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IWASAWA MODULES ATTACHED TO CONGRUENCES OF CUSP FORMS

BY HARUZO HIDA

§ 0. Introduction

A general principle in the study of congruences mod p between primitive cusp forms is (not to consider them directly but) to analyse the ring theoretic properties of the associated Hecke algebras. This approach appears indirect, but in fact, is more accessible. Roughly speaking, the local rings of the Hecke algebra over $\mathbf{Z}_p$ correspond bijectively to the maximal classes of primitive cusp forms congruent each other modulo p. If one of its local rings splits after extending scalar to $\mathbf{Q}_p$, then there exists a non-trivial congruence between distinct Galois conjugacy classes of primitive forms.

In our previous papers [7] [8] and Ribet [24], this principle was applied to primitive forms of fixed level and fixed weight. The present purpose is to consider all primitive forms of all weights for a fixed level, simultaneously, and to apply this principle. For a technical reason, we have to *assume* $p \geqq 5$ *for the prime p throughout this paper*. Then, as a result, the Hecke algebra $\mathscr{h}$ of the space of all *ordinary* forms is proved to be free of finite rank over the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[X]]$ (for the definition of ordinary forms, see below). The local rings of $\mathscr{h}$ correspond bijectively to the maximal classes of infinitely many ordinary forms congruent each other modulo p, and if one of them splits after extending scalar to the quotient field $\mathscr{L}$ of Λ, then there exists non-trivial congruences between systems of infinitely many ordinary forms. Furthermore, to each simple component $\mathscr{K}$ of $\mathscr{h} \otimes_\Lambda \mathscr{L}$, a finite torsion Λ-module $\mathscr{C}(\mathscr{K})$ can be naturally associated. In terms of $\mathscr{C}(\mathscr{K})$, one can give a fairly complete description of congruences mod p occurring at each weight between cusp forms belonging to $\mathscr{K}$ and others.

To give a more explicit illustration, we consider, just for simplicity, the space $\mathscr{S}_k(\Gamma_1(p);\mathbf{Z})$ consisting of cusp forms for $\Gamma_1(p)$ of weight k with rational integral Fourier coefficients, and put $\mathscr{S}_k = \mathscr{S}_k(\Gamma_1(p);\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Let $\mathscr{h}_k$ be the subalgebra of $\mathrm{End}_{\mathbf{Z}_p}(\mathscr{S}_k)$ generated over $\mathbf{Z}_p$ by all the Hecke operators $T(n)$ of level p. For sufficiently large $r \in \mathbf{Z}$, the p-adic limit $e = \lim_{n\to\infty} T(p)^{p^n(p^r-1)}$ exists and gives an idempotent of $\mathscr{h}_k$. Any non-zero common eigenform of operators in $\mathscr{h}_k$ is called ordinary if $f \mid e = f$ and its first Fourier coefficients is equal to 1. We fix once and for all an embedding of the algebraic closure of $\mathbf{Q}$ into the

p-adic completion Ω of an algebraic closure of $\mathbf{Q}_p$. Then, a non-zero common eigenform f with $f \mid T(p)=a(p,f)f$ is *ordinary* if and only if its eigenvalue $a(p,f)$ is a p-adic unit in Ω. For any ordinary form $f \neq 0$, there is a unique simple direct summand K of $h_k^0(\mathbf{Q}_p)=(eh_k)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$, which acts non-trivially on f. Let A be the complementary direct summand of K, $\bar{h}_k^0$ be the direct sum of the projected images of $h_k^0=eh_k$ in K and A, and define a finite p-power-torsion module $C(f)$ by $\bar{h}_k^0/h_k^0$. The following fact is fundamental in the theory of congruences of primitive forms:

(0.1) $C(f)\neq 0$ *if and only if there exists an ordinary form* g *in* $\mathscr{S}_k(\Omega)=\mathscr{S}_k\otimes_{\mathbf{Z}_p}\Omega$ *such that* $g\equiv f \bmod \mathfrak{P}$ *and* g *is not conjugate to* f *under any automorphism of* Ω *over* $\mathbf{Q}_p$,

where $\mathfrak{P}$ is the maximal ideal of the p-adic integer ring of Ω.

Now we generalize this to the infinite dimensional spaces of all ordinary forms of all weights. Let us denote by $\mathscr{S}^j$ the subspace of $\bigoplus_{k=1}^{j}\mathscr{S}_k^0(\mathbf{Q}_p)$ $(\mathscr{S}_k^0(\mathbf{Q}_p)=e\mathscr{S}_k(\mathbf{Q}_p))$ consisting of all forms with p-adically integral Fourier coefficients, and put $\mathscr{S}=\bigcup_j \mathscr{S}^j$. Then $\mathscr{S}$ contains $\bigoplus_{k=1}^{\infty}\mathscr{S}_k^0$ $(\mathscr{S}_k^0=e\mathscr{S}_k)$, but is much bigger than that. The usual action of Hecke operators respects $\mathscr{S}$ and $\mathscr{S}^j$ (see § 1). Naturally the Hecke algebra h^j is defined as the $\mathbf{Z}_p$-subalgebra of $\mathrm{End}_{\mathbf{Z}_p}(\mathscr{S}^j)$ generated by all Hecke operators. The restriction of operators of h^j to the subspace $\mathscr{S}^{j'}$ for $j>j'$ gives a projection morphism of h^j onto $h^{j'}$, and their projective limit $h=\varprojlim_j h^j$ naturally acts on $\mathscr{S}$. Let $z\in\Gamma=1+p\mathbf{Z}_p$ act on $\mathscr{S}_k^0(\mathbf{Q}_p)$ through $f \mid z=z^k f$; then, this action on $\bigoplus_{k=0}^{\infty}\mathscr{S}_k^0(\mathbf{Q}_p)$ leaves $\mathscr{S}$ stable by a result of Kats (see (1.12)). For any prime $l\equiv 1 \bmod p$, the action of $l\in\Gamma$ on $\mathscr{S}$ coincides with that of the Hecke operator $l^2T(l,l)$. Thus the Iwasawa algebra $\Lambda=\varprojlim_n \mathbf{Z}_p[\Gamma/\Gamma_n]$ $(\Gamma_n=1+p^n\mathbf{Z}_p)$ can be regarded as a subalgebra of h. Then one of our main results is

(0.2) h *is free of finite rank over* Λ (Theorem 3.1).

We now identify Λ with the power series ring $\mathbf{Z}_p[[X]]$ through $\Gamma\ni 1+p\mapsto 1+X\in\mathbf{Z}_p[[X]]$, and put $P_k(X)=(X+1)-(1+p)^k\in\Lambda$ for $k\in\mathbf{Z}$. The restriction of operators in h to the subspace $\mathscr{S}_k^0$ induces an isomorphism:

(0.3) $h/P_k h\cong h_k^0$ *if* $k\geqq 2$ (Corollary 3.2).

Let $\mathscr{L}$ be the quotient field of Λ and put $\mathscr{Q}=h\otimes_\Lambda\mathscr{L}$. Then

(0.4) $\mathscr{Q}$ *is a finite dimensional semi-simple algebra over* $\mathscr{L}$ (Corollary 3.3).

Let us take a simple direct summand $\mathscr{K}$ of $\mathscr{Q}$, and let $\mathscr{A}$ be its complementary direct summand. Denote by $h(\mathscr{K})$ and $h(\mathscr{A})$ the projected images of h in $\mathscr{K}$ and $\mathscr{A}$, respectively. Let us further pur $\tilde{h}(\mathscr{K})=\bigcap_P h(\mathscr{K})_P$, $\tilde{h}(\mathscr{A})=\bigcap_P h(\mathscr{A})_P$, $h'=h(\mathscr{K})\oplus h(\mathscr{A})$ and $\bar{h}=\tilde{h}(\mathscr{K})\oplus\tilde{h}(\mathscr{A})$, where P runs over all prime ideals of Λ of height 1, the subscript " P " indicates the localization at P and the intersection is taken in $\mathscr{K}$ and $\mathscr{A}$, respectively. Then, the module $\mathscr{N}=\bar{h}/h'$ has only finitely many elements; i. e., pseudo-null, and

(0.5) $\mathscr{C}(\mathscr{K})=\bar{h}/h$ *is a finite torsion module over* Λ (Theorem 3.6).

Furthermore, if $k \geqq 2$,

(0.6) $\mathscr{h}_k(\mathscr{K}) = \mathscr{h}(\mathscr{K})/\mathrm{P}_k \mathscr{h}(\mathscr{K})$ *is isomorphically embedded into* $\mathscr{h}_k^0(\mathbf{Q}_p)$ (Corollary 3.7).

For simplicity, we now assume that $\mathscr{K}$ is reduced to $\mathscr{L}$. Then (0.6) shows that there is a unique ordinary form f_k in $\mathscr{S}_k^0$ for each $k \geqq 2$, on which $\mathscr{h}(\mathscr{K})$ acts non-trivially. Then if $k \geqq 2$, we have an exact sequence:

$$(0.7) \qquad 0 \to \mathrm{C}(f_k) \to \mathscr{C}(\mathscr{K})/\mathrm{P}_k \mathscr{C}(\mathscr{K}) \to \mathscr{N}/\mathrm{P}_k \mathscr{N} \to 0 \quad \text{(Corollary 3.8)},$$

and we know that $\mathscr{C}(\mathscr{K}) \neq 0$ if and only if $\mathrm{C}(f_k) \neq 0$ for at least one $k \geqq 2$ (this condition is also equivalent to knowing that $\mathrm{C}(f_k) \neq 0$ for all $k \geqq 2$). Since $|\mathscr{N}| < \infty$ and $\mathscr{N}$ is independent of k, $\mathscr{C}(\mathscr{K})$ may be said to interpolate all the modules $\mathrm{C}(f_k)$ (as in (0.1)) for the system $\{f_k\}$ of p-ordinary forms belonging to $\mathscr{K}$. The pseudo-null module $\mathscr{N}$ is expected to vanish, and some sufficient (not too restrictive) conditions for $\mathscr{N}=0$ will be given in Proposition 3.9.

In this paper, we shall deal with only the algebraic aspect of the theory of the Iwasawa modules $\mathscr{C}(\mathscr{K})$, but as seen in [7, Th. 6.1, Cor. 6.3] and [8, §§ 6, 7], the number of elements of $\mathrm{C}(f_k)$ can be expressed by the rational part of the special value at $s=k$ of a certain zeta function $\mathrm{L}(s, f_k)$ of f_k. Thus the characteristic power series of $\mathscr{C}(\mathscr{K})$ may be conjectured to interpolate the values $\mathrm{L}(k, f_k)$ p-adically (Conjecture 3.10). An affirmative but partial solution of this conjecture will be given in our subsequent paper [11]. Besides this, another proof of the above facts (0.2-7) by using cohomology groups as in [8, § 3] will be given in [11].

The precise statement of our results valid for any level and over any ground ring will be given in § 3. The proof of (0.2) heavily relies on the theory of p-adic modular functions of Katz [18] and the duality between p-adic modular forms and their Hecke algebras. An exposition of Katz's theory is given in § 1 and a duality theorem is proved in § 2. Another key point is a result of Jochnowitz [13] which guarantees the finiteness of ordinary forms modulo p. This together with a proof of finiteness of $\mathscr{h}$ over Λ will be given in § 4. Main theorems will be proved in the following sections §§ 5 and 6. In § 7, we discuss the Hecke algebras obtained from theta series of imaginary quadratic fields and Eisenstein series. They provide ample examples of irreducible components $\mathscr{K}$ and Iwasawa modules $\mathscr{C}(\mathscr{K})$. Some other examples, together with a detailed exposition on the relation between congruences and the module $\mathscr{C}(\mathscr{K})$, are discussed in [31].

Notation. — The group $\mathrm{GL}_2^+(\mathbf{R})$ of real matrices with positive determinant can be considered as the holomorphic transformation group of the upper half complex plane $\mathfrak{H} = \{ z \in \mathbf{C} \mid \mathrm{Im}(z) > 0 \}$. For any function $f(z)$ on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$ and $k \in \mathbf{Z}$, we can define another function $f|_k\gamma$ by

$$(f|_k\gamma)(z) = \det(\gamma)^{k/2} f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k} \quad (z \in \mathfrak{H}).$$

For any positive integer N, we define subgroups of $SL_2(\mathbf{Z})$ by

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{Z}) \mid c\equiv 0 \bmod N\right\},$$
$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \Gamma_0(N) \mid d\equiv 1 \bmod N\right\},$$
$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \Gamma_1(N) \mid b\equiv 0 \bmod N\right\}.$$

Any holomorphic function f on $\mathfrak{H}$ with $f\mid_k\gamma=f$ for all $\gamma\in\Gamma(N)$ has Fourier expansion of the form:

$$\sum_{n\in\mathbf{Z}} a\left(\frac{n}{N}, f\right)e(nz/N) \qquad (e(z)=\exp(2\pi i z)). \tag{0.8}$$

Hereafter, we always write the Fourier expansion of f as in (0.8). For any subgroup Δ of $SL_2(\mathbf{Z})$ containing $\Gamma(N)$ for some N, the space $\mathcal{M}_k(\Delta)$ of modular forms on Δ consists of holomorphic functions f on $\mathfrak{H}$ with the properties:

(0.9 *a*) $\qquad f\mid_k\delta=f \qquad$ *for all* $\delta\in\Delta$;

(0.9 *b*) $\qquad a\left(\frac{n}{N}, f\mid_k\alpha\right)=0 \qquad$ *for all* $n<0$ *and any* $\alpha\in SL_2(\mathbf{Z})$.

For any character ψ of Δ of finite order, we put

$$\mathcal{M}_k(\Delta, \psi)=\{ f\in\mathcal{M}_k(\Delta) \mid f\mid_k\delta=\psi(\delta)f \quad \text{for any} \quad \delta\in\Delta\}.$$

Especially, any Dirichlet character ψ modulo N induces a character of $\Gamma_0(N)$ (which is again denoted by ψ) through

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\psi(d) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N).$$

Thus the space $\mathcal{M}_k(\Gamma_0(N), \psi)$ is defined in this manner. Put

$$\mathcal{S}_k(\Delta)=\{ f\in\mathcal{M}_k(\Delta) \mid a(0, f\mid_k\alpha)=0 \quad \text{for all} \quad \alpha\in SL_2(\mathbf{Z})\},$$
$$\mathcal{S}_k(\Delta, \psi)=\{ f\in\mathcal{M}_k(\Delta, \psi) \mid a(0, f\mid_k\alpha)=0 \quad \text{for all} \quad \alpha\in SL_2(\mathbf{Z})\}.$$

For any automorphism σ of $\mathbf{C}$ and $f\in\mathcal{M}_k(\Gamma_0(N), \psi)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \psi)$), there is a modular form $f^\sigma\in\mathcal{M}_k(\Gamma_0(N), \psi^\sigma)$ (resp. $\mathcal{S}_k(\Gamma_0(N), \psi^\sigma)$) such that $a(n, f^\sigma)=a(n, f)^\sigma$ for all n, where $\psi^\sigma(m)=(\psi(m))^\sigma$ for all $m\in\mathbf{Z}$ ([27, Th. 2]). The modular form f^σ is called a conjugate of f.

We denote by Ω the p-adic completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$. The normalized norm of $x\in\Omega$ is denoted by $|x|_p$ ($|p|_p=p^{-1}$). The algebraic closure of the rational number field $\mathbf{Q}$ is regarded as a subfield of $\mathbf{C}$ and Ω. Every finite extension of $\mathbf{Q}_p$ is considered in the universal domain Ω.

§ 1. p-adic modular forms and their Hecke algebras

Firstly, we shall define spaces of p-adic modular forms in an elementary manner; then, we shall give another definition of them in the context of works of Katz. This enables us to define Hecke operators acting on these spaces.

Let K be a finite extension of $\mathbf{Q}_p$ in Ω and $\mathcal{O}_K$ be the ring of all p-adic integers. Let K_0 be a finite extension of $\mathbf{Q}$ dense in K under the p-adic topology. Let Δ be either of the congruence subgroups $\Gamma_1(N)$ or $\Gamma(N)$ for a positive integer N. Put

$$\mathcal{M}_k(\Delta; K_0) = \left\{ f \in \mathcal{M}_k(\Delta) \mid a(n, f) \in K_0 \quad \text{for all} \quad n \in \frac{1}{N}\mathbf{Z} \right\},$$
$$\mathcal{M}_k(\Delta; K) = \mathcal{M}_k(\Delta; K_0) \otimes_{K_0} K$$

We define a p-adic norm $|\ \ |_p$ on $\Omega[[q^{1/N}]]$ by

$$(1.1) \qquad \left| \sum_{n=0}^{\infty} a(n) q^{n/N} \right|_p = \operatorname{Sup}_n |a(n)|_p.$$

We write the Fourier expansion of $f \in \mathcal{M}_k(\Delta; K_0)$ as

$$\sum_{n=0}^{\infty} a\left(\frac{n}{N}, f\right) q^{n/N} \qquad \text{for} \quad q = e(z).$$

Then one can define the norm $|f|_p$ by (1.1). It is known (cf. [27, Th. 1]) that $|f|_p$ is finite for all $f \in \mathcal{M}_k(\Delta; K_0)$, and we may regard $\mathcal{M}_k(\Delta; K)$ as a completion of $\mathcal{M}_k(\Delta; K_0)$ under this norm. Thus $\mathcal{M}_k(\Delta; K)$ can be identified with the closure of the image of $\mathcal{M}_k(\Delta; K_0)$ in $K[[q^{\frac{1}{N}}]]$, and thus every element of $\mathcal{M}_k(\Delta; K)$ has a unique q-expansion. The space $\mathcal{M}_k(\Delta; K)$ is determined independently of the choice of the subfield K_0 (see below (1.5)). Let Φ be either of the congruence subgroups $\Gamma_0(N)$ or $\Gamma_1(N) \cap \Gamma_0(p^r)$. Any Dirichlet character $\psi \bmod N$ or $\bmod\, p^r$ (according as $\Phi = \Gamma_0(N)$ or $\Gamma_1(N) \cap \Gamma_0(p^r)$) gives a character of Φ by $\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi(d)$. Then we can define, if ψ has values in K_0,

$$\mathcal{M}_k(\Phi, \psi; K_0) = \{ f \in \mathcal{M}_k(\Phi, \psi) \mid a(n, f) \in K_0 \qquad \text{for all} \quad n \},$$
$$\mathcal{M}_k(\Phi, \psi; K) = \mathcal{M}_k(\Phi, \psi; K_0) \otimes_{K_0} K.$$

Put

$$\mathcal{M}_k(\Delta; \mathcal{O}_K) = \{ f \in \mathcal{M}_k(\Delta; K) \mid |f|_p \leqq 1 \} = \mathcal{M}_k(\Delta; K) \cap \mathcal{O}_K[[q^{\frac{1}{N}}]],$$
$$\mathcal{M}_k(\Phi, \psi; \mathcal{O}_K) = \{ f \in \mathcal{M}_k(\Phi, \psi; K) \mid |f|_p \leqq 1 \}.$$

The corresponding spaces can be similarly defined for cusp forms, and for the space of p-adic cusp forms, we shall use the notations: $\mathcal{S}(\Delta; K)$, $\mathcal{S}(\Phi, \psi; \mathcal{O}_K)$, etc.

For each positive integer $j > 0$, put

$$\mathcal{M}^j(\Delta; K) = \bigoplus_{k=0}^{j} \mathcal{M}_k(\Delta; K), \quad \mathcal{S}^j(\Delta; K) = \bigoplus_{k=1}^{j} \mathcal{S}_k(\Delta; K).$$

Since $\mathscr{M}_k(\Delta; \mathrm{K})$ is embedded into $\mathrm{K}[[q^{1/\mathrm{N}}]]$ for each k, we can consider the q-expansion of any $f = \bigoplus_{k=0}^{j} f_k \in \mathscr{M}^j(\Delta; \mathrm{K})$ given by $a(n, f) = \sum_{k=0}^{j} a(n, f_k)$. Then, it is plain that $\mathscr{M}^j(\Delta; \mathrm{K})$ is embedded into $\mathrm{K}[[q^{1/\mathrm{N}}]]$ by this q-expansion, and so we can define a p-adic norm on $\mathscr{M}^j(\Delta; \mathrm{K})$ by (1.1). Put

$$\mathscr{M}^j(\Delta; \mathcal{O}_\mathrm{K}) = \{ f \in \mathscr{M}^j(\Delta; \mathrm{K}) \mid |f|_p \leq 1 \} = \mathscr{M}^j(\Delta; \mathrm{K}) \cap \mathcal{O}_\mathrm{K}[[q^{1/\mathrm{N}}]],$$
$$\mathscr{S}^j(\Delta; \mathcal{O}_\mathrm{K}) = \mathscr{S}^j(\Delta; \mathrm{K}) \cap \mathscr{M}^j(\Delta; \mathcal{O}_\mathrm{K}).$$

Let A denote either of the ring K or $\mathcal{O}_\mathrm{K}$. Now we take the limit:

$$\mathscr{M}(\Delta; \mathrm{A}) = \bigcup_j \mathscr{M}^j(\Delta; \mathrm{A}) \qquad \text{in} \quad \mathrm{A}[[q^{1/\mathrm{N}}]],$$
$$\mathscr{S}(\Delta; \mathrm{A}) = \bigcup_j \mathscr{S}^j(\Delta; \mathrm{A}) \qquad \text{in} \quad \mathrm{A}[[q^{1/\mathrm{N}}]].$$

Let $\overline{\mathscr{M}}(\Delta; \mathrm{A})$ and $\overline{\mathscr{S}}(\Delta; \mathrm{A})$ be the completion of $\mathscr{M}(\Delta; \mathrm{A})$ and $\mathscr{S}(\Delta; \mathrm{A})$ in $\mathrm{A}[[q^{1/\mathrm{N}}]]$ under the norm (1.1). The elements in $\overline{\mathscr{M}}(\Delta; \mathrm{A})$ will be called p-adic modular forms. When $\Delta = \Gamma_1(\mathrm{N})$, we simply write $\mathscr{M}^j(\mathrm{N}; \mathrm{A})$, $\mathscr{M}(\mathrm{N}; \mathrm{A})$ and $\overline{\mathscr{M}}(\mathrm{N}; \mathrm{A})$ instead of $\mathscr{M}^j(\Gamma_1(\mathrm{N}); \mathrm{A})$, $\mathscr{M}(\Gamma_1(\mathrm{N}); \mathrm{A})$ and $\overline{\mathscr{M}}(\Gamma_1(\mathrm{N}); \mathrm{A})$. This simplification for the symbols also applies to the space of cusp forms.

Here we summarize some results of Katz in a manner suited for our later application. Firstly, we give another definition of the space of modular forms as a solution of certain moduli problems. The details are found in [18, Chap. II]. Let μ_N be the finite flat group scheme over $\mathbf{Z}$ realized as the kernel of the multiplication by N on the multiplicative group $\mathbf{G}_m$. For each commutative ring A with identity, we consider triples $(\mathrm{E}, \omega, i)/\mathrm{A}$ consisting of the following three objects: (i) E is an elliptic curve (i. e., an abelian scheme of dimension 1) over A, (ii) ω is a nowhere vanishing invariant differential on E rational over A, and (iii) i is an inclusion over A of μ_N into the schematic kernel $\mathrm{E}[\mathrm{N}]$ in E of the multiplication by N. As an example of such triples, we may offer the Tate curve Tate (q) over the ring $\mathbf{Z}((q))$ of formal Laurent series (*cf.* [3, VII]). If we regard Tate (q) as a quotient of $\mathbf{G}_{m/\mathbf{Z}((q))}$ by the subgroup generated by q, the canonical level N structure $i_{\mathrm{can}} : \mu_\mathrm{N} \hookrightarrow \mathrm{Tate}(q)[\mathrm{N}]$ is induced by the natural inclusion μ_N into $\mathbf{G}_m$. If one identifies $\mathbf{G}_m$ with Spec $(\mathbf{Z}[x, x^{-1}])$, then the invariant differential dx/x induces a canonical differential ω_{can} on E. Thus we have the triple $(\mathrm{Tate}(q), \omega_{\mathrm{can}}, i_{\mathrm{can}})$ defined over $\mathbf{Z}((q))$. The space of modular forms on $\Gamma_1(\mathrm{N})$ over A in this context consists of functions f which assigns the value $f(\mathrm{E}, \omega, i) \in \mathrm{A}'$ to any triple (E, ω, i) over any over-ring A′ of A and which satisfies the following conditions:

(1.2 *a*) $f(\mathrm{E}, \omega, i)$ depends only on the A′-isomorphism class of (E, ω, i);

(1.2 *b*) The function f is compatible with the base change;

(1.2 *c*) For any unit $a \in (\mathrm{A}')^\times$, $\quad f(\mathrm{E}, a^{-1}\omega, i) = a^k f(\mathrm{E}, \omega, i)$.

The space of these functions is denoted by $\mathrm{R}_k(\Gamma_1(\mathrm{N}); \mathrm{A})$.

The evaluation of $f \in \mathrm{R}_k(\Gamma_1(\mathrm{N}); \mathrm{A})$ at $(\mathrm{Tate}(q), \omega_{\mathrm{can}}, i_{\mathrm{can}})$ gives an embedding:

$$(1.3\,a) \qquad \mathrm{R}_k(\Gamma_1(\mathrm{N}); \mathrm{A}) \hookrightarrow \mathrm{A}((q))$$

and if $A' \supset A$, then

(1.3 *b*) $f \in R_k(\Gamma_1(N); A)$ *if and only if* $f(\mathrm{Tate}(q), \omega_{can}, i_{can}) \in A((q))$ *and* $f \in R_k(\Gamma_1(N); A')$.

This type of assertion will be called the q-expansion principle. We mainly deal with the level N structure $i : \mu_N \hookrightarrow E[N]$ in this paper, but to prove one of key lemmas for our later use, we need some other types of level N structure, which concerns the principal congruence subgroup $\Gamma(N)$: Let $(\mathbf{Z}/N\mathbf{Z})/_A$ denote the constant group scheme of order N over A and define a standard pairing $\langle\ ,\ \rangle : \mu_N \times \mathbf{Z}/N\mathbf{Z} \to \mu_N$ by $\langle (\zeta, m), (\xi, n) \rangle = \zeta^n/(\xi)^m$. Then we consider the following two other types of level N structure:

(i) $\beta : \mu_N \times \mathbf{Z}/N\mathbf{Z} \simeq E[N]$ such that $\langle\ ,\ \rangle$ coincides with the Weil pairing on $E[N]$ under β;

(ii) $$\alpha : (\mathbf{Z}/N\mathbf{Z})^2 \simeq E[N].$$

For the very existence of the structure α as above, N must be invertible in A. We will use the symbols α, β and i exclusively to indicate which type of level N structure we are dealing with. Similarly to $R_k(\Gamma_1(N); A)$, we define the space of modular forms $R_k(\Gamma(N); A)$ (resp. $\mathscr{R}_k(\Gamma(N); A)$) which classifies triples (E, ω, β) (resp. (E, ω, α)). Assume that $N^{-1} \in A$, and we consider naive level N structures (E, ω, α) over A. Let e_N be the Weil pairing on $E[N]$. We define a primitive N-th root of unity $\det(\alpha)$ by

$$\det(\alpha) = e_N(\alpha(1, 0), \alpha(0, 1)).$$

Then we can define an arithmetic $\Gamma(N)$-structure $\beta_\alpha : \mu_N \times \mathbf{Z}/N\mathbf{Z} \simeq E[N]$ out of the given naive $\Gamma(N)$-structure α by

$$\beta_\alpha(\det(\alpha)^m, n) = \alpha(m, n).$$

The correspondence: $(E, \omega, \alpha) \to ((E, \omega, \beta_\alpha), \det(\alpha))$ gives a bijection (*cf.* [18, 2.0.8])

(1.4 *a*) $\{\text{naive } \Gamma(N)\text{-structures}\} \simeq \{\zeta \in \mu_N(A) \mid \zeta : \text{primitive}\} \times \{\text{arithmetic } \Gamma(N)\text{-structures}\}$.

