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HITTING PROBABILITIES OF KILLED BROWNIAN MOTION; A STUDY ON GEOMETRIC REGULARITY

By CHRISTER BORELL

1. Introduction

Consider a Brownian motion X in n -space with first hitting times $\tau_A = \tau_A(X) = \inf \{ t > 0; X(t) \in A \}$ and let $\mathcal{U}(\mathbb{R}^n)$ denote the class of all non-empty, open, and convex subsets of $\mathbb{R}^n$. Then, if $x_0, x_1 \in \mathbb{R}^n$ and $A_0, A_1, B_0, B_1 \in \mathcal{U}(\mathbb{R}^n)$:

$$(1.1) \quad \mathbb{P}_{x_\lambda}(\tau_{B^c} \geq \tau_{A_\lambda} < +\infty) \\ \geq \mathbb{P}_{x_0}(\tau_{B_0^c} \geq \tau_{A_0} < +\infty) \wedge \mathbb{P}_{x_1}(\tau_{B_1^c} \geq \tau_{A_1} < +\infty), \quad 0 < \lambda < 1,$$

where $\xi_\lambda = (1-\lambda)\xi_0 + \lambda\xi_1$, $\xi = x$, A , B , and $B^c = \mathbb{R}^n \setminus B$, respectively (Borell [4]).

In this paper, the basic diffusion process is a Brownian motion Y in $\mathbb{R}^n \cup \{\varphi\}$, which starts in $\mathbb{R}^n$ and behaves as an ordinary Brownian motion up till a certain random point of time when it jumps to φ and remains there. More explicitly, conditioned on X , the event $Y(t) \in \mathbb{R}^n$, has the probability $\exp\left(-\int_0^t V(X(s)) ds\right)$, where $V: \mathbb{R}^n \rightarrow [0, +\infty]$ is such that $V|_{\text{dom } V}$ is concave and $\text{dom } V = \{V < +\infty\} \in \mathcal{U}(\mathbb{R}^n)$. Under these assumptions (1.1) still holds with $\tau = \tau(Y)$ (Theorem 3.1). In fact, the same result remains true if $\mathbb{R}^n$ is replaced by an arbitrary Banach space.

About half the paper deals with various interpretations of Theorem 3.1. Thus, we discuss convexity properties of:

- (i) V -harmonic measures (Section 6, Example 7.1);
- (ii) V -Newtonian potentials (Theorem 7.1);
- (iii) V -equilibrium measures (Example 7.2) and
- (iv) logarithmic and Newtonian capacities (examples 7.3-7.5).

In the above list, perhaps the single most interesting point is the following: let $B \in \mathcal{U}(\mathbb{R}^n)$, $n \geq 2$, be bounded and suppose $V: B \rightarrow [0, +\infty[$ is $-1/2$ -concave, that is, $V^{-1/2}: B \rightarrow]0, +\infty]$ is concave. Moreover, let g denote the Green function of

$-1/2\Delta + V$ in B with the Dirichlet boundary condition zero. Then g is quasi-concave if $n=2$, and if $n \geq 3$ the function $g^{-1/(n-2)}$ is convex. Theorem 7.1 expresses these facts as a Brunn-Minkowski inequality of appropriate potentials of g . For comparison, we here only mention that the 3-dimensional potential $g\mu$, μ being the uniform distribution of a line segment, turns out to have convex equipotential surfaces. The same thing is known to be true for a point mass if $V=0$ (Gabriel [15], [16]). Needless to say, the beautiful works of Gabriel have played a decisive role for this and some other closely related papers of the author ([4], [5]).

Finally, in this section, let us make some remarks on the potential V above.

Again, consider a Y -process in $\mathbb{R}^n$ now with a convex potential V . Moreover, suppose $\text{dom } V \in \mathcal{U}(\mathbb{R}^n)$ is bounded. Then, by Brascamp and Lieb ([9], [10]), the transition densities p_t , $t > 0$, of Y are log-concave for each fixed $t > 0$. From this we expect nice geometrical properties of the corresponding Green function:

$$g = \int_0^\infty p_t dt.$$

In fact, our fruitless attempts to understand this puzzling problem have finally led us to $-1/2$ -concave potentials. The reader should note that a $-1/2$ -concave function is convex. (The log-concavity of the p_t for convex V turns out to be an algebraic consequence of (1.1) but that is another uniformity!) Below we will also see that $-1/2$ -concave potentials enter quite naturally in the hyperbolic potential theory of plane convex domains (Example 3.1).

2. Definitions

Throughout, E denotes a separable Banach space and $\mathcal{C}_E([0, +\infty[)$ is the standard Fréchet space of all continuous maps of $[0, +\infty[$ into E . A centered Gaussian random vector X in $C_E([0, +\infty[)$ is called a Brownian motion in E or an E -valued Brownian motion if X possesses stochastically independent increments and if, for every $t > 0$, the law of $X_t = [X(\cdot)](t)$ equals the law of $t^{1/2}X_1$ (see Gross [18] (potential theory) and Chow [12] (noise theory)).

Example 2.1. — Suppose S is a compact metric space and let $G = (G(s), s \in S)$ be a real-valued, centered Gaussian stochastic process with continuous paths. Then there exists a unique real-valued centered Gaussian process X with time set $S \times [0, +\infty[$ and covariance $[E(G(s)G(s'))](t \wedge t')$. Moreover, a version of $X = (X(s, t), (s, t) \in S \times [0, +\infty[)$ has continuous paths with probability one and, accordingly, induces a Brownian motion in $\mathcal{C}(S)$ (for details, see Carmona [11]). $\square$

The above example brings out the most general form of a Banach space-valued Brownian motion.

An E -valued Brownian motion is said to be non-degenerated if $\text{supp } \mathcal{L}(X_1) = E$. If F is a separable Banach space and $A : E \rightarrow F$ is a bounded linear map, then each E -valued Brownian motion X defines an F -valued Brownian motion by the rule $[AX]_t = AX_t$.

In what follows, X is supposed to be a fixed *non-degenerated* Brownian motion in E and, as usual, we let $\mathbb{P}_x = \mathcal{L}(x + X)$ and $\mathbb{E}_x = \int (\cdot) d\mathbb{P}_x$.

