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# ON THE CHARACTERS OF EXPONENTIAL SOLVABLE LIE GROUPS

BY NIELS VIGAND PEDERSEN (*)

## Introduction

Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. In [6] it was shown that for any normal representation $\pi$ of G (*cf.* [11]) there exists a continuous homomorphism $\chi : G \to \mathbb{R}_+^*$ such that $\pi$ has a distribution $\chi$-semicharacter. Moreover, it was shown that one can find a semi-invariant element $u$ (with multiplier $\chi$, say) in $U(\mathfrak{g}_{\mathbb{C}})$, the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, such that any normal representation $\pi$ whose associated orbit of $\mathscr{R}$ in $\mathfrak{g}'$ ([10], [11]) is contained in a certain G-invariant Zariski open subset of $\mathfrak{g}'$, has a distribution $\chi$-semicharacter $f_{\pi,\chi}$ expressible by $f_{\pi,\chi}(\varphi)=\phi(\pi(u * \varphi))$ for $\varphi \in C_c^\infty(G)$, $\phi$ being the trace on the factor generated by $\pi$ (here it is understood, in particular, that the right hand side is well defined). In [3] J.-Y. Charbonnel showed that for each normal representation $\pi$ of G one can find a continuous homomorphism $\chi : G \to \mathbb{R}_+^*$ and an element $u \in U(\mathfrak{g}_{\mathbb{C}})$ such that $\pi$ has a distribution $\chi$-semicharacter $f_{\pi,\chi}$ expressible as before: $f_{\pi,\chi}(\varphi)=\phi(\pi(u * \varphi))$ for $\varphi \in C_c^\infty(G)$. Here $u$ is not necessarily semi-invariant; however, $d\pi(u)$ is semi-invariant, i. e.

$$\pi(s)d\pi(u)\pi(s^{-1})=\chi(s)^{-1}d\pi(u).$$

Suppose now that G is exponential (1) (and therefore, in particular, of type I, *cf.* [2]). In this paper we make a construction, depending only on the choice of a Jordan-Hölder sequence for $\mathfrak{g}_{\mathbb{C}}$, of a finite set of polynomial functions $Q_j \geqslant 0, j=1, \ldots, n$, on $\mathfrak{g}'$, a finite set of continuous homomorphisms $\chi_j : G \to \mathbb{R}_+^*$, $j=1, \ldots, n$, and a finite set $\alpha_j$, $j=1, \ldots, n$ of positive, G-invariant analytic functions on $\mathfrak{g}$ such that, setting

$$\Omega_j = \{ g \in \mathfrak{g}' \mid \quad Q_j(g) \neq 0, Q_k(g)=0 \text{ for } k<j \}$$

we have:

1) $\Omega_j$ is G-invariant and $\mathfrak{g}'=\bigcup_{j=1}^{n}\Omega_j$,
2) $Q_j(sg)=\chi_j(s)Q_j(g)$ for $s \in G$, $g \in \Omega_j$,

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(1) G is said to be exponential if the exponential map exp : $\mathfrak{g} \to$ G is diffeomorphism.

3) for any G-orbit O contained in $\Omega_j$ the measure $Q_j\beta_O$ is a non-zero, positive, tempered, relatively invariant Radon measure on O with multiplier $\chi_j$ (here $\beta_O$ is the canonical measure on O),

and such that, letting $u_j, j=1,\ldots,n$, be the element in $U(\mathfrak{g}_\mathbb{C})$ corresponding via symmetrization to the polynomial function $g \to Q_j(ig)$ on $\mathfrak{g}'_\mathbb{C}$, we have for the irreducible representation $\pi$ of G associated with the orbit O contained in $\Omega_j$,

4) the operator $d\pi(u_j)$ is a selfadjoint, positive, invertible operator, semi-invariant under $\pi$ with multiplier $\chi_j$,

5) the operator $\pi(u_j * \varphi)$ is traceclass for all $\varphi \in C_c^\infty(G)$,

6) the functional $\varphi \to \mathrm{Tr}(\pi(u_j * \varphi))$ is a non-zero, $\chi_j$-semi-invariant distribution on G of positive type (a $\chi_j$-distribution semicharacter for $\pi$), and

7) for all $\varphi \in C_c^\infty(G)$ we have

$$(*) \qquad \mathrm{Tr}(\pi(u_j * \varphi)) = \int_O (\alpha_j \cdot \varphi \circ \exp)^\wedge(l) Q_j(l) d\beta_O(l),$$

where « $\wedge$ » stands for the ordinary Euclidian Fourier transform.

This construction is carried out in sections 1.1, 1.2 and 1.3, the theorem is formulated in section 1.4, and section 2 is devoted to the proof of the theorem; in section 3 we give a few examples.

We would like to emphasize the following feature of the formula (*) shared by no other previously known character formula for (non-nilpotent) solvable Lie groups: once a Jordan-Hölder basis in $\mathfrak{g}_\mathbb{C}$ has been selected, *all* objects in the formula are explicitly constructible (for a given orbit O and associated representation $\pi$), i. e. there is no choice (in particular of the weight function $\alpha_j$, *cf.* [9], [4], [5], [6], [3]) involved in setting up the formula. This, in particular, opens the possibility of using the formula (*) as a starting point for the pairing between orbits and representations, first established by Bernat ([1]), for exponential groups, and thus extending to these groups Pukanszky's approach to the Kirillov theory of nilpotent groups, [7].

In the special case where $\mathfrak{g}$ is nilpotent $\chi_j \equiv 1$ and $\alpha_j \equiv 1$. Therefore $Q_j$ is invariant on $O \subset \Omega_j$, $d\pi(u_j)$ is a scalar, and the formula (*) then gives that $d\pi(u_j) = Q_j(O)I$ and

$$\mathrm{Tr}(\pi(\varphi)) = \int_O (\varphi \circ \exp)^\wedge(l) d\beta_O(l),$$

so (*) reduces in particular to the Kirillov character formula.

The main difference between the results obtained in [3] and the results obtained here can be subsumed under the following points: i. We exhibit a *finite* collection of elements $u_j \in U(\mathfrak{g}_\mathbb{C})$ to choose from so as to make a formula like (*) valid, ii. we *construct* such a finite collection explicitly, and iii. here the functions $g \to Q_j(ig)$ in (*) are (rather surprisingly) the polynomial functions corresponding to the $u_j$'s via *symmetrization.*

The polynomials $Q_j$ were first considered by Pukanszky in the nilpotent case ([8], [10]). We also use in an essential way the work of Pukanszky on exponential groups ([9]) and the work of Duflo-Raïs ([5]). Our methods are very different from those of [3].

We conjecture that our results can be extended to arbitrary connected, simply connected solvable Lie groups (with the usual condition on the support of the function $\varphi$ appearing in the formula analogous to (*), though; *cf.* e. g. [6]).

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## 1. Preliminaries and formulation of Theorem

In sections 1.1, 1.2 and 1.3 we introduce the notation necessary to formulate our Theorem in section 1.4.

1.1. — Let G be a connected, simply connected solvable Lie group with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{f}_j$, $j=0, \ldots, m$, be a Jordan-Hölder sequence in $\mathfrak{g}_{\mathbb{C}}$, i. e. a sequence of ideals such that $\mathfrak{f}_j \supset \mathfrak{f}_{j-1}$ and such that $\dim \mathfrak{f}_j = j$, $j=0, \ldots, m$.

Let $\lambda_j : \mathfrak{g} \to \mathbb{C}$ be the root associated with the irreducible $\mathfrak{g}$-module $\mathfrak{f}_j/\mathfrak{f}_{j-1}$ (i. e. $\operatorname{ad}X(Z) = \lambda_j(X)Z \pmod{\mathfrak{f}_{j-1}}$ for all $Z \in \mathfrak{f}_j$, $X \in \mathfrak{g}$), and let $\Lambda_j$: $G \to \mathbb{C}^*$ be the continuous homomorphism with $\Lambda_j(\exp X) = e^{\lambda_j(X)}$ for all $X \in \mathfrak{g}$. We have $\operatorname{Ad}(s)Z = \Lambda_j(s)Z \pmod{\mathfrak{f}_{j-1}}$ for all $Z \in \mathfrak{f}_j$, $s \in G$.

We let G act in $\mathfrak{g}'$ via the coadjoint representation. For $g \in \mathfrak{g}'$ we have the skewsymmetric bilinearform $B_g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B_g(X, Y) = \langle g, [X, Y] \rangle$, $X, Y \in \mathfrak{g}$. The radical of $B_g$ is equal to the Lie algebra $\mathfrak{g}_g$ of the stabilizer $G_g$ of $g$ : $\mathfrak{g}_g = \{ X \in \mathfrak{g} \mid B_g(X, Y) = 0 \text{ for all } Y \in \mathfrak{g} \}$. We let $\hat{B}_g : \mathfrak{g}/\mathfrak{g}_g \times \mathfrak{g}/\mathfrak{g}_g \to \mathbb{R}$ designate the symplectic form on $\mathfrak{g}/\mathfrak{g}_g$ arising from $B_g$ by factorization. We extend $g$, $B_g$, etc. in the natural way to $\mathfrak{g}_{\mathbb{C}}$ whenever convenient.

For $g \in \mathfrak{g}'$ we set $\mathfrak{f}_j(g) = \mathfrak{f}_j + (\mathfrak{g}_g)_{\mathbb{C}}$, $j=0, \ldots, m$. We then have a sequence of subalgebras:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}_m(g) \supset \mathfrak{f}_{m-1}(g) \supset \ldots \supset \mathfrak{f}_1(g) \supset \mathfrak{f}_0(g) = (\mathfrak{g}_g)_{\mathbb{C}},$$

and $\dim \mathfrak{f}_j(g)/\mathfrak{f}_{j-1}(g) = 0$ or $=1$.

For $g \in \mathfrak{g}'$ we define $J_g$ to be the set $\{ 1 \leqslant j \leqslant m \mid \mathfrak{f}_j(g) \supsetneqq \mathfrak{f}_{j-1}(g) \}$.

Let $Z_j \in \mathfrak{f}_j \backslash \mathfrak{f}_{j-1}$, $j=1, \ldots, m$. Then $Z_1, \ldots, Z_m$ is a basis in $\mathfrak{g}_{\mathbb{C}}$, and we have

$$j \in J_g \Leftrightarrow Z_j \notin \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} = \mathfrak{f}_{j-1}(g).$$

If $g \in \mathfrak{g}'$ and $J_g = \{ j_1 < \ldots < j_d \}$ we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{f}_{j_d}(g) \supsetneqq \mathfrak{f}_{j_{d-1}}(g) \supsetneqq \ldots \supsetneqq \mathfrak{f}_{j_1}(g) \supsetneqq \mathfrak{f}_0(g) = (\mathfrak{g}_g)_{\mathbb{C}}.$$

In particular $Z_{j_1}, \ldots, Z_{j_d}$ is a basis for $\mathfrak{g}_{\mathbb{C}} \pmod{(\mathfrak{g}_g)_{\mathbb{C}}}$, and $d = \dim \mathfrak{g}/\mathfrak{g}_g$.

Set $\mathscr{E} = \{ J_g \mid g \in \mathfrak{g}' \}$, and for $e \in \mathscr{E}$, set $\Omega_e = \{ g \in \mathfrak{g}' \mid J_g = e \}$. Then we have $\mathfrak{g}' = \bigcup_{e \in \mathscr{E}} \Omega_e$ as a (finite) disjoint union. Since clearly $J_{sg} = J_g$ for $s \in G$, $\Omega_e$ is a G-invariant subset of $\mathfrak{g}'$.

Let $e \in \mathscr{E}$. If $e \neq \varnothing$ with $e = \{ j_1 < \ldots < j_d \}$ we define the skewsymmetric $d \times d$-*matrix* $M_e(g)$, $g \in \mathfrak{g}'$, by

$$M_e(g) = [B_g(Z_{j_r}, Z_{j_s})]_{1 \leqslant r,s \leqslant d},$$

and let $P_e(g)$ denote the Pfaffian of $M_e(g)$. If $e=\varnothing$ we set $M_e(g)=1$, and $P_e(g)=1$. The map $g \to P_e(g)$ is a complex valued polynomial function on $\mathfrak{g}'$, and $P_e(g)$ depends only on the restriction of $g$ to $[\mathfrak{g}, \mathfrak{g}]$. $P_e$ has the property that $P_e(g)^2=\det M_e(g)$. We set $Q_e(g)=|\det M_e(g)|=|P_e(g)|^2$. $g \to Q_e(g)$ is a real valued non-negative polynomial function on $\mathfrak{g}'$.

For $e\in\mathscr{E}$ we set $\Lambda_e=\prod_{j\in e}\Lambda_j$.

LEMMA 1.1.1. — Let $e\in\mathscr{E}$. If $g\in\Omega_e$, then $P_e(g)\neq 0$ and $P_e(sg)=\Lambda_e(s)^{-1}P_e(g)$ for all $s\in G$.

*Proof.* — Write $e=\{j_1< \dots <j_d\}$. Since $Z_{j_1}, \dots Z_{j_d}$ is a basis for $\mathfrak{g}_{\mathbb{C}}$ (mod $(\mathfrak{g}_g)_{\mathbb{C}}$) we have that $M_e(g)$ is a regular matrix, hence $P_e(g)^2=\det M_e(g)\neq 0$.

Now writing

$$\mathrm{Ad}(s^{-1})Z_{j_p}=\sum_{u=1}^{d} a_{up}Z_{j_u}+c_p,$$

where $c_p\in(\mathfrak{g}_g)_{\mathbb{C}}$, we have $a_{up}=0$ for $u>p$ and $a_{pp}=\Lambda_{j_p}(s^{-1})$, and

$$B_{sg}(Z_{j_p}, Z_{j_q})=\langle sg, [Z_{j_p}, Z_{j_q}]\rangle=\langle g, [\mathrm{Ad}(s^{-1})Z_{j_p}, \mathrm{Ad}(s^{-1})Z_{j_q}]\rangle$$
$$=\sum_{u,v=1}^{d} a_{up}\langle g, [Z_{j_u}, Z_{j_v}]\rangle a_{vq}=({}^t AM_e(g)A)_{p,q},$$

where A is the matrix $[a_{pq}]_{1\leq p,q\leq d}$. This shows that $M_e(sg)={}^tAM_e(g)A$, and since $\det A=\prod_{p=1}^{d}\Lambda_{j_p}(s^{-1})=\Lambda_e(s^{-1})$ we find that

$$P_e(sg)=Pf(M_e(sg))=Pf({}^tAM_e(g)A)=(\det A)Pf(M_e(g))=\Lambda_e(s^{-1})P_e(g).$$

This ends the proof of the lemma.

COROLLARY 1.1.2. — If $g\in\Omega_e$, then $Q_e(g)>0$ and $Q_e(sg)=|\Lambda_e(s)|^{-2}Q_e(g)$ for all $s\in G$.

For $e\in\mathscr{E}$ we set $|e|=$ the number of elements in $e$. We define a total ordering $<$ on $\mathscr{E}$ in the following way: let $e, e'\in\mathscr{E}$. Then $e<e'$ if and only if either $|e|>|e'|$ or $d=|e|=|e'|$ and, writing $e=\{j_1<\dots<j_d\}$, $e'=\{j'_1<\dots<j'_d\}$, $j_p<j'_p$, where $p=\min\{1\leq r\leq d\,|\,j_r\neq j'_r\}$.

LEMMA 1.1.3. — $\Omega_e=\{g\in\mathfrak{g}'\,|\,Q_{e'}(g)=0$ for $e'<e$ and $Q_e(g)\neq 0\}$.

*Proof.* — If $g\in\Omega_e$ we saw in Corollary 1.1.2 that $Q_e(g)\neq 0$. If $e'<e$ and $|e'|>|e|$, then, if $e'=\{j'_1< \dots <j'_c\}$, $Z_{j'_1}, \dots, Z_{j'_c}$ are linearly dependent (mod $(\mathfrak{g}_g)_{\mathbb{C}}$), so $M_{e'}(g)$ is singular, and therefore $Q_{e'}(g)=0$. If $|e|=|e'|$, and $j_1=j'_1, \dots, j_p=j'_p$, $j'_{p+1}<j_{p+1}$, then $Z_{j'_{p+1}}\in\mathfrak{f}_{j_p}+(\mathfrak{g}_g)_{\mathbb{C}}$, and therefore $Z_{j'_1}, \dots, Z_{j'_{p+1}}$ are linearly dependent (mod $(\mathfrak{g}_g)_{\mathbb{C}}$), and again $Q_{e'}(g)=0$. This shows the lemma.

*Remark* 1.1.4. — If $\mathfrak{g}$ is nilpotent our definitions agree with those given by Pukanszky in [10], p. 525 f. f., *cf.* also [8]. In [6], section 4.2 a study of the completely solvable case was initiated.

1.2. — Recall the following facts: there exists an isomorphism $\omega$ (the symmetrization map) between the complex vector space $S(\mathfrak{g}_{\mathbb{C}})$ (the symmetric algebra of $\mathfrak{g}_{\mathbb{C}}$), and the complex vector space $U(\mathfrak{g}_{\mathbb{C}})$ (the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$), characterized by the following property: if $Y_1, \ldots, Y_p$ are elements in $\mathfrak{g}_{\mathbb{C}}$, then the image of the element $Y_1 \ldots Y_p$ in $S(\mathfrak{g}_{\mathbb{C}})$ by $\omega$ is the element $(p\,!)^{-1}\sum_{\sigma\in S_p} Y_{\sigma(1)} \ldots Y_{\sigma(p)}$ in $U(\mathfrak{g}_{\mathbb{C}})$, where $S_p$ is the group of permutations of $p$ elements. The following lemma is easily verified:

LEMMA 1.2.1. — If Z is a central element in $\mathfrak{g}_{\mathbb{C}}$, then $\omega(Zu)=Z\omega(u)$ for all $u\in S(\mathfrak{g}_{\mathbb{C}})$.

We can identify $S(\mathfrak{g}_{\mathbb{C}})$ with $\mathrm{Pol}_{\mathbb{C}}(\mathfrak{g}')$, the complex vector space of complex valued polynomial functions on $\mathfrak{g}'$. If $u\in U(\mathfrak{g}_{\mathbb{C}})$ we let $P_u$ be the polynomial on $\mathfrak{g}'$ corresponding to $\omega^{-1}(u)$. The lemma above then says that if Z is central in $\mathfrak{g}_{\mathbb{C}}$ and if $u\in U(\mathfrak{g}_{\mathbb{C}})$, then $P_{Zu}=P_Z P_u$.