This yields an important isomorphism

(1.4 *b*) $$\mathscr{R}_k(\Gamma(N); A) \simeq R_k(\Gamma(N); A) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{N}, \zeta\right],$$

where ζ is a primitive N-th root of unity. On the Tate curve, a canonical arithmetic $\Gamma(N)$-structure $\beta_{can} : \mu_N \times \mathbf{Z}/N\mathbf{Z} \simeq \mathrm{Tate}(q)[N]$ over $\mathbf{Z}((q^{1/N}))$ is given by $\beta_{can}(\zeta, n) = \zeta q^{n/N}$. Then, the q-expansion principle holds for $R_k(\Gamma(N); A)$ for the evaluation at $(\mathrm{Tate}(q), \omega_{can}, \beta_{can})$.

In order to compare these new definitions of modular forms with those given in the beginning of this section, we consider the space of modular forms over $\mathbf{C}$. To any point $z \in \mathfrak{H}$, one can associate a lattice in $\mathbf{C}$ given by $L_z = 2\pi i(\mathbf{Z} + \mathbf{Z}z)$. By means of $\wp$-function relative to this lattice L_z, we can regard the triple $(\mathbf{C}/L_z, du, i)$ with $i\left(e\left(\frac{2\pi i}{N}\right)\right) = \left(\frac{1}{N} \bmod L_z\right)$ as an arithmetic level N structure, where u is the variable on $\mathbf{C}$. For any $f \in \mathscr{M}_k(\Gamma_1(N))$, the correspondence:

$$(\mathbf{C}/L_z, du, i) \mapsto (2\pi i)^{-k} f(z) \qquad \text{for each} \quad z \in \mathfrak{H}$$

defines an element of $R_k(\Gamma_1(N); \mathbf{C})$. Since the evaluation of $f \in \mathscr{M}_k(\Gamma_1(N))$ as an element of $R_k(\Gamma_1(N); \mathbf{C})$ at $(\mathrm{Tate}(q), \omega_{\mathrm{can}}, i_{\mathrm{can}})$ coincides with the analytic q-expansion of f *via* the identification $q = e(z)$, we can regard $\mathscr{M}_k(\Gamma_1(N))$ as a subspace of $R_k(\Gamma_1(N); \mathbf{C})$. The correspondence: $(E, \omega, \beta) \mapsto (E, \omega, \beta\,|_{\mu_N})$ yields an inclusion $R_k(\Gamma_1(N); A)$ into $R_k(\Gamma(N); A)$, which preserves q-expansion. Then through the identification (1.4 *b*), we can evaluate an element $f \in R_k(\Gamma_1(N); A)$ at Tate (q) over $A[\zeta]((q^{\frac{1}{N}}))$ with an arbitrary naive $\Gamma(N)$-structure $\alpha : (\mathbf{Z}/N\mathbf{Z})^2 \simeq \mathrm{Tate}(q)[N]$. Then we have that

$$\mathscr{M}_k(\Gamma_1(N)) = \{ f \in R_k(\Gamma_1(N); \mathbf{C}) \mid f(\mathrm{Tate}(q), \omega_{\mathrm{can}}, \alpha) \in \mathbf{C}[[q^{\frac{1}{N}}]] \quad \text{for every } \alpha \}.$$

This combined with the q-expansion principle shows that, for each subring A of $\mathbf{C}$ or Ω,

$$(1.5) \quad \mathscr{M}_k(\Gamma_1(N); A) = \left\{ f \in R_k(\Gamma_1(N); A) \mid f(\mathrm{Tate}(q), \omega_{\mathrm{can}}, \alpha) \in A\left[\zeta, \frac{1}{N}\right][[q]] \quad \text{for all } \alpha \right\},$$

as a subspace of $A[[q]]$. By this fact, $\mathscr{M}_k(\Gamma_1(N); K)$ does not depend on the choice of the dense subfield K_0.

Now we shall review the p-adic theory. Main source of the results is [18, Chap. V] and [16, 17]. We mean by a p-adic ring A an algebra which coincides naturally with the projecture limit $\varprojlim_n A/p^n A$. Thus it is p-adically complete, but for example, the p-adic field $\mathbf{Q}_p$ is not an object of the category of p-adic rings. Let E be an elliptic curve over a p-adic ring A. By a trivialization ϕ on E, we mean an isomorphism $\phi : \hat{E} \cong \hat{\mathbf{G}}_m$ between the formal completion $\hat{E}$ of E along the origin and the formal multiplicative group $\hat{\mathbf{G}}_m$. Write $N = N_0 p^r$ with $(N_0, p) = 1$. We consider triples (E, ϕ, i), (E, ϕ, β) and (E, ϕ, α) over p-adic rings and we impose the following additional compatibility condition between ϕ and the level N structures:

$$(1.6) \qquad \textit{The composition: } \mu_{p^r} \overset{\beta \text{ or } i}{\hookrightarrow} \hat{E} \overset{\phi}{\cong} \hat{\mathbf{G}}_m \textit{ is a natural inclusion.}$$

As for the naive $\Gamma(N)$-structure α, one can discuss it only when N is invertible in A (therefore, necessarily, p is prime to N), and thus no additional compatibility condition is necessary. We denote by $V(N; A)$ (resp. $V(\Gamma(N); A)$ and $\mathscr{V}(\Gamma(N_0); A)$) the space of functions which assign an element of A′ to each A′-isomorphism class of a compatible triple (E, ϕ, i) (resp. (E, ϕ, β) and (E, ϕ, α) for a naive $\Gamma(N_0)$-structure α) over any p-adic A-algebra A′ and which are compatible with any base change in the category of p-adic A-algebras. Then, in the same fashion as in (1.4 *a*, *b*), one has an isomorphism

$$(1.7) \qquad \mathscr{V}(\Gamma(N_0), A) \cong V(\Gamma(N_0), A) \otimes_{\mathbf{Z}} \mathbf{Z}\left[\frac{1}{N_0}, \zeta_0\right],$$

where ζ_0 is a primitive N_0-th root of unity.

Let $\widehat{\mathbf{Z}_p((q))}$ be the p-adic completion of $\mathbf{Z}_p((q))$. Since the Tate curve $\mathrm{Tate}(q)/\widehat{\mathbf{Z}_p((q))}$ is a quotient of $\mathbf{G}_m$ by the subgroup generated by q, we have a canonical trivialization $\phi_{\mathrm{can}} : \widehat{\mathrm{Tate}(q)} \simeq \hat{\mathbf{G}}_m$. By definition, the triples $(\mathrm{Tate}(q), \phi_{\mathrm{can}}, i_{\mathrm{can}})/\widehat{\mathbf{Z}_p((q))}$ and

$(\mathrm{Tate}\,(q),\ \phi_{\mathrm{can}},\ \beta_{\mathrm{can}})/\widehat{\mathbf{Z}_p((q^{1/N}))}$ are compatible. It is known (for the proof, see [16, 1.4]) that the evaluation at the canonical Tate curves as above gives

(1.8 *a*) $$\mathrm{V}(\mathrm{N}\,;\mathrm{A}) \hookrightarrow \widehat{\mathrm{A}((q))},\quad \mathrm{V}(\Gamma(\mathrm{N}),\mathrm{A}) \hookrightarrow \widehat{\mathrm{A}((q^{1/\mathrm{N}}))},$$

(1.8 *b*) *If* $\mathrm{A}'\supset\mathrm{A},\quad \mathrm{V}(\mathrm{N}\,;\mathrm{A}')\cap\widehat{\mathrm{A}((q))}=\mathrm{V}(\mathrm{N}\,;\mathrm{A}),\quad \mathrm{V}(\Gamma(\mathrm{N})\,;\mathrm{A}')\cap\widehat{\mathrm{A}((q^{1/\mathrm{N}}))}=\mathrm{V}(\Gamma(\mathrm{N})\,;\mathrm{A}),$

(1.8 *c*) *The quotient of* $\widehat{\mathrm{A}((q))}$ (resp. $\widehat{\mathrm{A}((q^{1/\mathrm{N}}))}$) *by the image of* $\mathrm{V}(\mathrm{N}\,;\mathrm{A})$ (resp. $\mathrm{V}(\Gamma(\mathrm{N})\,;\mathrm{A})$) *is* A-*flat*.

Thus the q-expansion principle also holds in this case. The correspondence: $(\mathrm{E},\phi,\beta)\mapsto(\mathrm{E},\phi,\beta\,|_{\mu_\mathrm{N}})$ again yields a q-expansion preserving embedding

$$\mathrm{V}(\mathrm{N}\,;\mathrm{A}) \hookrightarrow \mathrm{V}(\Gamma(\mathrm{N})\,;\mathrm{A}) \qquad \text{for all } p\text{-adic ring A.}$$

By the compatibility condition (1.6), to give a level N-structure i on the trivialized curve (E,ϕ) is equivalent to give a level N_0-structure $i\,|\,\mu_{\mathrm{N}_0}$. This correspondence yields an equivalence between the category of (E,ϕ,i) of level N and that of level N_0. Moreover, this correspondence takes the canonical Tate curve to the corresponding one and thus preserves q-expansion. Then, as a subspace of $\widehat{\mathrm{A}((q))}$, one has an identity

$$\mathrm{V}(\mathrm{N}\,;\mathrm{A})=\mathrm{V}(\mathrm{N}_0\,;\mathrm{A}).$$

Thus, by (1.7), we can evaluate any element of $\mathrm{V}(\mathrm{N}\,;\mathrm{A})$ (resp. $\mathrm{V}(\Gamma(\mathrm{N}_0)\,;\mathrm{A})$) at any trivialized Tate curve $(\mathrm{Tate}\,(q),\phi,\alpha)$ over $\widehat{\mathbf{Z}_p[\zeta]((q^{1/\mathrm{N}_0}))}$ with any arbitrary naive $\Gamma(\mathrm{N}_0)$-structure α. Put for each p-adic algebra A,

(1.9 *a*) $$\begin{aligned}\mathrm{W}(\mathrm{N}\,;\mathrm{A})&=\{f\in\mathrm{V}(\mathrm{N}\,;\mathrm{A})\,|\,f(\mathrm{Tate}(q),\phi,\alpha)\in\mathrm{A}[\zeta_0][[q^{1/\mathrm{N}_0}]] \text{ for all } \phi \text{ and } \alpha\},\\ \mathscr{W}(\Gamma(\mathrm{N}_0);\mathrm{A})&=\{f\in\mathscr{V}(\Gamma(\mathrm{N}_0);\mathrm{A})\,|\,f(\mathrm{Tate}(q),\phi,\alpha)\in\mathrm{A}[\zeta_0][[q^{1/\mathrm{N}_0}]] \text{ for all } \phi \text{ and } \alpha\},\\ \mathrm{W}(\Gamma(\mathrm{N}_0);\mathrm{A})&=\{f\in\mathrm{V}(\Gamma(\mathrm{N}_0);\mathrm{A})\,|\,f(\mathrm{Tate}(q),\phi,\alpha)\in\mathrm{A}[\zeta_0][[q^{1/\mathrm{N}_0}]] \text{ for all } \phi \text{ and } \alpha\},\end{aligned}$$

where ζ_0 is a primitive N_0-th root of unity. Then we have

(1.9 *b*) $$\mathrm{W}(\mathrm{N}\,;\mathrm{A})=\mathrm{W}(\mathrm{N}_0\,;\mathrm{A}),$$

(1.9 *c*) $$\mathscr{W}(\Gamma(\mathrm{N}_0)\,;\mathrm{A})=\mathrm{W}(\Gamma(\mathrm{N}_0)\,;\mathrm{A})\otimes_{\mathbf{Z}}\mathbf{Z}[1/\mathrm{N}_0,\zeta_0].$$

Thus, $\mathrm{W}(\Gamma(\mathrm{N}_0)\,;\mathbf{Z}_p[\zeta_0])$ gives one irreducible component of $\mathscr{W}(\Gamma(\mathrm{N}_0)\,;\mathbf{Z}_p[\zeta_0])$ which may be interpreted as the space of functions which classify the triples (E,ϕ,α) with $\det(\alpha)=\zeta_0$ and are finite at cusps. Thus the space $\mathrm{W}(\Gamma(\mathrm{N}_0)\,;\mathbf{Z}_p[\zeta_0])$ coincides with the space $\mathrm{V}_{\infty,\infty}$ defined in [16, (1.11)] (see also [16, Appendix I]). Let (E,ϕ,α) be the trivialized naive $\Gamma(\mathrm{N}_0)$-structure over A. For any $\gamma\in\mathrm{GL}_2(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})=\mathrm{Aut}\,((\mathbf{Z}/\mathrm{N}_0\mathbf{Z})^2)$, we can let γ act on (E,ϕ,α) by $(\mathrm{E},\phi,\alpha)\mapsto(\mathrm{E},\phi,\alpha\circ\gamma)$. Thus $\mathrm{GL}_2(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})$ acts on $\mathscr{W}(\Gamma(\mathrm{N}_0)\,;\mathrm{A})$ on the right. One can calculate the effect of this $\mathrm{GL}_2(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})$-action on q-expansion and then verifies that under the identification (1.7)

(1.10) $$\mathrm{W}(\mathrm{N}_0\,;\mathbf{Z}_p)=\mathscr{W}(\Gamma(\mathrm{N}_0),\mathbf{Z}_p)^{\mathrm{U}},\quad \mathrm{W}(\mathrm{N}_0\,;\mathbf{Z}_p)\otimes_{\mathbf{Z}}\mathbf{Z}[\zeta_0]=\mathscr{W}(\Gamma(\mathrm{N}_0)\,;\mathbf{Z}_p)^{\mathrm{U}_0}$$

where $\mathrm{U}=\left\{\begin{pmatrix}1 & a\\ 0 & d\end{pmatrix}\in\mathrm{GL}_2(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})\right\}$ and $\mathrm{U}_0=\left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\in\mathrm{GL}_2(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})\right\}$. Put

$$\mathrm{Z}=\mathbf{Z}_p^\times\times(\mathbf{Z}/\mathrm{N}_0\mathbf{Z})^\times.$$

Naturally, the finite group $(\mathbf{Z}/N_0\mathbf{Z})^\times$ is a quotient group of Z. We define an action of $z=(z_p, z_0)\in Z(z_p\in\mathbf{Z}_p^\times, z_0\in(\mathbf{Z}/N_0\mathbf{Z})^\times)$ on (E, ϕ, i) by $(E, \phi, i)\mapsto(E, z_p^{-1}\phi, z\circ i)$. Thus the compact group Z acts on $V(N;A)$ and $W(N;A)$ p-adically continuously. Let ω_0 be the canonical invariant differential on $\hat{\mathbf{G}}_m$ induced by dx/x. With each trivialization (E, ϕ) over a p-adic ring A, we can associate an invariant differential ω/A on E by $\omega=\phi^*\omega_0$. This correspondence: $(E, \phi)\mapsto(E, \omega)$ yields q-expansion preserving embeddings

$$R_k(\Gamma_1(N);A)\to V(N;A),$$
$$\mathscr{R}_k(\Gamma(N_0);A)\to\mathscr{V}(\Gamma(N_0);A),$$
$$R_k(\Gamma(N);A)\to V(\Gamma(N_0);A).$$

The injectivity is guaranteed by the q-expansion principle. For any $f\in\mathscr{M}(N;A)$ with a p-adic subalgebra A of Ω, the above embedding sends f into $(V(N;A)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p)\cap A[[q]]$, which is a torsion element of $V(N;A)\backslash A((q))$. Then A-flatness assertion (1.8 *c*) guarantees that f is contained in $V(N;A)$. By (1.5) and (1.9 *a*), f is in fact contained in $W(N;A)$. Then the p-adic continuity affirms that, for any p-adic subalgebra A of Ω,

$$W(N, A)\supset\overline{\mathscr{M}}(N;A). \tag{1.11 a}$$

Similarly, we have a continuous embedding

$$W(\Gamma(N_0);A)\supset\overline{\mathscr{M}}(\Gamma(N_0);A). \tag{1.11 b}$$

To state one of the results of Katz which will be used repeatedly, we introduce some notation. Let E_{p-1} be an Eisenstein series in $\mathscr{M}_{p-1}(SL_2(\mathbf{Z});\mathbf{Z}_p)$ given by the q-expansion:

$$1-\frac{2(p-1)}{B_{p-1}}\sum_{n=1}^{\infty}\Big(\sum_{0<d|n}d^{p-2}\Big)q^n.$$

Then E_{p-1} satisfies the congruence $E_{p-1}\equiv 1 \bmod p\mathbf{Z}_p$.

For a primitive N_0-th root of unity ζ_0, put $\mathbf{F}=\mathbf{Z}_p[\zeta_0]/p\mathbf{Z}_p[\zeta_0]$. Then $\mathbf{F}$ is a finite extension of $\mathbf{F}_p$. Write $\mathscr{M}_k(\Gamma(N_0);\mathbf{F})$ for $\mathscr{M}_k(\Gamma(N_0);\mathbf{Z}_p[\zeta_0])\otimes_{\mathbf{Z}_p[\zeta_0]}\mathbf{F}$. Put

$$G(\Gamma(N_0);\mathbf{F})=\bigoplus_{k=0}^{\infty}\mathscr{M}_k(\Gamma(N_0);\mathbf{F}),$$

which is a graded algebra. Since $\mathscr{M}_k(\Gamma(N_0);\mathbf{F})$ can be embedded into $\mathbf{F}[[q^{1/N_0}]]$ for each k through q-expansion, there is a natural $\mathbf{F}$-algebra homomorphism of $G(\Gamma(N_0);\mathbf{F})$ to $\mathbf{F}[[q^{1/N_0}]]$, which sends any $\varphi\in\mathscr{M}_k(\Gamma(N_0);\mathbf{F})$ to its q-expansion for each k. The ideal $(E_{p-1}-1)$ in $G(\Gamma(N_0);\mathbf{F})$ is contained in the kernel of this morphism. On the other hand, the q-expansion principle says that $W(\Gamma(N_0);\mathbf{F})$ is embedded into $\mathbf{F}[[q^{1/N_0}]]$. We know that $W(\Gamma(N_0);\mathbf{F})=W(\Gamma(N_0);\mathbf{Z}_p[\zeta_0])\otimes_{\mathbf{Z}_p[\zeta_0]}\mathbf{F}$ (*cf.* the construction given in [16]).

THEOREM 1.1 (Katz). — *Assume that either $N_0\geq 3$ or $p\geq 5$. Let $\Gamma=1+p\mathbf{Z}_p$ as a subgroup of Z and let $W(\Gamma(N_0);\mathbf{F})^\Gamma$ be the subspace of Γ-invariants of $W(\Gamma(N_0);\mathbf{F})$. Then we have*

(i) $W(\Gamma(N_0);\mathbf{Z}_p[\zeta_0])=\overline{\mathscr{M}}(\Gamma(N_0);\mathbf{Z}_p[\zeta_0])$ *in* $\mathbf{Z}_p[\zeta_0][[q^{1/N_0}]]$;

(ii) $W(\Gamma(N_0);\mathbf{F})^\Gamma\simeq G(\Gamma(N_0);\mathbf{F})/(E_{p-1}-1)$ *and this isomorphism preserves q-expansion.*

This theorem follows from [16] § 2 for $N_0 \geq 3$ and § 4 for $N_0 = 1$ and 2, but some explanation may be necessary. We use the same notation as in [16]. As already mentioned, $W(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])$ coincides with $V_{\infty,\infty}$. Thus $W(\Gamma(N_0); \mathbf{F})$ coincides with $V_{1,\infty}$ in $\mathbf{F}[[q^{1/N_0}]]$. The action of $\mathbf{Z}_p^\times$ as a subgroup of Z on $W(\Gamma(N_0); \mathbf{F})$ coincides with the action of the fundamental group $\pi_1(S_1)$ in [16, § 0]. Since $V_{1,\infty}$ is the etale Galois covering of $V_{1,1}$ with the Galois group Γ, $W(\Gamma(N_0); \mathbf{F})^\Gamma$ coincides with $V_{1,1}$. Then the results of Theorem 2.1 and 2.2 in [16] yields the above theorem when $N_0 \geq 3$. When $N_0 = 1$, then $W(SL_2(\mathbf{Z}), \mathbf{F})$ is $W(\Gamma(p-1), \mathbf{F})^{SL_2(\mathbf{Z}/(p-1)\mathbf{Z})}$. Similar fact is also true for $N_0 = 2$ (see, [16, § 4]). Then the theorem follows from [16, § 2] even if $N_0 = 1$ or 2 when $p \geq 5$.

COROLLARY 1.2. — *Let* $\mathcal{M}_k(\Gamma_1(N); \mathbf{F}_p) = \mathcal{M}_k(\Gamma_1(N); \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{F}_p$, *and put*

$$G(\Gamma_1(N); \mathbf{F}_p) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma_1(N); \mathbf{F}_p).$$

Then we have

(i) $$W(N; \mathbf{Z}_p) = W(N_0; \mathbf{Z}_p) = \overline{\mathcal{M}}(N, \mathbf{Z}_p) \quad \textit{in} \quad \mathbf{Z}_p[[q]],$$

(ii) $$W(N_0; \mathbf{F}_p)^\Gamma = G(\Gamma_1(N_0); \mathbf{F}_p)/(E_{p-1} - 1) \quad \textit{in} \quad \mathbf{F}_p[[q]].$$

Proof. — As for the first assertion, it is proved in [16, Appendix III], but we deduce it from Theorem 1.1 in a more elementary manner. By (1.9 *c*) and Theorem 1.1, we can identify

$$\mathcal{W}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0]) = \overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0]) \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_0].$$

The action of $GL_2(\mathbf{Z}/N_0\mathbf{Z})$ is realized on the right-hand side as follows: (i) for $\bar\gamma \in SL_2(\mathbf{Z}/N_0\mathbf{Z})$, we take a lift $\gamma \in SL_2(\mathbf{Z})$ with $\gamma \equiv \bar\gamma \bmod N_0$. Then the action of $\bar\gamma$ coincides on $\mathcal{M}_k(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])$ with the action $f \mapsto f|_k\gamma$ (*cf.* [3, VII.3.12]) and (ii) if $\bar\gamma = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ for $d \in (\mathbf{Z}/N_0\mathbf{Z})^\times$, then $\bar\gamma$ acts on $\mathbf{Z}[\zeta]$ as an algebra automorphism which takes ζ_0 to ζ_0^d and on $\overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])$ trivially. This may be verified by checking the effect of $\bar\gamma$ on the Tate curve.

Let U and U_0 be as in (1.10). Then we see

$$\mathcal{W}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])^U = \overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])^{U_0} = W(N_0; \mathbf{Z}_p[\zeta_0]).$$

We claim that

(*) $$\overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])^{U_0} = \overline{\mathcal{M}}(N_0; \mathbf{Z}_p[\zeta_0]) \quad \text{in} \quad \mathbf{Z}_p[\zeta_0][[q]].$$

One can check this as follows: Define

$$\mathrm{Tr}: \overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0]) \to \overline{\mathcal{M}}(\Gamma(N_0); \mathbf{Z}_p[\zeta_0])$$

by

$$\mathrm{Tr}(f) = N_0^{-1} \sum_{u \in U_0} f \mid u.$$

This morphism gives at least a map of $\mathcal{M}(\Gamma(N_0); \mathbf{Q}_p[\zeta_0])$ onto $\mathcal{M}(N_0; \mathbf{Q}_p[\zeta_0])$ and we have the formula:

$$\mathrm{Tr}\left(\sum_{n \in \mathbf{Z}} a\left(\frac{n}{N}\right) q^{\frac{n}{N}}\right) = \sum_{n \in \mathbf{Z}} a(n) q^n.$$

Thus Tr is uniformly continuous under $|\ |_p$; then, Tr gives a projection of $\overline{\mathcal{M}}(\Gamma(\mathrm{N}_0); \mathbf{Z}_p[\zeta_0])$ onto $\overline{\mathcal{M}}(\mathrm{N}_0; \mathbf{Z}_p[\zeta_0])$. Thus (*) holds. On the other hand, we have by definition that

$$\overline{\mathcal{M}}(\mathrm{N}; \mathbf{Z}_p) = \overline{\mathcal{M}}(\mathrm{N}; \mathbf{Z}_p[\zeta_0]) \cap \mathbf{Z}_p[[q]].$$

Similarly, by the q-expansion principle (1.8 b), one has

$$\mathrm{W}(\mathrm{N}; \mathbf{Z}_p) = \mathrm{W}(\mathrm{N}; \mathbf{Z}_p[\zeta_0]) \cap \mathbf{Z}_p[[q]].$$

Then the first assertion for $\mathrm{N}=\mathrm{N}_0$ follows from (*). The assertion for general N follows from (1.9 b). Now we shall prove the second assertion.

By Theorem 1.1, we have that

$$\mathrm{W}(\Gamma(\mathrm{N}_0); \mathbf{F})^{\Gamma} = \mathrm{G}(\Gamma(\mathrm{N}_0); \mathbf{F})/(\mathrm{E}_{p-1}-1).$$

By (1.5) (or else [3, VII.3]), we see $\mathrm{G}(\Gamma(\mathrm{N}_0); \mathbf{F}) = \mathrm{G}(\Gamma(\mathrm{N}_0); \mathbf{F}_p) \otimes \mathbf{F}$ for

$$\mathrm{G}(\Gamma(\mathrm{N}_0); \mathbf{F}_p) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma(\mathrm{N}_0); \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{F}_p.$$

This shows that

$$\mathrm{W}(\Gamma(\mathrm{N}_0); \mathbf{F}_p)^{\Gamma} = \mathrm{G}(\Gamma(\mathrm{N}_0); \mathbf{F}_p)/(\mathrm{E}_{p-1}-1).$$

The argument using the operator Tr in the proof of the first assertion works well even in this case, since $\mathrm{Tr}(\mathrm{E}_{p-1}-1) = \mathrm{E}_{p-1}-1$. Then we have

$$\mathrm{W}(\mathrm{N}_0; \mathbf{F}_p)^{\Gamma} = \mathrm{G}(\Gamma_1(\mathrm{N}_0); \mathbf{F}_p)/(\mathrm{E}_{p-1}-1),$$

which is to be shown.

