

ANNALES SCIENTIFIQUES DE L'É.N.S.

P.-A. MEYER

A remark on F. B. Knight's paper : "A post-predictive view of gaussian processes"

Annales scientifiques de l'É.N.S. 4^e série, tome 16, n° 4 (1983), p. 567-569

http://www.numdam.org/item?id=ASENS_1983_4_16_4_567_0

© Gauthier-Villars (Éditions scientifiques et médicales Elsevier), 1983, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'É.N.S. » (<http://www.elsevier.com/locate/ansens>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

A REMARK ON F. KNIGHT'S PAPER

By P. A. MEYER

The aim of this remark is to discuss F. Knight's statement (in the last few lines of the introduction of the preceding paper) that his basic Theorem 1.4 is "almost" true for non-Gaussian processes. We are going to prove it completely, without any assumption on higher moments, by a *very slight* technical modification of Knight's own proof.

Since the reader can refer to the main paper for details and comments, our editing in this note is rather concise.

NOTATION. — (Ω, \mathcal{F}, P) is a complete probability space, with a filtration (\mathcal{F}_t) which satisfies the usual conditions. On this space, (X_t) is a real valued, progressively measurable process, such that:

$$(1) \quad E \left[\int_0^\infty e^{-\lambda s} |X_s| ds \right] < \infty \quad \text{for } \lambda > 0.$$

Since (1) is satisfied, we may consider the process $\left(E \left[\int_t^\infty e^{-\lambda s} X_s ds \mid \mathcal{F}_t \right] \right)$, which is a difference of two positive supermartingales (potentials), and therefore has a cadlag ⁽¹⁾ version. The same is true for the following semimartingale (Knight's P_λ is λY^λ in our notation):

$$(2) \quad Y_t^\lambda = e^{\lambda t} E \left[\int_t^\infty e^{-\lambda s} X_s ds \mid \mathcal{F}_t \right] = E \left[\int_0^\infty e^{-\lambda s} X_{t+s} ds \mid \mathcal{F}_t \right].$$

Formally, λY_t^λ is an approximation of X_t for large λ . We now denote by Z_t^λ the local martingale part in the canonical decomposition of Y_t^λ , with the understanding that $Z_0^\lambda = Y_0^\lambda$. Let us compute it.

⁽¹⁾ Cadlag means *right continuous with left limits* in Brodning language.

First, consider a process (g_t) , progressively measurable and such that:

$$E \left[\int_0^\infty |g_s| ds \right] < \infty \quad \text{and set } h_t = E \left[\int_t^\infty g_s ds \mid \mathcal{F}_t \right] \quad (\text{cadlag, as above}).$$

Then $k_t = \int_0^t g_s ds + h_t$ is a martingale, and $j_t = k_0 + \int_0^t e^{\lambda s} dk_s$ (stochastic integral) is a local martingale. A simple integration by parts gives an explicit expression for it:

$$j_t = e^{\lambda t} h_t + \int_0^t e^{\lambda s} (g_s - \lambda h_s) ds,$$

It is very easy to see that (j_t) belongs to the class (D) on finite intervals, hence is a true martingale. If we now take $g_t = e^{-\lambda t} X_t$, $e^{\lambda t} h_t$ is the process we called Y_t^λ above, and j_t turns out to be Z_t^λ :

$$(3) \quad Z_t^\lambda = Y_t^\lambda + \int_0^t (X_s - \lambda Y_s^\lambda) ds.$$

Knight's M_λ is λZ^λ in our notation.

THE EXTENSION OF KNIGHT'S THEOREM. — We denote by \mathbf{D} the Skorohod space of all cadlag mappings from \mathbb{R}_+ to \mathbb{R} , with its usual Borel structure, and by \mathbf{D}_λ the (Borel) subspace of \mathbf{D} consisting of all mappings $y(t)$ such that $e^{-\lambda t} y(t) \rightarrow 0$ as $t \rightarrow \infty$.

