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A POST-PREDICTIVE VIEW OF GAUSSIAN PROCESSES (*)

BY F. B. KNIGHT

0. Introduction

The prediction problem for stationary Gaussian processes with continuous covariance is well understood, and for the present purposes we may consider it to be solved. That is, letting $X(t)$ denote the process, $-\infty<t<\infty$, we assume known an expression for $E(X(t+s)\,|\,X(t-\tau),\ 0\leqq\tau),\ 0<s$, depending only on $X(t-\tau),\ 0\leqq\tau$. Such an expression, called the predictor, involves s but may be assumed to be independent of t, and is linear in $X(.)$ as function of $t-\tau,\ 0\leqq\tau$. The complete solution of this problem, going back to Wold (1938), Kolmogorov (1939), and Wiener (1949), may be found for example in Dym and McKean [6], and another method of solution is in Yaglom [17].

It $X(t)$ is not assumed stationary, then the corresponding problem can become very complicated, but general forms for the solution are known (T. Hida [8], H. Cramer [2]) going back to P. Lévy's canonical form (P. Lévy [12]). In the nonstationary situation we may, of course, still assume linearity of the predictor in $X(t-\tau),\ 0\leqq\tau$, but the predictor will depend explicitly on t.

A closely related problem is to obtain for $X(t)$ a "moving integral" representation in terms of Gaussian processes with independent increments. In the stationary case, if we assume that $X(t)$ has no deterministic component, we may write:

$$X(t)=\int_{-\infty}^{t} h(t-s)\,dW(s) \qquad \text{where} \quad h\in L^2(0,\ \infty),$$

is a fixed function and $W(s)$ is a standard Brownian motion, $-\infty<s<\infty$ (since only dW is involved, we may assume $W(0)=0$). Furthermore, W may be chosen so that:

$$(0.1) \qquad E(X(t+s)\,|\,X(\tau),\ \tau\leqq t)=\int_{-\infty}^{t} h(t+s-\tau)\,dW(\tau).$$

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(*See* [4], Theorem 5.3.) In the non-stationary case, a similar representation is again possible, but instead of only $dW(t)$ it may involve a countable sum of stochastic integrals with respect to independent Gaussian processes with independent. increments, and the corresponding integrands may depend on t (*see* [8], Theorem 1.5 and below, Theorem 1.12).

In either case, a natural question to ask is how, given the solution to the prediction problem, one can best obtain the stochastic integral representation. In both cases the known answers seem to us rather unsatisfactory, at least from the standpoint of calculation. For the stationary case the answer (in [4], p. 589 for instance) involves the spectral representation:

$$X(t)=\int e^{2\pi i t u}\, c(u)\, d\xi^*(u),$$

in which $c(u)$ is often difficult to obtain and the white noise $d\xi^*$ is impossible to obtain from $X(\tau), \tau \leqq t$. Using $c(u)$ the method provides $h(u)$ in (0.1) and also a Fourier integral formula for $W(t)$, but this may be hard to implement. In the non-stationary case, the construction of [8], § 1.3 utilizes a deus ex machina in the form of the Hellinger-Hahn multiplicity theory for self-adjoint operators on Hilbert space (*see*, for example, [7]). This construction is non-unique, and gives little if any idea of how it can be implemented in pratice.

Accordingly, the main purpose of this paper is to bridge the above gap between the solution of the prediction problem and the "moving integral" representations. Thus we obtain what is (in our opinion) a simpler access to the generating processes of independent increments (i. e., Gaussian martingales) than is found elsewhere, and no use is made of any spectral representations. In so far as actual prediction of $X(t)$ must probably involve more or less continuous up-dating of the predictors, the moving integral type of representation appears to have potential computational advantages over the solution carried out separately for each t, but this this is a direction in which we lack the necessary expertise to give a qualified opinion.

Our main result is stated in Theorem 1.4. This leads us to introduce a new index $N(t)$ which we call the index of stationarity [if $X(t)$ is stationary, $N(t)=1$]. We then relate $N(t)$ to the index of multiplicity $E(t)$ of [7] ([1]), and obtain the generalized canonical representation of $X(t)$ (Theorem 1.12 *a*). Of course, since this is a "wide sense" result (in the language of [4], p. 77) it then translates immediately to the non-Gaussian case if we replace "independent increments" by "orthogonal increments" and also replace $E(X(t+s) \mid \mathscr{F}(t+))$ by the orthogonal projection of $X(t+s)$ onto the corresponding closed linear manifold $H(t+)=\bigcap_{\varepsilon>0} H(t+\varepsilon)$, where $H(t+\varepsilon)$ is the Hilbert space closure of $\{X(s), s \leqq t+\varepsilon\}$ (the author is indebted to Professor J. L. Doob for reminding him of this). Each of our other results also has an immediate wide sense extension when the analogous replacements are made, so that in particular martingales are replaced by wide sense martingales.