Below $\mathcal{U}(E)$ denotes the class of all non-empty, open, and convex subsets of E . Moreover, $\bar{\mathcal{U}}(E) = \{\bar{A}; A \in \mathcal{U}(E)\}$, $\mathcal{U}_\infty(E) = \{A \in \mathcal{U}(E); A \text{ bounded}\}$, and $\bar{\mathcal{U}}_\infty(E) = \{\bar{A}; A \in \mathcal{U}_\infty(E)\}$, respectively. If $A_0, A_1 \subseteq E$, and $0 < \lambda < 1$, we write $A_\lambda = (1 - \lambda)A_0 + \lambda A_1$. The same convention will be used for vectors in E . Given $A_i \in \mathcal{U}(E)$, concave functions $f_i: A_i \rightarrow [0, +\infty]$, $i = 0, 1$, and $\lambda \in [0, 1]$, the so-called λ -supremum convolution :

$$f_0 \lfloor \lambda \rfloor f_1: A_\lambda \rightarrow [0, +\infty],$$

of f_0 and f_1 is defined by:

$$(f_0 \lfloor \lambda \rfloor f_1)(x_\lambda) = \sup \{ (1 - \lambda) f_0(x_0) + \lambda f_1(x_1); x_0 \in A_0, x_1 \in A_1 \}.$$

Here $0 \cdot (+\infty) = 0$. Of course, $f_0 \lfloor \lambda \rfloor f_1$ is concave and by simple means one verifies:

$$(2.1) \quad f_0 \lfloor \theta_\lambda \rfloor f_1 = (f_0 \lfloor \theta_0 \rfloor f_1) \lfloor \lambda \rfloor (f_0 \lfloor \theta_1 \rfloor f_1), \quad \theta_0, \theta_1 \in [0, 1].$$

Next suppose $\alpha \in \mathbb{R} \setminus \{0\}$. Using the conventions $0^\alpha = +\infty$ and $(+\infty)^\alpha = 0$, if $\alpha < 0$, a function $f: A \rightarrow [0, +\infty]$ ($A \subseteq E$) is said to be α -convex (α -concave) if f^α is convex (concave). For this reason, a quasi-concave (log-concave) function is sometimes called $-\infty$ -convex (0-convex or 0-concave). The same terminology is used for set functions on vector spaces. For future reference, recall that a Gaussian Radon measure on a locally convex Hausdorff vector space is log-concave (Borell [6]).

3. The main result

Consider the Feynman-Kac semi-group:

$$S_t f = \mathbb{E} \left(f(X(t)) \exp \left(- \int_0^t V(X(s)) ds \right) \right), \quad t > 0,$$

where the potential $V: E \rightarrow [0, +\infty]$ is Borel measurable. If, in addition, V is convex, the log-concavity of Gaussian measures may be used to show that each S_t preserves log-concavity. Indeed, this property has many nice consequences (Brascamp, Lieb [9], [10], Lions [20]). The reader should note that if $B = \text{dom } V \in \mathcal{U}(E)$, then:

$$S_t f = \mathbb{E} \left(f(X(t)) \exp \left(- \int_0^t V(X(s)) ds \right); \tau_{B^c} \geq t \right), \quad t > 0.$$

THEOREM 3.1. — For $i=0, 1$, suppose $A_i, B_i \in \mathcal{U}(E)$, $x_i \in B_i$, and let $V_i: B_i \rightarrow [0, +\infty[$ be $-1/2$ -concave. Set $V_\lambda = (V_0^{-1/2} |\lambda| V_1^{-1/2})^{-2}$ and:

$$M(\lambda) = \mathbb{E}_{x_\lambda} \left(\exp \left(- \int_0^{\tau_{A_\lambda}} V_\lambda(X(s)) ds \right); \tau_{B^c} \geq \tau_{A_\lambda} < +\infty \right), \quad 0 < \lambda < 1,$$

respectively. Then M is quasi-concave.

Interestingly enough, there are several relations between Theorem 3.1 and the Brunn-Minkowski theory of convex bodies but the interplay is not yet fully understood. In particular, one may ask if the log-concavity of Gaussian measures (on all measurable sets!) and Theorem 3.1 have a common source.

For some other geometrical estimates on Feynman-Kac semi-groups, see Borell [7] and Ehrhard [14].

Before giving the proof of Theorem 3.1, which is rather lengthy, we should like to discuss an example where $-1/2$ -concave potentials arise in a natural way.

First, however, recall that if X is the usual Brownian motion in $\mathbb{R}^n$, then the expectation:

$$u(x) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_A} V(X(s)) ds \right); \tau_{B^c} \geq \tau_A < +\infty \right), \quad x \in \bar{B},$$

solves the V -equilibrium potential equation:

$$\begin{cases} \frac{1}{2} \Delta u - V u = 0 & \text{in } B, \\ u = 1 & \text{on } \bar{A}, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where, for example, $A, B \in \mathcal{U}(\mathbb{R}^n)$, $\bar{A} \subseteq B$, and $V: B \rightarrow [0, +\infty[$ is continuous (see e. g. Dynkin [13], Chap. 13). (Here and elsewhere $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$.)

Example 3.1. — Consider a $B \in \mathcal{U}(\mathbb{C})$, $B \neq \mathbb{C}$, equipped with the hyperbolic metric:

$$ds = \left| \frac{f'(z)}{\operatorname{Im} f(z)} \right| |dz|,$$

f being an arbitrary one-to-one conformal map onto the upper half plane in $\mathbb{C}$. Note that:

$$\frac{1}{2} \left| \frac{f'(z)}{\operatorname{Im} f(z)} \right| = \lim_{\zeta \rightarrow z} d(z, \zeta) / |z - \zeta|,$$

where $d(z, \zeta) = |(f(z) - f(\zeta)) / (f(z) - \overline{f(\zeta)})|$, $z, \zeta \in B$, is a strictly increasing function of the hyperbolic distance in B (see e. g. Ahlfors [1], [2]).

The following discussion is based on the fact that the Green function $g(z, \zeta)$ of $-1/2\Delta$ in B with the Dirichlet boundary condition zero is quasi-concave in (z, ζ) (this may be known; for safety's sake the result is proved in Theorem 7.1). Equivalently, if $B(z; r)$ denotes the open d -ball with center $z \in B$ and radius $r > 0$, then:

$$(1-\lambda)B(z_0; r) + \lambda B(z_1; r) \subseteq B(z_\lambda; r), \quad 0 < \lambda < 1.$$

Accordingly, for reals $t \neq 0$ close to zero:

$$\frac{1}{|t|} \left| \frac{f(z_\lambda + th_\lambda) - f(z_\lambda)}{f(z_\lambda + th_\lambda) - \overline{f(z_\lambda)}} \right| \leq \frac{1}{|t|} \left\{ \left| \frac{f(z_0 + th_0) - f(z_0)}{f(z_0 + th_0) - \overline{f(z_0)}} \right| \vee \left| \frac{f(z_1 + th_1) - f(z_1)}{f(z_1 + th_1) - \overline{f(z_1)}} \right| \right\},$$

and in the limit as $t \rightarrow 0$:

$$\left| \frac{f'(z_\lambda) h_\lambda}{\operatorname{Im} f(z_\lambda)} \right| \leq \left| \frac{f'(z_0) h_0}{\operatorname{Im} f(z_0)} \right| \vee \left| \frac{f'(z_1) h_1}{\operatorname{Im} f(z_1)} \right|.$$

By choosing:

$$h_v = \left| \frac{\operatorname{Im} f(z_v)}{f'(z_v)} \right|, \quad v=0, 1,$$

the resulting inequality states that the function $|\operatorname{Im} f(z)|/|f'(z)|$ is concave.