For $e\in\mathscr{E}$, let $u_e$ be the element in $U(\mathfrak{g})$ corresponding to the real valued polynomial function $g \to i^d Q_e(g)$ on $\mathfrak{g}'$. Note that $u_e$ actually is contained in $U([\mathfrak{g}, \mathfrak{g}])$, since $Q_e(g)$ only depends on the restriction of $g$ to $[\mathfrak{g}, \mathfrak{g}]$.

1.3. — If $g\in\mathfrak{g}'$, the weights of $\mathfrak{g}_g$ in $\mathfrak{g}/\mathfrak{g}_g$ are of the form $\pm\mu_1, \ldots, \pm\mu_{d/2}$, where $d=\dim \mathfrak{g}/\mathfrak{g}_g$, and these weights $\mu_j$ extend to linear forms, also called $\mu_j$, on the ideal $\mathfrak{k}=\mathfrak{g}_g+[\mathfrak{g}, \mathfrak{g}]$ in such a manner that they are zero on $[\mathfrak{g}, \mathfrak{g}]$ (*v.* [4], p. 248).

Following *loc. cit.* we set

$$S_\lambda(X)=\frac{\sin h(\lambda(X)/2)}{\lambda(X)/2},\ X\in\mathfrak{g},$$

for a complex linear form $\lambda$ on $\mathfrak{g}$, and define the function $P'_O$ on $\mathfrak{k}$ by

$$P'_O(X)=\prod_{j=1}^{d/2} S_{\mu_j}(X),\ X\in\mathfrak{k},$$

where $O=Gg$ is the G-orbit through $g$. This definition of $P'_O$ does not depend on the choice of $g\in O$.

We set

$$j_G(X)=\left|\det\frac{1-e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}\right|,\ X\in\mathfrak{g}.$$

$j_G$ is a G-invariant analytic function on $\mathfrak{g}$, and if $dX$ is a Lebesgue measure on $\mathfrak{g}$ there exists a Haar measure $\mu$ on G such that $d\mu(\exp X)=j_G(X)dX$.

If G is exponential we set for $e\in\mathscr{E}$,

$$\Gamma_e(X)=\Big(\prod_{j\in e}|S_{\lambda_j}(X)|\Big)^{\frac{1}{2}},\ X\in\mathfrak{g}.$$

LEMMA 1.3.1. — (G exponential) $\Gamma_e$ is a positive, G-invariant analytic function on $\mathfrak{g}$, extending $P'_O$ for any G-orbit O contained in $\Omega_e$.

*Proof.* — The function $X \to S_{\lambda_j}(X)$ is a G-invariant analytic function on $\mathfrak{g}$, and since $\mathfrak{g}$ is exponential $\lambda_j(X)\notin i\mathbb{R}\setminus\{0\}$ for all $X\in\mathfrak{g}$, hence $S_{\lambda_j}(X)\neq 0$ for all $X\in\mathfrak{g}$. This shows that $\Gamma_e$ is positive, G-invariant and analytic. Now an easy argument shows that $P'_O(X)\geqslant 0$ for all $X\in\mathfrak{k}=\mathfrak{g}_g+[\mathfrak{g}, \mathfrak{g}]$ (*see* e. g. [4], p. 264 top; again we use that $\mathfrak{g}$ is exponential). Therefore

$$P'_O(X)=|P'_O(X)|=\prod_{j=1}^{d/2}|S_{\mu_j}(X)|=\Big(\prod_{j=1}^{d/2}|S_{\mu_j}(X)|^2\Big)^{\frac{1}{2}}=\Big(\prod_{j=1}^{d/2}(|S_{\mu_j}(X)|\,|S_{-\mu_j}(X)|)\Big)^{\frac{1}{2}},$$

and noting that $\lambda_j$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$ and that the weights of $\mathfrak{g}_g$ in $\mathfrak{g}/\mathfrak{g}_g$ are precisely

$$\{\lambda_{j_1}|\mathfrak{g}_g, \ldots, \lambda_{j_d}|\mathfrak{g}_g\}=\{\pm\mu_1|\mathfrak{g}_g, \ldots, \pm\mu_{d/2}|\mathfrak{g}_g\},$$

we get that $P'_O(X)=\Gamma_e(X)$ for $X\in\mathfrak{k}$. This proves the lemma.

We set

$$\alpha_e(X)=j_G(X)\Gamma_e(X)^{-1},\ X\in\mathfrak{g}$$

(still assuming that G is exponential). $\alpha_e$ is a positive, G-invariant analytic function on $\mathfrak{g}$.

REMARK 1.3.2. — Lemma 1.3.1 should be compared with [4], section 4, p. 262-264. In the exponential case the result *loc. cit.* is that there exists a G-invariant Zariski open subset $\Omega$ of $\mathfrak{g}'$ and a positive, G-invariant analytic function P on $\mathfrak{g}$, such that for any G-orbit O contained in $\Omega$ the restriction of P to $\mathfrak{k}=\mathfrak{g}_g+[\mathfrak{g},\mathfrak{g}]$, $g\in O$, is equal to $P'_O$. By Lemma 1.3.1 and Lemma 1.1.3 we can obtain this result by taking $\Omega$ to be $\Omega_e$ for the minimal element $e$ in $\mathscr{E}$, and taking P to be $\Gamma_e$. In general the P from *loc. cit.* will be different from the one exhibited here. Incidentally, by refining the methods used here can give a complete solution to the problem raised, and partially solved, by Duflo, *loc. cit.*, p. 263, mid. However, at present this will not be needed, so we shall postpone it to a later time.

1.4. — Suppose now that G is exponential, and suppose in addition that the Jordan-Hölder sequence $\mathfrak{g}_{\mathbb{C}}=\mathfrak{f}_m\supset\ldots\supset\mathfrak{f}_0=\{0\}$ has the property that if $\bar{\mathfrak{f}}_j\neq\mathfrak{f}_j$, then $\bar{\mathfrak{f}}_{j-1}=\mathfrak{f}_{j-1}$ and $\bar{\mathfrak{f}}_{j+1}=\mathfrak{f}_{j+1}$, $1\leqslant j\leqslant m-1$ (such a Jordan-Hölder sequence clearly exists). Set $\chi_e=|\Lambda_e|^{-2}$.

*Theorem* 1.4.1. — (G exponential) Let $\pi$ be an irreducible representation of G, and let O be the G-orbit in $\mathfrak{g}'$ associated with $\pi$. Let $e\in\mathscr{E}$ be the unique element such that $\Omega_e$ contains O. Then

1) The measure $Q_e\beta_O$ is a non-zero, positive, tempered, relatively invariant Radon measure on O with multiplier $\chi_e\cdot(\beta_O$ is the canonical measure on O.)

2) The operator $d\pi(u_e)$ is a selfadjoint, positive, invertible operator, semi-invariant under $\pi$ with multiplier $\chi_e$ (i. e. $\pi(s)d\pi(u_e)\pi(s^{-1})=\chi_e(s^{-1})d\pi(u_e)$).

3) For any $\varphi\in C_c^\infty(G)$ the operator $\pi(u_e*\varphi)$ is traceclass.

4) The functional $\varphi\to\mathrm{Tr}(\pi(u_e*\varphi))$ on $C_c^\infty(G)$ is a non-zero, $\chi_e$-semi-invariant distribution on G of positive type (a distribution semicharacter for $\pi$ (with multiplier $\chi_e$)).

5) For any $\varphi\in C_c^\infty(G)$ we have the formula

$$(*)\qquad \mathrm{Tr}(\pi(u_e*\varphi))=\int_O(\alpha_e\cdot\varphi\circ\exp)^\wedge(l)Q_e(l)d\beta_O(l).$$

Here we use the notation $\hat{\psi}(l)=\int_{\mathfrak{g}}\psi(X)e^{i\langle X,l\rangle}dX$ for $\psi\in C_c^\infty(\mathfrak{g})$, $l\in\mathfrak{g}'$, where $dX$ is the Lebesgue measure on $\mathfrak{g}$ with the property that $d\mu(\exp X)=j_G(X)dX$, $d\mu$ being a fixed Haar measure on G, and $\pi(\varphi)=\int_G\varphi(s)\pi(s)d\mu(s)$ for $\varphi\in L^1(G)$.

REMARK 1.4.2. — In the formula $(*)$ above we can instead of $\alpha_e$ use any $C^\infty$-function $\alpha$ on $\mathfrak{g}$ with the property that the restriction of $\alpha$ to $\mathfrak{k}=\mathfrak{g}_g+[\mathfrak{g},\mathfrak{g}]$, $g\in O$, is the same as the restriction of $\alpha_e$ to $\mathfrak{k}$.

REMARK 1.4.3. — It will follow from the proof of Theorem 1.4.1 that the distributions $\varphi \to \mathrm{Tr}(\pi(u_e * \varphi))$ have a finite order not exceeding $2d+1$, where $d=|e|$.

## 2. Proof of Theorem

Here we shall for brevity say that a Jordan-Hölder sequence $\mathfrak{g}_\mathbb{C}=\mathfrak{f}_m \supset \dots \supset \mathfrak{f}_0=\{0\}$ is of class (*b*) if it has the property required in 1.4 (i. e. that $\bar{\mathfrak{f}}_j \neq \mathfrak{f}_j$, $1 \leqslant j \leqslant m-1$, implies that $\bar{\mathfrak{f}}_{j-1}=\mathfrak{f}_{j-1}$ and $\bar{\mathfrak{f}}_{j+1}=\mathfrak{f}_{j+1}$), *cf.* [2] Définition 4.2.1, pp. 78.

2.1. — The purpose of this subsection is to prove the following lemma, from which part 1) of Theorem 1.4.1 follows immediately.

LEMMA 2.1.1. — The measure $P_e\beta_O$ is a non-zero, tempered, $\Lambda_e^{-1}$-relatively invariant (complex) Radon measure on O.

REMARK 2.1.2. — In the completely solvable case this was proved in [6], section 4.1.*d*. The proof *loc. cit.* does not carry over to the case at hand, so we have to modify our approach.

*Proof.* — We have only left to show that $P_e\beta_O$ is tempered, *cf.* Lemma 1.1.1.

(i) Let I be the set of indices $0 \leqslant j \leqslant m$ for which $\bar{\mathfrak{f}}_j=\mathfrak{f}_j$. For $j \in I$ there exists an ideal $\mathfrak{g}_j$ in $\mathfrak{g}$ such that $(\mathfrak{g}_j)_\mathbb{C}=\mathfrak{f}_j$.

Set $I'=\{j\in I \mid j-1 \in I\}$ and $I''=\{j \in I\setminus\{0\} \mid j-1 \notin I\}$. Then $I=\{0\}\cup I' \cup I''$ as a disjoint union, and for $j \in I''$ we have that $j-2 \in I$ (since $\mathfrak{f}_0, \dots, \mathfrak{f}_m$ is of class (*b*)).

Now since $\Lambda_e$ only depends on the Jordan-Hölder sequence $\mathfrak{f}_j$ and not on the basis $Z_j$ we can assume here that the $Z_j$'s are constructed in the following way: for $j\in I'$, let $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, and set $Z_j=X_j$. For $j \in I''$, pick $Z_{j-1} \in \mathfrak{f}_{j-1}\setminus\mathfrak{f}_{j-2}$. Since $\bar{\mathfrak{f}}_{j-1} \neq \mathfrak{f}_{j-1}$ we have that $\bar{Z}_{j-1} \in \mathfrak{f}_j \setminus \mathfrak{f}_{j-1}$. Set $Z_j=\bar{Z}_{j-1}$, and define $X_{j-1}$, $X_j$ by $Z_j=X_{j-1}+iX_j$. Then $X_{j-1}$, $X_j$ is a basis for $\mathfrak{g}_j$ (mod $\mathfrak{g}_{j-2}$), and $X_1, \dots, X_m$ is a basis for $\mathfrak{g}$. Let $g_1, \dots, g_m \in \mathfrak{g}'$ be the basis dual to $X_1, \dots, X_m$.

Fix an element $g \in O$, and write $e=J_g=\{j_1 < \dots < j_d\}$. Set $D_1=\{1 \leqslant k \leqslant d \mid j_k \in I'\}$, $D_2=\{1 \leqslant k \leqslant d \mid j_k \notin I, j_k+1 \notin J_g\}$, $D_3=\{1 \leqslant k \leqslant d \mid j_k \notin I, j_k+1 \in J_g\}$, $D_4=\{1 \leqslant k \leqslant d \mid j_k \in I''\}$. Clearly $\{1, \dots, d\}=D_1 \cup D_2 \cup D_3 \cup D_4$ as a disjoint union. Observe that if $k \in D_3$, then clearly $k+1 \in D_4$. Conversely, if $k \in D_4$, then $j=j_k \in I'' \cap J_g$, and therefore $j-1 \in J_g$; in fact, if $j-1 \notin J_g$, then $Z_{j-1} \in \mathfrak{f}_{j-2}+(\mathfrak{g}_g)_\mathbb{C}$, that is, $X_{j-1}-iX_j \in (\mathfrak{g}_{j-2})_\mathbb{C}+(\mathfrak{g}_g)_\mathbb{C}$, implying that $X_{j-1}$, $X_j \in \mathfrak{g}_{j-2}+\mathfrak{g}_g$; but then $Z_j=X_{j-1}+iX_j \in (\mathfrak{g}_{j-2})_\mathbb{C}+(\mathfrak{g}_g)_\mathbb{C}=\mathfrak{f}_{j-2}+(\mathfrak{g}_g)_\mathbb{C}$ and therefore $j \notin J_g$ which is a contradiction. The conclusion of this is that $D_4=\{k+1 \mid k \in D_3\}$.

For $j \in I$, set $G_g^j=\{s \in G \mid sg=g \,(\mathrm{mod}\ \mathfrak{g}_j^\perp)\}$. $G_g^j$ is a closed, connected subgroup with Lie algebra $\mathfrak{g}_g^j=\{X \in \mathfrak{g} \mid Xg \in \mathfrak{g}_j^\perp\}$ (*cf.* [9], p. 105, III). Clearly $j \to \mathfrak{g}_g^j$, $j \in I$, is a decreasing sequence of subalgebras with $\mathfrak{g}_g^0=\mathfrak{g}$ and $\mathfrak{g}_g^m=\mathfrak{g}_g$.

If $j\in I'$, then $\dim \mathfrak{g}_g^{j-1}/\mathfrak{g}_g^j=0$ or $=1$, and $\mathfrak{g}_g^{j-1} \supsetneqq \mathfrak{g}_g^j$ if and only if $j \in J_g$. If $j \in I''$, then $\dim \mathfrak{g}_g^{j-2}/\mathfrak{g}_g^j=0$, $=1$ or $=2$, and $\dim \mathfrak{g}_g^{j-2}/\mathfrak{g}_g^j=2$ if and only if $j, j-1 \in J_g$, $\dim \mathfrak{g}_g^{j-2}/\mathfrak{g}_g^j=1$ if and only if $j-1 \in J_g$, $j \notin J_g$.

(ii) The following is an adaptation of [9], p. 102-106, II-III to the present situation:

For $k \in D_1$ there exists an element $Y_k$ in $\mathfrak{g}_g^{j_k-1} \backslash \mathfrak{g}_g^{j_k}$ such that $Y_k$ is a coexponential basis to $\mathfrak{g}_g^{j_k}$ in $\mathfrak{g}_g^{j_k-1}$ and such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k}^{\perp}}$, and for $s \in G_g^{j_k}$ we have

$$\mathrm{Ad}(s) Y_k = \Lambda_{j_k}(s^{-1}) Y_k \pmod{\mathfrak{g}_g^{j_k}}.$$

For $k \in D_2$ there exists an element $Y_k$ in $\mathfrak{g}_g^{j_k-1} \backslash \mathfrak{g}_g^{j_k+1}$ such that $Y_k$ is a coexponential basis to $\mathfrak{g}_g^{j_k+1}$ in $\mathfrak{g}_g^{j_k-1}$, and such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k+1}^{\perp}}$ (to obtain this it can be necessary to change $X_{j_k}$, $X_{j_k+1}$ in a way that only affects $Z_{j_k}$, $Z_{j_k+1}$ by multiplying them by a factor of modulus one), and for $s \in G_g^{j_k+1}$ we have $\mathrm{Ad}(s) Y_k = \Lambda_{j_k}(s^{-1}) Y_k \pmod{\mathfrak{g}_g^{j_k+1}}$ (so in particular $\Lambda_{j_k}(s^{-1})$ is real).

For $k \in D_3$ there exists elements $Y_k$, $Y_{k+1}$ in $\mathfrak{g}_g^{j_k-1} \backslash \mathfrak{g}_g^{j_k+1}$ such that $Y_k g = g_{j_k} \pmod{\mathfrak{g}_{j_k+1}^{\perp}}$, $Y_{k+1} g = g_{j_{k+1}} \pmod{\mathfrak{g}_{j_k+1}^{\perp}}$, such that $Y_k$, $Y_{k+1}$ is a coexponential basis to $\mathfrak{g}_g^{j_k+1}$ in $\mathfrak{g}_g^{j_k-1}$, such that $\lambda_{j_k}(Y_k) = \lambda_{j_k}(Y_{k+1}) = 0$ and such that

$$\begin{aligned}\exp t_k Y_k \exp t_{k+1} Y_{k+1} &= \exp (t_k Y_k + t_{k+1} Y_{k+1}) \pmod{G_g^{j_k+1}} \\ &= \exp t_{k+1} Y_{k+1} \exp t_k Y_k \pmod{G_g^{j_k+1}}.\end{aligned}$$

For $s \in G_g^{j_k+1}$ we have $\mathrm{Ad}(s)(Y_k + i Y_{k+1}) = \Lambda_{j_k}(s^{-1})(Y_k + i Y_{k+1}) \pmod{(\mathfrak{g}_g^{j_k+1})_{\mathbb{C}}}$.

(iii) The map $\mathbb{R}^d \to O = Gg$ given by

$$(*) \qquad (t_1, \ldots, t_d) \to \exp t_1 Y_1 \ldots \exp t_d Y_d g$$

is a differeomorphism. We shall compute the canonical measure $\beta_O$ in terms of the coordinates $t = (t_1, \ldots, t_d)$.

Let $\omega$ be the canonical symplectic form on O. Via the natural correspondence between $\mathfrak{g}/\mathfrak{g}_g$ and the tangent space to O at $g$, $\omega_g$ corresponds to $\hat{B}_g$.

LEMMA 2.1.3. — For a $\beta_O$-integrable function $f$ on O we have

$$\int_O f(l) d\beta_O(l) = C \int_{\mathbb{R}^d} f(\exp t_1 Y_1 \ldots \exp t_d Y_d g) \prod_{k<r} |\Lambda_{j_k}(\exp t_r Y_r)| \, dt_1 \ldots dt_d,$$

where $C = ((2\pi)^d Q_e(g))^{-\frac{1}{2}}$.