In the second section of the paper, which actually does not depend on Theorem 1.4, we express the martingales of Section 1 in terms of an arbitrary generalized canonical

([1]) This $N(t)$ is not to be confused with the multiplicity function $N(t)$ of H. Cramer [3], which is simply a localization of the multiplicity $E(t)$ of [7].

representation of $X(t)$, with emphasis on the canonical case $E(t)=1$. In this case there is only one underlying martingale, and when $N(t)=1$ our method provides the canonical representation explicitly. The general case $E(t)=1$, however, presents a more difficult problem which we consider elsewhere (2). Our main concern is with characterizing $N(t)$ in terms of the generalized canonical representation. We are able to completely characterize it, even when $E(t)>1$, where it determines a specific form for the integrands (Theorems 2.2 and 2.4).

In the third section, we begin by showing that the statistics (i. e., the covariance) of the martingales obtained in Section 1 determine uniquely the covariance of the original process. Thus it is possible, in theory, to replace the study of $X(t)$ by that of a certain family of Gaussian processes with independent increments. But this seems quite unwieldy except in the case of multiplicity 1, treated in Section 2. The main result of Section 3 is to give the explicit expression for the covariance of these martingales in terms of that of the predictors, and hence (under the wide sense interpretation) in terms of the covariance of X.

After completing the present paper, it came to our attention that some of the results extend without difficulty in various other directions. Thus, if $X(t)$ is complex or vector valued, as for example in [2], the methods and results of Section 1 carry over without change. However, the results of Section 2 are more intricate, and we make no claims as to their extendibility in this case. On the other hand, some of the results also extend to non-Gaussian $X(t)$ if we omit the word "Gaussian" but do *not* replace conditional expectation by projection on $H(t+)$ (thus they remain strict sense results). This is true of Theorem 1.2, and it is "almost" true of the basic Theorem 1.4. In fact, before the proof of Corollary 1.8 the only place at which a special property of the Gaussian distribution really is used is in (1.6) in the form of a moment of order exceeding 2. Whether this can be avoided is an open question, but in any case Corollary 1.8 does not extend in this sense.

1. A Family of Martingales

As before, we let $X(t)$, $t\in T$, be a real-valued Gaussian process, with complete probability space $(\Omega, \mathscr{F}, P)$. To avoid details, we assume that $X(t)$ is continuous in quadratic mean. We allow either $T=[0, \infty)$ or $T=(-\infty, \infty)$, with the understanding that $T=(-\infty, \infty)$ whenever $X(t)$ is assumed stationary. Without loss of generality (except when the statistics of $X(t)$ are not known) we assume that $EX(t)=0$. The covariance is denoted, as usual, by $\Gamma(s, t)=E(X(s)X(t))$, and we assume that:

$$\int_0^\infty e^{-\lambda s}\sigma^2(s)\,ds<\infty \quad \text{for } \lambda>0 \quad \text{where } \sigma^2(s)=\Gamma(s, s)$$

(otherwise we would simply replace $X(t)$ by $\sigma^{-1}(t)X(t)$, or by any other appropriate continuous multiple). For $\lambda>0$ and $t\geqq 0$ the expression $\int_0^\infty e^{-\lambda s}X(t+s)\,ds$ exists in the

(2) The solution is given in a paper of the author in Séminaire de Probabilités XVII.

sense of an integral in quadratic mean. Indeed, by Schwartz' inequality $|\Gamma(s, t)| \leqq \sigma(s)\sigma(t)$, whence

$$\mathrm{E}\left(\int_0^\infty e^{-\lambda s}|\mathrm{X}(t+s)|\,ds)^2 \leqq \left(\int_0^\infty e^{-\lambda s}\sigma(s+t)\,ds\right)^2,$$

which is finite by our hypothesis.

Next, let $\mathscr{F}^0(t)$ denote the σ-field generated by $\mathrm{X}(s)$, $s \leqq t$ let $\mathscr{F}(t)$ denote $\mathscr{F}^0(t)$ completed by adjoining all P-null sets in $\mathscr{F}$, and set $\mathscr{F}(t+) = \bigcap_{\varepsilon>0} \mathscr{F}(t+\varepsilon)$. Then the family $\mathscr{F}(t+)$ is right-continuous, and each contains all P-null sets in $\mathscr{F}$. We now introduct the martingales which are our main concern, beginning with:

LEMMA 1.1. — *For $\lambda > 0$, the expressions*

$$(1.1)\quad \mathrm{M}_\lambda(t) = \lambda\Bigg[\mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(t+s)\,ds \,\Big|\, \mathscr{F}(t+)\right) - \mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(s)\,ds \,\Big|\, \mathscr{F}(0+)\right) + \int_0^t \left(\mathrm{X}(u) - \lambda\,\mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u+s)\,ds \,\Big|\, \mathscr{F}(u+)\right)\right) du\Bigg],$$

are Gaussian martingales in t with respect to $\mathscr{F}(t+)$, where the integrals are in quadratic mean.

