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ON THE CHARACTERS OF EXPONENTIAL SOLVABLE LIE GROUPS

BY NIELS VIGAND PEDERSEN (*)

Introduction

Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. In [6] it was shown that for any normal representation π of G (cf. [11]) there exists a continuous homomorphism $\chi : G \rightarrow \mathbb{R}_+^*$ such that π has a distribution χ -semicharacter. Moreover, it was shown that one can find a semi-invariant element u (with multiplier χ , say) in $U(\mathfrak{g}_{\mathbb{C}})$, the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, such that any normal representation π whose associated orbit of $\mathcal{R}$ in $\mathfrak{g}'$ ([10], [11]) is contained in a certain G -invariant Zariski open subset of $\mathfrak{g}'$, has a distribution χ -semicharacter $f_{\pi, \chi}$ expressible by $f_{\pi, \chi}(\varphi) = \phi(\pi(u * \varphi))$ for $\varphi \in C_c^\infty(G)$, ϕ being the trace on the factor generated by π (here it is understood, in particular, that the right hand side is well defined). In [3] J.-Y. Charbonnel showed that for each normal representation π of G one can find a continuous homomorphism $\chi : G \rightarrow \mathbb{R}_+^*$ and an element $u \in U(\mathfrak{g}_{\mathbb{C}})$ such that π has a distribution χ -semicharacter $f_{\pi, \chi}$ expressible as before: $f_{\pi, \chi}(\varphi) = \phi(\pi(u * \varphi))$ for $\varphi \in C_c^\infty(G)$. Here u is not necessarily semi-invariant; however, $d\pi(u)$ is semi-invariant, i. e.

$$\pi(s)d\pi(u)\pi(s^{-1}) = \chi(s)^{-1}d\pi(u).$$

Suppose now that G is exponential ⁽¹⁾ (and therefore, in particular, of type I, cf. [2]). In this paper we make a construction, depending only on the choice of a Jordan-Hölder sequence for $\mathfrak{g}_{\mathbb{C}}$, of a finite set of polynomial functions $Q_j \geq 0$, $j=1, \dots, n$, on $\mathfrak{g}'$, a finite set of continuous homomorphisms $\chi_j : G \rightarrow \mathbb{R}_+^*$, $j=1, \dots, n$, and a finite set α_j , $j=1, \dots, n$ of positive, G -invariant analytic functions on $\mathfrak{g}$ such that, setting

$$\Omega_j = \{g \in \mathfrak{g}' \mid Q_j(g) \neq 0, Q_k(g) = 0 \text{ for } k < j\}$$

we have:

- 1) Ω_j is G -invariant and $\mathfrak{g}' = \bigcup_{j=1}^n \Omega_j$,
- 2) $Q_j(sg) = \chi_j(s)Q_j(g)$ for $s \in G, g \in \Omega_j$,

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⁽¹⁾ G is said to be exponential if the exponential map $\exp : \mathfrak{g} \rightarrow G$ is diffeomorphism.

3) for any G -orbit O contained in Ω_j the measure $Q_j\beta_O$ is a non-zero, positive, tempered, relatively invariant Radon measure on O with multiplier χ_j (here β_O is the canonical measure on O),

and such that, letting $u_j, j=1, \dots, n$, be the element in $U(\mathfrak{g}_C)$ corresponding via symmetrization to the polynomial function $g \rightarrow Q_j(ig)$ on $\mathfrak{g}'_C$, we have for the irreducible representation π of G associated with the orbit O contained in Ω_j ,

4) the operator $d\pi(u_j)$ is a selfadjoint, positive, invertible operator, semi-invariant under π with multiplier χ_j ,

5) the operator $\pi(u_j * \varphi)$ is traceclass for all $\varphi \in C_c^\infty(G)$,

6) the functional $\varphi \rightarrow \text{Tr}(\pi(u_j * \varphi))$ is a non-zero, χ_j -semi-invariant distribution on G of positive type (a χ_j -distribution semicharacter for π), and

7) for all $\varphi \in C_c^\infty(G)$ we have

$$(*) \quad \text{Tr}(\pi(u_j * \varphi)) = \int_0 (\alpha_j \cdot \varphi \circ \exp)^\wedge(l) Q_j(l) d\beta_O(l),$$

where « $\wedge$ » stands for the ordinary Euclidian Fourier transform.

This construction is carried out in sections 1.1, 1.2 and 1.3, the theorem is formulated in section 1.4, and section 2 is devoted to the proof of the theorem; in section 3 we give a few examples.

We would like to emphasize the following feature of the formula (*) shared by no other previously known character formula for (non-nilpotent) solvable Lie groups: once a Jordan-Hölder basis in $\mathfrak{g}_C$ has been selected, *all* objects in the formula are explicitly constructible (for a given orbit O and associated representation π), i. e. there is no choice (in particular of the weight function α_j , cf. [9], [4], [5], [6], [3]) involved in setting up the formula. This, in particular, opens the possibility of using the formula (*) as a starting point for the pairing between orbits and representations, first established by Bernat ([1]), for exponential groups, and thus extending to these groups Pukanszky's approach to the Kirillov theory of nilpotent groups, [7].

In the special case where $\mathfrak{g}$ is nilpotent $\chi_j \equiv 1$ and $\alpha_j \equiv 1$. Therefore Q_j is invariant on $O \subset \Omega_j$, $d\pi(u_j)$ is a scalar, and the formula (*) then gives that $d\pi(u_j) = Q_j(O)I$ and

$$\text{Tr}(\pi(\varphi)) = \int_0 (\varphi \circ \exp)^\wedge(l) d\beta_O(l),$$

so (*) reduces in particular to the Kirillov character formula.

The main difference between the results obtained in [3] and the results obtained here can be subsumed under the following points: i. We exhibit a *finite* collection of elements $u_j \in U(\mathfrak{g}_C)$ to choose from so as to make a formula like (*) valid, ii. we *construct* such a finite collection explicitly, and iii. here the functions $g \rightarrow Q_j(ig)$ in (*) are (rather surprisingly) the polynomial functions corresponding to the u_j 's via *symmetrization*.

The polynomials Q_j were first considered by Pukanszky in the nilpotent case ([8], [10]). We also use in an essential way the work of Pukanszky on exponential groups ([9]) and the work of Duflo-Rais ([5]). Our methods are very different from those of [3].

We conjecture that our results can be extended to arbitrary connected, simply connected

solvable Lie groups (with the usual condition on the support of the function ϕ appearing in the formula analogous to (*), though; cf. e. g. [6]).

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1. Preliminaries and formulation of Theorem

In sections 1.1, 1.2 and 1.3 we introduce the notation necessary to formulate our Theorem in section 1.4.

1.1. — Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{f}_j, j=0, \dots, m$, be a Jordan-Hölder sequence in $\mathfrak{g}_{\mathbb{C}}$, i. e. a sequence of ideals such that $\mathfrak{f}_j \supset \mathfrak{f}_{j-1}$ and such that $\dim \mathfrak{f}_j = j, j=0, \dots, m$.

Let $\lambda_j : \mathfrak{g} \rightarrow \mathbb{C}$ be the root associated with the irreducible $\mathfrak{g}$ -module $\mathfrak{f}_j/\mathfrak{f}_{j-1}$ (i. e. $\text{ad}X(Z) = \lambda_j(X)Z \pmod{\mathfrak{f}_{j-1}}$ for all $Z \in \mathfrak{f}_j, X \in \mathfrak{g}$), and let $\Lambda_j : G \rightarrow \mathbb{C}^*$ be the continuous homomorphism with $\Lambda_j(\exp X) = e^{\lambda_j(X)}$ for all $X \in \mathfrak{g}$. We have $\text{Ad}(s)Z = \Lambda_j(s)Z \pmod{\mathfrak{f}_{j-1}}$ for all $Z \in \mathfrak{f}_j, s \in G$.

We let G act in $\mathfrak{g}'$ via the coadjoint representation. For $g \in g'$ we have the skewsymmetric bilinearform $B_g : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ given by $B_g(X, Y) = \langle g, [X, Y] \rangle, X, Y \in \mathfrak{g}$. The radical of B_g is equal to the Lie algebra $\mathfrak{g}_g$ of the stabilizer G_g of $g : \mathfrak{g}_g = \{X \in \mathfrak{g} \mid B_g(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$. We let $\hat{B}_g : \mathfrak{g}/\mathfrak{g}_g \times \mathfrak{g}/\mathfrak{g}_g \rightarrow \mathbb{R}$ designate the symplectic form on $\mathfrak{g}/\mathfrak{g}_g$ arising from B_g by factorization. We extend g, B_g , etc. in the natural way to $\mathfrak{g}_{\mathbb{C}}$ whenever convenient.

For $g \in g'$ we set $\mathfrak{f}_j(g) = \mathfrak{f}_j + (\mathfrak{g}_g)_{\mathbb{C}}, j=0, \dots, m$. We then have a sequence of subalgebras:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}_m(g) \supset \mathfrak{f}_{m-1}(g) \supset \dots \supset \mathfrak{f}_1(g) \supset \mathfrak{f}_0(g) = (\mathfrak{g}_g)_{\mathbb{C}},$$

and $\dim \mathfrak{f}_j(g)/\mathfrak{f}_{j-1}(g) = 0$ or $=1$.

For $g \in g'$ we define J_g to be the set $\{1 \leq j \leq m \mid \mathfrak{f}_j(g) \not\supset \mathfrak{f}_{j-1}(g)\}$.

Let $Z_j \in \mathfrak{f}_j \setminus \mathfrak{f}_{j-1}, j=1, \dots, m$. Then $Z_1, \dots, Z_m$ is a basis in $\mathfrak{g}_{\mathbb{C}}$, and we have

$$j \in J_g \Leftrightarrow Z_j \notin \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} = \mathfrak{f}_{j-1}(g).$$

If $g \in g'$ and $J_g = \{j_1 < \dots < j_d\}$ we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}_{j_d}(g) \not\supset \mathfrak{f}_{j_d-1}(g) \not\supset \dots \not\supset \mathfrak{f}_{j_1}(g) \not\supset \mathfrak{f}_0(g) = (\mathfrak{g}_g)_{\mathbb{C}}.$$

In particular $Z_{j_1}, \dots, Z_{j_d}$ is a basis for $\mathfrak{g}_{\mathbb{C}} \pmod{(\mathfrak{g}_g)_{\mathbb{C}}}$, and $d = \dim \mathfrak{g}/\mathfrak{g}_g$.

Set $\mathcal{E} = \{J_g \mid g \in g'\}$, and for $e \in \mathcal{E}$, set $\Omega_e = \{g \in g' \mid J_g = e\}$. Then we have $g' = \bigcup_{e \in \mathcal{E}} \Omega_e$ as a (finite) disjoint union. Since clearly $J_{sg} = J_g$ for $s \in G$, Ω_e is a G -invariant subset of g' .

Let $e \in \mathcal{E}$. If $e \neq \emptyset$ with $e = \{j_1 < \dots < j_d\}$ we define the skewsymmetric $d \times d$ -matrix $M_e(g), g \in g'$, by

$$M_e(g) = [B_g(Z_{j_r}, Z_{j_s})]_{1 \leq r, s \leq d}$$

and let $P_e(g)$ denote the Pfaffian of $M_e(g)$. If $e = \emptyset$ we set $M_e(g) = 1$, and $P_e(g) = 1$. The map $g \rightarrow P_e(g)$ is a complex valued polynomial function on $\mathfrak{g}'$, and $P_e(g)$ depends only on the restriction of g to $[\mathfrak{g}, \mathfrak{g}]$. P_e has the property that $P_e(g)^2 = \det M_e(g)$. We set $Q_e(g) = |\det M_e(g)| = |P_e(g)|^2$. $g \rightarrow Q_e(g)$ is a real valued non-negative polynomial function on $\mathfrak{g}'$.

For $e \in \mathcal{E}$ we set $\Lambda_e = \prod_{j \in e} \Lambda_j$.

LEMMA 1.1.1. — Let $e \in \mathcal{E}$. If $g \in \Omega_e$, then $P_e(g) \neq 0$ and $P_e(sg) = \Lambda_e(s)^{-1} P_e(g)$ for all $s \in G$.

Proof. — Write $e = \{j_1 < \dots < j_d\}$. Since $Z_{j_1}, \dots, Z_{j_d}$ is a basis for $\mathfrak{g}_{\mathbb{C}} \pmod{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}}$ we have that $M_e(g)$ is a regular matrix, hence $P_e(g)^2 = \det M_e(g) \neq 0$.

Now writing

$$\text{Ad}(s^{-1})Z_{j_p} = \sum_{u=1}^d a_{up} Z_{j_u} + c_p,$$

where $c_p \in (\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}$, we have $a_{up} = 0$ for $u > p$ and $a_{pp} = \Lambda_{j_p}(s^{-1})$, and

$$\begin{aligned} B_{sg}(Z_{j_p}, Z_{j_q}) &= \langle sg, [Z_{j_p}, Z_{j_q}] \rangle = \langle g, [\text{Ad}(s^{-1})Z_{j_p}, \text{Ad}(s^{-1})Z_{j_q}] \rangle \\ &= \sum_{u,v=1}^d a_{up} \langle g, [Z_{j_u}, Z_{j_v}] \rangle a_{vq} = ({}^t A M_e(g) A)_{p,q} \end{aligned}$$

where A is the matrix $[a_{pq}]_{| \leq p, q \leq d}$. This shows that $M_e(sg) = {}^t A M_e(g) A$, and since $\det A = \prod_{p=1}^d \Lambda_{j_p}(s^{-1}) = \Lambda_e(s^{-1})$ we find that

$$P_e(sg) = \text{Pf}(M_e(sg)) = \text{Pf}({}^t A M_e(g) A) = (\det A) \text{Pf}(M_e(g)) = \Lambda_e(s^{-1}) P_e(g).$$

This ends the proof of the lemma.

COROLLARY 1.1.2. — If $g \in \Omega_e$, then $Q_e(g) > 0$ and $Q_e(sg) = |\Lambda_e(s)|^{-2} Q_e(g)$ for all $s \in G$.

For $e \in \mathcal{E}$ we set $|e|$ = the number of elements in e . We define a total ordering $<$ on $\mathcal{E}$ in the following way: let $e, e' \in \mathcal{E}$. Then $e < e'$ if and only if either $|e| > |e'|$ or $d = |e| = |e'|$ and, writing $e = \{j_1 < \dots < j_d\}$, $e' = \{j'_1 < \dots < j'_d\}$, $j_p < j'_p$, where $p = \min\{| \leq r \leq d | j_r \neq j'_r \}$.

LEMMA 1.1.3. — $\Omega_e = \{g \in \mathfrak{g}' | Q_{e'}(g) = 0 \text{ for } e' < e \text{ and } Q_e(g) \neq 0\}$.

Proof. — If $g \in \Omega_e$ we saw in Corollary 1.1.2 that $Q_e(g) \neq 0$. If $e' < e$ and $|e'| > |e|$, then, if $e' = \{j'_1 < \dots < j'_c\}$, $Z_{j'_1}, \dots, Z_{j'_c}$ are linearly dependent $\pmod{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}}$, so $M_{e'}(g)$ is singular, and therefore $Q_{e'}(g) = 0$. If $|e| = |e'|$, and $j_1 = j'_1, \dots, j_p = j'_p$, $j'_{p+1} < j_{p+1}$, then $Z_{j'_{p+1}} \in \mathfrak{f}_{j_p} + (\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}$, and therefore $Z_{j'_1}, \dots, Z_{j'_{p+1}}$ are linearly dependent $\pmod{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{C}}}$, and again $Q_{e'}(g) = 0$. This shows the lemma.

Remark 1.1.4. — If $\mathfrak{g}$ is nilpotent our definitions agree with those given by Pukanszky in [10], p. 525 f. f., cf. also [8]. In [6], section 4.2 a study of the completely solvable case was initiated.

1.2. — Recall the following facts: there exists an isomorphism ω (the symmetrization map) between the complex vector space $S(\mathfrak{g}_{\mathbb{C}})$ (the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$), and the complex vector space $U(\mathfrak{g}_{\mathbb{C}})$ (the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$), characterized by the following

property: if $Y_1, \dots, Y_p$ are elements in $\mathfrak{g}_{\mathbb{C}}$, then the image of the element $Y_1 \dots Y_p$ in $S(\mathfrak{g}_{\mathbb{C}})$ by ω is the element $(p!)^{-1} \sum_{\sigma \in S_p} Y_{\sigma(1)} \dots Y_{\sigma(p)}$ in $U(\mathfrak{g}_{\mathbb{C}})$, where S_p is the group of permutations of p elements. The following lemma is easily verified:

LEMMA 1.2.1. — If Z is a central element in $\mathfrak{g}_{\mathbb{C}}$, then $\omega(Zu) = Z\omega(u)$ for all $u \in S(\mathfrak{g}_{\mathbb{C}})$.

We can identify $S(\mathfrak{g}_{\mathbb{C}})$ with $\text{Pol}_{\mathbb{C}}(\mathfrak{g}')$, the complex vector space of complex valued polynomial functions on $\mathfrak{g}'$. If $u \in U(\mathfrak{g}_{\mathbb{C}})$ we let P_u be the polynomial on $\mathfrak{g}'$ corresponding to $\omega^{-1}(u)$. The lemma above then says that if Z is central in $\mathfrak{g}_{\mathbb{C}}$ and if $u \in U(\mathfrak{g}_{\mathbb{C}})$, then $P_{Zu} = P_Z P_u$.

