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ON THE CHARACTERS
OF EXPONENTIAL SOLVABLE LIE GROUPS

BY Niels Vigand PEDERSEN (*)

Introduction

Let \( G \) be a connected, simply connected solvable Lie group with Lie algebra \( g \). In [6] it was shown that for any normal representation \( \pi \) of \( G \) (cf. [11]) there exists a continuous homomorphism \( \chi : G \to \mathbb{R}^*_+ \) such that \( \pi \) has a distribution \( \chi \)-semicharacter. Moreover, it was shown that one can find a semi-invariant element \( u \) (with multiplier \( \chi \), say) in \( U(g_\mathbb{C}) \), the universal enveloping algebra of the complexification \( g_\mathbb{C} \) of \( g \), such that any normal representation \( \pi \) whose associated orbit of \( u \) in \( g' \) ([10], [11]) is contained in a certain \( G \)-invariant Zariski open subset of \( g' \), has a distribution \( \chi \)-semicharacter \( f_{\pi,\chi} \) expressible by \( f_{\pi,\chi}(\phi) = \phi(\pi(u \ast \phi)) \) for \( \phi \in C_c^\infty(G) \), \( \phi \) being the trace on the factor generated by \( \pi \) (here it is understood, in particular, that the right hand side is well defined). In [3] J.-Y. Charbonnel showed that for each normal representation \( \pi \) of \( G \) one can find a continuous homomorphism \( \chi : G \to \mathbb{R}^*_+ \) and an element \( u \in U(g_\mathbb{C}) \) such that \( \pi \) has a distribution \( \chi \)-semicharacter \( f_{\pi,\chi} \) expressible as before: \( f_{\pi,\chi}(\phi) = \phi(\pi(u \ast \phi)) \) for \( \phi \in C_c^\infty(G) \). Here \( u \) is not necessarily semi-invariant; however, \( d\pi(u) \) is semi-invariant, i.e.

\[
\pi(s) d\pi(u) (s^{-1}) = \chi(s)^{-1} d\pi(u).
\]

Suppose now that \( G \) is exponential (\(^1\)) (and therefore, in particular, of type I, cf. [2]). In this paper we make a construction, depending only on the choice of a Jordan-Hölder sequence for \( g_\mathbb{C} \), of a finite set of polynomial functions \( Q_j \geq 0, j=1, \ldots, n \), on \( g' \), a finite set of continuous homomorphisms \( \chi_j : G \to \mathbb{R}^*_+ \), \( j=1, \ldots, n \), and a finite set \( \alpha_j \), \( j=1, \ldots, n \) of positive, \( G \)-invariant analytic functions on \( g \) such that, setting

\[
\Omega_j = \{ g \in g' \mid Q_j(g) \neq 0, Q_k(g) = 0 \text{ for } k < j \}
\]

we have:

1) \( \Omega_j \) is \( G \)-invariant and \( g' = \bigcup_{j=1}^{n} \Omega_j \)
2) \( Q_j(sg) = \chi_j(s) Q_j(g) \) for \( s \in G, g \in \Omega_j \)

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(\(^1\)) \( G \) is said to be exponential if the exponential map \( \exp : g \to G \) is diffeomorphism.
3) for any $\text{G}$-orbit $O$ contained in $\Omega_j$, the measure $Q_j\beta_0$ is a non-zero, positive, tempered, relatively invariant Radon measure on $O$ with multiplier $\chi_j$ (here $\beta_0$ is the canonical measure on $O$),

and such that, letting $u_j, j = 1, \ldots, n$, be the element in $\text{U}(g_c)$ corresponding via symmetrization to the polynomial function $g \rightarrow Q_j(ig)$ on $g_c$, we have for the irreducible representation $\pi$ of $\text{G}$ associated with the orbit $O$ contained in $\Omega_j$,

4) the operator $d\pi(u_j)$ is a selfadjoint, positive, invertible operator, semi-invariant under $\pi$ with multiplier $\chi_j$,

5) the operator $\pi(u_j \ast \varphi)$ is traceclass for all $\varphi \in C_c^\infty(\text{G})$,

6) the functional $\varphi \rightarrow \text{Tr}(\pi(u_j \ast \varphi))$ is a non-zero, $\chi_j$-semi-invariant distribution on $\text{G}$ of positive type (a $\chi_j$-distribution semicharacter for $\pi$), and

7) for all $\varphi \in C_c^\infty(\text{G})$ we have

\[
\text{Tr}(\pi(u_j \ast \varphi)) = \int O (\alpha_j \cdot \varphi \circ \exp)^\wedge(l)Q_j(l)d\beta_0(l),
\]

where $\langle \cdot \rangle$ stands for the ordinary Euclidian Fourier transform.

This construction is carried out in sections 1.1, 1.2 and 1.3, the theorem is formulated in section 1.4, and section 2 is devoted to the proof of the theorem; in section 3 we give a few examples.

We would like to emphasize the following feature of the formula $(\ast)$ shared by no other previously known character formula for (non-nilpotent) solvable Lie groups: once a Jordan-Hölder basis in $g_c$ has been selected, all objects in the formula are explicitly constructible (for a given orbit $O$ and associated representation $\pi$), i.e. there is no choice (in particular of the weight function $\alpha_j$, cf. [9], [4], [5], [6], [3]) involved in setting up the formula. This, in particular, opens the possibility of using the formula $(\ast)$ as a starting point for the pairing between orbits and representations, first established by Bernat ([1]), for exponential groups, and thus extending to these groups Pukanszky’s approach to the Kirillov theory of nilpotent groups. [7].

In the special case where $g$ is nilpotent $\chi_j \equiv 1$ and $\alpha_j \equiv 1$. Therefore $Q_j$ is invariant on $O \subset \Omega_j$, $d\pi(u_j)$ is a scalar, and the formula $(\ast)$ then gives that $d\pi(u_j) = Q_j(O)I$ and

\[
\text{Tr}(\pi(\varphi)) = \int O (\varphi \circ \exp)^\wedge(l)d\beta_0(l),
\]

so $(\ast)$ reduces in particular to the Kirillov character formula.

The main difference between the results obtained in [3] and the results obtained here can be subsumed under the following points: i. We exhibit a finite collection of elements $u_j \in \text{U}(g_c)$ to choose from so as to make a formula like $(\ast)$ valid, ii. we construct such a finite collection explicitly, and iii. here the functions $g \rightarrow Q_j(ig)$ in $(\ast)$ are (rather surprisingly) the polynomial functions corresponding to the $u_j$’s via symmetrization.

The polynomials $Q_j$ were first considered by Pukanszky in the nilpotent case ([8], [10]). We also use in an essential way the work of Pukanszky on exponential groups ([9]) and the work of Duflo-Rais ([5]). Our methods are very different from those of [3].

We conjecture that our results can be extended to arbitrary connected, simply connected
solvable Lie groups (with the usual condition on the support of the function \( \varphi \) appearing in the formula analogous to (*), though; cf. e. g. [6]).

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1. Preliminaries and formulation of Theorem

In sections 1.1, 1.2 and 1.3 we introduce the notation necessary to formulate our Theorem in section 1.4.

1.1. — Let \( G \) be a connected, simply connected solvable Lie group with Lie algebra \( \mathfrak{g} \).

Let \( f_j, j = 0, \ldots, m \), be a Jordan-Hölder sequence in \( \mathfrak{g}_C \), i.e. a sequence of ideals such that \( f_j \supset f_{j-1} \) and such that \( \dim f_j = j, \) \( j = 0, \ldots, m \).

Let \( \lambda_j : \mathfrak{g} \to \mathbb{C} \) be the root associated with the irreducible \( \mathfrak{g} \)-module \( f_j/f_{j-1} \) (i.e. \( \text{ad} X(Z) = \lambda_j(X)Z (\mod f_{j-1}) \) for all \( Z \in f_j, X \in \mathfrak{g} \)), and let \( \Lambda_j : G \to \mathbb{C}^* \) be the continuous homomorphism with \( \Lambda_j(\exp X) = e^{\lambda_j(X)} \) for all \( X \in \mathfrak{g} \). We have \( \text{Ad} (s)Z = \Lambda_s Z (\mod f_{j-1}) \) for all \( Z \in f_j, s \in G \).

We let \( G \) act in \( \mathfrak{g}' \) via the coadjoint representation. For \( g \in \mathfrak{g}' \) we have the skewsymmetric bilinearform \( B_g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \) given by \( B_g(X, Y) = \langle X, [X, Y] \rangle \), \( X, Y \in \mathfrak{g} \). The radical of \( B_g \) is equal to the Lie algebra \( \mathfrak{g}_g \) of the stabilizer \( G_g \) of \( g \) : \( \mathfrak{g}_g = \{ X \in \mathfrak{g} \mid B_g(X, Y) = 0 \text{ for all } Y \in \mathfrak{g} \} \). We let \( \mathfrak{g} : \mathfrak{g}/\mathfrak{g}_g \times \mathfrak{g}/\mathfrak{g}_g \to \mathbb{R} \) designate the symplectic form on \( \mathfrak{g}/\mathfrak{g}_g \) arising from \( B_g \) by factorization. We extend \( g, B_g, \) etc. in the natural way to \( \mathfrak{g}_C \) whenever convenient.

For \( g \in \mathfrak{g}' \) we set \( f_j(g) = f_j + (\mathfrak{g}_g)_C, j = 0, \ldots, m \). We then have a sequence of subalgebras:

\[
\mathfrak{g}_C = f_m(g) \supset f_{m-1}(g) \supset \cdots \supset f_1(g) \supset f_0(g) = (\mathfrak{g}_g)_C,
\]

and \( \dim f_j(g)/f_{j-1}(g) = 0 \) or = 1.

For \( g \in \mathfrak{g}' \) we define \( J_g \) to be the set \( \{ 1 \leq j \leq m \mid f_j(g) \supset f_{j-1}(g) \} \).

Let \( Z_j \in f_j \setminus f_{j-1}, j = 1, \ldots, m \). Then \( Z_1, \ldots, Z_m \) is a basis in \( \mathfrak{g}_C \), and we have

\[
J_g \leftrightarrow Z_j \not\in f_{j-1} + (\mathfrak{g}_g)_C = f_{j-1}(g).
\]

If \( g \in \mathfrak{g}' \) and \( J_g = \{ j_1 < \ldots < j_d \} \) we have

\[
\mathfrak{g}_C = f_{j_d}(g) \supset f_{j_{d-1}}(g) \supset \cdots \supset f_{j_1}(g) \supset f_0(g) = (\mathfrak{g}_g)_C.
\]

In particular \( Z_{j_1}, \ldots, Z_{j_d} \) is a basis for \( \mathfrak{g}_C \) (mod \( (\mathfrak{g}_g)_C \)), and \( d = \dim \mathfrak{g}/\mathfrak{g}_g \).

Set \( \mathcal{E} = \{ J_g \mid g \in \mathfrak{g}' \} \), and for \( e \in \mathcal{E} \), set \( \Omega_e = \{ g \in \mathfrak{g}' \mid J_g = e \} \). Then we have \( \mathfrak{g}' = \bigcup_{e \in \mathcal{E}} \Omega_e \) as a (finite) disjoint union. Since clearly \( J_{sG} = J_g \) for \( s \in G \), \( \Omega_e \) is a \( G \)-invariant subset of \( \mathfrak{g}' \).

Let \( e \in \mathcal{E} \). If \( e \neq \emptyset \) with \( e = \{ j_1 < \ldots < j_d \} \) we define the skewsymmetric \( d \times d \)-matrix \( M_e(g), g \in \mathfrak{g}' \), by

\[
M_e(g) = \left[ B_g(Z_{j_r}, Z_{j_s}) \right]_{1 \leq r, s \leq d}.
\]
and let \( P_e(g) \) denote the Pfaffian of \( M_e(g) \). If \( e = \emptyset \) we set \( M_e(g) = 1 \), and \( P_e(g) = 1 \). The map \( g \rightarrow P_e(g) \) is a complex valued polynomial function on \( g' \), and \( P_e(g) \) depends only on the restriction of \( g \) to \( \langle g, g \rangle \). \( P_e \) has the property that \( P_e(g)^2 = \det M_e(g) \). We set \( Q_e(g) = | \det M_e(g) | = | P_e(g) |^2 \). \( g \rightarrow Q_e(g) \) is a real valued non-negative polynomial function on \( g' \).

For \( e \in \mathcal{E} \) we set \( \Lambda_e = \prod_{j \in \mathcal{E}} \Lambda_j \).

**Lemma 1.1.1.** — Let \( e \in \mathcal{E} \). If \( g \in \mathcal{O}_e \), then \( P_e(g) \neq 0 \) and \( P_e(\mathcal{O}_e) = \Lambda_e(s)^{-1} P_e(g) \) for all \( s \in G \).

**Proof.** — Write \( e = \{ j_1 < \ldots < j_d \} \). Since \( Z_{j_1}, \ldots, Z_{j_d} \) is a basis for \( g_c (\text{mod} \ (g_c)_c) \) we have that \( M_e(g) \) is a regular matrix, hence \( P_e(g)^2 = \det M_e(g) + 0 \).

Now writing

\[
\text{Ad}(s^{-1})Z_{j_p} = \sum_{u=1}^{d} a_{up} Z_{j_u} + c_{p},
\]

where \( c_{p} \in \mathbb{C} \), we have \( a_{up} = 0 \) for \( u > p \) and \( a_{pp} = \Lambda_j(s^{-1}) \), and

\[
B_{sg}(Z_{j_p}, Z_{j_q}) = \langle g, [Z_{j_p}, Z_{j_q}] \rangle = \langle g, [\text{Ad}(s^{-1})Z_{j_p}, \text{Ad}(s^{-1})Z_{j_q}] \rangle
= \sum_{u=1}^{d} a_{up} \langle g, [Z_{j_u}, Z_{j_p}] \rangle a_{pq} = (\Lambda M_{sg}(A)A)^{p,q},
\]

where \( A \) is the matrix \( [a_{pq}]_{p,q=1}^{d} \). This shows that \( M_{sg}(A)A \), and since \( \det A = \prod_{j=1}^{d} \Lambda_j(s^{-1}) = \Lambda \) we find that

\[
P_e(\mathcal{O}_e) = P(\det M_{sg}(A))A(\det A)P(\mathcal{O}_e) = \Lambda(s^{-1})P_e(g).
\]

This ends the proof of the lemma.

**Corollary 1.1.2.** — If \( g \in \mathcal{O}_e \) then \( Q_e(g) > 0 \) and \( Q_e(\mathcal{O}_e) = | \Lambda_e(s) |^{-2} Q_e(g) \) for all \( s \in G \).

For \( e \in \mathcal{E} \) we set \( | e | = \text{the number of elements in } e \). We define a total ordering \( < \) on \( \mathcal{E} \) in the following way: let \( e, e' \in \mathcal{E} \). Then \( e < e' \) if and only if either \( | e | > | e' | \) or \( e = | e' | \) and, writing \( e = \{ j_1 < \ldots < j_d \} \), \( e' = \{ j'_1 < \ldots < j'_d \}, j_p < j'_p \), where \( p = \min \{| < d | j_p + j'_p \}
\]

**Lemma 1.1.3.** — \( \Omega_e = \{ g \in g_c | P_e(g) = 0 \} \).

**Proof.** — If \( g \in \mathcal{O}_e \) we saw in Corollary 1.1.2 that \( Q_e(g) = 0 \). If \( e' < e \) and \( | e' | > | e | \), then, if \( e' = \{ j'_1 < \ldots < j'_e \}, Z_{j_1}, \ldots, Z_{j_e} \) are linearly dependent \( \text{mod} \ (g_c)_c \), so \( M_{e}(g) \) is singular, and therefore \( Q_e(g) = 0 \). If \( | e | = | e' | \), and \( j_1 = j'_1, \ldots, j_p = j'_p \), \( j_{p+1} < j_{p+1} \), then \( Z_{j_p} + e(\mathcal{O}_e) \), and therefore \( Z_{j_1}, \ldots, Z_{j_e} \), are linearly dependent \( \text{mod} \ (g_c)_c \), and again \( Q_e(g) = 0 \). This shows the lemma.