Proof. — The existence of the last integral is easily seen when we recall that the conditional expectation given $\mathscr{F}(u+)$ which appears in the integrand is simply the mean-square projection of the Gaussian random variable $\int_0^\infty e^{-\lambda s}\mathrm{X}(u+s)\,ds$ onto a certain Gaussian linear space (namely $\mathrm{H}(t+)$; however, even in the non-Gaussian case it may be interpreted as a projection). Now for $u_1 < u_2$ we have

$$\begin{aligned}
&\mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u_2+s)\,ds \,\Big|\, \mathscr{F}(u_2+)\right) - \mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u_1+s)\,ds \,\Big|\, \mathscr{F}(u_1+)\right) \\
&\quad = \mathrm{E}\left(\int_0^\infty e^{-\lambda s}(\mathrm{X}(u_2+s) - \mathrm{X}(u_1+s))\,ds \,\Big|\, \mathscr{F}(u_2+)\right) \\
&\quad + \mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u_1+s)\,ds \,\Big|\, \mathscr{F}(u_2+)\right) - \mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u_1+s)\,ds \,\Big|\, \mathscr{F}(u_1+)\right)
\end{aligned}$$

and using the continuity in quadratic mean of $\mathrm{X}(u)$, it follows that $\mathrm{E}\left(\int_0^\infty e^{-\lambda s}\mathrm{X}(u+s)\,ds \,\Big|\, \mathscr{F}(u+)\right)$ is right-continous in quadratic mean.

The remainder of the proof is by direct verification. Setting for brevity $E_t(s)=E(X(s) \mid \mathscr{F}(t+))$ we have

$$
\begin{aligned}
& E(M_\lambda(t_2)-M_\lambda(t_1) \mid \mathscr{F}(t_1+)) \\
&\qquad =\lambda\, E\left(\int_0^\infty e^{-\lambda s}(X(t_2+s)-X(t_1+s))\, ds \mid \mathscr{F}(t_1+)\right) \\
&\qquad\quad +\lambda \int_{t_2}^{t_2} E(X(u)-\lambda \int_0^\infty e^{-\lambda s} X(u+s)\, ds \mid \mathscr{F}(t_1+))\, du \\
&\qquad =\lambda\left(\int_{t_2}^\infty e^{-\lambda(s-t_2)} R_{t_1}(s)\, ds-\int_{t_1}^\infty e^{-\lambda(s-t_1)} E_{t_1}(s)\, ds\right) \\
&\qquad\quad +\lambda \int_{t_1}^{t_2} E_{t_1}(u)\, du-\lambda^2 \int_{t_1}^{t_2}\int_u^{t_2} e^{-\lambda(s-u)} E_{t_1}(s)\, ds\, du \\
&\qquad\qquad -\lambda^2 \int_{t_1}^{t_2}\int_{t_2}^\infty e^{-\lambda(s-u)} E_{t_1}(s)\, ds\, du \\
&\qquad =\lambda \int_{t_2}^\infty (e^{-\lambda(t-t_2)}-e^{-\lambda(s-t_1)}-\lambda \int_{t_1}^{t_2} e^{-\lambda(s-u)}\, du)\, E_{t_1}(s)\, ds \\
&\qquad\qquad +\lambda \int_{t_2}^{t_2} (1-e^{-\lambda(s-t_1)}-\lambda \int_{t_1}^{s} e^{-\lambda(s-u)}\, du)\, E_{t_1}(s)\, ds=0,
\end{aligned}
$$

where we used several times the fact that integration in quadratic mean commutes with projection on fixed subspace.

From the above comments, we also see that the $M_\lambda(t)$ are square integrable and right-continuous in quadratic mean. It is well-known (*see* [4], Theorem 11.5) that for each λ we may choose a standard modification of $M_\lambda(t)$ which is right-continuous, with left limits for $t>0$ [the possibility of identifying $M_\lambda(t)$ with its right-hand limit along rationals $r>t$ following since $M_\lambda(t)$ is right-continuous in quadratic mean]. We thus introduce

DEFINITION 1.1. — Let $M_\lambda(t)$, $M_\lambda(0)=0$, denote henceforth a standard modification of (1.1) which is right-continuous with left limits for $t>0$.