*Proof.* — Denote by $\sigma$ the inverse of the map (*). $\sigma$ is a global chart and

$$\int_O f(l) d\beta_O(l) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\sigma^{-1}(t)) \theta(\sigma^{-1}(t)) dt,$$

where $\theta(l) = (\det S_l)^{\frac{1}{2}}$, $S_l$ being the skewsymmetric matrix $S_l = [\omega_l(\partial/\partial t_u, \partial/\partial t_v)]_{1 \leq u,v \leq d}$ ([9] Proposition 4, p. 99).

Now $\omega$ is G-invariant. Therefore, writing $s = \exp t_1 Y_1 \ldots \exp t_d Y_d$ and $l = sg$, we have

$$\begin{aligned}\omega_l((\partial/\partial t_u)_l, (\partial/\partial t_v)_l) &= \omega_{sg}((\partial/\partial t_u)_{sg}, (\partial/\partial t_v)_{sg}) \\ &\omega_g(\gamma(s^{-1})_* (\partial/\partial t_u)_{sg}, \gamma(s^{-1})_* (\partial/\partial t_v)_{sg}),\end{aligned}$$

where $\gamma(s)$ ; $l \to sl$. Let us then compute $\gamma(s^{-1})_* (\partial/\partial t_u)_{sg}$:

For a differentiable function $\varphi$ we have

$$
\begin{aligned}
\gamma(s^{-1})_*(\partial/\partial t_u)_{sg}\varphi &= (\partial/\partial t_u)_{sg}\varphi \circ \gamma(s^{-1}) \\
&= \frac{d}{d\tau}\varphi(s^{-1}\sigma^{-1}(t+\tau^u))\,|_{\tau=0} \qquad (\tau^u=(\delta_{uv}\tau)_{1\leqslant v\leqslant d}) \\
&= \frac{d}{d\tau}\varphi(\exp -t_d Y_d \ldots \exp -t_1 Y_1 \exp t_1 Y_1 \ldots \exp (t_u+\tau)Y_u \ldots \exp t_d Y_d g\,|_{\tau=0} \\
&= \frac{d}{d\tau}\varphi(\exp -t_d Y_d \ldots \exp -t_{u+1}Y_{u+1} \exp \tau Y_u \exp t_{u+1}Y_{u+1} \ldots \exp t_d Y_d g)\,|_{\tau=0} \\
&= \frac{d}{d\tau}\varphi(s_u^{-1} \exp \tau Y_u s_u g)\,|_{\tau=0} = \frac{d}{d\tau}\varphi(\exp \tau \operatorname{Ad}(s_u^{-1})Y_u g)\,|_{\tau=0},
\end{aligned}
$$

where we have set $s_u = \exp t_{u+1}Y_{u+1} \ldots \exp t_d Y_d$, $u<d$, $s_d=e$.

The conclusion of this is that $S_l=[B_g(\operatorname{Ad}(s_u^{-1})Y_u, \operatorname{Ad}(s_v^{-1})Y_v)]_{1\leqslant u,v\leqslant d}$. Since $Y_1, \ldots, Y_d$ is a basis for $\mathfrak{g} \,(\mathrm{mod}\ \mathfrak{g}_g)$ we can write

$$\operatorname{Ad}(s_u^{-1})Y_u = \textstyle\sum_{p=1}^{d} a_{pu} Y_p + c_u,$$

where $c_u \in \mathfrak{g}_g$, and then $S_l = {}^t A S_g A$, where A is the matrix $[a_{uv}]_{1\leqslant u,v\leqslant d}$, so that $\theta(l) = |\det A|\,\theta(g)$.

We shall then find $\det A$: for $u\in D_1$ we have that $s_u \in G_g^{j_u}$, so $\operatorname{Ad}(s_u^{-1})Y_u = \Lambda_{j_u}(s_u)Y_u \,(\mathrm{mod}\ \mathfrak{g}_g^{j_u})$, implying that $a_{uu}=\Lambda_{j_u}(s_u)$, while $a_{uv}=0$ for $u<v$. For $u\in D_2$ we have

$$\operatorname{Ad}(s_u^{-1})Y_u = \Lambda_{j_u}(s_u)Y_u \,(\mathrm{mod}\ \mathfrak{g}_g^{j_u+1})$$

implying that $a_{uu}=\Lambda_{j_u}(s_u)=|\Lambda_{j_u}(s_u)|$, while $a_{uv}=0$ for $u<v$. For $u\in D_3$ we have

$$\operatorname{Ad}(s_u^{-1})(Y_u+iY_{u+1}) = \Lambda_{j_u}(s_u)(Y_u+iY_{u+1})\,(\mathrm{mod}\ \mathfrak{g}_g^{j_u+1}),$$

implying that

$$\det \begin{bmatrix} a_{uu} & a_{uu+1} \\ a_{u+1u} & a_{u+1u+1} \end{bmatrix} = |\Lambda_{j_u}(s_u)|^2,$$

while $a_{uv}=0$ and $a_{u+1v}=0$ for $v>u+1$. It follows that

$$\det A = \prod_{u\in D_1\cup D_2} |\Lambda_{j_u}(s_u)| \cdot \prod_{u\in D_3} |\Lambda_{j_u}(s_u)|^2.$$

Now for $u\in D_3$ we have

$$
\begin{aligned}
\Lambda_{j_u}(s_u) &= \Lambda_{j_u}(\exp t_{u+1}Y_{u+1} \ldots \exp t_d Y_d) \\
&= \Lambda_{j_u}(\exp t_{u+2}Y_{u+2} \ldots \exp t_d Y_d) = \overline{\Lambda_{j_{u+1}}(\exp t_{u+2}Y_{u+2} \ldots \exp t_d Y_d)} = \overline{\Lambda_{j_{u+1}}(s_{u+1})},
\end{aligned}
$$

so $|\Lambda_{j_u}(s_u)| = |\Lambda_{j_{u+1}}(s_{u+1})|$, hence $\det A = \prod_{u=1}^{d} |\Lambda_{j_u}(s_u)| = \prod_{1\leqslant u<r\leqslant d} |\Lambda_{j_u}(\exp t_r Y_r)|$.

Finally, a simple computation shows that $\det S_g = \det [B_g(Y_r, Y_s)]_{1 \leqslant r,s \leqslant d} = Q_e(g)^{-1}$. This ends the proof of the lemma.

(iv) For $1 \leqslant j \leqslant m$ we define the function $S_j$ by

$$S_j(t_1, \ldots, t_d) = \langle \exp t_1 Y_1 \ldots \exp t_d Y_d g, Z_j \rangle .$$

We consider $S_{j_k}$: arguing like in [9], p. 106 we find for $k \in D_1 \cup D_2$:

$$S_{j_k}(t_1, \ldots, t_d) = \frac{e^{-t_k \lambda_{j_k}(Y_k)^{-1}}}{-\lambda_{j_k}(Y_k)} \prod_{r<k} \Lambda_{j_k}(\exp t_r Y_r)^{-1} + S_{j_k}(t_1, \ldots, t_{k-1}, 0, \ldots, 0),$$

and for $k \in D_4$ we find

$$S_{j_k}(t_1, \ldots, t_d) = (t_{k-1} + it_k) \prod_{r<k-1} \Lambda_{j_k}(\exp t_r Y_r)^{-1} + S_{j_k}(t_1, \ldots, t_{k-2}, 0, \ldots, 0).$$

(v) For a real number $n>0$ we set $M(n) = \int_{\mathbb{R}} (1+x^2)^{-n/2} dx$. We have $0 < M(n) \leq +\infty$ and $M(n) < +\infty$ if and only if $n>1$.

**Lemma** 2.1.4. — Let $a$, $\alpha$, $\beta$ be real numbers with $a>0$, $\alpha \neq 0$, and let $c$, $k$ be complex numbers with $k \neq 0$. We have

$$(*) \qquad \int_{\mathbb{R}} (a + |ke^{(\alpha+i\beta)t} - c|^2)^{-n/2} e^{\alpha t} dt < \frac{M(n)}{|\alpha|\,|k|\, a^{(n-1)/2}},$$

$$(**) \qquad \int_{\mathbb{R}_2} (a + |k(s+it) - c|^2)^{-n/2} ds dt = \frac{M(n) M(n-1)}{|k|^2 a^{(n-2)/2}}.$$

*Proof.* — Obviously we can assume that $k>0$. Writing $k^{-1}c = be^{i\gamma}$, $b \geqslant 0$, $\gamma \in \mathbb{R}$ we have

$$\begin{aligned}
\int_{\mathbb{R}} (a + |ke^{(\alpha+i\beta)t} - c|^2)^{-n/2} e^{\alpha t} dt &= \int_{\mathbb{R}} (a + k^2 |e^{\alpha t + i(t\beta - \gamma)} - b|^2)^{-n/2} e^{\alpha t} dt \\
&\leqslant \int_{\mathbb{R}} (a + k^2 |e^{\alpha t} - b|^2)^{-n/2} e^{\alpha t} dt \\
&= |\alpha|^{-1} \int_0^\infty (a + k^2 |x - b|^2)^{-n/2} dx \\
&< |\alpha|^{-1} \int_{\mathbb{R}} (a + k^2 |x - b|^2)^{-n/2} dx \\
&= |\alpha|^{-1} \int_{\mathbb{R}} (a + k^2 x^2)^{-n/2} dx \\
&= \frac{M(n)}{|\alpha|\, k a^{(n-1)/2}}.
\end{aligned}$$

This proves $(*)$. Similarly for $(**)$.

(vi) We shall then prove the temperedness of the measure $P_e \beta_0$. First observe that

$l \to (\sum_{j=1}^{m} |\langle l, Z_j \rangle|^2)^{\frac{1}{2}} = \|l\|$ is a norm on $\mathfrak{g}'$. We must show that we can find $n>0$ such that $\int_{O}(1+\|l\|^2)^{-n/2}|P_e(l)|\,d\beta_O(l)$ is finite. We have, using Lemma 2.1.3:

$$\int_{O}(1+\|l\|^2)^{-n/2}|P_e(l)|\,d\beta_O(l)$$

$$=C\int_{\mathbb{R}^d}\frac{|P_e(\exp t_1Y_1 \ldots \exp t_dY_dg)|}{(1+\|\exp t_1Y_1 \ldots \exp t_dY_dg\|^2)^{n/2}}\prod_{k<r}|\Lambda_{j_k}(\exp t_rY_r)|\,dt_1 \ldots dt_d$$

$$=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\frac{\prod_{r\leqslant k}|\Lambda_{j_k}(\exp t_rY_r)|^{-1}}{(1+\sum_{j=1}^{m}|S_j(t_1, \ldots, t_d)|2)^{n/2}}\,dt_1 \ldots dt_d$$

$$\leqslant(2\pi)^{-d/2}\int_{\mathbb{R}^d}\frac{\prod_{r\leqslant k}|\Lambda_{j_k}(\exp t_rY_r)|^{-1}}{(1+\sum_{k=1, k\notin D_3}^{d}|S_{j_k}(t_1, \ldots, t_d)|^2)^{n/2}}\,dt_1 \ldots dt_d.$$

Suppose first that $d \in D_1 \cup D_2$. Then (assuming that $\lambda_{j_d}(Y_d) \neq 0$)

$$S_{j_d}(t_1, \ldots, t_d)=\frac{e^{-t_d\lambda_{j_d}(Y_d)}-1}{-\lambda_{j_d}(Y_d)}\prod_{r<d}\Lambda_{j_d}(\exp t_rY_r)^{-1}+S_{j_d}(t_1, \ldots, t_{d-1}, 0),$$

and the last integral is equal to

$$(2\pi)^{-d/2}\int_{\mathbb{R}^{d-1}}\prod_{r\leqslant k\leqslant d-1}|\Lambda_{j_k}(\exp t_rY_r)|^{-1}dt_1 \ldots dt_{d-1}\int_{\mathbb{R}}F(t_1, \ldots, t_d)dt_d,$$

where

$$F(t_1, \ldots, t_d)=\frac{\prod_{r=1}^{d}|\Lambda_{j_d}(\exp t_rY_r)|^{-1}}{(1+\sum_{k=1, k\notin D_3}^{d-1}|S_{j_k}(t_1, \ldots, t_{d-1}, 0)|^2+|S_{j_d}(t_1, \ldots, t_d)|^2)^{n/2}}.$$

Applying Lemma 2.1.4 with $a=1+\sum_{k=1, k\notin D_3}^{d-1}|S_{j_k}(t_1, \ldots, t_{d-1}, 0)|^2$, $\alpha+i\beta=-\lambda_{j_d}(Y_d)$, $k=-\lambda_{j_d}(Y_d)^{-1}\prod_{r<d}\Lambda_{j_d}(\exp t_rY_r)^{-1}$, $c=-S_{j_d}(t_1, \ldots, t_{d-1}, 0)-\lambda_{j_d}(Y_d)^{-1}\prod_{r<d}\Lambda_{j_d}(\exp t_rY_r)^{-1}$ we find that

$$\int_{\mathbb{R}}F(t_1, \ldots, t_d)dt_d\leqslant\frac{C_d\cdot M(n)}{(2\pi)^{d/2}}\cdot\frac{1}{(1+\sum_{k=1, k\notin D_3}^{d-1}|S_{j_k}(t_1, \ldots, t_{d-1}, 0)|^2)^{(n-1)/2}},$$

where $C_d=|\lambda_{j_d}(Y_d)|(|\operatorname{Re}\lambda_{j_d}(Y_d)|)^{-1}$ (note that since $\mathfrak{g}$ is exponential the non-vanishing of $\lambda_{j_d}(Y_d)$ implies the non-vanishing of $\operatorname{Re}(\lambda_d(Y_d))$, and therefore

$$\int_{O}(1+\|l\|^2)^{-n/2}|P_e(l)|\,d\beta_O(l)$$

$$(\#)\qquad \leqslant\frac{C_d\cdot M(n)}{(2\pi)^d}\cdot\int_{\mathbb{R}^{d-1}}\frac{\prod_{r\leqslant k\leqslant d-1}|\Lambda_{j_k}(\exp t_rY_r)|^{-1}}{(1+\sum_{k=1, k\notin D_3}^{d-1}|S_{j_k}(t_1, \ldots, t_{d-1}, 0)|^2)^{(n-1)/2}}\,dt_1 \ldots dt_{d-1}.$$

If $\lambda_{j_d}(Y_d)=0$ a simple change in the argument shows that the same relation is valid with $C_d=1$ (*cf.* below).

Suppose next that $d\in D_4$. Then

$$S_{j_d}(t_1, \ldots, t_d)=(t_{d-1}+it_d)\prod_{r\leqslant d-2}\Lambda_{j_d}(\exp t_r Y_r)^{-1}+S_{j_d}(t_1, \ldots, t_{d-2}, 0, 0),$$

and therefore we find as above that

$$\int_O (1+\|l\|^2)^{-n/2}\,|\,P_e(l)\,|\,d\beta_O(l)$$
$$\leqslant(2\pi)^{-d/2}\int_{\mathbb{R}^{d-2}}\prod_{r\leqslant k\leqslant d-2}\Lambda_{j_k}(\exp t_r Y_r)^{-1}dt_1\ldots dt_{d-2}\int_{\mathbb{R}^2}F(t_1, \ldots, t_d)dt_{d-1}dt_d,$$

where now

$$F(t_1, \ldots, t_d)=\frac{\prod_{r=1}^{d-2}|\,\Lambda_{j_d}(\exp t_r Y_r)\,|^{-2}}{\big(1+\sum_{\substack{k=1\\k\notin D_3}}^{d-2}|\,S_{j_k}(t_1, \ldots, t_{d-2}, 0, 0)\,|^2+|\,S_{j_d}(t_1, \ldots, t_d)\,|^2\big)^{n/2}}$$

(here we have used that $|\,\Lambda_{j_d}\,|=|\,\Lambda_{j_{d-1}}\,|$ and that

$$\lambda_{j_{d-1}}(Y_{d-1})=\lambda_{j_d}(Y_{d-1})=\lambda_{j_{d-1}}(Y_d)=\lambda_{j_d}(Y_d)=0).$$

Applying the relation (**) in Lemma 2.1.4 with $a=1+\sum_{\substack{k=1\\k\notin D_3}}^{d-2}|\,S_{j_k}(t_1, \ldots, t_{d-2}, 0, 0)\,|^2$, $k=\prod_{r=1}^{d-2}\Lambda_{j_d}(\exp t_r Y_r)^{-1}$, and $c=-S_{j_d}(t_1, \ldots, t_{d-2}, 0, 0)$ we find that

$$\int_{\mathbb{R}^2}F(t_1,\ldots,t_d)dt_{d-1}dt_d\leqslant\frac{M(n)M(n-1)}{(2\pi)^{d/2}}\cdot\frac{1}{\big(1+\sum_{k\notin D_3}{}_{k=1}^{d-2}|S_{j_k}(t_1,\ldots,t_{d-2},0,0)|^2\big)^{(n-2)/2}},$$

and therefore

$$\int_O (1+\|l\|^2)^{-n/2}\,|\,P_e(l)\,|\,d\beta_O(l)$$
$$(\#\,\#)\qquad\leqslant\frac{M(n)M(n-1)}{(2\pi)^{d/2}}\cdot\int_{\mathbb{R}^{d-2}}\frac{\prod_{r\leqslant k\leqslant d-2}|\,\Lambda_{j_k}(\exp t_r Y_r)\,|^{-1}}{\big(1+\sum_{\substack{k=1\\k\notin D_3}}^{d-2}|\,S_{j_k}(t_1, \ldots, t_{d-2}, 0, 0)\,|^2\big)^{(n-2)/2}}dt_1\ldots dt_{d-2}.$$

Repeating these two methods of estimation on the new integral (#) or (# #) we find that

$$\int_O (1+\|l\|^2)^{-n/2}\,|\,P_e(l)\,|\,d\beta_O(l)\leqslant(2\pi)^{-d/2}M(n)\ldots M(n-d+1)C_d\ldots C_1<+\infty$$

for $n>d$. Here $C_k=|\,\lambda_{j_k}(Y_k)\,|\,(\,|\,\mathrm{Re}\,\lambda_{j_k}(Y_k)\,|\,)^{-1}$ if $\lambda_{j_k}(Y_k)\neq 0$, and $C_k=1$ if $\lambda_{j_k}(Y_k)=0$. This ends the proof of Lemma 2.1.1.

2.2. — The purpose of this subsection is to prove Proposition 2.2.1 below.

Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, and let N be the analytic subgroup corresponding to $\mathfrak{n}$. We have $[\mathfrak{g}, \mathfrak{g}]\subset\mathfrak{n}\subset\mathfrak{g}$, and therefore $u_e\in U(\mathfrak{n})$.