Now we are ready to define the Hecke operators acting on $\overline{\mathcal{M}}(\mathrm{N}; \mathcal{O}_\mathrm{K})$. Recall that $z=(z_p, z_0) \in \mathrm{Z}$ acts on $f \in \overline{\mathcal{M}}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ as

$$(f \mid z)(\mathrm{E}, \phi, i) = f(\mathrm{E}, z_p^{-1}\phi, z \circ i).$$

Naturally $a \in (\mathbf{Z}/\mathrm{N}\mathbf{Z})^\times$ acts on $\mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathrm{K})$ through

$$(f \mid a)(\mathrm{E}, \omega, i) = f(\mathrm{E}, \omega, ai).$$

If we take $\gamma \in \Gamma_0(\mathrm{N})$ with $\gamma \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \bmod \mathrm{N}$, then we see easily $f \mid a = f|_k\gamma$.

Since $(z_p^{-1}\phi)^*(dx/x) = z_p^{-1}(\phi^*(dx/x))$, we know

$$(1.12) \qquad f \mid z = z_p^k f|_k\gamma \quad \text{for} \quad f \in \mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathcal{O}_\mathrm{K}),$$

where $\gamma \in \Gamma_0(\mathrm{N})$ with $\gamma \equiv \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \bmod \mathrm{N}$. Thus the image $f \mid z$ for any $f \in \mathcal{M}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ under the action of $z \in \mathrm{Z}$ is contained in $\mathrm{V}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ and hence has a p-integral q-expansion; namely, $f \mid z$ again belongs to $\mathcal{M}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ because $f \mid z$ is an element in $\mathcal{M}(\mathrm{N}; \mathrm{K})$ by (1.12). Thus the action of $z \in \mathrm{Z}$ is uniformly continuous under the norm (1.1); hence, it can be extended to $\overline{\mathcal{M}}(\mathrm{N}; \mathcal{O}_\mathrm{K})$. This extended action coincides with that induced from $\mathrm{V}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ through the inclusion (1.11 a).

Now assume that N is divisible by p $(r>0)$, and define Hecke operators $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ on $\overline{\mathscr{M}}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ for primes l by

$$(1.13\,a)\qquad a(n, f \mid \mathrm{T}(l))=\begin{cases} a(ln, f)+l^{-1}a\left(\dfrac{n}{l}, f \,\middle|\, l\right) & \text{if } l\nmid \mathrm{N}p, \\ a(ln, f) & \text{if } l\mid \mathrm{N}p, \end{cases}$$

$$(1.13\,b)\qquad a(n, f \mid \mathrm{T}(l, l))=\begin{cases} l^{-2}a(n, f \mid l) & \text{if } l\nmid \mathrm{N}p, \\ 0 & \text{if } l\mid \mathrm{N}p, \end{cases}$$

where we understand $a(m, f)=0$ if m is not an integer. By (1.12), this action coincides with the usual action of these operators on $\mathscr{M}_k(\Gamma_1(\mathrm{N}); \mathscr{O}_\mathrm{K})$ under the assumption of the divisibility of N by p. When $f\in\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})\subset \mathrm{V}(\mathrm{N}; \mathscr{O}_\mathrm{K})$, $f \mid l$ has a p-integral q-expansion; and hence, $f \mid \mathrm{T}(l)$ and $f \mid \mathrm{T}(l, l)$ have p-integral q-expansions. Since $\mathscr{M}(\mathrm{N}; \mathrm{K})$ is stable under the operators $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$, $f \mid \mathrm{T}(l)$ and $f \mid \mathrm{T}(l, l)$ are elements of $\mathscr{M}(\mathrm{N}; \mathrm{K})$ with p-integral q-expansion; hence, belong to $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$. Therefore, $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ is stable under the action of these operators. For any element f of $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$, we see $| f \mid \mathrm{T}(l)|_p\leqq | f|_p$ and $| f \mid \mathrm{T}(l, l)|_p \leqq | f|_p$ from (1.13 a, b). Thus these operators act naturally on $\overline{\mathscr{M}}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ through their uniform continuity on $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$.

One can give a more geometric and direct definition of the operators $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ on the bigger space $\mathrm{V}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ by using the modular meaning of $\mathrm{V}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ stated above (*cf.* Katz [14, 1.11 and 3.12]), but we rather prefer being elementary and the definition we adopted is sufficient for our later use.

Now we are going to define Hecke algebras by assuming the divisibility of N by p. Since $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ and $\mathscr{M}^j(\mathrm{N}; \mathrm{K})$ are stable under $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all primes l, $\mathscr{M}^j(\mathrm{N}; \mathscr{O}_\mathrm{K})=\mathscr{M}^j(\mathrm{N}; \mathrm{K})\cap\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ is also stable under these operators. For $\mathrm{A}=\mathrm{K}$ or $\mathscr{O}_\mathrm{K}$, let us denote by $\mathscr{H}^j(\mathrm{N}; \mathrm{A})$ the A-subalgebra of the A-linear endomorphism ring of $\mathscr{M}^j(\mathrm{N}; \mathrm{A})$ generated by $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all l. Since $\mathscr{M}^j(\mathrm{N}; \mathrm{A})$ is naturally embedded into $\mathscr{M}^{j'}(\mathrm{N}; \mathrm{A})$ for $j'>j$, we have a natural ring homomorphism of $\mathscr{H}^{j'}(\mathrm{N}; \mathrm{A})$ onto $\mathscr{H}^j(\mathrm{N}; \mathrm{A})$ by the restriction of operators to the subspace $\mathscr{M}^j(\mathrm{N}; \mathrm{A})$. Let us take the projective limit of these morphisms:

$$(1.14)\qquad \mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K})=\varprojlim_j \mathscr{H}^j(\mathrm{N}; \mathscr{O}_\mathrm{K}).$$

This naturally acts on the inductive limit $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})=\varinjlim_j \mathscr{M}^j(\mathrm{N}; \mathscr{O}_\mathrm{K})$. Since the action of $\mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ on $\mathscr{M}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ is uniformly continuous, it can be naturally extended to that on $\overline{\mathscr{M}}(\mathrm{N}; \mathscr{O}_\mathrm{K})$. We know that $\mathscr{H}^j(\mathrm{N}; \mathscr{O}_\mathrm{K})$ is free of finite rank over $\mathscr{O}_\mathrm{K}$ (*cf.* [29, Th. 3.51]); hence, is compact; therefore, their projective limit $\mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ is a compact ring. By our definition, the restriction morphism of $\mathscr{H}^j(\mathrm{N}; \mathscr{O}_\mathrm{K})$ onto $\mathscr{H}^j(\mathrm{N}_0 p; \mathscr{O}_\mathrm{K})$ indices a morphism of $\mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ onto $\mathscr{H}(\mathrm{N}_0 p; \mathscr{O}_\mathrm{K})$. Then, by Corollary 1.2, it must be injective and we know

$$(1.15\,a)\qquad \mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K})=\mathscr{H}(\mathrm{N}_0 p; \mathscr{O}_\mathrm{K}),$$

and

$$(1.15\,b)\qquad \mathscr{H}(\mathrm{N}; \mathscr{O}_\mathrm{K}) \quad \textit{acts on} \quad \overline{\mathscr{M}}(\mathrm{N}_0; \mathscr{O}_\mathrm{K}).$$

Even if we define $\mathscr{H}(\mathrm{N};\mathrm{K})$ by (1.14) for K in place of $\mathcal{O}_{\mathrm{K}}$, these assertions are false for $\mathscr{H}(\mathrm{N};\mathrm{K})$. The point of this fact is that the natural action of $\mathscr{H}(\mathrm{N};\mathrm{K})$ on $\mathscr{M}(\mathrm{N};\mathrm{K})$ is certainly not uniformly continuous.

For any finite extension M/K, we see that

$$\mathscr{M}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})\otimes_{\mathcal{O}_{\mathrm{K}}}\mathcal{O}_{\mathrm{M}}\subset\mathscr{M}^j(\mathrm{N};\mathcal{O}_{\mathrm{M}}),$$

and these two spaces are of finite index. This shows that

$$\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{M}})=\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})\otimes_{\mathcal{O}_{\mathrm{K}}}\mathcal{O}_{\mathrm{M}},$$

since the right-hand side contains all generators $\mathrm{T}(l)$ and $\mathrm{T}(l,l)$. Then, by its definition, we know

$$\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{M}})=\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})\otimes_{\mathcal{O}_{\mathrm{K}}}\mathcal{O}_{\mathrm{M}}. \tag{1.16}$$

Now we recall the definition of the idempotent e attached to the Hecke operator $\mathrm{T}(p)$ given in [10, § 4]. The natural image $\tilde{\mathrm{T}}(p)$ of $\mathrm{T}(p)$ in $\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})/p\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ can be decomposed uniquely into the sum $s+n$ of a semi-simple element s and a nilpotent element n with $ns=sn$. Then, for a sufficiently large t, $\tilde{\mathrm{T}}(p)^{p^t}$ becomes semi-simple, and therefore, we can find a positive integer m so that $\tilde{e}=\tilde{\mathrm{T}}(p)^{p^t m}$ gives an idempotent of $\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})/p\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})$. This idempotent can be lifted to a unique idempotent e_j of $\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ (*cf.* [1, III.4.6]). In $\mathscr{H}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})$, e_j can be explicitly given by [8, p. 236] as

$$e_j=\lim_{t\to\infty}\mathrm{T}(p)^{p^t m}\quad\textit{for a suitable}\quad 0<m\in\mathbf{Z}. \tag{1.17 a}$$

Obviously, the formation of e_j is compatible with the projective limit (1.14); therefore, we can define an idempotent $e\in\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ by

$$e=\varprojlim_j e_j. \tag{1.17 b}$$

This idempotent will be called the idempotent attached to $\mathrm{T}(p)$. For any module M over $\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$, the eigenspace $\mathrm{M}^0=e\mathrm{M}$ of e with eigenvalue 1 will be called the ordinary part of M.

So far, we have discussed only on the space of modular forms $\overline{\mathscr{M}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$, but by defining

$$h(\mathrm{N};\mathcal{O}_{\mathrm{K}})=\varprojlim_j h^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})\qquad\textit{for N divisible by } p \tag{1.18}$$

with the $\mathcal{O}_{\mathrm{K}}$-subalgebra $h^j(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ of $\mathrm{End}_{\mathcal{O}_{\mathrm{K}}}(\mathscr{S}^j(\mathrm{N};\mathcal{O}_{\mathrm{K}}))$ generated by $\mathrm{T}(\ell)$ and $\mathrm{T}(\ell,\ell)$ for all ℓ, we see that all the assertions proved for $\overline{\mathscr{M}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ can be naturally generalized to $\overline{\mathscr{S}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ with suitable modification. Especially, we have

$$\overline{\mathscr{S}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})=\overline{\mathscr{S}}(\mathrm{N}_0 p;\mathcal{O}_{\mathrm{K}}); \tag{1.19 a}$$

$$h(\mathrm{N};\mathcal{O}_{\mathrm{K}})=h(\mathrm{N}_0 p;\mathcal{O}_{\mathrm{K}}). \tag{1.19 b}$$

By restricting operators of $\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ to $\overline{\mathscr{S}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$, we know that

(1.20) $h(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ *is the quotient ring of* $\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ *by the annihilator of* $\overline{\mathscr{S}}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$ *in* $\mathscr{H}(\mathrm{N};\mathcal{O}_{\mathrm{K}})$.

We can similarly define the idempotent e attached to $\mathrm{T}(p)$ of $h(\mathrm{N};\mathcal{O}_{\mathrm{K}})$.

A normalized eigenform f of $\mathcal{M}_k(\Gamma_0(\mathrm{N}), \psi)$ is called ordinary if

(1.21 *a*) $\quad$ N *is divisible by* p;

(1.21 *b*) $\quad f \mid e = f$ *for the idempotent* e *attached to* $\mathrm{T}(p)$.

The condition (1.21 *b*) is equivalent to the fact $|a(p, f)|_p = 1$ and if $k \geqq 2$, we can attach to any primitive form f with $|a(p, f)|_p = 1$ a unique ordinary form f_0 by $f_0 = cf \mid e$ for $c \in \Omega$ with $|c|_p = 1$ (*cf.* [10, Lemma 3.3 and Lemma 4.2]). The associated ordinary form f_0 is characterized by the property:

(1.22) $\quad a(n, f_0) = a(n, f)$ *for all* n *outside* p.

§ 2. Duality between p-adic modular forms and their Hecke algebras

In this section, we continue to use the same notation as in § 1; especially, K is a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}_\mathrm{K}$ is its p-adic integer ring. We now define a bilinear form on $\mathcal{H}^j(\mathrm{N}; \mathrm{A}) \times \mathcal{M}^j(\mathrm{N}; \mathrm{A})$ for $\mathrm{A} = \mathrm{K}$ or $\mathcal{O}_\mathrm{K}$ by

(2.1) $\quad (h, f)_\mathrm{A} = a(1, f \mid h)$ *for* $h \in \mathcal{H}^j(\mathrm{N}; \mathrm{A})$ *and* $f \in \mathcal{M}^j(\mathrm{N}; \mathrm{A})$,

where we write the q-expansion of f as $\sum_{n=0}^{\infty} a(n, f) q^n$ for all $f \in \mathcal{M}^j(\mathrm{N}; \mathrm{A})$. This pairing obviously induces a pairing between $h^j(\mathrm{N}; \mathrm{A})$ and $\mathcal{S}^j(\mathrm{N}; \mathrm{A})$. Now we define

(2.2 *a*) $\quad m^j(\mathrm{N}; \mathcal{O}_\mathrm{K}) = (\mathrm{K} + \mathcal{M}^j(\mathrm{N}; \mathcal{O}_\mathrm{K}))/\mathrm{K}$ and $m^j(\mathrm{N}; \mathrm{K}) = \mathcal{M}^j(\mathrm{N}; \mathrm{K})/\mathrm{K}$.

Since for any $c \in \mathrm{K}$, we know $(h, c)_\mathrm{K} = 0$ for all $h \in \mathcal{H}^j(\mathrm{N}; \mathrm{K})$, the pairing (2.1) induces a pairing on $\mathcal{H}^j(\mathrm{N}; \mathrm{A}) \times m^j(\mathrm{N}; \mathrm{A})$ with values in A. We see easily that

(2.2 *b*) $$m^j(\mathrm{N}; \mathcal{O}_\mathrm{K}) \cong \left\{ f \in \bigoplus_{k=1}^{j} \mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathrm{K}) \mid a(n, f) \in \mathcal{O}_\mathrm{K} \text{ for all } n \geqq 1 \right\}.$$

We will identify the both sides of (2.2 *b*).

Let us define an auxiliary Hecke algebra $\mathcal{H}_i^j(\mathrm{N}; \mathrm{A})$ for $\mathrm{A} = \mathcal{O}_\mathrm{K}$ or K and $j > i > 0$ by the A-subalgebra of $\mathrm{End}_\mathrm{A}(\mathcal{M}_i^j(\mathrm{N}; \mathrm{A}))$ generated by $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all primes l, where

$$\mathcal{M}_i^j(\mathrm{N}; \mathrm{K}) = \bigoplus_{k=i}^{j} \mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathrm{K})$$

and

$$\mathcal{M}_i^j(\mathrm{N}; \mathcal{O}_\mathrm{K}) = \{ f \in \mathcal{M}_i^j(\mathrm{N}; \mathrm{K}) \mid |f|_p \leqq 1 \}.$$

By definition (2.2 *a*), $m^j(\mathrm{N}; \mathcal{O}_\mathrm{K})$ is stable under $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all l, and by (2.2 *b*), we know

$$m^j(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\mathcal{O}_\mathrm{K}} \mathrm{K} = \mathcal{M}_1^j(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\mathcal{O}_\mathrm{K}} \mathrm{K} = m^j(\mathrm{N}; \mathrm{K}).$$

Thus $\mathcal{H}_1^j(\mathrm{N}; \mathcal{O}_\mathrm{K})$ faithfully acts on $m^j(\mathrm{N}; \mathcal{O}_\mathrm{K})$ and may be regarded as the $\mathcal{O}_\mathrm{K}$-subalgebra of $\mathrm{End}_{\mathcal{O}_\mathrm{K}}(m^j(\mathrm{N}; \mathcal{O}_\mathrm{K}))$ generated by $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all l. The following result was essentially proved in Lang [19, Th. 4.4, p. 43], where the result is attributed to V. Miller:

PROPOSITION 2.1. — *For* $\mathrm{A} = \mathrm{K}$ *or* $\mathcal{O}_\mathrm{K}$ *and* $j > 0$, $m^j(\mathrm{N}; \mathrm{A})$ (resp. $\mathcal{S}^j(\mathrm{N}; \mathrm{A})$) *is dual to*

$\mathcal{H}_1^j(\mathrm{N}\,;\mathrm{A})$ (resp. $h^j(\mathrm{N}\,;\mathrm{A})$) *over* A *under the pairing* (2.1). *Thus we have natural isomorphisms:*

$$\mathcal{H}_1^j(\mathrm{N}\,;\mathrm{A}) \cong \mathrm{Hom}_{\mathrm{A}}(m^j(\mathrm{N}\,;\mathrm{A}), \mathrm{A}) \quad \textit{and} \quad h^j(\mathrm{N}\,;\mathrm{A}) \cong \mathrm{Hom}_{\mathrm{A}}(\mathcal{S}^j(\mathrm{N}\,;\mathrm{A}), \mathrm{A}).$$

Proof. — We only discuss the duality between $\mathcal{H}_1^j(\mathrm{N}\,;\mathrm{A})$ and $m^j(\mathrm{N}\,;\mathrm{A})$, since the case of cusp forms is similar and much easier to prove. For simplicity, for a fixed j, we write $\mathcal{H}(\mathrm{A})$ and $m(\mathrm{A})$ for $\mathcal{H}_1^j(\mathrm{N}\,;\mathrm{A})$ and $m^j(\mathrm{N}\,;\mathrm{A})$, respectively. First we consider the case $\mathrm{A}=\mathrm{K}$. Since these spaces are finite dimensional over K, we are going to prove only the non-degeneracy of the pairing (2.1). By the definition of the action of $\mathrm{T}(l)$ as in (1.13 a, b), we know, for the Hecke operator $\mathrm{T}(m)$ for general $0<m\in\mathbf{Z}$, $(\mathrm{T}(m), f)=a(m, f)$ (for the expression of $\mathrm{T}(m)$ as a polynomial of $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ with coefficients in $\mathbf{Z}$, see [29, Th. 3.24 and Th. 3.34]). Thus, if $(h, f)=0$ for all $h\in\mathcal{H}(\mathrm{K})$, f must be constant; therefore, $f=0$, since f is a linear combination of forms of positive weight. Conversely, if $(h, f)=0$ for all $f\in m(\mathrm{K})$, then we see

$$a(m, f \mid h)=(\mathrm{T}(m), f \mid h)=(\mathrm{T}(m)h, f)=(h, f \mid \mathrm{T}(m))=0$$

for any $0<m\in\mathbf{Z}$. This shows $f \mid h=0$ for all $f\in m(\mathrm{K})$ and thus $h=0$ by definition of $\mathcal{H}(\mathrm{K})$.

Now we prove the proposition when $\mathrm{A}=\mathcal{O}_{\mathrm{K}}$. It is sufficient to prove

$$m(\mathcal{O}_{\mathrm{K}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathrm{K}}}(\mathcal{H}(\mathcal{O}_{\mathrm{K}}), \mathcal{O}_{\mathrm{K}}),$$

since the desired assertion: $\mathcal{H}(\mathcal{O}_{\mathrm{K}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathrm{K}}}(m(\mathcal{O}_{\mathrm{K}}), \mathcal{O}_{\mathrm{K}})$ follows from this. Take any $\mathcal{O}_{\mathrm{K}}$-linear form ϕ on $\mathcal{H}(\mathcal{O}_{\mathrm{K}})$. Since $m(\mathrm{K})=m(\mathcal{O}_{\mathrm{K}})\otimes_{\mathcal{O}_{\mathrm{K}}}\mathrm{K}$, ϕ can be extended to a K-linear form on $\mathcal{H}(\mathrm{K})$, which we again denote by ϕ. Then, we can find $f\in m(\mathrm{K})$ so that $\phi(h)=(h, f)$ for all $h\in\mathcal{H}(\mathrm{K})$. Then, we see $a(m, f)=(\mathrm{T}(m), f)=\phi(\mathrm{T}(m))\in\mathcal{O}_{\mathrm{K}}$, since $\mathrm{T}(m)$ belongs to $\mathcal{H}(\mathcal{O}_{\mathrm{K}})$ for all m. This shows that $f\in m(\mathcal{O}_{\mathrm{K}})$ by (2.2 b) and the assertion has been proved.

Let us put for positive even integer $k>2$

$$\mathrm{G}_k=-\frac{\mathrm{B}_k}{2k}+\sum_{n=1}^{\infty}\Big(\sum_{0<d\mid n} d^{k-1}\Big)q^n\in\mathbf{Q}[[q]],$$

where B_k is the k-th Bernoulli number. Then, it is well known that G_k belongs to $\mathcal{M}_k(\mathrm{SL}_2(\mathrm{Z}), \mathbf{Q})$ and for $\mathrm{E}_k=-\dfrac{2k}{\mathrm{B}_k}\mathrm{G}_k$,

$$|\,\mathrm{E}_{p^n(p-1)}-1\,|_p \leqq p^{-n-1} \qquad \text{if} \quad p\geqq 5.$$

Thus $\mathcal{M}'(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})=\bigcup_j \mathcal{M}_i^j(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$ is dense in $\mathcal{M}(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$ for any i and hence is dense in $\overline{\mathcal{M}}(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$. This shows that we can construct $\mathcal{H}(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})(=\varprojlim_j \mathcal{H}^j(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$ as in § 1) by taking $\mathcal{H}_i^j(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$ instead of $\mathcal{H}^j(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}})$, and after doing this, we see

$$(2.3) \qquad \mathcal{H}(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}}) \cong \varprojlim_j \mathcal{H}_i^j(\mathrm{N}\,;\mathcal{O}_{\mathrm{K}}).$$

Now we are going to discuss duality between $\mathscr{H}(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$ and $\mathscr{M}(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$. Let us put, for $\mathbf{T}_p = \mathbf{Q}_p/\mathbf{Z}_p$,

(2.4 *a*) $$m^j(\mathrm{N}; \mathbf{T}_p) = m^j(\mathrm{N}; \mathbf{Q}_p)/m^j(\mathrm{N}; \mathbf{Z}_p),$$

(2.4 *b*) $$m(\mathrm{N}; \mathbf{T}_p) = \mathscr{M}(\mathrm{N}; \mathbf{Q}_p)/(\mathscr{M}(\mathrm{N}; \mathbf{Z}_p) + \mathbf{Q}_p) = \varinjlim_j m^j(\mathrm{N}; \mathbf{T}_p),$$

(2.4 *c*) $$\mathscr{M}(\mathrm{N}; \mathbf{T}_p) = \mathscr{M}(\mathrm{N}; \mathbf{Q}_p)/\mathscr{M}(\mathrm{N}; \mathbf{Z}_p),$$

(2.4 *d*) $$\mathscr{S}(\mathrm{N}; \mathbf{T}_p) = \mathscr{S}(\mathrm{N}; \mathbf{Q}_p)/\mathscr{S}(\mathrm{N}; \mathbf{Z}_p).$$

More generally, we put, for any finite extension K of $\mathbf{Q}_p$,

(2.5 *a*) $$\mathscr{M}(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}}) = \mathscr{M}(\mathrm{N}; \mathrm{K})/\mathscr{M}(\mathrm{N}; \mathcal{O}_{\mathrm{K}}),$$

(2.5 *b*) $$m(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}}) = \mathscr{M}(\mathrm{N}; \mathrm{K})/(\mathscr{M}(\mathrm{N}; \mathcal{O}_{\mathrm{K}}) + \mathrm{K}),$$

(2.5 *c*) $$\mathscr{S}(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}}) = \mathscr{S}(\mathrm{N}; \mathrm{K})/\mathscr{S}(\mathrm{N}; \mathcal{O}_{\mathrm{K}}).$$

We equip these $\mathscr{H}(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$-modules with the discrete topology. Then we can naturally define a pairing

(2.6 *a*) $$(\ ,\ )_{\mathbf{T}_p} : \mathscr{H}^j(\mathrm{N}; \mathbf{Z}_p) \times m^j(\mathrm{N}; \mathbf{T}_p) \to \mathbf{T}_p$$

by $$(h, \bar{f})_{\mathbf{T}_p} = (h, f)_{\mathbf{Q}_p} \bmod \mathbf{Z}_p \in \mathbf{T}_p,$$

where $\bar{f}$ is the class in $m^j(\mathrm{N}; \mathbf{T}_p)$ containing $f \in \mathscr{M}^j(\mathrm{N}; \mathbf{Q}_p)$. The projective limit (2.3) is naturally compatible with the inductive limit (2.4 *b*) under the pairing (2.6 *a*); thus, (2.6 *a*) naturally induces a pairing:

(2.6 *b*) $$(\ ,\ )_{\mathbf{T}_p} : \mathscr{H}(\mathrm{N}; \mathbf{Z}_p) \times m(\mathrm{N}; \mathbf{T}_p) \to \mathbf{T}_p.$$

THEOREM 2.2. — *If* $p \geqq 5$, *then* $\mathscr{H}(\mathrm{N}; \mathbf{Z}_p)$ *and* $m(\mathrm{N}; \mathbf{T}_p)$ (resp. $h(\mathrm{N}; \mathbf{Z}_p)$ *and* $\mathscr{S}(\mathrm{N}; \mathbf{T}_p)$) *are mutually compact-discrete dual in the sense of Pontrjagin* (*cf.* [30, § 28]) *under the pairing* (2.6 *b*).

Proof. — For simplicity, let us write $\mathscr{H}^j$ for $\mathscr{H}^j_1(\mathrm{N}; \mathbf{Z}_p)$. By definition $\mathscr{H}^j$ is $\mathbf{Z}_p$-free of finite rank; thus, by applying the functor $\mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{H}^j, *)$ to the exact sequence:

$$0 \to \mathbf{Z}_p \to \mathbf{Q}_p \to \mathbf{T}_p \to 0,$$

we have the upper exact sequence of the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{H}^j, \mathbf{Z}_p) & \to & \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{H}^j, \mathbf{Q}_p) & \to & \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{H}^j, \mathbf{T}_p) & \to & 0 \\
 & & \alpha \wr\uparrow & & \beta \wr\uparrow & & \gamma \wr\uparrow & & \\
0 & \longrightarrow & m^j(\mathrm{N}; \mathbf{Z}_p) & \longrightarrow & m^j(\mathrm{N}; \mathbf{Q}_p) & \longrightarrow & m^j(\mathrm{N}; \mathbf{T}_p) & \longrightarrow & 0.
\end{array}$$

The existence of the isomorphisms α and β follows from Proposition 2.1 and shows the existence of the isomorphism γ. Note that $\mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{H}^j, \mathbf{T}_p) = \mathrm{Hom}(\mathscr{H}^j, \mathbf{T}_p)$. Then we know that $\mathscr{H}^j$ is the Pontrjagin dual space of $m^j(\mathrm{N}; \mathbf{T}_p)$. Then the assertion is obvious from (2.3) and (2.4 *b*). The assertion for cusp forms can be also proved in the same manner as above.

