cases} 1 & \textit{for} \quad z \in R_{a,c}, \\ 0 & \textit{otherwise}. \end{cases}$$

If there is a virtual degeneration $X \succ Y$, we have the following consequence (with the notations of 3.6).

COROLLARY. — *If* $R_{a,c}$ *contains no direct summand of* Y, *then* $\delta_{X,Y}(a) \leq \sum_{b'} \delta_{X,Y}(b')$.

Proof. — For any vertex z, we have an exact sequence

$$0 \to \operatorname{Hom}(\operatorname{coker} f_{a,c}, z) \to \bigoplus_{b'} \operatorname{Hom}(b', z) \to \operatorname{Hom}(a, z) \to k^{\delta'_{a,c}(z)} \to 0,$$

which yields

$$\langle \operatorname{coker} f_{a,c}, z \rangle - \sum_{b'} \langle b', z \rangle + \langle a, z \rangle = \delta'_{a,c}(z).$$

The value of $\delta'_{a,c}$ is zero for all vertices z which are direct summands of Y, and therefore

$$\delta_{X,Y}(\operatorname{coker} f_{a,c}) - \sum_{b'} \delta_{X,Y}(b') + \delta_{X,Y}(a) \leq 0.$$

4.5 We are now ready to carry out the program explained in 3.6. We fix X and Y in $\mathscr{M}(\underline{d})$ non-isomorphic and such that there exists a virtual degeneration $X \succ Y$.

The set $\mathscr{S}$ of exact sequences consists of the $\Sigma_{a,c}$ constructed in 4.3. We call a rectangle $R_{a,c}$ admissible if $\Sigma_{a,c}$ is admissible for (X, Y). We have to prove:

Claim. — There exists an admissible rectangle $R_{a,c}$ for which τa and c are direct summands of Y.

We choose the vertex a as in lemma 3.6; i. e., such that $\delta_{X,Y}(a)>0$ and $\delta_{X,Y}(z)=0$ for all proper predecessors of a. Then $\tau a \notin Y$ and $R_{a,a}$ is admissible. Let $R_{a,d}$ be a biggest rectangle, with respect to inclusion, among all admissible rectangles « starting in a ». It suffices to show that $R_{a,d}$ contains some vertex $c \notin Y$.

Suppose this is not the case. Then corollary 4.4 applies to all subrectangles of $R_{a,d}$, and it is enough to find some $R_{a',d} \subset R_{a,d}$ such that $\delta_{X,Y}(b')=0$ for all $b' \in \mathscr{B}'_{a',d}$. It is easy to see that for this we can choose the smallest rectangle $R_{a',d}$ stopping in d with the property that no bigger rectangle $R_{a',d'} \supsetneqq R_{a',d}$ is admissible.

5. The case $\beta(Q, I) \leq 2$ and $\Gamma_{Q,I}$ simply connected

Let (Q, I) be of finite representation type, and assume that Q (and hence $\Gamma_{Q,I}$) is connected, that $\beta(Q, I) \leq 2$, and that $\Gamma = \Gamma_{Q,I}$ is simply connected. We will generalize the proof for $\overline{Q} = A_n$ given in the preceding chapter. In 5.5, we will indicate how to weaken the topological condition on Γ: It suffices to assume that Γ contains no oriented cycle.

5.1 As mentioned in 3.1, an Auslander-Reiten sequence containing a projective-injective indecomposable P as a direct summand of its middle term is of the form

$$0 \to \text{rad } P \to P \oplus \text{rad } P/\text{soc } P \to P/\text{soc } P \to 0.$$

Since rad P determines uniquely its injective envelope P, the condition $\beta(Q, I) \leq 2$ implies that card $x^+ \leq 3$ for all vertices x. We denote by $\mathscr{D}$ the set of vertices x with card $x^+ = 3$, and for $x \in \mathscr{D}$ we let x^* be the injective envelope of x.

We call the vertices x^* with $x \in \mathscr{D}$ special and the other ones ordinary. Note that a projective-injective vertex may still be ordinary. A path $v: x \to y$ in Γ is ordinary if it does not pass through special vertices.

Since Γ is simply connected, the following rules divide the ordinary arrows of Γ into two disjoint classes. We will say that the arrows in one class go up, the other ones down.

(i) α and $\sigma\alpha$ belong to distinct classes.

(ii) If two ordinary arrows start or land in a vertex, they belong to distinct classes.

With each vertex x of Γ we associate its height $h(x) \in \mathbb{Z}$: We choose $h(x_0)=0$ for some ordinary vertex x_0, and for each ordinary arrow $\alpha: x \to y$ we set $h(y)=h(x)+1$ or $h(x)-1$ according as α goes up or down. Finally, we define $h(x^*)=h(x)$ for $x \in \mathscr{D}$.