For $e \in \mathcal{E}$, let u_e be the element in $U(\mathfrak{g})$ corresponding to the real valued polynomial function $g \rightarrow i^d Q_e(g)$ on $\mathfrak{g}'$. Note that u_e actually is contained in $U([\mathfrak{g}, \mathfrak{g}])$, since $Q_e(g)$ only depends on the restriction of g to $[\mathfrak{g}, \mathfrak{g}]$.

1.3. — If $g \in \mathfrak{g}'$, the weights of $\mathfrak{g}_g$ in $\mathfrak{g}/\mathfrak{g}_g$ are of the form $\pm \mu_1, \dots, \pm \mu_{d/2}$, where $d = \dim \mathfrak{g}/\mathfrak{g}_g$, and these weights μ_j extend to linear forms, also called μ_j , on the ideal $\mathfrak{f} = \mathfrak{g}_g + [\mathfrak{g}, \mathfrak{g}]$ in such a manner that they are zero on $[\mathfrak{g}, \mathfrak{g}]$ (v. [4], p. 248).

Following *loc. cit.* we set

$$S_{\lambda}(X) = \frac{\sin h(\lambda(X)/2)}{\lambda(X)/2}, \quad X \in \mathfrak{g},$$

for a complex linear form λ on $\mathfrak{g}$, and define the function P'_O on $\mathfrak{f}$ by

$$P'_O(X) = \prod_{j=1}^{d/2} S_{\mu_j}(X), \quad X \in \mathfrak{f},$$

where $O = Gg$ is the G -orbit through g . This definition of P'_O does not depend on the choice of $g \in O$.

We set

$$j_G(X) = \left| \det \frac{1 - e^{-\text{ad } X}}{\text{ad } X} \right|, \quad X \in \mathfrak{g}.$$

j_G is a G -invariant analytic function on $\mathfrak{g}$, and if dX is a Lebesgue measure on $\mathfrak{g}$ there exists a Haar measure μ on G such that $d\mu(\exp X) = j_G(X) dX$.

If G is exponential we set for $e \in \mathcal{E}$,

$$\Gamma_e(X) = \left(\prod_{j \in e} |S_{\lambda_j}(X)| \right)^{\frac{1}{2}}, \quad X \in \mathfrak{g}.$$

LEMMA 1.3.1. — (G exponential) Γ_e is a positive, G -invariant analytic function on $\mathfrak{g}$, extending P'_O for any G -orbit O contained in Ω_e .

Proof. — The function $X \rightarrow S_{\lambda_j}(X)$ is a G -invariant analytic function on $\mathfrak{g}$, and since $\mathfrak{g}$ is exponential $\lambda_j(X) \notin i\mathbb{R} \setminus \{0\}$ for all $X \in \mathfrak{g}$, hence $S_{\lambda_j}(X) \neq 0$ for all $X \in \mathfrak{g}$. This shows that Γ_e is positive, G -invariant and analytic. Now an easy argument shows that $P'_O(X) \geq 0$ for all $X \in \mathfrak{f} = \mathfrak{g}_g + [\mathfrak{g}, \mathfrak{g}]$ (see e. g. [4], p. 264 top; again we use that $\mathfrak{g}$ is exponential). Therefore

$$P'_O(X) = |P'_O(X)| = \prod_{j=1}^{d/2} |S_{\mu_j}(X)| = \left(\prod_{j=1}^{d/2} |S_{\mu_j}(X)|^2 \right)^{\frac{1}{2}} = \left(\prod_{j=1}^{d/2} (|S_{\mu_j}(X)| |S_{-\mu_j}(X)|) \right)^{\frac{1}{2}},$$

and noting that λ_j vanishes on $[\mathfrak{g}, \mathfrak{g}]$ and that the weights of $\mathfrak{g}_g$ in $\mathfrak{g}/\mathfrak{g}_g$ are precisely

$$\{\lambda_{j_1} | \mathfrak{g}_g, \dots, \lambda_{j_d} | \mathfrak{g}_g\} = \{\pm \mu_1 | \mathfrak{g}_g, \dots, \pm \mu_{d/2} | \mathfrak{g}_g\},$$

we get that $P'_O(X) = \Gamma_e(X)$ for $X \in \mathfrak{f}$. This proves the lemma.

We set

$$\alpha_e(X) = j_G(X) \Gamma_e(X)^{-1}, X \in \mathfrak{g}$$

(still assuming that G is exponential). α_e is a positive, G -invariant analytic function on $\mathfrak{g}$.

REMARK 1.3.2. — Lemma 1.3.1 should be compared with [4], section 4, p. 262-264. In the exponential case the result *loc. cit.* is that there exists a G -invariant Zariski open subset Ω of $\mathfrak{g}'$ and a positive, G -invariant analytic function P on $\mathfrak{g}$, such that for any G -orbit O contained in Ω the restriction of P to $\mathfrak{f} = \mathfrak{g}_g + [\mathfrak{g}, \mathfrak{g}]$, $g \in O$, is equal to P'_O . By Lemma 1.3.1 and Lemma 1.1.3 we can obtain this result by taking Ω to be Ω_e for the minimal element e in $\mathcal{E}$, and taking P to be Γ_e . In general the P from *loc. cit.* will be different from the one exhibited here. Incidentally, by refining the methods used here can give a complete solution to the problem raised, and partially solved, by Duflo, *loc. cit.*, p. 263, mid. However, at present this will not be needed, so we shall postpone it to a later time.

1.4. — Suppose now that G is exponential, and suppose in addition that the Jordan-Hölder sequence $\mathfrak{g}_C = \mathfrak{f}_m \supset \dots \supset \mathfrak{f}_0 = \{0\}$ has the property that if $\bar{\mathfrak{f}}_j \neq \mathfrak{f}_j$, then $\bar{\mathfrak{f}}_{j-1} = \mathfrak{f}_{j-1}$ and $\bar{\mathfrak{f}}_{j+1} = \mathfrak{f}_{j+1}$, $1 \leq j \leq m-1$ (such a Jordan-Hölder sequence clearly exists). Set $\chi_e = |\Lambda_e|^{-2}$.

Theorem 1.4.1. — (G exponential) Let π be an irreducible representation of G , and let O be the G -orbit in $\mathfrak{g}'$ associated with π . Let $e \in \mathcal{E}$ be the unique element such that Ω_e contains O . Then

- 1) The measure $Q_e \beta_O$ is a non-zero, positive, tempered, relatively invariant Radon measure on O with multiplier $\chi_e \cdot (\beta_O$ is the canonical measure on O).
- 2) The operator $d\pi(u_e)$ is a selfadjoint, positive, invertible operator, semi-invariant under π with multiplier χ_e (i. e. $\pi(s)d\pi(u_e)\pi(s^{-1}) = \chi_e(s^{-1})d\pi(u_e)$).
- 3) For any $\varphi \in C_c^\infty(G)$ the operator $\pi(u_e * \varphi)$ is traceclass.
- 4) The functional $\varphi \rightarrow \text{Tr}(\pi(u_e * \varphi))$ on $C_c^\infty(G)$ is a non-zero, χ_e -semi-invariant distribution on G of positive type (a distribution semicharacter for π (with multiplier χ_e)).
- 5) For any $\varphi \in C_c^\infty(G)$ we have the formula

$$(*) \quad \text{Tr}(\pi(u_e * \varphi)) = \int_O (\alpha_e \cdot \varphi \circ \exp)^{\wedge}(l) Q_e(l) d\beta_O(l).$$

Here we use the notation $\hat{\psi}(l) = \int_{\mathfrak{g}} \psi(X) e^{i\langle X, l \rangle} dX$ for $\psi \in C_c^\infty(\mathfrak{g})$, $l \in \mathfrak{g}'$, where dX is the Lebesgue measure on $\mathfrak{g}$ with the property that $d\mu(\exp X) = j_G(X) dX$, $d\mu$ being a fixed Haar measure on G , and $\pi(\varphi) = \int_G \varphi(s) \pi(s) d\mu(s)$ for $\varphi \in L^1(G)$.

REMARK 1.4.2. — In the formula (*) above we can instead of α_e use any C^∞ -function α on $\mathfrak{g}$ with the property that the restriction of α to $\mathfrak{f} = \mathfrak{g}_g + [\mathfrak{g}, \mathfrak{g}]$, $g \in O$, is the same as the restriction of α_e to $\mathfrak{f}$.

REMARK 1.4.3. — It will follow from the proof of Theorem 1.4.1 that the distributions $\varphi \rightarrow \text{Tr}(\pi(u_e * \varphi))$ have a finite order not exceeding $2d+1$, where $d = |e|$.

2. Proof of Theorem

Here we shall for brevity say that a Jordan-Hölder sequence $\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}_m \supset \dots \supset \mathfrak{f}_0 = \{0\}$ is of class (b) if it has the property required in 1.4 (i. e. that $\bar{\mathfrak{f}}_j \neq \mathfrak{f}_j$, $1 \leq j \leq m-1$, implies that $\bar{\mathfrak{f}}_{j-1} = \mathfrak{f}_{j-1}$ and $\bar{\mathfrak{f}}_{j+1} = \mathfrak{f}_{j+1}$), cf. [2] Définition 4.2.1, pp. 78.

2.1. — The purpose of this subsection is to prove the following lemma, from which part 1) of Theorem 1.4.1 follows immediately.

LEMMA 2.1.1. — The measure $P_e \beta_0$ is a non-zero, tempered, Λ_e^{-1} -relatively invariant (complex) Radon measure on O .

REMARK 2.1.2. — In the completely solvable case this was proved in [6], section 4.1.d. The proof *loc. cit.* does not carry over to the case at hand, so we have to modify our approach.

Proof. — We have only left to show that $P_e \beta_0$ is tempered, cf. Lemma 1.1.1.

(i) Let I be the set of indices $0 \leq j \leq m$ for which $\bar{\mathfrak{f}}_j = \mathfrak{f}_j$. For $j \in I$ there exists an ideal $\mathfrak{g}_j$ in $\mathfrak{g}$ such that $(\mathfrak{g}_j)_{\mathbb{C}} = \mathfrak{f}_j$.

Set $I' = \{j \in I \mid j-1 \in I\}$ and $I'' = \{j \in I \setminus \{0\} \mid j-1 \notin I\}$. Then $I = \{0\} \cup I' \cup I''$ as a disjoint union, and for $j \in I''$ we have that $j-2 \in I$ (since $\mathfrak{f}_0, \dots, \mathfrak{f}_m$ is of class (b)).

Now since Λ_e only depends on the Jordan-Hölder sequence $\mathfrak{f}_j$ and not on the basis Z_j we can assume here that the Z_j 's are constructed in the following way: for $j \in I'$, let $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, and set $Z_j = X_j$. For $j \in I''$, pick $Z_{j-1} \in \mathfrak{f}_{j-1} \setminus \mathfrak{f}_{j-2}$. Since $\bar{\mathfrak{f}}_{j-1} \neq \mathfrak{f}_{j-1}$ we have that $\bar{Z}_{j-1} \in \mathfrak{f}_j \setminus \mathfrak{f}_{j-1}$. Set $Z_j = \bar{Z}_{j-1}$, and define X_{j-1}, X_j by $Z_j = X_{j-1} + iX_j$. Then X_{j-1}, X_j is a basis for $\mathfrak{g}_j \pmod{\mathfrak{g}_{j-2}}$, and $X_1, \dots, X_m$ is a basis for $\mathfrak{g}$. Let $g_1, \dots, g_m \in \mathfrak{g}'$ be the basis dual to $X_1, \dots, X_m$.

Fix an element $e \in O$, and write $e = J_g = \{j_1 < \dots < j_d\}$. Set $D_1 = \{1 \leq k \leq d \mid j_k \in I'\}$, $D_2 = \{1 \leq k \leq d \mid j_k \notin I, j_k + 1 \notin J_g\}$, $D_3 = \{1 \leq k \leq d \mid j_k \notin I, j_k + 1 \in J_g\}$, $D_4 = \{1 \leq k \leq d \mid j_k \in I''\}$. Clearly $\{1, \dots, d\} = D_1 \cup D_2 \cup D_3 \cup D_4$ as a disjoint union. Observe that if $k \in D_3$, then clearly $k+1 \in D_4$. Conversely, if $k \in D_4$, then $j = j_k \in I'' \cap J_g$, and therefore $j-1 \in J_g$; in fact, if $j-1 \notin J_g$, then $Z_{j-1} \in \mathfrak{f}_{j-2} + (\mathfrak{g}_j)_{\mathbb{C}}$, that is, $X_{j-1} - iX_j \in (\mathfrak{g}_{j-2})_{\mathbb{C}} + (\mathfrak{g}_j)_{\mathbb{C}}$, implying that $X_{j-1}, X_j \in \mathfrak{g}_{j-2} + \mathfrak{g}_j$; but then $Z_j = X_{j-1} + iX_j \in (\mathfrak{g}_{j-2})_{\mathbb{C}} + (\mathfrak{g}_j)_{\mathbb{C}} = \mathfrak{f}_{j-2} + (\mathfrak{g}_j)_{\mathbb{C}}$ and therefore $j \notin J_g$ which is a contradiction. The conclusion of this is that $D_4 = \{k+1 \mid k \in D_3\}$.

For $j \in I$, set $G_g^j = \{s \in G \mid sg = g \pmod{\mathfrak{g}_j^1}\}$. G_g^j is a closed, connected subgroup with Lie algebra $\mathfrak{g}_g^j = \{X \in \mathfrak{g} \mid Xg \in \mathfrak{g}_j^1\}$ (cf. [9], p. 105, III). Clearly $j \rightarrow \mathfrak{g}_g^j$, $j \in I$, is a decreasing sequence of subalgebras with $\mathfrak{g}_g^0 = \mathfrak{g}$ and $\mathfrak{g}_g^m = \mathfrak{g}_g$.

If $j \in I'$, then $\dim \mathfrak{g}_g^{j-1} / \mathfrak{g}_g^j = 0$ or $=1$, and $\mathfrak{g}_g^{j-1} \supsetneq \mathfrak{g}_g^j$ if and only if $j \in J_g$. If $j \in I''$, then $\dim \mathfrak{g}_g^{j-2} / \mathfrak{g}_g^j = 0, =1$ or $=2$, and $\dim \mathfrak{g}_g^{j-2} / \mathfrak{g}_g^j = 2$ if and only if $j, j-1 \in J_g$, $\dim \mathfrak{g}_g^{j-2} / \mathfrak{g}_g^j = 1$ if and only if $j-1 \in J_g$, $j \notin J_g$.

(ii) The following is an adaptation of [9], p. 102-106, II-III to the present situation:

For $k \in D_1$ there exists an element Y_k in $\mathfrak{g}_g^{j_k-1} \setminus \mathfrak{g}_g^{j_k}$ such that Y_k is a coexponential basis to $\mathfrak{g}_g^{j_k}$ in $\mathfrak{g}_g^{j_k-1}$ and such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k}^\perp}$, and for $s \in G_g^{j_k}$ we have

$$\text{Ad}(s)Y_k = \Lambda_{j_k}(s^{-1})Y_k \pmod{\mathfrak{g}_g^{j_k}}.$$

For $k \in D_2$ there exists an element Y_k in $\mathfrak{g}_g^{j_k-1} \setminus \mathfrak{g}_g^{j_k+1}$ such that Y_k is a coexponential basis to $\mathfrak{g}_g^{j_k+1}$ in $\mathfrak{g}_g^{j_k-1}$, and such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k+1}^\perp}$ (to obtain this it can be necessary to change X_{j_k}, X_{j_k+1} in a way that only affects Z_{j_k}, Z_{j_k+1} by multiplying them by a factor of modulus one), and for $s \in G_g^{j_k+1}$ we have $\text{Ad}(s)Y_k = \Lambda_{j_k}(s^{-1})Y_k \pmod{\mathfrak{g}_g^{j_k+1}}$ (so in particular $\Lambda_{j_k}(s^{-1})$ is real).

For $k \in D_3$ there exists elements Y_k, Y_{k+1} in $\mathfrak{g}_g^{j_k-1} \setminus \mathfrak{g}_g^{j_k+1}$ such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k+1}^\perp}$, $Y_{k+1} g = g_{j_{k+1}} \pmod{\mathfrak{g}_{j_{k+1}+1}^\perp}$, such that Y_k, Y_{k+1} is a coexponential basis to $\mathfrak{g}_g^{j_k+1}$ in $\mathfrak{g}_g^{j_k-1}$, such that $\lambda_{j_k}(Y_k) = \lambda_{j_k}(Y_{k+1}) = 0$ and such that

$$\begin{aligned} \exp t_k Y_k \exp t_{k+1} Y_{k+1} &= \exp (t_k Y_k + t_{k+1} Y_{k+1}) \pmod{G_g^{j_k+1}} \\ &= \exp t_{k+1} Y_{k+1} \exp t_k Y_k \pmod{G_g^{j_k+1}}. \end{aligned}$$

For $s \in G_g^{j_k+1}$ we have $\text{Ad}(s)(Y_k + iY_{k+1}) = \Lambda_{j_k}(s^{-1})(Y_k + iY_{k+1}) \pmod{(\mathfrak{g}_g^{j_k+1})_C}$.