From the definition (2.6 *a*, *b*), we see easily that

(2.7) $$(h, f \mid h')_{\mathbf{T}_p} = (hh', f)_{\mathbf{T}_p} \quad \text{for} \quad h, h' \in \mathscr{H}(\mathrm{N}; \mathbf{Z}_p) \quad \text{and} \quad f \in m(\mathrm{N}; \mathbf{T}_p).$$

By this duality theorem combined with (1.15 *a*), we know $m(\mathrm{N}; \mathbf{T}_p) \cong m(\mathrm{N}_0 p; \mathbf{T}_p)$.

Furthermore, since $\mathcal{M}(\mathrm{N}_0; \mathbf{Q}_p)$ is dense in $\overline{\mathcal{M}}(\mathrm{N}_0; \mathbf{Q}_p)$, we know

$$(2.8) \qquad m(\mathrm{N}; \mathbf{T}_p) = m(\mathrm{N}_0; \mathbf{T}_p) \, (= \mathcal{M}(\mathrm{N}_0; \mathbf{Q}_p)/(\mathcal{M}(\mathrm{N}_0; \mathbf{Z}_p) + \mathbf{Q}_p)).$$

COROLLARY 2.3. — *Put, for the idempotent e attached to* $\mathrm{T}(p)$, $m_0(\mathrm{N}; \mathbf{T}_p) = e m(\mathrm{N}; \mathbf{T}_p)$, $\mathcal{S}_0(\mathrm{N}; \mathbf{T}_p) = e\mathcal{S}(\mathrm{N}; \mathbf{T}_p)$, $\mathcal{H}_0(\mathrm{N}; \mathbf{Z}_p) = e\mathcal{H}(\mathrm{N}; \mathbf{Z}_p)$ *and* $h_0(\mathrm{N}; \mathbf{Z}_p) = e h(\mathrm{N}; \mathbf{Z}_p)$. *Then, if* $p \geqq 5$, $m_0(\mathrm{N}; \mathbf{T}_p)$ *and* $\mathcal{H}_0(\mathrm{N}; \mathbf{Z}_p)$ (resp. $\mathcal{S}_0(\mathrm{N}; \mathbf{T}_p)$ *and* $h_0(\mathrm{N}; \mathbf{Z}_p)$) *are mutually dual under the pairing* (2.6 *b*).

This follows easily from (2.7) and Theorem 2.2. These results imply that

(2.9) *the modules* $m(\mathrm{N}; \mathbf{T}_p)$ *and* $m_0(\mathrm{N}; \mathbf{T}_p)$ (resp. $\mathcal{S}(\mathrm{N}; \mathbf{T}_p)$ *and* $\mathcal{S}_0(\mathrm{N}; \mathbf{T}_p)$) *are equipped with a continuous action of the compact ring* $\mathcal{H}(\mathrm{N}; \mathbf{Z}_p)$ (resp. $h(\mathrm{N}; \mathbf{Z}_p)$).

Note that the definition of the action of $\mathcal{H}(\mathrm{N}; \mathbf{Z}_p)$ on $\mathcal{M}(\mathrm{N}; \mathbf{Z}_p)$ as in § 1 only guarantees its continuity under the p-adic topology, which is much stronger than the topology defined by the projective limit.

§ 3. Statement of Main Results

We begin by explaining how to consider the space of p-adic modular forms and their Hecke algebras as modules over the Iwasawa algebra. Put

$$\Gamma = \Gamma_1 = 1 + p\mathbf{Z}_p \qquad \text{and} \qquad \Gamma_n = 1 + p^n \mathbf{Z}_p$$

as subgroups of $\mathbf{Z}_p^\times$. Let K be a finite extension of $\mathbf{Q}_p$, and let $\mathcal{O}_\mathrm{K}$ denote its p-adic integer ring. The Iwasawa algebra Λ_K of Γ is defined by the following projective limit:

$$\Lambda_\mathrm{K} = \varprojlim_n \mathcal{O}_\mathrm{K}[\Gamma/\Gamma_n],$$

which can be identified with the formal power series ring $\mathcal{O}_\mathrm{K}[[\mathrm{X}]]$. We specify this identification by assigning $\gamma - 1$ for $\gamma = 1 + p \in \Gamma$ to the indeterminate X (*cf.* [25] or [20, Chapter 5]). We simply write Λ for $\Lambda_{\mathbf{Q}_p}$. As in the previous sections, we fix a positive integer N divisible by p and write $\mathrm{N} = \mathrm{N}_0 p^r$ with $(\mathrm{N}_0, p) = 1$. As seen in § 1, the group $\mathrm{Z} = \mathbf{Z}_p^\times \times (\mathbf{Z}/\mathrm{N}_0\mathbf{Z})^\times$ naturally acts on $\mathcal{M}(\mathrm{N}; \mathrm{K})$; hence $\mathbf{Z}_p^\times$ as a subgroup of Z acts on $\mathcal{M}(\mathrm{N}; \mathrm{K})$. If $f \in \mathcal{M}(\mathrm{N}; \mathrm{K})$ is a sum $\sum_k f_k$ of $f_k \in \mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathrm{K})$, this action of $\mathbf{Z}_p^\times$ is explicitly given by

$$(3.1) \qquad f \mid z = \sum_k z^k f_k \mid \sigma_z \qquad \text{for} \quad z \in \mathbf{Z}_p^\times,$$

where σ_z is an element of $\Gamma_0(\mathrm{N})$ with congruence $\sigma_z \equiv \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \bmod p^r$ and $\sigma_z \equiv 1 \bmod \mathrm{N}_0$.

By (1.12), this action preserves the space $\mathcal{M}(\mathrm{N}; \mathcal{O}_\mathrm{K})$ and therefore induces an action on $\mathcal{M}(\mathrm{N}; \mathrm{K}/\mathcal{O}_\mathrm{K})$. Take $f = \sum_k f_k \in \mathcal{M}(\mathrm{N}; \mathrm{K})$ with $f_k \in \mathcal{M}_k(\Gamma_1(\mathrm{N}); \mathrm{K})$ and assume that $p^n f_k$ is $\mathcal{O}_\mathrm{K}$-integral for all k. Then, for $z \in \Gamma_m$ with $m \geq r$, we see that σ_z acts on each f_k trivially, and we have an inequality:

$$| f \mid z - f |_p = | \sum_k (z^k - 1) f_k |_p \leq p^{n-m}.$$

This shows that the class of f in $\mathscr{M}(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}})$ is invariant under Γ_m if $m \geq n$. Thus the discrete modules $m(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}})$, $\mathscr{M}(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}})$ and $\mathscr{S}(\mathrm{N}; \mathrm{K}/\mathcal{O}_{\mathrm{K}})$ are unions of Γ_m-invariants. Then they are equipped with a continuous Λ_{K}-action (*cf.* [12, § 2]). By the duality in § 2, the universal Hecke algebras $\mathscr{H}(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$ and $h(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$ are also continuous Λ_{K}-modules. We now show that the Λ_{K}-module structure of the Hecke algebras is compatible with their ring structure. In fact, if l is a positive integer with $l \equiv 1 \bmod \mathrm{N}_0$, then l may be regarded as an element of $\mathbf{Z}_p^\times$ and the action of $l \in \mathbf{Z}_p^\times$ on the Hecke algebras coincides with the multiplication of the Hecke operator $l^2\mathrm{T}(l, l)$. Then the density of such integers l and the continuity of the action show

(3.2) $\mathscr{H}(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$ *and* $h(\mathrm{N}; \mathcal{O}_{\mathrm{K}})$ *are continuous* Λ_{K}*-algebras*.

Let μ be the subgroup of $\mathbf{Z}_p^\times$ consisting of $(p-1)$-th roots of unity. Via (3.1), μ also acts on the Hecke algebras. Thus, we can decompose the Hecke algebras according to the character of μ; namely, we have the following decomposition of the ordinary parts of Hecke algebras:

$$\mathscr{H}_0(\mathrm{N}; \mathcal{O}_{\mathrm{K}}) = \bigoplus_{a=0}^{p-2} \mathscr{H}_0(\mathrm{N}, a; \mathcal{O}_{\mathrm{K}}),$$
$$h_0(\mathrm{N}; \mathcal{O}_{\mathrm{K}}) = \bigoplus_{a=0}^{p-2} h_0(\mathrm{N}, a; \mathcal{O}_{\mathrm{K}}),$$

where μ acts on the component corresponding to a by the character: $\mu \ni \zeta \mapsto \zeta^a \in \mathbf{Z}_p$. Let A be either K or $\mathcal{O}_{\mathrm{K}}$. Then, we have a similar decomposition at each weight k:

$$\mathscr{M}_k(\Gamma_1(\mathrm{N}_0 p); \mathrm{A}) = \bigoplus_{a=0}^{p-2} \mathscr{M}_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}),$$
$$\mathscr{S}_k(\Gamma_1(\mathrm{N}_0 p); \mathrm{A}) = \bigoplus_{a=0}^{p-2} \mathscr{S}_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}).$$

Define the Hecke algebra $\mathscr{H}_k(\Gamma_1(\mathrm{N}); \mathrm{A})$ (resp. $h_k(\Gamma_1(\mathrm{N}); \mathrm{A})$) to be an A-subalgebra of $\mathrm{End}_{\mathrm{A}}(\mathscr{M}_k(\Gamma_1(\mathrm{N}); \mathrm{A}))$ (resp. $\mathrm{End}_{\mathrm{A}}(\mathscr{S}_k(\Gamma_1(\mathrm{N}); \mathrm{A}))$ generated by $\mathrm{T}(l)$ and $\mathrm{T}(l, l)$ for all primes l (we allow A even to be a subfield of **C** for this definition of Hecke algebras). Then one can decompose the Hecke algebras accordingly to the decomposition of the spaces of modular forms:

$$\mathscr{H}_k(\Gamma_1(\mathrm{N}_0 p); \mathrm{A}) = \bigoplus_{a=0}^{p-2} \mathscr{H}_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}),$$
$$h_k(\Gamma_1(\mathrm{N}_0 p); \mathrm{A}) = \bigoplus_{a=0}^{p-2} h_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}).$$

Let Φ denote the congruence subgroup $\Gamma_0(p) \cap \Gamma_1(\mathrm{N}_0)$. Then the Teichmüller character ω of $\mathbf{Z}_p^\times$ can be considered as a character of Φ *via* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \omega(d)$. Then, we know that

$$\mathscr{M}_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}) = \mathscr{M}_k(\Phi, \omega^{a-k}; \mathrm{A}),$$
$$\mathscr{S}_k(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A}) = \mathscr{S}_k(\Phi, \omega^{a-k}; \mathrm{A}).$$

Naturally, we can define the ordinary part of these spaces and they will be denoted by $\mathscr{M}_k^0(\Phi, \omega^{a-k}; \mathrm{A})$ and $\mathscr{H}_k^0(\Gamma_1(\mathrm{N}_0 p), a; \mathrm{A})$, etc.

THEOREM 3.1. — *Assume that* $p \geqq 5$ *and that* N *is divisible by p. Then the ordinary parts* $\mathscr{H}_0(N;\mathcal{O}_K)$ *and* $h_0(N;\mathcal{O}_K)$ *of the universal Hecke algebras are free of finite rank over* Λ_K. *Moreover, we have that*

$$\operatorname{rank}_{\Lambda_K}(\mathscr{H}_0(N, a;\mathcal{O}_K)) = \operatorname{rank}_{\mathbf{Z}_p}(\mathscr{M}_k^0(\Phi;\mathbf{Z}_p)),$$
$$\operatorname{rank}_{\Lambda_K}(h_0(N, a;\mathcal{O}_K)) = \operatorname{rank}_{\mathbf{Z}_p}(\mathscr{S}_k^0(\Phi;\mathbf{Z}_p))$$

if $k \geq 3$ *and* $k \equiv a \bmod p-1$.

The restriction of operators in the universal Hecke algebra $\mathscr{H}(N;\mathcal{O}_K)$ to the subspace $\mathscr{M}_k(\Gamma_1(N_0 p), a, \mathcal{O}_K)$ of $\mathscr{M}(N;\mathcal{O}_K)$ induces an algebra homomorphism of $\mathscr{H}_0(N, a;\mathcal{O}_K)$ onto the ordinary $\mathscr{H}_k^0(\Gamma_1(N_0 p), a;\mathcal{O}_K)$. A similar morphism also exists between $h_0(N, a;\mathcal{O}_K)$ and $h_k^0(N, a;\mathcal{O}_K)$. Define a polynomial $P_k = (1+X)-(1+p)^k \in \Lambda$ for each integer k and associate an integer $j(n)$ in the interval $[3, p+1]$ with each integer n by $j(n) \equiv n \bmod p-1$. Then we have

COROLLARY 3.2. — *The natural morphism induces the following isomorphism:*

(3.3 *a*) $\quad \mathscr{H}_0(N, a;\mathcal{O}_K)/P_k\mathscr{H}_0(N, a;\mathcal{O}_K) \cong \mathscr{H}_k^0(\Gamma_1(N_0 p), a;\mathcal{O}_K) \quad$ if $k \geqq j(a)$,

(3.3 *b*) $\quad h_0(N, a;\mathcal{O}_K)/P_k h_0(N, a;\mathcal{O}_K) \cong h_k^0(\Gamma_1(N_0 p), a;\mathcal{O}_K) \quad$ if $k \geqq 2$.

Now we shall bring the notion of primitive forms in the language of Hecke algebras. For each positive integer t, define a morphism:

$$[t] : \mathscr{M}(N;K) \to \mathscr{M}(Nt;K)$$

by $f \mid [t] = \sum_{n=0}^{\infty} a(n, f) q^{nt}$. Clearly, this morphism takes $\mathcal{O}_K$-integral forms into itself, and induces an injection of $\mathscr{M}(N;K/\mathcal{O}_K)$ into $\mathscr{M}(Nt;K/\mathcal{O}_K)$, which will be denoted by the same symbol. Let us define subspaces of old forms by

(3.4 *a*) $\quad \mathcal{O}(N;K/\mathcal{O}_K) = \sum_{1<t|N_0} m(N/t;K/\mathcal{O}_K) \mid [t] + \sum_{1<t|N_0} m(N/t;K/\mathcal{O}_K)$,

(3.4 *b*) $\quad \mathcal{O}_s(N;K/\mathcal{O}_K) = \sum_{1<t|N_0} \mathscr{S}(N/t;K/\mathcal{O}_K) \mid [t] + \sum_{1<t|N_0} \mathscr{S}(N/t;K/\mathcal{O}_K)$.

Since t in (3.4 *a*, *b*) is prime to p, T(p) and e acts on $\mathcal{O}(N;K/\mathcal{O}_K)$ and $\mathcal{O}_s(N;K/\mathcal{O}_K)$. Thus we can define the ordinary parts:

(3.4 *c*) $\quad \mathcal{O}_0(N;K/\mathcal{O}_K) = e\mathcal{O}(N;K/\mathcal{O}_K) \subset m_0(N;K/\mathcal{O}_K)$,

(3.4 *d*) $\quad \mathcal{O}_s^0(N;K/\mathcal{O}_K) = e\mathcal{O}_s(N;K/\mathcal{O}_K) \subset \mathscr{S}_0(N;K/\mathcal{O}_K)$.

DEFINITION. — *We define ideals* $\mathscr{P}(N;\mathcal{O}_K)$ (resp. $\wp(N;\mathcal{O}_K)$) *in* $\mathscr{H}_0(N;\mathcal{O}_K)$ (resp. $h_0(N;\mathcal{O}_K)$) *by the* annihilator *of* $\mathcal{O}_0(N;K/\mathcal{O}_K)$ (resp. $\mathcal{O}_s^0(N;K/\mathcal{O}_K)$). *By Corollary 1.2 and* (2.8), *the space* $\mathcal{O}(N;K/\mathcal{O}_K)$ *and* $\mathcal{O}_s(N;K/\mathcal{O}_K)$ *are independant of the exponent of p in* N. *Thus* $\mathscr{P}(N;\mathcal{O}_K)$ *and* $\wp(N;\mathcal{O}_K)$ *are also independent of the power of p dividing* N.

Then we have

COROLLARY 3.3. — *The intersection of the ideal* $\mathscr{P}(N;\mathcal{O}_K)$ (resp. $\wp(N;\mathcal{O}_K)$) *with the nilradical of* $\mathscr{H}_0(N;\mathcal{O}_K)$ (resp. $h_0(N;\mathcal{O}_K)$) *is reduced to null. Moreover* $\mathscr{P}(N;\mathcal{O}_K)$ *and* $\wp(N;\mathcal{O}_K)$ *are free of finite rank over* Λ_K.

Theorem 3.1 and Corollaries 3.2 and 3.3 will be proved in § 5.

Let us now explain the relation between Theorem 3.1 and congruences mod p among ordinary forms. By Theorem 3.1, the Hecke algebra $\mathscr{H}_0(\mathrm{N}, a; \mathcal{O}_\mathrm{K})$ is a direct sum of local rings. Each local ring is free over Λ_K. Take one of local rings of $\mathscr{H}_0(\mathrm{N}, a; \mathcal{O}_\mathrm{K})$ and denote it by R. Then, for each weight $k \geqq j(a)$, the residue ring $\mathrm{R}_k = \mathrm{R}/\mathrm{P}_k\mathrm{R}$ is local and is a direct summand of the Hecke algebra $\mathscr{H}_k(\Gamma_1(\mathrm{N}_0 p); \mathcal{O}_k)$. Now we fix one weight $k \geqq j(a)$. Let 1_{R_k} be the idempotent of $\mathscr{H}_k(\Gamma_1(\mathrm{N}_0 p); \mathcal{O}_\mathrm{K})$ corresponding to R_k, and let S be the semi-simplification of $\mathrm{R}_k \otimes_{\mathcal{O}_\mathrm{K}} \mathrm{K}$ (i. e. the quotient of $\mathrm{R}_k \otimes_{\mathcal{O}_\mathrm{K}} \mathrm{K}$ by its nilradical). Then, the arguments in [29, p. 82] show that there are d-normalized eigenforms $f_1, \ldots, f_d$ in $\mathscr{M}_k(\Gamma_1(\mathrm{N}_0 p))$ ($d = [\mathrm{S} : \mathrm{K}]$) such that $f_i \mid 1_{\mathrm{R}_k} = f_i$. Moreover, every $\mathcal{O}_\mathrm{K}$-linear homomorphism λ of S into Ω can be realized by one of f_i's, say f, as $f \mid \mathrm{T} = \lambda(\mathrm{T}) f$ for all $\mathrm{T} \in \mathrm{R}_k$. Since R_k is local, any pair f_i, f_j satisfies either of the following two conditions:

(3.5 *a*) *There is an automorphism* σ *of* Ω *over* K *such that* $f_i \equiv f_j^\sigma \bmod \mathfrak{P}$;

(3.5 *b*) $a(m, f_i) \equiv a(m, f_j) \bmod \mathfrak{P}$ *for all* $m > 0$,

where $\mathfrak{P}$ is the maximal ideal of the p-adic integer ring of Ω. Furthermore,

(3.6) *the set* $\{f_1, \ldots, f_d\}$ *is maximal among subsets of normalized eigenforms in* $\mathscr{M}_k(\Gamma_1(\mathrm{N}_0 p))$ *satisfying one of the properties* (3.5 *a*, *b*).

Let π be a prime element of $\mathcal{O}_\mathrm{K}$ and $\mathfrak{m}$ be the maximal ideal of Λ_K. Then the algebra $\mathrm{R}_k/\pi\mathrm{R}_k$ is isomorphic to $\mathrm{R}/\mathfrak{m}\mathrm{R}$ and is independent of k. Thus we have

(3.7) *If* (3.5 *b*) *is satisfied for all pairs* (i, j) *at one weight* $k \geq j(a)$, *then this is true for all weights* $(\geq j(a))$.

If the field K contains the numbers $a(m, f_i)$ for all m and i, we have the congruence:

$$a(m, f_1) \equiv a(m, f_2) \equiv \ldots \equiv a(m, f_d) \bmod \mathscr{P} \qquad \text{for all} \quad m > 0.$$

Then (3.7) shows that

(3.8) *By replacing* K *by a suitable finite extension* M, *we can make the condition* (3.5 *b*) *hold for all pairs* (i, j) *at every weight* $(\geqq j(a))$. *We obtain such an extension by adjoining to* K *the numbers* $a(n, f_j)$ *for all* j *and* $m > 0$ *at one weight* $k \geqq j(a)$. *Every finite extension of* M *has the same property.*

Therefore, each local ring of $\mathscr{H}_0(\mathrm{N}; \mathcal{O}_\mathrm{K})$ corresponds to a unique maximal class of ordinary forms which belong to the same eigenvalues modulo $\mathfrak{P}$ of Hecke operators. We cannot separate ordinary forms congruent with each other modulo $\mathfrak{P}$ only by using the decomposition of the universal Hecke algebra into the sum of local rings.

In order to distinguish ordinary forms with mod $\mathfrak{P}$ congruences, we need to have a finer decomposition of the Hecke algebra. Let $\mathscr{L}_\mathrm{K}$ be the quotient field of Λ_K and put

$$\begin{aligned}
\mathscr{Q} = \mathscr{Q}(\mathrm{N}; \mathrm{K}) &= \mathscr{H}_0(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\Lambda_\mathrm{K}} \mathscr{L}_\mathrm{K}, \\
\mathscr{q} = \mathscr{q}(\mathrm{N}; \mathrm{K}) &= \mathscr{h}_0(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\Lambda_\mathrm{K}} \mathscr{L}_\mathrm{K}, \\
\mathscr{P}(\mathrm{N}; \mathrm{K}) &= \mathscr{P}(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\Lambda_\mathrm{K}} \mathscr{L}_\mathrm{K}, \\
\mathscr{p}(\mathrm{N}; \mathrm{K}) &= \mathscr{p}(\mathrm{N}; \mathcal{O}_\mathrm{K}) \otimes_{\Lambda_\mathrm{K}} \mathscr{L}_\mathrm{K}.
\end{aligned}$$

Then, $\mathscr{Q}$ and $\mathscr{q}$ are finite dimensional artinian algebra over $\mathscr{L}_K$, and they are direct sums of local artinian algebras. Since $\mathscr{P}(N;K)$ and $\mathscr{p}(N;K)$ are ideals without intersection with nilradicals, they are semi-simple algebras, which are thus a product of finite extensions of $\mathscr{L}_K$.

Terminology. — A local ring $\mathscr{K}$ of $\mathscr{Q}(N;K)$ is called *primitive* of conductor N_0 if $\mathscr{K}$ is a direct algebra summand of $\mathscr{P}(N;K)$. When $\mathscr{K}$ is contained in $\mathscr{q}$, we say that $\mathscr{K}$ is *cuspidal.*

PROPOSITION 3.4. — *Let $\mathscr{K}$ be a simple direct algebra summand of $\mathscr{Q}$. Then, the algebraic closure* M *of* $\mathbf{Q}_p$ *in* $\mathscr{K}$ *is a finite extension of* K. *Furthermore the index* $[M:K]$ *divides the index* $[\mathscr{K}:\mathscr{L}_K]$.

Proof. — Since $\Lambda_K=\mathcal{O}_K[[X]]$ is a subalgebra of $\mathscr{K}$, the indeterminate X is transcendental over M. For any subfield M′ of M finite over K, $\mathcal{O}_{M'}[[X]]=\Lambda_K\otimes_{\mathcal{O}_K}\mathcal{O}_{M'}$ is contained in $\mathscr{K}$, and hence its quotient field $\mathscr{L}_{M'}$ is a subfield of $\mathscr{K}$. This shows

$$[M':K]=[\mathscr{L}_{M'}:\mathscr{L}_K]\mid[\mathscr{K}:\mathscr{L}_K].$$

Thus M must be finite over K, and its degree over K divides that of $\mathscr{K}$ over $\mathscr{L}_K$.

When the algebraic closure of $\mathbf{Q}_p$ in $\mathscr{K}$ coincides with K, we say that $\mathscr{K}$ is *defined* over K.

THEOREM 3.5. — *For any finite extension* M/K, *we have that* $\mathscr{Q}(N;K)\otimes_K M\simeq\mathscr{Q}(N;M)$ *and* $\mathscr{q}(N;K)\otimes_K M\simeq\mathscr{q}(N;M)$. *If* $\mathscr{K}$ *is a primitive component of* $\mathscr{Q}(N;K)$ (resp. $\mathscr{q}(N;K)$) *defined over* K, *then* $\mathscr{K}_M=\mathscr{K}\otimes_K M$ *is a field and can be regarded uniquely as a primitive component of* $\mathscr{Q}(N;M)$ (resp. $\mathscr{q}(N;M)$).

Proof. — By (1.16), we have that $\mathscr{H}_0(N;\mathcal{O}_M)=\mathscr{H}_0(N;\mathcal{O}_K)\hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_M$, but we know from the finiteness of $\mathscr{H}_0(N;\mathcal{O}_K)$ over Λ_K that $\mathscr{H}_0(N;\mathcal{O}_M)=\mathscr{H}_0(N;\mathcal{O}_K)\otimes_{\mathcal{O}_K}\mathcal{O}_M$. This shows that $\mathscr{Q}(N;M)=\mathscr{Q}(N;K)\otimes_K M$. The second assertion obviously follows from this identity.

Let V be a finite dimensional vector space over $\mathscr{L}_K$ and let $\mathscr{X}$ be a Λ_K-lattice of V in the sense of [1, VII.4.1]. Define another lattice $\tilde{\mathscr{X}}$ containing $\mathscr{X}$ by the intersection in V:

$$\tilde{\mathscr{X}}=\bigcap_P\mathscr{X}_P,$$

where P runs over all prime ideals of Λ_K of height 1 (i. e. prime divisors) and $\mathscr{X}_P$ denotes the localization of $\mathscr{X}$ at P. We call $\tilde{\mathscr{X}}$ the *free closure* of $\mathscr{X}$, because it is the smallest free module in V containing $\mathscr{X}$. In fact, it is known that $\tilde{\mathscr{X}}/\mathscr{X}$ has only finitely many elements; namely, it is pseudo-null [1, III.4.4] and $\mathscr{X}=\tilde{\mathscr{X}}$ if and only if $\mathscr{X}$ is Λ_K-free [1, VII.4.2, Th. 2].

We shall now apply this formation of free closures to our algebras $\mathscr{Q}$ and $\mathscr{q}$. Let $\mathscr{K}$ be a *primitive* component of $\mathscr{Q}(N;K)$ (resp. $\mathscr{q}(N;K)$). Since $\mathscr{K}$ is by definition a direct summand of $\mathscr{Q}$ (resp. $\mathscr{q}$), we can decompose $\mathscr{Q}=\mathscr{K}\oplus\mathscr{A}$ (resp. $\mathscr{q}=\mathscr{K}\oplus\mathscr{B}$) as an algebra direct sum. We write simply $\mathscr{H}$ (resp. $\mathscr{h}$) for $\mathscr{H}_0(N;\mathcal{O}_K)$ (resp. $\mathscr{h}_0(N;\mathcal{O}_K)$), and we denote by $\mathscr{H}(\mathscr{K})$ and $\mathscr{H}(\mathscr{A})$ (resp. $\mathscr{h}(\mathscr{K})$ and $\mathscr{h}(\mathscr{B})$) the projected images of $\mathscr{H}$ (resp. $\mathscr{h}$) in $\mathscr{K}$ and $\mathscr{A}$ (resp. $\mathscr{K}$ and $\mathscr{B}$). Put $\mathscr{H}'=\mathscr{H}(\mathscr{K})\oplus\mathscr{H}(\mathscr{A})$, $\mathscr{h}'=\mathscr{h}(\mathscr{K})\oplus\mathscr{h}(\mathscr{A})$ and denote by $\overline{\mathscr{H}}$ and $\overline{\mathscr{h}}$ the free closures of $\mathscr{H}'$ and $\mathscr{h}'$, respectively.