PROPOSITION 2.2.1. — If $g \in \Omega_e$ and if $\pi$ is the irreducible representation of N corresponding to the orbit $O = Nf$, where $f = g \mid \mathfrak{n}$, then

$$d\pi(u_e) = Q_e(g)I.$$

REMARK 2.2.2. — Even in the special case where $\mathfrak{g}$ is assumed to be nilpotent (and therefore $\mathfrak{g} = \mathfrak{n}$), Proposition 2.2.1 provides a new result.

*Proof.* — The proof is by induction on the dimension of $\mathfrak{g}$. The proposition is clearly valid for $\dim \mathfrak{g} = 1$ (in which case $e = \emptyset$, and $Q_e \equiv 1$, $u_e = 1$). Assume then that the proposition has been proved for all dimensions of $\mathfrak{g}$ less than or equal to $m-1$, and that $\dim \mathfrak{g} = m$. The case $e = \emptyset$ being trivial we can assume that $e \neq \emptyset$, and write $e = \{ j_1 < \ldots < j_d \}$.

*Case* (a) : Suppose that there exists a non-trivial abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ such that $g \mid \mathfrak{a} = 0$. Let A be the analytic subgroup of G corresponding to $\mathfrak{a}$. We have $\mathfrak{a} \subset \mathfrak{n}$ and setting $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{a}$, $\tilde{\mathfrak{n}} = \mathfrak{n}/\mathfrak{a}$ is the nilradical of $\tilde{\mathfrak{g}}$. We set $\hat{\mathfrak{f}}_j = \mathfrak{f}_j + a_{\mathbb{C}}/a_{\mathbb{C}}$, $0 \leqslant j \leqslant m$, and let $c : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ denote the coset map. Then we have the diagram

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = \hat{\mathfrak{f}}_m \supset \hat{\mathfrak{f}}_{m-1} \supset \ldots \supset \hat{\mathfrak{f}}_1 \supset \hat{\mathfrak{f}}_0 = \{ 0 \},$$

and $\dim \hat{\mathfrak{f}}_j/\hat{\mathfrak{f}}_{j-1} = 0$ or $= 1$. Set $I = \{ 1 \leqslant j \leqslant m \mid \hat{\mathfrak{f}}_j \supsetneqq \hat{\mathfrak{f}}_{j-1} \}$, write $I = \{ i_1 < \ldots < i_{m'} \}$, and set $\tilde{\mathfrak{f}}_j = \hat{\mathfrak{f}}_{i_j}$, $1 \leqslant j \leqslant m'$. We then have a Jordan-Hölder sequence in $\tilde{\mathfrak{g}}_{\mathbb{C}}$:

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = \tilde{\mathfrak{f}}_{m'} \supset \tilde{\mathfrak{f}}_{m'-1} \supset \ldots \supset \tilde{\mathfrak{f}}_1 \supset \tilde{\mathfrak{f}}_0 = \{ 0 \}$$

which is immediately seen to be of class (*b*), and setting $\tilde{Z}_j = c(Z_{i_j})$ we have that

$$\tilde{Z}_j \in \tilde{\mathfrak{f}}_j \backslash \tilde{\mathfrak{f}}_{j-1}, j = 1, \ldots, m'.$$

Define $\tilde{g} \in \tilde{\mathfrak{g}}'$ by $\tilde{g} \circ c = g$ and $\tilde{f} = \tilde{g} \mid \tilde{\mathfrak{n}}$. We have $\mathfrak{a} \subset \mathfrak{g}_g$ and $\tilde{\mathfrak{g}}_{\tilde{g}} = \mathfrak{g}_g/\mathfrak{a}$. Moreover, $j \in J_g \Rightarrow j \in I$, since $j \notin I \Rightarrow \mathfrak{f}_j \subset \mathfrak{f}_{j-1} + \mathfrak{a}_{\mathbb{C}} \subset \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Rightarrow j \notin J_g$. Writing

$$\tilde{e} = J_{\tilde{g}} = \{ \tilde{j}_1 < \ldots < \tilde{j}_d \}$$

we have $J_g = \{ i_{\tilde{j}_1} < \ldots < i_{\tilde{j}_d} \} = \{ j_1 < \ldots < j_d \}$. For $\tilde{l} \in \tilde{\mathfrak{g}}'$ we then have with $l = \tilde{l} \circ c$:

$$Q_e(l) = | \det [B_l(Z_{j_r}, Z_{j_s})]_{1 \leqslant r,s \leqslant d} | = | \det [B_l(Z_{i_{\tilde{j}_r}}, Z_{i_{\tilde{j}_s}})]_{1 \leqslant r,s \leqslant d} |$$
$$= | \det [B_{\tilde{l}}(\tilde{Z}_{\tilde{j}_r}, \tilde{Z}_{\tilde{j}_s})]_{1 \leqslant r,s \leqslant d} | = Q_{\tilde{e}}(\tilde{l}).$$

This shows that the canonical image of $u_e$ in $U(\tilde{\mathfrak{g}})$ is precisely $u_{\tilde{e}}$ ($\in U(\tilde{\mathfrak{n}})$). Now the representation $\pi$ is trivial on A, so there exists an irreducible representation $\tilde{\pi}$ of $\tilde{N} = N/A$ such that $\tilde{\pi} \circ (c \mid N) = \pi$, and the orbit of $\tilde{\pi}$ is $\tilde{N}\tilde{f}$. But since $\tilde{g} \in \Omega_{\tilde{e}}$ we have $d\tilde{\pi}(u_{\tilde{e}}) = Q_e(\tilde{g})I$ by the induction hypothesis, and therefore $d\pi(u_e) = d\tilde{\pi}(c(u_e)) = d\tilde{\pi}(u_{\tilde{e}}) = Q_{\tilde{e}}(\tilde{g})I = Q_e(g)I$. This ends case (a).

*Case* (b) : Suppose that we are not in case (a) and that $\lambda_1 \neq 0$.

Write $Z_1 = X_1 + iY_1$ and set $\mathfrak{a} = \mathbb{R}X_1 + \mathbb{R}Y_1$. Then $\mathfrak{a}$ is an abelian ideal (of dimension 1 or 2), and $g \mid \mathfrak{a} \neq 0$ (since otherwise we would be in case (a)), and therefore $\langle g, Z_1 \rangle \neq 0$.

Since G is exponential we can write $\lambda_1(X) = \alpha_1(X)(1 + ik_1)$, where $\alpha_1$ is a real linear form on $\mathfrak{g}$, and where $k_1$ is a real number.

Set $\mathfrak{h} = \ker \lambda_1$ $(= \ker \alpha_1)$. $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension 1 with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{h}$, so the nilradical of $\mathfrak{h}$ is $\mathfrak{n}$. Clearly $Z_1 \in \mathfrak{h}_{\mathbb{C}}$. Set $p = \min\{1 \leqslant j \leqslant m \mid Z_j \notin \mathfrak{h}_{\mathbb{C}}\}$. $p$ is well-defined and $p \geqslant 2$. We observe that $p \in J_g = e$. In fact, suppose $p \notin J_g$. Then $Z_p \in \mathfrak{f}_{p-1} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore $0 \neq \langle g, [Z_p, \mathfrak{f}_1] \rangle = \langle Z_p g, \mathfrak{f}_1 \rangle = \langle \mathfrak{f}_{p-1} g, \mathfrak{f}_1 \rangle = \langle g, [\mathfrak{f}_{p-1}, \mathfrak{f}_1] \rangle = 0$, which is a contradiction. Also $1 \in J_g$, since otherwise $Z_1 \in (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$0 = \langle g, [\mathfrak{g}, \mathfrak{f}_1] \rangle = \langle g, \mathfrak{f}_1 \rangle \neq 0.$$

We also note that $\mathfrak{g}_g \subset \mathfrak{h}$, since otherwise $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_g$ and therefore $0 = \langle \mathfrak{g} g, \mathfrak{f}_1 \rangle = \langle g, \mathfrak{f}_1 \rangle \neq 0$.

Set $\hat{Z}_j = Z_j$ for $1 \leqslant j \leqslant p-1$, $\hat{Z}_j = Z_{j+1} + c_{j+1} Z_p$ for $p \leqslant j \leqslant m-1$ and $\hat{Z}_m = Z_p$. Here $c_j$, $p+1 \leqslant j \leqslant m$, is defined such that $Z_j + c_j Z_p \in \mathfrak{h}_{\mathbb{C}}$. This is possible since $\mathbb{C} Z_p \oplus \mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$. Clearly $\hat{Z}_1, \ldots, \hat{Z}_m$ is a basis in $\mathfrak{g}_{\mathbb{C}}$.

Set $\hat{\mathfrak{f}}_j = \mathbb{C}\hat{Z}_1 \oplus \ldots \oplus \mathbb{C}\hat{Z}_j$. For $0 \leqslant j \leqslant p-1$ we have that $\hat{\mathfrak{f}}_j = \mathfrak{f}_j$. For $p-1 \leqslant j \leqslant m-1$ we have $\hat{\mathfrak{f}}_j \oplus \mathbb{C} Z_p = \mathfrak{f}_{j+1}$, hence

$$\begin{aligned}
&\hat{\mathfrak{f}}_j = \mathfrak{f}_j \quad \text{for} \quad 0 \leqslant j \leqslant p-1,\\
&\hat{\mathfrak{f}}_j = \mathfrak{f}_{j+1} \cap \mathfrak{h}_{\mathbb{C}} \quad \text{for} \quad p-1 \leqslant j \leqslant m-1,\\
&\hat{\mathfrak{f}}_m = \mathfrak{g}_{\mathbb{C}}.
\end{aligned}$$

From this it follows that $\hat{\mathfrak{f}}_j$, $j = 0, \ldots, m$, is a Jordan-Hölder sequence for $\mathfrak{g}_{\mathbb{C}}$ with $\hat{\mathfrak{f}}_{m-1} = \mathfrak{h}_{\mathbb{C}}$. We claim it is of class (*b*). In fact, since $\hat{\mathfrak{f}}_{p-1} = \mathfrak{f}_p \cap \mathfrak{h}_{\mathbb{C}}$ and $\hat{\mathfrak{f}}_{p-1} = \mathfrak{f}_{p-1}$ it follows that $\bar{\hat{\mathfrak{f}}}_{p-1} = \hat{\mathfrak{f}}_{p-1}$, and from this it is immediate that the claim is true. We thus have a new diagram

$$\begin{array}{c}
\mathfrak{g}_{\mathbb{C}} = \hat{\mathfrak{f}}_m \supset \hat{\mathfrak{f}}_{m-1} \supset \ldots \supset \hat{\mathfrak{f}}_1 \supset \hat{\mathfrak{f}}_0 = \{0\}. \\
\qquad\qquad\qquad \| \\
\qquad\qquad\qquad \mathfrak{h}_{\mathbb{C}}
\end{array}$$

The objects defined relative to this new Jordan-Hölder sequence are designated $\hat{J}_g$, $\hat{e}$, etc.

For $1 \leqslant j \leqslant p-1$ we clearly have $j \in J_g \Leftrightarrow j \in \hat{J}_g$. Furthermore $p \in J_g$ (*see* above) and $m \in \hat{J}_g$. In fact, if $m \notin \hat{J}_g$, then $Z_p = \hat{Z}_m \in \hat{\mathfrak{f}}_{m-1} + (\mathfrak{g}_g)_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$0 \neq \langle Z_p g, \mathfrak{f}_1 \rangle = \langle \mathfrak{h} g, \mathfrak{f}_1 \rangle = 0.$$

For $p+1 \leqslant j \leqslant m$ we have

$$\begin{aligned}
j \notin J_g \Leftrightarrow Z_j \in \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} &\Leftrightarrow Z_j \in \hat{\mathfrak{f}}_{j-2} + \mathbb{C} Z_p + (\mathfrak{g}_g)_{\mathbb{C}}\\
&\Leftrightarrow \hat{Z}_{j-1} \in \hat{\mathfrak{f}}_{j-2} + \mathbb{C} Z_p + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-1} \in \hat{\mathfrak{f}}_{j-2} + (\mathfrak{g}_g)_{\mathbb{C}}
\end{aligned}$$

(since $\mathfrak{g}_g \subset \mathfrak{h}$) $\Leftrightarrow j-1 \notin \hat{J}_g$. Therefore, if $j_\alpha = p$ we have $\hat{j}_h = j_h$ for $1 \leqslant h \leqslant \alpha - 1$, $\hat{j}_h + 1 = j_{h+1}$ for $\alpha \leqslant h \leqslant d-1$ and $\hat{j}_d = m$, so

$$\begin{aligned}
&\hat{Z}_{\hat{j}_h} = Z_{j_h} \quad \text{for} \quad 1 \leqslant h \leqslant \alpha - 1,\\
&\hat{Z}_{\hat{j}_h} = Z_{j_{h+1}} + c_{j_{h+1}} Z_{j_\alpha} \quad \text{for} \quad \alpha \leqslant h \leqslant d-1,\\
&\hat{Z}_{\hat{j}_d} = Z_{j_\alpha}.
\end{aligned}$$

Therefore, letting $C=[c_{rs}]_{1\leqslant r,s\leqslant d}$ be the $d\times d$-matrix:

$$C=\left[\begin{array}{c|c|c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & \\ \hline & c_{j_{\alpha+1}}\ \dots\ c_{j_d} & 1 \\ \hline & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \end{array}\right],$$

where the empty entries are 0, we have $\hat{Z}_{\hat{j}_s}=\Sigma_{r=1}^{d} c_{rs} Z_{j_r}$, and therefore $M_{\hat{e}}(l)={}^tCM_e(l)C$, with $\hat{e}=\hat{J}_g$. Now $\det C=(-1)^{\alpha}$, so $\det M_e(l)=\det M_{\hat{e}}(l)$, and $Q_e(l)=Q_{\hat{e}}(l)$, and therefore $u_e=u_{\hat{e}}$. The conclusion of this is then that we can assume that $\mathfrak{k}_{m-1}=\mathfrak{h}_{\mathbb{C}}$, and this assumption will be in effect from now on. We then have:

$$\begin{aligned} Q_e(l) &= |\det [B_l(Z_{j_r}, Z_{j_s})]_{1\leqslant r,s\leqslant d}| \\ &= \Big|\sum_{\sigma\in S_d} \operatorname{sign}\sigma \langle l, [Z_{j_1}, Z_{j_{\sigma(1)}}]\rangle \dots \langle l, [Z_{j_d}, Z_{j_{\sigma(d)}}]\rangle\Big| \\ &= |\langle l, [Z_1, Z_m]\rangle|^2 \cdot \Big|\sum_{\sigma\in S_d^*} \operatorname{sign}\sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)}}]\rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-1)}}]\rangle\Big| \end{aligned}$$

where $S_d^*$ is the set of elements $\sigma\in S_d$ with $\sigma(1)=d$, $\sigma(d)=1$.

Set $g_0=g\,|\,\mathfrak{h}$. Then $f=g_0\,|\,\mathfrak{n}$. We designate the objects associated with the group $H=\exp\mathfrak{h}$, and the class (*b*) Jordan-Hölder sequence $\mathfrak{h}_{\mathbb{C}}=\mathfrak{k}_{m-1}\supset\dots\supset\mathfrak{k}_1\supset\mathfrak{k}_0=\{0\}$ by $J^0_{g_0}$, etc. We have $(\mathfrak{h}_{g_0})_{\mathbb{C}}=(\mathfrak{g}_g)_{\mathbb{C}}\oplus\mathbb{C}Z_1$, so $J^0_{g_0}=J_g\setminus\{1, m\}$, and therefore

$$J^0_{g_0}=\{j^0_1<\dots<j^0_{d-2}\}$$

with $j^0_h=j_{h+1}$ for $1\leqslant h\leqslant d-2$, so we have for $l\in\mathfrak{h}'$:

$$\begin{aligned} Q_{e^0}(l) &= |\det [B_l(Z_{j^0_r}, Z_{j^0_s})]_{1\leqslant r,s\leqslant d-2}| \\ &= \Big|\sum_{\sigma\in S_{d-2}} \operatorname{sign}\sigma \langle l, [Z_{j^0_1}, Z_{j^0_{\sigma(1)}}]\rangle \dots \langle l, [Z_{j^0_{d-2}}, Z_{j^0_{\sigma(d-2)}}]\rangle\Big| \\ &= \Big|\sum_{\sigma\in S_{d-2}} \operatorname{sign}\sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)+1}}]\rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-2)+1}}]\rangle\Big| \\ &= \Big|\sum_{\sigma\in S_d^*} \operatorname{sign}\sigma \langle l, [Z_{j_2}, Z_{j_{\sigma(2)}}]\rangle \dots \langle l, [Z_{j_{d-1}}, Z_{j_{\sigma(d-1)}}]\rangle\Big|, \end{aligned}$$

and comparing with the result above we get $Q_e(l)=|\langle l, W\rangle|^2 Q_{e^0}(l_0)$, where $W=[Z_1, Z_m]$ and $l_0=l\,|\,\mathfrak{h}$. Now since W is central in $\mathfrak{h}_{\mathbb{C}}$ and since $P_W(l)=\langle l, W\rangle$, $P_{\overline{W}}(l)=\langle l, \overline{W}\rangle$ we find that $i^d Q_e(l)=-P_W(l)P_{\overline{W}}(l)i^{d-2}Q_{e^0}(l_0)$, and therefore $u_e=-W\overline{W}u_{e^0}$ by Lemma 1.2.1. By the induction hypothesis we have that $d\pi(u_{e^0})=Q_{e^0}(g)I$, and noting that

$$d\pi(W)=i\langle g, W\rangle I,\ d\pi(\overline{W})=i\langle g, \overline{W}\rangle I$$

we finally get $d\pi(u_e)=|\langle g, W\rangle|^2 d\pi(u_{e^0})=|\langle g, W\rangle|^2 Q_{e^0}(g)I=Q_e(g)I$. This settles case (*b*).

*Case* (c) : Suppose we are not in case (a) and (b) and that $\lambda_2 \neq 0$.

Again we have $\langle g, Z_1 \rangle \neq 0$ and, moreover, $\overline{\mathfrak{f}}_1 = \mathfrak{f}_1$ (since $\mathfrak{f}_1$ is a central ideal in $\mathfrak{g}_{\mathbb{C}}$). We write $[X, Z_2] = \lambda_2(X)Z_2 + \gamma(X)Z_1$, $X \in \mathfrak{g}$, where $\gamma$ is a (complex valued) linear form on $\mathfrak{g}$. The linear form $\gamma$ has the form $\gamma(X) = \gamma_1(X) + i\gamma_2(X)$, where $\gamma_1, \gamma_2$ are real linear forms on $\mathfrak{g}$. We extend $\lambda_2$, $\gamma$ to complex linear forms on $\mathfrak{g}_{\mathbb{C}}$ such that we have

$$[Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1 \quad \text{for} \quad Z \in \mathfrak{g}_{\mathbb{C}}.$$

We note the formula

$$\gamma([Z, W]) = \gamma(Z)\lambda_2(W) - \gamma(W)\lambda_2(Z) \tag{2.2.2}$$

for $Z, W \in \mathfrak{g}_{\mathbb{C}}$, which we get by a simple application of the Jacobi identity.