We denote by \mathbf{A} the subspace of $L_{loc}^1(\mathbb{R}_+)$ consisting of those classes $x(t)$ such that:

$$\int_0^\infty e^{-\lambda s} |x(s)| ds < \infty \quad \text{for every } \lambda > 0.$$

\mathbf{A} is given the Borel structure inherited from the (Polish space) L_{loc}^1 .

For simplicity of notation we set $\mathbb{N} = \{1, 2, \dots\}$, not $\{0, 1, \dots\}$ as usual. We define a Borel mapping Φ from $\mathbf{B} = \mathbf{A} \times \prod_{\lambda \in \mathbb{N}} \mathbf{D}_\lambda$ to $\mathbf{D}^\mathbb{N}$: it associates to the class $x \in \mathbf{A}$ and the functions $y_\lambda \in \mathbf{D}_\lambda$ the functions in \mathbf{D} :

$$(4) \quad z_\lambda(t) = y_\lambda(t) + \int_0^t (x(s) - \lambda y_\lambda(s)) ds, \quad \lambda = 1, 2, \dots$$

The crucial step is the following:

LEMMA. — *There exists a Borel mapping Ψ from $\mathbf{D}^\mathbb{N}$ to \mathbf{B} such that $\Psi \circ \Phi$ is the identity mapping on \mathbf{B} .*

Proof. — We need only show that Φ is injective. Indeed, it is well known that \mathbf{A} and \mathbf{D} (hence \mathbf{D}_λ) are Lusin measurable spaces, and so is their product \mathbf{B} . Then Lusin's theorem implies that $\Phi(\mathbf{B})$ is Borel in $\mathbf{D}^\mathbb{N}$, and Φ is a Borel isomorphism from \mathbf{B} to $\Phi(\mathbf{B})$. Then the construction of Ψ is trivial (Φ^{-1} on $\Phi(\mathbf{B})$, some constant function on its complement).

To prove injectivity, we need only show that the vanishing of all functions z_λ in (4) implies the vanishing of all y_λ , and the a. s. vanishing of x . Now if $z_\lambda = 0$, (4) can be solved explicitly in y_λ :

$$(5) \quad y_\lambda(t) = -e^{\lambda t} \int_0^t e^{-\lambda s} x(s) ds.$$

We have assumed that this function belongs to D_λ , i. e. $e^{-\lambda t} y_\lambda(t)$ tends to 0 at infinity. This means that the Laplace transform of x vanishes at λ . Since λ may take all values $1, 2, \dots$, x must vanish a. e. Coming back to (5) we see that $y_\lambda = 0$.

Returning to the probabilistic situation, we get Knight's result without assumptions on the existence of moments:

THEOREM. — *There is a deterministic procedure which, for a. e. ω , reconstructs the class of $X_\cdot(\omega)$ from the paths $Z^\lambda(\omega)$, $\lambda = 1, 2, \dots$*

Proof. — According to (1), we a. s. have:

$$\int_0^\infty e^{-\lambda s} |X_s(\omega)| ds < \infty \quad \text{for all } \lambda > 0.$$

According to (2), we a. s. have for $\lambda = 1, 2, \dots$:

$$\lim_{t \rightarrow \infty} e^{-\lambda t} Y_t^\lambda(\omega) = 0,$$

since these processes are differences of two potentials.

According to (3), we a. s. have:

$$(Z^\lambda(\omega))_{\lambda=1, 2, \dots} = \Phi(X_\cdot(\omega), Y^1(\omega), Y^2(\omega), \dots)$$

We now apply Ψ to the sequence $(Z^\lambda(\omega))_{\lambda=1, 2, \dots} \in D^{\mathbb{N}}$, and the main lemma tells us that we a. s. recover the class of $X_\cdot(\omega)$, and moreover all mappings $Y^\lambda(\omega)$, $\lambda = 1, 2, \dots$

(Manuscrit reçu le 27 novembre 1982.)

P. A. MEYER,
Université Louis-Pasteur,
Département de Mathématiques,
7, rue René-Descartes,
67084 Strasbourg Cedex.