DEFINITION. — *We define Λ_K-modules $\mathscr{C}$, $\mathscr{C}$, $\mathscr{N}$ and $\mathscr{N}_s$ by*

(3.9 *a*) $$\mathscr{C}=\mathscr{C}(\mathscr{K};K)=\overline{\mathscr{H}}/\mathscr{H},\qquad \mathscr{N}=\mathscr{N}(\mathscr{K};K)=\overline{\mathscr{H}}/\mathscr{H}',$$

(3.9 *b*) $$\mathscr{C}=\mathscr{C}(\mathscr{K};K)=\overline{h}/h,\qquad \mathscr{N}_s=\mathscr{N}_s(\mathscr{K};K)=\overline{h}/h'.$$

THEOREM 3.6. — *Let $\mathscr{R}$ (resp. $\mathscr{R}_s$) be the local ring of $\mathscr{H}_0(N;\mathcal{O}_K)$ (resp. $h_0(N;\mathcal{O}_K)$) such that $\mathscr{R}\otimes_{\Lambda_K}\mathscr{L}_K$ (resp. $\mathscr{R}_s\otimes_{\Lambda_K}\mathscr{L}_K$) contains $\mathscr{K}$. We assume $\mathscr{K}$ to be cuspidal in the following statements concerning $\mathscr{C}(\mathscr{K};K)$ and $\mathscr{N}_s(\mathscr{K}:K)$. Then we have:*

(3.10 *a*) *$\mathscr{C}(\mathscr{K};K)$ and $\mathscr{C}(\mathscr{K};K)$ are finite torsion Λ_K-modules.*

(3.10 *b*) *If $\mathscr{K}$ is defined over K and M/K is any finite extension, then $\mathscr{K}_M=\mathscr{K}\otimes_K M$ is a primitive local ring of $\mathscr{Q}(N;M)$ and*

$$\mathscr{C}(\mathscr{K}_M;M)=\mathscr{C}(\mathscr{K};K)\otimes_{\Lambda_K}\Lambda_M,$$
$$\mathscr{C}(\mathscr{K}_M;M)=\mathscr{C}(\mathscr{K};K)\otimes_{\Lambda_K}\Lambda_M.$$

(3.10 *c*) *$\mathscr{C}(\mathscr{K};K)=0$ (resp. $\mathscr{C}(\mathscr{K};K)=0$) if and only if $\mathscr{R}\otimes_{\Lambda_K}\mathscr{L}_K=\mathscr{K}$ (resp. $\mathscr{R}_s\otimes_{\Lambda_K}\mathscr{L}_K=\mathscr{K}$).*

(3.10 *d*) *The annihilator of $\mathscr{C}(\mathscr{K};K)$ and $\mathscr{C}(\mathscr{K};K)$ in Λ_K are principal ideals of Λ_K.*

(3.10 *e*) *$\mathscr{N}(\mathscr{K};K)$ and $\mathscr{N}_s(\mathscr{K};K)$ have only finitely many elements.*

Let $\mathscr{K}$ be a local ring of $\mathscr{Q}$, and $\mathscr{H}(\mathscr{K})$ be the image of $\mathscr{H}$ in $\mathscr{K}$. Then, we have a natural projection map of $\mathscr{H}$ into the free closure $\tilde{\mathscr{H}}(\mathscr{K})$ of $\mathscr{H}(\mathscr{K})$. By tensoring $\Lambda_K/P_k\Lambda_K$ to this morphism, we have a natural morphism of $\mathscr{H}_k(\Gamma_1(N_0p);\mathcal{O}_K)$ into $\tilde{\mathscr{H}}(\mathscr{K})/P_k\tilde{\mathscr{H}}(\mathscr{K})$ by Corollary 3.2. We write λ_k for this morphism at each meight k. On the other hand, to give an $\mathcal{O}_K$-algebra homomorphism λ of $\mathscr{H}_k(\Gamma_1(N_0p);\mathcal{O}_K)$ into Ω is equivalent to giving a normalized eigenform f in $\mathscr{M}_k(\Gamma_1(N_0p))$ such that $f\,|\,T(n)=\lambda(T(n))f$ for all $n>0$ (*cf.* § 2).

DEFINITION. — *Let f be a normalized eigenform in $\mathscr{M}_k(\Gamma_1(N_0p))$ and let λ be the $\mathcal{O}_K$-algebra homomorphism of $\mathscr{H}_k(\Gamma_1(N_0p);\mathcal{O}_K)$ into Ω corresponding to f. Then, we say that f (and λ) belongs to $\mathscr{K}$ if λ factors through λ_k.*

Any ordinary form belongs by definition to some local ring of $\mathscr{Q}$. By the theory of primitive forms, one can associate with any normalized eigenform in $\mathscr{M}_k(\Gamma_1(N))$ a primitive form whose n-th coefficient gives that of the original eigenform if n is prime to the level N.

COROLLARY 3.7. — *Assume $\mathscr{K}$ to be primitive and cuspidal and write d for $[\mathscr{K}:\mathscr{L}_K]$. Then, for each $k\geq 2$, the residue algebra $\tilde{\mathscr{H}}_k(\mathscr{K})=\tilde{\mathscr{H}}(\mathscr{K})\otimes_{\Lambda_K}\Lambda_K/P_k\Lambda_K$ can be naturally embedded into $\mathscr{H}_k^0(\Gamma_1(N_0p);K)$ as a subalgebra, and there exist exactly d ordinary forms in $\mathscr{S}_k(\Gamma_1(N_0p))$ belonging to $\mathscr{K}$. Moreover, the primitive forms associated with forms belonging to $\mathscr{K}$ has conductor divisible by N_0. Conversely, if f is an ordinary form in $\mathscr{S}_k(\Gamma_1(N_0p))$ for $k\geq 2$ and if f is associated with a primitive form whose conductor is divisible by N_0, then the local ring of $\mathscr{Q}(N;K)$ to which f belongs is unique and primitive.*

Even if $\mathscr{K}$ is non-cuspidal, the whole assertion of this Corollary holds under the condition that $k\geq j(a)$ if $\mathscr{K}$ is contained in $\mathscr{H}_0(N,a;\mathcal{O}_K)\otimes_{\Lambda_K}\mathscr{L}_K$.

Let $\mathscr{K}$ be a primitive component of $\mathscr{H}_0(N,a;\mathcal{O}_K)\otimes_{\Lambda_K}\mathscr{L}_K$, and put

$$F=F_k=(\tilde{\mathscr{H}}(\mathscr{K})/P_k\tilde{\mathscr{H}}(\mathscr{K}))\otimes_{\mathcal{O}_K}K.$$

Then F is semi-simple and is an algebra direct summand of $\mathcal{H}_k(\Gamma_1(\mathrm{N}_0 p)\,;\mathrm{K})$ by Corollary 3.7 and [10, (4.4 *c*)]. Thus we can decompose

$$\mathcal{H}_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathrm{K})=\mathrm{F}_k\oplus \mathrm{A}_k \quad \text{as an algebra direct sum}.$$

If $\mathcal{K}$ is cuspidal, one also has an algebra direct sum decomposition:

$$h_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathrm{K})=\mathrm{F}_k\oplus \mathrm{B}_k.$$

Clearly, these decompositions are unique. Let $\mathcal{K}(\mathrm{F}_k)$ and $\mathcal{K}(\mathrm{A}_k)$ (resp. $h(\mathrm{F}_k)$ and $h(\mathrm{B}_k)$) be the projections of $\mathcal{H}_k(\Gamma_1(\mathrm{N}_0 p)\,;\mathcal{O}_\mathrm{K})$ (resp. $h_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathcal{O}_\mathrm{K})$) in F_k and A_k (resp. F_k and B_k).

DEFINITION. — Define finite p-power torsion modules $\mathrm{T}_k(\mathcal{K})$ and $\mathrm{C}_k(\mathcal{K})$ for each k by

(3.11 *a*) $\mathrm{T}_k(\mathcal{K})=(\mathcal{H}(\mathrm{F}_k)\oplus\mathcal{H}(\mathrm{A}_k))/\mathcal{K}_k^0(\Gamma_1(\mathrm{N}_0 p);\mathcal{O}_\mathrm{K})$ if $(k\geq j(a)$,

(3.11 *b*) $\mathrm{C}_k(\mathcal{K})=(h(\mathrm{F}_k)\oplus h(\mathrm{B}_k))/h_k^0(\Gamma_1(\mathrm{N}_0 p);\mathcal{O}_\mathrm{K})$ if $k\geq 2$.

They will be called the *module of congruence* of $\mathcal{K}$ at weight k. The reason for this appellation will be clarified later. These modules give the fibres of $\mathcal{T}$ and $\mathcal{C}$ at each weight k:

COROLLARY 3.8. — *Assume $\mathcal{K}$ to be cuspidal in the statement for $\mathcal{C}(\mathcal{K}\,;\mathrm{K})$. Then we have canonical exact sequences:*

$$0\to \mathrm{T}_k(\mathcal{K})\to \mathcal{T}(\mathcal{K}\,;\mathrm{K})/\mathrm{P}_k\mathcal{T}(\mathcal{K}\,;\mathrm{K})\to \mathcal{N}(\mathcal{K}\,;\mathrm{K})/\mathrm{P}_k\mathcal{N}(\mathcal{K}\,;\mathrm{K})\to 0 \quad \textit{for each}\quad k\geq j(a);$$
$$0\to \mathrm{C}_k(\mathcal{K})\to \mathcal{C}(\mathcal{K}\,;\mathrm{K})/\mathrm{P}_k\mathcal{C}(\mathcal{K}\,;\mathrm{K})\to \mathcal{N}_s(\mathcal{K}\,;\mathrm{K})/\mathrm{P}_k\mathcal{N}_s(\mathcal{K}\,;\mathrm{K})\to 0 \quad \textit{for each}\quad k\geq 2.$$

Theorem 3.6 and its corollaries will be proved in § 6. Now let us give a heuristic explanation of these modules. For simplicity, take a primitive cuspidal component $\mathcal{K}$ of $\varphi(\mathrm{N}\,;\mathrm{K})$ with $[\mathcal{K}:\mathcal{L}_\mathrm{K}]=1$. Then, the image $h(\mathcal{K})$ coincides with Λ_K, because $h(\mathcal{K})$ and Λ_K have the same quotient field and $h(\mathcal{K})$ is integral over Λ_K. At each k, we can identify $\Lambda_\mathrm{K}/\mathrm{P}_k\Lambda_\mathrm{K}$ with $\mathcal{O}_\mathrm{K}$ by assigning to each power series its value at $(1+p)^k-1$. On the other hand, the projected image of the Hecke operator $\mathrm{T}(n)$ in $h(\mathcal{K})$ is a power series $\mathrm{A}(n\,;\mathrm{X})\in\Lambda_\mathrm{K}$. Then, the following formal Fourier expansion for each $k\geq 2$:

$$f_k=\sum_{n=1}^{\infty}\mathrm{A}(n\,;(1+p)^k-1)e(nz)$$

gives in fact the actual Fourier expansion of a unique ordinary form in $\mathcal{S}_k(\Gamma_1(\mathrm{N}_0 p))$ belonging to $\mathcal{K}$. Note that the conductor of the subalgebra $h_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathcal{O}_\mathrm{K})$ of the algebra $h(\mathrm{F}_k)\oplus h(\mathrm{B}_k)$ is the annihilator of the module of congruences $\mathrm{C}_k(\mathcal{K})$. By this fact, we can conclude that

(3.12) *There is a normalized eigenform g in $\mathcal{S}_k(\Gamma_1(\mathrm{N}_0 p))$ such that $g\equiv f_k \bmod \mathfrak{P}$ if and only if $\mathrm{C}_k(\mathcal{K})$ is non-trivial.*

Moreover, the module of congruences may be considered as an invariant which gives us a quantitative measure of congruences satisfied by the ordinary forms belonging to $\mathcal{K}$. A remarkable fact is by Corollary 3.8 that the module $\mathrm{C}_k(\mathcal{K})$ depends p-adic analytically on the weight k. In fact, the Iwasawa module $\mathcal{C}(\mathcal{K}\,;\mathrm{K})$, whose annihilator is a principal ideal $\mathrm{H}(\mathrm{X})\Lambda_\mathrm{K}$ by Theorem 3.6, is pseudo-isomorphic to the quotient $\Lambda_\mathrm{K}/\mathrm{H}(\mathrm{X})\Lambda_\mathrm{K}$. Then, Corollary 3.8 says that $C_k(\mathscr{X})$ can be approximated by the module $\mathcal{O}_K/H((1+p)^k-1)\mathcal{O}_K$ up to finite errors, which are bounded independently of k. If $\mathscr{N}_s(\mathscr{X};K)$ is null, one can show (see below) that $\mathscr{C}(\mathscr{X};K)$ exactly coincides with $\Lambda_K/H(X)\Lambda_K$ and thus, there is no error at all. In fact, if $\mathscr{N}_s=0$, we have an isomorphism induced by the projection map:

$$\mathscr{C}(\mathscr{X};K)\simeq h(\mathscr{X})/(\mathscr{X}\cap h).$$

By assumption, we have that $h(\mathscr{X})=\Lambda_K$. On the other hand, the intersection $\mathscr{X}\cap h$ is reflexive. Thus $\mathscr{X}\cap h$ is a principal ideal which coincides with $H(X)\Lambda_K$. Therefore, our next goal is to know when the module of errors $\mathscr{N}_s$ vanishes. There is an effective criterion for this: let $\mathscr{X}$ be a primitive cuspidal component of $\varphi(N;K)$, and let $\mathscr{R}$ be the local ring of $h_0(N;\mathcal{O}_K)$ such that $\mathscr{X}$ is contained in $\mathscr{R}\otimes_{\Lambda_K}\mathscr{L}_K$. We do not assume here that $[\mathscr{X}:\mathscr{L}_K]=1$. Decompose

$$\mathscr{R}\otimes_{\Lambda_K}\mathscr{L}_K=\mathscr{X}\oplus\mathscr{B}\quad\text{as algebra direct sum},$$

and denote by $\mathscr{R}(\mathscr{X})$ and $\mathscr{R}(\mathscr{B})$ the projections of $\mathscr{R}$ in $\mathscr{X}$ and $\mathscr{B}$. Then, $\mathscr{C}(\mathscr{X};K)$ is by definition isomorphic to $(\tilde{\mathscr{R}}(\mathscr{X})\oplus\tilde{\mathscr{R}}(\mathscr{B}))/\mathscr{R}$, where " $\sim$ " indicates the free closure. At each weight k, $\mathscr{R}/P_k\mathscr{R}$ is a local ring of $h_k^0(\Gamma_1(N_0 p);\mathcal{O}_K)$. Put $F_k=(\tilde{h}(\mathscr{X})/P_k\tilde{h}(\mathscr{X}))\otimes_{\mathcal{O}_K}K$, and decompose

$$(\mathscr{R}/P_k\mathscr{R})\otimes_{\mathcal{O}_K}K=F_k\oplus B_k.$$

Denote by $R(F_k)$ and $R(B_k)$ the projections of $(\mathscr{R}/P_k\mathscr{R})$ in F_k and B_k. Again by definition, $C_k(\mathscr{X})$ is isomorphic to $(R(F_k)\oplus R(B_k))/(\mathscr{R}/P_k\mathscr{R})$. Then our criterion is

PROPOSITION 3.9. — *If one of the following two conditions is satisfied by $\mathscr{R}/P_k\mathscr{R}$ for at least one $k\geq 2$, then $\mathscr{N}_s(\mathscr{X};K)$ vanishes. The conditions are:*

(i) $R(F_k)\oplus R(B_k)$ *is integrally closed in* $(\mathscr{R}/P_k\mathscr{R})\otimes_{\mathcal{O}_K}K$;

(ii) $\mathscr{R}/P_k\mathscr{R}\simeq \mathrm{Hom}_{\mathcal{O}_K}((\mathscr{R}/P_k\mathscr{R}),\mathcal{O}_K)$ *as* $(\mathscr{R}/P_k\mathscr{R})$*-module, and* $R(F_k)$ *is integrally closed in* F_k.

Here are some remarks about this criterion, whose proof will be given in § 6. Firstly, the same criterion is also valid for $\mathscr{N}(\mathscr{X};K)$. Secondly, the first part of (ii) is equivalent to saying that $\mathscr{R}/P_k\mathscr{R}$ is a Gorenstein ring [22, II.15], and the second part of (ii) is always true if $[\mathscr{X}:\mathscr{L}_K]=1$. Thirdly, it is known by Mazur [22, Cor. 15.2, p. 124] that $\mathscr{R}/P_2\mathscr{R}$ is a Gorenstein ring if $\mathscr{R}/P_2\mathscr{R}$ is contained in $h_2^0(\Gamma_0(q);\mathcal{O}_K)$ for a prime q. This case can happen only when $p=q$ and $N_0=1$, or $N_0=q$. Thus, if $\mathscr{R}/P_2\mathscr{R}$ is contained in $h_2^0(\Gamma_0(q);\mathcal{O}_K)$ and $[\mathscr{X}:\mathscr{L}_K]=1$, then we know the vanishing $\mathscr{N}_s(\mathscr{X};K)=0$. This is the only known case where the vanishing of $\mathscr{N}_s(\mathscr{X};K)$ is theoretically proved. In § 7, we shall discuss some numerical examples of $\mathscr{C}(\mathscr{X};K)$ with $\mathscr{N}_s(\mathscr{X};K)=0$. Some other examples can be also found in [31, § 4]. No example of $\mathscr{X}$ with non-trivial $\mathscr{N}_s(\mathscr{X};K)$ is so far known.

Now we shall discuss briefly about a (conjectured) relation between the modules of congruences and the special values of a certain L-function associated with ordinary forms. Let f be a primitive form in $\mathscr{S}_k(\Gamma_0(N),\psi)$ with $k\geq 2$, and let F be a finite extension of $\mathbf{Q}$ containing all Fourier coefficients of f. Then, the Hecke algebra $h_k(\Gamma_1(N);F)$ is decomposed into an algebra direct sum $F\oplus B$, where F is the local ring of $h_k(\Gamma_1(N);F)$ corresponding to f. As introduced in [29, Chap. 8], there is a cohomological interpretation of spaces of cusp forms. To describe this, let $L_n(\mathbf{Z})=\mathbf{Z}^{n+1}$ for each non-negative integer n, and let $M_2(\mathbf{Z})$ act on $L_n(\mathbf{Z})$ through n-th symmetric tensor representation as specified in [29, 8.2] and [8, § 1]. Define a $M_2(\mathbf{Z})$-module $L_n(A)$ by $L_n(\mathbf{Z})\otimes_{\mathbf{Z}}A$ for any algebra A. As shown in [29, 8.3], the Hecke algebra $h_k(\Gamma_1(N); F)$ for $k=n+2$ naturally acts on the parabolic cohomology group $H^1_p(\Gamma_1(N); L_n(F))$ defined in [29, Chap. 8]. Therefore, we can decompose $H^1_p(\Gamma_1(N), L_n(F))=H(F)\oplus H(B)$ accordingly to the decomposition of the Hecke algebra. Let $\mathcal{O}_F$ be the valuation ring of F corresponding to the fixed embedding of $\overline{\mathbf{Q}}$ into Ω. Let $V(k)$ be the image of $H^1_p(\Gamma_1(N), L_n(\mathcal{O}_F))$ in $H^1_p(\Gamma_1(N), L_n(F))$. Write $W(F)$ and $W(B)$ for the projections of $V(k)$ in $H(F)$ and $H(B)$, and define $W(f)$ and $V(F)$ by

$$W(f)=W(F)\oplus W(B), \qquad V(F)=V(k)\cap H(F).$$

Then, we can relate the index $[W(f):V(k)]=[W(F):V(F)]$ with the special value at weight k of the L-function of f defined by the Euler product:

$$L(s, f)=\prod_l [(1-\overline{\psi}_0(l)\alpha_l^2 l^{-s})(1-\overline{\psi}_0(l)\alpha_l\beta_l l^{-s})(1-\overline{\psi}_0(l)\beta_l^2 l^{-s})]^{-1},$$

where ψ_0 is the primitive character associated with ψ and α_l, $\beta_l\in\mathbf{C}$ are the two roots of the equation:

$$X^2-a(l, f)X+\psi(l)l^{k-1}=0.$$

Here we shall explain this relation only when

(3.13) F *is contained in* $\mathbf{R}$;

otherwise, we have to consider the cohomology groups with coefficients in $L_n(\mathbf{C})$, which require further technical complexity (*cf.* [28, § 4]). We will discuss this in full generality in a subsequent paper [11]. By (3.13), we can extend scalar to $\mathbf{R}$ and know from a result of Shimura [29, Th. 8.4] that

$$H^1_p(\Gamma_1(N); L_n(F))\otimes_F\mathbf{R}\simeq H^1_p(\Gamma_1(N); L_n(\mathbf{R})),$$

which can be identified with $\mathscr{S}_{n+2}(\Gamma_1(N))$ as vector space over $\mathbf{R}$. Then, the subspace $H(F)\otimes_F\mathbf{R}$ is spanned by f and $\sqrt{-1}f$. Thus $W(F)$ and $V(F)$ are $\mathcal{O}_F$-lattices of $H(F)$. Take a basis δ_1 and δ_2 of $V(F)$ over $\mathcal{O}_F$ and define a real matrix $X\in GL_2(\mathbf{R})$ by

$$(\delta_1, \delta_2)X=(f, \sqrt{-1}f).$$

Put

$$U_\infty(f)=\pi^{k+1}\,|\det X\,|/(k-1)!NC(\psi)\phi(N/C(\psi)),$$

where $C(\psi)$ is the conductor of ψ and ϕ is the Euler function. The number $U_\infty(f)$ is determined independently of the basis (δ_1, δ_2) up to units in $\mathcal{O}_F$. Let K be the closure of F in Ω. Then, a fundamental identity is

$$(3.14)\qquad |\,L(k, f)/U_\infty(f)\,|_p^{-2[K:\mathbf{Q}_p]}=[W(f):V(k)]=[W(F):V(F)].$$

This can be shown in a similar manner in [7, § 6], and the transcendental factor $U_\infty(f)$ can be also given by the period determinant of f as in [7, (6.17)] if $k=2$. Now we shall state a conjecture:

CONJECTURE 3.10. — *Let* $\mathscr{K}$ *be a primitive cuspidal component of* $\mathscr{g}(N; K)$. *Assume* *that* $[\mathscr{K}:\mathscr{L}_K]=1$. *Let* ϕ_K *be the unique ordinary form in* $\mathscr{S}_k(\Gamma_1(N_0p))$ *belonging to* $\mathscr{K}$ *for each* $k\geq 2$, *and let* f_k *be the primitive form associated with* ϕ_k. *Then we have*

$$|C_k(\mathscr{K})|_p^{-2[K:\mathbf{Q}_p]}=[W(f_k):V(k)].$$

Here are some remarks about this conjecture: Firstly, if the p-part of the character of f_2 is non-trivial or $N_0=1$, the conjecture is true for $k=2$. When $N_0=1$, this follows from [22, II.15], and the other case is a consequence of [9, (3.17)]. Secondly, one can define a canonical transcendental factor $U_\infty(f)$ without the assumption (3.13) and show that (3.14) still holds. Finally, assume that $\mathscr{N}_s(\mathscr{K};K)=0$. Then the combination of the conjecture and (3.14) shows the existence of a p-adic unit $U_p(f_k)$ for each $k\geq 2$ such that

$$L(k, f_k)/U_\infty(f_k)U_p(f_k)=H((1+p)^k-1)\qquad\text{for every}\quad k\geq 2,$$

where H(X) is the characteristic power series in Λ_K of $\mathscr{C}(\mathscr{K};K)$. Thus, the function $s\mapsto H((1+p)^s-1)$ gives a p-adic interpolation of L-values $L(k, f_k)/U_\infty(f_k)$.

§ 4. Finiteness of $\mathscr{H}_0(N;\mathscr{O}_K)$ over Λ_K

In this section, we firstly review a result of Jachnowitz [13], and then, we shall prove the finiteness of $\mathscr{H}_0(N;\mathscr{O}_K)$ over Λ_K by using her result. As in § 1, we write $\mathscr{M}_k(\Gamma_1(N);\mathbf{F}_p)$ (resp. $\mathscr{S}_k(\Gamma_1(N);\mathbf{F}_p)$) for $\mathscr{M}_k(\Gamma_1(N);\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p$ (resp. $\mathscr{S}_k(\Gamma_1(N);\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p$). By regarding $\mathscr{M}_k(\Gamma_1(N_0);\mathbf{Z}_p)$ as a subspace of $R_k(\Gamma_1(N_0);\mathbf{Z}_p)$ through (1.5), one can let the finite group $(\mathbf{Z}/N_0\mathbf{Z})^\times$ act on $\mathscr{M}_k(\Gamma_1(N_0);\mathbf{Z}_p)$ by $(f\mid a)(E,\omega,i)=f(E,\omega,ai)$ for $a\in(\mathbf{Z}/N_0\mathbf{Z})^\times$. Then the action of Hecke operator $T_{N_0}(p)$ on $\mathscr{M}_k(\Gamma_1(N_0);\mathbf{Z}_p)$ is given by

$$\text{(4.1 a)}\qquad a(n, f\mid T_{N_0}(p))=a(np, f)+p^{k-1}a\left(\frac{n}{p}, f\,\middle|\, p\right),$$

where p acts on f as an element of $(\mathbf{Z}/N_0\mathbf{Z})^\times$. On the other hand, on $\mathscr{M}_k(\Gamma_1(N_0p);\mathbf{Z}_p)$, $T(p)$ acts as

$$\text{(4.1 b)}\qquad a(n, f\mid T(p))=a(np, f).$$

Thus, the actions (4.1 *a*, *b*) are generally different, but they coincide on $\mathscr{M}_k(\Gamma_1(N_0p);\mathbf{F}_p)$ if $k\geq 2$. In this manner, the universal Hecke algebra $\mathscr{H}(N_0p;\mathbf{Z}_p)$ acts on $\mathscr{M}_k(\Gamma_1(N_0);\mathbf{F}_p)$ for $k\geq 2$. The Eisenstein series $E_{p-1}\in\mathscr{M}_{p-1}(SL_2(\mathbf{Z});\mathbf{Z}_p)$ has congruence $E_{p-1}\equiv 1 \bmod p\mathbf{Z}_p$; therefore, the multiplication by E_{p-1} defines an isomorphism of $\mathscr{M}_k(\Gamma_1(N_0);\mathbf{F}_p)$ into $\mathscr{M}_{k+(p-1)}(\Gamma_1(N_0);\mathbf{F}_p)$, which is a morphism of $\mathscr{H}(N_0p;\mathbf{Z}_p)$-modules if $k\geq 2$. For each $a\in\mathbf{Z}/(p-1)\mathbf{Z}$, define an injective limit of these morphisms

$$G_a(\mathbf{F}_p)=\varinjlim_n \mathscr{M}_{j(a)+n(p-1)}(\Gamma_1(N_0);\mathbf{F}_p),$$

where $j(a)$ is an integer satisfying $3\leq j(a)\leq p+1$ and $j(a)\equiv a \bmod p-1$. By definition, we have, with the notation of Corollary 1.2,

$$\text{(4.2)}\qquad G(\Gamma_1(N_0);\mathbf{F}_p)/(E_{p-1}-1)\simeq\bigoplus_{a \bmod p-1} G_a(\mathbf{F}_p)$$

as $\mathscr{H}(N_0p;\mathbf{Z}_p)$-modules. The following fact is proved in [13, Lemma 1.9].