Now recall that the Laplace-Beltrami operator Δ_B in the hyperbolic B equals:

$$\Delta_B = \left| \frac{\operatorname{Im} f(z)}{f'(z)} \right|^2 \Delta.$$

Consequently, if $A \in \mathcal{U}_\infty(\mathbb{C})$ and $\bar{A} \subseteq B$, Theorem 3.1 applies to the 1-equilibrium potential equation:

$$\begin{cases} \Delta_B u - u = 0 & \text{in } B \setminus \bar{A}, \\ u|_{\bar{A}} = 1, \end{cases}$$

and we conclude that u is quasi-concave. Moreover, if u_z denotes the 1-equilibrium potential of $B(z; r)$, then the map $(z, \zeta) \mapsto u_z(\zeta)$ is quasi-concave too. $\square$

4. Reduction of Theorem 3.1 to finite dimension

To begin with, we list a series of Lemmas, which are all well-known and easy to prove.

LEMMA 4.1. — Suppose F_n , $n \in \mathbb{N}$, are closed and $F_n \downarrow F$. Then $\mathbb{P}(\tau_{F_n} \downarrow \tau_F) = 1$ on $F^r \cup F^c$.

Here $F^r = \{\mathbb{P}(\tau_F = 0) = 1\}$ is the set of all regular points for F . Recall that $\mathbb{P}(\tau_F = 0)$ vanishes on $(F^r)^c$ by Blumenthal's zero-one law (see e. g. Port and Stone [22]).

LEMMA 4.2. — If $A \in \mathcal{U}(E)$, then $A^r = \bar{A}^r = \bar{A}$. In addition, $\tau_A = \tau_{\bar{A}}$ a. s. $\mathbb{P}$.

The reader should note that the last part of Lemma 4.2 depends on the strong Markov property of X . The next Lemma is a consequence of continuity of paths only.

LEMMA 4.3. — Let F_n , $n \in \mathbb{N}$, be closed and $F_n \downarrow F$. If $B_n \in \mathcal{B}(E)$, $n \in \mathbb{N}$, and $B_n \downarrow B$, then:

$$\{\tau_{B_n^c} \geq \tau_{F_n} < +\infty\} \downarrow \{\tau_{B^c} \geq \tau_F < +\infty\}, \quad \text{a. s. } \mathbb{P}.((\quad) \cap \{\tau_{B^c} < +\infty\}),$$

on $F^r \cup F^c$.

Here $\mathcal{B}(E)$ denotes the Borel field in E .

LEMMA 4.4. — Suppose $0 < \lambda < 1$:

(a) If $A_0, A_1 \in \mathcal{U}_\infty(E)$ and A_λ is contained in an open affine half-space H , then there exist open affine half-spaces H_0, H_1 , satisfying $H \supseteq H_\lambda$, $H_0 \supseteq A_0$, and $H_1 \supseteq A_1$.

(b) Let $B_i \in \mathcal{U}_\infty(E)$ and suppose $f_i : B_i \rightarrow [0, +\infty[$, $i=0, 1$, are concave. If ζ is a continuous affine function on E and $\zeta|_{B_\lambda} \geq f_0 \wedge \lambda \wedge f_1$, then there exist continuous affine functions ζ_0, ζ_1 on E satisfying $\zeta \geq \zeta_0 \wedge \lambda \wedge \zeta_1$, $\zeta_0|_{B_0} \geq f_0$, and $\zeta_1|_{B_1} \geq f_1$.

LEMMA 4.5:

(a) $\bar{A}_0 + \bar{A}_1 \subseteq \overline{A_0 + A_1}$, $A_0, A_1 \subseteq E$;

(b) If $A_n, A \in \mathcal{U}(E)$, $n \in \mathbb{N}$, and $A_n \downarrow A$, then $\bar{A}_n \downarrow \bar{A}$.

Proof of Theorem 3.1, $\dim E < +\infty \Rightarrow$ Theorem 3.1. In view of (2.1) it is enough to establish the following inequality:

$$M(\lambda) \geq M(0) \wedge M(1),$$

where $0 < \lambda < 1$ is fixed. Furthermore, we may assume $B_0, B_1 \in \mathcal{U}_\infty(E)$.

Let $j=0, 1$, or λ and set $f_j = V_j^{-1/2}$. By monotone convergence, there is no loss of generality if we only treat the case when the f_j are finite-valued. Suppose:

$$f_j = \inf_{n \in \mathbb{N}} \zeta_{jn}|_{B_j}$$

and $\zeta_{\lambda n} \geq \zeta_{0n} \wedge \lambda \wedge \zeta_{1n}$, where the ζ_{jn} are finite infimums of continuous affine functions on E . This construction is possible due to Lemma 4.4. By the same Lemma here exist open polyhedrons C_{jn} , $n \in \mathbb{N}$, $C=A, B$, satisfying:

$$C_{jn} \downarrow C_j \quad \text{as } n \rightarrow +\infty$$

and $C_{\lambda n} \supseteq (1-\lambda)C_{0n} + \lambda C_{1n}$.

We now introduce:

$$f_{jn} = \inf_{0 \leq k \leq n} \zeta_{jk}|_{B_{jn}} \cap \{\zeta_{j0} > 0, \dots, \zeta_{jn} > 0\}$$

and:

$$M_n(j) = \mathbb{E}_{x_j} \left(\exp \left(- \int_0^{\tau_{A_{jn}}} f_{jn}^{-2}(X(s)) ds \right); \tau_{B_{jn}^c} \geq \tau_{A_{jn}} < +\infty \right).$$

Granted the validity of Theorem 3.1 in the finite-dimensional case, we have:

$$M_n(\lambda) \geq M_n(0) \wedge M_n(1)$$

and (4.1) follows from Lemmas 4.1-4.4 and monotone convergence. $\square$

5. Proof of Theorem 3.1, $\dim E < +\infty$

In the following lemma, the V_j , $j=0, 1$, or λ , are as in Theorem 3.1.