5.2 Let x be a non-injective vertex with card $x^+ = 2$. Then we say that the two paths of length 2 from x to $\tau^{-1}x$ are homotopic. We call two ordinary paths $v, w: x \to y$ homotopic if they are equivalent with respect to the equivalence relation generated by these mesh homotopies. We say that an ordinary path $v: x \to y$ is essential if v is not homotopic to a path containing some $\tau z \to z' \to z$ with $z^- = \{z'\}$.

LEMMA. — *For any two ordinary vertices x and y, $k(\Gamma)(x, y) = \oplus k\overline{v}$, where v ranges over representatives of the homotopy classes of essential paths from x to y.*

Proof. — Modulo the mesh relations for meshes containing a special vertex, any path from x to y in Γ can be replaced by a unique linear combination of ordinary paths. The remaining mesh relations express that homotopic paths yield the same morphism, up to a factor ± 1, and that an inessential path $w: x \to y$ yields $\overline{w}=0$.

REMARK. — For a special vertex x^*, the arrows $\iota : x \to x^*$ and $\varkappa : x^* \to \tau^{-1}x$ induce isomorphisms

$$k(\Gamma)(z, x) \xrightarrow{\sim} k(\Gamma)(z, x^*)$$

and

$$k(\Gamma)(\tau^{-1}x, z) \xrightarrow{\sim} k(\Gamma)(x^*, z)$$

for all vertices $z \neq x^*$ (see [4]).

For any ordinary path $v: x \to y$ there is a highest path $v_{\max}$ and a lowest path $v_{\min}$ homotopic to v, which are defined as follows: If $v_{\max}$ or $v_{\min}$ contains a subpath $\tau z \overset{\sigma\alpha}{\to} z' \overset{\alpha}{\to} z$ with card $z^- = 2$, then α goes down or up, respectively. The path v is essential if and only if neither $v_{\max}$ nor $v_{\min}$ contains a subpath $\tau z \overset{\sigma\alpha}{\to} z' \overset{\alpha}{\to} z$ with $z^- = \{ z' \}$.

5.3 Let $\underline{k(\Gamma)}$ be the residue category of $k(\Gamma)$ modulo the morphisms factoring through projectives; i. e., $\underline{k(\Gamma)}$ has the same objects as $k(\Gamma)$, and $\underline{k(\Gamma)}(x, y) = k(\Gamma)(x, y)/\text{proj}\,(x, y)$, where proj (x, y) is the subspace of morphisms $\sum g_p \circ f_p$ with $f_p \in k(\Gamma)(x, p)$ and $g_p \in k(\Gamma)(p, y)$ and p projective.

It is easy to see that $\dim_k \underline{k(\Gamma)}(x, y) \leq 1$ for any two vertices x, y of Γ. We claim that, for $x, y \in \mathscr{D}$,

$$\underline{k(\Gamma)}(x, y) = \begin{cases} k & \text{for } x=y, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, any morphism $f: x \to y$ can be extended to a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & x & \to & x^* & \to & s & \to & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \to & y & \to & y^* & \to & t & \to & 0, \end{array}$$

where s and t are the simple tops of x^* and y^*, respectively. If f is not an isomorphism, $\bar{f}=0$, and f factors through f^*.

Let a, c be two vertices of Γ such that $\underline{k(\Gamma)}(a, c) \neq 0$, and define the rectangle $R_{a,c}$ to be the following set of vertices:

$$R_{a,c} = \{ d : \underline{k(\Gamma)}(a, d) \neq 0 \neq \underline{k(\Gamma)}(d, c) \}$$

(see Fig. 3). Denote by $\mathscr{B}_{a,c}$ the set of vertices of Γ that belong to a mesh stopping in $R_{a,c}$ but neither to $R_{a,c}$ nor to $\tau(R_{a,c})$. The set $\mathscr{B}_{a,c}$ is not empty, since

$$\sum_{z \in R_{a,c}} (\underline{\dim}\, z + \underline{\dim}\, \tau z - \sum_{z' \in z^-} \underline{\dim}\, z') = \underline{\dim}\, \tau a + \underline{\dim}\, c - \sum_{b \in \mathscr{B}_{a,c}} \underline{\dim}\, b = 0.$$

For $b \in \mathscr{B}_{a,c}$, let $v_b : \tau a \to b$ and $w_b : b \to c$ be arbitrary paths between these vertices.