(iii) The map $\mathbb{R}^d \rightarrow O = Gg$ given by

$$(*) \quad (t_1, \dots, t_d) \rightarrow \exp t_1 Y_1 \dots \exp t_d Y_d g$$

is a diffeomorphism. We shall compute the canonical measure β_O in terms of the coordinates $t = (t_1, \dots, t_d)$.

Let ω be the canonical symplectic form on O . Via the natural correspondence between $\mathfrak{g}/\mathfrak{g}_g$ and the tangent space to O at g , ω_g corresponds to $\hat{B}_g$.

LEMMA 2.1.3. — For a β_O -integrable function f on O we have

$$\int_O f(l) d\beta_O(l) = C \int_{\mathbb{R}^d} f(\exp t_1 Y_1 \dots \exp t_d Y_d g) \prod_{k < r} |\Lambda_{j_k}(\exp t_r Y_r)| dt_1 \dots dt_d,$$

where $C = ((2\pi)^d Q_e(g))^{-\frac{1}{2}}$.

Proof. — Denote by σ the inverse of the map $(*)$. σ is a global chart and

$$\int_O f(l) d\beta_O(l) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\sigma^{-1}(t)) \theta(\sigma^{-1}(t)) dt,$$

where $\theta(l) = (\det S_l)^{\frac{1}{2}}$, S_l being the skewsymmetric matrix $S_l = [\omega_l(\partial/\partial t_u, \partial/\partial t_v)]_{1 \leq u, v \leq d}$ ([9] Proposition 4, p. 99).

Now ω is G -invariant. Therefore, writing $s = \exp t_1 Y_1 \dots \exp t_d Y_d$ and $l = sg$, we have

$$\begin{aligned} \omega_l((\partial/\partial t_u)_l, (\partial/\partial t_v)_l) &= \omega_{sg}((\partial/\partial t_u)_{sg}, (\partial/\partial t_v)_{sg}) \\ &= \omega_g(\gamma(s^{-1}) * (\partial/\partial t_u)_{sg}, \gamma(s^{-1}) * (\partial/\partial t_v)_{sg}), \end{aligned}$$

where $\gamma(s); l \rightarrow sl$. Let us then compute $\gamma(s^{-1}) * (\partial/\partial t_u)_{sg}$:

For a differentiable function φ we have

$$\begin{aligned}
\gamma(s^{-1}) * (\partial/\partial t_u)_{sg} \varphi &= (\partial/\partial t_u)_{sg} \varphi \circ \gamma(s^{-1}) \\
&= \frac{d}{d\tau} \varphi(s^{-1} \sigma^{-1}(t + \tau^u))|_{\tau=0} \quad (\tau^u = (\delta_{uv}\tau)_{1 \leq v \leq d}) \\
&= \frac{d}{d\tau} \varphi(\exp -t_d Y_d \dots \exp -t_1 Y_1 \exp t_1 Y_1 \dots \exp (t_u + \tau) Y_u \dots \exp t_d Y_d g)|_{\tau=0} \\
&= \frac{d}{d\tau} \varphi(\exp -t_d Y_d \dots \exp -t_{u+1} Y_{u+1} \exp \tau Y_u \exp t_{u+1} Y_{u+1} \dots \exp t_d Y_d g)|_{\tau=0} \\
&= \frac{d}{d\tau} \varphi(s_u^{-1} \exp \tau Y_u s_u g)|_{\tau=0} = \frac{d}{d\tau} \varphi(\exp \tau \operatorname{Ad}(s_u^{-1}) Y_u g)|_{\tau=0},
\end{aligned}$$

where we have set $s_u = \exp t_{u+1} Y_{u+1} \dots \exp t_d Y_d$, $u < d$, $s_d = e$.

The conclusion of this is that $S_l = [B_g(\operatorname{Ad}(s_u^{-1}) Y_u, \operatorname{Ad}(s_v^{-1}) Y_v)]_{1 \leq u, v \leq d}$. Since $Y_1, \dots, Y_d$ is a basis for $\mathfrak{g} \pmod{\mathfrak{g}_g}$ we can write

$$\operatorname{Ad}(s_u^{-1}) Y_u = \sum_{p=1}^d a_{pu} Y_p + c_u$$

where $c_u \in \mathfrak{g}_g$, and then $S_l = A S_g A$, where A is the matrix $[a_{uv}]_{1 \leq u, v \leq d}$, so that $\theta(l) = |\det A| \theta(g)$.

We shall then find $\det A$: for $u \in D_1$ we have that $s_u \in G_g^{j_u}$, so $\operatorname{Ad}(s_u^{-1}) Y_u = \Lambda_{j_u}(s_u) Y_u \pmod{\mathfrak{g}_g^{j_u}}$, implying that $a_{uu} = \Lambda_{j_u}(s_u)$, while $a_{uv} = 0$ for $u < v$. For $u \in D_2$ we have

$$\operatorname{Ad}(s_u^{-1}) Y_u = \Lambda_{j_u}(s_u) Y_u \pmod{\mathfrak{g}_g^{j_u+1}}$$

implying that $a_{uu} = \Lambda_{j_u}(s_u) = |\Lambda_{j_u}(s_u)|$, while $a_{uv} = 0$ for $u < v$. For $u \in D_3$ we have

$$\operatorname{Ad}(s_u^{-1})(Y_u + i Y_{u+1}) = \Lambda_{j_u}(s_u)(Y_u + i Y_{u+1}) \pmod{\mathfrak{g}_g^{j_u+1}},$$

implying that

$$\det \begin{bmatrix} a_{uu} & a_{uu+1} \\ a_{u+1u} & a_{u+1u+1} \end{bmatrix} = |\Lambda_{j_u}(s_u)|^2,$$

while $a_{uv} = 0$ and $a_{u+1v} = 0$ for $v > u+1$. It follows that

$$\det A = \prod_{u \in D_1 \cup D_2} |\Lambda_{j_u}(s_u)| \cdot \prod_{u \in D_3} |\Lambda_{j_u}(s_u)|^2.$$

Now for $u \in D_3$ we have

$$\begin{aligned}
\Lambda_{j_u}(s_u) &= \Lambda_{j_u}(\exp t_{u+1} Y_{u+1} \dots \exp t_d Y_d) \\
&= \Lambda_{j_u}(\exp t_{u+2} Y_{u+2} \dots \exp t_d Y_d) = \overline{\Lambda_{j_{u+1}}(\exp t_{u+2} Y_{u+2} \dots \exp t_d Y_d)} = \overline{\Lambda_{j_{u+1}}(s_{u+1})},
\end{aligned}$$

so $|\Lambda_{j_u}(s_u)| = |\Lambda_{j_{u+1}}(s_{u+1})|$, hence $\det A = \prod_{u=1}^d |\Lambda_{j_u}(s_u)| = \prod_{1 \leq u < r \leq d} |\Lambda_{j_u}(\exp t_r Y_r)|$.

Finally, a simple computation shows that $\det S_g = \det [B_g(Y_r, Y_s)]_{1 \leq r, s \leq d} = Q_e(g)^{-1}$. This ends the proof of the lemma.

(iv) For $1 \leq j \leq m$ we define the function S_j by

$$S_j(t_1, \dots, t_d) = \langle \exp t_1 Y_1 \dots \exp t_d Y_d g, Z_j \rangle.$$

We consider S_{j_k} : arguing like in [9], p. 106 we find for $k \in D_1 \cup D_2$:

$$S_{j_k}(t_1, \dots, t_d) = \frac{e^{-t_k \lambda_{j_k}(Y_k)^{-1}}}{-\lambda_{j_k}(Y_k)} \prod_{r < k} \Lambda_{j_k}(\exp t_r Y_r)^{-1} + S_{j_k}(t_1, \dots, t_{k-1}, 0, \dots, 0),$$

and for $k \in D_4$ we find

$$S_{j_k}(t_1, \dots, t_d) = (t_{k-1} + it_k) \prod_{r < k-1} \Lambda_{j_k}(\exp t_r Y_r)^{-1} + S_{j_k}(t_1, \dots, t_{k-2}, 0, \dots, 0).$$

(v) For a real number $n > 0$ we set $M(n) = \int_{\mathbb{R}} (1+x^2)^{-n/2} dx$. We have $0 < M(n) \leq +\infty$ and $M(n) < +\infty$ if and only if $n > 1$.

LEMMA 2.1.4. — Let a, α, β be real numbers with $a > 0, \alpha \neq 0$, and let c, k be complex numbers with $k \neq 0$. We have

$$(*) \quad \int_{\mathbb{R}} (a + |ke^{(\alpha+i\beta)t} - c|^2)^{-n/2} e^{\alpha t} dt < \frac{M(n)}{|\alpha| |k| a^{(n-1)/2}},$$

$$(**) \quad \int_{\mathbb{R}_2} (a + |k(s+it) - c|^2)^{-n/2} ds dt = \frac{M(n)M(n-1)}{|k|^2 a^{(n-2)/2}}.$$

Proof. — Obviously we can assume that $k > 0$. Writing $k^{-1}c = be^{i\gamma}$, $b \geq 0, \gamma \in \mathbb{R}$ we have

$$\begin{aligned} \int_{\mathbb{R}} (a + |ke^{(\alpha+i\beta)t} - c|^2)^{-n/2} e^{\alpha t} dt &= \int_{\mathbb{R}} (a + k^2 |e^{\alpha t + i(t\beta - \gamma)} - b|^2)^{-n/2} e^{\alpha t} dt \\ &\leq \int_{\mathbb{R}} (a + k^2 |e^{\alpha t} - b|^2)^{-n/2} e^{\alpha t} dt \\ &= |\alpha|^{-1} \int_0^\infty (a + k^2 |x - b|^2)^{-n/2} dx \\ &< |\alpha|^{-1} \int_{\mathbb{R}} (a + k^2 |x - b|^2)^{-n/2} dx \\ &= |\alpha|^{-1} \int_{\mathbb{R}} (a + k^2 x^2)^{-n/2} dx \\ &= \frac{M(n)}{|\alpha| k a^{(n-1)/2}}. \end{aligned}$$

This proves (*). Similarly for (**).

(vi) We shall then prove the temperedness of the measure $P_e \beta_0$. First observe that

$l \rightarrow (\sum_{j=1}^m |\langle l, Z_j \rangle|^2)^{\frac{1}{2}} = \|l\|$ is a norm on $\mathfrak{g}'$. We must show that we can find $n > 0$ such that $\int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_0(l)$ is finite. We have, using Lemma 2.1.3:

$$\begin{aligned} & \int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_0(l) \\ &= C \int_{\mathbb{R}^d} \frac{|P_e(\exp t_1 Y_1 \dots \exp t_d Y_d g)|}{(1 + \|\exp t_1 Y_1 \dots \exp t_d Y_d g\|^2)^{n/2}} \prod_{k < r} |\Lambda_{j_k}(\exp t_r Y_r)| dt_1 \dots dt_d \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\prod_{r \leq k} |\Lambda_{j_k}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{j=1}^m |S_j(t_1, \dots, t_d)|^2)^{n/2}} dt_1 \dots dt_d \\ &\leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} \frac{\prod_{r \leq k} |\Lambda_{j_k}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{\substack{k=1 \\ k \notin D_3}}^d |S_{j_k}(t_1, \dots, t_d)|^2)^{n/2}} dt_1 \dots dt_d. \end{aligned}$$

Suppose first that $d \in D_1 \cup D_2$. Then (assuming that $\lambda_{j_d}(Y_d) \neq 0$)

$$S_{j_d}(t_1, \dots, t_d) = \frac{e^{-t_d \lambda_{j_d}(Y_d)} - 1}{-\lambda_{j_d}(Y_d)} \prod_{r < d} \Lambda_{j_d}(\exp t_r Y_r)^{-1} + S_{j_d}(t_1, \dots, t_{d-1}, 0),$$

and the last integral is equal to

$$(2\pi)^{-d/2} \int_{\mathbb{R}^{d-1}} \prod_{r \leq k \leq d-1} |\Lambda_{j_k}(\exp t_r Y_r)|^{-1} dt_1 \dots dt_{d-1} \int_{\mathbb{R}} F(t_1, \dots, t_d) dt_d,$$

where

$$F(t_1, \dots, t_d) = \frac{\prod_{r=1}^d |\Lambda_{j_d}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{\substack{k=1 \\ k \notin D_3}}^{d-1} |S_{j_k}(t_1, \dots, t_{d-1}, 0)|^2 + |S_{j_d}(t_1, \dots, t_d)|^2)^{n/2}}.$$

Applying Lemma 2.1.4 with $a = 1 + \sum_{\substack{k=1 \\ k \notin D_3}}^{d-1} |S_{j_k}(t_1, \dots, t_{d-1}, 0)|^2$, $\alpha + i\beta = -\lambda_{j_d}(Y_d)$, $k = -\lambda_{j_d}(Y_d)^{-1} \prod_{r < d} \Lambda_{j_d}(\exp t_r Y_r)^{-1}$, $c = -S_{j_d}(t_1, \dots, t_{d-1}, 0) - \lambda_{j_d}(Y_d)^{-1} \prod_{r < d} \Lambda_{j_d}(\exp t_r Y_r)^{-1}$ we find that

$$\int_{\mathbb{R}} F(t_1, \dots, t_d) dt_d \leq \frac{C_d \cdot M(n)}{(2\pi)^{d/2}} \cdot \frac{1}{(1 + \sum_{\substack{k=1 \\ k \notin D_3}}^{d-1} |S_{j_k}(t_1, \dots, t_{d-1}, 0)|^2)^{(n-1)/2}},$$

where $C_d = |\lambda_{j_d}(Y_d)| (|\operatorname{Re} \lambda_{j_d}(Y_d)|)^{-1}$ (note that since $\mathfrak{g}$ is exponential the non-vanishing of $\lambda_{j_d}(Y_d)$ implies the non-vanishing of $\operatorname{Re}(\lambda_{j_d}(Y_d))$), and therefore

$$\begin{aligned} & \int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_0(l) \\ (\#) \quad & \leq \frac{C_d \cdot M(n)}{(2\pi)^d} \cdot \int_{\mathbb{R}^{d-1}} \frac{\prod_{r \leq k \leq d-1} |\Lambda_{j_k}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{\substack{k=1 \\ k \notin D_3}}^{d-1} |S_{j_k}(t_1, \dots, t_{d-1}, 0)|^2)^{(n-1)/2}} dt_1 \dots dt_{d-1}. \end{aligned}$$

If $\lambda_{j_d}(Y_d)=0$ a simple change in the argument shows that the same relation is valid with $C_d=1$ (cf. below).

Suppose next that $d \in D_4$. Then

$$S_{j_d}(t_1, \dots, t_d) = (t_{d-1} + it_d) \prod_{r \leq d-2} \Lambda_{j_d}(\exp t_r Y_r)^{-1} + S_{j_d}(t_1, \dots, t_{d-2}, 0, 0),$$

and therefore we find as above that

$$\begin{aligned} \int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_O(l) \\ \leq (2\pi)^{-d/2} \int_{\mathbb{R}^{d-2}} \prod_{r \leq k \leq d-2} \Lambda_{j_k}(\exp t_r Y_r)^{-1} dt_1 \dots dt_{d-2} \int_{\mathbb{R}^2} F(t_1, \dots, t_d) dt_{d-1} dt_d, \end{aligned}$$

where now

$$F(t_1, \dots, t_d) = \frac{\prod_{r=1}^{d-2} |\Lambda_{j_d}(\exp t_r Y_r)|^{-2}}{(1 + \sum_{k=1}^{d-2} |S_{j_k}(t_1, \dots, t_{d-2}, 0, 0)|^2 + |S_{j_d}(t_1, \dots, t_d)|^2)^{n/2}}$$

(here we have used that $|\Lambda_{j_d}| = |\Lambda_{j_{d-1}}|$ and that

$$\lambda_{j_{d-1}}(Y_{d-1}) = \lambda_{j_d}(Y_{d-1}) = \lambda_{j_{d-1}}(Y_d) = \lambda_{j_d}(Y_d) = 0).$$

Applying the relation (**) in Lemma 2.1.4 with $a = 1 + \sum_{k=1}^{d-2} |S_{j_k}(t_1, \dots, t_{d-2}, 0, 0)|^2$, $k = \prod_{r=1}^{d-2} \Lambda_{j_d}(\exp t_r Y_r)^{-1}$, and $c = -S_{j_d}(t_1, \dots, t_{d-2}, 0, 0)$ we find that

$$\int_{\mathbb{R}^2} F(t_1, \dots, t_d) dt_{d-1} dt_d \leq \frac{M(n)M(n-1)}{(2\pi)^{d/2}} \cdot \frac{1}{(1 + \sum_{k=1}^{d-2} |S_{j_k}(t_1, \dots, t_{d-2}, 0, 0)|^2)^{(n-2)/2}},$$

and therefore

$$\begin{aligned} \int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_O(l) \\ (\# \#) \leq \frac{M(n)M(n-1)}{(2\pi)^{d/2}} \cdot \int_{\mathbb{R}^{d-2}} \frac{\prod_{r \leq k \leq d-2} |\Lambda_{j_k}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{k=1}^{d-2} |S_{j_k}(t_1, \dots, t_{d-2}, 0, 0)|^2)^{(n-2)/2}} dt_1 \dots dt_{d-2}. \end{aligned}$$

Repeating these two methods of estimation on the new integral (#) or (# #) we find that

$$\int_0 (1 + \|l\|^2)^{-n/2} |P_e(l)| d\beta_O(l) \leq (2\pi)^{-d/2} M(n) \dots M(n-d+1) C_d \dots C_1 < +\infty$$

for $n > d$. Here $C_k = |\lambda_{j_k}(Y_k)| (|\operatorname{Re} \lambda_{j_k}(Y_k)|)^{-1}$ if $\lambda_{j_k}(Y_k) \neq 0$, and $C_k = 1$ if $\lambda_{j_k}(Y_k) = 0$. This ends the proof of Lemma 2.1.1.