LEMMA 5.1. — If $J(r)=r$, $r>0$, then:

$$J^3 \otimes V_\lambda \leq (J^3 \otimes V_0) \lfloor \lambda \rfloor (J^3 \otimes V_1), \quad 0 < \lambda < 1.$$

Proof. — By the Hölder inequality the function $(J \otimes 1)^3 / (1 \otimes J)^2$ is convex and the result follows at once. $\square$

LEMMA 5.2. — Suppose $A, B \in \mathcal{U}_\infty(\mathbb{R}^n)$ and $0 \in \bar{A} \subseteq B$. Let $f: B \rightarrow]0, \infty[$ be $\mathcal{C}^\infty$ and concave and set $V=f^{-2}$. Then the solution of the Dirichlet problem:

$$\begin{cases} \Delta u - V u = 0 & \text{in } B \setminus \bar{A}, \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial B, \quad u \in \mathcal{C}(\bar{B}), \end{cases}$$

has a non-vanishing gradient in $B \setminus \bar{A}$.

Proof. — The solution u is $\mathcal{C}^\infty$ (see e. g. Gilbarg and Trudinger [17], Theorem 6.17).

We first prove that the function $v(x) = x; \nabla u(x)$, $x \in B \setminus \bar{A}$, is non-positive.

To see this, let $\alpha > 1$ satisfy $\alpha \bar{A} \subseteq B$ and note that:

$$\Delta[u(x/\alpha)] - \alpha^{-2} V(x/\alpha) u(x/\alpha) = 0 \quad \text{in } B \setminus \alpha \bar{A}.$$

Moreover, as:

$$f(x/\alpha) \geq \alpha^{-1} f(x) + (1 - \alpha^{-1}) f(0) \quad \text{in } B,$$

we have $\alpha f(x/\alpha) \geq f(x)$, $x \in B$, and hence:

$$\Delta[u(x/\alpha)] - V(x) u(x/\alpha) \leq 0 \quad \text{in } B \setminus \alpha \bar{A}.$$

Thus:

$$\Delta[u(x) - u(x/\alpha)] - V(x)[u(x) - u(x/\alpha)] \geq 0 \quad \text{in } B \setminus \bar{A}$$

and as $(u - u(\cdot/\alpha))|_{\partial(B \setminus \alpha \bar{A})} \leq 0$, the maximum principle ([17], cor. 3.2) gives

$$(u - u(\cdot/\alpha))|_{B \setminus \alpha \bar{A}} \leq 0.$$

But then $v \leq 0$.

In the next step we show that v is strictly negative.

A computation yields:

$$\Delta v = x; \nabla(\Delta u) + 2 \Delta u = x; \nabla(Vu) + 2 Vu = (x; \nabla V)u + V(x; \nabla u) + 2 Vu,$$

that is:

$$\Delta v - Vu = (2V + x; \nabla V)u.$$

But:

$$2V + x; \nabla V = \frac{2}{f^3} (f - x; \nabla f) \geq \frac{2}{f^3} f(0) > 0$$

and so $\Delta v - Vu > 0$. Since $v \leq 0$, the strong maximum principle ([17], Th. 35) gives $v < 0$ and accordingly $v \neq 0$ in $B \setminus \bar{A}$. $\square$

The main points in the proof which follows are due to Gabriel ([15], [16]). The Brunn-Minkowski aspect was added for the first time in [4]. The Gabriel differential method also applies to certain time-dependent [5] and non-linear (Lewis [19]) problems.

Proof of Theorem 3.1, $\dim E < +\infty$. — There is no loss of generality in assuming:

- (i) X is the usual Brownian motion in $\mathbb{R}^n$, $n \geq 1$;
- (ii) $0 \in \bar{A}_0 \cap \bar{A}_1$, $B_0, B_1 \in \mathcal{U}_\infty(\mathbb{R}^n)$;
- (iii) the functions $f_i = V_i^{-1/2}$ have concave $\mathcal{C}^\infty$ extensions $\tilde{f}_i: B_i + B(0; \delta) \rightarrow]0, +\infty[$, $i=0,1$ ($\delta > 0$ fixed) and from (iii) and Lemma 4.3;
- (iv) $\bar{A}_i \subseteq B_i$, $i=0,1$.

Next let $0 < \lambda < 1$ be fixed. Moreover, suppose:

$\tilde{V}_\lambda: B_\lambda \rightarrow]0, +\infty[$ is $-1/2$ -concave and $\mathcal{C}^\infty$ and $\tilde{V}_\lambda \leq V_\lambda$.

Set:

$$u_i(x) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_{A_i}} \tilde{V}_i(X(s)) ds \right); \tau_{B_i^c} \geq \tau_{A_i} \right), \quad x \in \bar{B}_i, \quad i=0,1,$$

and:

$$u_\lambda(x) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_{A_\lambda}} \tilde{V}_\lambda(X(s)) ds \right); \tau_{B_\lambda^c} \geq \tau_{A_\lambda} \right), \quad x \in \bar{B}_\lambda.$$

It now only remains to prove that:

$$u_\lambda(x_\lambda) \geq u_0(x_0) \wedge u_1(x_1), \quad x_0 \in \bar{B}_0, \quad x_1 \in \bar{B}_1.$$

Let $u_\lambda^*(x_\lambda) = \sup \{ u_0(x_0) \wedge u_1(x_1); x_0 \in \bar{B}_0, x_1 \in \bar{B}_1 \}$. If $\neg(u_\lambda^* \leq u_\lambda)$, then:

$$\sup(u_\lambda^* - u_\lambda) = u_\lambda^*(\hat{x}_\lambda) - u_\lambda(\hat{x}_\lambda) > 0,$$

for a suitable $\hat{x}_\lambda \in \bar{B}_\lambda$. Suppose $u_\lambda^*(\hat{x}_\lambda) = u_0(\hat{x}_0) \wedge u_1(\hat{x}_1)$, where $\hat{x}_\lambda = (1-\lambda)\hat{x}_0 + \lambda\hat{x}_1$. Certainly, $(\hat{x}_0, \hat{x}_1) \in (B_0 \times B_1) \setminus (\bar{A}_0 \times \bar{A}_1)$. Also it is easy to see that the relation $\hat{x}_0 \notin A_0$,

$\hat{x}_1 \in A_1$ is contradictory. Indeed, arbitrarily close to $\hat{x}_0$ there are points where u_0 exceeds $u_0(\hat{x}_0)$, by the maximum principle. Thus, by symmetry, $(\hat{x}_0, \hat{x}_1) \in (B_0 \setminus \bar{A}_0) \times (B_1 \setminus \bar{A}_1)$.