**Remark 1.1.4.** — If \( g \) is nilpotent our definitions agree with those given by Pukanszky in [10], p. 525 f. f., cf. also [8]. In [6], section 4.2 a study of the completely solvable case was initiated.

1.2. — Recall the following facts: there exists an isomorphism \( \omega \) (the symmetrization map) between the complex vector space \( \mathcal{S}(g_c) \) (the symmetric algebra of \( g_c \)), and the complex vector space \( U(g_c) \) (the universal enveloping algebra of \( g_c \)), characterized by the following
property: if \( Y_1, \ldots, Y_p \) are elements in \( g_<c \), then the image of the element \( Y_1 \ldots Y_p \) in \( S(g_<c) \) by \( \omega \) is the element \( (p!)^{-1} \sum_{\sigma \in S_p} Y_{\sigma(1)} \ldots Y_{\sigma(p)} \) in \( U(g_<c) \), where \( S_p \) is the group of permutations of \( p \) elements. The following lemma is easily verified:

**Lemma 1.2.1.** — If \( Z \) is a central element in \( g_<c \), then \( \omega(ZM) = Z\omega(M) \) for all \( M \in S(g_<c) \).

We can identify \( S(g_<c) \) with \( P'_c(g') \), the complex vector space of complex valued polynomial functions on \( g' \). If \( u \in U(g_<c) \) we let \( P_u \) be the polynomial on \( g' \) corresponding to \( \omega^{-1}(u) \).

The lemma above then says that if \( Z \) is central in \( g_<c \) and if \( u \in U(g_<c) \) then \( P_Z = P_Z P_u \).

For \( e \in \mathfrak{e} \), let \( u_e \) be the element in \( U(g) \) corresponding to the real valued polynomial function \( g \rightarrow \mu Q_\epsilon(g) \) on \( g' \). Note that \( u_e \) actually is contained in \( U([g, g]) \), since \( Q_\epsilon(g) \) only depends on the restriction of \( g \) to \([g, g]\).

1.3. — If \( g \in g' \), the weights of \( g_\epsilon \) in \( g/g_\epsilon \) are of the form \( \pm \mu_1, \ldots, \pm \mu_{d/2} \), where \( d = \text{dim } g/g_\epsilon \) and these weights \( \mu_j \) extend to linear forms, also called \( \mu_j \), on the ideal \( g_\epsilon = g_\epsilon + [g, g] \) in such a manner that they are zero on \([g, g]\) (v. [4], p. 248).

Following loc. cit. we set

\[
S_\lambda(X) = \frac{\sin h(\lambda(X)/2)}{\lambda(X)/2}, \quad X \in g_<g,
\]

for a complex linear form \( \lambda \) on \( g_\epsilon \), and define the function \( P'_\epsilon \) on \( g_\epsilon \) by

\[
P'_\epsilon(X) = \prod_{\lambda \in \Omega_\epsilon} S_\lambda(X), \quad X \in g_\epsilon,
\]

where \( \Omega = G g \) is the \( G \)-orbit through \( g \). This definition of \( P'_\epsilon \) does not depend on the choice of \( g \in O \).

We set

\[
j_\epsilon(X) = \det \left| \frac{1 - e^{-a d X}}{a d X} \right|, \quad X \in g_<g.
\]

\( j_\epsilon \) is a \( G \)-invariant analytic function on \( g_\epsilon \), and if \( dX \) is a Lebesgue measure on \( g \) there exists a Haar measure \( \mu \) on \( G \) such that \( d\mu(\exp X) = j_\epsilon(X)dX \).

If \( G \) is exponential we set for \( e \in \mathfrak{e} \),

\[
\Gamma_\epsilon(X) = \left( \prod_{\xi \in \Omega_\epsilon} |S_\xi(X)| \right)^{1/2}, \quad X \in g_<g.
\]

**Lemma 1.3.1.** — (\( G \) exponential) \( \Gamma_\epsilon \) is a positive, \( G \)-invariant analytic function on \( g_\epsilon \), extending \( P'_\epsilon \) for any \( G \)-orbit \( O \) contained in \( \Omega_\epsilon \).

**Proof.** — The function \( X \rightarrow S_\lambda(X) \) is a \( G \)-invariant analytic function on \( g_\epsilon \), and since \( g \) is exponential \( \lambda_\epsilon(X) \neq \emptyset \) for all \( X \in g_<g \), hence \( S_\lambda(X) \neq 0 \) for all \( X \in g_<g \). This shows that \( \Gamma_\epsilon \) is positive, \( G \)-invariant and analytic. Now an easy argument shows that \( P'_\epsilon(X) \geq 0 \) for all \( X \in g_\epsilon + [g, g] \) (see e. g. [4], p. 264 top; again we use that \( g \) is exponential). Therefore

\[
P'_\epsilon(X) = |P'_\epsilon(X)| = \prod_{\lambda \in \Omega_\epsilon} S_\lambda(X) = \left( \prod_{\lambda \in \Omega_\epsilon} |S_\lambda(X)| \right)^{1/2} = \left( \prod_{\lambda \in \Omega_\epsilon} |S_\lambda(X) - S_{-\lambda}(X)| \right)^{1/2},
\]

and noting that \( \lambda_j \) vanishes on \([g, g]\) and that the weights of \( g_\epsilon \) in \( g/g_\epsilon \) are precisely

\[
\{ \lambda_{j_1} \mid g_{j_1}, \ldots, \lambda_{j_d} \mid g_{j_1} \} = \{ \pm \mu_1 \mid g_{j_1}, \ldots, \pm \mu_{d/2} \mid g_{j_1} \},
\]

we get that \( P'_\epsilon(X) = \Gamma_\epsilon(X) \) for \( X \in g_\epsilon \). This proves the lemma.
We set
\[ \alpha_e(X) = j_G(X) \Gamma_e(X)^{-1}, \quad X \in \mathfrak{g} \]
(still assuming that \( G \) is exponential). \( \alpha_e \) is a positive, \( G \)-invariant analytic function on \( \mathfrak{g} \).

**Remark 1.3.2.** — Lemma 1.3.1 should be compared with [4], section 4, p. 262-264. In the exponential case the result *loc. cit.* is that there exists a \( G \)-invariant Zariski open subset \( \Omega \) of \( \mathfrak{g}' \) and a positive, \( G \)-invariant analytic function \( P \) on \( \mathfrak{g} \), such that for any \( G \)-orbit \( O \) contained in \( \Omega \) the restriction of \( P \) to \( \mathfrak{f} = \mathfrak{g}_e + [\mathfrak{g}_e, \mathfrak{g}] \), \( e \in O \), is equal to \( P_0 \). By Lemma 1.3.1 and Lemma 1.1.3 we can obtain this result by taking \( \Omega \) to be \( \Omega_e \) for the minimal element \( e \) in \( \mathcal{E} \), and taking \( P \) to be \( \Gamma_e \). In general the \( P \) from *loc. cit.* will be different from the one exhibited here. Incidentally, by refining the methods used here can give a complete solution to the problem raised, and partially solved, by Duflo, *loc. cit.*, p. 263, mid. However, at present this will not be needed, so we shall postpone it to a later time.

1.4. — Suppose now that \( G \) is exponential, and suppose in addition that the Jordan-Hölder sequence \( \mathfrak{g}_c = f_m \supseteq \ldots \supseteq f_0 = \{0\} \) has the property that if \( f_j \neq f_i \), then \( f_{j-1} = f_{j-1} \) and \( f_{j+1} = f_{j+1}, \quad 1 \leq j \leq m-1 \) (such a Jordan-Hölder sequence clearly exists). Set \( \chi_e = |\Lambda_e|^{-2} \).

**Theorem 1.4.1.** — (\( G \) exponential) Let \( \pi \) be an irreducible representation of \( G \), and let \( O \) be the \( G \)-orbit in \( \mathfrak{g}' \) associated with \( \pi \). Let \( e \in \mathcal{E} \) be the unique element such that \( \Omega_g \) contains \( O \). Then

1) The measure \( \Omega_e \beta_0 \) is a non-zero, positive, tempered, relatively invariant Radon measure on \( O \) with multiplier \( \chi_e \) (\( \beta_0 \) is the canonical measure on \( O \)).

2) The operator \( d\pi(u_e) \) is a selfadjoint, positive, invertible operator, semi-invariant under \( \pi \) with multiplier \( \chi_e \) (i.e. \( \pi(s)d\pi(u_e)\pi(s^{-1}) = \chi_e(s^{-1})d\pi(u_e) \)).

3) For any \( \varphi \in \mathcal{C}_c^\infty(G) \) the operator \( \pi(u_e \ast \varphi) \) is traceclass.

4) The functional \( \varphi \rightarrow \text{Tr} \, (\pi(u_e \ast \varphi)) \) on \( \mathcal{C}_c^\infty(G) \) is a non-zero, \( \chi_e \)-semi-invariant distribution on \( G \) of positive type (a distribution semicharacter for \( \pi \) (with multiplier \( \chi_e \)).

5) For any \( \varphi \in \mathcal{C}_c^\infty(G) \) we have the formula

\[
\text{Tr} \, (\pi(u_e \ast \varphi)) = \int_O (\alpha_e \ast \varphi \circ \exp)^\wedge(l)Q_e(l)d\beta_0(l).
\]

Here we use the notation \( \Psi(l) = \int_{\mathfrak{g}'} \psi(X)e^{i(l, X)}dX \) for \( \psi \in \mathcal{C}_c^\infty(g), \ l \in \mathfrak{g}' \), where \( dX \) is the Lebesgue measure on \( g \) with the property that \( d\mu(\exp X) = j_G(X)dX, \ d\mu \) being a fixed Haar measure on \( G \), and \( \pi(\varphi) = \int_G \varphi(s)\pi(s)d\mu(s) \) for \( \varphi \in L^1(G) \).

**Remark 1.4.2.** — In the formula (\ast) above we can instead of \( \alpha_e \) use any \( \mathcal{C}_c^\infty \)-function \( \alpha \) on \( \mathfrak{g} \) with the property that the restriction of \( \alpha \) to \( \mathfrak{f} = \mathfrak{g}_e + [\mathfrak{g}_e, \mathfrak{g}] \), \( e \in O \), is the same as the restriction of \( \alpha_e \) to \( \mathfrak{f} \).
REMARK 1.4.3. — It will follow from the proof of Theorem 1.4.1 that the distributions \( \phi \rightarrow \text{Tr}(\pi(u_\phi \circ \phi)) \) have a finite order not exceeding \( 2d+1 \), where \( d = |e| \).

2. Proof of Theorem

Here we shall for brevity say that a Jordan-Hölder sequence \( g_c = f_m \supset \ldots \supset f_0 = \{0\} \) is of class \( (b) \) if it has the property required in 1.4 (i.e. that \( f_{j+1} \neq f_j \), \( 1 \leq j \leq m-1 \), implies that \( f_{j+1} = f_j \cdot f_{j-1} \) and \( f_{j+1} = f_{j-1} \)), cf. [2] Définition 4.2.1, pp. 78.

2.1. — The purpose of this subsection is to prove the following lemma, from which part 1) of Theorem 1.4.1 follows immediately.

LEMMA 2.1.1. — The measure \( P^0 \beta_0 \) is a non-zero, tempered, \( \Lambda^{-1} \)-relatively invariant (complex) Radon measure on \( G \).

REMARK 2.1.2. — In the completely solvable case this was proved in [6], section 4.1. The proof loc. cit. does not carry over to the case at hand, so we have to modify our approach.

Proof. — We have only left to show that \( P^0 \beta_0 \) is tempered, cf. Lemma 1.1.1.

(i) Let \( I \) be the set of indices \( 0 \leq j \leq m \) for which \( f_j = f_j \cdot f_j \). For \( i \in I \) there exists an ideal \( g_j \) in \( g \) such that \( (g_j)_C = f_j \).

Set \( I' = \{ j \in I | j-1 \in \mathbb{I} \} \) and \( I'' = \{ j \in I | j-1 \notin \mathbb{I} \} \). Then \( I = \{ 0 \} \cup I' \cup I'' \) as a disjoint union, and for \( i \in I'' \) we have that \( j-2 \in I \) (since \( f_0, \ldots, f_m \) is of class \( (b) \)).

Now since \( \Lambda_X \) onlydepends on the Jordan-Hölder sequence \( f_j \) and not on the basis \( Z_j \) we can assume here that the \( Z_j \)'s are constructed in the following way: for \( i \in I' \), let \( X_j \in g \cdot g_j \cdot \mathbb{I} \), and set \( Z_j = X_j \). For \( j \in I'' \), pick \( Z_j = (X_j)^{-1} - (X_j)^{-1} \). Since \( f_j = f_j \cdot f_j \cdot f_j \) we have that \( Z_j = f_j \cdot f_j \cdot f_j \). Set \( Z_j = (X_j)^{-1} - (X_j)^{-1} \). Define \( X_j, X_j \) by \( X_j = Z_j \cdot X_j \cdot X_j \cdot X_j \). Then \( X_j, X_j \) is a basis for \( g_j \) (mod \( g_j^{-2} \)), and \( X_1, \ldots, X_m \) is a basis for \( g \). Let \( g_1, \ldots, g_m \in g \) be the basis dual to \( X_1, \ldots, X_m \).

Fix an element \( g \in G \), and write \( e = j \rightarrow g \cdot \{ j \leq k \leq j \} \). Set \( D_1 = \{ 1 \leq k \leq d | j \in I' \} \), \( D_2 = \{ 1 \leq k \leq d | j \in I' \}, D_3 = \{ 1 \leq k \leq d | j \in I'' \} \), \( D_4 = \{ 1 \leq k \leq d | j \in I'' \} \). Clearly \( I \cdot D_1 \cdot D_2 \cdot D_3 \cdot D_4 \) is a disjoint union. Observe that if \( k \in D_3 \), then clearly \( k+1 \in D_4 \). Conversely, if \( k \in D_4 \), then \( j \in I'' \cap J \), and therefore \( j \notin I' \); in fact, if \( j \notin I' \), then \( Z_j = (X_j)^{-1} + (g_j)_C \), that is, \( X_j = X_j^{-1} \cdot (X_j)^{-1} \cdot (g_j)_C \), implying that \( X_j = X_j^{-1} \cdot (X_j)^{-1} \cdot (g_j)_C \), but then \( Z_j = X_j^{-1} \cdot (X_j)^{-1} \cdot (g_j)_C = f_j \cdot f_j \cdot (g_j)_C \), and therefore \( f_j = f_j \cdot f_j \), which is a contradiction. The conclusion of this is that \( D_4 = \{ k \in k \cap D_3 \} \).

For \( j \in I' \), set \( g_j = \{ s \in G | sg = g \cdot \{ j \} \} \). \( G_j \) is a closed, connected subgroup with Lie algebra \( g_j = \{ X \in g \cdot g_j \} \) (cf. [9], p. 105, III). Clearly \( j \rightarrow g_j, j \in I' \), is a decreasing sequence of subalgebras with \( g_0 = g \) and \( g_m = g_m \).

If \( j \in I' \), then \( \dim g_j^{-1} - g_j = 0 \) or \( = 1 \), and \( g_j^{-1} g_j \neq g_j \) if and only if \( j \in I' \). If \( j \in I'' \), then \( \dim g_j^{-1} - g_j = 0 \), \( = 1 \) or \( = 2 \), and \( g_j^{-1} g_j = 2 \) if and only if \( j, j-1 \in I' \), \( \dim g_j^{-1} g_j = 2 \) if and only if \( j \in I' \).

(ii) The following is an adaptation of [9], p. 102-106, II-III to the present situation:
For $k \in D_1$ there exists an element $Y_k$ in $g_h^{-1} g_h^{k+1}$ such that $Y_k$ is a coexponential basis to $g_h^k$ in $g_h^{k-1}$ and such that $Y_k g = g_{i_h}(\mod g_h^{k+1})$, and for $s \in G_h^{k+1}$ we have
\[ \text{Ad}(s)Y_k = \Lambda_{i_h}(s^{-1})Y_k (\mod g_h^{k+1}). \]

For $k \in D_2$ there exists an element $Y_k$ in $g_h^{-1} g_h^{k+1}$ such that $Y_k$ is a coexponential basis to $g_h^k$ in $g_h^{k-1}$, and such that $Y_k g = g_{i_h}(\mod g_h^{k+1})$ (to obtain this it can be necessary to change $X_{i_h}, X_{i_h+1}$ in a way that only affects $Z_{i_h}, Z_{i_h+1}$ by multiplying them by a factor of modulus one), and for $s \in G_h^{k+1}$ we have $\text{Ad}(s)Y_k = \Lambda_{i_h}(s^{-1})Y_k (\mod g_h^{k+1})$ (so in particular $\Lambda_{i_h}(s^{-1})$ is real).