2.2. — The purpose of this subsection is to prove Proposition 2.2.1 below.

Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and let N be the analytic subgroup corresponding to $\mathfrak{n}$. We have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{g}$, and therefore $u_e \in U(\mathfrak{n})$.

PROPOSITION 2.2.1. — If $g \in \Omega_e$ and if π is the irreducible representation of N corresponding to the orbit $O = Nf$, where $f = g|n$, then

$$d\pi(u_e) = Q_e(g)I.$$

REMARK 2.2.2. — Even in the special case where g is assumed to be nilpotent (and therefore $g = n$), Proposition 2.2.1 provides a new result.

Proof. — The proof is by induction on the dimension of g . The proposition is clearly valid for $\dim g = 1$ (in which case $e = \emptyset$, and $Q_e \equiv 1$, $u_e = 1$). Assume then that the proposition has been proved for all dimensions of g less than or equal to $m-1$, and that $\dim g = m$. The case $e = \emptyset$ being trivial we can assume that $e \neq \emptyset$, and write $e = \{j_1 < \dots < j_d\}$.

Case (a): Suppose that there exists a non-trivial abelian ideal a in g such that $g|a = 0$. Let A be the analytic subgroup of G corresponding to a . We have $a \subset n$ and setting $\tilde{g} = g/a$, $\tilde{n} = n/a$ is the nilradical of $\tilde{g}$. We set $\hat{f}_j = \tilde{f}_j + a_{\mathbb{C}}/a_{\mathbb{C}}$, $0 \leq j \leq m$, and let $c : g \rightarrow g/a$ denote the coset map. Then we have the diagram

$$\tilde{g}_{\mathbb{C}} = \hat{f}_m \supset \hat{f}_{m-1} \supset \dots \supset \hat{f}_1 \supset \hat{f}_0 = \{0\},$$

and $\dim \hat{f}_j/\hat{f}_{j-1} = 0$ or $= 1$. Set $I = \{1 \leq j \leq m \mid \hat{f}_j \not\supset \hat{f}_{j-1}\}$, write $I = \{i_1 < \dots < i_{m'}\}$, and set $\tilde{f}_j = \hat{f}_{i_j}$, $1 \leq j \leq m'$. We then have a Jordan-Hölder sequence in $\tilde{g}_{\mathbb{C}}$:

$$\tilde{g}_{\mathbb{C}} = \tilde{f}_{m'} \supset \tilde{f}_{m'-1} \supset \dots \supset \tilde{f}_1 \supset \tilde{f}_0 = \{0\}$$

which is immediately seen to be of class (b), and setting $\tilde{Z}_j = c(Z_{i_j})$ we have that

$$\tilde{Z}_j \in \tilde{f}_j \setminus \tilde{f}_{j-1}, j = 1, \dots, m'.$$

Define $\tilde{g} \in \tilde{g}'$ by $\tilde{g} \circ c = g$ and $\tilde{f} = \tilde{g}| \tilde{n}$. We have $a \subset g_{\tilde{g}}$ and $\tilde{g}_{\tilde{g}} = g_{\tilde{g}}/a$. Moreover, $j \in J_{\tilde{g}} \Rightarrow j \in I$, since $j \notin I \Rightarrow \tilde{f}_j \subset \tilde{f}_{j-1} + a_{\mathbb{C}} \subset \tilde{f}_{j-1} + (g_{\tilde{g}})_{\mathbb{C}} \Rightarrow j \notin J_{\tilde{g}}$. Writing

$$\tilde{e} = J_{\tilde{g}} = \{j_1 < \dots < j_d\}$$

we have $J_{\tilde{g}} = \{i_{j_1} < \dots < i_{j_d}\} = \{j_1 < \dots < j_d\}$. For $\tilde{l} \in \tilde{g}'$ we then have with $l = \tilde{l} \circ c$:

$$\begin{aligned} Q_e(l) &= |\det [B_l(Z_{j_r}, Z_{j_s})]_{1 \leq r, s \leq d}| = |\det [B_{\tilde{l}}(Z_{i_{j_r}}, Z_{i_{j_s}})]_{1 \leq r, s \leq d}| \\ &= |\det [B_{\tilde{l}}(\tilde{Z}_{j_r}, \tilde{Z}_{j_s})]_{1 \leq r, s \leq d}| = Q_{\tilde{e}}(\tilde{l}). \end{aligned}$$

This shows that the canonical image of u_e in $U(\tilde{g})$ is precisely $u_{\tilde{e}} (\in U(\tilde{n}))$. Now the representation π is trivial on A , so there exists an irreducible representation $\tilde{\pi}$ of $\tilde{N} = N/A$ such that $\tilde{\pi} \circ (c|N) = \pi$, and the orbit of $\tilde{\pi}$ is $\tilde{N}\tilde{f}$. But since $\tilde{g} \in \Omega_{\tilde{e}}$ we have $d\tilde{\pi}(u_{\tilde{e}}) = Q_{\tilde{e}}(\tilde{g})I$ by the induction hypothesis, and therefore $d\pi(u_e) = d\tilde{\pi}(c(u_e)) = d\tilde{\pi}(u_{\tilde{e}}) = Q_{\tilde{e}}(\tilde{g})I = Q_e(g)I$. This ends case (a).

Case (b): Suppose that we are not in case (a) and that $\lambda_1 \neq 0$.

Write $Z_1 = X_1 + iY_1$ and set $a = \mathbb{R}X_1 + \mathbb{R}Y_1$. Then a is an abelian ideal (of dimension 1 or 2), and $g|a \neq 0$ (since otherwise we would be in case (a)), and therefore $\langle g, Z_1 \rangle \neq 0$.

Since G is exponential we can write $\lambda_1(X) = \alpha_1(X)(1 + ik_1)$, where α_1 is a real linear form on g , and where k_1 is a real number.

Set $\mathfrak{h} = \ker \lambda_1 (= \ker \alpha_1)$. $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension 1 with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{h}$, so the nilradical of $\mathfrak{h}$ is $\mathfrak{n}$. Clearly $Z_1 \in \mathfrak{h}_{\mathbb{C}}$. Set $p = \min \{1 \leq j \leq m \mid Z_j \notin \mathfrak{h}_{\mathbb{C}}\}$. p is well-defined and $p \geq 2$. We observe that $p \in J_g = e$. In fact, suppose $p \notin J_g$. Then $Z_p \in \mathfrak{f}_{p-1} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore $0 \neq \langle g, [Z_p, \mathfrak{f}_1] \rangle = \langle Z_p g, \mathfrak{f}_1 \rangle = \langle \mathfrak{f}_{p-1} g, \mathfrak{f}_1 \rangle = \langle g, [\mathfrak{f}_{p-1}, \mathfrak{f}_1] \rangle = 0$, which is a contradiction. Also $1 \in J_g$, since otherwise $Z_1 \in (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$0 = \langle g, [\mathfrak{g}, \mathfrak{f}_1] \rangle = \langle g, \mathfrak{f}_1 \rangle \neq 0.$$

We also note that $\mathfrak{g}_g \subset \mathfrak{h}$, since otherwise $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_g$ and therefore $0 = \langle \mathfrak{g} g, \mathfrak{f}_1 \rangle = \langle g, \mathfrak{f}_1 \rangle \neq 0$.

Set $\hat{Z}_j = Z_j$ for $1 \leq j \leq p-1$, $\hat{Z}_j = Z_{j+1} + c_{j+1} Z_p$ for $p \leq j \leq m-1$ and $\hat{Z}_m = Z_p$. Here c_j , $p+1 \leq j \leq m$, is defined such that $Z_j + c_j Z_p \in \mathfrak{h}_{\mathbb{C}}$. This is possible since $\mathbb{C} Z_p \oplus \mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. Clearly $\hat{Z}_1, \dots, \hat{Z}_m$ is a basis in $\mathfrak{g}_{\mathbb{C}}$.

Set $\hat{\mathfrak{f}}_j = \mathbb{C} \hat{Z}_1 \oplus \dots \oplus \mathbb{C} \hat{Z}_j$. For $0 \leq j \leq p-1$ we have that $\hat{\mathfrak{f}}_j = \mathfrak{f}_j$. For $p-1 \leq j \leq m-1$ we have $\hat{\mathfrak{f}}_j \oplus \mathbb{C} Z_p = \mathfrak{f}_{j+1}$, hence

$$\begin{aligned} \hat{\mathfrak{f}}_j &= \mathfrak{f}_j \quad \text{for } 0 \leq j \leq p-1, \\ \hat{\mathfrak{f}}_j &= \mathfrak{f}_{j+1} \cap \mathfrak{h}_{\mathbb{C}} \quad \text{for } p-1 \leq j \leq m-1, \\ \hat{\mathfrak{f}}_m &= \mathfrak{g}_{\mathbb{C}}. \end{aligned}$$

From this it follows that $\hat{\mathfrak{f}}_j$, $j=0, \dots, m$, is a Jordan-Hölder sequence for $\mathfrak{g}_{\mathbb{C}}$ with $\hat{\mathfrak{f}}_{m-1} = \mathfrak{h}_{\mathbb{C}}$. We claim it is of class (b). In fact, since $\hat{\mathfrak{f}}_{p-1} = \mathfrak{f}_p \cap \mathfrak{h}_{\mathbb{C}}$ and $\hat{\mathfrak{f}}_{p-1} = \mathfrak{f}_{p-1}$ it follows that $\hat{\mathfrak{f}}_{p-1} = \hat{\mathfrak{f}}_{p-1}$, and from this it is immediate that the claim is true. We thus have a new diagram

$$\mathfrak{g}_{\mathbb{C}} = \hat{\mathfrak{f}}_m \supset \hat{\mathfrak{f}}_{m-1} \supset \dots \supset \hat{\mathfrak{f}}_1 \supset \hat{\mathfrak{f}}_0 = \{0\}.$$

$\parallel$
 $\mathfrak{h}_{\mathbb{C}}$

The objects defined relative to this new Jordan-Hölder sequence are designated $\hat{J}_g, \hat{e}$, etc.

For $1 \leq j \leq p-1$ we clearly have $j \in J_g \Leftrightarrow j \in \hat{J}_g$. Furthermore $p \in J_g$ (see above) and $m \in \hat{J}_g$. In fact, if $m \notin \hat{J}_g$, then $Z_p = \hat{Z}_m \in \hat{\mathfrak{f}}_{m-1} + (\mathfrak{g}_g)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$0 \neq \langle Z_p g, \mathfrak{f}_1 \rangle = \langle \mathfrak{h} g, \mathfrak{f}_1 \rangle = 0.$$

For $p+1 \leq j \leq m$ we have

$$\begin{aligned} j \notin J_g &\Leftrightarrow Z_j \notin \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \notin \hat{\mathfrak{f}}_{j-2} + \mathbb{C} Z_p + (\mathfrak{g}_g)_{\mathbb{C}} \\ &\Leftrightarrow \hat{Z}_{j-1} \notin \hat{\mathfrak{f}}_{j-2} + \mathbb{C} Z_p + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-1} \notin \hat{\mathfrak{f}}_{j-2} + (\mathfrak{g}_g)_{\mathbb{C}} \end{aligned}$$

(since $\mathfrak{g}_g \subset \mathfrak{h}$) $\Leftrightarrow j-1 \notin \hat{J}_g$. Therefore, if $j_{\alpha} = p$ we have $\hat{j}_h = j_h$ for $1 \leq h \leq \alpha-1$, $\hat{j}_h + 1 = j_{h+1}$ for $\alpha \leq h \leq d-1$ and $\hat{j}_d = m$, so

$$\begin{aligned} \hat{Z}_{\hat{j}_h} &= Z_{j_h} \quad \text{for } 1 \leq h \leq \alpha-1, \\ \hat{Z}_{\hat{j}_h} &= Z_{j_{h+1}} + c_{j_{h+1}} Z_{j_{\alpha}} \quad \text{for } \alpha \leq h \leq d-1, \\ \hat{Z}_{\hat{j}_d} &= Z_{j_{\alpha}}. \end{aligned}$$

Therefore, letting $C = [c_{rs}]_{1 \leq r,s \leq d}$ be the $d \times d$ -matrix:

$$C = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c_{j_{\alpha+1}} & \dots & c_{j_d} \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

where the empty entries are 0, we have $\hat{Z}_{j_s} = \sum_{r=1}^d c_{rs} Z_{j_r}$, and therefore $M_{\hat{e}}(l) = {}^t C M_e(l) C$, with $\hat{e} = \hat{J}_g$. Now $\det C = (-1)^\alpha$, so $\det M_e(l) = \det M_{\hat{e}}(l)$, and $Q_e(l) = Q_{\hat{e}}(l)$, and therefore $u_e = u_{\hat{e}}$. The conclusion of this is then that we can assume that $\mathfrak{f}_{m-1} = \mathfrak{h}_C$, and this assumption will be in effect from now on. We then have:

$$\begin{aligned} Q_e(l) &= |\det [B_l(Z_{j_r}, Z_{j_s})]_{1 \leq r,s \leq d}| \\ &= \left| \sum_{\sigma \in S_d} \text{sign } \sigma \langle l, [Z_{j_1}, Z_{j_{\sigma(1)}}] \rangle \dots \langle l, [Z_{j_d}, Z_{j_{\sigma(d)}}] \rangle \right| \\ &= |\langle l, [Z_1, Z_m] \rangle|^2 \cdot \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)}}] \rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-1)}}] \rangle \right| \end{aligned}$$

where S_d^* is the set of elements $\sigma \in S_d$ with $\sigma(1) = d$, $\sigma(d) = 1$.

Set $g_0 = g|_{\mathfrak{h}}$. Then $f = g_0|_{\mathfrak{n}}$. We designate the objects associated with the group $H = \exp \mathfrak{h}$, and the class (b) Jordan-Hölder sequence $\mathfrak{h}_C = \mathfrak{f}_{m-1} \supset \dots \supset \mathfrak{f}_1 \supset \mathfrak{f}_0 = \{0\}$ by $J_{g_0}^0$, etc. We have $(\mathfrak{h}_{g_0})_C = (\mathfrak{g}_g)_C \oplus CZ_1$, so $J_{g_0}^0 = J_g \setminus \{1, m\}$, and therefore

$$J_{g_0}^0 = \{j_1^0 < \dots < j_{d-2}^0\}$$

with $j_h^0 = j_{h+1}$ for $1 \leq h \leq d-2$, so we have for $l \in \mathfrak{h}'$:

$$\begin{aligned} Q_{e^0}(l) &= |\det [B_l(Z_{j_r^0}, Z_{j_s^0})]_{1 \leq r,s \leq d-2}| \\ &= \left| \sum_{\sigma \in S_{d-2}} \text{sign } \sigma \langle l, [Z_{j_1^0}, Z_{j_{\sigma(1)}^0}] \rangle \dots \langle l, [Z_{j_{d-2}^0}, Z_{j_{\sigma(d-2)}^0}] \rangle \right| \\ &= \left| \sum_{\sigma \in S_{d-2}} \text{sign } \sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)+1}}] \rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-2)+1}}] \rangle \right| \\ &= \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)}}] \rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-1)}}] \rangle \right|, \end{aligned}$$

and comparing with the result above we get $Q_e(l) = |\langle l, W \rangle|^2 Q_{e^0}(l_0)$, where $W = [Z_1, Z_m]$ and $l_0 = l|_{\mathfrak{h}}$. Now since W is central in $\mathfrak{h}_C$ and since $P_W(l) = \langle l, W \rangle$, $P_{\bar{W}}(l) = \langle l, \bar{W} \rangle$ we find that $i^d Q_e(l) = -P_W(l) P_{\bar{W}}(l) i^{d-2} Q_{e^0}(l_0)$, and therefore $u_e = -W \bar{W} u_{e^0}$ by Lemma 1.2.1. By the induction hypothesis we have that $d\pi(u_{e^0}) = Q_{e^0}(g)I$, and noting that

$$d\pi(W) = i \langle g, W \rangle I, \quad d\pi(\bar{W}) = i \langle g, \bar{W} \rangle I$$

we finally get $d\pi(u_e) = |\langle g, W \rangle|^2 d\pi(u_{e^0}) = |\langle g, W \rangle|^2 Q_{e^0}(g)I = Q_e(g)I$. This settles case (b).

Case (c) : Suppose we are not in case (a) and (b) and that $\lambda_2 \neq 0$.