In the following, let $i=0$ or 1 and $j=0, 1$, or λ .

Suppose $h \in \mathbb{R}^n$ and $h; \nabla u_i(\hat{x}_i) > 0$ (i fixed). Then, if $s > 0$ is small, $u_i(\hat{x}_i + sh) > u_i(\hat{x}_i)$ and, hence, $u_\lambda^*(\hat{x}_\lambda + s\lambda_i h) > u_\lambda^*(\hat{x}_\lambda)$, where $\lambda_i = (2i-1)\lambda + 1 - i$, so that $u_\lambda(\hat{x}_\lambda + s\lambda_i h) \geq u_\lambda(\hat{x}_\lambda)$. Accordingly, $h; \nabla u_\lambda(\hat{x}_\lambda) \geq 0$ and it follows that the non-zero vectors $\nabla u_i(\hat{x}_i)$ and $\nabla u_\lambda(\hat{x}_\lambda)$ are parallel. Let $a_j = |\nabla u_j(\hat{x}_j)|$ and $v = \nabla u_j(\hat{x}_j)/a_j$.

From now on we assume that $u_\lambda^*(\hat{x}_\lambda) = u_0(\hat{x}_0)$. The case $u_\lambda^*(\hat{x}_\lambda) = u_1(\hat{x}_1)$ may be treated in a similar way.

Let $h \in \mathbb{R}^n$ be such that $\kappa = h; v \neq 0$. For each s close to 0 there exists a unique $r = r(s)$, with $|r|$ minimal, satisfying the equation :

$$u_0(\hat{x}_0 + sh/a_0) - u_0(\hat{x}_0) = u_1(\hat{x}_1 + rh/a_1) - u_1(\hat{x}_1).$$

Writing:

$$\hat{x}_\lambda(s) = (1-\lambda)(\hat{x}_0 + sh/a_0) + \lambda(\hat{x}_1 + r(s)h/a_1) = \hat{x}_\lambda + [(1-\lambda)s/a_0 + \lambda r(s)/a_1]h,$$

we have:

$$u_0(\hat{x}_0 + sh/a_0) - u_\lambda(\hat{x}_\lambda(s)) \leq u_\lambda^*(\hat{x}_\lambda(s)) - u_\lambda(\hat{x}_\lambda(s)) \leq u_0(\hat{x}_0) - u_\lambda(\hat{x}_\lambda)$$

and, in particular:

$$D_s^k(u_0(\hat{x}_0 + sh/a_0) - u_\lambda(\hat{x}_\lambda(s)))|_{s=0} = \begin{cases} 0, & k=1 \\ \leq 0, & k=2. \end{cases}$$

Next suppose:

$$u_j(\hat{x}_j + sh/a_j) = u_j(\hat{x}_j) + \kappa s + b_j s^2 + o(s^2) \quad \text{as } s \rightarrow 0.$$

Then:

$$r(s) = s + \kappa^{-1}(b_0 - b_1)s^2 + o(s^2) \quad \text{as } s \rightarrow 0$$

and introducing $p = (1-\lambda)/a_0 + \lambda/a_1$ we have:

$$\begin{cases} a_\lambda p = 1, \\ \left(1 - \lambda \frac{a_\lambda}{a_1}\right)b_0 + \lambda \frac{a_\lambda}{a_1}b_1 - b_\lambda \leq 0. \end{cases}$$

Thus:

$$\sum_{1 \leq \alpha, \beta \leq n} \left[\frac{1-\lambda}{a_0^3} D_{\alpha\beta} u_0(\hat{x}_0) + \frac{\lambda}{a_1^3} D_{\alpha\beta} u_1(\hat{x}_1) - \frac{1}{a_\lambda^3} D_{\alpha\beta} u_\lambda(\hat{x}_\lambda) \right] h_\alpha h_\beta \leq 0$$

and, accordingly:

$$\frac{1-\lambda}{a_0^3} V_0(\hat{x}_0) u_0(\hat{x}_0) + \frac{\lambda}{a_1^3} V_1(\hat{x}_1) u_1(\hat{x}_1) \leq p^3 \tilde{V}_\lambda(\hat{x}_\lambda) u_\lambda(\hat{x}_\lambda).$$

Finally, noting that $u_\lambda(\hat{x}_\lambda) < u_0(\hat{x}_0) \wedge u_1(\hat{x}_1)$ we get:

$$\frac{1-\lambda}{a_0^3} V_0(\hat{x}_0) + \frac{\lambda}{a_1^3} V_1(\hat{x}_1) < p^3 V_\lambda(\hat{x}_\lambda),$$

which contradicts Lemma 5.1. Hence $u_\lambda^* \leq u_\lambda$. $\square$

6. Quasi-concavity of V-harmonic measures restricted to supporting hyperplanes

We first recall some known properties of quasi-concave measures on Banach spaces. All the results may be found in the author's papers [6] and [8].

A non-negative finite Borel measure μ on E is quasi-concave if:

$$(6.1) \quad \mu(A_\lambda) \geq \mu(A_0) \wedge \mu(A_1),$$

for all $0 < \lambda < 1$ and all $A_0, A_1 \in \mathcal{B}(E)$ = the Borel field in E . It turns out that a non-negative finite Borel measure μ on E is quasi-concave if (6.1) holds for all $0 < \lambda < 1$ and all $A_0, A_1 \in \mathcal{U}(E)$.

Next suppose $0 < \lambda < 1$ is fixed and suppose $\mu_0, \mu_1, \mu_\lambda$ are quasi-concave measures on E . If:

$$(6.2) \quad \mu_\lambda(A_\lambda) \geq \mu_0(A_0) \wedge \mu_1(A_1),$$

for all $A_0, A_1 \in \mathcal{U}(E)$, then (6.2) is true for all $A_0, A_1 \in \mathcal{B}(E)$. Moreover, if $E = \mathbb{R}^n$ and $d\mu_j = f_j dx$, $j = 0, 1, \lambda$, where the $f_j: E \rightarrow [0, +\infty]$ are semi-continuous from below, then (6.2) holds for all Borel sets A_0, A_1 in $\mathbb{R}^n$ if and only if:

$$f_\lambda^{-1/n}(x_\lambda) \leq (1-\lambda) f_0^{-1/n}(x_0) + \lambda f_1^{-1/n}(x_1), \quad x_0, x_1 \in \mathbb{R}^n.$$

The above makes it possible to pass from convex bodies to Borel sets in a very special but still interesting case of Theorem 3.1.