The use of $e^{-\lambda s}$ in Lemma 1.1 turns out to have real advantages over other choices of integrands. However, for the sake of completeness we may extend as follows.

THEOREM 1.2. — *For any bounded Borel f, and any $\lambda>0$, the expression:*

$$
\begin{aligned}
& E\left(\int_0^\infty I_\lambda f(s) X(t+s)\, ds \mid \mathscr{F}(t+)\right)-E\left(\int_0^\infty I_\lambda f(s) X(s)\, ds \mid \mathscr{F}(0+)\right) \\
&\qquad +\int_0^t (I_\lambda f(0) X(u)-E\left(\int_0^\infty e^{-\lambda s} f(s) X(s+u)\, ds \mid \mathscr{F}(u+)\right))\, du,
\end{aligned}
$$

where $I_\lambda f(s)=\int_s^\infty e^{-\lambda v} f(v)\, dv$, *is a Gaussian martingale with respect to* $\mathscr{F}(t+)$. *For fixed* t, *the family of all such martingales generates the same (completed) σ-field as* $\{M_\lambda(t), \lambda>0\}$.

Proof. — For $f=e^{-\mu t}$, $\mu \geqq 0$, the above expression is just $(\lambda+\mu)^{-2} M_{\lambda+\mu}(t)$. Since it is linear in f, the Stone-Weierstrass approximation theorem extends it to all continuous f with limit 0 at ∞. Finally, by the monotone class theorem and monotone convergence of conditional expectations, it extends as stated.

Thus far, we have not given any pathwise meaning to the separate terms appearing in $M_\lambda(t)$. This can be done as follows. Since $X(t)$ is continuous in quadratic mean we may choose a measurable and integrable standard modification, in the sense of [4], Theorems 2.6 and 2.7. We use this to define $\int_0^t X(u)\,du$ as a Lebesgue integral, which is then continuous in t. Now we set

$$E\left(\int_0^\infty e^{-\lambda s} X(t+s)\,ds \,\Big|\, \mathscr{F}(t+)\right) - \lambda \int_0^t E\left(\int_0^\infty e^{-\lambda s} X(u+s)\,ds \,\Big|\, \mathscr{F}(u+)\right) du$$
$$= \lambda^{-1} M_\lambda(t) + E\left(\int_0^\infty e^{-\lambda s} X(s)\,ds \,\Big|\, \mathscr{F}(0+)\right) - \int_0^t X(u)\,du,$$

in accordance with the definition of $M_\lambda(t)$. The right side of this expression is now right-continuous with left limits. Denoting it by $K(t)(=K(t, w))$, we can solve the equation uniquely in t for the function

$$P_\lambda(t) = \lambda\, E\left(\int_0^\infty e^{-\lambda s} X(t+s)\,ds \,\Big|\, \mathscr{F}(t+)\right).$$

Indeed, the solution is given by

$$P_\lambda(t) = \lambda K(t) + \lambda^2 \int_0^t K(s_1)\,ds_1 + \lambda^3 \int_0^t \int_0^{s_1} K(s_2)\,ds_2\,ds_1 + \ldots$$

where the terms of the series converge uniformly in t for t in compact sets because $K(s)$ is then bounded. The uniqueness follows by forming the difference $D(t)$ of any two solutions, and observing that we have $D(t) - \lambda \int_0^t D(s)\,ds = 0$ which implies that $D(t)$ is continuous, hence differentiable. Therefore $D'(t) = \lambda D(t)$ with $D(0) = 0$, so $D(t) = 0$. Summing up, we may introduce

DEFINITION 1.3. — Let $P_\lambda(t)$ denote the above choice of

$\lambda E\left(\int_0^\infty e^{-\lambda s} X(t+s)\,ds \,\Big|\, \mathscr{F}(t+)\right)$. Thus $P_\lambda(t)$ is right-continuous (3), with left limits for

(3) The existence of a right continuous version of $P_\lambda(t)$ follows easily from the fact that $e^{-\lambda t} P_\lambda(t)$ is a difference of two positive supermartingales. However the above proof shows more, namely the possibility of constructing the prediction P_λ from M_λ and X.

$t>0$, and we have the pathwise identity

$$(1.2) \qquad M_\lambda(t)=P_\lambda(t)-P_\lambda(0)+\lambda\int_0^t (X(u)-P_\lambda(u))\,du,$$

where $X(t)$ is the version used above.