For $k \in D_3$ there exists elements $Y_k, Y_{k+1}$ in $g_h^{-1} g_h^{k+1}$ such that $Y_k g = g_{i_h}(\mod g_h^{k+1})$, $Y_{k+1} g = g_{i_h+1}(\mod g_h^{k+1})$, such that $Y_k, Y_{k+1}$ is a coexponential basis to $g_h^{k+1}$ in $g_h^{k-1}$, such that $\Lambda_{i_h}(Y_k) = \Lambda_{i_h}(Y_{k+1}) = 0$ and such that
\[ \exp t_k Y_k \exp t_{k+1} Y_{k+1} = \exp (t_k Y_k + t_{k+1} Y_{k+1}) (\mod G_h^{k+1}) \]

For $s \in G_h^{k+1}$ we have $\text{Ad}(s)(Y_k + iY_k) = \Lambda_{i_h}(s^{-1})(Y_k + iY_k) (\mod (g_h^{k+1})^c)$.

(iii) The map $\mathbb{R}^d \to O = G g$ given by
\[
\left( t_1, \ldots, t_d \right) \to \exp t_1 Y_1 \ldots \exp t_d Y_d
\]
is a diffeomorphism. We shall compute the canonical measure $\beta_0$ in terms of the coordinates $t = (t_1, \ldots, t_d)$.

Let $\omega$ be the canonical symplectic form on $O$. Via the natural correspondence between $g/q$ and the tangent space to $O$ at $g$, $\omega$ corresponds to $B_g$.

**Lemma 2.1.3.** — For a $\beta_0$-integrable function $f$ on $O$ we have
\[
\int_O f(l)d\beta_0(l) = C \int_{\mathbb{R}^d} f(\exp t_1 Y_1 \ldots \exp t_d Y_d) \prod_{k<r} |\Lambda_{i_h}(\exp t_k Y_k)| dt_1 \ldots dt_d,
\]
where $C = (2\pi)^d Q_e(g)$. 

**Proof.** — Denote by $\sigma$ the inverse of the map $(\ast)$. $\sigma$ is a global chart and
\[
\int_O f(l)d\beta_0(l) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\sigma^{-1}(t))|\theta(t)\theta^{-1}(t)|dt,
\]
where $\theta(t) = (\det S_t)^t$, $S_t$ being the skew-symmetric matrix $S_t = [\omega(\partial/\partial t_w, \partial/\partial t_v)]_{1 \leq w,v \leq d}$ (19) Proposition 4, p. 99).

Now $\omega$ is $G$-invariant. Therefore, writing $s = \exp t_1 Y_1 \ldots \exp t_d Y_d$ and $l = sg$, we have
\[
\omega_k(\partial/\partial t_w) = \omega_k(\partial/\partial t_w)_{sg}, \quad \omega_k(\partial/\partial t_v)_{sg}
\]
where $\gamma(s) ; l \to sl$. Let us then compute $\gamma(s^{-1}) \star (\partial/\partial t_w)_{sg}$.

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For a differentiable function $\varphi$ we have

$$\gamma(s^{-1}) * (\partial/\partial u)_{x^g} \varphi = (\partial/\partial u)_{z^g} \varphi * \gamma(s^{-1})$$

$$= \frac{d}{dt} \varphi(s^{-1} \sigma^{-1}(t + \tau^n)) \mid_{t=0} \quad (\tau^n = (\delta, \tau)_{1 \leq v \leq d})$$

$$= \frac{d}{dt} \varphi(\exp -t_d Y_d \ldots \exp -t_1 Y_1 \exp t_1 Y_1 \ldots \exp (t_u + \tau) Y_u \ldots \exp t_d Y_d g) \mid_{t=0}$$

$$= \frac{d}{dt} \varphi(\exp -t_d Y_d \ldots \exp -t_u+1 Y_{u+1} \exp \tau Y_u \exp t_{u+1} Y_{u+1} \ldots \exp t_d Y_d g) \mid_{t=0}$$

$$= \frac{d}{dt} \varphi(s^{-1}_u \exp \tau Y_u s_u g) \mid_{t=0} = \frac{d}{dt} \varphi(\exp \tau \text{Ad}(s^{-1}_u Y_u g)) \mid_{t=0},$$

where we have set $s_u = \exp t_{u+1} Y_{u+1} \ldots \exp t_d Y_d \ u < d, \ s_d = e$.

The conclusion of this is that $S_1 = [B_g(\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)]_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u,$$

where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

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where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

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where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

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where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u,$$

where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u,$$

where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u,$$

where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u,$$

where $c_u \in g_1$, and then $S_1 = (\text{Ad}(s^{-1}_u) Y_u \ Ad(s^{-1}_v) Y_v)_{1 \leq u, v \leq d}$. Since $Y_1, \ldots, Y_d$ is a basis for $g \ (\text{mod } g_1)$ we can write

$$\text{Ad}(s^{-1}_u) Y_u = \sum_{r=1}^{d} a_p Y_p + c_u.$$
Finally, a simple computation shows that \( \det S_g = \det [B_g(Y_r, Y_s)]_{1 \leq r, s \leq d} = Q_d(g)^{-1} \). This ends the proof of the lemma.

(iv) For \( 1 \leq j \leq m \) we define the function \( S_j \) by
\[
S_j(t_1, \ldots, t_d) = \langle \exp t_1 Y_1 \ldots \exp t_d Y_d, Z_j \rangle.
\]
We consider \( S_{\lambda} \): arguing like in [9], p. 106 we find for \( k \in D_1 \cup D_2 \):
\[
S_{\lambda}(t_1, \ldots, t_d) = \frac{e^{-t_1 \lambda(Y_k)^{-1}}}{-\lambda(Y_k)} \prod_{r < k} \Lambda_{ir}(\exp t_r Y_r)^{-1} + S_{\lambda}(t_1, \ldots, t_{k-1}, 0, \ldots, 0),
\]
and for \( k \in D_4 \) we find
\[
S_{\lambda}(t_1, \ldots, t_d) = (t_{k-1} + it_k) \prod_{r < k-1} \Lambda_{ir}(\exp t_r Y_r)^{-1} + S_{\lambda}(t_1, \ldots, t_{k-2}, 0, \ldots, 0).
\]

(v) For a real number \( n > 0 \) we set \( M(n) = \int_\mathbb{R} (1 + x^2)^{-n/2} dx \). We have \( 0 < M(n) \leq +\infty \) and \( M(n) < +\infty \) if and only if \( n > 1 \).

**Lemma 2.1.4.** — Let \( a, \alpha, \beta \) be real numbers with \( a > 0, \alpha \neq 0 \), and let \( c, k \) be complex numbers with \( k \neq 0 \). We have
\[
\begin{align*}
\text{(*)} & \quad \int_\mathbb{R} (a + |k| e^{(\alpha + i \beta)t} - c |^2)^{-n/2} e^a dt < \frac{M(n)}{\alpha | | k | | d^{(n-1)/2}}, \\
\text{(**)} & \quad \int_{\mathbb{R}^2} (a + |k| (s + it) - c |^2)^{-n/2} ds dt = \frac{M(n) M(n - 1)}{|k|^2 d^{(n-2)/2}}.
\end{align*}
\]

**Proof.** — Obviously we can assume that \( k > 0 \). Writing \( k^{-1} c = be^{iy}, b > 0, y \in \mathbb{R} \) we have
\[
\int_\mathbb{R} (a + |k| e^{(\alpha + i \beta)t} - c |^2)^{-n/2} e^a dt = \int_\mathbb{R} (a + k^2 | e^{a + i(\beta - y)} - b |^2)^{-n/2} e^a dt
\]
\[
\leq \int_\mathbb{R} (a + k^2 | e^{a} - b |^2)^{-n/2} e^a dt
\]
\[
= |\alpha|^{-1} \int_0^\infty (a + k^2 | x - b |^2)^{-n/2} dx
\]
\[
< |\alpha|^{-1} \int_\mathbb{R} (a + k^2 | x - b |^2)^{-n/2} dx
\]
\[
= |\alpha|^{-1} \int_\mathbb{R} (a + k^2 x^2)^{-n/2} dx
\]
\[
= \frac{M(n)}{|\alpha| d^{(n-1)/2}}.
\]

This proves (\text{*}). Similarly for (\text{**}).

(vi) We shall then prove the temperedness of the measure \( P_g \beta_0 \). First observe that
Suppose first that $d \in D_1 \cup D_2$. Then (assuming that $\lambda_{jd}(Y_d) \neq 0$)

$$S_{jd}(t_1, \ldots, t_d) = \frac{e^{-t_d \lambda_{jd}(Y_d)} - 1}{-\lambda_{jd}(Y_d)} \prod_{r < d} \Lambda_{kr}(\exp t_r Y_r)^{-1} + S_{jd}(t_1, \ldots, t_{d-1}, 0),$$

and the last integral is equal to

$$(2\pi)^{-d/2} \int_{R^{d-1}} \prod_{r < k < d-1} |\Lambda_{kr}(\exp t_k Y_k)|^{-1} dt_1 \ldots dt_{d-1} \int_{R} F(t_1, \ldots, t_d) dt_d,$$

where

$$F(t_1, \ldots, t_d) = \frac{\prod_{r < d} |\Lambda_{jd}(\exp t_r Y_r)|^{-1}}{(1 + \sum_{k \in D_3} |S_{jk}(t_1, \ldots, t_d-1, 0)|^2 + |S_{jd}(t_1, \ldots, t_d)|^2)^{n/2}}.$$

Applying Lemma 2.1.4 with $a = 1 + \sum_{k \in D_3} |S_{jk}(t_1, \ldots, t_d-1, 0)|^2$, $\alpha + i\beta = -\lambda_{jd}(Y_d)$, $k = -\lambda_{jd}(Y_d)^{-1} \prod_{r < d} \Lambda_{kr}(\exp t_r Y_r)^{-1}$, $c = -S_{jd}(t_1, \ldots, t_{d-1}, 0) - \lambda_{jd}(Y_d)^{-1} \prod_{r < d} \Lambda_{kr}(\exp t_r Y_r)^{-1}$, we find that

$$\int_{R} F(t_1, \ldots, t_d) dt_d \leq \frac{C_d \cdot M(n)}{(2\pi)^{n/2}} \cdot \frac{1}{(1 + \sum_{k \in D_3} |S_{jk}(t_1, \ldots, t_{d-1}, 0)|^2)^{(n-1)/2}},$$

where $C_d = |\lambda_{jd}(Y_d)| \left( |\text{Re } \lambda_{jd}(Y_d)| \right)^{-1}$ (note that since $g$ is exponential the non-vanishing of $\lambda_{jd}(Y_d)$ implies the non-vanishing of $\text{Re } \lambda_{jd}(Y_d)$), and therefore

$$\int_{O} (1 + ||l||^2)^{-n/2} |P_e(l)| d\beta_0(l) \leq \frac{C_d \cdot M(n)}{(2\pi)^{d}} \cdot \int_{R^{d-1}} \prod_{r < k < d-1} |\Lambda_{kr}(\exp t_k Y_k)|^{-1} \left( \frac{1}{(1 + \sum_{k \in D_3} |S_{jk}(t_1, \ldots, t_{d-1}, 0)|^2)^{(n-1)/2}} \right) dt_1 \ldots dt_{d-1}.$$
If $\lambda_{j}(Y_{d})=0$ a simple change in the argument shows that the same relation is valid with $C_{d}=1$ (cf. below).

Suppose next that $d=D_{d}$. Then

$$S_{j}(t_{1}, \ldots, t_{d})=(t_{d-1}+it_{d}) \prod_{r \leq d-2} \Lambda_{j}(\exp t_{r}Y_{r})^{-1}+S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0),$$

and therefore we find as above that

$$\int_{\Omega} \left(1+||l||^{2}\right)^{-n/2} |P_{d}(l)| \omega_{d}(l) \leq(2\pi)^{-d/2} \prod_{r \leq d-2} \Lambda_{j}(\exp t_{r}Y_{r})^{-1} \int_{\mathbb{R}^{d-2}} F(t_{1}, \ldots, t_{d}) dt_{d-1} dt_{d},$$

where now

$$F(t_{1}, \ldots, t_{d})=\frac{\prod_{r=k}^{d-2} |\Lambda_{j}(\exp t_{r}Y_{r})|^{-2}}{(1+\sum_{k=d}^{d-2} |S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0)|^{2}+|S_{j}(t_{1}, \ldots, t_{d})|^{2})^{d/2}}.$$

(here we have used that $|\Lambda_{j}|=|\Lambda_{j-1}|$ and that $\lambda_{j-1}(Y_{d})=\lambda_{j}(Y_{d-1})=\lambda_{j-1}(Y_{d})=\lambda_{j}(Y_{d})=0$).

Applying the relation ($**$) in Lemma 2.1.4 with $a=1+\sum_{k=d}^{d-2} |S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0)|^{2}$, $k=\prod_{r=k}^{d-2} |\Lambda_{j}(\exp t_{r}Y_{r})|^{-1}$, and $c=-S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0)$ we find that

$$\int_{\mathbb{R}^{d-2}} F(t_{1}, \ldots, t_{d}) dt_{d-1} dt_{d} \leq \frac{M(n)M(n-1)}{(2\pi)^{d/2}} \frac{1}{(1+\sum_{k=d}^{d-2} |S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0)|^{2})^{(n-2)d/2}},$$

and therefore

$$\int_{\Omega} \left(1+||l||^{2}\right)^{-n/2} |P_{d}(l)| \omega_{d}(l) \leq \frac{M(n)M(n-1)}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-2}} \left(1+\sum_{k=d}^{d-2} |S_{j}(t_{1}, \ldots, t_{d-2}, 0, 0)|^{2}\right)^{d/2} dt_{1} \ldots dt_{d-2}.$$

Repeating these two methods of estimation on the new integral $(\#)$ or $(\#\#)$ we find that

$$\int_{\Omega} \left(1+||l||^{2}\right)^{-n/2} |P_{d}(l)| \omega_{d}(l) \leq (2\pi)^{-d/2}M(n) \ldots M(n-d+1)C_{d} \ldots C_{1} < +\infty$$

for $n>d$. Here $C_{d} = |\lambda_{j}(Y_{d})|^{-1}$ if $\lambda_{j}(Y_{d}) \neq 0$, and $C_{d} = 1$ if $\lambda_{j}(Y_{d}) = 0$. This ends the proof of Lemma 2.1.1.

2.2. — The purpose of this subsection is to prove Proposition 2.2.1 below.

Let $n$ be the nilradical of $g$, and let $N$ be the analytic subgroup corresponding to $n$. We have $[g, g] \subset n \subset g$, and therefore $u \in U(n)$.
PROPOSITION 2.2.1. — If $g \in \Omega_\pi$ and if $\pi$ is the irreducible representation of $N$ corresponding to the orbit $O = Nf$, where $f = g \mid n$, then

$$d\pi(u_g) = Q_{\pi}^\ast(g)I.$$

REMARK 2.2.2. — Even in the special case where $g$ is assumed to be nilpotent (and therefore $g \equiv n$), Proposition 2.2.1 provides a new result.

Proof. — The proof is by induction on the dimension of $g$. The proposition is clearly valid for $\dim g = 1$ (in which case $e = \emptyset$, and $Q_{\pi} \equiv 1, u_g = 1$). Assume then that the proposition has been proved for all dimensions of $g$ less than or equal to $m - 1$, and that $\dim g = m$.

The case $e = \emptyset$ being trivial we can assume that $e \neq \emptyset$, and write $e = \{j_1 < \ldots < j_d\}$.