Since G is exponential we can write $\lambda_2(X) = \alpha_2(X)(1 + ik_2)$, where $\alpha_2$ is a real linear form on $\mathfrak{g}$ and where $k_2$ is a real number.

We then distinguish three subcases: (c1): rank $(\alpha_2, \gamma_1, \gamma_2) = 3$, (c2): rank $(\alpha_2, \gamma_1, \gamma_2) = 2$ and (c3): rank $(\alpha_2, \gamma_1, \gamma_2) = 1$.

*Case* (c1): Set $\mathfrak{h} = \ker \gamma_1 \cap \ker \gamma_2$ $(= \ker \gamma \mid \mathfrak{g})$. It follows from the formula (2.2.2) that $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$, and its codimension is 2. We observe that $Z \in \mathfrak{h}_{\mathbb{C}}$ if and only if $\gamma(Z) = 0$ and $\gamma(\overline{Z}) = 0$. Set $\mathfrak{h}_0 = \ker \lambda_2 \mid \mathfrak{h} = \ker \alpha_2 \mid \mathfrak{h} = \ker \operatorname{ad} Z_2 \mid \mathfrak{g}$. $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$ of codimension 3. That $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$ follows from the fact that

$$\mathfrak{h}_0 = \ker \gamma \cap \ker \lambda_2 \cap \mathfrak{g}$$

and by applying the formula (2.2.2).

Let $\mathfrak{m}$ be the nilradical of $\mathfrak{h}_0$. Since $\mathfrak{h}_0$ is an ideal we have that $\mathfrak{m} = \mathfrak{n} \cap \mathfrak{h}_0 = \mathfrak{n} \cap \mathfrak{h}$. Observe that $\dim \mathfrak{n}/\mathfrak{m} = 2$. In fact, pick $W \in \mathfrak{h} \backslash \mathfrak{h}_0$. Then we have that

$$\gamma([Z, W]) = \lambda_2(W)\gamma(Z) \quad \text{for} \quad Z \in \mathfrak{g}_{\mathbb{C}},$$

and therefore $\gamma(\overline{[Z, W]}) = \lambda_2(\overline{W})\gamma(\overline{Z})$. Choosing Z such that $\gamma(Z) = 1$, $\gamma(\overline{Z}) = 0$ and $Z'$ such that $\gamma(Z') = 0$, $\gamma(\overline{Z'}) = 1$ we get that

$$\gamma([Z, W]) = \lambda_2(W) \neq 0, \ \gamma(\overline{[Z, W]}) = 0, \ \gamma(\overline{[Z', W]}) = \lambda_2(\overline{W}) \neq 0, \ \gamma([Z', W]) = 0,$$

and this shows that $[Z, W]$, $[Z', W]$ is a basis in $\mathfrak{n}_{\mathbb{C}}$ (mod $\mathfrak{m}_{\mathbb{C}}$).

We claim that $\overline{\mathfrak{f}}_2 \neq \mathfrak{f}_2$. In fact, we have $[Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1$ for all $Z \in \mathfrak{g}_{\mathbb{C}}$, and therefore $[Z, \overline{Z}_2] = \overline{\lambda_2(\overline{Z})}\overline{Z}_2 + \overline{\gamma(\overline{Z})}\overline{Z}_1$. Since $\lambda_2$ does not vanish on $\mathfrak{h}_{\mathbb{C}}$ we have that $[\mathfrak{h}_{\mathbb{C}}, \mathfrak{f}_2] = \mathbb{C}Z_2$ and $[\mathfrak{h}_{\mathbb{C}}, \overline{\mathfrak{f}}_2] = \mathbb{C}\overline{Z}_2$. Therefore, if $\overline{\mathfrak{f}}_2 = \mathfrak{f}_2$, then $\mathbb{C}Z_2 = \mathbb{C}\overline{Z}_2$, hence $\gamma(Z) = 0$ implies that $\gamma(\overline{Z}) = 0$, so $\mathfrak{h}_{\mathbb{C}}$ is the set of $Z \in \mathfrak{g}_{\mathbb{C}}$ such that $\gamma(Z) = 0$, contradicting the fact that codim $\mathfrak{h} = 2$. We conclude that $\overline{\mathfrak{f}}_2 \neq \mathfrak{f}_2$, and therefore that $\overline{\mathfrak{f}}_1 = \mathfrak{f}_1$ and $\overline{\mathfrak{f}}_3 = \mathfrak{f}_3$. In particular $\overline{Z}_2 \notin \mathfrak{f}_2$.

We have seen that $Z_1$, $Z_2$, $\overline{Z}_2$ span $\mathfrak{f}_3$. Now since $\lambda_2(Z_2) = 0$ we have that $\alpha_2(Z_2) = 0$, and this means that $[\mathfrak{f}_3, \mathfrak{f}_2] \subset \mathfrak{f}_1$. We then distinguish two possibilities: case (c11): $[\mathfrak{f}_3, \mathfrak{f}_2] = 0$ and case (c12): $[\mathfrak{f}_3, \mathfrak{f}_2] = \mathfrak{f}_1$.

Set $f_0 = f \mid \mathfrak{m} = g \mid \mathfrak{m}$, and let $\pi_0$ be the irreducible representation of $M = \exp \mathfrak{m}$ corresponding to $Mf_0$.

*Case* (c11): (i) It is our first aim to show that $u_e \in U(\mathfrak{m}_{\mathbb{C}})$, and that $d\pi_0(u_e) = Q_e(g)I$. We start by noting that we can assume that $\langle g, Z_2 \rangle = 0$; in fact, if necessary replace $Z_2$ by $Z_2 - cZ_1$; this does not change $e$, $Q_e$, etc. (it changes $\gamma = \gamma_1 + i\gamma_2$, though, but does not affect $\mathfrak{h}_0$ and rank $(\alpha_2, \gamma_1, \gamma_2)$).

Set $p = \min \{ 1 \leqslant j \leqslant m \mid Z_j \notin \mathfrak{h}_{\mathbb{C}} \}$. $p$ is well-defined, and $4 \leqslant p \leqslant m-1$, since $Z_1, Z_2, Z_3 \in \mathfrak{h}_{\mathbb{C}}$, and since the codimension of $\mathfrak{h}_{\mathbb{C}}$ is 2. Set $q = \min \{ 1 \leqslant j \leqslant m \mid Z_j \notin \mathbb{C}Z_p \oplus \mathfrak{h}_{\mathbb{C}} \}$. $q$ is well-defined and $5 \leqslant p+1 \leqslant q \leqslant m$ (so $\dim \mathfrak{g} \geq 6$).

We first note $2, 3 \in J_g$. In fact, if $2 \notin J_g$, then $Z_2 \in \mathfrak{f}_1 + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$\gamma(Z) \langle g, Z_1 \rangle = \langle g, [Z, Z_2] \rangle = \langle Z_2 g, Z \rangle = 0 \quad \text{for all} \quad Z \in \mathfrak{g}_{\mathbb{C}}$$

which is a contradiction. So $2 \in J_g$. If $3 \notin J_g$, then $\overline{Z}_2 \in \mathfrak{f}_2 + (\mathfrak{g}_g)_{\mathbb{C}}$, i. e. $\overline{Z}_2 = aZ_2 \pmod{(g_g)_{\mathbb{C}}}$, $a \in \mathbb{C}$. But then

$$\overline{\gamma(\overline{Z})}\,\overline{\langle g, Z_1 \rangle} = \langle g, [Z, \overline{Z}_2] \rangle = \langle \overline{Z}_2 g, Z \rangle = a \langle Z_2 g, Z \rangle = a\gamma(Z) \langle g, Z_1 \rangle$$

which contradicts the fact that codim $\mathfrak{h} = 2$, so $3 \in J_g$. We also note that $1 \notin J_g$, since $\mathfrak{f}_1 \subset (\mathfrak{g}_g)_{\mathbb{C}}$.

Next we note that $p, q \in J_g$. In fact, if $p \notin J_g$, then $Z_p \in \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$ and $\overline{Z}_p \in \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, and therefore

$$\begin{aligned} -\gamma(Z_p) \langle g, Z_1 \rangle = \langle g, [Z_2, Z_p] \rangle = \langle Z_p g, Z_2 \rangle &\subset \langle \mathfrak{h}_{\mathbb{C}} g, Z_2 \rangle \\ &= \langle g, [\mathfrak{h}_{\mathbb{C}}, Z_2] \rangle = \langle g, \mathbb{C}Z_2 \rangle = 0, \end{aligned}$$

so $\gamma(Z_p) = 0$ and similarly $\gamma(\overline{Z}_p) = 0$ implying that $Z_p \in \mathfrak{h}_{\mathbb{C}}$, which is a contradiction. Therefore $p \in J_g$. Suppose then that $q \notin J_g$. Then $Z_q \in \mathbb{C}Z_p + \mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}}$, i. e. there exists $a \in \mathbb{C}$ with $Z_q = aZ_p \pmod{(\mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_g)_{\mathbb{C}})}$. But then

$$-\gamma(Z_q) \langle g, Z_1 \rangle = \langle g, [Z_2, Z_q] \rangle = \langle Z_q g, Z_2 \rangle = a \langle Z_p g, Z_2 \rangle = -a\gamma(Z_p) \langle g, Z_1 \rangle,$$

from which $\gamma(Z_q) = a\gamma(Z_p)$. Similarly $\overline{\gamma(\overline{Z}_q)} = \overline{a\gamma(\overline{Z}_p)}$. Now consider the linear map from $\mathfrak{g}_{\mathbb{C}}$ to $\mathbb{C}^2$ given by $Z \to (\gamma(Z), \overline{\gamma(\overline{Z})})$. The kernel is $\mathfrak{h}_{\mathbb{C}}$, so it is surjective since codim $\mathfrak{h} = 2$. But $Z_p, Z_q$ is a basis for $\mathfrak{g}_{\mathbb{C}} \pmod{\mathfrak{h}_{\mathbb{C}}}$, and we have just shown that the images of $Z_p$ and of $Z_q$ are linearly dependent; in fact, $(\gamma(Z_q), \overline{\gamma(\overline{Z}_q)}) = a(\gamma(Z_p), \overline{\gamma(\overline{Z}_p)})$. But this is a contradiction, and we conclude that $q \in J_g$.

Define $\hat{Z}_j = Z_j$ for $1 \leqslant j \leqslant p-1$, $\hat{Z}_j = Z_{j+1} + a_{j+1} Z_p$ for $p \leqslant j \leqslant q-2$ (empty if $q = p+1$), $\hat{Z}_j = Z_{j+2} + a_{j+2} Z_p + b_{j+2} Z_q$ for $q-1 \leqslant j \leqslant m-2$, $Z_{m-1} = aZ_p + bZ_q$, $Z_m = a'Z_p + b'Z_q$, where $a_{p+1}, \ldots, a_{q-1}, a_{q+1}, \ldots, a_m, b_{q+1}, \ldots, b_m$ has been picked such that $\hat{Z}_j \in \mathfrak{h}_{\mathbb{C}}$, $1 \leqslant j \leqslant m-2$; this is possible since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}Z_p \oplus \mathbb{C}Z_q$. The numbers $a, b, a', b' \in \mathbb{C}$ has been selected such that $ab' - a'b = 1$, and such that $\langle g, [\hat{Z}_{m-1}, Z_2] \rangle = 0$, $\langle g, [\hat{Z}_{m-1}, Z_3] \rangle \neq 0$, $\langle g, [\hat{Z}_m, Z_3] \rangle = 0$, $\langle g, [\hat{Z}_m, Z_2] \rangle \neq 0$ which is possible by a reasoning as above. Clearly $\hat{Z}_1, \ldots, \hat{Z}_m$ is a basis for $\mathfrak{g}_{\mathbb{C}}$. Set $\hat{\mathfrak{f}}_j = \mathbb{C}\hat{Z}_1 \oplus \ldots \oplus \mathbb{C}\hat{Z}_j$. For $0 \leqslant j \leqslant p-1$ we have that $\hat{\mathfrak{f}}_j=\mathfrak{f}_j$. For $p-1\leqslant j\leqslant q-2$ we have that $\hat{\mathfrak{f}}_j\oplus\mathbb{C}Z_p=\mathfrak{f}_{j+1}$ and for $q-2\leqslant j\leqslant m-2$ we have that $\hat{\mathfrak{f}}_j\oplus\mathbb{C}Z_p\oplus\mathbb{C}Z_q=\mathfrak{f}_{j+2}$. Also $\hat{\mathfrak{f}}_{m-2}=\mathfrak{h}_{\mathbb{C}}$, $\hat{\mathfrak{f}}_m=\mathfrak{g}_{\mathbb{C}}$. We thus have

$$\begin{aligned}
&\hat{\mathfrak{f}}_j=\mathfrak{f}_j \quad \text{for} \quad 0\leqslant j\leqslant p-1,\\
&\hat{\mathfrak{f}}_j=\mathfrak{f}_{j+1}\cap\mathfrak{h}_{\mathbb{C}} \quad \text{for} \quad p-1\leqslant j\leqslant q-2,\\
&\hat{\mathfrak{f}}_j=\mathfrak{f}_{j+2}\cap\mathfrak{h}_{\mathbb{C}} \quad \text{for} \quad q-2\leqslant j\leqslant m-2,\\
&\hat{\mathfrak{f}}_m=\mathfrak{g}_{\mathbb{C}}.
\end{aligned}$$

From this it follows that

$$\mathfrak{h}_{\mathbb{C}}=\hat{\mathfrak{f}}_{m-2}\supset \ldots \supset\hat{\mathfrak{f}}_1\supset\hat{\mathfrak{f}}_0=\{0\}$$

is a Jordan-Hölder sequence for $\mathfrak{h}_{\mathbb{C}}$ (but note that $\hat{\mathfrak{f}}_0, \ldots, \hat{\mathfrak{f}}_m$ is not necessarily a Jordan-Hölder sequence for $\mathfrak{g}_{\mathbb{C}}$, since $\mathfrak{h}$ is not necessarily an ideal in $\mathfrak{g}$). We claim it is a Jordan-Hölder sequence of class (*b*). To see this, observe that $\hat{\mathfrak{f}}_{p-1}=\bar{\hat{\mathfrak{f}}}_{p-1}$, since $\hat{\mathfrak{f}}_{p-1}=\mathfrak{f}_{p-1}$ and $\hat{\mathfrak{f}}_{p-1}=\mathfrak{f}_p\cap\mathfrak{h}_{\mathbb{C}}$, and $\hat{\mathfrak{f}}_{q-2}=\bar{\hat{\mathfrak{f}}}_{q-2}$, since $\hat{\mathfrak{f}}_{q-2}=\mathfrak{f}_{q-1}\cap\mathfrak{h}_{\mathbb{C}}$ and $\hat{\mathfrak{f}}_{q-2}=\mathfrak{f}_q\cap\mathfrak{h}_{\mathbb{C}}$, and from this it follows easily that $\hat{\mathfrak{f}}_j$, $j=0, \ldots, m-2$ is of class (*b*).

Write $e=\{j_1<\ldots<j_d\}$, and let $j_\alpha=p$, $j_\beta=q$ with $1\leqslant\alpha<\beta\leqslant d$. Define the set $\hat{\mathrm{J}}_g=\{\hat{j}_1<\ldots<\hat{j}_d\}$ by setting $\hat{j}_1=j_1, \ldots, \hat{j}_{\alpha-1}=j_{\alpha-1}$, $\hat{j}_h=j_{h+1}-1$ for $\alpha\leqslant h\leqslant\beta-2$, $\hat{j}_h=j_{h+2}-2$ for $\beta-1\leqslant h\leqslant d-2$, $\hat{j}_{d-1}=m-1$, $\hat{j}_d=m$. We then have

$$\begin{aligned}
&\hat{Z}_{\hat{j}_h}=Z_{j_h} \quad \text{for} \quad 1\leqslant h\leqslant\alpha-1,\\
&\hat{Z}_{\hat{j}_h}=Z_{j_{h+1}}+a_{j_{h+1}}Z_{j_\alpha} \quad \text{for} \quad \alpha\leqslant h\leqslant\beta-2,\\
&\hat{Z}_{\hat{j}_h}=Z_{j_{h+2}}+a_{j_{h+2}}Z_{j_\alpha}+b_{j_{h+2}}Z_{j_\beta} \quad \text{for} \quad \beta-1\leqslant h\leqslant d-2,\\
&\hat{Z}_{\hat{j}_{d-1}}=aZ_{j_\alpha}+bZ_{j_\beta},\\
&\hat{Z}_{\hat{j}_d}=a'Z_{j_\alpha}+b'Z_{j_\beta}.
\end{aligned}$$

Therefore, letting $C=[c_{rs}]_{1\leqslant r,s\leqslant d}$ be the $d\times d$-matrix:

$$\begin{array}{r}
\\ \\ \alpha\rightarrow \\ C= \\ \beta\rightarrow \\ \\
\end{array}
\begin{array}{c}
\qquad\qquad\ \alpha\downarrow \qquad \beta-2\downarrow \quad \beta-1\downarrow \quad d-2\downarrow \quad d-1\downarrow \quad d\downarrow\\
\left[\begin{array}{c|c|c|c|c}
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & & & \\ \hline
 & a_{j_{\alpha+1}}\ \ldots\ a_{j_{\beta-1}} & a_{j_{\beta+1}}\ \ldots\ a_{j_d} & a & a' \\ \hline
 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & & \\ \hline
 & & b_{j_{\beta+1}}\ \ldots\ b_{j_d} & b & b' \\ \hline
 & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & &
\end{array}\right]
\end{array}$$

where the empty entries are zero, we have $\hat{Z}_{\hat{j}_s}=\Sigma_{r=1}^{d}c_{rs}Z_{j_r}$, and $\hat{M}_e(l)={}^tCM_e(l)C$, where

$\hat{M}_e(l)$ is the matrix $[B_l(\hat{Z}_{\hat{j}_r}, \hat{Z}_{\hat{j}_s})]_{1 \leq r,s \leq d}$. Now det $C=(-1)^{\alpha+\beta}$, and therefore we have for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$:

$$
\begin{aligned}
Q_e(l) = |\det M_e(l)| = |\det \hat{M}_e(l)| \\
&= \Big|\sum_{\sigma \in S_d} \operatorname{sign} \sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_d}, \hat{Z}_{\hat{j}_{\sigma(d)}}] \rangle \Big| \\
&= |\langle l, [\hat{Z}_2, \hat{Z}_m] \rangle|^2 \, |\langle l, [\hat{Z}_3, \hat{Z}_{m-1}] \rangle|^2 \\
&\quad \cdot \Big|\sum_{\sigma \in S_d^*} \operatorname{sign} \sigma \langle l, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(3)}}] \rangle \dots \langle l, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \Big|,
\end{aligned}
$$

where $S_d^*$ is the set of permutations $\sigma$ in $S_d$ such that $\sigma(1)=d$, $\sigma(2)=d-1$, $\sigma(d-1)=2$, $\sigma(d)=1$.