It is easy to see that $P_\lambda(t)$, and hence $M_\lambda(t)$, is continuous in quadratic mean as function of λ for fixed t. Further regularity of the dependence on λ will not be needed below, except for the fact that $\lim_{\lambda\to\infty} E(P_\lambda(t)-X(t))^2=0$ uniformly in finite intervals of t. This is an easy consequence of

$$(1.3) \quad E(P_\lambda(t)-X(t))^2=E\left(E\int_0^\infty \lambda e^{-\lambda s}(X(t+s)-X(t)\,ds\,|\,\mathscr{F}(t+)\right)^2$$
$$\leqq E\left(\int_0^\infty \lambda e^{-\lambda s}(X(t+s)-X(t))\,ds\right)^2,$$

by Jensen's inequality, where the right side tends to 0 by continuity of $\Gamma(s,\ t)$. Thus we see that the σ-field generated by $\{P_r(s), 0\leqq s\leqq t, r>0 \text{ rational}\}$ contains that generated by $X(s)$, $0\leqq s\leqq t$, up to P-null sets. Since it also is contained in $\mathscr{F}(t+)$ we see that (in the case when the parameter set is $[0,\ \infty)$) its completion by all P-null sets in $\mathscr{F}$ lies somewhere between $\mathscr{F}(t)$ and $\mathscr{F}(t+)$. We will see below (Corollary 1.8) that it equals $\mathscr{F}(t+)$.

What we wish to show next, and it is then main result of the paper, is that $\{P_\lambda(0), M_\lambda(s), s\leqq t, \lambda$ in a suitable countable set$\}$ generates the same completed σ-fields as $\{P_\lambda(s), 0\leqq s\leqq t, \lambda>0\}$. We note the difficulty: as $\lambda\to\infty$ it does not in general hold that $\lambda\int_0^t (X(u)-P_\lambda(u))\,du$ tends to zero, hence one cannot easily obtain $X(t)-X(0)$ from the $M_\lambda(t)$. For example, let $B(t)$ be a standard Brownian motion and let N be an independent standard normal random variable. If $X(t)=N+\int_0^t B(s)\,ds$, then we obtain easily

$$P_\lambda(t)=N+\lambda\int_0^\infty e^{-\lambda s}\left(\int_0^t B(u)\,du+s\,B(t)\right)ds=X(t)+\lambda^{-1}B(t),$$

and

$$M_\lambda(t)=P_\lambda(t)-P_\lambda(0)-\lambda\int_0^t \lambda^{-1}B(u)\,du=\lambda^{-1}B(t).$$

Thus it is true that $\{P_\lambda(0)$ and $M_\lambda(s), 0<s\leqq t\}$ generates the same completed σ-field as $\{P_\lambda(s), 0\leqq s\leqq t\}$, but not entirely trivial even in this simple case. In the general case our methods are of the "existential" type: no explicit general method of obtaining $P_\lambda(t)$ from $P_\lambda(0)$ and $M_\lambda(s), 0<s\leqq t$, has been found. One might hope to learn how to solve (1.2) for P_λ (and for $X=\lim_{\lambda\to\infty} P_\lambda$) by replacing t by a discrete parameter n, and using geometric sums instead of integrals to define the martingales in (1.1). However, it turns out that the discrete analogue of (1.2) then does not determine P_λ and X uniquely. Hence we apparently

have a situation in which a result holds in the continuous parameter case, but the discrete analog does not hold. We note also that, in this example, it suffices to know the quantities for only a single λ. This turns out to be true whenever $X(t)$ is stationary, or more generally has a Lévy canonical representation with $F=F(t-u)$ [*see* (2.10)].

THEOREM 1.4. — *For each* $0<t$, *and integer* $K>0$, *the* σ*-fields generated by* $\{P_\lambda(s),\ 0\leqq s\leqq t,\ \lambda>0\}$ *and by* $\{P_k(0),\ M_k(s),\ 0<s\leqq t, \text{ integer } k>K\}$ *differ only by P-null sets* [*i. e., they have the same completion in* $\mathscr{F}(t+)$].

Proof. — The idea of the proof is to transfer the pathwise identity (1.2) to a canonical path space on which it can be shown that the paths $(M_k(s))$ determine $[P_r(s),\ r \text{ rational}]$ uniquely for a given $\{P_k(0)\}$. Then the correspondence is one-to-one and Borel measurable hence a well-known theorem of D. Blackwell [1, Chap. III, Theorem 26] implies that they generate the same σ-fields.

Added in proof. — We omit our original attempted proof. It has been completed and simplified by P. A. Meyer. His proof is given in the note at the end the paper, for which we are extremely grateful.

We do not in general have $\mathscr{F}(t)=\mathscr{F}(t+)$, but it next will be shown that the completion in Theorem 1.4 always equals $\mathscr{F}(t+)$.