Case (a): Suppose that there exists a non-trivial abelian ideal $a$ in $g$ such that $g \mid a = 0$. Let $A$ be the analytic subgroup of $G$ corresponding to $a$. We have $a \subset n$ and setting $\tilde{g} = g/a, \tilde{n} = n/a$ is the nilradical of $\tilde{g}$. We set $\tilde{f}_j = f_j + a_c/a_c, 0 \leq j \leq m$, and let $c : g \to g/a$ denote the coset map. Then we have the diagram

$$\tilde{g}_c = \tilde{f}_m \Rightarrow \tilde{f}_{m-1} \Rightarrow \ldots \Rightarrow \tilde{f}_1 \Rightarrow \tilde{f}_0 = \{0\},$$

and $\dim \tilde{f}_j = 0$ or $= 1$. Set $I = \{0 \leq j \leq d : \tilde{f}_j = \tilde{f}_j \}$, write $I = \{i_1 < \ldots < i_m'\}$, and set $\tilde{f}_j = \tilde{f}_{i_j}, 1 \leq j \leq m'$. We then have a Jordan-Hölder sequence in $\tilde{g}_c$:

$$\tilde{g}_c = \tilde{f}_{i_m'} \Rightarrow \tilde{f}_{i_{m-1}} \Rightarrow \ldots \Rightarrow \tilde{f}_1 \Rightarrow \tilde{f}_0 = \{0\}$$

which is immediately seen to be of class (b), and setting $\tilde{Z}_j = c(Z_{i_j})$ we have that

$$\tilde{Z}_{\tilde{f}_j} = \tilde{f}_{i_j} \Rightarrow \tilde{f}_{i_{j-1}} \Rightarrow \ldots \Rightarrow \tilde{f}_1 \Rightarrow \tilde{f}_0 = \{0\}.$$

Define $\tilde{g} \in \tilde{g}'$ by $\tilde{g} \circ c = g$ and $\tilde{f} = \tilde{g} \mid \tilde{n}$. We have $a \subset g$ and $\tilde{g} = g/a$. Moreover, $j \in J_a \Rightarrow j \in I$, since $j \notin I \Rightarrow \tilde{f}_j \subset \tilde{f}_{i_{j-1}} + a_c \subset \tilde{f}_{i_{j-1}} + (g_{\tilde{f}_j})_c \Rightarrow j \notin J_a$. Writing

$$\hat{e} = J_{\tilde{g}} = \{\tilde{j}_1 < \ldots < \tilde{j}_d\}$$

we have $J_a = \{i_{j_1} < \ldots < i_{j_d}\} = \{j_1 < \ldots < j_d\}$. For $\tilde{f} \in \tilde{g}'$ we then have with $l = \tilde{f} \circ c$:

$$Q_{\pi}(l) = |\det [B_d(Z_{i_r}, Z_{i_s})]_{1 \leq r, s \leq d}| = |\det [B_d(Z_{\tilde{i}_r}, Z_{\tilde{i}_s})]_{1 \leq r, s \leq d}| = |\det [B_d(\tilde{Z}_{\tilde{i}_r}, \tilde{Z}_{\tilde{i}_s})]_{1 \leq r, s \leq d}| = Q_{\pi}(\tilde{f}).$$

This shows that the canonical image of $u_g$ in $U(\tilde{g})$ is precisely $u_g (\in U(\tilde{n}))$. Now the representation $\pi$ is trivial on $A$, so there exists an irreducible representation $\tilde{\pi}$ of $\tilde{N} = N/A$ such that $\tilde{\pi} \circ (c \mid N) = \pi$, and the orbit of $\tilde{\pi}$ is $\tilde{N} \tilde{f}$. But since $\tilde{g} \in \Omega_{\tilde{g}}$ we have $d\tilde{\pi}(u_g) = Q_{\pi}(\tilde{g})I$ by the induction hypothesis, and therefore $d\pi(u_g) = d\tilde{\pi}(c(u_g)) = d\tilde{\pi}(u_g) = Q_{\pi}(\tilde{g})I = Q_{\pi}(g)I$.

This ends case (a).

Case (b): Suppose that we are not in case (a) and that $\lambda_1 \neq 0$.

Write $Z_1 = X_1 + iY_1$ and set $a = iRX_1 + RX_1$. Then $a$ is an abelian ideal (of dimension 1 or 2), and $g \mid a = 0$ (since otherwise we would be in case (a)), and therefore $\langle g, Z_1 \rangle \neq 0$.

Since $G$ is exponential we can write $\lambda_1(X) = \alpha_1(X)(1 + ik_1)$, where $\alpha_1$ is a real linear form on $g$, and where $k_1$ is a real number.
Set \( \mathfrak{h} = \ker \lambda = \ker \alpha \). \( \mathfrak{h} \) is an ideal in \( \mathfrak{g} \) of codimension 1 with \( [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n} \subseteq \mathfrak{h} \), so the nilradical of \( \mathfrak{h} \) is \( \mathfrak{n} \). Clearly \( Z_{j} \in \mathfrak{h}_{\mathfrak{c}} \). Set \( p = \min \{ 1 \leq j \leq m \mid Z_{j} \notin \mathfrak{h}_{\mathfrak{c}} \} \). \( p \) is well-defined and \( p \geq 2 \). We observe that \( \mathfrak{p} \in J_{\mathfrak{c}} \). In fact, suppose \( \mathfrak{p} \notin J_{\mathfrak{c}} \). Then \( \mathfrak{z}_{p} \notin \mathfrak{g}_{p-1} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \), and therefore \( 0 = \langle g, [Z_{p}, f_{1}] \rangle = \langle Z_{g}, f_{1} \rangle = \langle f_{p-1}, g, f_{1} \rangle = \langle g, [f_{p-1}, f_{1}] \rangle = 0 \), which is a contradiction. Also \( 1 \mid J_{\mathfrak{c}} \) since otherwise \( Z_{1} \in (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \), and therefore

\[
0 = \langle g, [g, f_{1}] \rangle = \langle g, f_{1} \rangle = 0.
\]

We also note that \( \mathfrak{g}_{\mathfrak{c}} \subseteq \mathfrak{h} \), since otherwise \( \mathfrak{g} = \mathfrak{h} + \mathfrak{g}_{\mathfrak{c}} \) and therefore \( 0 = \langle \mathfrak{g} \mathfrak{g}, f_{1} \rangle = \langle \mathfrak{g}, f_{1} \rangle = 0 \).

Set \( \mathfrak{Z}_{j} = \mathfrak{Z}_{j} \) for \( 1 \leq j \leq p-1 \), \( \mathfrak{Z}_{j} = \mathfrak{Z}_{j+1} + c_{j+1} \mathfrak{Z}_{p} \) for \( p \leq j \leq m-1 \) and \( \mathfrak{Z}_{m} = \mathfrak{Z}_{p} \). Here \( c_{p}, p+1 \leq j \leq m \), is defined such that \( \mathfrak{Z}_{j} + c_{j} \mathfrak{Z}_{p} \in \mathfrak{h}_{\mathfrak{c}} \). This is possible since \( \mathfrak{C} \mathfrak{Z}_{p} \oplus \mathfrak{h}_{\mathfrak{c}} = \mathfrak{g}_{\mathfrak{c}} \).

Clearly \( \mathfrak{Z}_{1}, \ldots, \mathfrak{Z}_{m} \) is a basis in \( \mathfrak{g}_{\mathfrak{c}} \).

Set \( \mathfrak{f}_{j} = \mathfrak{C} \mathfrak{Z}_{1} \oplus \ldots \oplus \mathfrak{C} \mathfrak{Z}_{j} \). For \( 0 \leq j \leq p-1 \) we have that \( \mathfrak{f}_{j} = \mathfrak{f}_{j} \). For \( p-1 \leq j \leq m-1 \) we have \( \mathfrak{f}_{j} \oplus \mathfrak{C} \mathfrak{Z}_{p} = \mathfrak{f}_{j+1} \), hence

\[
\begin{align*}
\mathfrak{f}_{j} &= \mathfrak{f}_{j} \quad \text{for} \ 0 \leq j \leq p-1, \\
\mathfrak{f}_{j} &= \mathfrak{f}_{j+1} \cap \mathfrak{h}_{\mathfrak{c}} \quad \text{for} \ p-1 \leq j \leq m-1, \\
\mathfrak{f}_{m} &= \mathfrak{g}_{\mathfrak{c}}.
\end{align*}
\]

From this it follows that \( \mathfrak{f}_{p}, \mathfrak{f}_{0} = 0, \ldots, \mathfrak{m}, \) is a Jordan-Hölder sequence for \( \mathfrak{g}_{\mathfrak{c}} \) with \( \mathfrak{f}_{m-1} = \mathfrak{h}_{\mathfrak{c}} \). We claim it is of class \( (b) \). In fact, since \( \mathfrak{f}_{p-1} = \mathfrak{f}_{p} \cap \mathfrak{h}_{\mathfrak{c}} \) and \( \mathfrak{f}_{p-1} = \mathfrak{f}_{p-1} \) it follows that \( \mathfrak{f}_{p-1} = \mathfrak{f}_{p-1} \), and from this it is immediate that the claim is true. We thus have a new diagram

\[
\begin{align*}
\mathfrak{g}_{\mathfrak{c}} &= \mathfrak{f}_{m} = \mathfrak{f}_{m-1} = \cdots = \mathfrak{f}_{1} = \mathfrak{f}_{0} = \{ 0 \}.
\end{align*}
\]

The objects defined relative to this new Jordan-Hölder sequence are designated \( \mathfrak{f}_{p}, \mathfrak{e}, \) etc.

For \( 1 \leq j \leq p-1 \) we clearly have \( j \mathfrak{f}_{\mathfrak{e}} = j \mathfrak{e} \). Furthermore \( p \mathfrak{e} = \mathfrak{e} \) (see above) and \( m \mathfrak{e} = \mathfrak{h} \). In fact, if \( m \mathfrak{e} = \mathfrak{e} \), then \( Z_{p} = \mathfrak{Z}_{m} \mathfrak{e} \mathfrak{f}_{m-1} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} = \mathfrak{h}_{\mathfrak{c}} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \) and therefore

\[
0 = \langle Z_{g}, f_{1} \rangle = \langle h_{g}, f_{1} \rangle = 0.
\]

For \( p+1 \leq j \leq m \) we have

\[
\begin{align*}
\mathfrak{j} \mathfrak{e} &= Z_{g} \mathfrak{e} \mathfrak{f}_{j-1} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \Rightarrow Z_{g} \mathfrak{e} \mathfrak{f}_{j-2} + \mathfrak{C} Z_{p} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \\
&\Rightarrow Z_{j-1} \mathfrak{e} \mathfrak{f}_{j-2} + \mathfrak{C} Z_{p} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}} \Rightarrow Z_{j-1} \mathfrak{e} \mathfrak{f}_{j-2} + (\mathfrak{g}_{\mathfrak{c}})_{\mathfrak{c}}
\end{align*}
\]

(since \( \mathfrak{g}_{\mathfrak{c}} \subseteq \mathfrak{h} \)) \( \Rightarrow j-1 \mathfrak{e} \mathfrak{f}_{\mathfrak{e}} \). Therefore, if \( j_{e} = p \) we have \( j_{e} = j_{h} \) for \( 1 \leq h \leq \alpha - 1, j_{h} + 1 = j_{h+1} \) for \( \alpha \leq h \leq d - 1 \) and \( j_{d} = m \), so

\[
\begin{align*}
\mathfrak{Z}_{j_{h}} &= Z_{j_{h}} \quad \text{for} \ 1 \leq h \leq \alpha - 1, \\
\mathfrak{Z}_{j_{h+1}} &= Z_{j_{h+1}} + c_{j_{h+1}} Z_{j_{h}} \quad \text{for} \ \alpha \leq h \leq d - 1, \\
\mathfrak{Z}_{j_{d}} &= Z_{j_{d}}.
\end{align*}
\]
Therefore, letting $C = [c_{rs}]_{1 \leq r, s \leq d}$ be the $d \times d$-matrix:

$$C = \begin{bmatrix}
1 \\
0 & 1 \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix},$$

where the empty entries are 0, we have $Z_j = \sum_{r=1}^d c_{rs} Z_{j, r}$, and therefore $M_\delta(l) = CM_\delta(l)C$, with $\delta = J_\epsilon$. Now $\det C = (-1)^d$, so $\det M_\delta(l) = \det M_\delta(l)$, and $Q_\delta(l) = Q_\delta(l)$, and therefore $u_{\delta} = u_{\delta}$. The conclusion of this is then that we can assume that $f_{m-1} = b_\epsilon$, and this assumption will be in effect from now on. We then have:

$$Q_{\epsilon}(l) = | \det \left[ B(\delta, Z_{j, r}, Z_{j, s}) \right]_{1 \leq r, s \leq d} |$$

$$= | \sum_{\sigma \in S_\delta} \text{sign} \sigma < l, [Z_{j_1}, \ldots, Z_{j_{\delta+1}}] > \ldots < l, [Z_{j_{\delta-1}}, \ldots, Z_{j_{\epsilon}}] > |$$

where $S_\delta$ is the set of elements $\sigma \in S_\delta$ with $\sigma(1) = d$, $\sigma(d) = 1$.

Set $g_0 = g \mid h$. Then $g = g_0 \mid n$. We designate the objects associated with the group $H = \exp b$, and the class (b) Jordan-Hölder sequence $b_\epsilon = f_{m-1} \rightarrow \ldots \rightarrow f_1 \rightarrow f_0 = \{0\}$ by $j_0^0$, etc. We have $(b_{g_0}h) = (g \mid h) \oplus CZ_1$, so $j_0^0 = J_\epsilon \setminus \{1, m\}$, and therefore

$$j_0^0 = \{ j_0^0 < \ldots < j_{d-2}^0 \}$$

with $j_h^0 = j_{h+1}$ for $1 \leq h \leq d-2$, so we have for left:

$$Q_{\epsilon}(l) = | \det \left[ B(\delta, Z_{j, r}, Z_{j, s}) \right]_{1 \leq r, s \leq d-2} |$$

$$= | \sum_{\sigma \in S_{d-2}} \text{sign} \sigma < l, [Z_{j_1}, \ldots, Z_{j_{\delta+1}}] > \ldots < l, [Z_{j_{d-1}}, Z_{j_{d-2}}] > |$$

and comparing with the result above we get $Q_\delta(l) = | < l, W > |^2 Q_{\epsilon}(l)$, where $W = [Z_1, Z_m]$ and $l_0 = l \mid h$. Now since $W$ is central in $b_\epsilon$ and since $P_{l_0} = \langle l, W \rangle$ we find that $\delta Q_\delta(l) = -P_{l_0}P_{l_0}(l)^{d-2} Q_{\epsilon}(l)$, and therefore $u_{\delta} = -WW_{\epsilon}u_{\delta}$ by Lemma 1.2.1. By the induction hypothesis we have that $d\pi(u_{\epsilon}) = Q_\delta(l)$, and noting that

$$d\pi(W) = i < g, W > I, d\pi(W) = i < g, W > I$$

we finally get $d\pi(u_{\epsilon}) = | < g, W > |^2 d\pi(u_{\epsilon}) = | < g, W > |^2 Q_{\epsilon}(g)I = Q_\delta(g)I$. This settles case (b).
Case (c): Suppose we are not in case (a) and (b) and that \( \lambda_2 \neq 0 \).

Again we have \( \langle g, Z_1 \rangle \neq 0 \) and, moreover, \( \overline{f}_1 = f_1 \) (since \( f_1 \) is a central ideal in \( g_c \)). We write \( [X, Z_2] = \lambda_2(X)Z_2 + \gamma(X)Z_1 \), \( X \in g \), where \( \gamma \) is a (complex valued) linear form on \( g \). The linear form \( \gamma \) has the form \( \gamma(X) = \gamma_1(X) + \gamma_2(X) \), where \( \gamma_1, \gamma_2 \) are real linear forms on \( g \). We extend \( \lambda_2, \gamma \) to complex linear forms on \( g_c \) such that we have

\[
[Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1 \quad \text{for} \quad Z \in g_c.
\]

We note the formula

\[
(2.2.2) \quad \gamma([Z, W]) = \gamma(Z)\lambda_2(W) - \gamma(W)\lambda_2(Z)
\]

for \( Z, W \in g_c \), which we get by a simple application of the Jacobi identity.