LEMMA 4.1 (Jochnowitz). — *If* $k \geqq 3$ *and* $p \geq 5$, $\mathrm{T}(p)$ *annihilates the quotients:* $\mathscr{M}_{k+(p-1)}(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)/\mathscr{M}_k(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$ and $\mathscr{S}_{k+(p-1)}(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)/\mathscr{S}_k(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$.

In [13, Lemma 1.9], this lemma is proved for the forms on $\Gamma_0(\mathrm{N}_0)$, but the proof given there works well without any change even for $\Gamma_1(\mathrm{N}_0)$.

For each Λ-module $\mathscr{M}$, put

$$\mathscr{M}[p^r]=\{x\in\mathscr{M}\mid p^r x=0\},\qquad \mathscr{M}^\Gamma=\{x\in\mathscr{M}\mid \gamma.x=x\quad\text{for all}\quad \gamma\in\Gamma\}.$$

Let $\mathscr{M}_0(\mathrm{N}\,;\mathbf{T}_p)$ (resp. $\mathscr{M}_k^0(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$, $\mathscr{S}_0(\mathrm{N}\,;\mathbf{T}_p)$ and $\mathscr{S}_k^0(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$) be the ordinary part of $\mathscr{M}(\mathrm{N}\,;\mathbf{T}_p)$ (resp. $\mathscr{M}_k(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$, $\mathrm{S}(\mathrm{N}\,;\mathbf{T}_p)$ and $\mathscr{S}_k(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)$) as $\mathscr{H}(\mathrm{N}_0 p\,;\mathbf{Z}_p)$-modules.

THEOREM 4.2. — *Assume that* $p\geq 5$. *Then we have q-expansion preserving isomorphisms of* $\mathscr{H}(\mathrm{N}_0 p\,;\mathbf{Z}_p)$*-modules:*

$$(\mathscr{M}_0(\mathrm{N}\,;\mathbf{T}_p)[p])^\Gamma\simeq\bigoplus_{a \bmod p-1}\mathscr{M}_{j(a)}^0(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p),$$
$$(\mathscr{S}_0(\mathrm{N}\,;\mathbf{T}_p)[p])^\Gamma\simeq\bigoplus_{a \bmod p-1}\mathscr{S}_{j(a)}^0(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p).$$

Proof. — By the multiplication by p, one has an isomorphism:

$$\mathscr{M}(\mathrm{N}\,;\mathbf{T}_p)[p]\simeq\mathscr{M}(\mathrm{N}\,;\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p.$$

By Corollary 1.2 and (4.2), we have that

$$\mathscr{M}(\mathrm{N}\,;\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p\cong \mathrm{W}(\mathrm{N}_0\,;\mathbf{F}_p)\cong \mathrm{G}(\Gamma_1(\mathrm{N}_0)\,;\mathbf{F}_p)/(\mathrm{E}_{p-1}-1)\cong\bigoplus_{a \bmod p-1}\mathrm{G}_a(\mathbf{F}_p).$$

Then, the assertion for modular forms is clear from Lemma 4.1 and the definition of the idempotent e in (1.17 *a*). The assertion for cusp forms can be proved similarly, since $\mathrm{T}(p)$ takes cusp forms into itself.

COROLLARY 4.2. — *If* $p\geqq 5$, *then* $\mathscr{H}_0(\mathrm{N}\,;\mathscr{O}_\mathrm{K})$ *and* $h_0(\mathrm{N}\,;\mathscr{O}_\mathrm{K})$ *are finite over* Λ_K. *Furthermore, we see that*

$$\mathscr{H}_0(\mathrm{N}\,;\mathscr{O}_\mathrm{K})=\mathscr{H}_0(\mathrm{N}\,;\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathscr{O}_\mathrm{K}\qquad\textit{and}\qquad h_0(\mathrm{N}\,;\mathscr{O}_\mathrm{K})=h_0(\mathrm{N}\,;\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathscr{O}_\mathrm{K}.$$

Proof. — Since $\mathscr{H}_0(\mathrm{N}\,;\mathscr{O}_\mathrm{K})=\mathscr{H}_0(\mathrm{N}\,;\mathbf{Z}_p)\hat{\otimes}_{\mathbf{Z}_p}\mathscr{O}_\mathrm{K}$ (*cf.* (1.16)), the second assertion follows from the first one and we may assume $\mathscr{O}_\mathrm{K}=\mathbf{Z}_p$ in order to prove the first assertion. Let $\mathfrak{m}$ be the maximal ideal of Λ. We have an exact sequence of Λ-modules:

$$0\to\mathbf{T}_p\to\mathscr{M}(\mathrm{N}\,;\mathbf{T}_p)\to m(\mathrm{N}\,;\mathbf{T}_p)\to 0.$$

Since on $\mathbf{T}_p$, $\mathrm{T}(p)$ acts trivially (*cf.* (1.13 *a*)), by applying the idempotent e to this sequence, we have another exact sequence:

$$0\to\mathbf{T}_p\to\mathscr{M}_0(\mathrm{N}\,;\mathbf{T}_p)\to m_0(\mathrm{N}\,;\mathbf{T}_p)\to 0.$$

Let M be the Pontrjagin dual module of $\mathscr{M}_0(\mathrm{N}\,;\mathbf{T}_p)$. Then by Theorem 2.2, we have an exact sequence of Λ-modules:

$$0\to\mathscr{H}_0(\mathrm{N}\,;\mathbf{Z}_p)\to\mathrm{M}\to\mathbf{Z}_p\to 0,$$

where Γ acts trivially on $\mathbf{Z}_p$. By definition, $\mathrm{M}/\mathfrak{m}\mathrm{M}$ is a dual space of $(\mathcal{M}(\mathrm{N};\mathbf{T}_p)[p])^\Gamma$, which has only a finitely many elements by Theorem 4.2. Since M is a continuous module over a compact ring Λ with the $\mathfrak{m}$-adic topology, M is finite over Λ. In fact, as seen in § 3, if we put

$$\mathcal{M}_0(\mathrm{N};\mathbf{T}_p)[\mathfrak{m}^j]=\{x\in\mathcal{M}_0(\mathrm{N};\mathbf{T}_p)\mid\lambda x=0\quad\text{for all}\quad\lambda\in\mathfrak{m}^j\},$$

then $\mathcal{M}_0(\mathrm{N};\mathbf{T}_p)=\bigcup_j\mathcal{M}_0(\mathrm{N};\mathbf{T}_p)[\mathfrak{m}^j]$. By duality, this shows

$$\mathrm{M}=\varprojlim_j \mathrm{M}/\mathfrak{m}^j\mathrm{M}.$$

Then, by [1, III.2.11, Prop. 14], we know that M is finite over Λ and any subset S of M whose image in $\mathrm{M}/\mathfrak{m}\mathrm{M}$ generates $\mathrm{M}/\mathfrak{m}\mathrm{M}$ generates M over Λ. Since Λ is noetherian, $\mathcal{H}_0(\mathrm{N};\mathbf{Z}_p)$ must be finite over Λ. Since $h_0(\mathrm{N};\mathbf{Z}_p)$ is a quotient ring of $\mathcal{H}_0(\mathrm{N};\mathbf{Z}_p)$, the finiteness of $h_0(\mathrm{N};\mathbf{Z}_p)$ follows from that of $\mathcal{H}_0(\mathrm{N};\mathbf{Z}_p)$.

§ 5. Proof of Theorem 3.1 and its corollaries

Let the notation be as in the theorem. We only prove the theorem for $\mathcal{O}_\mathrm{K}=\mathbf{Z}_p$, since the general case follows from this by Corollary 4.2. As seen in § 3, the group $\mathbf{Z}_p^\times=\mu\times\Gamma$ naturally acts on the spaces $\mathcal{M}_0(\mathrm{N}_0;\mathbf{T}_p)$ and $\mathcal{S}_0(\mathrm{N}_0;\mathbf{T}_p)$. Thus we can decompose them into the sum of eigenspaces of μ:

$$\mathcal{M}_0(\mathrm{N}_0;\mathbf{T}_p)=\bigoplus_{a=0}^{p-2}\mathcal{M}_0(\mathrm{N}_0,a;\mathbf{T}_p)\qquad\text{and}\qquad\mathcal{S}_0(\mathrm{N}_0;\mathbf{T}_p)=\bigoplus_{a=0}^{p-2}\mathcal{S}_0(\mathrm{N}_0,a;\mathbf{T}_p),$$

where $\mathcal{M}_0(\mathrm{N}_0,a;\mathbf{T}_p)=\{f\in\mathcal{M}_0(\mathrm{N}_0;\mathbf{T}_p)\mid f\mid\zeta=\zeta^a f\text{ for }\zeta\in\mu\}$ and

$$\mathcal{S}_0(\mathrm{N}_0,a;\mathbf{T}_p)=\mathcal{M}_0(\mathrm{N}_0,a;\mathbf{T}_p)\cap\mathcal{S}_0(\mathrm{N}_0;\mathbf{T}_p).$$

For any $\mathbf{Z}_p$-module M, put $\mathrm{M}[p^r]=\{x\in\mathrm{M}\mid p^r x=0\}$, and for simplicity, we write $\mathcal{M}$, $\mathcal{S}$, $\mathcal{M}(a)$ and $\mathcal{S}(a)$ for $\mathcal{M}_0(\mathrm{N}_0;\mathbf{T}_p)$, $\mathcal{S}_0(\mathrm{N}_0;\mathbf{T}_p)$, $\mathcal{M}_0(\mathrm{N}_0,a;\mathbf{T}_p)$ and $\mathcal{S}_0(\mathrm{N}_0,a;\mathbf{T}_p)$, respectively.

First we treat the case of $\mathcal{H}_0(\mathrm{N};\mathbf{Z}_p)$. By Theorem 4.2, we know, as μ-modules,

$$(\mathcal{M}[p])^\Gamma\cong\bigoplus_{k=3}^{p+1}\mathcal{M}_k^0(\Gamma_1(\mathrm{N}_0);\mathbf{F}_p).$$

By decomposing this through the action of μ, we have by (3.1)

$$(5.1)\qquad(\mathcal{M}(a)[p])^\Gamma\cong\mathcal{M}_{j(a)}^0(\Gamma_1(\mathrm{N}_0);\mathbf{F}_p)$$

where $j(a)\in\mathbf{Z}$ with $3\leqq j(a)\leqq p+1$ and $j(a)\equiv a\bmod(p-1)$. Let us put

$$\mathcal{M}_k=\{f\in\mathcal{M}\mid f\mid z=z^k f\qquad\text{for any}\quad z\in\mathbf{Z}_p^\times\}.$$

Then, we know that if $k\equiv a\bmod(p-1)$,

$$(5.2)\qquad\mathcal{M}_k[p]\hookrightarrow(\mathcal{M}(a)[p])^\Gamma=\mathcal{M}_{j(a)}^0(\Gamma_1(\mathrm{N}_0);\mathbf{F}_p).$$

Naturally, $\mathscr{M}'_k = \mathscr{M}^0_k(\Gamma_1(N_0); \mathbf{Q}_p)/\mathscr{M}^0_k(\Gamma_1(N_0); \mathbf{Z}_p)$ is injected into $\mathscr{M}_k$. By Lemma 4.1, if $k \geq 3$ and $k \equiv a \bmod (p-1)$, we have

$$\mathscr{M}'_k[p] \simeq \mathscr{M}^0_k(\Gamma_1(N_0); \mathbf{F}_p) \simeq \mathscr{M}^0_{j(a)}(\Gamma_1(N_0); \mathbf{F}_p).$$

By counting the number of elements in $\mathscr{M}_k[p]$ and $\mathscr{M}'_k[p]$, we conclude $\mathscr{M}'_k[p] = \mathscr{M}_k[p]$. As already seen, $\mathscr{M}'_k$ is contained in $\mathscr{M}_k$ and is p-divisible. Therefore, $\mathscr{M}'_k$ must coincide with $\mathscr{M}_k$. Namely, we have

$$(5.3) \qquad \mathscr{M}_k = \mathscr{M}^0_k(\Gamma_1(N_0); \mathbf{Q}_p)/\mathscr{M}^0_k(\Gamma_1(N_0); \mathbf{Z}_p) \qquad \text{if} \quad k \geqq 3,$$

and for any $k \geqq 3$ with $k \equiv a \bmod (p-1)$,

$$(5.4) \qquad \mathscr{M}_k \cong \mathbf{T}^r_p \qquad \text{for} \quad r = r(a) = \operatorname{rank}_{\mathbf{Z}_p} \mathscr{M}^0_{j(a)}(\Gamma_1(N_0); \mathbf{Z}_p).$$

Now fix $0 \leqq a < p-1$ and let $M(a)$ be the Pontrjagin dual space of $\mathscr{M}(a)$. Note that for any positive integer k with $k \equiv a \bmod (p-1)$, $\mathscr{M}_k$ is contained in $\mathscr{M}(a)$. Thus, by (5.4), there is an exact sequence of Λ-modules:

$$(5.5) \qquad 0 \to \mathrm{Ker} \to \Lambda^r \to M(a) \to 0.$$

Let us put $P_k(X) = (X+1) - (1+p)^k \in \Lambda$ for any $k \in \mathbf{Z}$. Then, for any $k \geqq 3$ with $k \equiv a \bmod (p-1)$, the exact sequence (5.5) induces a surjection: $\Lambda^r/P_k\Lambda^r \to M(a)/P_kM(a)$. Since $M(a)/P_kM(a)$ is the dual space of $\mathscr{M}_k$, $M(a)/P_kM(a)$ is isomorphic to $\mathbf{Z}^r_p$ by (5.4). On the other hand, we know $\Lambda/P_k\Lambda \cong \mathbf{Z}_p$ as $\mathbf{Z}_p$-modules; hence, we have a commutative diagram:

$$\begin{array}{ccc} \mathbf{Z}^r_p & \longrightarrow & \mathbf{Z}^r_p \\ \wr\uparrow & & \wr\uparrow \\ (\Lambda/P_k\Lambda)^r & \to & M(a)/P_kM(a) \to 0. \end{array}$$

This shows that $(\Lambda/P_k\Lambda)^r = \Lambda^r/P_k\Lambda^r \cong M(a)/P_kM(a)$. Thus we know that Ker is contained in $P_k\Lambda^r$ for all $k \geqq 3$ with $k \equiv a \bmod (p-1)$. This shows that $\mathrm{Ker} \subset \bigcap_k P_k\Lambda^r = \{0\}$, since $\{P_k\}_{k \geqq 3, k \equiv a \bmod (p-1)}$ are mutually different prime elements in Λ. This implies that $M(a)$ is free of rank $r = r(a)$. Since $\mathbf{Z}^\times_p$ acts on $\mathscr{M}_0(\Gamma_1(N); \mathbf{Z}_p) (= \mathbf{Z}_p)$ trivially, we see $\mathscr{M}(a) = m_0(N_0, a; \mathbf{T}_p) = \{ f \in m_0(N; \mathbf{T}_p) \mid f \mid \zeta = \zeta^a f \text{ for all } \zeta \in \mu \}$ if $a \neq 0$. Thus, by the duality as in Corollary 4.2, we see that

$$\mathscr{H}_0(N, a; \mathbf{Z}_p) \cong M(a) \cong \Lambda^{r(a)} \qquad \text{if} \quad a \neq 0.$$

Thus, the assertion for $\mathscr{H}_0(N, a; \mathbf{Z}_p)$ has already been proved for $a \neq 0$. Now assume $a = 0$. Then we have an exact sequence of discrete Λ-modules:

$$0 \to \mathbf{T}_p \to \mathscr{M}(0) \to m(0) \to 0,$$

where $m(0) = m_0(N_0, 0; \mathbf{T}_p)$. By duality, we obtain another exact sequence of compact Λ-modules:

$$(5.6) \qquad 0 \to \mathscr{H}(0) \to M(0) \to \mathbf{Z}_p \to 0,$$

where $\mathscr{H}(0) = \mathscr{H}_0(N, 0; \mathbf{Z}_p)$ and Γ acts trivially on $\mathbf{Z}_p$ $(\cong \Lambda/X\Lambda)$. We have already seen that $M(0) \cong \Lambda^r$ $(r = r(0))$, and (5.6) induces a surjection $\Phi : M(0)/XM(0)$ $(\cong \mathbf{Z}^r_p) \to \mathbf{Z}_p$.

By choosing a suitable basis of M(0)/XM(0) ($\cong \mathbf{Z}_p^r$), we know that this morphism can be realized as

$$\begin{array}{ccc} \mathbf{Z}_p^r & \xrightarrow{\Phi} & \mathbf{Z}_p \\ \Psi & & \Psi \\ {}^t(x_1, \ldots, x_r) & \mapsto & x_1 . \end{array}$$

Write this choice of basis as $\bar{e}_1, \ldots, \bar{e}_r \in \mathrm{M}(0)/\mathrm{XM}(0)$. Then we see $\Phi(\bar{e}_1)=1$ and $\Phi(\bar{e}_i)=0$ for $i>1$. Take a free basis $e_1, \ldots, e_r$ of M(0) so that $e_i \equiv \bar{e}_i \bmod \mathrm{XM}(0)$. In fact, if $e_1, \ldots, e_r$ satisfy the above congruence, $\Lambda e_1 + \ldots + \Lambda e_r$ is dense and complete in M(0); hence, coincides with M(0). Thus $\{ e_1, \ldots, e_r \}$ must be a free basis of M(0).

Then, Φ can be explicitly given by the commutative diagram:

$$\begin{array}{ccc} \mathrm{M}(0) & \xrightarrow{\Phi} & \mathbf{Z}_p \\ \wr\!\uparrow & & \wr\!\uparrow \\ \Lambda^r & \longrightarrow & \mathbf{Z}_p \\ \Psi & & \Psi \\ {}^t(x_1, \ldots, x_r) & \mapsto & x_1 \bmod \mathrm{XM}(0) . \end{array}$$

This shows that $\mathscr{H}_0(\mathrm{N}, 0; \mathbf{Z}_p) = \mathrm{Ker}(\Phi) \cong \Lambda^{r-1} \oplus \mathrm{X}\Lambda \cong \Lambda^r$; hence, the assertion for $\mathscr{H}_0(\mathrm{N}, a; \mathbf{Z}_p)$ has been proved.

The assertion for cusp forms can be proved in exactly the same fashion as in the case of modular forms.

Proof of Corollary 3.2. — By the reason already mensioned, we may assume that $\mathscr{O}_{\mathrm{K}} = \mathbf{Z}_p$. For any $0<k\in\mathbf{Z}$, put

$$\mathscr{M}_k(a) = \{ f \in \mathscr{M}(a) \mid f \mid z = z^k f \quad \text{for any} \quad z \in \Gamma \} .$$

Then, Theorem 3.1 asserts that

$$(5.7\,a) \qquad \mathscr{M}_k(a) \cong \mathbf{T}_p^r \qquad \text{for} \quad r = r(a) \quad \text{as in (5.4)} .$$

Thus, if $k \equiv a \bmod (p-1)$ and $k \geqq j(a)$, $\mathscr{M}_k(a)$ coincides with $\mathscr{M}_k$. (This fact is essentially proved in the proof of Theorem 3.1 and in fact, independent of Theorem 3.1, since $\mathscr{M}_k$ is by definition the same as $\mathscr{M}_k(a)$ when $k \equiv a \bmod (p-1)$). Thus we may assume that $k \not\equiv a \bmod (p-1)$.

On the other hand, by the definition of the action of Γ as in (3.1), we know that

$$\mathscr{M}_k'(a) = \mathscr{M}_k^0(\Gamma_1(\mathrm{N}_0 p), a ; \mathbf{Q}_p) / \mathscr{M}_k^0(\Gamma_1(\mathrm{N}_0 p), a ; \mathbf{Z}_p)$$

is contained in $\mathscr{M}_k(a)$. The surjectivity of the natural morphism of $\mathscr{H}_0(\mathrm{N}, a; \mathbf{Z}_p)$ into $\mathscr{H}_k^0(\Gamma_1(\mathrm{N}_0 p), a; \mathbf{Z}_p)$ is obvious from the definition of these algebras. Note that

$$\mathscr{M}_k(a) = \{ f \in \mathscr{M}(a) \mid f \mid \mathrm{P}_k = 0 \} .$$

This shows that if we prove $\mathscr{M}_k'(a) = \mathscr{M}_k(a)$, by the duality in Corollary 2.3, we know that

$$\mathscr{H}_k^0(\Gamma_1(\mathrm{N}_0 p), a ; \mathbf{Z}_p) = \mathscr{H}_0(\mathrm{N}, a ; \mathbf{Z}_p) / \mathrm{P}_k \mathscr{H}_0(\mathrm{N}, a ; \mathbf{Z}_p) .$$

To see $\mathscr{M}'_k(a)=\mathscr{M}_k(a)$, it is sufficient to know that

$$\operatorname{rank}(\mathscr{M}'_k(a))\geqq \operatorname{rank}(\mathscr{M}_k(a)),$$

since both modules are p-divisible. By (5.7 a), this follows from the inequality:

$$\dim_{\mathbf{F}_p}(\mathscr{M}'_k(a)[p])\geqq r(a). \tag{5.7 b}$$

Now we are going to prove (5.7 b). We may assume $k\not\equiv a \bmod (p-1)$ as already mentioned and further assume $k>j(a)$ as in the corollary. For any $0<\lambda\in\mathbf{Z}$ with $\lambda\not\equiv 0 \bmod (p-1)$, put

$$G'_\lambda=-\frac{B_{\lambda,\omega^{-\lambda}}}{2\lambda}+\sum_{n=1}^{\infty}\Big(\sum_{0<d\mid n} d^{\lambda-1}\omega^{-\lambda}(d)\Big)q^n,$$

where ω is a character of $\mathbf{Z}_p^\times$ defined by $\omega(x)=\lim_{t\to\infty} x^{p^t}$ and $B_{\lambda,\omega^{-\lambda}}$ is the generalized Bernoulli number for the character $\omega^{-\lambda}$. It is known by Hecke [6] (see also [10, (6.5 b)]) that G'_λ belongs to $\mathscr{M}_\lambda(\Gamma_0(p),\omega^{-\lambda};\mathbf{Q}_p)$. Let us further put

$$E'_\lambda=-\frac{2\lambda}{B_{\lambda,\omega^{-\lambda}}}G'_\lambda.$$

Then, by Carlitz [2, Th. 3], we know that

$$E'_\lambda\equiv 1 \bmod p\mathbf{Z}_p.$$

Thus, the multiplication of $E'_{k-j(a)}$ for $k>j(a)$ and $k\not\equiv j(a) \bmod (p-1)$ defines an isomorphism of $\mathscr{M}_{j(a)}[p]$ $(=\mathscr{M}'_{j(a)}(a)[p])$ into $\mathscr{M}'_k(a)[p]$ (*cf.* the definition of $\mathscr{M}_k(\Gamma_1(N_0p),a;\mathbf{Z}_p)$ in § 3). This shows

$$r(a)=\dim_{\mathbf{F}_p}(\mathscr{M}_{j(a)}[p])\leqq \dim_{\mathbf{F}_p}(\mathscr{M}'_k(a)[p]).$$

Thus the assertion for $\mathscr{H}_0(N,a;\mathbf{Z}_p)$ has been proved.

Now let us discuss the case of cusp forms: What we have to show is

$$\dim_{\mathbf{F}_p}(\mathscr{S}'_k(a)[p])\geq \dim_{\mathbf{F}_p}(\mathscr{S}^0_{j(a)}(\Gamma_1(N_0);\mathbf{F}_p))\qquad \text{for}\quad k\geq 2, \tag{5.8}$$

where $\mathscr{S}'_k(a)=\mathscr{S}^0_k(\Gamma_1(N_0p),a;\mathbf{Q}_p)/\mathscr{S}^0_k(\Gamma_1(N_0p),a;\mathbf{Z}_p)$.

Write simply $s(a)$ for the right-hand side of (5.8). By [8, Prop. 6.2] (see also, Ribet [24]), we know that

$$\dim_{\mathbf{F}_p}(\mathscr{S}'_2(a)[p])=s(a).$$

Then for any $k\geqq 2$, the multiplication of E'_{k-2} (or E^n_{p-1} for $n=(k-2)/(p-1)$ if $k\equiv 2 \bmod (p-1)$) induces an inclusion of $\mathscr{S}'_2(a)[p]$ into $\mathscr{S}'_k(a)[p]$. This proves (5.8), which was to be shown.

Proof of Corollary 3.3. — Let us first examine the intersection of $\mathscr{P}(N;\mathcal{O}_K)$ with the nilradical $\mathscr{N}$ of $\mathscr{H}_0(N;\mathcal{O}_K)$. We may assume that $K=\mathbf{Q}_p$. Let us put

$$\mathscr{P}'=\{h\in\mathscr{H}_0(N;\mathbf{Z}_p)\mid (h,f)_{\mathbf{T}_p}=0\quad\text{for all}\quad f\in\mathcal{O}_0(N;\mathbf{T}_p)\}.$$

By definition, $\mathcal{O}_0(N;\mathbf{T}_p)$ is stable under $\mathscr{H}=\mathscr{H}_0(N;\mathbf{Z}_p)$. Thus, for any $h\in\mathscr{P}'$ and $h'\in\mathscr{H}$, we see

$$(hh',f)_{\mathbf{T}_p}=(h',f\mid h)_{\mathbf{T}_p}=(h,f\mid h')_{\mathbf{T}_p}=0\quad\text{for any}\quad f\in\mathcal{O}_0(N;\mathbf{T}_p).$$

This shows that $\mathscr{P}'$ is an ideal and $\mathscr{P}' \subset \mathscr{P} = \mathscr{P}(\mathrm{N}; \mathbf{Z}_p)$. The converse inclusion $\mathscr{P}' \supset \mathscr{P}$ is obvious, and therefore $\mathscr{P} = \mathscr{P}'$. Let ϕ_j be the projection map of $\mathscr{H}$ onto ${}_e\mathscr{H}_2^j(\mathrm{N}_0 p; \mathbf{Z}_p)$ as in (2.3). Then $\phi_j(\mathscr{P})$ can be characterized by

$$\phi_j(\mathscr{P}) = \{ h \in {}_e\mathscr{H}_2^j(\mathrm{N}_0 p; \mathbf{Z}_p) \mid (h, f)_{\mathbf{T}_p} = 0 \quad \text{for all} \quad f \in \mathscr{O}_0(\mathrm{N}_0 p; \mathbf{T}_p) \cap \mathscr{M}_2^j(\mathrm{N}_0 p; \mathbf{T}_p) \}.$$

Then, in a similar manner as above, we know

$$\phi_j(\mathscr{P}) \quad \textit{annihilates} \quad \mathscr{O}_0(\mathrm{N}_0 p; \mathbf{T}_p) \cap {}_e\mathscr{M}_2^j(\mathrm{N}_0 p; \mathbf{T}_p).$$

On the other hand, we know

$$\mathscr{O}(\mathrm{N}_0 p; \mathbf{T}_p) \cap \mathscr{M}_2^j(\mathrm{N}_0 p; \mathbf{T}_p) \supset \sum_{1 < t \mid \mathrm{N}_0} \mathscr{M}_2^j(\mathrm{N}_0 p/t; \mathbf{T}_p) \mid [t] + \sum_{1 < t \mid \mathrm{N}_0} \mathscr{M}_2^j(\mathrm{N}/t; \mathbf{T}_p).$$

Thus

$$\phi_j(\mathscr{P}) \quad \textit{annihilates old forms in} \quad \mathscr{M}_2^j(\mathrm{N}_0 p; \mathbf{T}_p).$$

Then the theory of primitive forms ([23]) combined with [10, Prop. 4.4] shows that $\phi_j(\mathscr{P}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a semi-simple subalgebra of $\mathscr{H}_2^j(\mathrm{N}_0 p; \mathbf{Q}_p)$. If $h \in \mathscr{P}$ is nilpotent, $\phi_j(h)$ is nilpotent for all j, and hence, $\phi_j(h) = 0$ for all j. This shows $h = 0$ by (2.3). The assertion for $\not{p}(\mathrm{N}; \mathscr{O}_\mathrm{K})$ can be proved in exactly the same manner as above.