COROLLARY 1.8. — *The* σ*-fields* $\mathscr{F}^*(t)$ *generated by* $\{P_\lambda(s),\ 0\leqq s\leqq t,\ \lambda>0\}$ *have completions* $\mathscr{F}(t+)$.

Proof. — For $t_1<t_2$, the completed σ-field generated by $\{P_\lambda(s),\ s\leqq t_2\}$ contains $\mathscr{F}(t_1+)$ [since it contains $\mathscr{F}(t_2)$] hence it suffices to show that these σ-fields are right-continuous in t_2. By Theorem 1.4 they are generated by $\{P_k(0),\ M_k(s),\ 0<s\leqq t\}$, where $M_k(s)$ for each integer k is a right-continuous Gaussian martingale relative to $\mathscr{F}(s+)$. It follows that the increments $M_k(t_2)-M_k(t_1)$ are orthogonal to the Gaussian space generated by $P_k(0)$, $M_k(s)$, $0<s\leqq t_1$. Therefore, they are jointly independent of this subspace, and it follows that, if $S\in \bigcap_{t_2>t_1} \mathscr{F}^*(t_2)$ for fixed t_1, then S is independent of $M_k(t_2)-M_k(t_1+\varepsilon)$ for every k and $\varepsilon>0$. By right-continuity it is therefore independent of $M_k(t_2)-M_k(t_1)$. But for any $t_2>t_1$ we can write

$$I_S=f(M_k(s_i),\ s_i\leqq t_1;\ M_k(s_j)-M_k(t_1),\ t_1<s_j\leqq t_2),$$

where f is a Borel function of countably many variables. Letting $\sigma(X_k)$ denote the σ-field generated by $\{X_k\}$, it follows that for $A\in\sigma(M_k(s_j)-M_k(t_1);\ t_1<s_j\leqq t_2)$ and $B\in\sigma(P_\lambda(0),\ M_\lambda(s_i);\ s_i\leqq t_1)$ we have

$$\begin{aligned} P(S\cap A\cap B)=P(A)\,P(S\cap B)&=P(A)\,E(P(S\mid P_k(0),\ M_k(s_i);\ s_i\leqq t_1);\ B)\\ &=E(P(S\mid P_k(0),\ M_k(s_i);\ s_i\leqq t_1);\ A\cap B).\end{aligned}$$

Therefore, by the monotone extension theorem $I_S=P(S\mid P_k(0),\ M_k(s_i);\ s_i\leqq t_1)$, and hence $S\in\mathscr{F}^*(t_1)$ as required for right-continuity.

We now introduce for $X(t)$ an index $N(t)$ which measures, in effect, the number of translation invariant averages of Gaussian processes with independent increments needed, to represent $X(t)$ (Theorems 2.4 and 2.5). This index is quite different from the index of multiplicity $E(t)$ introduced by T. Hida ([8], Definition I.5) for the case $T=[0, \infty)$. Subsequently, we will define $E(t)$ in an equivalent way, and make some comparison of the two.

Definition 1.9. – The index of stationary of $X(t)$, $0<t<\infty$, denoted by $N(t)$, is the dimension of the linear (Gaussian) space generated by $\{M_\lambda(t), 0<\lambda\}$ (*see* also Theorem 1.2).

We note explicitly that $N(t)=\infty$ is allowed in Definition 1.9. In fact simple examples show that $N(t)=\infty$ can actually occur (for instance below, Example 1.15) and from the standpoint of Gaussian Markov processes, $N(t)=\infty$ is perhaps the rule rather than the exception. We will see that $N(t)$ need not be either right or left-continuous (Example 1.16). However, we have:

Proposition 1.10. – *The index* $N(t)$ *is non-decreasing. If* $0<t_1<t_2$ *and* $N(t_1)<\infty$, *then the Gaussian process* $X(t_1+t)$, $t>0$, *has index at* $t=t_2-t_1$ *bounded below by* $N(t_2)-N(t_1)$, *and above by* $N(t_2)$.

Proof. – Suppose first (contrary to what is asserted) that $N(t_2)<N(t_1)$. Then by the Gram-Schmidt orthogonalization procedure it is seen that there exist integers $0<k_1<\ldots<k_N$ for some $N>N(t_2)$ such that $\{M_{k_i}(t_1), 1\leqq i\leqq N\}$ generates a Gaussian subspace of dimension N. By definition of $N(t_2)$, there must be a non-trivial linear dependence at $t=t_2$

$$0=\sum_{i=1}^{N} c_i M_{k_i}(t_2); \qquad \sum_{i=1}^{N} c_i^2\neq 0.$$

Writing this in the form

$$\sum_{i=1}^{N} c_i M_{k_i}(t_1)=\sum_{i=1}^{N} c_i(M_{k_i}(t_1)-M_{k_i}(t_2)),$$

the right and left sides are independent Gaussian variables with mean 0. It follows that both are 0, contradicting the choice of N.