Again we have $\langle g, Z_1 \rangle \neq 0$ and, moreover, $\bar{f}_1 = f_1$ (since f_1 is a central ideal in $g_{\mathbb{C}}$). We write $[X, Z_2] = \lambda_2(X)Z_2 + \gamma(X)Z_1$, $X \in g$, where γ is a (complex valued) linear form on g . The linear form γ has the form $\gamma(X) = \gamma_1(X) + i\gamma_2(X)$, where γ_1, γ_2 are real linear forms on g . We extend λ_2, γ to complex linear forms on $g_{\mathbb{C}}$ such that we have

$$[Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1 \quad \text{for } Z \in g_{\mathbb{C}}.$$

We note the formula

$$(2.2.2) \quad \gamma([Z, W]) = \gamma(Z)\lambda_2(W) - \gamma(W)\lambda_2(Z)$$

for $Z, W \in g_{\mathbb{C}}$, which we get by a simple application of the Jacobi identity.

Since G is exponential we can write $\lambda_2(X) = \alpha_2(X)(1 + ik_2)$, where α_2 is a real linear form on g and where k_2 is a real number.

We then distinguish three subcases: (c1): $\text{rank}(\alpha_2, \gamma_1, \gamma_2) = 3$, (c2): $\text{rank}(\alpha_2, \gamma_1, \gamma_2) = 2$ and (c3): $\text{rank}(\alpha_2, \gamma_1, \gamma_2) = 1$.

Case (c1): Set $\mathfrak{h} = \ker \gamma_1 \cap \ker \gamma_2 (= \ker \gamma | g)$. It follows from the formula (2.2.2) that $\mathfrak{h}$ is a subalgebra in g , and its codimension is 2. We observe that $Z \in \mathfrak{h}_{\mathbb{C}}$ if and only if $\gamma(Z) = 0$ and $\gamma(\bar{Z}) = 0$. Set $\mathfrak{h}_0 = \ker \lambda_2 | \mathfrak{h} = \ker \alpha_2 | \mathfrak{h} = \ker \text{ad } Z_2 | g$. $\mathfrak{h}_0$ is an ideal in g of codimension 3. That $\mathfrak{h}_0$ is an ideal in g follows from the fact that

$$\mathfrak{h}_0 = \ker \gamma \cap \ker \lambda_2 \cap g$$

and by applying the formula (2.2.2).

Let $\mathfrak{m}$ be the nilradical of $\mathfrak{h}_0$. Since $\mathfrak{h}_0$ is an ideal we have that $\mathfrak{m} = \mathfrak{n} \cap \mathfrak{h}_0 = \mathfrak{n} \cap \mathfrak{h}$. Observe that $\dim \mathfrak{n}/\mathfrak{m} = 2$. In fact, pick $W \in \mathfrak{h} \setminus \mathfrak{h}_0$. Then we have that

$$\gamma([Z, W]) = \lambda_2(W)\gamma(Z) \quad \text{for } Z \in g_{\mathbb{C}},$$

and therefore $\gamma(\overline{[Z, W]}) = \lambda_2(\bar{W})\gamma(\bar{Z})$. Choosing Z such that $\gamma(Z) = 1$, $\gamma(\bar{Z}) = 0$ and Z' such that $\gamma(Z') = 0$, $\gamma(\bar{Z}') = 1$ we get that

$$\gamma([Z, W]) = \lambda_2(W) \neq 0, \gamma(\overline{[Z, W]}) = 0, \gamma(\overline{[Z', W]}) = \lambda_2(\bar{W}) \neq 0, \gamma([Z', W]) = 0,$$

and this shows that $[Z, W], [Z', W]$ is a basis in $\mathfrak{n}_{\mathbb{C}} \pmod{\mathfrak{m}_{\mathbb{C}}}$.

We claim that $\bar{f}_2 \neq f_2$. In fact, we have $[Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1$ for all $Z \in g_{\mathbb{C}}$, and therefore $[Z, \bar{Z}_2] = \lambda_2(\bar{Z})\bar{Z}_2 + \gamma(\bar{Z})\bar{Z}_1$. Since λ_2 does not vanish on $\mathfrak{h}_{\mathbb{C}}$ we have that $[\mathfrak{h}_{\mathbb{C}}, f_2] = \mathbb{C}Z_2$ and $[\mathfrak{h}_{\mathbb{C}}, \bar{f}_2] = \mathbb{C}\bar{Z}_2$. Therefore, if $\bar{f}_2 = f_2$, then $\mathbb{C}Z_2 = \mathbb{C}\bar{Z}_2$, hence $\gamma(Z) = 0$ implies that $\gamma(\bar{Z}) = 0$, so $\mathfrak{h}_{\mathbb{C}}$ is the set of $Z \in g_{\mathbb{C}}$ such that $\gamma(Z) = 0$, contradicting the fact that $\text{codim } \mathfrak{h} = 2$. We conclude that $\bar{f}_2 \neq f_2$, and therefore that $\bar{f}_1 = f_1$ and $\bar{f}_3 = f_3$. In particular $\bar{Z}_2 \notin f_2$.

We have seen that $Z_1, Z_2, \bar{Z}_2$ span f_3 . Now since $\lambda_2(Z_2) = 0$ we have that $\alpha_2(Z_2) = 0$, and this means that $[f_3, f_2] \subset f_1$. We then distinguish two possibilities: case (c11): $[f_3, f_2] = 0$ and case (c12): $[f_3, f_2] = f_1$.

Set $f_0 = f|_{\mathfrak{m}=\mathfrak{g}|\mathfrak{m}}$, and let π_0 be the irreducible representation of $M = \exp \mathfrak{m}$ corresponding to Mf_0 .

Case (c11): (i) It is our first aim to show that $u_e \in U(\mathfrak{m}_{\mathbb{C}})$, and that $d\pi_0(u_e) = Q_e(g)I$. We start by noting that we can assume that $\langle g, Z_2 \rangle = 0$; in fact, if necessary replace Z_2 by $Z_2 - cZ_1$; this does not change e , Q_e , etc. (it changes $\gamma = \gamma_1 + i\gamma_2$, though, but does not affect $\mathfrak{h}_0$ and rank $(\alpha_2, \gamma_1, \gamma_2)$).

Set $p = \min \{ 1 \leq j \leq m \mid Z_j \notin \mathfrak{h}_{\mathbb{C}} \}$. p is well-defined, and $4 \leq p \leq m-1$, since $Z_1, Z_2, Z_3 \in \mathfrak{h}_{\mathbb{C}}$, and since the codimension of $\mathfrak{h}_{\mathbb{C}}$ is 2. Set $q = \min \{ 1 \leq j \leq m \mid Z_j \notin \mathbb{C}Z_p \oplus \mathfrak{h}_{\mathbb{C}} \}$. q is well-defined and $5 \leq p+1 \leq q \leq m$ (so $\dim \mathfrak{g} \geq 6$).

We first note $2, 3 \in J_g$. In fact, if $2 \notin J_g$, then $Z_2 \in \mathfrak{f}_1 + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$\gamma(Z) \langle g, Z_1 \rangle = \langle g, [Z, Z_2] \rangle = \langle Z_2 g, Z \rangle = 0 \quad \text{for all } Z \in \mathfrak{g}_{\mathbb{C}}$$

which is a contradiction. So $2 \in J_g$. If $3 \notin J_g$, then $\bar{Z}_2 \in \mathfrak{f}_2 + (\mathfrak{g}_g)_{\mathbb{C}}$, i. e. $\bar{Z}_2 = aZ_2 \pmod{(\mathfrak{g}_g)_{\mathbb{C}}}$, $a \in \mathbb{C}$. But then

$$\overline{\gamma(\bar{Z})} \langle g, \bar{Z}_1 \rangle = \langle g, [Z, \bar{Z}_2] \rangle = \langle \bar{Z}_2 g, Z \rangle = a \langle Z_2 g, Z \rangle = a \gamma(Z) \langle g, Z_1 \rangle$$

which contradicts the fact that $\text{codim } \mathfrak{h} = 2$, so $3 \in J_g$. We also note that $1 \notin J_g$, since $\mathfrak{f}_1 \subset (\mathfrak{g}_g)_{\mathbb{C}}$.

Next we note that $p, q \in J_g$. In fact, if $p \notin J_g$, then $Z_p \in \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$ and $\bar{Z}_p \in \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$\begin{aligned} -\gamma(Z_p) \langle g, Z_1 \rangle &= \langle g, [Z_2, Z_p] \rangle = \langle Z_p g, Z_2 \rangle \subset \langle \mathfrak{h}_{\mathbb{C}} g, Z_2 \rangle \\ &= \langle g, [\mathfrak{h}_{\mathbb{C}}, Z_2] \rangle = \langle g, \mathbb{C}Z_2 \rangle = 0, \end{aligned}$$

so $\gamma(Z_p) = 0$ and similarly $\gamma(\bar{Z}_p) = 0$ implying that $Z_p \in \mathfrak{h}_{\mathbb{C}}$, which is a contradiction. Therefore $p \in J_g$. Suppose then that $q \notin J_g$. Then $Z_q \in \mathbb{C}Z_p + \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, i. e. there exists $a \in \mathbb{C}$ with $Z_q = aZ_p \pmod{(\mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}})}$. But then

$$-\gamma(Z_q) \langle g, Z_1 \rangle = \langle g, [Z_2, Z_q] \rangle = \langle Z_q g, Z_2 \rangle = a \langle Z_p g, Z_2 \rangle = -a \gamma(Z_p) \langle g, Z_1 \rangle,$$

from which $\gamma(Z_q) = a\gamma(Z_p)$. Similarly $\overline{\gamma(\bar{Z}_q)} = a\overline{\gamma(\bar{Z}_p)}$. Now consider the linear map from $\mathfrak{g}_{\mathbb{C}}$ to $\mathbb{C}^2$ given by $Z \rightarrow (\gamma(Z), \overline{\gamma(\bar{Z})})$. The kernel is $\mathfrak{h}_{\mathbb{C}}$, so it is surjective since $\text{codim } \mathfrak{h} = 2$. But Z_p, Z_q is a basis for $\mathfrak{g}_{\mathbb{C}} \pmod{\mathfrak{h}_{\mathbb{C}}}$, and we have just shown that the images of Z_p and of Z_q are linearly dependent; in fact, $(\gamma(Z_q), \overline{\gamma(\bar{Z}_q)}) = a(\gamma(Z_p), \overline{\gamma(\bar{Z}_p)})$. But this is a contradiction, and we conclude that $q \in J_g$.

Define $\hat{Z}_j = Z_j$ for $1 \leq j \leq p-1$, $\hat{Z}_j = Z_{j+1} + a_{j+1}Z_p$ for $p \leq j \leq q-2$ (empty if $q = p+1$), $\hat{Z}_j = Z_{j+2} + a_{j+2}Z_p + b_{j+2}Z_q$ for $q-1 \leq j \leq m-2$, $Z_{m-1} = aZ_p + bZ_q$, $Z_m = a'Z_p + b'Z_q$, where $a_{p+1}, \dots, a_{q-1}, a_{q+1}, \dots, a_m, b_{q+1}, \dots, b_m$ has been picked such that $\hat{Z}_j \in \mathfrak{h}_{\mathbb{C}}$, $1 \leq j \leq m-2$; this is possible since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}Z_p \oplus \mathbb{C}Z_q$. The numbers $a, b, a', b' \in \mathbb{C}$ has been selected such that $ab' - a'b = 1$, and such that $\langle g, [\hat{Z}_{m-1}, Z_2] \rangle = 0$, $\langle g, [\hat{Z}_{m-1}, Z_3] \rangle \neq 0$, $\langle g, [\hat{Z}_m, Z_3] \rangle = 0$, $\langle g, [\hat{Z}_m, Z_2] \rangle \neq 0$ which is possible by a reasoning as above. Clearly $\hat{Z}_1, \dots, \hat{Z}_m$ is a basis for $\mathfrak{g}_{\mathbb{C}}$. Set $\hat{\mathfrak{f}}_j = \mathbb{C}\hat{Z}_1 \oplus \dots \oplus \mathbb{C}\hat{Z}_j$. For $0 \leq j \leq p-1$ we have

$$\begin{aligned}\hat{f}_j &= f_j && \text{for } 0 \leq j \leq p-1, \\ \hat{f}_j &= f_{j+1} \cap h_C && \text{for } p-1 \leq j \leq q-2, \\ \hat{f}_j &= f_{j+2} \cap h_C && \text{for } q-2 \leq j \leq m-2, \\ \hat{f}_m &= g_C.\end{aligned}$$
$$h_C = \hat{f}_{m-2} \supset \dots \supset \hat{f}_1 \supset \hat{f}_0 = \{0\}$$

Write $e = \{j_1 < \dots < j_d\}$, and let $j_\alpha = p$, $j_\beta = q$ with $1 \leq \alpha < \beta \leq d$. Define the set $\hat{j}_g = \{\hat{j}_1 < \dots < \hat{j}_d\}$ by setting $\hat{j}_1 = j_1, \dots, \hat{j}_{\alpha-1} = j_{\alpha-1}, \hat{j}_h = j_{h+1} - 1$ for $\alpha \leq h \leq \beta - 2$, $\hat{j}_h = j_{h+2} - 2$ for $\beta - 1 \leq h \leq d - 2$, $\hat{j}_{d-1} = m - 1$, $\hat{j}_d = m$. We then have

$$\begin{aligned}\hat{Z}_{j_h} &= Z_{j_h} \quad \text{for } 1 \leq h \leq \alpha - 1, \\ \hat{Z}_{j_h} &= Z_{j_{h+1}} + a_{j_{h+1}} Z_{j_\alpha} \quad \text{for } \alpha \leq h \leq \beta - 2, \\ \hat{Z}_{j_h} &= Z_{j_{h+2}} + a_{j_{h+2}} Z_{j_\alpha} + b_{j_{h+2}} Z_{j_\beta} \quad \text{for } \beta - 1 \leq h \leq d - 2, \\ \hat{Z}_{j_{d-1}} &= a Z_{j_\alpha} + b Z_{j_\beta}, \\ \hat{Z}_{j_d} &= a' Z_{j_\alpha} + b' Z_{j_\beta}.\end{aligned}$$

$$\begin{array}{c}
\alpha \rightarrow \\
C = \\
\beta \rightarrow
\end{array}
\left[\begin{array}{c|c|c|c|c|c}
1 & & & & & \\
& \ddots & & & & \\
& & 1 & & & \\
\hline
& a_{j_{\alpha+1}} \dots a_{j_{\beta-1}} & & a_{j_{\beta+1}} \dots a_{j_d} & a & a' \\
\hline
& 1 & & & & \\
& & \ddots & & & \\
& & & 1 & & \\
\hline
& & & b_{j_{\beta+1}} \dots b_{j_d} & b & b' \\
\hline
& & & 1 & & \\
& & & & \ddots & \\
& & & & & 1
\end{array} \right]
\begin{array}{c}
\alpha \\
\downarrow \\
\beta-2 \\
\downarrow \\
\beta-1 \\
\downarrow \\
d-2 \\
\downarrow \\
d-1 \\
\downarrow \\
d \\
\downarrow
\end{array}$$
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$\hat{M}_e(l)$ is the matrix $[B_l(\hat{Z}_{\hat{j}_r}, \hat{Z}_{\hat{j}_s})]_{1 \leq r, s \leq d}$. Now $\det C = (-1)^{\alpha+\beta}$, and therefore we have for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$:

$$\begin{aligned} Q_e(l) &= |\det M_e(l)| = |\det \hat{M}_e(l)| \\ &= \left| \sum_{\sigma \in S_d} \text{sign } \sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_d}, \hat{Z}_{\hat{j}_{\sigma(d)}}] \rangle \right| \\ &= |\langle l, [\hat{Z}_2, \hat{Z}_m] \rangle|^2 |\langle l, [\hat{Z}_3, \hat{Z}_{m-1}] \rangle|^2 \\ &\quad \cdot \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(3)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \right|, \end{aligned}$$

where S_d^* is the set of permutations σ in S_d such that $\sigma(1)=d$, $\sigma(2)=d-1$, $\sigma(d-1)=2$, $\sigma(d)=1$.