Set $g_0 = g \,|\, \mathfrak{h}$, and let $\hat{J}^0_{g_0}$ etc. designate the objects defined relative to the Jordan-Hölder sequence $\hat{\mathfrak{f}}_0 \subset \hat{\mathfrak{f}}_1 \subset \dots \subset \hat{\mathfrak{f}}_{m-2} = \mathfrak{h}_{\mathbb{C}}$. Since clearly $\mathfrak{g}_g \subset \mathfrak{h}$, and $(\mathfrak{h}_{g_0})_{\mathbb{C}} = (\mathfrak{g}_g)_{\mathbb{C}} + \mathbb{C}Z_2 + \mathbb{C}Z_3$ we find that $1, 2, 3 \notin \hat{J}^0_{g_0}$, and for $4 \leq j \leq p-1$ we find $j \in \hat{J}^0_{g_0} \Leftrightarrow j \in J_g$. For $p+1 \leq j \leq q-2$ we have

$$
j \notin J_g \Leftrightarrow Z_j \in \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \in \hat{\mathfrak{f}}_{j-2} + \mathbb{C}Z_p + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-1} \in \hat{\mathfrak{f}}_{j-2} + (\mathfrak{h}_{g_0})_{\mathbb{C}} \Leftrightarrow j-1 \in \hat{J}^0_{g_0},
$$

so $j \in J_g \Leftrightarrow j-1 \in \hat{J}^0_{g_0}$. For $q+1 \leq j \leq m$ we have

$$
j \notin J_g \Leftrightarrow Z_j \in \mathfrak{f}_{j-1} + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \in \hat{\mathfrak{f}}_{j-3} + \mathbb{C}Z_p + \mathbb{C}Z_q + (\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-2} \in \hat{\mathfrak{f}}_{j-3} + (\mathfrak{h}_{g_0})_{\mathbb{C}} \Leftrightarrow j-2 \notin \hat{J}^0_{g_0},
$$

so $j \in J_g \Leftrightarrow j-2 \in \hat{J}^0_{g_0}$. Therefore, if $\hat{e}^0 = \hat{J}^0_{g_0} = \{\hat{j}^0_1 < \dots < \hat{j}^0_{d-4}\}$ we find that $\hat{j}^0_h = j_{h+2}$ for $1 \leq h \leq \alpha-3$, $\hat{j}^0_h + 1 = j_{h+3}$ for $\alpha-2 \leq h \leq \beta-4$, $\hat{j}^0_h + 2 = j_{h+4}$ for $\beta-3 \leq h \leq d-4$, and comparing with the definition of $\hat{j}_h$ we find that $\hat{j}^0_h = \hat{j}_{h+2}$ for $1 \leq h \leq d-4$. Using this we get for $l_0 \in \mathfrak{h}'$:

$$
\begin{aligned}
Q_{e_0}(l_0) &= \Big|\sum_{\sigma \in S_{d-4}} \operatorname{sign} \sigma \langle l_0, [\hat{Z}_{\hat{j}^0_1}, \hat{Z}_{\hat{j}^0_{\sigma(1)}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}^0_{d-4}}, \hat{Z}_{\hat{j}^0_{\sigma(d-4)}}] \rangle \Big| \\
&= \Big|\sum_{\sigma \in S_{d-4}} \operatorname{sign} \sigma \langle l_0, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(1)+2}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)+2}}] \rangle \Big| \\
&= \Big|\sum_{\sigma \in S_d^*} \operatorname{sign} \sigma \langle l_0, [\hat{Z}_{\hat{j}_3}, \hat{Z}_{\hat{j}_{\sigma(3)}}] \rangle \dots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}] \rangle \Big|,
\end{aligned}
$$

and comparing with what we saw above we find for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$ and $l_0 = l \,|\, \mathfrak{h}$:

$$
(*) \qquad Q_e(l) = |\langle l, [\hat{Z}_2, \hat{Z}_m] \rangle|^2 \, |\langle l, [\hat{Z}_3, \hat{Z}_{m-1}] \rangle|^2 Q_{e^0}(l_0).
$$

Let us now observe that the nilradical of $\mathfrak{h}$ is $\mathfrak{m}$. In fact, since $Z_2 \in \mathfrak{h}$, $\lambda_2 \,|\, \mathfrak{h}$ is a root for $\mathfrak{h}$, and therefore the nilradical of $\mathfrak{h}$ is contained in $\mathfrak{h}_0$ and consequently it is precisely $\mathfrak{m}$.

Write $Z_2 = X_2 + iY_2$ and set $\mathfrak{b} = \mathbb{R}X_2 \oplus \mathbb{R}Y_2$. Then $\mathfrak{b}$ is an ideal in $\mathfrak{h}$, and $g \,|\, \mathfrak{b} = 0$. Let $c : \mathfrak{h} \to \mathfrak{h}/\mathfrak{b} = \tilde{\mathfrak{h}}$ be the coset map and define $\tilde{g}_0 \in \tilde{\mathfrak{h}}'$ by $\tilde{g}_0 \circ c = g_0$.

We now claim that $u_e \in U(\mathfrak{m})$, i. e. that $Q_e$ only depends on its restriction to $\mathfrak{h}$ (and therefore to $\mathfrak{m}$). Assuming for a moment this claim to be true, we consider $Q_e$ as a polynomial function on $\mathfrak{h}'$ and get for $\tilde{l}_0 \in \tilde{\mathfrak{h}}'$ (using the formula $(*)$):

$$
Q_e(\tilde{l}_0 \circ c) = |\langle l_0, W_1 \rangle|^2 \, |\langle l_0, W_2 \rangle|^2 Q_{\hat{e}^0}(\tilde{l}_0 \circ c),
$$

where $W_1 = c([\hat{Z}_2, \hat{Z}_m])$, $W_2 = c([\hat{Z}_3, \hat{Z}_{m-1}])$. Now since $W_1, \overline{W}_1, W_2, \overline{W}_2$ are central in $\tilde{\mathfrak{h}}$ and since $P_{W_1}(\tilde{l}_0)=\langle W_1, \tilde{l}_0\rangle$, $P_{\overline{W}_1}(\tilde{l}_0)=\overline{\langle W_1, \tilde{l}_0\rangle}$, and similarly for $W_2$, we find that $i^d Q_e(\tilde{l}_0 \circ c)=P_{W_1}(\tilde{l}_0)P_{\overline{W}_1}(\tilde{l}_0)P_{W_2}(\tilde{l}_0)P_{\overline{W}_2}(\tilde{l}_0)i^{d-4}Q_{\hat{e}^0}(\tilde{l}_0 \circ c)$, and therefore

$$c(u_e)=W_1\overline{W}_1W_2\overline{W}_2c(u_{\hat{e}^0})$$

by Lemma 1.2.1.

Since $\mathfrak{b}$ is an abelian ideal in $\mathfrak{h}_0$ and since $\langle f_0, \mathfrak{b}\rangle=0$ there exists a representation $\tilde{\pi}_0$ of $\tilde{M}=M/B$, $B=\exp\mathfrak{b}$, such that $\tilde{\pi}_0 \circ c=\pi_0$. Using the induction hypothesis we then get

$$\begin{aligned} d\pi_0(u_e)=d\tilde{\pi}_0(c(u_e))=d\tilde{\pi}_0(W_1\overline{W}_1W_2\overline{W}_2c(u_{\hat{e}^0}))&=|\langle f_0, W_1\rangle|^2|\langle f_0, W_2\rangle|^2 d\pi_0(u_{\hat{e}_0}) \\ &=|\langle g, [\hat{Z}_2, \hat{Z}_m]\rangle|^2|\langle g, [\hat{Z}_3, \hat{Z}_{m-1}]\rangle|^2 Q_{\hat{e}^0}(g_0)I=Q_e(g)I. \end{aligned}$$

We have thus shown that $d\pi_0(u_e)=Q_e(g)I$. This ends case (c11) (i), except for the fact that we have to prove the claim from above:

*Proof of claim:* We shall prove that $Q_e(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$. If all $\hat{Z}_{\hat{j}_r}$, $3\leq r\leq d-2$ belong to $(\mathfrak{h}_0)_{\mathbb{C}}$, then the result is clear (because $\mathfrak{h}_0$ is an ideal). Suppose then that there exists $3\leq r\leq d-2$ such that $\hat{Z}_{\hat{j}_r}\notin(\mathfrak{h}_0)_{\mathbb{C}}$, and let $\rho$ be the smallest such $r$. We then have $\mathfrak{h}_{\mathbb{C}}=\mathbb{C}\hat{Z}_{\hat{j}_\rho}\oplus(\mathfrak{h}_0)_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}=\mathbb{C}\hat{Z}_{\hat{j}_\rho}\oplus\ker\alpha_2$. Set $Y_r=\hat{Z}_{\hat{j}_r}+c_r\hat{Z}_{\hat{j}_\rho}$, $r=1, \ldots, d$, where $c_r$ is defined such that $Y_r\in\ker\alpha_2$ for $r\neq\rho$, and where $c_\rho=0$. We then have $Y_r\in(\mathfrak{h}_0)_{\mathbb{C}}$ for $1\leq r\leq d-2$, $r\neq\rho$, while $Y_\rho\notin(h_0)_{\mathbb{C}}$ and $Y_d$, $Y_{d-1}\in\ker\alpha_2$. We also have $[Y_d, Z_2]\in\mathbb{C}Z_1$, $[Y_d, Z_3]=0$, $[Y_{d-1}, Z_3]\in\mathbb{C}Z_1$, $[Y_{d-1}, Z_2]=0$.

Letting $C=[c_{rs}]_{1\leq r,s\leq d}$ be the $d\times d$-matrix given by:

$$\left[\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & 1 & c_{\rho+1} & \cdots & c_d \\ \hline & & & & 1 & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right]$$

the empty entries meaning zero, we have $Y_s=\Sigma_{r=1}^{d}c_{rs}\hat{Z}_{\hat{j}_r}$, so, setting $N(l)=[B_l(Y_r, Y_s)]_{1\leq r,s\leq d}$ we get $N(l)={}^tC\hat{M}_e(l)C$, from which $Q_e(l)=|\det M_e(l)|=|\det\hat{M}_e(l)|=|\det N(l)|$. Therefore

$$Q_e(l)=|\textstyle\sum_{\sigma\in S_d}\operatorname{sign}\sigma P_\sigma(l)|,$$

where we have set $P_\sigma(l)=\langle l, [Y_1, Y_{\sigma(1)}]\rangle \ldots \langle l, [Y_d, Y_{\sigma(d)}]\rangle$.

Define the following subsets of $S_d$:

$$S_d^{(1)}=\{\sigma\mid\sigma(1)=\rho,\ \sigma(2)=d-1,\ \sigma(\rho)=d,\ \sigma(d-1)=2,\ \sigma(d)=1\},$$

$$S_d^{(2)}=\{\sigma\mid\sigma(1)=d,\ \sigma(2)=\rho,\ \sigma(\rho)=d-1,\ \sigma(d-1)=2,\ \sigma(d)=1\},$$

$$S_d^{(3)}=\{\sigma\mid\sigma(1)=d,\ \sigma(2)=d-1,\ \sigma(\rho)=1,\ \sigma(d-1)=2,\ \sigma(d)=\rho\},$$

$$S_d^{(4)}=\{\sigma\mid\sigma(1)=d,\ \sigma(2)=d-1,\ \sigma(\rho)=2,\ \sigma(d-1)=\rho,\ \sigma(d)=1\},$$

$$S_d^{(5)}=\{\sigma\mid\sigma(\rho)\neq d\wedge\sigma(\rho)\neq d-1\wedge\rho\neq\sigma(d)\wedge\rho\neq\sigma(d-1)\},$$

$$S_d^{(6)}=S_d\setminus\textstyle\bigcup_{j=1}^{5}S_d^{(j)}.$$

We then assert that $P_\sigma=0$ if $\sigma\in S_d^{(6)}$. In fact, observe first that

$$P_\sigma\neq 0 \Rightarrow (\sigma(1)=\rho \vee \sigma(1)=d)\wedge(\sigma(2)=\rho\vee\sigma(2)=d-1)$$
$$\wedge(1=\sigma(\rho)\vee 1=\sigma(d))\wedge(2=\sigma(\rho)\vee 2=\sigma(d-1)).$$

Therefore, if $P_\sigma\neq 0$ and if $\sigma\notin S_d^{(5)}$ with e. g. $\sigma(\rho)=d$, then $\sigma(1)=\rho$, $\sigma(2)=d-1$, $\sigma(d)=1$, $\sigma(d-1)=2$, so $\sigma\in S_d^{(1)}$. Similarly, if $\sigma\notin S_d^{(5)}$ with $\sigma(\rho)=d-1$, then $P_\sigma\neq 0 \Rightarrow \sigma\in S_d^{(2)}$, etc. This shows our assertion.

We next assert that $P(l)=\sum_{j=1}^4\sum_{\sigma\in S_d^j}\operatorname{sign}\sigma P_\sigma(l)=0$. To see this, define the permutations $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$ in $S_d$ by $\tau_1=$identity, $\tau_2(1)=\rho$, $\tau_2(2)=1$, $\tau_2(\rho)=2$, $\tau_3(1)=\rho$, $\tau_3(\rho)=d$, $\tau_3(d)=1$, $\tau_4(1)=\rho$, $\tau_4(\rho)=d-1$, $\tau_4(d-1)=1$, all other elements left fixed. It is then immediate to verify that the map $\sigma\to\sigma\circ\tau_j$, $j=1, 2, 3, 4$, defines a bijection between $S_d^{(1)}$ and $S_d^{(j)}$, and since $\tau_j$ are even permutations we get

$$P(l)=\sum_{\sigma\in S_d^{(1)}}\operatorname{sign}\sigma\sum_{j=1}^4 P_{\sigma\circ\tau_j}(l).$$

Now for $\sigma\in S_d^{(1)}$ we have

$$\sum_{j=1}^4 P_{\sigma\circ\tau_j}(l)=\prod_{\substack{i=1\\ i\neq 1,2,\rho,\\ d-1,d}}^{d}\langle l, [Y_i, Y_{\sigma(i)}]\rangle\Big(\sum_{j=1}^4\prod_{\substack{i=1,2,\rho,\\ d-1,d}}\langle l, [Y_i, Y_{\sigma(\tau_j(i))}]\rangle\Big),$$

and a direct computation shows that

$$\sum_{j=1}^4\prod_{\substack{i=1,2,\rho,\\ d-1,d}}\langle l, [Y_i, Y_{\sigma(\tau_j(i))}]\rangle=0$$

for all $l\in\mathfrak{g}'$. This shows that $P\equiv 0$, and therefore we have

$$Q_e(l)=\Big|\sum_{\sigma\in S_d^{(5)}}\operatorname{sign}\sigma P_\sigma(l)\Big|.$$

But we clearly have that $P_\sigma(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$ if $\sigma\in S_d^{(5)}$, because all $[Y_r, Y_{\sigma(r)}]$, $r=1, \ldots, d$, then belong to $\mathfrak{h}$ (we use here that $\mathfrak{h}$ is a subalgebra and that $\mathfrak{h}_0$ is an ideal). This proves our claim and ends (i).

(ii) We now apply (i) to the same Jordan-Hölder sequence $\mathfrak{f}_j$, but to another basis $Z'_j\in\mathfrak{f}_j\backslash\mathfrak{f}_{j-1}$ (whereby $\mathfrak{h}_0$ and therefore $\mathfrak{m}$ are not changed), and we get similarly that $d\pi_0(u'_e)=Q'_e(g)I$, where $Q'_e$, $u'_e$ are the objects associated with this new basis. Setting in particular $Z'_j=\operatorname{Ad}(s)Z_j$, we get $u'_e=\operatorname{Ad}(s)u_e$, and $Q'_e(l)=Q_e(s^{-1}l)$ for $s\in G$, and therefore $d\pi_0(\operatorname{Ad}(s)u_e)=Q_e(s^{-1}g)I=|\Lambda_e(s)|^2Q_e(g)I$.

Now since $Z_1$, $Z_2$, $\overline{Z}_2\in\mathfrak{m}_{\mathbb{C}}$ it follows that $\mathfrak{n}_f\subset\mathfrak{m}$ and from this we get that

$$(\mathfrak{m}_{f_0})_{\mathbb{C}}=(\mathfrak{n}_f)_{\mathbb{C}}\oplus\mathbb{C}Z_2\oplus\mathbb{C}\overline{Z}_2.$$

It follows that a polarization in $\mathfrak{m}$ at $f_0$ is also a polarization in $\mathfrak{n}$ at $f$, hence $\pi=\operatorname{ind}_{M\uparrow N}\pi_0$. Let then $\varphi$ be a differentiable vector in $L^2(N, \pi_0)$, the space of the induced representation $\pi=\operatorname{ind}_{M\uparrow N}\pi_0$. We have $d\pi(u_e)\varphi(s)=d\pi_0(\operatorname{Ad}(s^{-1})u_e)\varphi(s)=Q_e(g)\varphi(s)$, $s\in N$, so $d\pi(u_e)=Q_e(g)I$. This ends case (c11).

*Case* (c12): (i) As in case (c11) we start by showing that $u_e \in U(\mathfrak{m})$ and that $d\pi_0(u_e)=Q_e(g)I$, and we can assume that $\langle g, Z_2 \rangle = 0$.