Since \( G \) is exponential we can write \( \lambda_2(X) = \alpha_2(X)(1 + ik_2) \), where \( \alpha_2 \) is a real linear form on \( g \) and where \( k_2 \) is a real number.

We then distinguish three subcases: (c1): rank \( (\alpha_2, \gamma_1, \gamma_2) = 3 \), (c2): rank \( (\alpha_2, \gamma_1, \gamma_2) = 2 \) and (c3): rank \( (\alpha_2, \gamma_1, \gamma_2) = 1 \).

Case (c1): Set \( h = \ker \gamma_1 \cap \ker \gamma_2 \) (= ker \( \gamma | g \)). It follows from the formula (2.2.2) that \( h \) is a subalgebra in \( g \) and its codimension is 2. We observe that \( Z \in b_c \) if and only if \( \gamma(Z) = 0 \) and \( \gamma(Z) = 0 \). Set \( b_0 = \ker \lambda_2 | b = \ker \alpha_2 | b = \ker \text{ad} Z_2 | g \). \( b_0 \) is an ideal in \( g \) of codimension 3. That \( b_0 \) is an ideal in \( g \) follows from the fact that

\[
b_0 = \ker \gamma \cap \ker \lambda_2 \cap g
\]

and by applying the formula (2.2.2).

Let \( m \) be the nilradical of \( b_0 \). Since \( b_0 \) is an ideal we have that \( m = n \cap b_0 = n \cap h \). Observe that \( \dim n/m = 2 \). In fact, pick \( W \in b \setminus b_0 \). Then we have that

\[
\gamma([Z, W]) = \lambda_2(W)\gamma(Z) \quad \text{for} \quad Z \in g_c,
\]

and therefore \( \gamma([Z, W]) = \lambda_2(W)\gamma(Z) \). Choosing \( Z \) such that \( \gamma(Z) = 1 \), \( \gamma(Z) = 0 \) and \( Z' \) such that \( \gamma(Z') = 0 \), \( \gamma(Z') = 1 \) we get that

\[
\gamma([Z, W]) = \lambda_2(W) = 0, \quad \gamma([Z, W]) = 0, \quad \gamma([Z', W]) = \lambda_2(W) = 0, \quad \gamma([Z', W]) = 0,
\]

and this shows that \( [Z, W], [Z', W] \) is a basis in \( n_c \) (mod \( m_c \)).

We claim that \( \overline{f}_2 + f_2 \). In fact, we have \( [Z, Z_2] = \lambda_2(Z)Z_2 + \gamma(Z)Z_1 \) for all \( Z \in g_c \), and therefore \( [Z, \overline{Z}_2] = \lambda_2(Z)\overline{Z}_2 + \gamma(Z)\overline{Z}_1 \). Since \( \lambda_2 \) does not vanish on \( h_c \) we have that \( [b_c, \overline{f}_2] = \overline{CZ}_2 \) and \( [b_c, f_2] = CZ_2 \). Therefore, if \( \overline{f}_2 = f_2 \), then \( CZ_2 = \overline{CZ}_2 \), hence \( \gamma(Z) = 0 \) implies that \( \gamma(Z) = 0 \), so \( k_2 \) is the set of \( Z \in g_c \) such that \( \gamma(Z) = 0 \), contradicting the fact that \( \text{codim } h = 2 \). We conclude that \( \overline{f}_2 \neq f_2 \), and therefore that \( \overline{f}_1 = f_1 \) and \( f_3 = \overline{f}_3 \). In particular \( \overline{Z}_2 \neq f_2 \).

We have seen that \( Z_1, Z_2, \overline{Z}_2 \) span \( f_3 \). Now since \( \lambda_2(Z_2) = 0 \) we have that \( \alpha_2(Z_2) = 0 \), and this means that \( [f_3, f_2] \subset f_1 \). We then distinguish two possibilities: case (c11): \( [f_3, f_2] = 0 \) and case (c12): \( [f_3, f_2] = f_1 \).
Set $f_0 = f | m = g | m$, and let $\pi_0$ be the irreducible representation of $M = \exp m$ corresponding to $M_0$.

Case (c11): (i) It is our first aim to show that $u_\epsilon \in \mathcal{U}(m_c)$, and that $d\pi_0(u_\epsilon) = Q_\epsilon(g)I$. We start by noting that we can assume that $\langle g, Z_2 \rangle = 0$; in fact, if necessary replace $Z_2$ by $Z_2 - cZ_1$; this does not change $e$, $Q_\epsilon$, etc. (it changes $\gamma = \gamma_1 + i\gamma_2$, though, but does not affect $h_0$ and rank $(\alpha_2, \gamma_1, \gamma_2)$).

Set $p = \min \{ 1 \leq k \leq m | Z_k \notin h_C \}$. $p$ is well-defined, and $4 \leq p \leq m - 1$, since $Z_1, Z_2, Z_3 \in h_C$, and since the codimension of $h_C$ is 2. Set $q = \min \{ 1 \leq k \leq m | Z_k \notin h_C \oplus h_C \}$, which is well-defined and $4 \leq p + 1 \leq q \leq m$ (so dim $g \geq 6$).

We first note $2, 3 \in I_\epsilon$. In fact, if $2 \notin I_\epsilon$, then $Z_2 \in f_1 + (g_q)_C$, and therefore

$$\gamma(Z) \langle g, Z_1 \rangle = \langle g, [Z, Z_2] \rangle = \langle Z_2g, Z \rangle = 0 \text{ for all } Z \in g_C$$

which is a contradiction. So $2 \in I_\epsilon$. If $3 \notin I_\epsilon$, then $Z_2 \in f_1 + (g_q)_C$, i.e. $Z_2 = aZ_2$ (mod $(g_q)_C$), $a \in C$. But then

$$\overline{\gamma(Z)} \langle g, Z_1 \rangle = \langle g, [Z, Z_2] \rangle = \langle Z_2g, Z \rangle = a \langle Z_2g, Z \rangle = \alpha\gamma(Z) \langle g, Z_1 \rangle$$

which contradicts the fact that codim $h = 2$, so $3 \in I_\epsilon$. We also note that $1 \notin I_\epsilon$, since $f_1 \subseteq (g_q)_C$.

Next we note that $p, q \in I_\epsilon$. In fact, if $p \notin I_\epsilon$, then $Z_p \in h_C + (g_q)_C$ and $Z_q \in h_C + (g_q)_C$, and therefore

$$-\gamma(Z_p) \langle g, Z_1 \rangle = \langle g, [Z_2, Z_p] \rangle = \langle Z_pg, Z_2 \rangle = \langle b_Cg, Z_2 \rangle$$

$$= \langle g, [b_C, Z_2] \rangle = \langle g, CZ_2 \rangle = 0,$$

so $\gamma(Z_p) = 0$ and similarly $\gamma(Z_q) = 0$ implying that $Z_p \notin h_C$, which is a contradiction. Therefore $p \in I_\epsilon$. Suppose then that $q \notin I_\epsilon$. Then $Z_q \in h_C + (g_q)_C$, i.e. there exists $a \in C$ with $Z_q = aZ_p$ (mod $(h_C + (g_q)_C))$. But then

$$-\gamma(Z_q) \langle g, Z_1 \rangle = \langle g, [Z_2, Z_q] \rangle = \langle Z_qg, Z_2 \rangle = a \langle Z_qg, Z_2 \rangle = -a\alpha\gamma(Z_p) \langle g, Z_1 \rangle,$$

from which $\gamma(Z_q) = \alpha\gamma(Z_p)$. Similarly $\overline{\gamma(Z_q)} = \alpha\overline{\gamma(Z_p)}$. Now consider the linear map from $g_C$ to $C^2$ given by $Z \rightarrow (\gamma(Z), \overline{\gamma(Z)})$. The kernel is $h_C$, so it is surjective since codim $h = 2$. But $Z_p, Z_q$ is a basis for $g_C$ (mod $h_C$), and we have just shown that the images of $Z_p$ and of $Z_q$ are linearly dependent; in fact, $(\gamma(Z_q), \overline{\gamma(Z_q)}) = a(\gamma(Z_p), \overline{\gamma(Z_p)})$. But this is a contradiction, and we conclude that $q \notin I_\epsilon$.

Define $Z_j = Z_j$ for $1 \leq j \leq p - 1$, $Z_j = Z_{j+1} + a_{j+1}Z_p$ for $p \leq j \leq q - 2$ (empty if $q = p + 1$), $Z_j = Z_{j+2} + a_{j+2}Z_p + b_{j+2}Z_q$ for $q - 1 \leq j \leq m - 2$, $Z_{m-1} = aZ_p + bZ_q$, $Z_m = aZ_p + bZ_q$, where $a_{p+1}, \ldots, a_{q-1}, a_{q+1}, \ldots, a_m, b_{q+1}, \ldots, b_m$ has been picked such that $Z_j \in h_C$, $1 \leq j \leq m - 2$; this is possible since $g_C = h_C \oplus CZ_q \oplus CZ_p$. The numbers $a, b, a', b' \in C$ has been selected such that $ab' - a'b = 1$, and such that $\langle g, [Z_{m-1}, Z_2] \rangle = 0$, $\langle g, [Z_{m-1}, Z_3] \rangle = 0$, $\langle g, [Z_m, Z_2] \rangle + 0$ which is possible by a reasoning as above. Clearly $Z_1, \ldots, Z_m$ is a basis for $g_C$. Set $f_j = CZ_1 \oplus \cdots \oplus CZ_p$. For $0 \leq j \leq p - 1$ we have
that \( \hat{f}_j \leq f_j \) for \( p - 1 \leq j \leq q - 2 \) we have that \( \hat{f}_j \oplus CZ_p = f_{j+1} \) and for \( q - 2 \leq j \leq m - 2 \) we have that \( \hat{f}_j \oplus CZ_p \oplus CZ_q = f_{j+2} \). Also, \( \hat{f}_{m-2} = h_C \), \( \hat{f}_m = g_C \). We thus have

\[
\begin{align*}
\hat{f}_j & = f_j \quad \text{for} \quad 0 \leq j \leq p - 1, \\
\hat{f}_j & = f_{j+1} \cap h_C \quad \text{for} \quad p - 1 \leq j \leq q - 2, \\
\hat{f}_j & = f_{j+2} \cap h_C \quad \text{for} \quad q - 2 \leq j \leq m - 2, \\
\hat{f}_m & = g_C.
\end{align*}
\]

From this it follows that

\[
b_C = \hat{f}_{m-2} \supset \ldots \supset \hat{f}_1 \supset \hat{f}_0 = \{ 0 \}
\]

is a Jordan-Hölder sequence for \( h_C \) (but note that \( \hat{f}_0, \ldots, \hat{f}_m \) is not necessarily a Jordan-Hölder sequence for \( g_C \), since \( h \) is not necessarily an ideal in \( g \)). We claim it is a Jordan-Hölder sequence of class \( (b) \). To see this, observe that \( \hat{f}_{p-1} = \hat{f}_{p-1} \), since \( \hat{f}_{p-1} = \hat{f}_{p-1} \) and \( \hat{f}_{p-1} = f_p \cap h_C \), and \( \hat{f}_{q-2} = \hat{f}_{q-2} \), since \( \hat{f}_{q-2} = \hat{f}_{q-1} \cap h_C \) and \( \hat{f}_{q-2} = f_p \cap h_C \), and from this it follows easily that \( \hat{f}_p = 0, \ldots, m - 2 \) is of class \( (b) \).

Write \( e = \{ j_1 < \ldots < j_d \} \), and let \( j_\alpha = p \), \( j_\beta = q \) with \( 1 \leq \alpha < \beta \leq d \). Define the set \( \hat{J}_e = \{ \hat{j}_1 < \ldots < \hat{j}_d \} \) by setting \( \hat{j}_1 = j_1, \ldots, \hat{j}_{\alpha-1} = j_{\alpha-1}, \hat{j}_\alpha = j_{\alpha+1} - 1 \) for \( \alpha \leq h \leq \beta - 2 \), \( \hat{j}_\alpha = j_{\alpha+2} - 2 \) for \( \beta - 1 \leq h \leq d - 2 \), \( \hat{j}_{d-1} = m - 1 \), \( \hat{j}_d = m \). We then have

\[
\begin{align*}
\hat{Z}_{j_h} & = Z_{j_h} \quad \text{for} \quad 1 \leq h \leq \alpha - 1, \\
\hat{Z}_{j_h} & = Z_{j_{h+1}} + a_{j_\alpha} Z_{j_h} \quad \text{for} \quad \alpha \leq h \leq \beta - 2, \\
\hat{Z}_{j_h} & = Z_{j_{h+1}} + a_{j_\alpha} Z_{j_h} + b_{j_\beta} Z_{j_h} \quad \text{for} \quad \beta - 1 \leq h \leq d - 2, \\
\hat{Z}_{j_{d-1}} & = a Z_{j_d} + b Z_{j_d}, \\
\hat{Z}_{j_d} & = a' Z_{j_d} + b' Z_{j_d}.
\end{align*}
\]

Therefore, letting \( C = [c_{rs}]_{1 \leq r, s \leq d} \) be the \( d \times d \)-matrix:

\[
\begin{bmatrix}
1 & \alpha & \beta - 2 & \beta - 1 & d - 2 & d - 1 & d \\
\alpha & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{bmatrix}
\]

where the empty entries are zero, we have \( \hat{Z}_{j_s} = \Sigma_{r=1}^d c_{rs} Z_{j_s} \) and \( \hat{M}_d(l) = C M_d(l) C \), where
\( \tilde{M}_d(l) \) is the matrix \([B_j(\tilde{Z}_j, \tilde{Z}_j)]_{1 \leq r, s \leq d} \). Now \( \det C = (-1)^{s+q} \), and therefore we have for \( \langle l, Z_2 \rangle = 0 \):

\[
Q_e(l) = | \det M_d(l) | = | \det \tilde{M}_d(l) | = | \sum_{\sigma \in S_d} \text{sign } \sigma \langle l, [\tilde{Z}_j, \tilde{Z}_{j(d)}] \rangle | = | \sum_{\sigma \in S_d} \text{sign } \sigma \langle l, [\tilde{Z}_j, \tilde{Z}_{j(1)}] \rangle | = | \langle l, [\tilde{Z}_j, \tilde{Z}_m] \rangle |^2 \]

where \( S_d^+ \) is the set of permutations \( \sigma \) in \( S_d \) such that \( \sigma(1) = d \), \( \sigma(2) = d-1 \), \( \sigma(d-1) = 2 \), and \( \sigma(d) = 1 \).