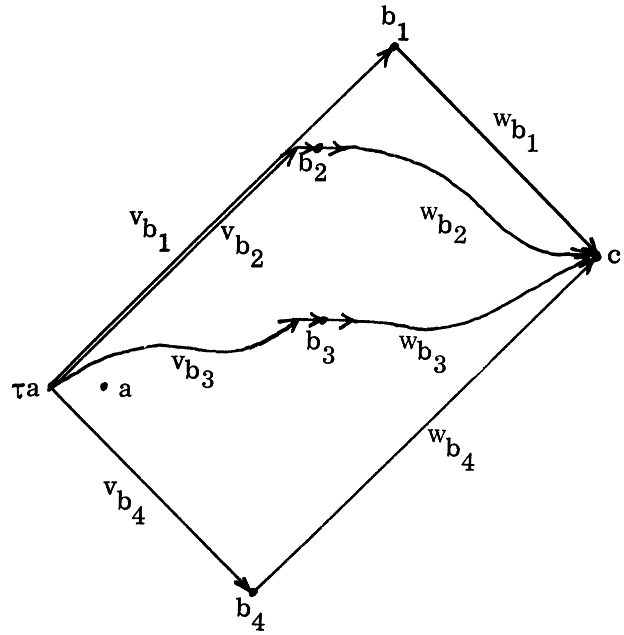

FIG. 3.

LEMMA. — *There exist numbers* $\varepsilon_b = \pm 1$, *for* $b \in \mathscr{B}_{a,c}$, *such that the sequence*

$$\Sigma_{a,c} : 0 \to \tau a \overset{[\bar{v}_b]}{\to} \oplus b \xrightarrow{[\varepsilon_b \bar{w}_b]} c \to 0$$

is exact. In addition,

$$\delta_{\Sigma_{a,c}}(z) = \begin{cases} 1 & \text{for} \quad z \in R_{a,c}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. — We claim that there exists no path $v : \tau^{-1}x \to y$ in Γ for $x, y \in \mathscr{D} \cap \tau(R_{a,c})$. If there is such a path, y must lie in $R_{a,c}$, and then v is essential. Since $\underline{k(\Gamma)}\,(x, y) = 0$, there exists an ordinary arrow $\alpha : x' \to \tau^{-1}x$ such that $v\alpha(\sigma\alpha)$ is not essential. Suppose α goes down. Since $R_{a,c}$ contains no projective vertices, the highest path $v_{\max}$ homotopic to v has to pass by the « upper corner of $R_{a,c}$ »; i. e., the highest vertex in $R_{a,c}$. But then y cannot lie in $\tau(R_{a,c})$.

The claim implies that any path from τa to $x \in \mathscr{D} \cap \tau(R_{a,c})$ is ordinary, and that all such paths are homotopic. Similarly, any path from $y \in R_{a,c} \cap \tau^{-1}(\mathscr{D})$ to c is ordinary, and all such paths are homotopic. In addition, no two vertices $x, y \in \mathscr{D} \cap \tau(R_{a,c})$ have the same height. We number the vertices of $\mathscr{B}_{a,c}$, setting $\mathscr{B}_{a,c} = \{ b_1, \ldots, b_t \}$ with $h(b_i) > h(b_j)$ for $i < j$ (see Fig. 3).

For $b = x^* \in \mathscr{B}_{a,c}$, with $x \in \mathscr{D}$, we let $v_{b,0}$ and $w_{b,0}$ be the subpaths of v_b and w_b obtained by deleting b, and we set

$$(w_b v_b)^+ = w_{b,0}\alpha(\sigma\alpha)v_{b,0},$$
$$(w_b v_b)^- = w_{b,0}\beta(\sigma\beta)v_{b,0},$$

where α and β are the arrows with head $\tau^{-1}x$ going down and up, respectively (see Fig. 4).

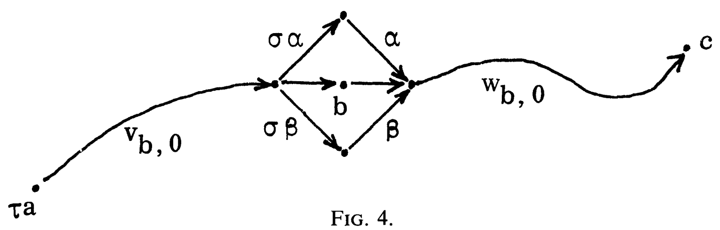

FIG. 4.