Since $[\mathfrak{f}_3, \mathfrak{f}_2]=\mathfrak{f}_1$ we have that $Z_2, Z_3 \notin \mathfrak{h}_{\mathbb{C}}$. Therefore $\mathfrak{g}_{\mathbb{C}}=\mathbb{C}Z_2 \oplus \mathbb{C}Z_3 \oplus \mathfrak{h}_{\mathbb{C}}$. Just like in case (c11) we see that $2, 3 \in J_g$. Define $\hat{Z}_1=Z_1$, $\hat{Z}_j=Z_{j+2}+a_{j+2}Z_2+b_{j+2}Z_3$ for $2 \leqslant j \leqslant m-2$, $\hat{Z}_{m-1}=Z_2$, $\hat{Z}_m=Z_3$, where $a_4, \ldots, a_m, b_4, \ldots, b_m$ have been picked such that $\hat{Z}_j \in \mathfrak{h}_{\mathbb{C}}$, $1 \leqslant j \leqslant m-2$. Clearly $\hat{Z}_1, \ldots, \hat{Z}_m$ is a basis for $\mathfrak{g}_{\mathbb{C}}$. Set $\hat{\mathfrak{f}}_j=\mathbb{C}\hat{Z}_1 \oplus \ldots \oplus \mathbb{C}\hat{Z}_j$. We have that $\hat{\mathfrak{f}}_1=\mathfrak{f}_1$ and $\hat{\mathfrak{f}}_j \oplus \mathbb{C}Z_2 \oplus \mathbb{C}Z_3=\mathfrak{f}_{j+2}$ for $1 \leqslant j \leqslant m-2$. Also $\hat{\mathfrak{f}}_{m-2}=\mathfrak{h}_{\mathbb{C}}$, $\hat{\mathfrak{f}}_m=\mathfrak{g}_{\mathbb{C}}$. We thus have

$$\hat{\mathfrak{f}}_1=\mathfrak{f}_1,$$
$$\hat{\mathfrak{f}}_j=\mathfrak{f}_{j+2}\cap\mathfrak{h}_{\mathbb{C}} \quad \text{for} \quad 1 \leqslant j \leqslant m-2,$$
$$\hat{\mathfrak{f}}_m=\mathfrak{g}_{\mathbb{C}}.$$

From this it follows that $\mathfrak{h}_{\mathbb{C}}=\hat{\mathfrak{f}}_{m-2} \supset \ldots \supset \hat{\mathfrak{f}}_1 \supset \hat{\mathfrak{f}}_0=\{0\}$ is a Jordan-Hölder sequence for $\mathfrak{h}_{\mathbb{C}}$. We claim it is of class (*b*). But this follows easily from the fact that $\overline{\hat{\mathfrak{f}}}_1=\hat{\mathfrak{f}}_1$.

Write $e=\{j_1< \ldots j_d\}$, and define the set $\hat{J}_g=\{\hat{j}_1< \ldots <\hat{j}_d\}$ by setting $\hat{j}_h=j_{h+2}-2$ for $1 \leqslant h \leqslant d-2$, $\hat{j}_{d-1}=m-1$, $\hat{j}_d=m$. We then have

$$\hat{Z}_{\hat{j}_h}=Z_{j_{h+2}}+a_{j_{h+2}}Z_{j_1}+b_{j_{h+2}}Z_{j_2} \quad \text{for} \quad 1 \leqslant h \leqslant d-2,$$
$$\hat{Z}_{\hat{j}_{d-1}}=Z_{j_1},$$
$$\hat{Z}_{\hat{j}_d}=Z_{j_2}.$$

Therefore, letting $C=[c_{rs}]_{1 \leqslant r,s \leqslant d}$ be the $d \times d$-matrix:

$$C=\left[\begin{array}{ccc|c|c} a_{j_3} & \ldots & a_{j_d} & 1 & \\ \hline b_{j_3} & \ldots & b_{j_d} & & 1 \\ \hline 1 & & & & \\ & \ddots & & & \\ & & 1 & & \end{array}\right],$$

where the empty entries are zero, we have $\hat{Z}_{\hat{j}_s}=\Sigma_{r=1}^d c_{rs}Z_{j_r}$, and $\hat{M}_e(l)={}^tCM_e(l)C$, where $\hat{M}_e(l)$ is the matrix $[B_l(\hat{Z}_{\hat{j}_r}, \hat{Z}_{\hat{j}_s})]_{1 \leqslant r,s \leqslant d}$. Now $\det C=1$, and therefore we have for $l \in \mathfrak{g}'$ with $\langle l, Z_2 \rangle = 0$:

$$\begin{aligned} Q_e(l) &= |\det \hat{M}_e(l)| \\ &= \Big|\sum_{\sigma\in S_d} \operatorname{sign}\sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}]\rangle \ldots \langle l, [\hat{Z}_{\hat{j}_d}, \hat{Z}_{\hat{j}_{\sigma(d)}}]\rangle\Big| \\ &= |\langle l, [\hat{Z}_{m-1}, \hat{Z}_m]\rangle|^2 \Big|\sum_{\sigma\in S_d^*} \operatorname{sign}\sigma \langle l, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}]\rangle \ldots \langle l, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}]\rangle\Big|, \end{aligned}$$

where $S_d^*$ is the set of permutations $\sigma$ in $S_d$ such that $\sigma(d-1)=d$, $\sigma(d)=d-1$.

Set $g_0=g \mid \mathfrak{h}$, and let $\hat{J}_{g_0}^0$, etc. designate the objects defined relative to the Jordan-Hölder sequence $\hat{\mathfrak{f}}_0 \subset \hat{\mathfrak{f}}_1 \subset \ldots \subset \hat{\mathfrak{f}}_{m-2}=\mathfrak{h}_{\mathbb{C}}$. Since clearly $\mathfrak{g}_g \subset \mathfrak{h}$, and $\mathfrak{h}_{g_0}=\mathfrak{g}_g$ we find that $1 \notin \hat{J}_{g_0}^0$, and for $4 \leqslant j \leqslant m$ we have

$$j \notin J_g \Leftrightarrow Z_j \in \mathfrak{f}_{j-1}+(\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow Z_j \in \hat{\mathfrak{f}}_{j-3}+\mathbb{C}Z_2+\mathbb{C}Z_3+(\mathfrak{g}_g)_{\mathbb{C}} \Leftrightarrow \hat{Z}_{j-2} \in \hat{\mathfrak{f}}_{j-3}+(\mathfrak{h}_{g_0})_{\mathbb{C}} \Leftrightarrow j-2 \notin \hat{J}_{g_0}^0,$$

so $j\in J_{\mathfrak{g}} \Leftrightarrow j-2\in \hat{J}^0_{\mathfrak{g}_0}$. Therefore, if $\hat{e}^0=\hat{J}^0_{\mathfrak{g}_0}=\{\hat{j}^0_1<\ldots<\hat{j}^0_{d-2}\}$ we find that $\hat{j}^0_h+2=j_{h+2}$ for $1\leqslant h\leqslant d-2$, and comparing with the definition of $\hat{j}_h$ we find that $\hat{j}^0_h=\hat{j}_h$ for $1\leqslant h\leqslant d-2$. Using this we get for $l_0\in h'$:

$$\begin{aligned} Q_{\hat{e}^0}(l_0) &= |\det [B_{l_0}(\hat{Z}_{\hat{j}^0_r}, \hat{Z}_{\hat{j}^0_s})]_{1\leqslant r,s\leqslant d-2}| \\ &= \Big|\sum_{\sigma\in S_{d-2}} \operatorname{sign}\sigma \langle l_0, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}]\rangle \ldots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}]\rangle\Big| \\ &= \Big|\sum_{\sigma\in S^*_d} \operatorname{sign}\sigma \langle l_0, [\hat{Z}_{\hat{j}_1}, \hat{Z}_{\hat{j}_{\sigma(1)}}]\rangle \ldots \langle l_0, [\hat{Z}_{\hat{j}_{d-2}}, \hat{Z}_{\hat{j}_{\sigma(d-2)}}]\rangle\Big|, \end{aligned}$$

and comparing with what we saw above we find for $l\in\mathfrak{g}'$ with $\langle l, Z_2\rangle=0$:

$$(*)\qquad Q_e(l)=|\langle l, [\hat{Z}_m, \hat{Z}_{m-1}]\rangle|^2 Q_{\hat{e}^0}(l_0),$$

where $l_0=l\,|\,\mathfrak{h}$.

We then claim that $Q_e(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$. Assuming for a moment this to be true, we find that the formula (*) is valid for all $l\in\mathfrak{g}'$. The conclusion of this is that $i^dQ_e(l)=-\langle l, W\rangle\langle l, \overline{W}\rangle\, i^{d-2}Q_{\hat{e}^0}(l_0)$, where $W=[\hat{Z}_m, \hat{Z}_{m-1}]=[Z_2, Z_3]\in\mathfrak{f}_1$, and therefore that $u_e=-W\overline{W}u_{\hat{e}^0}$.

Let us then prove our claim: if all $\hat{Z}_{\hat{j}_r}$, $1\leqslant r\leqslant d-2$ belong to $(\mathfrak{h}_0)_{\mathbb{C}}$, then the result is clear (since $\mathfrak{h}_0$ is an ideal). Suppose then that there exists $1\leqslant r\leqslant d-2$ such that $\hat{Z}_{\hat{j}_r}\notin(\mathfrak{h}_0)_{\mathbb{C}}$, and let $\rho$ be the smallest such number $r$. We then have $\mathfrak{h}_{\mathbb{C}}=\mathbb{C}\hat{Z}_{\hat{j}_\rho}\oplus(\mathfrak{h}_0)_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}=\mathbb{C}\hat{Z}_{\hat{j}_\rho}\oplus\ker\alpha_2$. Set $Y_r=\hat{Z}_{\hat{j}_r}+c_r\hat{Z}_{\hat{j}_\rho}$, $r=1,\ldots,d$, where $c_r$ is defined such that $Y_r\in\ker\alpha_2$ for $r\neq\rho$ and where $c_\rho=0$. We then have $Y_r\in(\mathfrak{h}_0)_{\mathbb{C}}$ for all $1\leqslant r\leqslant d-2$, $r\neq\rho$, while $Y_\rho\in(\mathfrak{h})_{\mathbb{C}}$, $Y_d=\hat{Z}_m=\hat{Z}_{\hat{j}_d}$, $Y_{d-1}=\hat{Z}_{m-1}=\hat{Z}_{\hat{j}_{d-1}}\in\ker\alpha_2$.

Letting $C=[c_{rs}]_{1\leqslant r,s\leqslant d}$ be the $d\times d$-matrix given by:

$$C=\left[\begin{array}{ccc|c|ccccc} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ \hline & & & 1 & c_{\rho+1} & \ldots & c_{d-2} & 0 & 0 \\ \hline & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & & & 1 \end{array}\right]$$

the empty entries meaning zero, we have $Y_s=\sum_{r=1}^d c_{rs}\hat{Z}_{\hat{j}_r}$, so, setting $N(l)=[B_l(Y_r, Y_s)]_{1\leqslant r,s\leqslant d}$ we get $N(l)={}^tC\hat{M}_e(l)C$, from which $Q_e(l)=|\det M_e(l)|=|\det\hat{M}_e(l)|=|\det N(l)|$. Therefore $Q_e(l)=\left|\sum_{\sigma\in S_d}\operatorname{sign}\sigma P_\sigma(l)\right|$, where we have set $P_\sigma(l)=\langle l, [Y_1, Y_{\sigma(1)}]\rangle\ldots\langle l, [Y_d, Y_{\sigma(d)}]\rangle$. Define then the following subsets of $S_d$:

$$\begin{aligned} S_d^{(1)} &= \{\sigma\mid\sigma(d-1)=\rho\wedge\sigma(d)=d-1\wedge\sigma(\rho)=d\}, \\ S_d^{(2)} &= \{\sigma\mid\sigma(d-1)=d\wedge\sigma(d)=\rho\wedge\sigma(\rho)=d-1\}, \\ S_d^{(3)} &= \{\sigma\mid\sigma(d-1)\neq\rho\wedge\sigma(d)\neq\rho\}, \\ S_d^{(4)} &= S_d\backslash(S_d^{(1)}\cup S_d^{(2)}\cup S_d^{(3)}). \end{aligned}$$

We then assert that: $\sigma \in S_d^{(4)} \Rightarrow P_\sigma = 0$. In fact, observe first that since:

$$[Y_{d-1}, Y_r] \neq 0 \Rightarrow r = d \vee r = \rho,$$

and since: $[Y_d, Y_r] \neq 0 \Rightarrow r = d-1 \vee r = \rho$ we have:

$$P_\sigma \neq 0 \Rightarrow (\sigma(d-1) = d \vee \sigma(d-1) = \rho) \wedge (\sigma(d) = d-1 \vee \sigma(d) = \rho).$$

Moreover, if $r \notin \{\rho, d-1, d\}$, then: $[Y_r, Y_{\sigma(r)}] \neq 0 \Rightarrow \sigma(r) \neq d-1 \wedge \sigma(r) \neq d$, hence:

$$P_\sigma \neq 0 \Rightarrow (d = \sigma(d-1) \vee d = \sigma(\rho)) \wedge (d-1 = \sigma(d) \vee d-1 = \sigma(\rho)).$$

Therefore, if $P_\sigma \neq 0$ and $\sigma \notin S_d^{(3)}$ with e. g. $\sigma(d-1) = \rho$, then $\sigma(d) = d-1$ and $\sigma(\rho) = d$, and therefore $\sigma \in S_d^{(1)}$. Similarly, if $\sigma \notin S_d^{(3)}$ with $\sigma(d) = \rho$, then $P_\sigma \neq 0 \Rightarrow \sigma \in S_d^{(2)}$. This shows our assertion.

We next assert that $P(l) = \sum_{\sigma \in S'_d} \text{sign}\, \sigma P_\sigma(l) = 0$, where $S'_d = S_d^{(1)} \cup S_d^{(2)}$. To see this, define the permutation $\tau$ in $S_d$ by $\tau(\rho) = d$, $\tau(d-1) = \rho$, $\tau(d) = d-1$, all other elements left fixed. It is then immediate to verify that the map $\sigma \to \sigma. \circ \tau$ defines a bijection between $S_d^{(1)}$ and $S_d^{(2)}$, and since $\tau$ is an even permutation we get $P(l) = \sum_{\sigma \in S_d^{(1)}} \text{sign}\, \sigma (P_\sigma + P_{\sigma \circ \tau})$. Now for $\sigma \in S_d^{(1)}$ we have

$$P_\sigma(l) + P_{\sigma\circ\tau}(l) = \prod_{\substack{i \neq \rho,\\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle \Big( \prod_{\substack{i = \rho,\\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle + \prod_{\substack{i = \rho,\\ d-1, d}} \langle l, [Y_i, Y_{\sigma(\tau(i))}] \rangle \Big),$$

and

$$\begin{aligned} &\prod_{\substack{i = \rho,\\ d-1, d}} \langle l, [Y_i, Y_{\sigma(i)}] \rangle + \prod_{\substack{i = \rho,\\ d-1, d}} \langle l, [Y_i, Y_{\sigma(\tau(i))}] \rangle \\ &= \langle l, [Y_\rho, Y_d] \rangle \langle l, [Y_{d-1}, Y_\rho] \rangle \langle l, [Y_d, Y_{d-1}] \rangle \\ &+ \langle l, [Y_\rho, Y_{d-1}] \rangle \langle l, [Y_{d-1}, Y_d] \rangle \langle l, [Y_d, Y_\rho \rangle = 0. \end{aligned}$$

This shows that $P \equiv 0$, and therefore we have

$$Q_e(l) = \Big| \sum_{\sigma \in S_d^{(3)}} \text{sign}\, \sigma P_\sigma(l) \Big|,$$

and since $P_\sigma(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$ when $\sigma \in S_d^{(3)}$ we have proved our assertion.

Now if $\mathfrak{m}$ is the nilradical of $\mathfrak{h}$ it follows from the induction hypothesis that $d\pi_0(u_{\hat{e}_0}) = Q_{\hat{e}^0}(g_0)\text{I}$, and since $d\pi_0(\text{W}) = i \langle f_0, \text{W} \rangle$, and $u_e = -\text{W}\overline{\text{W}} u_{\hat{e}^0}$ we get that

$$d\pi_0(u_e) = |\langle f_0, \text{W} \rangle|^2 Q_{\hat{e}^0}(g_0)\text{I} = Q_e(g)\text{I},$$

and this proves (i) in this case.

Suppose then that $\mathfrak{m}$ is not the nilradical $\mathfrak{m}_1$ of $\mathfrak{h}$. Then $\mathfrak{m} = \mathfrak{h}_0 \cap \mathfrak{m}_1$, and $\dim \mathfrak{m}_1/\mathfrak{m} = 1$. Setting $M_1 = \exp \mathfrak{m}_1$ we now face two possibilities (1) either $\pi_0$ extends to an irreducible representation $\pi'_0$ of $M_1$ or (2) $\text{ind}_{M \uparrow M_1} \pi_0 = \pi'_0$ is an irreducible representation of $M_1$. In the first case we obviously get as above that $d\pi'_0(u_e) = Q_e(g)\text{I}$, and therefore $d\pi_0(u_e) = Q_e(g)\text{I}$. In the second case we have $\pi'_0 \,|\, M = \int_{M_1/M}^{\oplus} s\pi_0 \, d\dot{s}$, and therefore we get by the induction hypothesis that $Q_e(g)I = d\pi'_0(u_e) = \int_{M_1/M}^{\oplus} d(s\pi_0)(u_e)d\dot{s}$, from which $d(s\pi_0)(u_e) = Q_e(g)I$ for almost all $\dot{s}$, hence for all $\dot{s}$ by continuity. This shows that $d\pi_0(u_e) = Q_e(g)I$, and ends (i).

(ii) Just like in case (c11) (ii) we conclude from (i) that

$$d\pi_0(\mathrm{Ad}\,(s)u_e) = Q_e(s^{-1}g)I = |\Lambda_e(s)|^2 Q_e(g)I \quad \text{for all} \quad s\in G.$$

Now since $Z_2$, $Z_3 \in \mathfrak{n}_\mathbb{C}$, and since $[\mathfrak{m}, Z_2] = [\mathfrak{m}, Z_3] = 0$ we see at once that $\mathfrak{m}_{f_0} = \mathfrak{n}_f$. Suppose then that $\mathfrak{p}$ is a polarization in $\mathfrak{m}$ at $f_0$. Then, writing $Z_2 = X_2 + iY_2$, $\mathfrak{p}_1 = \mathfrak{p} \oplus \mathbb{R}Y_2$ is a polarization in $\mathfrak{n}$ at $f$, and therefore $\pi = \mathrm{ind}_{P_1\uparrow N}\, \eta_1$, where $\eta_1$ is the unitary character on $P_1 = \exp \mathfrak{p}_1$ corresponding to $f \mid \mathfrak{p}_1$. Similarly $\pi_0 = \mathrm{ind}_{P\uparrow M}\, \eta$, where $P = \exp \mathfrak{p}$, $\eta = \eta_1 \mid P$. We then set $\mathfrak{n}_1 = \mathbb{R}Y_2$, and note that $\mathfrak{n}_1$ is a direct product of $\mathfrak{m}$ and $\mathbb{R}Y_2$. Let $\pi_1$ be the irreducible representation of $N_1 = \exp \mathfrak{n}_1$ with $\pi_1 \mid M = \pi_0$,

$$\pi_1(\exp tY_2) = e^{it\langle f, Y_2\rangle}.$$

Then $\pi = \mathrm{ind}_{P_1\uparrow N}\, \eta_1 = \mathrm{ind}_{N_1\uparrow N}\, (\mathrm{ind}_{P_1\uparrow N_1}\, \eta_1) = \mathrm{ind}_{N_1\uparrow N}\pi_1$. Now noting that $N_1$ is a normal subgroup in N and that clearly $d\pi_1(\mathrm{Ad}\,(s)u_e) = Q_e(g)I$ for $s\in N$, we can end this case just like case (c11) (ii). This ends case (c12).