Set \( g_0 = g | h \), and let \( \tilde{J}^0_{g_0} \) etc. designate the objects defined relative to the Jordan-Hölder sequence \( \tilde{f}_0 < \tilde{f}_1 < \ldots < \tilde{f}_{d-2} = b \). Since clearly \( g_0 \subset b \), and \( (b_{g_0})_C = (g_0)_C + CZ_2 + CZ_3 \) we find that \( 1, 2, 3 \notin \tilde{J}^0_{g_0} \); and for \( 4 \leq j \leq p-1 \) we find \( j e \tilde{J}^0_{g_0} \Rightarrow j \in J_g \). For \( p+1 \leq j \leq q-2 \) we have

\[
J_h \Leftrightarrow Z_\tilde{f}_j = \tilde{f}_j - (a_j)(g_0)_C \Leftrightarrow Z_\tilde{f}_j - (a_j)(g_0)_C \Leftrightarrow \tilde{f}_j - 1 \in \tilde{S}_{g_0},
\]

so \( j \in J_g \Leftrightarrow j - 1 \in \tilde{J}^0_{g_0} \). For \( q+1 \leq j \leq q \) we have

\[
J_h \Leftrightarrow Z_\tilde{f}_j = \tilde{f}_j - (a_j)(g_0)_C \Leftrightarrow Z_\tilde{f}_j - 1 = \tilde{f}_j - 1 = \tilde{f}_j \cdot (b_0)_C \Leftrightarrow j - 1 \in \tilde{J}^0_{g_0},
\]

so \( j \in J_g \Leftrightarrow j - 1 \in \tilde{J}^0_{g_0} \). Therefore, if \( \tilde{g}_0 = \tilde{J}^0_{g_0} = \{ \tilde{f}_1 < \ldots < \tilde{f}_d \} \) we find that \( \tilde{J}^0_h = \tilde{f}_{h+2} \) for \( 1 \leq h \leq \alpha - 3 \), \( \tilde{J}^0_h + 1 = \tilde{f}_{h+3} \) for \( \alpha - 2 \leq h \leq \beta - 4 \), \( \tilde{J}^0_h + 2 = \tilde{f}_{h+4} \) for \( \beta - 3 \leq h \leq d - 4 \), and comparing with the definition of \( \tilde{f}_h \) we find that \( \tilde{J}^0_h = \tilde{f}_{h+2} \) for \( 1 \leq h \leq d - 4 \). Using this we get for \( l_0 \in \mathfrak{h}' \):

\[
Q_{e_0}(l_0) = | \sum_{\sigma \in S_{d-4}} \text{sign } \sigma \langle l_0, [\tilde{Z}_{\tilde{f}_0}, \tilde{Z}_{\tilde{f}_1}] \rangle | \ldots \langle l_0, [\tilde{Z}_{\tilde{f}_{d-2}}, \tilde{Z}_{\tilde{f}_{d-1}}] \rangle | = | \sum_{\sigma \in S_{d-4}} \text{sign } \sigma \langle l_0, [\tilde{Z}_{\tilde{f}_j}, \tilde{Z}_{\tilde{f}_{j(1)}}] \rangle | \ldots \langle l_0, [\tilde{Z}_{\tilde{f}_{d-2}}, \tilde{Z}_{\tilde{f}_{d-1}}] \rangle | = | \sum_{\sigma \in S_{d-4}} \text{sign } \sigma \langle l_0, \tilde{Z}_{\tilde{f}_j}, \tilde{Z}_{\tilde{f}_{j(1)}} \rangle | \ldots \langle l_0, \tilde{Z}_{\tilde{f}_{d-2}}, \tilde{Z}_{\tilde{f}_{d-1}} \rangle |,
\]

and comparing with what we saw above we find for \( \langle l, Z_2 \rangle = 0 \) and \( l_0 = l | h \):

\[
(*) \quad Q_e(l) = | \langle l, [\tilde{Z}_2, \tilde{Z}_m] \rangle |^2 | \langle l, [\tilde{Z}_3, \tilde{Z}_{m-1}] \rangle |^2 Q_{e_0}(l_0).
\]

Let us now observe that the nilradical of \( h \) is \( m \). In fact, since \( Z_2 \in h, \lambda_2 | h \) is a root for \( h \), and therefore the nilradical of \( h \) is contained in \( b_0 \) and consequently it is precisely \( m \).

Write \( Z_2 = X_2 + iY_2 \) and set \( b = \mathbb{R}X_2 \oplus \mathbb{R}Y_2 \). Then \( b \) is an ideal in \( h \), and \( g | b = 0 \). Let \( c : b \to h | b = \tilde{h} \) be the coset map and define \( \tilde{g}_0 \in \tilde{h} \) by \( \tilde{g}_0 \cdot c = g_0 \).

We now claim that \( u e U(m) \), i.e. that \( Q_e \) only depends on its restriction to \( h \) (and therefore to \( m \)). Assuming for a moment this claim to be true, we consider \( Q_e \) as a polynomial function on \( \tilde{h} \) and get for \( \tilde{l}_0 \in \tilde{h} \) (using the formula \( (*) \)):

\[
Q_{e_0}(\tilde{l}_0) = | \langle l_0, W_1 \rangle |^2 | \langle l_0, W_2 \rangle |^2 Q_{e_0}(\tilde{l}_0),
\]

where \( W_1 = c([\tilde{Z}_2, \tilde{Z}_m]), W_2 = c([\tilde{Z}_3, \tilde{Z}_{m-1}]) \). Now since \( W_1, W_1, W_2, W_2 \) are central.
in $\bar{b}$ and since $P_{W_1}(\bar{b}) = \langle W_1, \bar{b} \rangle$, $P_{W_2}(\bar{b}) = \langle W_2, \bar{b} \rangle$, and similarly for $W_3$, we find that $Q_e(\bar{b} \circ c) = P_{W_1}(\bar{b})P_{W_2}(\bar{b})P_{W_3}(\bar{b})Q_e(\bar{b} \circ c)$, and therefore
\[ c(u_e) = W_1 W_2 W_3 c(u_{e_0}) \]
by Lemma 1.2.1.

Since $b$ is an abelian ideal in $h_0$ and since $\langle f_0, b \rangle = 0$ there exists a representation $\pi_0$ of $M = M/B$, $B = \text{exp} b$, such that $\pi_0 \circ c = \pi_0$. Using the induction hypothesis we then get
\[ d\pi_0(u_e) = d\pi_0(c(u_e)) = d\pi_0(W_1 W_2 W_3 c(u_{e_0})) = |\langle f_0, W_1 \rangle|^2 |\langle f_0, W_2 \rangle|^2 d\pi_0(u_{e_0}) = |\langle g, [Z_2, Z_m] \rangle|^2 |\langle g, [Z_3, Z_{m-1}] \rangle|^2 Q_e(g_{e_0})I = Q_e(g)I. \]

We have thus shown that $d\pi_0(u_e) = Q_e(g)I$. This ends case (c11) (i), except for the fact that we have to prove the claim from above:

\textbf{Proof of claim:} We shall prove that $Q_e(l)$ only depends on the restriction of $l$ to $h$. If all $Z_{r \leq p \leq d-2}$ belong to $(h_0)c$, then the result is clear (because $h_0$ is an ideal). Suppose then that there exists $3 \leq r \leq d-2$ such that $Z_r \notin (h_0)c$, and let $\rho$ be the smallest such $r$. We then have $h_c = C(Z_r \oplus (h_0)c)$, and $g_e = C(Z_r \oplus \ker \alpha_2$. Set $Y_r = Z_r + c_r Z_{r-1}, r = 1, \ldots, d$, where $c_r$ is defined such that $Y_r \in \ker \alpha_2$ for $r \neq \rho$, and where $c_\rho = 0$. We then have $Y_r \notin (h_0)c$ for $1 \leq r \leq d-2$, $r \neq \rho$, while $Y_\rho \notin (h_0)c$ and $Y_\rho Y_{d-1} \in \ker \alpha_2$. We also have $[Y_\rho, Z_2] \in C_1, [Y_\rho, Z_3] = 0, [Y_{d-1}, Z_3] \in C_1, [Y_{d-1}, Z_2] = 0$. Letting $C = [c_{\rho s}]$, where $c_{\rho s}$ is the $\rho \times s$-matrix given by:
\[
\begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & \cdots & & \\
& & & 1 & \\
& & & & 1
\end{pmatrix}
\]
the empty entries meaning zero, we have $Y_s = \sum_{r=1}^d c_{r\rho} Z_r$, so, setting $N(l) = [B(Y_\rho, Y_s)]$, we get $N(l) = C_M(l)c$, from which $Q_e(l) = |\det M_e(l)| = |\det M_e(l)| = |\det N(l)|$. Therefore
\[ Q_e(l) = |\sum_{(e, s)} \text{sgn } P_{\sigma}(l) |, \]
where we have set $P_{\sigma}(l) = \langle l, [Y_1, Y_{e(1)}] \rangle \cdots \langle l, [Y_\rho, Y_{e(\rho)}] \rangle$.

Define the following subsets of $S_4$:
\[
S_4^{(1)} = \{ \sigma | \sigma(1) = \rho, \sigma(2) = d-1, \sigma(d-1) = 2, \sigma(d) = 1 \}, \\
S_4^{(2)} = \{ \sigma | \sigma(1) = d, \sigma(2) = \rho, \sigma(d-1) = 2, \sigma(d) = 1 \}, \\
S_4^{(3)} = \{ \sigma | \sigma(1) = d, \sigma(2) = d-1, \sigma(\rho) = 1, \sigma(d-1) = 2, \sigma(d) = \rho \}, \\
S_4^{(4)} = \{ \sigma | \sigma(1) = 1, \sigma(2) = 1, \sigma(d) = \rho, \sigma(\rho) = \rho \}, \\
S_4^{(5)} = \{ \sigma | \sigma(\rho) + d \wedge \sigma(d) - 1 \wedge \rho \neq \sigma(d-1) \}, \\
S_4^{(6)} = S_4 \setminus \bigcup_{j=1}^5 S_4^{(j)}. \]
We then assert that $P_{\sigma}=0$ if $\sigma \in \mathcal{S}_d^d$. In fact, observe first that

$$P_{\sigma} \neq 0 \implies (\sigma(1) = \rho \lor \sigma(1) = d) \land (\sigma(2) = \rho \lor \sigma(2) = d-1) \land (1 = \sigma(\rho) \lor 1 = \sigma(d)) \land (2 = \sigma(\rho) \lor 2 = \sigma(d-1)).$$

Therefore, if $P_{\sigma} \neq 0$ and if $\sigma \in \mathcal{S}_d^d$ with e. g. $\sigma(\rho) = d$, then $\sigma(1) = \rho$, $\sigma(2) = d-1$, $\sigma(d) = 1$, $\sigma(d-1) = 2$, so $\sigma \in \mathcal{S}_d^d$. Similarly, if $\sigma \in \mathcal{S}_d^d$ with $\sigma(\rho) = d-1$, then $P_{\sigma} \neq 0 \implies \sigma \in \mathcal{S}_d^d$, etc. This shows our assertion.

We next assert that $P(l) = \sum_{\sigma \in \mathcal{S}_d^d} \sum_{i=1}^d \text{sign} \sigma P_{\sigma}(l) = 0$. To see this, define the permutations $\tau_1, \tau_2, \tau_3, \tau_4$ in $S_d$ by $\tau_1 = \text{identity}$, $\tau_2(1) = \rho$, $\tau_2(2) = 1$, $\tau_2(\rho) = 2$, $\tau_3(1) = \rho$, $\tau_3(\rho) = d$, $\tau_4(1) = \rho$, $\tau_4(\rho) = d-1$, $\tau_4(d-1) = 1$, all other elements left fixed. It is then immediate to verify that the map $\sigma \to \sigma \circ \tau_j, j = 1, 2, 3, 4$, defines a bijection between $\mathcal{S}_d^1$ and $\mathcal{S}_d^4$, and since $\tau_j$ are even permutations we get

$$P(l) = \sum_{\sigma \in \mathcal{S}_d^1} \sum_{i=1}^d \text{sign} \sigma P_{\sigma}(l).$$

Now for $\sigma \in \mathcal{S}_d^1$ we have

$$\sum_{i=1}^d \sum_{\sigma \in \mathcal{S}_d^1} \text{sign} \sigma P_{\sigma}(l) = \prod_{i=1}^d \frac{1}{\rho_i} \langle l, [Y_{\rho_i}, Y_{\sigma}(l)] \rangle \prod_{i=1}^{d-1} \frac{1}{\rho_i} \langle l, [Y_{\rho_i}, Y_{\sigma(\tau_i(l))}] \rangle,$$

and a direct computation shows that

$$\sum_{i=1}^d \prod_{i=1}^{d-1} \langle l, [Y_{\rho_i}, Y_{\sigma(\tau_i(l))}] \rangle = 0$$

for all $l \in \mathfrak{g}$. This shows that $P \equiv 0$, and therefore we have

$$Q_{\sigma}(l) = \sum_{\sigma \in \mathcal{S}_d^1} \text{sign} \sigma P_{\sigma}(l).$$

But we clearly have that $P_{\sigma}(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$ if $\sigma \in \mathcal{S}_d^d$, because all $[Y_{\rho_i}, Y_{\sigma}(l)]$, $r = 1, \ldots, d$, then belong to $\mathfrak{h}$ (we use here that $\mathfrak{h}$ is a subalgebra and that $\mathfrak{h}_0$ is an ideal). This proves our claim and ends (i).

(ii) We now apply (i) to the same Jordan-Hölder sequence $\{\mathfrak{h}\}$ but to another basis $Z_j \in \mathfrak{g}_j \backslash \mathfrak{h}_j$ (whereby $\mathfrak{h}_0$ and therefore $\mathfrak{m}$ are not changed), and we get similarly that $\left. d\pi_0(u'_e) = Q'_e(g)I \right|$, where $Q'_e, u'_e$ are the objects associated with this new basis. Setting in particular $Z_j' = \text{Ad}(s)Z_j$, we get $u'_e = \text{Ad}(s)u_e$ and $Q'_e(l) = Q_e(s^{-1}l)$ for $s \in G$, and therefore $\left. d\pi_0(\text{Ad}(s)u_e) = Q_e(s^{-1}g)I = |\Lambda_0(s)|^2 Q_e(g)I \right|$. Now since $Z_1, Z_2, Z_2 \in \mathfrak{g}$ it follows that $\mathfrak{n} < \mathfrak{m}$ and from this we get that

$$(m_{\rho}(e))_c = (m_{\rho})_c \oplus C\mathbb{Z}_2 \oplus C\mathbb{Z}_2.$$

It follows that a polarization in $\mathfrak{m}$ at $f_0$ is also a polarization in $\mathfrak{n}$ at $f$, hence $\pi = \text{ind}_{\mathfrak{m} \cap \mathfrak{n}} \pi_0$.

Let then $\varphi$ be a differentiable vector in $L^2(N, \pi_0)$, the space of the induced representation $\pi = \text{ind}_{\mathfrak{m} \cap \mathfrak{n}} \pi_0$. We have $d\pi_0(\varphi)(s) = d\pi_0(\text{Ad}(s^{-1})u_e)\varphi(s) = Q_e(g)\varphi(s), s \in \mathfrak{N}$, so $d\pi_0(\varphi) = Q_e(g)I$. This ends case (c11).
Case (c12): (i) As in case (c11) we start by showing that $u \in U(m)$ and that $d \pi_0 (u) = Q_d (g) I$, and we can assume that $\langle g, Z_2 \rangle = 0$.

Since $[f_3, f_2] = f_1$, we have that $Z_2$, $Z_3 \notin h_C$. Therefore $g_C = C Z_2 \oplus C Z_3 \oplus h_C$. Just like in case (c11) we see that 2, 3 are $F_d$. Define $\xi_1 = Z_1$, $\xi_j = Z_j + a_j Z_2 + b_j Z_3$ for $2 \leq j \leq m - 2$, $\xi_{m-1} = Z_2$, $\xi_m = Z_3$, where $a_1$, $a_2$, $\ldots$, $a_m$, $b_1$, $b_2$, $\ldots$, $b_m$ have been picked such that $\xi_j \in h_C, 1 \leq j \leq m - 2$. Clearly $\xi_1$, $\ldots$, $\xi_m$ is a basis for $g_C$. Set $\xi_j = C Z_1 \oplus \ldots \oplus C Z_j$. We have that $\tilde{f}_1 = f_1$ and $\tilde{f}_j \oplus C Z_2 \oplus C Z_3 = f_{j+2}$ for $1 \leq j \leq m - 2$. Also $f_{m-2} = h_C$, $f_m = g_C$. We have that $\xi_1 = f_1$ and $\xi_j \oplus C Z_2 \oplus C Z_3 = f_{j+2}$ for $1 \leq j \leq m - 2$. Also $f_{m-2} = h_C$, $f_m = g_C$.

We thus have

\[ \tilde{f}_1 = f_1, \]

\[ \tilde{f}_j = f_{j+2} \cap h_C \text{ for } 1 \leq j \leq m - 2, \]

\[ \tilde{f}_m = g_C. \]

From this it follows that $h_C = \tilde{f}_{m-2} \supset \ldots \supset \tilde{f}_1 \supset \tilde{f}_0 = \{0\}$ is a Jordan-Hölder sequence for $h_C$. We claim it is of class (b). But this follows easily from the fact that $\tilde{f}_1 = f_1$.