If $b \in \mathscr{B}_{a,c}$ is ordinary, we set

$$v_{b,0}=v_b, \quad w_{b,0}=w_b, \qquad \text{and} \qquad (w_b v_b)^+ = (w_b v_b)^- = w_b v_b.$$

It is easy to see, using again the claim above, that $(w_{b_i} v_{b_i})^-$ is homotopic to $(w_{b_{i+1}} v_{b_{i+1}})^+$ for $i=1, \ldots, t-1$. Therefore we can define ε_{b_i} by induction on i starting with $\varepsilon_{b_1}=1$ in such a way that

$$\sum_{i=1}^{t} \varepsilon_{b_i} \overline{w}_{b_i} \overline{v}_{b_i} = 0.$$

As in 4.3, we have that

$$\operatorname{coker}\big(\bigoplus_b k(\Gamma)(z, b) \xrightarrow{[k(\Gamma)(z, \varepsilon_b \overline{w}_b)]} k(\Gamma)(z, c)\big) = \begin{cases} k & \text{for } z \in \mathbf{R}_{a,c}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\operatorname{coker}\big(\bigoplus_b k(\Gamma)(b, z) \xrightarrow{[k(\Gamma)(\overline{v}_b, z)]} k(\Gamma)(\tau a, z)\big) = \begin{cases} k & \text{for } z \in \tau(\mathbf{R}_{a,c}), \\ 0 & \text{otherwise}. \end{cases}$$

So the statement about $\delta_{\Sigma_{a,c}}$ follows from the exactness of $\Sigma_{a,c}$.

We show that for any vertex z the induced sequence

$$0 \to k(\Gamma)(z, \tau a) \to \bigoplus_b k(\Gamma)(z, b) \to k(\Gamma)(z, c)$$

is exact. We may suppose z ordinary. The proof for the injectivity of $[k(\Gamma)(z, \overline{v}_b)]$ is the same as in 4.3.

It remains to be shown that for any family $f_b \in k(\Gamma)(z, b)$ with $\sum_b \varepsilon_b \overline{w}_b f_b = 0$ there exists a $g \in k(\Gamma)(z, \tau a)$ such that $f_b = \overline{v}_b g$. It suffices to examine families $[f_b]$ of the following kind (see Fig. 5): There is a path

$$v'_i : z \to b_i \quad \text{for} \quad 1 \leq r \leq i \leq s \leq t$$

such that $v'_{i,0}$ is essential, $(w_{b_r} v'_r)^+$ is not essential, $(w_{b_i} v'_i)^-$ is homotopic to $(w_{b_{i+1}} v'_{i+1})^+$ for $i=r, \ldots, s-1$, $(w_{b_s} v'_s)^-$ is not essential, and

$$f_{b_i} = \begin{cases} \pm \overline{v'_i} & \text{for } r \leq i \leq s, \\ 0 & \text{otherwise}. \end{cases}$$

We use the same notations as above; the signs $\pm$ in the definition of f_i are chosen in such a way that

$$\sum_{i=r}^{s} \varepsilon_{b_i} \overline{w}_{b_i} f_{b_i} = 0.$$

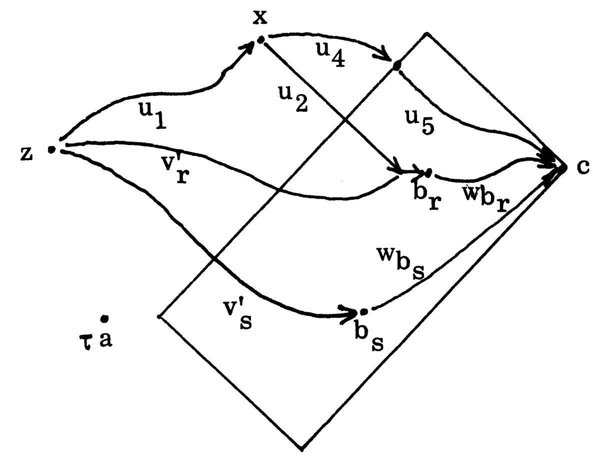

FIG. 5.

Consider the highest path homotopic to $v'_{r,0}$, and decompose it as $u_2 u_1$, where u_2 consists of arrows going down and where the last arrow of $u_1 : z \to x$ goes up unless u_1 is trivial. The highest path homotopic to $(w_{b_r} v'_r)^+$ contains u_1 as a subpath, and we decompose it as $u_3 u_1$. Since u_1 is essential, u_3 is not. Therefore x lies outside of $\mathrm{R}_{a,c}$, higher than τa, and we can decompose $u_3 = u_5 u_4$, where no vertex of u_4 but the last one lies in $\mathrm{R}_{a,c}$, whereas all vertices of u_5 do. The path u_4 cannot be essential.