Set $g_0 = g|_{\mathfrak{h}}$, and let $\hat{J}_{g_0}^0$ etc. designate the objects defined relative to the Jordan-Hölder sequence $\hat{f}_0 \subset \hat{f}_1 \subset \dots \subset \hat{f}_{m-2} = \mathfrak{h}_{\mathbb{C}}$. Since clearly $\mathfrak{g}_g \subset \mathfrak{h}$, and $(\mathfrak{h}_{g_0})_{\mathbb{C}} = (\mathfrak{g}_g)_{\mathbb{C}} + \mathbb{C}Z_2 + \mathbb{C}Z_3$ we find that $1, 2, 3 \notin \hat{J}_{g_0}^0$, and for $4 \leq j \leq p-1$ we find $j \in \hat{J}_{g_0}^0 \Leftrightarrow j \in J_g$. For $p+1 \leq j \leq q-2$ we have

$$j \notin J_g \Leftrightarrow Z_j \in \hat{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \in \hat{f}_{j-2} + \mathbb{C}Z_p + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-1} \in \hat{f}_{j-2} + (\mathfrak{h}_{g_0})_{\mathbb{C}} \Leftrightarrow j-1 \in \hat{J}_{g_0}^0,$$

so $j \in J_g \Leftrightarrow j-1 \in \hat{J}_{g_0}^0$. For $q+1 \leq j \leq m$ we have

$$j \notin J_g \Leftrightarrow Z_j \in \hat{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \in \hat{f}_{j-3} + \mathbb{C}Z_p + \mathbb{C}Z_q + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-2} \in \hat{f}_{j-3} + (\mathfrak{h}_{g_0})_{\mathbb{C}} \Leftrightarrow j-2 \notin \hat{J}_{g_0}^0,$$

so $j \in J_g \Leftrightarrow j-2 \in \hat{J}_{g_0}^0$. Therefore, if $\hat{e}^0 = \hat{J}_{g_0}^0 = \{\hat{j}_1^0 < \dots < \hat{j}_{d-4}^0\}$ we find that $\hat{j}_h^0 = j_{h+2}$ for $1 \leq h \leq \alpha-3$, $\hat{j}_h^0 + 1 = j_{h+3}$ for $\alpha-2 \leq h \leq \beta-4$, $\hat{j}_h^0 + 2 = j_{h+4}$ for $\beta-3 \leq h \leq d-4$, and comparing with the definition of $\hat{j}_h$ we find that $\hat{j}_h^0 = \hat{j}_{h+2}$ for $1 \leq h \leq d-4$. Using this we get for $l_0 \in \mathfrak{h}'$:

$$\begin{aligned} Q_{e_0}(l_0) &= \left| \sum_{\sigma \in S_{d-4}} \text{sign } \sigma \langle l_0, [\hat{Z}_{\hat{j}_1^0}, \hat{Z}_{\hat{j}_{\sigma(1)}^0}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-4}^0}, \hat{Z}_{\hat{j}_{\sigma(d-4)}^0}] \rangle \right| \\ &= \left| \sum_{\sigma \in S_{d-4}} \text{sign } \sigma \langle l_0, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(1)+2}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)+2}}] \rangle \right| \\ &= \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l_0, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(3)}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \right|, \end{aligned}$$

and comparing with what we saw above we find for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$ and $l_0 = l|_{\mathfrak{h}}$:

$$(*) \quad Q_e(l) = |\langle l, [\hat{Z}_2, \hat{Z}_m] \rangle|^2 |\langle l, [\hat{Z}_3, \hat{Z}_{m-1}] \rangle|^2 Q_{e_0}(l_0).$$

Let us now observe that the nilradical of $\mathfrak{h}$ is $\mathfrak{m}$. In fact, since $Z_2 \in \mathfrak{h}$, $\lambda_2|_{\mathfrak{h}}$ is a root for $\mathfrak{h}$, and therefore the nilradical of $\mathfrak{h}$ is contained in $\mathfrak{h}_0$ and consequently it is precisely $\mathfrak{m}$.

Write $Z_2 = X_2 + iY_2$ and set $\mathfrak{b} = \mathbb{R}X_2 \oplus \mathbb{R}Y_2$. Then $\mathfrak{b}$ is an ideal in $\mathfrak{h}$, and $g|_{\mathfrak{b}} = 0$. Let $c: \mathfrak{h} \rightarrow \mathfrak{h}/\mathfrak{b} = \tilde{\mathfrak{h}}$ be the coset map and define $\tilde{g}_0 \in \tilde{\mathfrak{h}}'$ by $\tilde{g}_0 \circ c = g_0$.

We now claim that $u_e \in U(\mathfrak{m})$, i. e. that Q_e only depends on its restriction to $\mathfrak{h}$ (and therefore to $\mathfrak{m}$). Assuming for a moment this claim to be true, we consider Q_e as a polynomial function on $\mathfrak{h}'$ and get for $\tilde{l}_0 \in \tilde{\mathfrak{h}}'$ (using the formula (*)):

$$Q_e(\tilde{l}_0 \circ c) = |\langle l_0, W_1 \rangle|^2 |\langle l_0, W_2 \rangle|^2 Q_{e_0}(\tilde{l}_0 \circ c),$$

where $W_1 = c([\hat{Z}_2, \hat{Z}_m])$, $W_2 = c([\hat{Z}_3, \hat{Z}_{m-1}])$. Now since $W_1, \bar{W}_1, W_2, \bar{W}_2$ are central

in $\tilde{\mathfrak{h}}$ and since $P_{W_1}(\tilde{l}_0) = \langle W_1, \tilde{l}_0 \rangle$, $P_{\bar{W}_1}(\tilde{l}_0) = \overline{\langle W_1, \tilde{l}_0 \rangle}$, and similarly for W_2 , we find that $i^d Q_e(\tilde{l}_0 \circ c) = P_{W_1}(\tilde{l}_0) P_{\bar{W}_1}(\tilde{l}_0) P_{W_2}(\tilde{l}_0) P_{\bar{W}_2}(\tilde{l}_0) i^{d-4} Q_{\tilde{e}^0}(\tilde{l}_0 \circ c)$, and therefore

$$c(u_e) = W_1 \bar{W}_1 W_2 \bar{W}_2 c(u_{\tilde{e}^0})$$

by Lemma 1.2.1.

Since $\mathfrak{b}$ is an abelian ideal in $\mathfrak{h}_0$ and since $\langle f_0, \mathfrak{b} \rangle = 0$ there exists a representation $\tilde{\pi}_0$ of $\tilde{M} = M/B$, $B = \exp \mathfrak{b}$, such that $\tilde{\pi}_0 \circ c = \pi_0$. Using the induction hypothesis we then get

$$\begin{aligned} d\pi_0(u_e) &= d\tilde{\pi}_0(c(u_e)) = d\tilde{\pi}_0(W_1 \bar{W}_1 W_2 \bar{W}_2 c(u_{\tilde{e}^0})) = |\langle f_0, W_1 \rangle|^2 |\langle f_0, W_2 \rangle|^2 d\pi_0(u_{\tilde{e}^0}) \\ &= |\langle g, [\hat{Z}_2, \hat{Z}_m] \rangle|^2 |\langle g, [\hat{Z}_3, \hat{Z}_{m-1}] \rangle|^2 Q_{\tilde{e}^0}(g_0) I = Q_e(g) I. \end{aligned}$$

We have thus shown that $d\pi_0(u_e) = Q_e(g) I$. This ends case (c11) (i), except for the fact that we have to prove the claim from above:

Proof of claim: We shall prove that $Q_e(l)$ only depends on the restriction of l to $\mathfrak{h}$. If all $\hat{Z}_{j_r}$, $3 \leq r \leq d-2$ belong to $(\mathfrak{h}_0)_{\mathbb{C}}$, then the result is clear (because $\mathfrak{h}_0$ is an ideal). Suppose then that there exists $3 \leq r \leq d-2$ such that $\hat{Z}_{j_r} \notin (\mathfrak{h}_0)_{\mathbb{C}}$, and let ρ be the smallest such r . We then have $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}\hat{Z}_{j_\rho} \oplus (\mathfrak{h}_0)_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\hat{Z}_{j_\rho} \oplus \ker \alpha_2$. Set $Y_r = \hat{Z}_{j_r} + c_r \hat{Z}_{j_\rho}$, $r = 1, \dots, d$, where c_r is defined such that $Y_r \in \ker \alpha_2$ for $r \neq \rho$, and where $c_\rho = 0$. We then have $Y_r \in (\mathfrak{h}_0)_{\mathbb{C}}$ for $1 \leq r \leq d-2$, $r \neq \rho$, while $Y_\rho \notin (\mathfrak{h}_0)_{\mathbb{C}}$ and $Y_d, Y_{d-1} \in \ker \alpha_2$. We also have $[Y_d, Z_2] \in \mathbb{C}Z_1$, $[Y_d, Z_3] = 0$, $[Y_{d-1}, Z_3] \in \mathbb{C}Z_1$, $[Y_{d-1}, Z_2] = 0$.

Letting $C = [c_{rs}]_{1 \leq r, s \leq d}$ be the $d \times d$ -matrix given by:

$$\left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & c_{\rho+1} & \dots & c_d \\ \hline & & & & 1 & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{array} \right]$$

the empty entries meaning zero, we have $Y_s = \sum_{r=1}^d c_{rs} \hat{Z}_{j_r}$, so, setting $N(l) = [B_l(Y_r, Y_s)]_{1 \leq r, s \leq d}$ we get $N(l) = {}^t \hat{C} \hat{M}_e(l) C$, from which $Q_e(l) = |\det M_e(l)| = |\det \hat{M}_e(l)| = |\det N(l)|$. Therefore

$$Q_e(l) = \left| \sum_{\sigma \in S_d} \text{sign } \sigma P_\sigma(l) \right|,$$

where we have set $P_\sigma(l) = \langle l, [Y_1, Y_{\sigma(1)}] \rangle \dots \langle l, [Y_d, Y_{\sigma(d)}] \rangle$.

Define the following subsets of S_d :

$$\begin{aligned} S_d^{(1)} &= \{ \sigma \mid \sigma(1) = \rho, \sigma(2) = d-1, \sigma(\rho) = d, \sigma(d-1) = 2, \sigma(d) = 1 \}, \\ S_d^{(2)} &= \{ \sigma \mid \sigma(1) = d, \sigma(2) = \rho, \sigma(\rho) = d-1, \sigma(d-1) = 2, \sigma(d) = 1 \}, \\ S_d^{(3)} &= \{ \sigma \mid \sigma(1) = d, \sigma(2) = d-1, \sigma(\rho) = 1, \sigma(d-1) = 2, \sigma(d) = \rho \}, \\ S_d^{(4)} &= \{ \sigma \mid \sigma(1) = d, \sigma(2) = d-1, \sigma(\rho) = 2, \sigma(d-1) = \rho, \sigma(d) = 1 \}, \\ S_d^{(5)} &= \{ \sigma \mid \sigma(\rho) \neq d \wedge \sigma(\rho) \neq d-1 \wedge \rho \neq \sigma(d) \wedge \rho \neq \sigma(d-1) \}, \\ S_d^{(6)} &= S_d \setminus \bigcup_{j=1}^5 S_d^{(j)}. \end{aligned}$$

We then assert that $P_\sigma = 0$ if $\sigma \in S_d^{(6)}$. In fact, observe first that

$$P_\sigma \neq 0 \Rightarrow (\sigma(1) = \rho \vee \sigma(1) = d) \wedge (\sigma(2) = \rho \vee \sigma(2) = d-1) \\ \wedge (1 = \sigma(\rho) \vee 1 = \sigma(d)) \wedge (2 = \sigma(\rho) \vee 2 = \sigma(d-1)).$$

Therefore, if $P_\sigma \neq 0$ and if $\sigma \notin S_d^{(5)}$ with e. g. $\sigma(\rho) = d$, then $\sigma(1) = \rho$, $\sigma(2) = d-1$, $\sigma(d) = 1$, $\sigma(d-1) = 2$, so $\sigma \in S_d^{(1)}$. Similarly, if $\sigma \notin S_d^{(5)}$ with $\sigma(\rho) = d-1$, then $P_\sigma \neq 0 \Rightarrow \sigma \in S_d^{(2)}$, etc. This shows our assertion.

We next assert that $P(l) = \sum_{j=1}^4 \sum_{\sigma \in S_d^{(j)}} \text{sign } \sigma P_\sigma(l) = 0$. To see this, define the permutations $\tau_1, \tau_2, \tau_3, \tau_4$ in S_d by $\tau_1 = \text{identity}$, $\tau_2(1) = \rho$, $\tau_2(2) = 1$, $\tau_2(\rho) = 2$, $\tau_3(1) = \rho$, $\tau_3(\rho) = d$, $\tau_3(d) = 1$, $\tau_4(1) = \rho$, $\tau_4(\rho) = d-1$, $\tau_4(d-1) = 1$, all other elements left fixed. It is then immediate to verify that the map $\sigma \rightarrow \sigma \circ \tau_j$, $j = 1, 2, 3, 4$, defines a bijection between $S_d^{(1)}$ and $S_d^{(j)}$, and since τ_j are even permutations we get

$$P(l) = \sum_{\sigma \in S_d^{(1)}} \text{sign } \sigma \sum_{j=1}^4 P_{\sigma \circ \tau_j}(l).$$

Now for $\sigma \in S_d^{(1)}$ we have

$$\sum_{j=1}^4 P_{\sigma \circ \tau_j}(l) = \prod_{\substack{i=1 \\ i \neq 1, 2, \rho, \\ d-1, d}}^d \langle l, [Y_i, Y_{\sigma(i)}] \rangle \left(\sum_{j=1}^4 \prod_{\substack{i=1, 2, \rho, \\ d-1, d}}^d \langle l, [Y_i, Y_{\sigma(\tau_j(i))}] \rangle \right),$$

and a direct computation shows that

$$\sum_{j=1}^4 \prod_{\substack{i=1, 2, \rho, \\ d-1, d}}^d \langle l, [Y_i, Y_{\sigma(\tau_j(i))}] \rangle = 0$$

for all $l \in \mathfrak{g}'$. This shows that $P \equiv 0$, and therefore we have

$$Q_e(l) = \left| \sum_{\sigma \in S_d^{(5)}} \text{sign } \sigma P_\sigma(l) \right|.$$

But we clearly have that $P_\sigma(l)$ only depends on the restriction of l to $\mathfrak{h}$ if $\sigma \in S_d^{(5)}$, because all $[Y_r, Y_{\sigma(r)}]$, $r = 1, \dots, d$, then belong to $\mathfrak{h}$ (we use here that $\mathfrak{h}$ is a subalgebra and that $\mathfrak{h}_0$ is an ideal). This proves our claim and ends (i).

(ii) We now apply (i) to the same Jordan-Hölder sequence $\mathfrak{f}_j$, but to another basis $Z'_j \in \mathfrak{f}_j \setminus \mathfrak{f}_{j-1}$ (whereby $\mathfrak{h}_0$ and therefore $\mathfrak{m}$ are not changed), and we get similarly that $d\pi_0(u'_e) = Q'_e(g)I$, where Q'_e, u'_e are the objects associated with this new basis. Setting in particular $Z'_j = \text{Ad}(s)Z_j$, we get $u'_e = \text{Ad}(s)u_e$, and $Q'_e(l) = Q_e(s^{-1}l)$ for $s \in G$, and therefore $d\pi_0(\text{Ad}(s)u_e) = Q_e(s^{-1}g)I = |\Lambda_e(s)|^2 Q_e(g)I$.

Now since $Z_1, Z_2, \bar{Z}_2 \in \mathfrak{m}_{\mathbb{C}}$ it follows that $\mathfrak{n}_f \subset \mathfrak{m}$ and from this we get that

$$(\mathfrak{m}_{f_0})_{\mathbb{C}} = (\mathfrak{n}_f)_{\mathbb{C}} \oplus \mathbb{C}Z_2 \oplus \mathbb{C}\bar{Z}_2.$$

It follows that a polarization in $\mathfrak{m}$ at f_0 is also a polarization in $\mathfrak{n}$ at f , hence $\pi = \text{ind}_{\mathfrak{M} \uparrow \mathfrak{N}} \pi_0$. Let then φ be a differentiable vector in $L^2(\mathfrak{N}, \pi_0)$, the space of the induced representation $\pi = \text{ind}_{\mathfrak{M} \uparrow \mathfrak{N}} \pi_0$. We have $d\pi(u_e)\varphi(s) = d\pi_0(\text{Ad}(s^{-1})u_e)\varphi(s) = Q_e(g)\varphi(s)$, $s \in \mathfrak{N}$, so $d\pi(u_e) = Q_e(g)I$. This ends case (c11).

Case (c12): (i) As in case (c11) we start by showing that $u_e \in U(m)$ and that $d\pi_0(u_e) = Q_e(g)I$, and we can assume that $\langle g, Z_2 \rangle = 0$.

Since $[f_3, f_2] = f_1$ we have that $Z_2, Z_3 \notin \mathfrak{h}_C$. Therefore $\mathfrak{g}_C = \mathbb{C}Z_2 \oplus \mathbb{C}Z_3 \oplus \mathfrak{h}_C$. Just like in case (c11) we see that $2, 3 \in J_g$. Define $\hat{Z}_1 = Z_1$, $\hat{Z}_j = Z_{j+2} + a_{j+2}Z_2 + b_{j+2}Z_3$ for $2 \leq j \leq m-2$, $\hat{Z}_{m-1} = Z_2$, $\hat{Z}_m = Z_3$, where $a_4, \dots, a_m, b_4, \dots, b_m$ have been picked such that $\hat{Z}_j \in \mathfrak{h}_C$, $1 \leq j \leq m-2$. Clearly $\hat{Z}_1, \dots, \hat{Z}_m$ is a basis for $\mathfrak{g}_C$. Set $\hat{f}_j = \mathbb{C}\hat{Z}_1 \oplus \dots \oplus \mathbb{C}\hat{Z}_j$. We have that $\hat{f}_1 = f_1$ and $\hat{f}_j \oplus \mathbb{C}Z_2 \oplus \mathbb{C}Z_3 = f_{j+2}$ for $1 \leq j \leq m-2$. Also $f_{m-2} = \mathfrak{h}_C$, $f_m = \mathfrak{g}_C$. We thus have

$$\begin{aligned}\hat{f}_1 &= f_1, \\ \hat{f}_j &= f_{j+2} \cap \mathfrak{h}_C \quad \text{for } 1 \leq j \leq m-2, \\ \hat{f}_m &= \mathfrak{g}_C.\end{aligned}$$

From this it follows that $\mathfrak{h}_C = \hat{f}_{m-2} \supset \dots \supset \hat{f}_1 \supset \hat{f}_0 = \{0\}$ is a Jordan-Hölder sequence for $\mathfrak{h}_C$. We claim it is of class (b). But this follows easily from the fact that $\bar{f}_1 = f_1$.