Next, we are going to prove that $\mathscr{P}$ and $\not{p}$ are free over Λ. Let us put $\mathrm{W} = \mathscr{P} \otimes_\Lambda \mathscr{L}_\mathrm{K}$. Then, for any $w \in \mathrm{W} \cap \mathscr{H}$, we can find $h \in \Lambda$ $(h \neq 0)$ so that $hw \in \mathscr{P}$. Then we see that

$$w\mathscr{O}_0(\mathrm{N}; \mathbf{T}_p) = wh\mathscr{O}_0(\mathrm{N}; \mathbf{T}_p) = 0,$$

since $\mathscr{O}_0(\mathrm{N}; \mathbf{T}_p)$ is Λ-divisible by its definition and Theorem 3.1. This shows that $\mathscr{P} = \mathrm{W} \cap \mathscr{H}$. Since $\mathscr{H}$ is free, $\mathscr{P}$ must be reflexive ([1, VII.4, Prop. 6]). By a result of Serre [25, Lemme 6], any reflexive Λ-module is free; hence, $\mathscr{P}$ is Λ-free. The assertion for $\not{p}$ can be proved in exactly the same manner as for $\mathscr{P}$.

§ 6. Proof of Theorem 3.6 and its corollaries

Proof of Theorem 3.6. — We only prove the result for $\mathscr{H}_0(\mathrm{N}; \mathscr{O}_\mathrm{K})$, because the other case concerning $h_0(\mathrm{N}; \mathscr{O}_\mathrm{K})$ can be verified in exactly the same manner. By definition, we know $\mathscr{Q} = \mathscr{Q}(\mathrm{N}; \mathrm{K}) = \mathscr{H} \otimes_{\Lambda_\mathrm{K}} \mathscr{L}_\mathrm{K}$ for $\mathscr{H} = \mathscr{H}_0(\mathrm{N}; \mathscr{O}_\mathrm{K})$, and by Theorem 3.1, $\mathscr{H}$ is a lattice over Λ_K of the $\mathscr{L}_\mathrm{K}$-vector space $\mathscr{Q}$ in the sense of [1, VII.4.1]. This shows that $\mathscr{C}(\mathscr{K}; \mathrm{K}) = \overline{\mathscr{H}}/\mathscr{H}$ is a torsion Λ_K-module with the notation of (3.9 *a*), since $\mathscr{Q}/\mathscr{H}$ is a torsion Λ_K-module. Since $\mathscr{H}$ is free of finite rank over Λ_K, its images $\mathscr{H}(\mathscr{K})$ and $\mathscr{H}(\mathscr{A})$ are Λ_K-lattices. This shows that $\overline{\mathscr{H}} = \tilde{\mathscr{H}}(\mathscr{K}) \oplus \tilde{\mathscr{H}}(\mathscr{A})$ is a Λ_K-free lattice of $\mathscr{Q}$; therefore, $\mathscr{C}(\mathscr{K}; \mathrm{K}) = \overline{\mathscr{H}}/\mathscr{H}$ is a finite torsion Λ_K-module. This shows (3.10 *a*).

The first part of (3.10 *b*) is a restatement of Theorem 3.5. Now we prove the rest of (3.10 *b*). For a finite extension M of K, define $\overline{\mathscr{H}}_\mathrm{M}$ and $\mathscr{H}_\mathrm{M}(\mathscr{K}_\mathrm{M})$ and $\mathscr{H}_\mathrm{M}(\mathscr{A}_\mathrm{M})$ for $\mathscr{K}_\mathrm{M} = \mathscr{K} \otimes_\mathrm{K} \mathrm{M} \subset \mathscr{Q}(\mathrm{N}; \mathrm{M})$ and $\mathscr{A}_\mathrm{M} = \mathscr{A} \otimes_\mathrm{K} \mathrm{M}$ in the same fashion as for $\overline{\mathscr{H}}$, $\mathscr{H}(\mathscr{K})$ and $\mathscr{H}(\mathscr{A})$. Then we see easily by Corollary 4.2, $\overline{\mathscr{H}}_\mathrm{M} = \overline{\mathscr{H}} \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}$ and $\mathscr{H}_\mathrm{M} = \mathscr{H} \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}$. In fact, $\overline{\mathscr{H}} \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M} \cong \overline{\mathscr{H}} \otimes_{\Lambda_\mathrm{K}} \Lambda_\mathrm{M}$ is Λ_M-free and $(\overline{\mathscr{H}} \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M})/(\mathscr{H}' \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}) = \mathscr{N}(\mathscr{K}; \mathrm{K}) \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}$ is pseudo-null for $\mathscr{H}' = \mathscr{H}(\mathscr{K}) \oplus \mathscr{H}(\mathscr{A})$. Such a Λ_K-free module containing $\mathscr{H}' \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}$ is unique and coincides with the free closure of $\mathscr{H}' \otimes_{\mathscr{O}_\mathrm{K}} \mathscr{O}_\mathrm{M}$ (*cf.* [1, VII.4, Th. 2]). By definition, we know that $\mathcal{H}'\otimes_{\mathcal{O}_K}\mathcal{O}_M=\mathcal{H}_M(\mathcal{K}_M)\oplus\mathcal{H}_M(\mathcal{A}_M)$. This shows $\overline{\mathcal{H}}_M=\overline{\mathcal{H}}\otimes_{\mathcal{O}_K}\mathcal{O}_M$. Thus we know $\mathcal{C}(\mathcal{K}_M;M)=\mathcal{C}(\mathcal{K};K)\otimes_{\mathcal{O}_K}\mathcal{O}_M$. Note that $\Lambda_M=\Lambda_K\otimes_{\mathcal{O}_K}\mathcal{O}_M$. This shows that $\mathcal{C}(\mathcal{K}_M;M)=\mathcal{C}(\mathcal{K};K)\otimes_{\Lambda_K}\Lambda_M$, which proves (3.10 *b*). (3.10 *c*) is obvious from the definition of $\mathcal{C}(\mathcal{K};K)$.

To prove (3.10 *d*), we recall the divisor theory of integrally closed noetherian domains [1, VII]. First, note that Λ_K is a unique factorization domain; hence, is integrally closed noetherian [1, VII.4.9, Prop. 8]. For any ideal $\mathfrak{a}$ of Λ_K, put $\operatorname{div}(\mathfrak{a})=\bigcap_x x\Lambda_K$, where x runs over all elements of Λ_K such that $x\Lambda_K\supset\mathfrak{a}$. Then, $\operatorname{div}(\mathfrak{a})$ is again an ideal. When $\mathfrak{a}=\operatorname{div}(\mathfrak{a})$, we call $\mathfrak{a}$ a divisor of Λ_K. Any divisor can be expressed as a product of powers of prime divisors. Prime divisors coincide with prime ideals of height one of Λ_K. Since Λ_K is a unique factorization domain, any divisor of Λ_K is principal [1, VII.3.2, Th. 1]. Let $\mathfrak{a}$ be the annihilator of $\mathcal{C}(\mathcal{K};K)$ in Λ_K. We want to prove that $\mathfrak{a}$ is a divisor. By Theorem 3.1, $\mathcal{H}=\mathcal{H}_0(N;\mathcal{O}_K)$ is a Λ_K-lattice in the finite dimensional $\mathcal{L}_K$-vector space $\mathcal{Q}$ in the sense of [1, VII.4.1]. Since $\mathcal{H}$ is Λ_K-free, $\mathcal{H}$ is reflexive. By the definition of $\operatorname{div}(\mathfrak{a})$, $\operatorname{div}(\mathfrak{a})/\mathfrak{a}$ is pseudo-null in the sense of [1, VII.4.4, Def. 2]. Thus $\operatorname{div}(\mathfrak{a})\mathcal{C}(\mathcal{K};K)$ is also pseudo-null. For simplicity, we write X for $\operatorname{div}(\mathfrak{a})\mathcal{C}(\mathcal{K};K)$. We know $X=(\operatorname{div}(\mathfrak{a})\overline{\mathcal{H}}+\mathcal{H})/\mathcal{H}\subset\mathcal{Q}/\mathcal{H}$. By [1, VII.4.2, Th. 2], the associated primes of $\mathcal{Q}/\mathcal{H}$ are prime divisors since $\mathcal{H}$ is reflexive; hence, the associated primes of X are prime divisors if they exist ([1, IV.1.1, Prop. 3]). Since X is pseudo-null, the set of associated primes must be empty and hence $X=0$ (*cf.* [1, IV.1.1, Cor. 1 to Prop. 2]). Therefore, $\operatorname{div}(\mathfrak{a})$ annihilates $\mathcal{C}(\mathcal{K};K)$; hence, $\mathfrak{a}=\operatorname{div}(\mathfrak{a})$ because $\operatorname{div}(\mathfrak{a})\supset\mathfrak{a}$. This shows (3.10 *d*). By our definition of $\mathcal{N}(\mathcal{K};K)$, (3.10 *e*) follows from [1, VII.4.2, Th. 1 and 4.4, Ex. 3].

Proof of Corollary 3.7. — Let $\mathcal{K}$ be a primitive cuspidal component of $h=h_0(N;\mathcal{O}_K)$. Put $L=\mathcal{K}\cap h$. Then, by [1, VII.4, Prop. 6], L is reflexive, and hence, is free ([24, Lemme 6]). Thus, for $P=P_k$, L/PL is injected into L_P/PL_P for the localization L_P of L at P. We have an exact sequence:

$$0\to L\to h\to h/L\to 0.$$

Naturally, h/L is Λ-torsion free. By localizing at P, we have another exact sequence:

$$0\to L_P\to h_P\to h_P/L_P\to 0.$$

Since the local ring Λ_P of Λ at P is a discrete valuation ring, $h_P/L_P=(h/L)_P$ is Λ_P-free. Note that by Corollary 3.2,

$$h_P/Ph_P=h_k^0(\Gamma_1(N_0p);K)$$

and as in the proof of Corollary 3.3, L_P/PL_P is an ideal of h_P/Ph_P which annihilates old forms. Since h_P/Ph_P is an artinian ring over K, L_P/PL_P must be a semi-simple ring. This shows that L_P itself is a subring of $\mathcal{K}$ and L_P must coincides with the localization $\mathcal{H}(\mathcal{K})_P$ of $\mathcal{H}(\mathcal{K})$ at P, since both the rings over Λ_P have the same residue ring modulo P. Thus, the semi-simplicity of L_P/PL_P shows that $\mathcal{H}(\mathcal{K})_P$ is unramified over Λ_P. Since $\tilde{\mathcal{H}}(\mathcal{K})$ is Λ_K-free, $\tilde{\mathcal{H}}_k(\mathcal{K})=\tilde{\mathcal{H}}(\mathcal{K})/P\tilde{\mathcal{H}}(\mathcal{K})$ is $\mathcal{O}_K$-free. Since $\tilde{\mathcal{H}}(\mathcal{K})_P=\mathcal{H}(\mathcal{K})_P$ is unramified at P, L/PL is a $\mathcal{O}_K$-submodule of $\tilde{\mathcal{H}}_k(\mathcal{K})$ with the same $\mathcal{O}_K$-rank. This shows that $\tilde{\mathcal{H}}_k(\mathcal{K})$ is embedded into L_P/PL_P. We have already seen $F_k=\tilde{\mathcal{H}}_k(\mathcal{K})\otimes_{\mathcal{O}_K}K=L_P/PL_P$

is a ring direct summund of $h_k^0(\Gamma_1(N_0 p); K)$, and thus, $\tilde{\mathscr{H}}_k(\mathscr{K})$ is isomorphically embedded into $h_k^0(\Gamma_1(N_0 p); K)$.

Since F_k annihilates old forms and is a semi-simple direct summand of $h_k^0(\Gamma_1(N_0 p); K)$, there are d ordinary forms (for $d=[F_k : K]=[\mathscr{K} : \mathscr{L}_K]$) with the properties required by the corollary (*cf.* [10, Prop. 4.4]). Now we shall prove the converse. Let f be an ordinary form in $\mathscr{S}_k(\Gamma_1(N_0 p))$ and assume that the primitive form associated with f has conductor divisible by N_0. What we have to show is that the local ring to which f belongs is primitive. Our method is rather indirect; indeed, we shall show that the number of ordinary forms belonging to primitive components coincides with the number of ordinary forms with this property. The forms belonging to primitive components, as already shown, automatically satisfy this property concerning the divisibility of conductor by N_0, and hence, the assertion of the corollary follows. To see this, for each pair of divisors t, s of N_0 with $ts \mid N_0$, define an injection $[t]_s$ of $\mathscr{S}_0(sp; K/\mathscr{O}_K)$ into $\mathscr{S}_0(N_0 p; K/\mathscr{O}_K)$ by

$$f \mid [t]_s = \sum_{n=0}^{\infty} a(n, f) q^{nt}.$$

By duality, this morphism induces a surjective Λ_K-linear map of $h_0(N_0 p; \mathscr{O}_K)$ onto $h_0(sp; \mathscr{O}_K)$, which will be denoted by the same symbol $[t]_s$. Let $\phi_{t,s}$ be the combination of $[t]_s$ with the projection of $\mathscr{q}(sp; K)$ onto $\mathscr{p}(sp; K)$. Then, we can define a morphism ϕ of $\mathscr{q}(N_0 p; K)$ into $\bigoplus_{t,s} \mathscr{p}(sp; K)$ by $\phi(\mathscr{q}) = \bigoplus_{t,s} \phi_{t,s}(\mathscr{q})$, where the summation is taken over all pairs (s, t) of divirors of N_0 with $st \mid N_0$. By the definition of $\mathscr{p}(sp; K)$, ϕ is injective. Put $d(s) = \dim_{\mathscr{L}_K} \mathscr{p}(sp; K)$. Then, we have an inequality:

$$\dim_{\mathscr{L}_K} \mathscr{q}(N_0 p; K) \leq \sum_{0 < t \mid N_0} \nu(t) d(N_0/t),$$

where $\nu(t)$ is the number of positive divisors of t. Let $d_k(s)$ be the number of ordinary forms in $\mathscr{S}_k(\Gamma_1(sp))$ with which associated primitive forms have conductor divisible by s. Then, by [10, Prop. 4.4], we have an identity:

$$\dim_K \mathscr{S}_k^0(\Gamma_1(N_0 p); K) = \sum_{0 < t \mid N_0} \nu(t) d_k(N_0/t).$$

As already seen, we know an inequality:

$$d(t) \leq d_k(t) \qquad \text{for all} \quad 0 < t \mid N_0 \quad \text{and} \quad k \geq 2.$$

By Theorem 3.1, we know that

$$\dim_{\mathscr{L}_K} \mathscr{q}(N_0 p; K) = \dim_K \mathscr{S}_k^0(\Gamma_1(N_0 p); K) \qquad \text{for all} \quad k \geq 2.$$

Thus $d_k(N_0)$ is independent of k and coincides with $d(N_0)$ if $k \geq 2$. Note that $d(N_0)$ is the number of ordinary forms belonging to primitive local rings of $\mathscr{q}(N_0 p; K)$. This finishes the proof of Corollary 3.7.

Proof of Corollary 3.8. — We shall prove only the assertion concerning the module $\mathscr{C}(\mathscr{K}; K)$, because the other case can be dealt with in exactly the same fashion. We shall use the symbols defined in § 3. We recall some of them:

$$\bar{h} = \tilde{h}', \quad h' = h(\mathscr{K}) \oplus h(\mathscr{B}), \quad \mathscr{C} = \bar{h}/h \quad \text{and} \quad \mathscr{N}_s = \bar{h}/h',$$

where $\mathbf{h}=\mathbf{h}_0(\mathrm{N}\,;\mathcal{O}_\mathrm{K})$. Put $\mathscr{C}'=\mathbf{h}'/\mathbf{h}$. By definition, we have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{h} & = & \mathbf{h} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathbf{h}' & \to & \overline{\mathbf{h}} & \to & \mathscr{N}_s & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathscr{C}' & \to & \mathscr{C} & \to & \mathscr{N}_s & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whose vertical and horizontal lines are exact. After localizing this diagram at $\mathrm{P}=\mathrm{P}_k$, we have that $(\mathscr{N}_s)_\mathrm{P}=0$ (since $\mathscr{N}_s$ is pseudo-null) and $\overline{\mathbf{h}}_\mathrm{P}=\mathbf{h}_\mathrm{P}$ if $k\geqq 2$. Thus $\overline{\mathbf{h}}/\mathrm{P}\overline{\mathbf{h}}$ is a subalgebra of $\mathbf{h}_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathrm{K})$. By tensoring $\Lambda_\mathrm{K}/\mathrm{P}\Lambda_\mathrm{K}$ to the above diagram, we have another commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{h}/\mathrm{P}\mathbf{h} & = & \mathbf{h}/\mathrm{P}\mathbf{h} & & & & \\
\downarrow & & \downarrow & & & & \\
\mathbf{h}'/\mathrm{P}\mathbf{h}' & \xrightarrow{\alpha} & \overline{\mathbf{h}}/\mathrm{P}\overline{\mathbf{h}} & \to & \mathscr{N}_s/\mathrm{P}\mathscr{N}_s & \to & 0\\
\downarrow & & \downarrow{\scriptstyle\beta} & & \| & & \\
\mathscr{C}'/\mathrm{P}\mathscr{C}' & \xrightarrow[\gamma]{} & \mathscr{C}/\mathrm{P}\mathscr{C} & \to & \mathscr{N}_s/\mathrm{P}\mathscr{N}_s & \to & 0,\\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

whose vertical and horizontal lines are exact. Then the image of α (in the diagram) coincides with $h(\mathrm{F}_k)\oplus h(\mathrm{B}_k)$ in (3.11 *b*). Thus we see that

$$\gamma(\mathscr{C}'/\mathrm{P}\mathscr{C}')=\beta(\alpha(\mathbf{h}'/\mathrm{P}\mathbf{h}'))=(h(\mathrm{F}_k)\oplus h(\mathrm{B}_k))/\mathbf{h}_k^0(\Gamma_1(\mathrm{N}_0 p)\,;\mathcal{O}_\mathrm{K})=\mathrm{C}_k(\mathscr{K}).$$

This shows the exactitude of the sequence:

$$0\to\mathrm{C}_k(\mathscr{K})\to\mathscr{C}/\mathrm{P}_k\mathscr{C}\to\mathscr{N}_s/\mathrm{P}_k\mathscr{N}_s\to 0\quad\text{for each}\quad k\geq 2.$$

Proof of Proposition 3.9. — We shall assume that $\mathrm{R}(\mathrm{F}_k)\oplus\mathrm{R}(\mathrm{B}_k)$ is integrally closed in $(\mathscr{R}/\mathrm{P}_k\mathscr{R})\otimes_{\mathcal{O}_\mathrm{K}}\mathrm{K}$ for some $k\geq 2$ and show the vanishing of $\mathscr{N}_s$. For simplicity, write $\mathscr{R}'$ for $\mathscr{R}(\mathscr{K})\oplus\mathscr{R}(\mathscr{B})$ and $\overline{\mathscr{R}}$ for the free closure of $\mathscr{R}'$. In the same manner as in the proof of Corollary 3.8, we have an exact sequence:

$$0\to\mathrm{R}(\mathrm{F}_k)\oplus\mathrm{R}(\mathrm{B}_k)\to\overline{\mathscr{R}}/\mathrm{P}\overline{\mathscr{R}}\to\mathscr{N}_s/\mathrm{P}\mathscr{N}_s\to 0\quad\text{for}\quad \mathrm{P}=\mathrm{P}_k.$$

Meantime, $\overline{\mathscr{R}}/\mathrm{P}\overline{\mathscr{R}}$ is a subalgebra of $(\mathscr{R}/\mathrm{P}\mathscr{R})\otimes_{\mathcal{O}_\mathrm{K}}\mathrm{K}$ integral over $\mathrm{R}(\mathrm{F}_k)\oplus\mathrm{R}(\mathrm{B}_k)$. Thus, we know from the assumption of normality the vanishing: $\mathscr{N}_s/\mathrm{P}\mathscr{N}_s=0$, which implies the expected vanishing $\mathscr{N}_s=0$ by Nakayama's lemma.

Next, we shall assume the existence of an isomorphism:

$$(\mathscr{R}/\mathrm{P}_k\mathscr{R})\simeq\mathrm{Hom}_{\mathcal{O}_\mathrm{K}}((\mathscr{R}/\mathrm{P}_k\mathscr{R}),\mathcal{O}_\mathrm{K})\quad\text{as}\quad\mathscr{R}/\mathrm{P}_k\mathscr{R}\text{-modules}$$

and the normality of $\mathrm{R}(\mathrm{F}_k)$ in F_k for some $k\geq 2$. Since $\mathscr{R}$ is Λ_K-free, we know for $\mathrm{P}=\mathrm{P}_k$ that

$$\mathrm{Hom}_{\mathcal{O}_\mathrm{K}}((\mathscr{R}/\mathrm{P}\mathscr{R}),\mathcal{O}_\mathrm{K})\simeq\mathrm{Hom}_{\Lambda_\mathrm{K}}(\mathscr{R},\Lambda_\mathrm{K})\otimes_{\Lambda_\mathrm{K}}\Lambda_\mathrm{K}/\mathrm{P}\Lambda_\mathrm{K}$$

as $\mathscr{R}$-modules. Thus, by the assumption combined with Nakayama's lemma, we know that

$$\mathrm{Hom}_{\Lambda_K}(\mathscr{R}, \Lambda_K) \simeq \mathscr{R} \quad \text{as} \quad \mathscr{R}\text{-modules}.$$

Then, we can find a dual pairing $\langle\ ,\ \rangle$ on $\mathscr{R}$ with values in Λ_K so that $\langle ra, b \rangle = \langle a, rb \rangle$ for all a, b, $r \in \mathscr{R}$. On the other hand, a similar argument as in the proof of criterion (*i*) shows that $\tilde{\mathscr{R}}(\mathscr{K}) = \mathscr{R}(\mathscr{K})$ if $R(F_k)$ is normal in F_k. Thus, $\mathscr{R}(\mathscr{K})$ is Λ_K-free. Let $\mathfrak{a}$ be the kernel of the natural projection of $\mathscr{R}$ onto $\mathscr{R}(\mathscr{K})$. Then, by definition, $\mathfrak{a}$ is the intersection $\mathscr{B} \cap \mathscr{R}$ in $\mathscr{K} \oplus \mathscr{B}$, and thus, $\mathfrak{a}$ is reflexive [1, VII.4.2, Prop. 6]; hence, $\mathfrak{a}$ is Λ_K-free [25, Lemme 6]. Namely, we have an exact sequence of Λ_K-free modules:

$$0 \to \mathfrak{a} \to \mathscr{R} \to \mathscr{R}(\mathscr{K}) \to 0.$$

Write M^* for $\mathrm{Hom}_{\Lambda_K}(M, \Lambda_K)$ for any Λ_K-module M. Then, we have another exact sequence:

$$0 \to \mathscr{R}(\mathscr{K})^* \to \mathscr{R}^* \to \mathfrak{a}^* \to 0.$$

Since we have already identified $\mathscr{R}^*$ with $\mathscr{R}$ by a pairing $\langle\ ,\ \rangle$, we see that

$$\begin{aligned} \mathscr{R}(\mathscr{K})^* &= \{ r \in \mathscr{R} \mid \langle r, a \rangle = 0 \quad \text{for any} \quad a \in \mathfrak{a} \} \\ &= \{ r \in \mathscr{R} \mid \langle ra, 1 \rangle = 0 \quad \text{for any} \quad a \in \mathfrak{a} \} \\ &= \{ r \in \mathscr{R} \mid ar = 0 \quad \text{for any} \quad a \in \mathfrak{a} \} \\ &= \mathscr{K} \cap \mathscr{R}. \end{aligned}$$

Thus, we know that

$$\mathfrak{a}^* = \mathrm{Hom}_{\Lambda_K}(\mathfrak{a}, \Lambda_K) \simeq \mathscr{R}(\mathscr{B}) \quad \text{as} \quad \mathscr{R}\text{-modules}.$$

By [1, VII.4.2, Th. 1], we know that $\mathfrak{a}^*$ is reflexive; hence, $\mathfrak{a}^*$ is Λ_K-free. This shows by definition the vanishing: $\mathscr{N}_s = 0$.

§ 7. **Primitive components attached to imaginary quadratic fields and Eisenstein series**

Let M be an imaginary quadratic field with discriminant $-d$ such that

(7.1) *the fixed prime p ($\geqq 5$) is split.*

We consider M as a subfield of Ω and let $\mathfrak{p}$ be the prime factor associated with this embedding of M into Ω (i. e., $\mathfrak{p} = \mathfrak{P} \cap M$). Fix an integral ideal $\mathfrak{c}$ of M outside p, and let $I(\mathfrak{pc})$ be the group of the fractional ideals of M prime to $\mathfrak{pc}$ and $P(\mathfrak{pc})$ the subgroup of $I(\mathfrak{pc})$ consisting of all principal ideals (a) with $a \equiv 1 \bmod^\times \mathfrak{pc}$, where " $\bmod^\times \mathfrak{pc}$ " means the multiplicative congruence modulo $\mathfrak{pc}$. Fix a complete set S of representatives for $H = I(\mathfrak{pc})/P(\mathfrak{pc})$ consisting of integral ideals. For $\mathfrak{a} \in S$ and $0 \leqq \nu \in \mathbf{Z}$, put (according to Shimura [26, p. 203])

(7.2 *a*) $$\mathfrak{a}_0 = \{ a \in \mathfrak{a} \mid a \equiv 1 \bmod^\times \mathfrak{pc} \},$$

(7.2 *b*) $$g_\nu(\mathfrak{a}; z) = \sum_{a \in \mathfrak{a}_0} a^\nu e(N(a)z/N(\mathfrak{a})).$$

Then, it is known by Hecke [5] (see also [26, Lemma 3] and [10, Prop. 1.1]) that

(7.2 c) $$g_\nu(\mathfrak{a}\,;z)\in \mathcal{M}_{\nu+1}(\Gamma_1(p\mathrm{N}_0)) \qquad \text{for} \quad \mathrm{N}_0 = d\mathrm{N}(\mathfrak{c}).$$

We have a natural morphism of $(\mathbf{Z}/\mathrm{N}(\mathfrak{c})p\mathbf{Z})^\times$ into H. Thus the group $(\mathbf{Z}/\mathrm{N}(\mathfrak{c})p\mathbf{Z})^\times$ acts on H. Since $\Gamma_0(\mathrm{N})/\Gamma_1(\mathrm{N})\cong(\mathbf{Z}/\mathrm{N}\mathbf{Z})^\times$ through $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto d$, $\Gamma_0(\mathrm{N})/\Gamma_1(\mathrm{N})$ acts on H for $\mathrm{N}=\mathrm{N}_0 p$. Then the explicit action of $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(\mathrm{N})$ on $g_\nu(\mathfrak{a}\,;z)$ is given by

(7.3) $$g_\nu(\mathfrak{a})\mid\gamma=\chi(d)g_\nu(\gamma\mathfrak{a}),$$

where $\gamma\in\Gamma_0(\mathrm{N})$ acts on $\mathfrak{a}\in\mathrm{S}$ through its action on H as described above and χ is the quadratic residue symbol associated with M.