Similarly, $v'_{s,0}$ is homotopic to a path passing through a vertex y outside of $\mathrm{R}_{a,c}$ and lower than τa. Hence we find a path $u : z \to \tau a$ such that $v'_{i,0}$ is homotopic to $v_{b_i,0} u$ for $i = r, \ldots, s$, and we set $g = \bar{u}$. For $i < r$, $v_{b_i,0} u$ is homotopic to a path containing u_4 and thus is not essential. In the same way $v_{b_i,0} u$ is inessential for $i > s$.

5.4 Let $\mathscr{B}'_{a,c}$ be the set of vertices belonging to a mesh that starts in $\mathrm{R}_{a,c}$ but neither to $\mathrm{R}_{a,c}$ nor to $\tau^{-1}(\mathrm{R}_{a,c})$. Choose any path $v_{b'} : a \to b'$ for $b' \in \mathscr{B}'_{a,c}$, and set

$$f_{a,c} = [\bar{v}_{b'}] : a \to \oplus b'.$$

We define

$$\delta'_{a,c}(z) = \dim_k \operatorname{coker} \left(\bigoplus_{b'} k(\Gamma)(b', z) \to k(\Gamma)(a, z) \right).$$

Then lemma 4.4 and its corollary are true. The proof of proposition 3.3 given in 4.5 for $\overline{\mathrm{Q}} = \mathrm{A}_n$ carries over to the present situation.

5.5 We indicate briefly how to extend this proof to pairs (Q, I) of finite representation type with $\beta(\mathrm{Q}, \mathrm{I}) \leq 2$ and such that $\Gamma = \Gamma_{\mathrm{Q,I}}$ is connected and contains no oriented cycle.

Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal cover of Γ. Starting from a rectangle $\mathrm{R}_{a,c}$ in $\tilde{\Gamma}$, we obtain a sequence $\pi(\Sigma_{a,c})$ of representations of (Q, I):

$$\pi(\Sigma_{a,c}) : 0 \to \pi\tau a \xrightarrow{[\overline{\pi v_b}]} \oplus \pi b \xrightarrow{[\varepsilon_b \overline{\pi w_b}]} \pi c \to 0,$$

and our considerations in 5.3 together with the fact that π induces a covering functor (see [4])

$$\pi : k(\tilde{\Gamma}) \rightarrow k(\Gamma) \qquad \text{imply that } \pi(\Sigma_{a,c}) \text{ is exact.}$$

In addition,

$$\delta_{\pi(\Sigma_{a,c})}(\pi z) = \text{card}\,(R_{a,c} \cap \Pi z),$$

for any z in $\tilde{\Gamma}$, where Π is the fundamental group of Γ. If a rectangle $R_{a,c}$ contains two distinct vertices z_1, $z_2 = g z_1$ lying in the same Π-orbit, we can assume that z_1, z_2 are corners of $R_{a,c}$; i. e., the highest and the lowest vertex in $R_{a,c}$, respectively. Moreover, replacing $R_{a,c}$ by $\tau^{-r}(R_{a,c})$ if necessary we may suppose that $R_{a,c}$ contains an injective vertex j. But then we find a vertex x with

$$k(\tilde{\Gamma})(x, j) \neq 0 \neq k(\tilde{\Gamma})(x, gj),$$

which means that πx contains some simple twice as a composition factor. But the structure of the indecomposable representations of (Q, I) is known, and a simple cannot have multiplicity 2 in an indecomposable unless Γ contains an oriented cycle. In fact, here it suffices to know that the endomorphism ring of any indecomposable is k. So we have that two rectangles $R_{a,c}$ and $g(R_{a,c})$ with $g \in \Pi \setminus \{1\}$ do not intersect or equivalently that

$$\delta_{\pi(\Sigma_{a,c})}(\pi z) \leq 1 \qquad \text{for all } z.$$

Using the set of exact sequences of the form $\pi(\Sigma_{a,c})$ as our set $\mathcal{S}$, we carry out the strategy of 3.6 as in 4.5 for $\overline{Q} = A_n$. Note that here we do need that Γ contains no oriented cycle, first to have lemma 3.6 and then to conclude that if, for a rectangle $R_{a,c}$, $\pi\tau a$ and πc are direct summands of Y, their direct sum $\pi\tau a \oplus \pi c$ is, too.

6. The case $\overline{Q} = D_n$

Let Q be a quiver with underlying graph $\overline{Q} = D_n$, $I = 0$, and set $\Gamma = \Gamma_{Q,I}$.