Write $e = \{j_1 < \dots < j_d\}$, and define the set $\hat{J}_g = \{\hat{j}_1 < \dots < \hat{j}_d\}$ by setting $\hat{j}_h = j_{h+2} - 2$ for $1 \leq h \leq d-2$, $\hat{j}_{d-1} = m-1$, $\hat{j}_d = m$. We then have

$$\begin{aligned}\hat{Z}_{\hat{j}_h} &= Z_{j_{h+2}} + a_{j_{h+2}}Z_{j_1} + b_{j_{h+2}}Z_{j_2} \quad \text{for } 1 \leq h \leq d-2, \\ \hat{Z}_{\hat{j}_{d-1}} &= Z_{j_1}, \\ \hat{Z}_{\hat{j}_d} &= Z_{j_2}.\end{aligned}$$

Therefore, letting $C = [c_{rs}]_{1 \leq r,s \leq d}$ be the $d \times d$ -matrix:

$$C = \left[\begin{array}{ccc|c|c} a_{j_3} & \dots & a_{j_d} & 1 & \\ \hline b_{j_3} & \dots & b_{j_d} & & 1 \\ \hline 1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & 1 & \end{array} \right],$$

where the empty entries are zero, we have $\hat{Z}_{\hat{j}_s} = \sum_{r=1}^d c_{rs} Z_{j_r}$, and $\hat{M}_e(l) = \hat{C} M_e(l) C$, where $\hat{M}_e(l)$ is the matrix $[B_l(\hat{Z}_{\hat{j}_r}, \hat{Z}_{\hat{j}_s})]_{1 \leq r,s \leq d}$. Now $\det C = 1$, and therefore we have for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$:

$$\begin{aligned}Q_e(l) &= |\det \hat{M}_e(l)| \\ &= \left| \sum_{\sigma \in S_d} \text{sign } \sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_d}, \hat{Z}_{\hat{j}_{\sigma(d)}}] \rangle \right| \\ &= |\langle l, [\hat{Z}_{m-1}, \hat{Z}_m] \rangle|^2 \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \right|,\end{aligned}$$

where S_d^* is the set of permutations σ in S_d such that $\sigma(d-1) = d$, $\sigma(d) = d-1$.

Set $g_0 = g|_{\mathfrak{h}}$, and let $\hat{J}_{g_0}^0$, etc. designate the objects defined relative to the Jordan-Hölder sequence $\hat{f}_0 \subset \hat{f}_1 \subset \dots \subset \hat{f}_{m-2} = \mathfrak{h}_C$. Since clearly $g_g \subset \mathfrak{h}$, and $\mathfrak{h}_{g_0} = g_g$ we find that $1 \notin \hat{J}_{g_0}^0$, and for $4 \leq j \leq m$ we have

$$j \notin J_g \Leftrightarrow Z_j \in \hat{f}_{j-1} + (g_g)_C \Leftrightarrow Z_j \in \hat{f}_{j-3} + \mathbb{C}Z_2 + \mathbb{C}Z_3 + (g_g)_C \Leftrightarrow \hat{Z}_{j-2} \in \hat{f}_{j-3} + (\mathfrak{h}_{g_0})_C \Leftrightarrow j-2 \notin \hat{J}_{g_0}^0,$$

so $j \in J_g \Leftrightarrow j - 2 \in \hat{J}_{g_0}^0$. Therefore, if $\hat{e}^0 = \hat{J}_{g_0}^0 = \{ \hat{j}_1^0 < \dots < \hat{j}_{d-2}^0 \}$ we find that $\hat{j}_h^0 + 2 = j_{h+2}$ for $1 \leq h \leq d-2$, and comparing with the definition of $\hat{j}_h$ we find that $\hat{j}_h^0 = \hat{j}_h$ for $1 \leq h \leq d-2$. Using this we get for $l_0 \in h'$:

$$\begin{aligned} Q_{\hat{e}^0}(l_0) &= |\det [B_{l_0}(\hat{Z}_{\hat{j}_p}, \hat{Z}_{\hat{j}_q})]_{1 \leq r, s \leq d-2}| \\ &= \left| \sum_{\sigma \in S_{d-2}} \text{sign } \sigma \langle l_0, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \right| \\ &= \left| \sum_{\sigma \in S_d^*} \text{sign } \sigma \langle l_0, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \right|, \end{aligned}$$

and comparing with what we saw above we find for $l \in g'$ with $\langle l, Z_2 \rangle = 0$:

$$(*) \quad Q_e(l) = |\langle l, [\hat{Z}_m, \hat{Z}_{m-1}] \rangle|^2 Q_{\hat{e}^0}(l_0),$$

where $l_0 = l|_{\mathfrak{h}}$.

We then claim that $Q_e(l)$ only depends on the restriction of l to $\mathfrak{h}$. Assuming for a moment this to be true, we find that the formula $(*)$ is valid for all $l \in g'$. The conclusion of this is that $i^d Q_e(l) = -\langle l, W \rangle \langle l, \bar{W} \rangle i^{d-2} Q_{\hat{e}^0}(l_0)$, where $W = [\hat{Z}_m, \hat{Z}_{m-1}] = [Z_2, Z_3] \in \mathfrak{f}_1$, and therefore that $u_e = -W\bar{W}u_{\hat{e}^0}$.

Let us then prove our claim: if all $\hat{Z}_{\hat{j}_r}$, $1 \leq r \leq d-2$ belong to $(\mathfrak{h}_0)_{\mathbb{C}}$, then the result is clear (since $\mathfrak{h}_0$ is an ideal). Suppose then that there exists $1 \leq r \leq d-2$ such that $\hat{Z}_{\hat{j}_r} \notin (\mathfrak{h}_0)_{\mathbb{C}}$, and let ρ be the smallest such number r . We then have $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}\hat{Z}_{\hat{j}_\rho} \oplus (\mathfrak{h}_0)_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}\hat{Z}_{\hat{j}_\rho} \oplus \ker \alpha_2$. Set $Y_r = \hat{Z}_{\hat{j}_r} + c_r \hat{Z}_{\hat{j}_\rho}$, $r = 1, \dots, d$, where c_r is defined such that $Y_r \in \ker \alpha_2$ for $r \neq \rho$ and where $c_\rho = 0$. We then have $Y_r \in (\mathfrak{h}_0)_{\mathbb{C}}$ for all $1 \leq r \leq d-2$, $r \neq \rho$, while $Y_\rho \in (\mathfrak{h})_{\mathbb{C}}$. $Y_d = \hat{Z}_m = \hat{Z}_{\hat{j}_d}$, $Y_{d-1} = \hat{Z}_{m-1} = \hat{Z}_{\hat{j}_{d-1}} \in \ker \alpha_2$.

Letting $C = [c_{rs}]_{1 \leq r, s \leq d}$ be the $d \times d$ -matrix given by:

$$C = \left[\begin{array}{ccc|ccc|cc} 1 & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & 1 & & & \\ \hline & & & & & 1 & c_{\rho+1} & \dots & c_{d-2} & 0 & 0 \\ \hline & & & & & & 1 & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & \\ & & & & & & & & & & 1 \end{array} \right]$$

the empty entries meaning zero, we have $Y_s = \sum_{r=1}^d c_{rs} \hat{Z}_{\hat{j}_r}$, so, setting $N(l) = [B_l(Y_r, Y_s)]_{1 \leq r, s \leq d}$ we get $N(l) = {}^t C \hat{M}_e(l) C$, from which $Q_e(l) = |\det M_e(l)| = |\det \hat{M}_e(l)| = |\det N(l)|$. Therefore $Q_e(l) = |\sum_{\sigma \in S_d} \text{sign } \sigma P_\sigma(l)|$, where we have set $P_\sigma(l) = \langle l, [Y_1, Y_{\sigma(1)}] \rangle \dots \langle l, [Y_d, Y_{\sigma(d)}] \rangle$. Define then the following subsets of S_d :

$$\begin{aligned} S_d^{(1)} &= \{ \sigma \mid \sigma(d-1) = \rho \wedge \sigma(d) = d-1 \wedge \sigma(\rho) = d \}, \\ S_d^{(2)} &= \{ \sigma \mid \sigma(d-1) = d \wedge \sigma(d) = \rho \wedge \sigma(\rho) = d-1 \}, \\ S_d^{(3)} &= \{ \sigma \mid \sigma(d-1) \neq \rho \wedge \sigma(d) \neq \rho \}, \\ S_d^{(4)} &= S_d \setminus (S_d^{(1)} \cup S_d^{(2)} \cup S_d^{(3)}). \end{aligned}$$

We then assert that: $\sigma \in S_d^{(4)} \Rightarrow P_\sigma = 0$. In fact, observe first that since:

$$[Y_{d-1}, Y_r] \neq 0 \Rightarrow r = d \vee r = \rho,$$

and since: $[Y_d, Y_r] \neq 0 \Rightarrow r = d-1 \vee r = \rho$ we have:

$$P_\sigma \neq 0 \Rightarrow (\sigma(d-1) = d \vee \sigma(d-1) = \rho) \wedge (\sigma(d) = d-1 \vee \sigma(d) = \rho).$$

Moreover, if $r \notin \{\rho, d-1, d\}$, then: $[Y_r, Y_{\sigma(r)}] \neq 0 \Rightarrow \sigma(r) \neq d-1 \wedge \sigma(r) \neq d$, hence:

$$P_\sigma \neq 0 \Rightarrow (d = \sigma(d-1) \vee d = \sigma(\rho)) \wedge (d-1 = \sigma(d) \vee d-1 = \sigma(\rho)).$$

Therefore, if $P_\sigma \neq 0$ and $\sigma \notin S_d^{(3)}$ with e. g. $\sigma(d-1) = \rho$, then $\sigma(d) = d-1$ and $\sigma(\rho) = d$, and therefore $\sigma \in S_d^{(1)}$. Similarly, if $\sigma \notin S_d^{(3)}$ with $\sigma(d) = \rho$, then $P_\sigma \neq 0 \Rightarrow \sigma \in S_d^{(2)}$. This shows our assertion.

We next assert that $P(l) = \sum_{\sigma \in S_d'} \text{sign } \sigma P_\sigma(l) = 0$, where $S_d' = S_d^{(1)} \cup S_d^{(2)}$. To see this, define the permutation τ in S_d by $\tau(\rho) = d$, $\tau(d-1) = \rho$, $\tau(d) = d-1$, all other elements left fixed. It is then immediate to verify that the map $\sigma \rightarrow \sigma \circ \tau$ defines a bijection between $S_d^{(1)}$ and $S_d^{(2)}$, and since τ is an even permutation we get $P(l) = \sum_{\sigma \in S_d^{(1)}} \text{sign } \sigma (P_\sigma + P_{\sigma \circ \tau})$. Now for $\sigma \in S_d^{(1)}$ we have

$$P_\sigma(l) + P_{\sigma \circ \tau}(l) = \prod_{\substack{i \neq \rho, \\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle \left(\prod_{\substack{i \neq \rho, \\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle + \prod_{\substack{i \neq \rho, \\ d-1, d}} \langle l, [Y_i, Y_{\sigma(\tau(i))}] \rangle \right),$$

and

$$\begin{aligned} & \prod_{\substack{i \neq \rho, \\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle + \prod_{\substack{i \neq \rho, \\ d-1, d}} \langle l, [Y_i, Y_{\sigma(\tau(i))}] \rangle \\ &= \langle l, [Y_\rho, Y_d] \rangle \langle l, [Y_{d-1}, Y_\rho] \rangle \langle l, [Y_d, Y_{d-1}] \rangle \\ &+ \langle l, [Y_\rho, Y_{d-1}] \rangle \langle l, [Y_{d-1}, Y_d] \rangle \langle l, [Y_d, Y_\rho] \rangle = 0. \end{aligned}$$

This shows that $P \equiv 0$, and therefore we have

$$Q_e(l) = \left| \sum_{\sigma \in S_d^{(3)}} \text{sign } \sigma P_\sigma(l) \right|,$$

and since $P_\sigma(l)$ only depends on the restriction of l to $\mathfrak{h}$ when $\sigma \in S_d^{(3)}$ we have proved our assertion.

Now if $\mathfrak{m}$ is the nilradical of $\mathfrak{h}$ it follows from the induction hypothesis that $d\pi_0(u_{\hat{e}_0}) = Q_{\hat{e}_0}(g_0)I$, and since $d\pi_0(W) = i \langle f_0, W \rangle$, and $u_e = -W\bar{W}u_{\hat{e}_0}$ we get that

$$d\pi_0(u_e) = |\langle f_0, W \rangle|^2 Q_{\hat{e}_0}(g_0)I = Q_e(g)I,$$

and this proves (i) in this case.

Suppose then that $\mathfrak{m}$ is not the nilradical $\mathfrak{m}_1$ of $\mathfrak{h}$. Then $\mathfrak{m} = \mathfrak{h}_0 \cap \mathfrak{m}_1$, and $\dim \mathfrak{m}_1/\mathfrak{m} = 1$. Setting $M_1 = \exp \mathfrak{m}_1$ we now face two possibilities (1) either π_0 extends to an irreducible representation π'_0 of M_1 or (2) $\text{ind}_{M_1/M} \pi_0 = \pi'_0$ is an irreducible representation of M_1 . In the first case we obviously get as above that $d\pi'_0(u_e) = Q_e(g)I$, and therefore $d\pi_0(u_e) = Q_e(g)I$.

In the second case we have $\pi'_0|_M = \int_{M_1/M}^\oplus s\pi_0 ds$, and therefore we get by the induction

hypothesis that $Q_e(g)I = d\pi'_0(u_e) = \int_{M_1/M}^{\oplus} d(s\pi_0)(u_e)ds$, from which $d(s\pi_0)(u_e) = Q_e(g)I$ for almost all s , hence for all s by continuity. This shows that $d\pi_0(u_e) = Q_e(g)I$, and ends (i).

(ii) Just like in case (c11) (ii) we conclude from (i) that

$$d\pi_0(\text{Ad}(s)u_e) = Q_e(s^{-1}g)I = |\Lambda_e(s)|^2 Q_e(g)I \quad \text{for all } s \in G.$$

Now since $Z_2, Z_3 \in \mathfrak{n}_C$, and since $[\mathfrak{m}, Z_2] = [\mathfrak{m}, Z_3] = 0$ we see at once that $\mathfrak{m}_{f_0} = \mathfrak{n}_f$. Suppose then that $\mathfrak{p}$ is a polarization in $\mathfrak{m}$ at f_0 . Then, writing $Z_2 = X_2 + iY_2$, $\mathfrak{p}_1 = \mathfrak{p} \oplus \mathbb{R}Y_2$ is a polarization in $\mathfrak{n}$ at f , and therefore $\pi = \text{ind}_{\mathfrak{p}_1 \uparrow \mathfrak{N}} \eta_1$, where η_1 is the unitary character on $P_1 = \exp \mathfrak{p}_1$ corresponding to $f|_{\mathfrak{p}_1}$. Similarly $\pi_0 = \text{ind}_{\mathfrak{P} \uparrow \mathfrak{M}} \eta$, where $P = \exp \mathfrak{p}$, $\eta = \eta_1|_P$. We then set $\mathfrak{n}_1 = \mathbb{R}Y_2$, and note that $\mathfrak{n}_1$ is a direct product of $\mathfrak{m}$ and $\mathbb{R}Y_2$. Let π_1 be the irreducible representation of $N_1 = \exp \mathfrak{n}_1$ with $\pi_1|_M = \pi_0$,

$$\pi_1(\exp tY_2) = e^{it\langle f, Y_2 \rangle}.$$

Then $\pi = \text{ind}_{\mathfrak{p}_1 \uparrow \mathfrak{N}} \eta_1 = \text{ind}_{N_1 \uparrow \mathfrak{N}} (\text{ind}_{\mathfrak{p}_1 \uparrow N_1} \eta_1) = \text{ind}_{N_1 \uparrow \mathfrak{N}} \pi_1$. Now noting that N_1 is a normal subgroup in $\mathfrak{N}$ and that clearly $d\pi_1(\text{Ad}(s)u_e) = Q_e(g)I$ for $s \in N$, we can end this case just like case (c11) (ii). This ends case (c12).