THEOREM 6.1. — *Let $B \in \mathcal{U}(E)$ and suppose F is a supporting hyperplane ($0 \in F$) of $\bar{B}$. If $V: B \rightarrow [0, +\infty[$ is $-1/2$ -concave, then the V-harmonic measure:*

$$\kappa_x(A) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_{B^c}} V(X(s)) ds \right); X(\tau_{B^c}) \in A \right), \quad A \in \mathcal{B}(B^c),$$

at $x \in B$ satisfies:

$$\kappa_{x_\lambda}(A_\lambda) \geq \kappa_{x_0}(A_0) \wedge \kappa_{x_1}(A_1), \quad 0 < \lambda < 1, \quad A_0, A_1 \in \mathcal{B}(F).$$

In particular, $\kappa_x|_{\mathcal{B}(F)}$ is quasi-concave.

Proof. — First note that for any closed $A \subseteq B^c$:

$$\kappa_x(A) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_A} V(X(s)) ds \right); \tau_{B^c} \geq \tau_A < +\infty \right),$$

because $x \in B$ is non-regular for B^c . Hence the inequality we shall prove is true for all $A_0, A_1 \in \mathcal{U}(F)$ and Theorem 6.1 follows from what we said above. $\square$

Example 6.1. — Let G be a Borel measurable additive subgroup of F , where we abide by the various assumptions in Theorem 6.1. Then $\kappa_x(G)$ or $\kappa_x(F \setminus G) = 0$ from the zero-one law of quasi-concave measures [6]. A direct proof of this fact is rather simple but we do not know any proof independent of the zero-one law of quasi-concave measures. $\square$

Example 6.2. — Let $E = \mathbb{R}^n$ but otherwise assume the same conditions as in Theorem 6.1.

If $\bar{B} \cap F = C$ is $(n-1)$ -dimensional, then an appropriate version of the restricted Poisson kernel $(d\kappa_x/d\sigma_{\partial B})(y)$, $(x, y) \in B \times C$, is $-1/(n-1)$ -convex. $\square$

Example 6.3. — If $C_0, C_1 \in \mathcal{U}_\infty(\mathbb{R}^n)$, then the original Brunn-Minkowski inequality states that:

$$(6.3) \quad |C_0 + C_1|^{1/n} \geq |C_0|^{1/n} + |C_1|^{1/n}.$$

To deduce this estimate from (1.1) we let $B_0 = B_1 = \{x_{n+1} > 0\} \subseteq \mathbb{R}^{n+1}$, $x = x_0 = x_1 = (\alpha, \dots, \alpha, 1)$, and get:

$$|\alpha|^{n+1} \int_{C_\lambda} \frac{dy}{\|x-y\|^{n+1}} \geq |\alpha|^{n+1} \left(\int_{C_0} \frac{dy}{\|x-y\|^{n+1}} \wedge \int_{C_1} \frac{dy}{\|x-y\|^{n+1}} \right), \quad 0 < \lambda < 1.$$

As $|\alpha| \rightarrow +\infty$, we obtain $|C_\lambda| \geq |C_0| \wedge |C_1|$ or, due to homogeneity, (6.3). In fact, already Minkowski's ideas entail (6.3) for arbitrary Borel sets but the Gabriel differential method seems to collapse beyond star-shaped bodies. $\square$

7. Quasi-concavity of V-Newtonian potentials of very thin bodies

Consider, for $\dim E \geq 3$, the Newtonian potential of $A \in \mathcal{B}(E)$:

$$v_*(A) = \mathbb{E} \left(\int_0^\infty 1_A(X(t)) dt \right),$$

that is, the expected amount of time the Brownian motion spends in A . If $x \in E$ is fixed, the measure v_x is not quasi-concave, although, by ([8], Th. 5.1):

$$v_{x_0+x_1}(A_0 + A_1) \geq v_{x_0}(A_0) \wedge v_{x_1}(A_1),$$

or, stated otherwise:

$$v_{x_{1/2}}(A_{1/2}) \geq \frac{1}{4} [v_{x_0}(A_0) \wedge v_{x_1}(A_1)],$$

for all $x_0, x_1 \in E$ and all $A_0, A_1 \in \mathcal{B}(E)$. The convexity behaviour of $v_*(A)$, with $A \in \mathcal{U}(E)$ fixed, is unknown to us.

The main questions we focus on in this section have no direct meaning without restriction on $\dim E$. We therefore assume throughout that $E = \mathbb{R}^n$, $n \geq 2$.

Now suppose $B \in \mathcal{U}(\mathbb{R}^n)$ and that $V: B \rightarrow [0, +\infty[$ is $-1/2$ -concave. Moreover, we suppose $B \neq \mathbb{R}^2$ if $n=2$ and $V=0$ so that B becomes a Greenian domain for the operator $-1/2 \Delta + V$ with the Dirichlet boundary condition zero. Let:

$$v_x(A) = \mathbb{E}_x \left(\int_0^{\tau_B^c} 1_A(X(t)) \exp \left(- \int_0^t V(X(s)) ds \right) dt \right) = \int_A g(x, y) dy, \quad x \in B,$$

be the V -Newtonian potential of $A \in \mathcal{B}(B)$, g being the corresponding Green function. The reader should note that $g: B \times B \rightarrow [0, +\infty]$ is continuous (see e. g. [13], Chap. 13). In particular, given a k -dimensional affine manifold F in $\mathbb{R}^n$ possessing Lebesgue measure $m^F(m^{\{a\}} = \delta_a)$, the V -Newtonian potential of any $A \in \mathcal{B}(F \cap B)$, viz:

$$v_x^F(A) = \int_A g(x, y) dm^F(y), \quad x \in B,$$

becomes well-defined.

THEOREM 7.1. — *If $\dim F = n-2$, then:*

$$v_{x_\lambda}^{c_\lambda + F}(A_\lambda) \geq v_{x_0}^{c_0 + F}(A_0) \wedge v_{x_1}^{c_1 + F}(A_1), \quad 0 < \lambda < 1,$$

where $A_i \in \mathcal{B}((c_i + F) \cap B)$, $c_i \in \mathbb{R}^n$, and $x_i \in B$, $i=0, 1$, are arbitrary.

Before presenting the proof of Theorem 7.1, we recall some basic facts from potential theory.

Suppose $A \in \mathcal{U}_\infty(\mathbb{R}^n)$ and $\bar{A} \subseteq B$. Then there exists a unique non-negative measure μ_A in $\bar{A}$, called the V -equilibrium measure of A , such that:

$$\int g(x, y) d\mu_A(y) = \mathbb{E}_x \left(\exp \left(- \int_0^{\tau_A} V(X(s)) ds \right); \tau_B^c \geq \tau_A < +\infty \right), \quad x \in B.$$

The total mass $\mu_A(\bar{A}) = \mathcal{C}(A)$ is termed the V -capacity of A and, moreover, writing $g\mu = (\mu(g(x, \cdot)))_{x \in B}$ if μ is a non-negative measure in B :

$$\mathcal{C}(A) = \sup \{ \mu(B); \text{supp } \mu \subseteq A, g\mu \leq 1 \}$$

(see e. g. Blumental, Gettoor [3], Chap. 6.4).