Write $e = \{ j_1 < \ldots < j_d \}$, and define the set $\tilde{J}_e = \{ j_1 < \ldots < j_d \}$ by setting $\tilde{j}_h = j_{h+2} - 2$ for $1 \leq h \leq d - 2$, $\tilde{j}_{d-1} = m - 1$, $\tilde{j}_d = m$. We then have

\[ \xi_{j_h} = Z_{j_h} + a_{j_h} Z_{j_1} + b_{j_h} Z_{j_2} \text{ for } 1 \leq h \leq d - 2, \xi_{j_{d-1}} = Z_2, \xi_{j_d} = Z_3. \]

Therefore, letting $C = [c_{rs}]_{1 \leq r, s \leq d}$ be the $d \times d$-matrix:

\[
C = \begin{bmatrix}
a_{j_1} \ldots a_{j_m} & 1 \\
b_{j_1} \ldots b_{j_m} & 1 \\
\vdots & \vdots \\
1 & 1
\end{bmatrix},
\]

where the empty entries are zero, we have $\xi_{j_h} = \Sigma_{r=1}^{d} c_{rs} Z_j$, and $h_e = \{ C M_e (\xi_{j_1}) \}$, where $M_e (\xi)$ is the matrix $[B (\xi_{j_1}^{s}, \xi_{j_1})]_{1 \leq r, s \leq d}$. Now $det C = 1$, and therefore we have for $e \in \mathcal{E}$ with $\langle l, Z_2 \rangle = 0$:

\[
Q_e (l) = | det h_e | = | \sum_{i \in S_d} \text{sign} \ \sigma \ \langle l, [\xi_{j_1}^{1}, \xi_{j_{1+1}}] \rangle \ldots \langle l, [\xi_{j_{d-1}}^{1}, \xi_{j_{d+1}}] \rangle |
\]

\[
= | \langle [\xi_{j_1}^{1}, \xi_{j_{2-1}}^{1}], \xi_{j_m-1} \xi_m \rangle |^2 | \sum_{i \in S_d} \text{sign} \ \sigma \ \langle l, [\xi_{j_1}^{1}, \xi_{j_{1+1}}] \rangle \ldots \langle l, [\xi_{j_{d-1}}^{1}, \xi_{j_{d+1}}] \rangle |,
\]

where $S^*_d$ is the set of permutations $\sigma$ in $S_d$ such that $\sigma (d - 1) = d$, $\sigma (d) = d - 1$.

Set $g_0 = g \mid h$, and let $\bar{h}_{g_0}$, etc. designate the objects defined relative to the Jordan-Hölder sequence $\bar{f}_0 \subseteq \bar{f}_1 \subseteq \ldots \subseteq \bar{f}_{m-2} \subseteq h_C$. Since clearly $g_k \subseteq h$, and $h_{g_0} = g_k$ we find that $1 \notin \bar{f}_{g_0}$, and for $4 \leq j \leq m$ we have

\[ j \# J_e = Z_j e \bar{f}_{j-1} + (g_k) e \subseteq Z_j e \bar{f}_{j-3} + C Z_2 + C Z_3 + (g_k) e \subseteq Z_j e \bar{f}_{j-3} + (h_k) e \subseteq j - 2 \# \bar{f}_{g_0}. \]
so $j \in \mathbb{Z}$ if and only if $j - 2 \in \mathbb{Z}$. Therefore, if $\mathfrak{h} = \mathfrak{z} = \{ j_1 < \ldots < j_{d-2} \}$ we find that $j_h^2 = j_h + 2$ for $1 \leq h \leq d - 2$, and comparing with the definition of $\mathfrak{h}$ we find that $\mathfrak{h}^2 = \mathfrak{h}$ for $1 \leq h \leq d - 2$. Using this we get for $l \in \mathfrak{h}$:

$$Q(l) = \det \begin{bmatrix} \hat{Z}_{j_1} \hat{Z}_{j_2} & \cdots & \hat{Z}_{j_{d-1}} \end{bmatrix}_{r, s \leq d - 2}$$

and comparing with what we saw above we find for $l \in \mathfrak{h}'$ with $\langle l, Z_2 \rangle = 0$:

$$Q(l) = \langle l, [Z_m, Z_{m-1}] \rangle \cdot Q(l_0),$$

where $l_0 = l \mid \mathfrak{h}$. We then claim that $Q(l)$ only depends on the restriction of $l$ to $\mathfrak{h}$. Assuming for a moment this to be true, we find that the formula (*) is valid for all $l \in \mathfrak{h}$. The conclusion of this is that $t^*Q(l) = -\langle l, W \rangle$, $\langle l, W \rangle$ $t^*Q(l_0)$, where $W = [Z_m, Z_{m-1}] = [Z_2, Z_3] \in \mathfrak{h}$, and therefore that $\mathfrak{h} = -W^{\mathfrak{h}} W^{\mathfrak{h}}$.

Let us then prove our claim: if all $\hat{Z}_{j_r}$, $1 \leq r \leq d$ belong to $(\mathfrak{h}_0)_{\mathfrak{h}}$, then the result is clear (since $\mathfrak{h}_0$ is an ideal). Suppose then that there exists $1 \leq r \leq d$ such that $\hat{Z}_{j_r} \not\in (\mathfrak{h}_0)_{\mathfrak{h}}$, and let $\rho$ be the smallest such number $r$. We then have $\mathfrak{h}_0 = \mathbb{C} \hat{Z}_{j_r} \oplus (\mathfrak{h}_0)_{\mathfrak{h}}$, and $\mathfrak{g}_c = \mathbb{C} \hat{Z}_{j_r} \oplus \ker \alpha_2$. Set $Y_r = \hat{Z}_{j_r} + c_r \hat{Z}_{j_r}$, $r = 1, \ldots, d$, where $c_r$ is defined such that $Y_r \in \ker \alpha_2$ for $r \neq \rho$ and where $c_{\rho} = 0$. We then have $Y_\rho \in (\mathfrak{h}_0)_{\mathfrak{h}}$ for all $1 \leq r \leq d - 2$, $r \neq \rho$, while $Y_\rho \in (\mathfrak{h}_0)_{\mathfrak{h}}$, $Y_d = \hat{Z}_m = \hat{Z}_{j_1}$ $Y_{d-1} = \hat{Z}_m = \hat{Z}_{j_1}$ $Y_{d-1} = \ker \alpha_2$.

Letting $C = [C_{r,s}]_{r,s \leq d}$ be the $d \times d$-matrix given by:

$$C = \begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & c_{\rho+1} & \cdots & c_{d-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cdots & c_{d-1} & \cdots & 1 \end{bmatrix}$$

the empty entries meaning zero, we have $Y_s = \sum_{r=1}^d c_{r,s} \hat{Z}_{j_r}$, so, setting $N(l) = [B(Y_\rho, Y_s)]_{r,s \leq d}$ we get $N(l) = \mathbb{C} \hat{M}_d(l)C$, from which $Q_d(l) = \det \mathbb{M}_d(l) = \det \mathbb{M}_d(l) = \det N(l)$. Therefore $Q_d(l) = \sum_{\sigma \in S_d} \det \mathbb{M}_d(l)$, where we have set $P_d(l) = \langle l, [Y_1, Y_{d-1}] \rangle \cdots \langle l, [Y_{d-1}, Y_{d-1}] \rangle$. Define then the following subsets of $S_d$:

$$S_0 = \{ \sigma \mid \sigma(d-1) = \rho \land \sigma(d) = d - 1 \land \sigma(\rho) = d \},$$

$$S_1 = \{ \sigma \mid \sigma(d-1) = d \land \sigma(d) = \rho \land \sigma(\rho) = d - 1 \},$$

$$S_2 = \{ \sigma \mid \sigma(d-1) = \rho \land \sigma(d) = d - 1 \},$$

$$S_3 = S_0 \cup S_1 \cup S_2.$$
We then assert that: $\sigma \in S_d^{(4)} \Rightarrow P_\sigma = 0$. In fact, observe first that since:

$$[Y_{d-1}, Y_r] + 0 \Rightarrow r = d \lor r = \rho,$$

and since: $[Y_{\rho}, Y_r] + 0 \Rightarrow r = d - 1 \lor r = \rho$ we have:

$$P_\sigma + 0 \Rightarrow (\sigma(d - 1) = d \lor \sigma(d - 1) = \rho) \land (\sigma(d) = d - 1 \lor \sigma(d) = \rho).$$

Moreover, if $r \notin \{\rho, d - 1, d\}$, then: $[Y_r, Y_{\sigma(r)}] + 0 \Rightarrow \sigma(r) \neq d - 1 \land \sigma(r) \neq d$, hence:

$$P_\sigma + 0 \Rightarrow (d = \sigma(d - 1) \lor d = \sigma(d)) \land (d - 1 = \sigma(d) \lor d - 1 = \sigma(d)).$$

Therefore, if $P_\sigma + 0$ and $\sigma \notin S_d^{(3)}$ with e. g. $\sigma(d - 1) = \rho$, then $\sigma(d) = d - 1$ and $\sigma(\rho) = d$, and therefore $\sigma \in S_d^{(4)}$. Similarly, if $\sigma \notin S_d^{(3)}$ with $\sigma(d) = \rho$, then $P_\sigma + 0 \Rightarrow \sigma \in S_d^{(4)}$. This shows our assertion.

We next assert that $P(l) = \sum_{\sigma \in S_d^{(3)}} \text{sign} \sigma P_\sigma(l) = 0$, where $S_d^{(3)} = S_d^{(1)} \cup S_d^{(2)}$. To see this, define the permutation $\tau$ in $S_d$ by $\tau(\rho) = d$, $\tau(d - 1) = \rho$, $\tau(d) = d - 1$, all other elements left fixed. It is then immediate to verify that the map $\sigma \rightarrow \sigma \circ \tau$ defines a bijection between $S_d^{(1)}$ and $S_d^{(2)}$, and since $\tau$ is an even permutation we get $P(l) = \sum_{\sigma \in S_d^{(1)}} \text{sign} \sigma (P_\sigma + P_{\sigma \circ \tau})$. Now for $\sigma \in S_d^{(1)}$ we have

$$P_\sigma(l) + P_{\sigma \circ \tau}(l) = \prod_{i=\rho, \rho} \langle l, [Y_{\rho}, Y_{\sigma(i)}] \rangle \left( \prod_{i=d-\rho, d-\rho} \langle l, [Y_{d-\rho}, Y_{\sigma(i)}] \rangle \right) + \prod_{i=\rho, \rho} \langle l, [Y_{d-\rho}, Y_{\sigma(i)}] \rangle \left( \prod_{i=d-\rho, d-\rho} \langle l, [Y_{\rho}, Y_{\sigma(i)}] \rangle \right),$$

and

$$\prod_{i=\rho, \rho} \langle l, [Y_{\rho}, Y_{\sigma(i)}] \rangle + \prod_{i=\rho, \rho} \langle l, [Y_{d-\rho}, Y_{\sigma(i)}] \rangle = \langle l, [Y_{\rho}, Y_{d-\rho}] \rangle \langle l, [Y_{d-\rho}, Y_{\rho}] \rangle = \langle l, [Y_{\rho}, Y_{d-\rho}] \rangle \langle l, [Y_{d-\rho}, Y_{\rho}] \rangle = 0.$$

This shows that $P \equiv 0$, and therefore we have

$$Q_\sigma(l) = \left| \sum_{\sigma \in S_d^{(3)}} \text{sign} \sigma P_\sigma(l) \right|,$$

and since $P_\sigma(l)$ only depends on the restriction of $l$ to $h$ when $\sigma \in S_d^{(3)}$ we have proved our assertion.

Now if $m$ is the nilradical of $h$ it follows from the induction hypothesis that $d\pi_0(u_{\rho}) = Q_{\rho}(g_0)I$, and since $d\pi_0(W) = l \langle f_0, W \rangle$, and $u_\rho = -W^* W u_\rho$ we get that

$$d\pi_0(u_\rho) = l \langle f_0, W \rangle \langle f_0, W \rangle Q_{\rho}(g_0)I = Q_\rho(g)I,$$

and this proves (i) in this case.

Suppose then that $m$ is not the nilradical $m_1$ of $h$. Then $m = h_0 \cap m_1$, and dim $m_1/m = 1$. Setting $M_1 = \exp m_1$ we now face two possibilities (1) either $\pi_0$ extends to an irreducible representation $\pi'_0$ of $M_1$ or (2) $\text{ind}_{M_1/M} \pi_0 = \pi'_0$ is an irreducible representation of $M_1$. In the first case we obviously get as above that $d\pi'_0(u_\rho) = Q_\rho(g)I$, and therefore $d\pi'_0(u_\rho) = Q_\rho(g)I$.

In the second case we have $\pi'_0 \mid M = \int_{M_1/M} s \pi_0 ds$, and therefore we get by the induction
hypothesis that $Q_e(g)\mathbb{I} = d\pi_0(\mu_0) = \int_{M/M} d(s\pi_0(\mu_0))ds$, from which $d(s\pi_0(\mu_0))Q_e(g)\mathbb{I}$ for almost all $s$, hence for all $s$ by continuity. This shows that $d\pi_0(\mu_0) = Q_e(g)\mathbb{I}$, and ends (i).

(ii) Just like in case (c1) (ii) we conclude from (i) that

$$d\pi_0(Ad(s)\mu_0) = Q_e(s^{-1}g)\mathbb{I} = |A_e(s)|^2 Q_e(g)\mathbb{I} \quad \text{for all } s \in G.$$ 

Now since $Z_2, Z_3 \in \mathfrak{c}$, and since $[m, Z_2] = [m, Z_3] = 0$ we see at once that $m_{f_0} = n_{f}$. Suppose then that $p$ is a polarization in $m$ at $f_0$. Then, writing $Z_2 = X_2 + iY_2, p_1 = p \oplus \mathbb{R}Y_2$ is a polarization in $n$ at $f$, and therefore $\pi = \text{ind}_{p_1N} \eta_1$, where $\eta_1$ is the unitary character on $P_1 = \exp p_1$ corresponding to $f | p_1$. Similarly $\pi_0 = \text{ind}_{pM} \eta_1$, where $P = \exp p, \eta_1 = \eta_1 | P$. We then set $n_1 = \mathbb{R}Y_2, \text{ and note that } n_1 \text{ is a direct product of } m \text{ and } \mathbb{R}Y_2$.

Let $\pi_1$ be the irreducible representation of $N_1 = \exp n_1$ with $\pi_1 | M = \pi_0, \eta_1 | P$. Then $\pi = \text{ind}_{p_1N} \eta_1 = \text{ind}_{N_1N} (\text{ind}_{p_1N} \eta_1) = \text{ind}_{N_1N} \pi_1$. Now noting that $N_1$ is a normal subgroup in $N$ and that clearly $d\pi_1(Ad(s)\mu_0) = Q_e(g)\mathbb{I}$ for $s \in N$, we can end this case just like case (c1) (ii). This ends case (c12).

Case (c2): (i) Since $f_1 = f_1$ we can clearly assume that $Z_1 = X_1 \in \mathfrak{g}$. A standard argument shows that $\lambda_2$ must be a real root in this case, so $\lambda_2 = \alpha_2$. We claim that it is no loss of generality to assume that $\gamma_2 \equiv 0$. In fact, let $a_1, a_2, b$ be real numbers, not all equal to zero, such that $0 = a_1 \gamma_1 + a_2 \gamma_2 + b \beta_2$. Then $(a_1, a_2) = (0, 0)$, since $\alpha_2 \not\equiv 0$, and we can assume that $a_1^2 + a_2^2 = 1$. Replacing $Z_2$ by $Z_2' = (a_2 + ia)Z_2 - bZ_1$ does not change $Q_e$, and it is trivial to verify that $[X, X_2'] = \lambda_2(X)Z_2' + \gamma_1(X)Z_1$, where $\gamma_1' = a_2 \gamma_1 - a_1 \gamma_2$. This proves the claim. So, from now on we assume that $Z_1 = X_1 \in \mathfrak{g}, \gamma = \gamma_1$, and writing $Z_2 = X_2 + iY_2$ we then have

$$[X, X_2] = \lambda_2(X)X_2 + \gamma(X)X_1$$
$$[X, Y_2] = \lambda_2(X)Y_2.$$ 

Set $\mathfrak{h} = \ker \gamma$. It follows from the formula (2.2.2) that $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$, and its codimension is 1. Set $\mathfrak{h}_0 = \ker \alpha_2 | \mathfrak{h} = \ker \text{ad} Z_2 | \mathfrak{g}$. $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$ of codimension 2.