Case (c2): (i) Since $\bar{f}_1 = \bar{f}_1$ we can clearly assume that $Z_1 = X_1 \in \mathfrak{g}$. A standard argument shows that λ_2 must be a real root in this case, so $\lambda_2 = \alpha_2$. We claim that it is no loss of generality to assume that $\gamma_2 \equiv 0$. In fact, let a_1, a_2, b be real numbers, not all equal to zero, such that $0 = a_1\gamma_1 + a_2\gamma_2 + b\alpha_2$. Then $(a_1, a_2) \neq (0, 0)$, since $\alpha_2 \neq 0$, and we can assume that $a_1^2 + a_2^2 = 1$. Replacing Z_2 by $Z'_2 = (a_2 + ia)Z_2 - bZ_1$ does not change Q_e , and it is trivial to verify that $[X, X'_2] = \lambda_2(X)Z'_2 + \gamma'_1(X)Z_1$, where $\gamma'_1 = a_2\gamma_1 - a_1\gamma_2$. This proves the claim. So, from now on we assume that $Z_1 = X_1 \in \mathfrak{g}$, $\gamma = \gamma_1$, and writing $Z_2 = X_2 + iY_2$ we then have

$$[X, X_2] = \lambda_2(X)X_2 + \gamma(X)X_1$$

$$[X, Y_2] = \lambda_2(X)Y_2.$$

Set $\mathfrak{h} = \ker \gamma$. It follows from the formula (2.2.2) that $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$, and its codimension is 1. Set $\mathfrak{h}_0 = \ker \alpha_2|_{\mathfrak{h}} = \ker \text{ad } Z_2|_{\mathfrak{g}}$. $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$ of codimension 2.

Let $\mathfrak{m}$ be the nilradical of $\mathfrak{h}_0$. Since $\mathfrak{h}_0$ is an ideal we have that $\mathfrak{m} = \mathfrak{n} \cap \mathfrak{h}_0 = \mathfrak{n} \cap \mathfrak{h}$. Observe that $\dim \mathfrak{n}/\mathfrak{m} = 1$. In fact, pick $W \in \mathfrak{h} \setminus \mathfrak{h}_0$. We then have $\gamma([X, W]) = \lambda_2(W)\gamma(X)$ for $X \in \mathfrak{g}$. Choosing X such that $\gamma(X) = 1$ we get that $\gamma([X, W]) = \lambda_2(W) \neq 0$, and this shows that $[X, W]$ is a basis in $\mathfrak{n} \pmod{\mathfrak{m}}$.

Set $f_0 = f|_{\mathfrak{m} = \mathfrak{g}|_{\mathfrak{m}}}$, and let π_0 be the irreducible representation of $M = \exp \mathfrak{m}$ corresponding to Mf_0 .

(ii) We first show that $u_e \in U(\mathfrak{m})$, and that $d\pi_0(u_e) = Q_e(g)I$. We start by noting that we can assume that $\langle g, X_2 \rangle = 0$; in fact, if necessary replace X_2 by $X_2 - cX_1$; this does not change e , Q_e , etc. (it will change γ , $\mathfrak{h}$, though, but does not affect $\mathfrak{h}_0$, $\text{rank}(\alpha_2, \gamma_1, \gamma_2)$ and the fact that $\gamma_2 \equiv 0$).

Except for some obvious modifications we can now proceed just like in case (c11) (i).

(iii) Just like in case (c11) (ii) we conclude that $d\pi_0(\text{Ad}(s)u_e) = |\Lambda_e(s)|^2 Q_e(g)I$ for $s \in G$.

Now since $X_1, X_2 \in \mathfrak{m}$ it follows that $\mathfrak{n}_f \subset \mathfrak{m}$ and from this we get that $\mathfrak{m}_{f_0} = \mathfrak{n}_f \oplus \mathbb{R}X_2$. Therefore a polarization in $\mathfrak{m}$ at f_0 is also a polarization in $\mathfrak{n}$ at f , hence $\pi = \text{ind}_{M \uparrow N} \pi_0$. We can then end this case just like we did in case (c11) (ii).

Case (c3): It is no loss of generality to assume that $\gamma \equiv 0$. In fact, there exists real numbers a_1, a_2 such that $\gamma_1 = a_1 \alpha_2, \gamma_2 = a_2 \alpha_2$, and therefore $\gamma_1 = a_1(1 + ik_2)^{-1} \lambda_2, \gamma_2 = a_2(1 + ik_2)^{-1} \lambda_2$. Replacing Z_2 by $Z'_2 = Z_2 + (1 + ik_2)^{-1}(a_1 + ia_2)Z_1$ does not change Q_e , etc., and we have $[X, Z'_2] = \lambda_2(X)Z'_2$. This proves the assertion.

Set $\mathfrak{h} = \ker \lambda_2$. Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension 1, so $\mathfrak{n} \subset \mathfrak{h}$. We can now proceed here much like in case (b), so we omit the details.

Case (d): Suppose we are not in case (a), (b) or (c).

We have $[X, Z_2] = \gamma(X)Z_1$, where $\gamma \neq 0$ and $\langle g, Z_1 \rangle \neq 0$ (since otherwise we would be in case (a)), and also $\bar{f}_1 = \bar{f}_1$.

Writing $\gamma = \gamma_1 + i\gamma_2$ we distinguish two subcases: (d1): $\text{rank}(\gamma_1, \gamma_2) = 2$ and (d2): $\text{rank}(\gamma_1, \gamma_2) = 1$.

Case (d1): Set $\mathfrak{h} = \ker \gamma_1 \cap \ker \gamma_2$. Then $\mathfrak{h}$ is an ideal of codimension 2. We then distinguish two possibilities: case (d11): $[\bar{f}_3, \bar{f}_2] = 0$ and case (d12): $[\bar{f}_3, \bar{f}_2] = \bar{f}_1$. We can then proceed here much like in case (c1) (the case at hand is easier, since here $\mathfrak{h}$ is an ideal containing $[g, g]$). We omit the details.

Case (d2): Just like in case (c2) we see that we can assume that $\gamma_2 = 0$. Set $\mathfrak{h} = \ker \gamma$. Then $\mathfrak{h}$ is an ideal of codimension 1, and we can treat this case much like case (b). We also omit the details here. This ends the proof of Proposition 2.2.1.

2.3. — We shall now end the proof of Theorem 1.4.1. We use [4], 4.2.2 Théorème, p. 121 with $\psi(l) = |P_e(l)|$. It follows from Lemma 2.1.1 that the condition of the theorem *loc. cit.* is satisfied. The conclusion is that the operator $A\pi(\varphi)A$ is traceclass for all $\varphi \in C_c^\infty(G)$, that $\varphi \rightarrow \text{Tr}([A\pi(\varphi)A])$ is a distribution (of positive type) on G , and that

$$\text{Tr}([A\pi(\varphi)A]) = \int_0 (\alpha_e \cdot \varphi \circ \exp)^{\wedge}(l) Q_e(l) d\beta_0(l).$$

Here we have also used Lemma 1.3.1.

REMARK 2.3.1. — In [4], p. 248 and [5], p. 118 appear two different definitions of the function P'_0 (cf. section 1.3). Here we use the one from [4] (which is the most natural one), while the 4.2.2. Théorème in [5] uses the definition of P'_0 from [5]. There is no difficulty in proving 4.2.2. Théorème with the definition of P'_0 from [4] when ψ has the property that $\psi(l)$ only depends on the restriction of l to $[g, g]$ which is the case here (cf. [5] 4.2.3. Remarque).

We shall then identify the operator A : Set $G_0 = \ker \chi_e$, let $\mathfrak{g}_0$ be the Lie algebra of G_0 ,

and let π_0 be the irreducible representation associated with $g_0 = g|_{g_0}$. Then $\pi = \text{ind}_{G_0 \uparrow G} \pi_0$, and A is realized on $L^2(G, \pi_0)$, the space of the induced representation $\pi = \text{ind}_{G_0 \uparrow G} \pi_0$, by $Af(s) = \psi(sg)f(s) = |P_e(sg)|f(s)$. Now it follows from Proposition 2.2.1 that $d\pi_0(u_e) = Q_e(g)I$, and that $d\pi_0(\text{Ad}(s)u_e) = Q_e(s^{-1}g)I$ which implies that we have for a differentiable vector $f \in L^2(G, \pi_0)$:

$$d\pi(u_e)f(s) = d\pi_0(\text{Ad}(s^{-1})u_e)f(s) = Q_e(sg)f(s) = |P_e(sg)|^2 f(s) = A^2 f(s),$$

and thus $d\pi(u_e) = A^2$.

Now since $A\pi(\varphi) \subset \pi(\chi_e^{-1}\varphi)A$ we have that $A\pi(\varphi)A \subset A^2\pi(\chi_e^{-1}\varphi)$, and therefore $[A\pi(\varphi)A] = [A^2\pi(\chi_e^{-1}\varphi)]$ from which $[A^2\pi(\chi_e^{-1}\varphi)]$, hence $[A^2\pi(\varphi)]$, is traceclass for all $\varphi \in C_c^\infty(G)$, and $\text{Tr}([A^2\pi(\varphi)]) = \text{Tr}([A\pi(\chi_e\varphi)A]) = \text{Tr}([A\pi(\varphi)A])$, the last equality being valid because the distribution $\varphi \rightarrow \text{Tr}([A\pi(\varphi)A])$ is supported on G_0 (cf. [5], [6]). Observing finally that $[A^2\pi(\varphi)] = \pi(u_e * \varphi)$, we have proved the theorem.

3. Examples

We shall give a few examples of the calculation of $\mathcal{E}$, Q_e , u_e , Ω_e for an exponential solvable Lie algebra $\mathfrak{g}$. If $Z_1, \dots, Z_m$ is a Jordan-Hölder basis for $\mathfrak{g}_\mathbb{C}$ we denote by $M(g)$, $g \in \mathfrak{g}'$, the skewsymmetric $m \times m$ -matrix $[\langle g, [Z_i, Z_j] \rangle]_{1 \leq i, j \leq m}$ and we write $\zeta_j = \langle g, Z_j \rangle$. The matrices $M_e(g)$ are all submatrices of $M(g)$. Note that $Z = \sum_{j=1}^m z_j Z_j$ belongs to $(\mathfrak{g}_\mathfrak{g})_\mathbb{C}$ if and only if $M(g)z = 0$, where $z = (z_1, \dots, z_m)$. We write $\mathcal{E} = \{e_1 < \dots < e_p\}$.

3.1. — Let $\mathfrak{g}$ be the five dimensional real solvable Lie algebra with the following non-vanishing brackets: $[X_5, X_4] = -X_4$, $[X_5, X_3] = 2X_3$, $[X_5, X_2] = X_2$, $[X_4, X_3] = X_2$, $[X_4, X_2] = X_1$. Then $X_1, \dots, X_5$ is a Jordan-Hölder basis for $\mathfrak{g}$, so $\mathfrak{g}$ is completely solvable. We set $Z_j = X_j$ and $\xi_j = \langle g, X_j \rangle = \zeta_j$, $j = 1, \dots, 5$.

We have

$$M(g) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_1 & -\xi_2 \\ 0 & 0 & 0 & -\xi_2 & -2\xi_3 \\ 0 & \xi_1 & \xi_2 & 0 & -\xi_4 \\ 0 & \xi_2 & 2\xi_3 & \xi_4 & 0 \end{bmatrix}$$

i) If $\xi_2^2 - 2\xi_1\xi_3 \neq 0$, then $\mathfrak{g}_\mathfrak{g} = \mathbb{R}X_1$ and therefore $J_\mathfrak{g} = \{2, 3, 4, 5\}$.

ii) If $\xi_2^2 - 2\xi_1\xi_3 = 0$ and $\xi_1 \neq 0$ then

$$\mathfrak{g}_\mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}(-\xi_2X_2 + \xi_1X_3) \oplus \mathbb{R}(-\xi_4X_2 - \xi_2X_4 + \xi_1X_5), \quad J_\mathfrak{g} = \{3, 5\}.$$

iii) If $\xi_2^2 - 2\xi_1\xi_3 = 0$, $\xi_1 = 0$ and $\xi_3 \neq 0$, then $\mathfrak{g}_\mathfrak{g} = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}(-\xi_4X_3 + 2\xi_3X_4)$, $J_\mathfrak{g} = \{3, 5\}$.

iv) If $\xi_2^2 - 2\xi_1\xi_3 = 0$, $\xi_1 = 0$, $\xi_3 = 0$ and $\xi_4 \neq 0$, then

$$\mathfrak{g}_g = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3, \quad J_g = \{4, 5\}$$

v) If $\xi_2^2 - 2\xi_1\xi_3 = 0$, $\xi_1 = 0$, $\xi_3 = 0$, $\xi_4 = 0$, then $\mathfrak{g}_g = \mathfrak{g}$, $J_g = \emptyset$.

We can then write down:

$$\begin{aligned} e_1 &= \{2, 3, 4, 5\}, & \Omega_{e_1} &= \{g \mid \xi_2^2 - 2\xi_1\xi_3 \neq 0\}, \\ e_2 &= \{2, 4\}, & \Omega_{e_2} &= \{g \mid \xi_2^2 - 2\xi_1\xi_3 = 0, \xi_1 \neq 0\}, \\ e_3 &= \{3, 5\}, & \Omega_{e_3} &= \{g \mid \xi_1 = \xi_2 = 0, \xi_3 \neq 0\}, \\ e_4 &= \{4, 5\}, & \Omega_{e_4} &= \{g \mid \xi_1 = \xi_2 = \xi_3 = 0, \xi_4 \neq 0\}, \\ e_5 &= \emptyset & \Omega_{e_5} &= \{g \mid \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0\}, \end{aligned}$$

$$\begin{aligned} Q_{e_1}(g) &= (\xi_2^2 - 2\xi_1\xi_3)^2, & u_{e_1} &= (X_2^2 - 2X_1X_3)^2, \\ Q_{e_2}(g) &= \xi_1^2, & u_{e_2} &= -X_1^2, \\ Q_{e_3}(g) &= 4\xi_3^2, & u_{e_3} &= -4X_3^2, \\ Q_{e_4}(g) &= \xi_4^2, & u_{e_4} &= -X_4^2, \\ Q_{e_5}(g) &= 1, & u_{e_5} &= 1. \end{aligned}$$

3.2. — Let $\mathfrak{g}$ be the six dimensional real exponential solvable Lie algebra having a basis $X_1, \dots, X_6$ with the following non-vanishing brackets: $[X_6, X_5] = X_4 + X_5$, $[X_6, X_4] = X_4 - X_5$, $[X_6, X_2] = X_1 + X_2$, $[X_6, X_1] = X_1 - X_2$, $[X_5, X_4] = X_3$, $[X_5, X_3] = X_2$, $[X_4, X_3] = X_1$. Set $Z_1 = X_1 - iX_2$, $Z_2 = X_1 + iX_2$, $Z_3 = X_3$, $Z_4 = X_4 - iX_5$, $Z_5 = X_4 + iX_5$, $Z_6 = X_6$. Then $Z_1, \dots, Z_6$ is a Jordan-Hölder basis for $\mathfrak{g}_{\mathbb{C}}$, and

$$M(g) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -(1-i)\zeta_1 \\ 0 & 0 & 0 & 0 & 0 & -(1+i)\zeta_2 \\ 0 & 0 & 0 & -\zeta_1 & -\zeta_2 & 0 \\ 0 & 0 & \zeta_1 & 0 & -2i\zeta_3 & -(1-i)\zeta_4 \\ 0 & 0 & \zeta_2 & 2i\zeta_3 & 0 & -(1+i)\zeta_5 \\ (1-i)\zeta_1 & (1+i)\zeta_2 & 0 & (1-i)\zeta_4 & (1+i)\zeta_5 & 0 \end{bmatrix}$$

Writing $\xi_j = \langle g, X_j \rangle$, $j = 1, \dots, 6$, we have $\zeta_1 = \xi_1 - i\xi_2$, $\zeta_2 = \xi_1 + i\xi_2$, $\zeta_3 = \xi_3$, $\zeta_4 = \xi_4 - i\xi_5$, $\zeta_5 = \xi_4 + i\xi_5$, $\zeta_6 = \xi_6$.

i) If $\zeta_1 \neq 0$, then $J_g = \{1, 3, 4, 6\}$.

ii) If $\zeta_1 = 0$ ($\Rightarrow \zeta_2 = 0$), $\zeta_3 \neq 0$, then $J_g = \{4, 5\}$.

iii) If $\zeta_1 = 0$, $\zeta_3 = 0$, $\zeta_4 \neq 0$, then $J_g = \{4, 6\}$.

iv) If $\zeta_1 = 0$, $\zeta_3 = 0$, $\zeta_4 = 0$ ($\Rightarrow \zeta_5 = 0$), then $J_g = \emptyset$.

We can then write down:

$$\begin{aligned}
 e_1 &= \{1, 3, 4, 6\}, & \Omega_{e_1} &= \{g \mid \xi_1^2 + \xi_2^2 \neq 0\}, \\
 e_2 &= \{4, 5\}, & \Omega_{e_2} &= \{g \mid \xi_1^2 + \xi_2^2 = 0, \xi_3 \neq 0\}, \\
 e_3 &= \{4, 6\}, & \Omega_{e_3} &= \{g \mid \xi_1^2 + \xi_2^2 = 0, \xi_3 = 0, \xi_4^2 + \xi_5^2 \neq 0\}, \\
 e_4 &= \emptyset, & \Omega_{e_4} &= \{g \mid \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = 0\}, \\
 Q_{e_1}(g) &= 2(\xi_1^2 + \xi_2^2)^2, & u_{e_1} &= 2(X_1^2 + X_2^2)^2, \\
 Q_{e_2}(g) &= 4\xi_3^2, & u_{e_2} &= -4X_3^2, \\
 Q_{e_3}(g) &= 2(\xi_4^2 + \xi_5^2), & u_{e_3} &= -2(X_4^2 + X_5^2), \\
 Q_{e_4}(g) &= 1, & u_{e_4} &= 1.
 \end{aligned}$$

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