Proof of Theorem 7.1. — We shall prove that g is $-1/(n-2)$ -convex. By eventually diminishing V and using the Dini theorem, there is no loss of generality in assuming $\sup V = q < +\infty$.

In the following, we sometimes write $g^{B,V}$, $\mathcal{C}^{B,V}$, $\mu_A^{B,V}$ instead of g , $\mathcal{C}$, and μ_A , respectively, and assume, as we may, that X is the standard Brownian motion.

Case $n \geq 3$. — Letting $\mathcal{C}^{\mathbb{R}^n}(\mathbf{B}(0; r)) = c_n r^{n-2}$, we claim that:

$$(7.1) \quad \lim_{r \rightarrow 0^+} \frac{\mathcal{C}^{\mathbf{B}, \mathbf{V}}(\mathbf{B}(y; r))}{c_n r^{n-2}} = 1, \quad y \in \mathbf{B}.$$

To see this, let $y \in \mathbf{B}$ be fixed and write $\mathbf{B}_r = \mathbf{B}(y; r)$ for brevity. Then, if $\mathbf{B}_r \subseteq \bar{\mathbf{B}}_r \subseteq \mathbf{B}$, certainly:

$$c_n r^{n-2} \leq \mathcal{C}^{\mathbf{B}, \mathbf{V}}(\mathbf{B}_r) \leq \mathcal{C}^{\mathbf{B}_r, q}(\mathbf{B}_r).$$

We next integrate:

$$g^{\mathbf{B}_r, 0} = g^{\mathbf{B}_r, q} + q g^{\mathbf{B}_r, 0} g^{\mathbf{B}_r, q},$$

with respect to $\mu_{\mathbf{B}_r}^{\mathbf{B}_r, 0} \otimes \mu_{\mathbf{B}_r}^{\mathbf{B}_r, q}$, arriving at:

$$\mathcal{C}^{\mathbf{B}_r, q}(\mathbf{B}_r) \leq \mathcal{C}^{\mathbf{B}_r, 0}(\mathbf{B}_r) + q d_n R^n,$$

where $d_n = \text{Vol } \mathbf{B}(0; 1)$. Moreover, by integrating:

$$g^{\mathbf{B}_r, 0}(x, \xi) = g^{\mathbb{R}^n, 0}(x, \xi) - \mathbb{E}_x g^{\mathbb{R}^n, 0}(X(\tau_{\mathbf{B}_r^c}), \xi)$$

with respect to $\mu_{\mathbf{B}_r}^{\mathbf{B}_r, 0}(dx) \otimes \mu_{\mathbf{B}_r}^{\mathbb{R}^n, 0}(d\xi)$, we get:

$$\mathcal{C}^{\mathbf{B}_r, 0}(\mathbf{B}_r) = c_n r^{n-2} (1 - (r/R)^{n-2})^{-1}.$$

Finally, by choosing $R = r^{1-1/n}$ in the above estimates (7.1) follows at once.

Writing $g = g^{\mathbf{B}, \mathbf{V}}$ as above we have for all $r_0, r_1 > 0$, $0 < \lambda < 1$, and $\varepsilon > 0$:

$$\varepsilon^{2-n} (g \mu_{\mathbf{B}(y_\lambda; \varepsilon r_\lambda)})(x_\lambda) \geq \varepsilon^{2-n} [(g \mu_{\mathbf{B}(y_0; \varepsilon r_0)})(x_0) \wedge (g \mu_{\mathbf{B}(y_1; \varepsilon r_1)})(x_1)],$$

by Theorem 3.1, and in the limit as $\varepsilon \rightarrow 0^+$:

$$g(x_\lambda, y_\lambda) r_\lambda^{n-2} \geq g(x_0, y_0) r_0^{n-2} \wedge g(x_1, y_1) r_1^{n-2}.$$

Thus, choosing $r_i = (g(x_i, y_i))^{-1/(n-2)}$, if $x_i \neq y_i$, $i=0,1$, the resulting inequality becomes:

$$g^{-1/(n-2)}(x_\lambda, y_\lambda) \leq (1-\lambda) g^{-1/(n-2)}(x_0, y_0) + \lambda g^{-1/(n-2)}(x_1, y_1),$$

and it follows at once that g is $-1/(n-2)$ -convex.

Case $n=2$. — If Theorem 7.1 is true in $\mathbb{R}^{n_0+1}$, $n_0 \geq 2$, then we may use the theory of α -convex measures to prove Theorem 7.1 in $\mathbb{R}^{n_0}$. Indeed, set $\tilde{\mathbf{V}}(x, \xi) = \mathbf{V}(x)$, $(x, \xi) \in \mathbf{B} \times \mathbb{R}$ and note that:

$$g(x, y) = \int_{-\infty}^{\infty} g^{\mathbf{B} \times \mathbb{R}, \tilde{\mathbf{V}}}(x, 0, y, \eta) d\eta.$$

If $g^{\mathbf{B} \times \mathbb{R}, \tilde{\mathbf{V}}}$ is $-1/(n_0-1)$ -convex it follows from ([8], Th. 3.1) that g is $-1/(n_0-2)$ -convex. $\square$

In the following two examples we suppose in addition to the above assumptions that ∂B is $\mathcal{C}^\infty$ and that V has a $\mathcal{C}^\infty$ extension to a neighbourhood of $\bar{B}$.

Example 7.1. — For each $y \in \partial B$, let $n_i(y) = n_i^B(y)$ denote the inner unit normal of $\bar{B}$ at y and set:

$$p(x, y) = \lim_{\varepsilon \rightarrow 0^+} g(x, y + \varepsilon n_i(y)) / 2\varepsilon.$$

If $n_i(y_0) = n_i(y_1)$, then $n_i(y_\lambda) = n_i(y_0)$, $0 < \lambda < 1$, and the $-1/(n-2)$ -convexity of g gives:

$$p^{-1/(n-1)}(x_\lambda, y_\lambda) \leq (1-\lambda)p^{-1/(n-1)}(x_0, y_0) + \lambda p^{-1/(n-1)}(x_1, y_1),$$

employing the same type of argument as in the proof of Theorem 7.1. Noting that $p(x, y) d\sigma_{\partial B}(y)$ is the V -harmonic measure at x (use ([17], Th. 6.14) and the Green formula) we have thus complemented Example 6.2. $\square$

Example 7.2. — Let $A \in \mathcal{U}_\infty(\mathbb{R}^n)$, $\bar{A} \subseteq B$, and assume $\partial A \in \mathcal{C}^\infty$. Moreover, suppose F is a supporting hyperplane of $\bar{A}$ such that $\bar{A} \cap F = C$ is $(n-1)$ -dimensional. Then:

$$d\mu_{A|F(C)} = f d\sigma_C,$$

where f is -1 -concave.