Let $m$ be the nilradical of $\mathfrak{h}_0$. Since $\mathfrak{h}_0$ is an ideal we have that $m = n \cap \mathfrak{h}_0 = n \cap \mathfrak{h}$. Observe that $\dim n/m = 1$. In fact, pick $\mathfrak{w} \in \text{ad} \mathfrak{h}_0$. We then have $\gamma([X, \mathfrak{w}]) = \lambda_2(W)\gamma(X)$ for $X \in \mathfrak{g}$. Choosing $X$ such that $\gamma(X) = 1$ we get that $\gamma([X, \mathfrak{w}]) = \lambda_2(W) \not\equiv 0$, and this shows that $[X, \mathfrak{w}]$ is a basis in $n$ (mod $\mathfrak{m}$).

Set $f_0 = f | m = g | m,$ and let $\pi_0$ be the irreducible representation of $M = \exp m$ corresponding to $M_{f_0}$.

(ii) We first show that $u_e \in U(m)$, and that $d\pi_0(\mu_0) = Q_e(g)\mathbb{I}$. We start by noting that we can assume that $\langle g, X_2 \rangle = 0$; in fact, if necessary replace $X_2$ by $X_2 - cX_1$; this does not change $e, Q_e, \text{ etc.}$ (it will change $\gamma, \mathfrak{h}, \text{ though, but does not affect } \mathfrak{h}_0, \text{ rank } (\alpha_2, \gamma_1, \gamma_2)$ and the fact that $\gamma_2 \equiv 0$).
Except for some obvious modifications we can now proceed just like in case (c11) (i).

(iii) Just like in case (c11) (ii) we conclude that \( d\pi_\sigma(\text{Ad}(s)\mu_j) = |\Lambda_\sigma(s)|^2 \text{Q}_\sigma(g)I \) for \( s \in G \).

Now since \( X_1, X_2 \in m \) it follows that \( n_f \subset m \) and from this we get that \( m_{f_0} = n_f \oplus RX_2 \).
Therefore a polarization in \( m \) at \( f_0 \) is also a polarization in \( n \) at \( f \), hence \( \pi = \text{ind}_{M_1N}^{M_0N_0} \).
We can then end this case just like we did in case (c11) (ii).

Case (c3): It is no loss of generality to assume that \( \gamma \equiv 0 \). In fact, there exists real numbers \( a_1, a_2 \) such that \( \gamma_1 = a_1 \alpha_2, \gamma_2 = a_2 \alpha_2 \), and therefore \( \gamma_1 = a_1(1 + ik_2)^{-1}\lambda_2, \gamma_2 = a_2(1 + ik_2)^{-1}\lambda_2 \).
Replacing \( Z_2 \) by \( Z_2' = Z_2 + (1 + ik_2)^{-1}(a_1 + ia_2)Z_1 \) does not change \( Q_\sigma \) etc., and we have \( [X, Z_2'] = \lambda_2(X)Z_2' \). This proves the assertion.

Set \( \mathfrak{h} = \ker \lambda_2 \). Then \( \mathfrak{h} \) is an ideal in \( g \) of codimension 1, so \( n \subset \mathfrak{h} \). We can now proceed here much like in case (b), so we omit the details.

Case (d): Suppose we are not in case (a), (b) or (c).
We have \( [X, Z_2] = \gamma(X)Z_1 \), where \( \gamma \neq 0 \) and \( \langle g, Z_1 \rangle \neq 0 \) (since otherwise we would be in case (a)), and also \( f_1 = \overline{f}_1 \).

Writing \( \gamma = \gamma_1 + \gamma_2 \) we distinguish two subcases: (d1): \( \text{rank}(\gamma_1, \gamma_2) = 2 \) and (d2): \( \text{rank}(\gamma_1, \gamma_2) = 1 \).

Case (d1): Set \( \mathfrak{h} = \ker \gamma_1 \cap \ker \gamma_2 \). Then \( \mathfrak{h} \) is an ideal of codimension 2. We then distinguish two possibilities: case (d11): \( [f_3, f_2] = 0 \) and case (d12): \( [f_3, f_2] = f_1 \). We can then proceed here much like in case (c1) (the case at hand is easier, since here \( \mathfrak{h} \) is an ideal containing \( [g, g] \)). We omit the details.

Case (d2): Just like in case (c2) we see that we can assume that \( \gamma_2 = 0 \). Set \( \mathfrak{h} = \ker \gamma \).
Then \( \mathfrak{h} \) is an ideal of codimension 1, and we can treat this case much like case (b). We also omit the details here. This ends the proof of Proposition 2.2.1.

2.3. — We shall now conclude the proof of Theorem 1.4.1. We use [4], 4.2.2 Théorème, p. 121 with \( \psi(l) = |P_\sigma(l)| \). It follows from Lemma 2.1.1 that the condition of the theorem loc. cit. is satisfied. The conclusion is that the operator \( A\pi(\phi)A \) is traceclass for all \( \phi \in C_2^\infty(G) \), that \( \phi \to \text{Tr}([A\pi(\phi)A]) \) is a distribution (of positive type) on \( G \), and that

\[
\text{Tr}([A\pi(\phi)A]) = \int_0 (\sigma_e \cdot \phi \circ \exp)(l)Q_\sigma(l)\beta_\sigma(l)dl.
\]

Here we have also used Lemma 1.3.1.

Remark 2.3.1. — In [4], p. 248 and [5], p. 118 appear two different definitions of the function \( P_0 \) (cf. section 1.3). Here we use the one from [4] (which is the most natural one), while the 4.2.2. Théorème in [5] uses the definition of \( P_0 \) from [5]. There is no difficulty in proving 4.2.2. Théorème with the definition of \( P_0 \) from [4] when \( \psi \) has the property that \( \psi(l) \) only depends on the restriction of \( l \) to \( [g, g] \) which is the case here (cf. [5] 4.2.3. Remarque).

We shall then identify the operator \( A \): Set \( G_0 = \ker \chi_\sigma \), let \( g_0 \) be the Lie algebra of \( G_0 \),
and let $\pi_0$ be the irreducible representation associated with $g_0 = g | g_0$. Then $\pi = \text{ind}_{g_0 \rtimes \pi_0}$, and $A$ is realized on $L^2(G, \pi_0)$, the space of the induced representation, by $Af(s) = \psi(sg)f(s) = |P_e(sg)|^2 f(s)$. Now it follows from Proposition 2.2.1 that $d\pi_0(u_e) = Q_e(g)$, and that $d\pi_0(Ad(s)u_e) = Q_e(s^{-1}g)I$ which implies that we have for a differentiable vector $f \in L^2(G, \pi_0)$:

$$d\pi_0(u_e)f(s) = d\pi_0(Ad(s^{-1})u_e)f(s) = Q_e(sg)f(s) = |P_e(sg)|^2 f(s) = A^2 f(s),$$

and thus $d\pi_0(u_e) = A^2$.

Now since $A\pi(\varphi) \subset \pi(\chi_e^{-1}\varphi)A$ we have that $A\pi(\varphi)A \subset A^2\pi(\chi_e^{-1}\varphi)$, and therefore $[A\pi(\varphi)A] = [A^2\pi(\chi_e^{-1}\varphi)]$ from which $[A^2\pi(\varphi)]$, hence $[A\pi(\varphi)]$, is trace-class for all $\varphi \in C_c^\infty(G)$, and $\text{Tr} ([A^2\pi(\varphi)]) = \text{Tr} ([A\pi(\chi_e\varphi)A]) = \text{Tr} ([A\pi(\varphi)A])$, the last equality being valid because the distribution $\varphi \mapsto \text{Tr} ([A\pi(\varphi)A])$ is supported on $G_0$ (cf. [5], [6]). Observing finally that $[A^2\pi(\varphi)] = \pi(u_e \ast \varphi)$, we have proved the theorem.

3. Examples

We shall give a few examples of the calculation of $\delta, Q_e, u_e, \Omega_e$ for an exponential solvable Lie algebra $\mathfrak{g}$. If $Z_1, \ldots, Z_m$ is a Jordan-Hölder basis for $\mathfrak{g}_C$ we denote by $M(g)$, $\text{gg}'$, the skew-symmetric $m \times m$-matrix $[\langle g, [Z_i, Z_j] \rangle]_{1 \leq i, j \leq m}$, and we write $\zeta_j = \langle g, Z_j \rangle$. The matrices $M(g)$ are all submatrices of $M(g)$. Note that $Z = \sum_{j=1}^m z_j Z_j$ belongs to $(\mathfrak{g}_C)_C$ if and only if $M(g)z = 0$, where $z = (z_1, \ldots, z_m)$. We write $\delta = \{ e_1 < \ldots < e_p \}$.

3.1. — Let $\mathfrak{g}$ be the five dimensional real solvable Lie algebra with the following non-vanishing brackets: $[X_2, X_4] = -X_4$, $[X_5, X_3] = 2X_3$, $[X_5, X_2] = X_2$, $[X_4, X_3] = X_2$, $[X_4, X_2] = X_1$. Then $X_1, \ldots, X_5$ is a Jordan-Hölder basis for $\mathfrak{g}$, so $\mathfrak{g}$ is completely solvable. We set $Z_j = X_j$ and $\xi_j = \langle g, X_j \rangle = \zeta_j$, $j = 1, \ldots, 5$.

We have

$$M(g) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\xi_1 & -\xi_2 \\
0 & 0 & 0 & 0 & 2\xi_3 \\
0 & 0 & 0 & 2\xi_3 & \xi_4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

i) If $\xi_2^2 - 2\xi_1 \xi_3 = 0$, then $g_e = \mathbb{R}X_1$ and therefore $J_g = \{ 2, 3, 4, 5 \}$.

ii) If $\xi_2^2 - 2\xi_1 \xi_3 = 0$ and $\xi_2 = 0$ then

$$g_e = \mathbb{R}X_1 \oplus \mathbb{R}(-\xi_2 X_2 + \xi_2 X_3) \oplus \mathbb{R}(-\xi_2 X_2 - \xi_2 X_4 + \xi_3 X_3), \quad J_g = \{ 3, 5 \}.$$
iv) If \( \xi_2^2 - 2\xi_1\xi_3 = 0, \xi_1 = 0, \xi_3 = 0 \) and \( \xi_4 \neq 0 \), then
\[
\mathfrak{g}_4 = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}X_3, \quad \mathcal{J}_4 = \{4, 5\}
\]

v) If \( \xi_2^2 - 2\xi_1\xi_3 = 0, \xi_1 = 0, \xi_3 = 0, \xi_4 = 0 \), then \( \mathfrak{g}_5 = \mathfrak{g}, \mathcal{J}_5 = \emptyset \).

We can then write down:

\[
e_1 = \{2, 3, 4, 5\}, \quad \mathcal{O}_{e_1} = \{g \mid \xi_2^2 - 2\xi_1\xi_3 = 0\},
\]
\[
e_2 = \{2, 4\}, \quad \mathcal{O}_{e_2} = \{g \mid \xi_2^2 - 2\xi_1\xi_3 = 0, \xi_1 = 0\},
\]
\[
e_3 = \{3, 5\}, \quad \mathcal{O}_{e_3} = \{g \mid \xi_1 = \xi_2 = 0, \xi_3 = 0\},
\]
\[
e_4 = \{4, 5\}, \quad \mathcal{O}_{e_4} = \{g \mid \xi_1 = \xi_2 = \xi_3 = 0, \xi_4 = 0\},
\]
\[
e_5 = \emptyset \quad \mathcal{O}_{e_5} = \{g \mid \xi_1 = \xi_2 = \xi_3 = \xi_4 = 0\},
\]
\[
\mathcal{Q}_{e_1}(g) = (\xi_2^2 - 2\xi_1\xi_3)^2, \quad \mathcal{Q}_{e_2}(g) = \xi_2^2, \quad \mathcal{Q}_{e_3}(g) = 4\xi_3^2, \quad \mathcal{Q}_{e_4}(g) = \xi_4^2, \quad \mathcal{Q}_{e_5}(g) = 1,
\]
\[
u_{e_1} = (X_2^2 - 2X_1X_3)^2, \quad \nu_{e_2} = -X_1, \quad \nu_{e_3} = -4X_3^2, \quad \nu_{e_4} = -X_4^2, \quad \nu_{e_5} = 1.
\]

3.2. — Let \( \mathfrak{g} \) be the six dimensional real exponential solvable Lie algebra having a basis \( X_1, \ldots, X_6 \) with the following non-vanishing brackets: \( [X_6, X_4] = X_4 - X_5, [X_6, X_2] = X_1 + X_2, [X_6, X_1] = X_1 - X_2, [X_5, X_4] = X_3, [X_5, X_3] = X_2, [X_4, X_3] = X_1 \). Set \( Z_1 = X_1 - iX_2, Z_2 = X_1 + iX_2, Z_3 = X_3, Z_4 = X_4 - iX_5, Z_5 = X_4 + iX_5, Z_6 = X_6 \). Then \( Z_1, \ldots, Z_6 \) is a Jordan-Hölder basis for \( \mathfrak{g}_6 \), and

\[
M(g) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -(1 - i)\zeta_1 \\
0 & 0 & 0 & 0 & 0 & -(1 + i)\zeta_2 \\
0 & 0 & 0 & -\zeta_1 & -\zeta_2 & 0 \\
0 & 0 & \zeta_1 & 0 & -2i\zeta_3 & -(1 - i)\zeta_4 \\
0 & 0 & \zeta_2 & 2i\zeta_3 & 0 & -(1 + i)\zeta_5 \\
(1 - i)\zeta_1 & (1 + i)\zeta_2 & 0 & (1 - i)\zeta_4 & (1 + i)\zeta_5 & 0
\end{bmatrix}
\]

Writing \( \zeta_j = \langle g, X_j \rangle, j = 1, \ldots, 6 \), we have \( \zeta_1 = \xi_1 - i\xi_2, \zeta_2 = \xi_1 + i\xi_2, \zeta_3 = \xi_3, \zeta_4 = \xi_4 - i\xi_5, \zeta_5 = \xi_4 + i\xi_5, \zeta_6 = \xi_6 \).

i) If \( \zeta_1 \neq 0 \), then \( \mathcal{J}_g = \{1, 3, 4, 6\} \).

ii) If \( \zeta_1 = 0 \) (\( \Rightarrow \zeta_2 = 0 \)), \( \zeta_3 \neq 0 \), then \( \mathcal{J}_g = \{4, 5\} \).

iii) If \( \zeta_1 = 0, \zeta_3 = 0, \zeta_4 \neq 0 \), then \( \mathcal{J}_g = \{4, 6\} \).

iv) If \( \zeta_1 = 0, \zeta_3 = 0, \zeta_4 = 0 \) (\( \Rightarrow \zeta_5 = 0 \)), then \( \mathcal{J}_g = \emptyset \).
We can then write down:

\[ e_1 = \{1, 3, 4, 6\}, \quad \Omega_{e_1} = \{ g \mid \xi_1^2 + \xi_2^2 + 0 \}, \]
\[ e_2 = \{4, 5\}, \quad \Omega_{e_2} = \{ g \mid \xi_1^2 + \xi_3^2 = 0, \xi_3 + 0 \}, \]
\[ e_3 = \{4, 6\}, \quad \Omega_{e_3} = \{ g \mid \xi_4^2 + \xi_5^2 = 0, \xi_3 = 0, \xi_4 + \xi_5 + 0 \}, \]
\[ e_4 = \emptyset, \quad \Omega_{e_4} = \{ g \mid \xi_1 = \xi_2 = \xi_3 = \xi_4 = \xi_5 = 0 \}, \]
\[ Q_{e_1}(g) = 2(\xi_1^2 + \xi_2^2)^2, \quad u_{e_1} = 2(X_1^2 + X_2^2)^2, \]
\[ Q_{e_2}(g) = 4\xi_3^2, \quad u_{e_2} = -4X_3^2, \]
\[ Q_{e_3}(g) = 2(\xi_4^2 + \xi_5^2), \quad u_{e_3} = -2(X_4^2 + X_5^2), \]
\[ Q_{e_4}(g) = 1, \quad u_{e_4} = 1. \]

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