6.1 Let K be the quiver

$$K = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n-2 \begin{matrix} \nearrow n-1 \\ \searrow n \end{matrix}.$$

The translation quiver $\mathbb{Z}D_n$ is defined as follows [8]: Start from $\mathbb{Z} \times K$ and add an arrow $(i, j) \rightarrow (i+1, j-1)$, for $i \in \mathbb{Z}$ and $2 \leq j \leq n-1$, and an arrow $(i, n) \rightarrow (i+1, n-2)$ for $i \in \mathbb{Z}$. The translation is given by $\tau(i, j) = (i-1, j)$.

We call a vertex (i, j) of $\mathbb{Z}D_n$ low if $j \leq n-2$ and high otherwise. A high vertex (i, j) is said to be even or odd if $i+j$ is even or odd, respectively. Two high vertices (i, j) and (p, q) are called congruent if $i+j \equiv p+q \pmod 2$.

Embed the opposite quiver Q^{op} of Q into $\mathbb{Z}D_n$ in such a way that all τ-orbits of vertices in $\mathbb{Z}D_n$ are hit. Then the Auslander-Reiten quiver Γ can be identified with the full subquiver of $\mathbb{Z}D_n$ given by the vertices lying on or between Q^{op} and $\nu(Q^{op})$, where ν is the Nakayama permutation; if (i, j) is low, $\nu(i, j) = (i+n-2, j)$, and if (i, j) is high, $\nu(i, j)$ is the high vertex with first coordinate $i+n-2$ which is congruent to (i, j). From now on we denote by τ the translation of Γ.

6.2 We now describe the set of exact sequences which replace the sequences associated with rectangles in the previous chapters. Let $a=(i,j)$ and $c=(i',j')$ be vertices of Γ with $k(\Gamma)(a,c)\neq 0$, and suppose that a is not projective. Set

$$\mu_{a,c}=\begin{cases} 3 & \text{if } a \text{ and } c \text{ are low and } i+n-1\leq i'+j',\ i'\leq i+j-1, \\ 1 & \text{otherwise.}\end{cases}$$

Below we define a set $H_{a,c}$ of integralvalued functions $\eta_1, \ldots, \eta_{\mu_{a,c}}$ on the vertices of Γ for all pairs a, c. For $\eta\in H_{a,c}$, we set

$$\lambda_\eta=a, \qquad \rho_\eta=c,$$

and we define

$$\operatorname{supp}\eta=\{z:\eta(z)\neq 0\}.$$

we say that $\eta'\leq\eta$ for η, $\eta'\in H=\bigcup_{a,c} H_{a,c}$ if $\eta'(z)\leq\eta(z)$ for all vertices z.

Choose $\eta\in H_{a,c}$. We claim that:

1) There exists an exact sequence

$$\Sigma_\eta: 0\to \tau a \xrightarrow{[\bar v_b]} \oplus b \xrightarrow{[\varepsilon_b \bar w_b]} c\to 0$$

with $\delta_{\Sigma_\eta}=\eta$. In the middle term, b ranges over the set $\mathscr{B}_\eta$ of vertices belonging to a mesh stopping in $\operatorname{supp}\eta$ but neither to $\operatorname{supp}\eta$ nor to $\tau(\operatorname{supp}\eta)$. For $b\in\mathscr{B}_\eta$, $v_b:\tau a\to b$ and $w_b: b\to c$ are suitably chosen paths, and $\varepsilon_b=\pm 1$.

2) If $X\succ Y$ is a virtual degeneration and if no vertex in the support of η is a direct summand of Y, we have

$$\delta_{X,Y}(a)\leq\sum\delta_{X,Y}(b').$$

Here b' ranges over the set $\mathscr{B}'_\eta=\tau^{-1}(\mathscr{B}_\eta)$.

We leave it to the reader to verify this claim using the detailed description of morphisms in $k(\mathbb{Z}D_n)$ given in [9].

(i) *a and c are low and* $i'\leq i+j-1$, $i'+j'\leq i+n-2$:

$$\eta(p,q)=\begin{cases} 1 & \text{if } \ i\leq p\leq i',\ i+j\leq p+q\leq i'+j', \\ 0 & \text{otherwise.}\end{cases}$$