The basic observation needed to prove the second assertion is that the martingales $M_\lambda(t_1+t)-M_\lambda(t_1)$ are precisely the $M_\lambda(t)$ corresponding to the process $X(t_1+t)$. This is seen immediately from their definition. Then, if there are $k_1<\ldots<k_N$ such that $M_{k_i}(t_2)-M_{k_i}(t_1)$ are linearly independent (up to a P-null set), as in the first part of the proof $M_{k_i}(t_2)$, $1\leqq i\leqq N$, are also linearly independent, and so $N\leqq N(t_2)$. On the other hand, if $\{M_{k_i}(t_1), 1\leqq i\leqq N(t_1)\}$ generates the same Gaussian space as

$$\{M_\lambda(t_1), \lambda>0\} \quad \text{and if } \{M_{i_j}(t_2)-M_{i_j}(t_1), 1\leqq j\leqq N\},$$

generates the same Gaussian space as $\{M_\lambda(t_2)-M_\lambda(t_1), \lambda>0\}$, then for $\lambda>0$ we can write $M_\lambda(t_2)=M_\lambda(t_1)+(M_\lambda(t_2)-M_\lambda(t_1))$, i.e. as element of the space of dimension $N(t_1)+N$ generated by these two sets together. Therefore $N(t_2)-N(t_1)\leqq N$, as asserted.

We turn next to a basic structure theorem for $\mathscr{F}(t+)$. For this, we need to introduce the multiplicity index $E(t)$, first applied in our situation by T. Hida and H. Cramer. In relation to $N(t)$, we mention here only that $E(t) \leqq N(t)$ always holds. We will not repeat the definition of $E(t)$ from [8], which is complicated and derived from abstract Hilbert space theorems (4). Instead, we use

DEFINITION 1.11. – The index of multiplicity $E(t)$ $(0 \leqq E(t) \leqq \infty)$ of $X(t)$ is the smallest integer for which there exist right-continuous independent Gaussian processes $Y_1(s), Y_2(s), \ldots, Y_{E(t)}(s)$ with mean 0 and independent increments, whose Hilbert space closure in

$$0 < s \leqq u \quad \text{is } H\{M_k(s), k \geqq 1, s \leqq u\} \quad \text{for } u \leqq t \qquad (E(t)=0 \text{ if } M_k(t) \equiv 0).$$

The choice of $Y_1(s), \ldots, Y_{E(t)}(s)$ is obviously not unique. In the multiplicity theory approach, one obtains $Y_n(s)$ having the further property $d\mu_1 \gg d\mu_2 \gg \ldots$ where $d\mu_n(s) = dEY_n^2(s)$ and $\gg$ denotes absolute continuity from right to left. Then the $d\mu_n$ are unique but not the Y_n. A succint exposition of this theory is given in [10], Theorem 1, and it need not concern us here. All that is important for the present work is the fact that, given any sequence $Z_1(s), \ldots, Z_k(s), \ldots$, $(k < k_0 + 1 \leqq \infty)$ of right-continuous, square-integrable martingales, one may define as above the multiplicity $E_Z(t) \leqq k_0$ of the Hilbert space closures $H_Z(t) = H\{Z_k(s), s \leqq t, k < k_0 + 1\}$, and construct in a standard way an orthogonal sequence for $0 \leqq s \leqq 1$ of the form

$$Y_n(s) = \sum_{k=1}^{k_0} \int_0^s f_{n,k}(u)\, dZ_k(u), \qquad n < E_Z(1) + 1,$$

which generates the same $H_Z(t)$, $t \leqq 1$, and has the above absolute continuity. The construction may of course be repeated in $1 \leqq s \leqq 2$, etc., to extend Y_n for all t.

We can now state the basic result of Hida [8], and prove the inequality of indices mentioned. For this we fix a choice of Y_n in Definition 1.11.

THEOREM 1.12. – *(a) There are functions $F_n(t, u)$, $0 \leqq u \leqq t$, measurable in u and continuous in $t \geqq u$, such that*

$$X(t) = E(X(t) \mid \mathscr{F}(0+)) + \sum_{n=1}^{E(t)} \int_{0+}^{t} F_n(t, u)\, dY_n(u),$$

the integrals and sum being in quadratic mean.

For each (n, t), $F_n(t, u)$ is unique up to a $d\mu_n$-null set.

(b) The indices satisfy $E(t) \leqq N(t)$.