For any prime idele $\mathfrak{l}$ of M, let $\mathrm{M}_\mathfrak{l}$ be the $\mathfrak{l}$-adic completion of M and $\mathrm{U}_\mathfrak{l}$ be the group of all $\mathfrak{l}$-adic units in $\mathrm{M}_\mathfrak{l}$. Further let M_∞ be the completion of M at the infinite place ($\mathrm{M}_\infty\cong\mathbf{C}$). Let us put $\mathrm{U}(\mathfrak{c})=\{x=(x_\mathfrak{l})\in\prod_{\mathfrak{l}\neq\mathfrak{p}}\mathrm{U}_\mathfrak{l}\mid x\equiv 1 \bmod \mathfrak{c}\}$,

$$\mathrm{W}=\mathrm{M}_\mathrm{A}^\times/\overline{\mathrm{U}(\mathfrak{c})\mathrm{M}_\infty\mathrm{M}^\times},$$

where $\mathrm{M}_\mathrm{A}^\times$ is the idele group of M and $\overline{\mathrm{U}(\mathfrak{c})\mathrm{M}_\infty\mathrm{M}^\times}$ is the closure in $\mathrm{M}_\mathrm{A}^\times$ of $\mathrm{U}(\mathfrak{c})\mathrm{M}_\infty\mathrm{M}^\times$. Any $x\in\mathrm{M}_\mathrm{A}^\times$ with $x_\mathfrak{l}\in\mathrm{U}_\mathfrak{l}$ for $\mathfrak{l}\mid\mathfrak{p}\mathfrak{c}$ naturally acts on the fractional ideals of $\mathrm{I}(\mathfrak{p}\mathfrak{c})$; then, we put $il(x)=x\mathfrak{R}$ for the ring $\mathfrak{R}$ of integers in M. The inclusion $\Gamma\hookrightarrow\mathrm{U}_\mathfrak{p}\subset\mathrm{M}_\mathrm{A}^\times$ and the morphism $x\mapsto il(x)$ of $\mathrm{M}_\mathrm{A}^\times$ into $\mathrm{I}(\mathfrak{p}\mathfrak{c})$ gives an exact sequence of topological groups: $0\to\Gamma\to\mathrm{W}\to\mathrm{H}\to 0$. Hence, as topological spaces, we have $\mathrm{W}\cong\mathrm{H}\times\Gamma$. Any Hecke character λ modulo $\mathfrak{c}\mathfrak{p}^r$ $(0\leqq r\in\mathbf{Z})$ can be considered as a continuous character of W (e. g., [9, § 1] and (7.8) below in the text) with values in Ω.

Let Δ be the maximal p-profinite torsion free subgroup of W containing Γ and μ be the torsion part of W; thus, $\mathrm{W}\cong\Delta\times\mu$. Let L′ be the extension of $\mathbf{Q}_p$ containing the values of all characters of $(\Delta/\Gamma)\times\mu$. For any Hecke character λ modulo $\mathfrak{c}\mathfrak{p}$ with

(7.4) $$\lambda((a))=a \qquad \text{if} \quad a\equiv 1 \bmod^\times \mathfrak{c}\mathfrak{p}$$

λ automatically induces an isomorphism of Δ onto $\lambda(\Delta)\subset\Omega$. We fix such a character λ and adjoin all the values of λ on $\mathrm{I}(\mathfrak{p}\mathfrak{c})$ to L′ and denote by L the subfield of Ω obtained.

Then L is still finite over $\mathbf{Q}_p$ and contains all the values of all Hecke character satisfying (7.4). Put $q=[\Delta:\Gamma]$ and define the Iwasawa algebra A_L of Δ over $\mathcal{O}_\mathrm{L}$ (i. e., $\mathrm{A}_\mathrm{L}=\varprojlim_n\mathcal{O}_\mathrm{L}[\Delta/\Gamma_n]$). Take a generator δ of Δ so that $\delta^q=1+p$. Then A_L is isomorphic to $\mathcal{O}_\mathrm{L}[[\mathrm{Y}]]$ through $\delta\mapsto 1+\mathrm{Y}$ and $\mathrm{A}_\mathrm{L}\cong\Lambda_\mathrm{L}[\mathrm{Y}]/\mathrm{F}(\mathrm{Y})\Lambda_\mathrm{L}[\mathrm{Y}]$ for $\mathrm{F}(\mathrm{Y})=(1+\mathrm{Y})^q-(1+\mathrm{X})$. Then the quotient field $\mathcal{A}_\mathrm{L}$ of A_L is defined over L.

Theorem 7.1. — *Let the notation be as above. For any character ξ of μ, let $\mathfrak{C}$ be the conductor of $\lambda_0=\xi\hat{\lambda}$, where $\hat{\lambda}$ is the restriction of λ with* (7.4) *on Δ and put $\lambda_j=\xi\hat{\lambda}^j$ for $j>0$. Put $\mathrm{N}=d\mathrm{N}(\mathfrak{C})$ and write $\mathrm{N}=\mathrm{N}_0 p^t$ with $(\mathrm{N}_0,p)=1$. Then there exists a unique primitive cuspidal component $\mathcal{K}$ of conductor N_0 in $\varphi(\mathrm{N}\,;\mathrm{L})$ such that $\mathcal{K}\cong\mathcal{A}_\mathrm{L}$ and for each $j\geqq 0$ and all character ε of Δ/Γ, $f_{\lambda,j\varepsilon}=\sum_\mathfrak{a}\varepsilon\lambda_j(\mathfrak{a})q^{\mathrm{N}(\mathfrak{a})}\in\mathcal{S}_{j+1}(\Gamma_1(\mathrm{N})\,;\mathrm{L})$ belongs to $\mathcal{K}$ in the sense of Corollary* 3.7, *where $\mathfrak{a}$ runs over all integral ideals prime to $\mathfrak{p}\mathfrak{C}$.*

Note that the condition : " $q=1$ " is equivalent to " $d(\mathscr{K})=[\mathscr{K}:\mathscr{L}_L]=1$ " ; thus, the primitive cuspidal components as in Theorem 7.1 provide infinitely many examples of these with $d(\mathscr{K})=1$.

Proof. — By our assumption, the closure of M in Ω (that is, M_p) coincides with $\mathbf{Q}_p$. By a theorem of Mahler [21, § 10], every continuous function ϕ on $\mathbf{Z}_p$ with values in K for any finite extension K of $\mathbf{Q}_p$ can be expanded as

$$(7.5\,a)\qquad \phi(x)=\sum_{n=0}^{\infty} c(n)\binom{x}{n}\qquad (x\in\mathbf{Z}_p),$$

where $c(n)\in K$ with $\lim_{n\to\infty}|c(n)|_p=0$, $\binom{x}{n}=\dfrac{x(x-1)\dots(x-n+1)}{n!}$ and $\binom{x}{0}=1$. Since Γ is an open subset of $\mathbf{Z}_p$, every continuous function on Γ can be extended to $\mathbf{Z}_p$. Thus every function on W has an expansion of the form:

$$(7.5\,b)\qquad \phi(h,z)=\sum_{n=0}^{\infty} c(h,n)\binom{z}{n}\qquad (h\in H,\quad z\in\Gamma),$$

where we have identified W with $H\times\Gamma$ and $c(h,n)\in K$ with $\lim_{n\to\infty}|c(h,n)|_p=0$. For any topological spaces T and T′, let us write C(T ; T′) for the space of all continuous functions on T with values in T′. Let Γ act on C(W ; T) through

$$\phi(h,z)\,|\,z'=\phi(h,zz')\qquad \text{for}\quad \phi\in C(W;T).$$

It is well known (*cf.* [12, § 5]) that

$$(7.6\,a)\qquad C(\Gamma;\mathbf{T}_p)\quad\text{is the Pontrjagin dual }\Lambda\text{-module of }\Lambda,$$

$$(7.6\,b)\qquad C(\Gamma;K/\mathcal{O}_K)\cong C(\Gamma;K)/C(\Gamma;\mathcal{O}_K).$$

The latter assertion is equivalent to the density of locally constant functions in $C(\Gamma;K)$. We see easily as Λ-modules

(7.6 *c*) $C(W;T)\cong C(\Gamma;T)^H$ *for* $T=\mathbf{T}_p$, $\mathbf{Z}_p$ *or more generally*, K *or* $K/\mathcal{O}_K$.

Furthermore C(W ; K) is equipped with a p-adic Banach norm defined by

$$(7.7)\qquad \|\phi\|=\operatorname*{Sup}_{w\in W}|\phi(w)|_p.$$

For $\phi\in C(W;K)$, put

$$\theta(\phi)=\sum_{\mathfrak{a}\in S}\sum_{a\in\mathfrak{a}_0}\phi(h(\mathfrak{a}),a)q^{N(a)/N(\mathfrak{a})}\in K[[q]],$$

where $h(\mathfrak{a})$ is the class of $\mathfrak{a}$ in H. Any Hecke character φ modulo $\mathfrak{c}\mathfrak{p}$ with values in $\overline{\mathbf{Q}}\subset\Omega$ can be considered as a character of W. Especially, we know

(7.8) *if* $\varphi((a))=a^{\nu}$ *for* $a\equiv 1 \bmod^{\times}\mathfrak{c}\mathfrak{p}$, *then* $\varphi(x)=\varphi(il(x))^{-1}x_{\mathfrak{p}}^{\nu}$ *for* $x\in M_A^{\times}$ *with* $x_{\mathfrak{p}}\in\Gamma$ *and* $x_{\mathfrak{l}}\equiv 1 \bmod \mathfrak{c}$ *for* $\mathfrak{l}\,|\,\mathfrak{c}$.

Let $\Xi(\nu, \mathfrak{c})$ be the set of all characters satisfying the assumption of (7.8). Then, it is well known that $\theta(\varphi)$ is a normalized eigenform in $\mathscr{S}_{\nu+1}(\Gamma_0(\mathrm{N}), \tilde{\varphi}\chi)$ with

$$(7.9)\qquad a(l, \theta(\varphi))=\begin{cases}\varphi(\mathfrak{l})+\varphi(\bar{\mathfrak{l}}) & \text{if}\quad l=\mathfrak{l}\bar{\mathfrak{l}}\quad \text{in M}\quad (\mathfrak{l}\neq\bar{\mathfrak{l}}),\\ \varphi(\mathfrak{l}) & \text{if}\quad l=\mathfrak{l}^2\quad \text{in M},\\ 0 & \text{otherwise},\end{cases}$$

where l is a prime and $\tilde{\varphi}(m)=\varphi((m))m^{-\nu}$ for $0\neq m\in\mathbf{Z}$ (*cf.* [26]). Furthermore, as shown in [26, § 4],

(7.10) $\{\theta(\varphi)\}_{\varphi\in\Xi(\nu,\mathfrak{c})}$ *spans h-dimensional subspace of* $\mathscr{S}_{\nu+1}(\Gamma_1(\mathrm{N}))$ *for* $h=|\mathrm{H}|$, *if* $\nu\geqq 1$.

Let $\mathrm{P}(\mathrm{W}\,;\mathrm{K})$ be the subspace of $\mathrm{C}(\mathrm{W}\,;\mathrm{K})$ consisting of all polynomial functions on Γ. Since $\mathrm{P}(\mathrm{W}\,;\mathrm{K})$ is dense in $\mathrm{C}(\mathrm{W}\,;\mathrm{K})$ by Mahler's theorem already mentioned, we see from (7.5 *b*) that

$$(7.11)\qquad \mathrm{P}(\mathrm{W}\,;\mathrm{K})/(\mathrm{C}(\mathrm{W}\,;\mathcal{O}_\mathrm{K})\cap\mathrm{P}(\mathrm{W}\,;\mathrm{K}))\cong\mathrm{C}(\mathrm{W}\,;\mathrm{K}/\mathcal{O}_\mathrm{K}).$$

Now one may give a proof of (7.10) as follows: Let us put for $\mathfrak{a}\in\mathrm{S}$, $\mathrm{R}(\mathfrak{a})=\{a\in\mathfrak{a}_0\,|\,\mathrm{N}(a)/\mathrm{N}(\mathfrak{a})$ is a prime outside $\mathrm{N}(\mathfrak{c}\mathfrak{p})$, $a\equiv 1 \bmod \mathrm{N}(\mathfrak{c}\mathfrak{p})\}$, and for any topological module T and a function $\phi\in\mathrm{C}(\mathrm{W}\,;\mathrm{T})$, define a formal series $\theta(\phi)=\sum_{\mathfrak{a}\in\mathrm{S}}\sum_{a\in\mathfrak{a}_0}\phi(h(\mathfrak{a}), a)q^{\mathrm{N}(a)/\mathrm{N}(\mathfrak{a})}\in\mathrm{T}[[q]]$. We may assume that any $\mathfrak{a}\in\mathrm{S}$ is prime to $\mathrm{N}(\mathfrak{c}\mathfrak{p})$. Then for any $\mathfrak{a}\in\mathrm{S}$, we can find an element $\mathfrak{a}^*$ in S in the class of the ideal $\mathfrak{a}^\rho$. Then for $a\in\mathrm{R}(\mathfrak{a})$, $(a\mathfrak{a}^{-1})^\rho$ can be written uniquely as $a^*(\mathfrak{a}^*)^{-1}$ for a unique $a^*\in\mathfrak{a}_0^*$. We now embed $\mathrm{R}(\mathfrak{a})$ into $\Gamma\times\Gamma$ through $a\mapsto(a, a^*)\in\Gamma\times\Gamma$. The Tchebotarev density theorem shows that $\mathrm{R}(\mathfrak{a})$ is dense in $\Gamma\times\Gamma$. If $\theta(\phi)=0$, then $\phi(h(\mathfrak{a}), a)+\phi(h(\mathfrak{a}^*), a^*)=0$ for all $a\in\mathrm{R}(\mathfrak{a})$. Now we define a continuous function $\Phi(\gamma, \delta)$ on $\Gamma\times\Gamma$ by $\Phi(\gamma, \delta)=\phi(h(\mathfrak{a}), \gamma)+\phi(h(\mathfrak{a}^*), \delta)$ for any $\gamma, \delta\in\Gamma$. The density of $\mathrm{R}(\mathfrak{a})$ in $\Gamma\times\Gamma$ shows $\Phi(\gamma, \delta)=0$ on $\Gamma\times\Gamma$ and thus $\phi(h(\mathfrak{a}), \gamma)=-\phi(h(\mathfrak{a}^*), \delta)$ for all $\gamma, \delta\in\Gamma$. Therefore, $\phi(h(\mathfrak{a}), \gamma)$ is constant on Γ, and hence, is a function in $\mathrm{C}(\mathrm{H}\,;\mathrm{T})$. Especially, the kernel of $\theta:\mathrm{C}(\mathrm{W}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})\to\mathscr{M}(\mathrm{N}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})$ is contained in $\mathrm{C}(\mathrm{H}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})\cong(\mathrm{K}/\mathcal{O}_\mathrm{K})^\mathrm{H}$, which shows (7.10).

Comparing the definition of the action of Γ on $\mathscr{M}(\mathrm{N}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})$ with that on $\mathrm{C}(\mathrm{W}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})$, we see

$$(7.12\,a)\qquad z\theta(\phi\,|\,z)=\theta(\phi)\,|\,z.$$

We define a Λ-module $\mathbf{Z}_p(r)$ by $\Lambda/\mathrm{P}_r\Lambda$ for $\mathrm{P}_r(\mathrm{X})=(\mathrm{X}+1)-(1+p)^r$. Then, (7.12 *a*) shows that θ gives a morphism of Λ-modules:

$$(7.12\,b)\qquad \theta:\mathbf{Z}_p(1)\otimes_{\mathbf{Z}_p}\mathrm{C}(\mathrm{W}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})\to\mathscr{M}(\mathrm{N}\,;\mathrm{K}/\mathcal{O}_\mathrm{K})$$
$$\textit{with kernel contained in}\quad \mathbf{Z}_p(1)\otimes_{\mathbf{Z}_p}\mathrm{C}(\mathrm{H}\,;\mathrm{K}/\mathcal{O}_\mathrm{K}).$$

When $\mathrm{K}=\mathbf{Q}_p$, we see $\mathrm{C}(\mathrm{W}\,;\mathbf{T}_p)\cong\mathrm{E}(\infty)^\mathrm{H}$ for the Pontrjagin dual space $\mathrm{E}(\infty)$ of Λ from (7.6 *a*) and (7.11). Note that

$$(7.13)\qquad \mathbf{Z}_p(1)\otimes_{\mathbf{Z}_p}\mathrm{E}(\infty)\cong\mathrm{E}(\infty)\quad \textit{as}\quad \Lambda\textit{-modules}.$$

Since all $\theta(\phi)$ for $\phi \in C(W; L)$ is a linear combination of $\theta(\lambda)$ as in (7.9), $\theta(C(W; L/\mathcal{O}_L))$ is stable under the action of $\mathcal{H}(N; \mathcal{O}_L)$ by (7.9). Furthermore, we see from (7.9) that

(7.14 *a*) $$\theta(\lambda) \mid T(p) = \lambda(\bar{\mathfrak{p}})\theta(\lambda) \quad \textit{and} \quad |\lambda(\bar{\mathfrak{p}})|_p = 1.$$

In fact, for a generator a of $\bar{\mathfrak{p}}^h$ with $a \equiv 1 \bmod^\times \mathfrak{c}$, we have $\lambda(\bar{\mathfrak{p}})^h = \lambda(\bar{\mathfrak{p}}^h) = a^v$. Since $p = \mathfrak{p}\bar{\mathfrak{p}}$ and $\mathfrak{p} = \mathfrak{P} \cap M$, we see $|a|_p = 1$. This shows (7.14 *a*). Then, by [10, Lemma 4.2], we know that

(7.14 *b*) $\theta(\lambda)$ *belongs to* $\mathcal{M}_0(N; L)$ *and* $\theta(C(W; L/\mathcal{O}_L))$ *is a submodule of* $\mathcal{M}_0(N; L/\mathcal{O}_L)$ *stable under* $\mathcal{H}_0(N; \mathcal{O}_L)$.

Naturally, we can consider $C(W; L/\mathcal{O}_L)$ as a module over W through $(\phi \mid w')(w) = \phi(ww')$. Then, we have isomorphisms as W-modules:

$$C(W; L/\mathcal{O}_L) \cong P(W; L)/(P(W; L) \cap C(W; \mathcal{O}_L)) \cong C(\Delta; L/\mathcal{O}_L) \otimes_{\mathbf{Z}} \mathbf{Z}[\mu],$$

where $\mathbf{Z}[\mu]$ is the group algebra of μ. Decomposing this space into the sum of eigenspaces of the action of the finite group μ, we see that

$$C(\xi) = \{ f \in C(W; L/\mathcal{O}_L) \mid f \mid m = \xi(m) f \quad \text{for all} \quad m \in \mu \}$$

is isomorphic to $C(\Delta; L/\mathcal{O}_L)$ as a module of A_L. The restriction of the operators in $\mathcal{H}_0(N; \mathcal{O}_L)$ to the subspace $\Theta(\xi) = \theta(C(\xi))$ defines a homomorphism of $\mathcal{Q}(N; L)$ into $\mathcal{A}_L$ by (7.14 *b*) and (7.9). For any $k > 1$, by the definition of L, we have $A_L/P_{k-1}A_L \otimes_{\mathcal{O}_L} L \cong L^q$. Since the image of $\mathcal{H}_k^0(\Gamma_1(N); \mathcal{O}_L) \otimes_{\mathcal{O}_L} L$ in $\mathrm{End}_{\mathcal{O}_L}(\Theta(\xi) \cap \mathcal{M}_k(\Gamma_1(N); L/\mathcal{O}_L)) \otimes_{\mathcal{O}_L} L$ has dimension q over L (*cf.* [29, Th. 3.51]) if $k > 1$, it must coincide with $(A_L/P_{k-1}A_L) \otimes_{\mathcal{O}_L} L$. In fact, we see $\Theta(\xi) \cap \mathcal{M}_k(\Gamma_1(N); L/\mathcal{O}_L) \supset \sum_{\varepsilon} (L/\mathcal{O}_L) f_{\lambda_j \varepsilon}$ by definition, where ε runs over all the characters of Δ/Γ. Therefore the homomorphism of $\mathcal{Q}(N; L)$ into $\mathcal{A}_L$ is surjective. The primitivity of the component $\mathcal{A}_L$ follows from the fact that the conductor of $f_{\lambda_j \varepsilon}$ is divisible by N_0 (*cf.* Cor. 3.7). This shows the theorem.

Now we consider the irreducible component corresponding to the Eisenstein series. Let χ and ψ be primitive characters of $(\mathbf{Z}/M_1\mathbf{Z})^\times$ and $(\mathbf{Z}/M_2\mathbf{Z})^\times$. Then, for any positive integer $k \geq 3$ with $(-1)^k = \chi\psi(-1)$, put

(7.15) $$E_k(\chi, \psi) = -\delta(\psi)\frac{B_{k,\chi}}{2k} + \sum_{n=1}^{\infty} \left(\sum_{0 < d \mid n} \chi(d)\psi\left(\frac{n}{d}\right) d^{k-1} \right) q^n,$$

where $\delta(\psi) = 1$ or 0 according as ψ is trivial or not, and $B_{k,\chi}$ is the k-th generalized Bernoulli number with character χ.

Let us denote by $\mathbf{Q}(\chi, \psi)$ (resp. $\mathbf{Q}_p(\chi, \psi)$) the field generated over $\mathbf{Q}$ (resp. $\mathbf{Q}_p$) by all the values of χ and ψ. Then, it is well known by [6] (see also [4, Th. 4.7.1]) that $E_k(\chi, \psi)$ belongs to $\mathcal{M}_k(\Gamma_0(N), \chi\psi; \mathbf{Q}(\chi, \psi))$ for $N = M_1 M_2$. Furthermore, $E_k(\chi, \psi)$ is a normalized eigenform. The eigenvalue of $T(p)$ at $E_k(\chi, \psi)$ is given by

(7.16) $$\psi(p) + \chi(p)p^{k-1}.$$

Thus, if M_2 is prime to p, then $(1 - \bar{\psi}\chi(p)p^{k-1}) . E_k(\chi, \psi) \mid e$ is a p-ordinary form for the idempotent e attached to $T(p)$ [10, Lemma 4.2]. We write simply $\mathcal{E}_k(\chi, \psi)$ for

$(1-\bar{\psi}\chi(p)p^{k-1})E_k(\chi, \psi) \mid e$. When p divides M_1, $\mathscr{E}_k(\chi, \psi)=E_k(\chi, \psi)$, and when M_1 is prime to p, $\mathscr{E}_k(\chi, \psi)$ can be given by (7.15) by putting $\chi(d)=0$ if p divides d.

THEOREM 7.2. — *Let χ and ψ be primitive Dirichlet characters modulo M_1 and M_2, respectively. Assume that M_1 and M_2 are prime to p. Then, for $N_0=M_1M_2$, $0 \leqq a < p-1$ and $K=\mathbf{Q}_p(\chi, \psi)$, there exists a unique irreducible component $\mathscr{K}$ in $\mathscr{Q}(N ; K)$ of conductor N_0 such that $\mathscr{K}$ is isomorphic to the quotient field $\mathscr{L}_K$ of Λ_K and $\mathscr{E}_k(\chi\omega^{a-k}, \psi)$ belongs to $\mathscr{K}$ for any $k \geqq 3$, where ω is a character of $\mathbf{Z}_p^\times$ defined by $\omega(x)=\lim_{r\to\infty} x^{p^r}$.*

This can be proved without any conflicts if one understands the essential points of the proof of Theorem 7.1; so we leave it to the reader as an exercise.

Now let us discuss some numerical examples of the Iwasawa module $\mathscr{C}(\mathscr{K} ; K)$. For a while, we use the same notation as in Th. 7.1. By Theorem 7.1, we can separate the cusp forms belonging to a primitive component $\mathscr{A}_L$ obtained from the imaginary quadratic field M and others even if they are congruent mod p. If there exists a congruence $f_\lambda \equiv f \bmod \mathfrak{P}$ for $\lambda \in \Xi(\nu, \mathfrak{c})$ and another p-ordinary form f with exact level $dN(\mathfrak{c})p$ and of weight $\nu+1$, which is not obtained as a theta series of M, the torsion Λ_L-module $\mathscr{C}(\mathscr{A}_L ; L)$ of Theorem 3.6 must be non-trivial by Corollary 3.8 and (3.12). We can list some primes p with non-trivial $\mathscr{C}(\mathscr{A}_L ; L)$ from Maeda's table given in [7, (8.11)] when $M=\mathbf{Q}(\sqrt{-3})$ and $\mathfrak{c}=1$: $p=13$, 30842593. In this case, we can take $\mathbf{Q}_p$ as L and check the criterion (i) of Proposition 3.9 numerically at $k=13$ for $p=13$ and $k=31$ for $p=30842593$. Thus, for these primes, we know $\mathscr{N}_s(\mathscr{A}_L ; L)=0$. Note that the latter prime is an irregular prime for $\mathbf{Q}(\sqrt{-3})$ (*cf.* [9, p. 438]). In view of [9, (7.5)], one may conjecture as a special case of Conjecture 3.10 that the characteristic power series $\mathscr{C}(\mathscr{A}_L ; L)$ interpolates p-adically the special values (or more precisely, their algebraic part) of Hecke L-functions of M at certain integers. By [9, (7.5)], one can of course specify the evaluation points and the Hecke L-functions to be interpolated. Hence, if the conjecture is true, the characteristic power series gives a part of the p-adic measure constructed in Katz [18] associated with the imaginary quadratic field M.

A similar conjecture can be made for the primitive components attached to Eisenstein series as in Theorem 7.2. In this case, the corresponding p-adic L-functions must be those of Kubota-Leopoldt, which interpolate generalized Bernoulli numbers.

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