*Case* (c2): (i) Since $\mathfrak{k}_1 = \bar{\mathfrak{k}}_1$ we can clearly assume that $Z_1 = X_1 \in \mathfrak{g}$. A standard argument shows that $\lambda_2$ must be a real root in this case, so $\lambda_2 = \alpha_2$. We claim that it is no loss of generality to assume that $\gamma_2 \equiv 0$. In fact, let $a_1$, $a_2$, $b$ be real numbers, not all equal to zero, such that $0 = a_1\gamma_1 + a_2\gamma_2 + b\alpha_2$. Then $(a_1, a_2) \neq (0, 0)$, since $\alpha_2 \neq 0$, and we can assume that $a_1^2 + a_2^2 = 1$. Replacing $Z_2$ by $Z'_2 = (a_2 + ia)Z_2 - bZ_1$ does not change $Q_e$, and it is trivial to verify that $[X, X'_2] = \lambda_2(X)Z'_2 + \gamma'_1(X)Z_1$, where $\gamma'_1 = a_2\gamma_1 - a_1\gamma_2$. This proves the claim. So, from now on we assume that $Z_1 = X_1 \in \mathfrak{g}$, $\gamma = \gamma_1$, and writing $Z_2 = X_2 + iY_2$ we then have

$$[X, X_2] = \lambda_2(X)X_2 + \gamma(X)X_1$$
$$[X, Y_2] = \lambda_2(X)Y_2.$$

Set $\mathfrak{h} = \ker \gamma$. It follows from the formula (2.2.2) that $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$, and its codimension is 1. Set $\mathfrak{h}_0 = \ker \alpha_2 \mid \mathfrak{h} = \ker \mathrm{ad}\, Z_2 \mid \mathfrak{g}$. $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$ of codimension 2.

Let $\mathfrak{m}$ be the nilradical of $\mathfrak{h}_0$. Since $\mathfrak{h}_0$ is an ideal we have that $\mathfrak{m} = \mathfrak{n} \cap \mathfrak{h}_0 = \mathfrak{n} \cap \mathfrak{h}$. Observe that $\dim \mathfrak{n}/\mathfrak{m} = 1$. In fact, pick $W \in \mathfrak{h}\backslash\mathfrak{h}_0$. We then have $\gamma([X, W]) = \lambda_2(W)\gamma(X)$ for $X\in\mathfrak{g}$. Choosing X such that $\gamma(X) = 1$ we get that $\gamma([X, W]) = \lambda_2(W) \neq 0$, and this shows that $[X, W]$ is a basis in $\mathfrak{n}$ (mod $\mathfrak{m}$).

Set $f_0 = f \mid \mathfrak{m} = g \mid \mathfrak{m}$, and let $\pi_0$ be the irreducible representation of $M = \exp \mathfrak{m}$ corresponding to $Mf_0$.

(ii) We first show that $u_e \in U(\mathfrak{m})$, and that $d\pi_0(u_e) = Q_e(g)I$. We start by noting that we can assume that $\langle g, X_2 \rangle = 0$; in fact, if necessary replace $X_2$ by $X_2 - cX_1$; this does not change $e$, $Q_e$, etc. (it will change $\gamma$, $\mathfrak{h}$, though, but does not affect $\mathfrak{h}_0$, rank $(\alpha_2, \gamma_1, \gamma_2)$ and the fact that $\gamma_2 \equiv 0$).

Except for some obvious modifications we can now proceed just like in case (c11) (i).

(iii) Just like in case (c11) (ii) we conclude that $d\pi_0(\mathrm{Ad}(s)u_e)=|\Lambda_e(s)|^2 Q_e(g) I$ for $s\in G$.

Now since $X_1, X_2 \in \mathfrak{m}$ it follows that $\mathfrak{n}_f \subset \mathfrak{m}$ and from this we get that $\mathfrak{m}_{f_0}=\mathfrak{n}_f \oplus \mathbb{R}X_2$. Therefore a polarization in $\mathfrak{m}$ at $f_0$ is also a polarization in $\mathfrak{n}$ at $f$, hence $\pi=\mathrm{ind}_{M\uparrow N}\pi_0$. We can then end this case just like we did in case (c11) (ii).

*Case* (c3): It is no loss of generality to assume that $\gamma\equiv 0$. In fact, there exists real numbers $a_1, a_2$ such that $\gamma_1=a_1\alpha_2, \gamma_2=a_2\alpha_2$, and therefore $\gamma_1=a_1(1+ik_2)^{-1}\lambda_2, \gamma_2=a_2(1+ik_2)^{-1}\lambda_2$. Replacing $Z_2$ by $Z_2'=Z_2+(1+ik_2)^{-1}(a_1+ia_2)Z_1$ does not change $Q_e$, etc., and we have $[X, Z_2']=\lambda_2(X)Z_2'$. This proves the assertion.

Set $\mathfrak{h}=\ker\lambda_2$. Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension 1, so $\mathfrak{n}\subset\mathfrak{h}$. We can now proceed here much like in case (b), so we omit the details.

*Case* (d): Suppose we are not in case (a), (b) or (c).

We have $[X, Z_2]=\gamma(X)Z_1$, where $\gamma\neq 0$ and $\langle g, Z_1\rangle \neq 0$ (since otherwise we would be in case (a)), and also $\mathfrak{f}_1=\overline{\mathfrak{f}}_1$.

Writing $\gamma=\gamma_1+i\gamma_2$ we distinguish two subcases: (d1): $\mathrm{rank}(\gamma_1, \gamma_2)=2$ and (d2): $\mathrm{rank}(\gamma_1, \gamma_2)=1$.

*Case* (d1): Set $\mathfrak{h}=\ker\gamma_1\cap\ker\gamma_2$. Then $\mathfrak{h}$ is an ideal of codimension 2. We then distinguish two possibilities: case (d11): $[\mathfrak{f}_3, \mathfrak{f}_2]=0$ and case (d12): $[\mathfrak{f}_3, \mathfrak{f}_2]=\mathfrak{f}_1$. We can then proceed here much like in case (c1) (the case at hand is easier, since here $\mathfrak{h}$ is an ideal containing $[\mathfrak{g}, \mathfrak{g}]$). We omit the details.

*Case* (d2): Just like in case (c2) we see that we can assume that $\gamma_2=0$. Set $\mathfrak{h}=\ker\gamma$. Then $\mathfrak{h}$ is an ideal of codimension 1, and we can treat this case much like case (b). We also omit the details here. This ends the proof of Proposition 2.2.1.

2.3. — We shall now end the proof of Theorem 1.4.1. We use [4], 4.2.2 Théorème, p. 121 with $\psi(l)=|P_e(l)|$. It follows from Lemma 2.1.1 that the condition of the theorem *loc. cit.* is satisfied. The conclusion is that the operator $A\pi(\varphi)A$ is traceclass for all $\varphi\in C_c^\infty(G)$, that $\varphi\to \mathrm{Tr}([A\pi(\varphi)A])$ is a distribution (of positive type) on G, and that

$$\mathrm{Tr}([A\pi(\varphi)A])=\int_O (\alpha_e\cdot\varphi\circ\exp)^\wedge(l)Q_e(l)\,d\beta_O(l).$$

Here we have also used Lemma 1.3.1.

REMARK 2.3.1. — In [4], p. 248 and [5], p. 118 appear two different definitions of the function $P_O'$ (*cf.* section 1.3). Here we use the one from [4] (which is the most natural one), while the 4.2.2. Théorème in [5] uses the definition of $P_O'$ from [5]. There is no difficulty in proving 4.2.2. Théorème with the definition of $P_O'$ from [4] when $\psi$ has the property that $\psi(l)$ only depends on the restriction of $l$ to $[\mathfrak{g}, \mathfrak{g}]$ which is the case here (*cf.* [5] 4.2.3. Remarque).

We shall then identify the operator A: Set $G_0=\ker\chi_e$, let $\mathfrak{g}_0$ be the Lie algebra of $G_0$, and let $\pi_0$ be the irreducible representation associated with $g_0=g\,|\,\mathfrak{g}_0$. Then $\pi=\operatorname{ind}_{G_0\uparrow G}\pi_0$, and A is realized on $L^2(G, \pi_0)$, the space of the induced representation $\pi=\operatorname{ind}_{G_0\uparrow G}\pi_0$, by $Af(s)=\psi(sg)f(s)=|P_e(sg)|\,f(s)$. Now it follows from Proposition 2.2.1 that $d\pi_0(u_e)=Q_e(g)I$, and that $d\pi_0(\operatorname{Ad}(s)u_e)=Q_e(s^{-1}g)I$ which implies that we have for a differentiable vector $f\in L^2(G, \pi_0)$:

$$d\pi(u_e)f(s)=d\pi_0(\operatorname{Ad}(s^{-1})u_e)f(s)=Q_e(sg)f(s)=|P_e(sg)|^2 f(s)=A^2 f(s),$$

and thus $d\pi(u_e)=A^2$.

Now since $A\pi(\varphi)\subset\pi(\chi_e^{-1}\varphi)A$ we have that $A\pi(\varphi)A\subset A^2\pi(\chi_e^{-1}\varphi)$, and therefore $[A\pi(\varphi)A]=[A^2\pi(\chi_e^{-1}\varphi)]$ from which $[A^2\pi(\chi_e^{-1}\varphi)]$, hence $[A^2\pi(\varphi)]$, is traceclass for all $\varphi\in C_c^\infty(G)$, and $\operatorname{Tr}([A^2\pi(\varphi)])=\operatorname{Tr}([A\pi(\chi_e\varphi)A])=\operatorname{Tr}([A\pi(\varphi)A])$, the last equality being valid because the distribution $\varphi\to\operatorname{Tr}([A\pi(\varphi)A])$ is supported on $G_0$ (*cf.* [5], [6]). Observing finally that $[A^2\pi(\varphi)]=\pi(u_e*\varphi)$, we have proved the theorem.

## 3. Examples

We shall give a few examples of the calculation of $\mathscr{E}$, $Q_e$, $u_e$, $\Omega_e$ for an exponential solvable Lie algebra $\mathfrak{g}$. If $Z_1, \ldots, Z_m$ is a Jordan-Hölder basis for $\mathfrak{g}_{\mathbb{C}}$ we denote by $M(g)$, $g\in\mathfrak{g}'$, the skewsymmetric $m\times m$-matrix $[\langle g, [Z_i, Z_j]\rangle]_{1\leq i,j\leq m}$, and we write $\zeta_j=\langle g, Z_j\rangle$. The matrices $M_e(g)$ are all submatrices of $M(g)$. Note that $Z=\sum_{j=1}^m z_jZ_j$ belongs to $(\mathfrak{g}_g)_{\mathbb{C}}$ if and only if $M(g)\underline{z}=\underline{0}$, where $\underline{z}=(z_1, \ldots, z_m)$. We write $\mathscr{E}=\{e_1<\ldots<e_p\}$.

3.1. — Let $\mathfrak{g}$ be the five dimensional real solvable Lie algebra with the following non-vanishing brackets: $[X_5, X_4]=-X_4$, $[X_5, X_3]=2X_3$, $[X_5, X_2]=X_2$, $[X_4, X_3]=X_2$, $[X_4, X_2]=X_1$. Then $X_1, \ldots, X_5$ is a Jordan-Hölder basis for $\mathfrak{g}$, so $\mathfrak{g}$ is completely solvable. We set $Z_j=X_j$ and $\xi_j=\langle g, X_j\rangle=\zeta_j$, $j=1, \ldots, 5$.

We have

$$M(g)=\begin{bmatrix}0&0&0&0&0\\0&0&0&-\xi_1&-\xi_2\\0&0&0&-\xi_2&-2\xi_3\\0&\xi_1&\xi_2&0&-\xi_4\\0&\xi_2&2\xi_3&\xi_4&0\end{bmatrix}$$

i) If $\xi_2^2-2\xi_1\xi_3\neq 0$, then $\mathfrak{g}_g=\mathbb{R}X_1$ and therefore $J_g=\{2, 3, 4, 5\}$.

ii) If $\xi_2^2-2\xi_1\xi_3=0$ and $\xi_1\neq 0$ then

$$\mathfrak{g}_g=\mathbb{R}X_1\oplus\mathbb{R}(-\xi_2X_2+\xi_1X_3)\oplus\mathbb{R}(-\xi_4X_2-\xi_2X_4+\xi_1X_5),\quad J_g=\{3, 5\}.$$

iii) If $\xi_2^2-2\xi_1\xi_3=0$, $\xi_1=0$ and $\xi_3\neq 0$, then $\mathfrak{g}_g=\mathbb{R}X_1\oplus\mathbb{R}X_2\oplus\mathbb{R}(-\xi_4X_3+2\xi_3X_4)$, $J_g=\{3, 5\}$.

iv) If $\xi_2^2-2\xi_1\xi_3=0$, $\xi_1=0$, $\xi_3=0$ and $\xi_4\neq 0$, then

$$\mathfrak{g}_g=\mathbb{R}X_1\oplus\mathbb{R}X_2\oplus\mathbb{R}X_3,\quad J_g=\{4,5\}$$

v) If $\xi_2^2-2\xi_1\xi_3=0$, $\xi_1=0$, $\xi_3=0$, $\xi_4=0$, then $\mathfrak{g}_g=\mathfrak{g}$, $J_g=\varnothing$.

We can then write down:

$$\begin{array}{ll} e_1=\{2,3,4,5\}, & \Omega_{e_1}=\{g\mid \xi_2^2-2\xi_1\xi_3\neq 0\},\\ e_2=\{2,4\}, & \Omega_{e_2}=\{g\mid \xi_2^2-2\xi_1\xi_3=0,\ \xi_1\neq 0\},\\ e_3=\{3,5\}, & \Omega_{e_3}=\{g\mid \xi_1=\xi_2=0,\ \xi_3\neq 0\},\\ e_4=\{4,5\}, & \Omega_{e_4}=\{g\mid \xi_1=\xi_2=\xi_3=0,\ \xi_4\neq 0\},\\ e_5=\varnothing & \Omega_{e_5}=\{g\mid \xi_1=\xi_2=\xi_3=\xi_4=0\}, \end{array}$$

$$\begin{array}{ll} Q_{e_1}(g)=(\xi_2^2-2\xi_1\xi_3)^2, & u_{e_1}=(X_2^2-2X_1X_3)^2,\\ Q_{e_2}(g)=\xi_1^2, & u_{e_2}=-X_1^2,\\ Q_{e_3}(g)=4\xi_3^2, & u_{e_3}=-4X_3^2,\\ Q_{e_4}(g)=\xi_4^2, & u_{e_4}=-X_4^2,\\ Q_{e_5}(g)=1, & u_{e_5}=1. \end{array}$$

3.2. — Let $\mathfrak{g}$ be the six dimensional real exponential solvable Lie algebra having a basis $X_1, \ldots, X_6$ with the following non-vanishing brackets: $[X_6, X_5]=X_4+X_5$, $[X_6, X_4]=X_4-X_5$, $[X_6, X_2]=X_1+X_2$, $[X_6, X_1]=X_1-X_2$, $[X_5, X_4]=X_3$, $[X_5, X_3]=X_2$, $[X_4, X_3]=X_1$. Set $Z_1=X_1-iX_2$, $Z_2=X_1+iX_2$, $Z_3=X_3$, $Z_4=X_4-iX_5$, $Z_5=X_4+iX_5$, $Z_6=X_6$. Then $Z_1, \ldots, Z_6$ is a Jordan-Hölder basis for $\mathfrak{g}_{\mathbb{C}}$, and

$$M(g)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -(1-i)\zeta_1\\ 0 & 0 & 0 & 0 & 0 & -(1+i)\zeta_2\\ 0 & 0 & 0 & -\zeta_1 & -\zeta_2 & 0\\ 0 & 0 & \zeta_1 & 0 & -2i\zeta_3 & -(1-i)\zeta_4\\ 0 & 0 & \zeta_2 & 2i\zeta_3 & 0 & -(1+i)\zeta_5\\ (1-i)\zeta_1 & (1+i)\zeta_2 & 0 & (1-i)\zeta_4 & (1+i)\zeta_5 & 0 \end{bmatrix}$$

Writing $\xi_j=\langle g, X_j\rangle$, $j=1, \ldots, 6$, we have $\zeta_1=\xi_1-i\xi_2$, $\zeta_2=\xi_1+i\xi_2$, $\zeta_3=\xi_3$, $\zeta_4=\xi_4-i\xi_5$, $\zeta_5=\xi_4+i\xi_5$, $\zeta_6=\xi_6$.

i) If $\zeta_1\neq 0$, then $J_g=\{1,3,4,6\}$.

ii) If $\zeta_1=0$ ($\Rightarrow \zeta_2=0$), $\zeta_3\neq 0$, then $J_g=\{4,5\}$.

iii) If $\zeta_1=0$, $\zeta_3=0$, $\zeta_4\neq 0$, then $J_g=\{4,6\}$.

iv) If $\zeta_1=0$, $\zeta_3=0$, $\zeta_4=0$ ($\Rightarrow \zeta_5=0$), then $J_g=\varnothing$.

We can then write down:

$$e_1=\{1,3,4,6\},\qquad \Omega_{e_1}=\{g\mid \xi_1^2+\xi_2^2\neq 0\},$$
$$e_2=\{4,5\},\qquad \Omega_{e_2}=\{g\mid \xi_1^2+\xi_2^2=0,\ \xi_3\neq 0\},$$
$$e_3=\{4,6\},\qquad \Omega_{e_3}=\{g\mid \xi_1^2+\xi_2^2=0,\ \xi_3=0,\ \xi_4^2+\xi_5^2\neq 0\},$$
$$e_4=\varnothing,\qquad \Omega_{e_4}=\{g\mid \xi_1=\xi_2=\xi_3=\xi_4=\xi_5=0\},$$
$$Q_{e_1}(g)=2(\xi_1^2+\xi_2^2)^2,\qquad u_{e_1}=2(X_1^2+X_2^2)^2,$$
$$Q_{e_2}(g)=4\xi_3^2,\qquad u_{e_2}=-4X_3^2,$$
$$Q_{e_3}(g)=2(\xi_4^2+\xi_5^2),\qquad u_{e_3}=-2(X_4^2+X_5^2),$$
$$Q_{e_4}(g)=1,\qquad u_{e_4}=1.$$

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