(ii) *a and c low and* $i'\leq i+j-1$, $i+n-1\leq i'+j'$:

$$\eta_1(p,q)=\begin{cases} 1 & \text{if } \ (p,q) \text{ is low and } i\leq p\leq i',\ i+j\leq p+q\leq i'+j', \\ 1 & \text{if } \ (p,q) \text{ is high and even and } i\leq p\leq i'+j'+1-n, \\ 0 & \text{otherwise.}\end{cases}$$

$$\eta_2(p,q)=\begin{cases} 1 & \text{if } \ (p,q) \text{ low and } i\leq p\leq i',\ i+j\leq p+q\leq i'+j', \\ 1 & \text{if } \ (p,q) \text{ is high and odd and } i\leq p\leq i'+j'+1-n, \\ 0 & \text{otherwise.}\end{cases}$$

$$\eta_3(p,q)=\begin{cases} 2 & \text{if } \ (p,q) \text{ is low and } p\leq i'+j'+1-n,\ i+n-1\leq p+q, \\ 1 & \text{if } \ (p,q) \text{ is low and } \ i\leq p\leq i'+j'+1-n,\ \ i+j\leq p+q\leq i+n-2, \\ 1 & \text{if } \ (p,q) \text{ is low and } i'+j'+2-n\leq p\leq i',\ i+n-1\leq p+q\leq i'+j', \\ 1 & \text{if } \ (p,q) \text{ is high and } i\leq p\leq i'+j'+1-n, \\ 0 & \text{otherwise.}\end{cases}$$

(iii) *a and c are low and* $i+j \leq i'$, $i+n-1 \leq i'+j'$:
η is defined as η_3 in (ii).

(iv) *a is low, c is high, and* $i' \leq i+j-1$:

$$\eta(p, q) = \begin{cases} 1 & \text{if } (p, q) \text{ is low and } i \leq p \leq i',\ i+j \leq p+q, \\ 1 & \text{if } (p, q) \text{ is high, congruent to } (i', j') \text{ and } i \leq p \leq i', \\ 0 & \text{otherwise}. \end{cases}$$

(v) *a is high, c is low and* $i' \leq i+n-1$:

$$\eta(p, q) = \begin{cases} 1 & \text{if } (p, q) \text{ is low and } p \leq i',\ i+n-1 \leq p+q \leq i'+j', \\ 1 & \text{if } (p, q) \text{ is high, congruent to } (i, j) \text{ and } i \leq p \leq i'+j'+1-n, \\ 0 & \text{otherwise}. \end{cases}$$

(vi) *a and c are high, congruent, and* $i' \leq i+n-1$:

$$\eta(p, q) = \begin{cases} 1 & \text{if } (p, q) \text{ is low, } p \leq i',\ i+n-1 \leq p+q, \\ 1 & \text{if } (p, q) \text{ is high, congruent to } (i, j), \text{ and } i \leq p \leq i', \\ 0 & \text{otherwise}. \end{cases}$$

Note that for $\eta \in H_{a,c}$ and $d \in \operatorname{supp} \eta$, there is a function $\eta' \in H_{a,d}$ with $\eta' \leq \eta$.

6.3 Let us carry out the strategy explained in 3.6. The set $\mathscr{S}$ of exact sequences consists of the Σ_η with $\eta \in H$.

Let $X \not\cong Y$ be in $\mathscr{M}(\underline{d})$ such that there is a virtual degeneration $X \succ Y$. We say that a function $\eta \in H$ is admissible for (X, Y) if Σ_η is. We choose the vertex a as in lemma 3.6; i. e., $\delta_{X,Y}(a) > 0$ and $\delta_{X,Y}(z) = 0$ for all proper predecessors of a. Then $\tau a \in\!\!\!\mid Y$ and the only function $\eta \in H_{a,a}$ is admissible.

Let η' be maximal among the functions in H with $\lambda_{\eta'} = \lambda_\eta = a$ which are admissible for (X, Y). Again it suffices to show that supp η' contains a direct summand of Y. If not, part 2) of the claim in 6.2 applies to all $\eta'' \leq \eta'$ with $\eta'' \in H$, and it suffices to find such an η'' with $\sum_{b'} \delta_{X,Y}(b') = 0$, $b' \in \mathscr{B}'_{\eta''}$. For this we can take the minimal element in the set of all functions $\varepsilon \in H$ with $\varepsilon \leq \eta'$, $\rho_\varepsilon = \rho_{\eta'}$ and such that no $\varepsilon' \geq \varepsilon$ with $\varepsilon' \neq \varepsilon$ and $\lambda_{\varepsilon'} = \lambda_\varepsilon$ is admissible for (X, Y).

We explain this in detail in the most complicated case: Suppose that $a = \lambda_{\eta'} = (i, j)$ and $c = \rho_{\eta'} = (i', j')$ are low with $i' \leq i+j-1$, $i+n-1 \leq i'+j'$ and that $\eta' = \eta_3 \in H_{a,c}$. The same argument works if $a = \lambda_{\eta'} = (i, j)$ and $c = \rho_{\eta'} = (i', j')$ are low with $i+j \leq i'$, $i+n-1 \leq i'+j'$ and if η' is the only function in $H_{a,c}$. For simplicity we assume that c is not injective.