To see this, we apply the Green formula once more to get:

$$-\frac{1}{2} \frac{\partial u_A}{\partial n_e} d\sigma_{\partial A} = d\mu_A - 1_A \vee dm,$$

where m is Lebesgue measure, $u_A = g \mu_A$, and $n_e = -n_i^A$. However, as u_A is quasi-concave $-\partial u_A / \partial n_e$ is -1 -concave on C . $\square$

In the planar case, we shall complement Theorem 7.1 in the following way.

THEOREM 7.2. — Let for $A \in \bar{\mathcal{U}}_\infty(\mathbb{C})$, $g_A \leq 0$ be the Green function of Δ in $\mathbb{C} \setminus A$ with pole at ∞ and with the Dirichlet boundary condition zero. Then:

$$g_{A_\lambda}(z_\lambda) \geq g_{A_0}(z_0) \wedge g_{A_1}(z_1), \quad 0 < \lambda < 1.$$

Proof. — Assuming $0 \in A$, $g = g_A$ possesses the following characteristic properties:

- (i) g is harmonic in $\mathbb{C} \setminus A$;
- (ii) g is continuous in $\mathbb{C}$ and $g|_A = 0$,
- (iii) $g(z) = 1/n \frac{1}{|z|} - 1/n \frac{1}{\mathcal{C}_2(A)} + \mathcal{O}\left(\frac{1}{|z|}\right)$ as $|z| \rightarrow +\infty$.

The constant $\mathcal{C}_2(A)$ is the logarithmic capacity of A [1]. If $B(0; R) \supseteq A$ and $u_A^{B(0; R)}$ denotes the equilibrium potential of A relative to $B(0; R)$ we thus have:

$$(u_A^{B(0; R)}(z) - 1/n \frac{R}{\mathcal{C}_2(A)} - g_A(z)) = \mathcal{O}\left(\frac{1}{R}\right) \text{ as } R \rightarrow +\infty$$

and, consequently:

$$g_A(z) = \lim_{R \rightarrow +\infty} (u_A^{B(0; R)}(z) - 1) \frac{1}{n} \frac{R}{\mathcal{C}_2(A)}.$$

From this representation formula Theorem 7.2 follows at once using Theorem 3.1. $\square$

Example 7.3. — $\mathcal{C}_2$ is concave on $\bar{\mathcal{U}}_\infty(\mathbb{C})$:

$$(7.2) \quad \mathcal{C}_2(A_0 + A_1) \geq \mathcal{C}_2(A_0) + \mathcal{C}_2(A_1), \quad A_0, A_1 \in \bar{\mathcal{U}}_\infty(\mathbb{C}).$$

Indeed, as:

$$1/n \mathcal{C}_2(A) = \lim_{|z| \rightarrow +\infty} (g_A(z) + 1/n |z|).$$

Theorem 7.2 gives:

$$\mathcal{C}_2(A_i) \geq \mathcal{C}_2(A_0) \wedge \mathcal{C}_2(A_1),$$

and (7.2) follows by homogeneity. $\square$

The next example is mainly a preparation for Example 7.5.

Example 7.4. — By an exercise in Pólya and Szegő [21], Aufg. [124] :

$$(7.3) \quad \mathcal{C}_2(A) \leq \frac{1}{2\pi} \text{length } \partial A, \quad A \in \bar{\mathcal{U}}_\infty(\mathbb{C}).$$

A possible solution reads as follows.

Let H_A be the support function of A :

$$H_A(\xi) = \sup_{x \in A} \langle x, \xi \rangle, \quad \xi \in \mathbb{C},$$

and remember that:

$$(7.4) \quad \int_0^{2\pi} H_A(e^{i\theta} \xi) d\theta / 2\pi = \frac{\|\xi\|}{2\pi} \text{length } \partial A, \quad \xi \in \mathbb{C}.$$

We next approximate the average in the left-hand side by:

$$\sum_{k=1}^p H_A(e^{i\theta_k} \xi) \lambda_k \quad (0 < \lambda_k < 1, \lambda_1 + \dots + \lambda_p = 1),$$

that is, by the support function of $\sum_{k=1}^p \lambda_k e^{-i\theta_k} A$. However,

$$\mathcal{C}_2\left(\sum_{k=1}^p \lambda_k e^{-i\theta_k} A\right) \geq \mathcal{C}_2(A)$$

from Example 7.3 and as the right-hand side of (7.4) is the support function of a ball of radius $1/2\pi \text{length } \partial A$, we have (7.3). $\square$

Example 7.5. — Consider an $A \in \bar{\mathcal{U}}_\infty(\mathbb{R}^3)$ with principal radius R_1 and R_2 and mean curvature:

$$\mathcal{M}(A) = \frac{1}{2} \int_{\partial A} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) d\sigma(\xi).$$

Then by a Theorem of Szegő [23], Satz III:

$$(7.5) \quad \mathcal{C}_3(A) \leq \frac{1}{4\pi} \mathcal{M}(A),$$

where $\mathcal{C}_3$ is the Newtonian capacity normalized so that $\mathcal{C}_3(B(0; 1)) = 1$. A very important ingredient in Szegő's proof is the following inequality for mixed volumes due to Minkowski:

$$\mathcal{M}^2(A) \geq 4\pi \text{ area } \partial A.$$

Noting that $\mathcal{C}_3$ is concave on $\bar{\mathcal{U}}_\infty(\mathbb{R}^3)$ [4] due to (1.1) we, alternatively, obtain (7.5) as in the previous example. The n -dimensional counterpart of (7.5) is now obvious: if $\mathcal{C}_n$ denotes the Newtonian capacity in $\mathbb{R}^n$ ($n \geq 3$, $\mathcal{C}_n(B(0; 1)) = 1$) and if Z_n is a uniformly distributed random vector on S^{n-1} , then:

$$\mathbb{E} H_A(Z_n) \geq \mathcal{C}_n^{1/(n-2)}(A), \quad A \in \bar{\mathcal{U}}_\infty(\mathbb{R}^n).$$

Certainly, the Szegő line of reasoning leads to the same estimate. $\square$

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