By construction, η' is maximal among the functions $\eta \in H$ with $\lambda_\eta = a$ which are admissible for (X, Y). This implies that

$$\delta_{X,Y}(i'+1, q) = 0 \quad \text{for some } q \text{ with } i+n-2-i' \leq q \leq j'-1$$

and that $\delta_{X,Y}$ satisfies one of the following sets of inequalities (see Fig. 6).

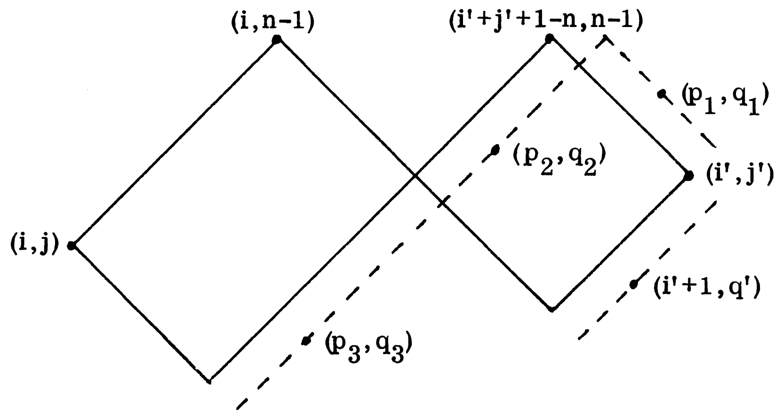

FIG. 6.

Set

$$Z_1=\{(p_1,q_1): p_1+\min(q_1,n-1)=i'+j'+1,\ q_1\geq j'+1\},$$
$$Z_2=\{(p_2,q_2): p_2=i'+j'+2-n,\ i+n-1\leq q_2\leq n-2\},$$
$$Z_3=\{(p_3,q_3): p_3=i'+j'+2-n,\ i+j\leq q_3\leq i+n-2\}.$$

(*a*) $\delta_{X,Y}(p_1,q_1)=0$ for some $(p_1,q_1)\in Z_1$, $\delta_{X,Y}(p'_1,q'_1)\geq 1$ for all $(p'_1,q'_1)\in Z_1$ with $q'_1<\min(q_1,n-1)$.

(*b*) $\delta_{X,Y}(p_2,q_2)\leq 1$ for some $(p_2,q_2)\in Z_2$, $\delta_{X,Y}(p'_2,q'_2)\geq 2$ for all $(p'_2,q'_2)\in Z_2$ with $q'_2>q_2$, and $\delta_{X,Y}\geq 1$ on Z_1.

(*c*) $\delta_{X,Y}(p_3,q_3)=0$ for some $(p_3,q_3)\in Z_3$, $\delta_{X,Y}(p'_3,q'_3)\geq 1$ for all $(p'_3,q'_3)\in Z_3$ with $q'_3>q_3$, $\delta_{X,Y}\geq 2$ on Z_2, and $\delta_{X,Y}\geq 1$ on Z_1.

In case (*a*), we have $\lambda_{\eta''}=(p_1, i'+1+q-p_1)$, and we obtain $\delta_{X,Y}(\lambda_{\eta''})=0$, a contradiction.

In case (*c*), $\lambda_{\eta''}=(i'+q+2-n,\ p_3+q_3-i'-q+n-2)$ and $\delta_{X,Y}(\lambda_{\eta''})=0$.

Finally, we show that case *b*) cannot occur. Indeed, set $c'=(i'+j'+2-n,\ n-1)\in Z_1$, and let $a'=(p_2+q_2+2-n,\ r)$ be the high vertex congruent to c'. Consider the exact sequence

$$\Sigma_\eta: 0\to \tau a'=(p_2+q_2+1-n,\ r)\to b=(p_2,q_2)\to c'=(i'+j'+2-n,\ r)\to 0$$

associated with the only function $\eta\in H_{a',c'}$. Since supp η' contains no direct summand of Y, the sequence

$$0\to \mathrm{Hom}(c',Y)\to \mathrm{Hom}(b,Y)\to \mathrm{Hom}(\tau a',Y)\to 0$$

is exact, and therefore

$$\delta_{X,Y}(c')+\delta_{X,Y}(\tau a')\leq \delta_{X,Y}(b)\leq 1.$$

But $\delta_{X,Y}(\tau a')\geq 1$ and thus $\delta_{X,Y}(c')=0$ although $c'\in Z_1$.

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(Manuscrit reçu le 29 janvier 1985
révisé le 18 novembre 1985).

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