Proof. – We observe first that $X(t) - E(X(t) \mid \mathscr{F}(0+))$ is in the Gaussian subspace generated by $\{M_k(s), s \leqq t, k \geqq 1\}$. Otherwise it would have a component orthogonal to,

(4) In [8] it is assumed that $\{P_\lambda(0), \lambda > 0\}$ generates only the null subspace. This may be obtained here by replacing $X(t)$ by its projection on the orthogonal complement.

hence independent of, that subspace, contrary to Theorem 1.4. Since the operations defining the Y_k are all linear, the $Y_k(s)$, $s \leqq t$, generate this same subspace, hence $X(t)$ is the sum of its projections onto the subspaces of $Y_k(s)$, $s \leqq t$. But the projections are simply

$$\int_0^t \frac{dE(X(t)Y_k(u))}{d\mu_k(u)}\,dY_k(u),$$

as noted previously.

By Schwartz's inequality, for $u_1 < u_2 \leqq t_1 < t_2$

$$\frac{|E(X(t_2)-X(t_1))(Y_k(u_2)-Y_k(u_1))|}{E(Y_k(u_2)-Y_k(u_1))^2} \leqq E^{1/2}(X(t_2)-X(t_1))^2,$$

where the right side tends to zero as $t_2 \to t_1+$. It follows that

$$\frac{dE(X(r_2)Y_k(u))}{d\mu_k(u)} - \frac{dE(X(r_1)Y_k(u))}{d\mu_k(u)},$$

may be chosen to be bounded in absolute value by $E^{1/2}(X(r_2)-X(r_1))^2$ for all $0 < u \leqq r_1$, when $0 \leqq r_1 < r_2$ are rationals. Then we may define for all $t \geqq 0$;

$$F_k(t, u) = \lim_{r \to t+} \frac{dE(X(r)Y_k(u))}{d\mu_k(u)}, \qquad \text{and} \qquad F_k(t, u)$$

is continuous in t, uniformly in $u \leqq t$ for t in bounded sets [at $t=u$, of course, we only have $F_k(t+, u) = F_k(t, u)$].

Turning to part (*b*), we recall that $N(t)$ may be defined by ortogonalizing $\{M_r(t)\}$. Thus if $\{M_k;\, k < N(t)+1\}$ denotes for fixed t an orthonormal set generating the same subspace as $\{M_r(t),\ r \text{ rational}\}$, then the martingales $E(M_k \mid \mathscr{F}(s+))$, $0 < s \leqq t$, $1 \leqq k < N(t)+1$, generate for each s the same σ-field as $\{M_r(s)\}$ since each $M_r(t)$ is a (finite or infinite) linear combination of $\{M_k\}$, and $M_r(s) = E(M_r(t) \mid \mathscr{F}(s+))$. Consequently, in defining $E(t)$ we may use these $N(t)$ martingales and proceed as in the original construction to obtain at most $N(t)$ independent processes generating the same σ-fields. By the minimality property of $E(t)$, we therefore have $E(t) \leqq N(t)$.

Before going further, we will give a few examples of $E(t)$ and $N(t)$ in the simplest cases.

Example 1.13. — Let $T = (-\infty, \infty)$ and $X(t)$ be the stationary Ornstein-Uhlenbeck process with parameter $\beta > 0$. Such a process may be written in the form

$$X(t) = \int_{-\infty}^t e^{-\beta(t-u)}\,dW(u) \qquad \text{where} \quad dW(u)$$

is a process of Brownian increments, or "white noise". We have for $t \geqq 0$

$$E(X(t+s) \mid \mathscr{F}(t+))) = \int_{-\infty}^t e^{-\beta(t+s-u)}\,dW(u).$$

Thus

$$M_\lambda(t)=\frac{\lambda}{\lambda+\beta}\left(\int_{-\infty}^{t} e^{-\beta(t-u)}\,dW(u)-\int_{-\infty}^{0} e^{\beta u}\,dW(u)\right)$$
$$+\lambda\int_0^t\left(\int_{-\infty}^{s} e^{-\beta(s-u)}\,dW(u)-\frac{\lambda}{\lambda+\beta}\int_{-\infty}^{s} e^{-\beta(s-u)}\,dW(u)\right)ds$$
$$=\frac{\lambda}{\lambda+\beta}(e^{-\beta t}-1)\int_{-\infty}^{0} e^{\beta u}\,dW(u)+\frac{\lambda}{\lambda+\beta}\int_0^t e^{-\beta(t-u)}\,dW(u)$$
$$+\frac{\lambda\beta}{\lambda+\beta}\left(\int_{-\infty}^{0}\int_0^t e^{-\beta s}\,ds\,e^{\beta u}\,dW(u)+\int_0^t\int_u^t e^{-\beta s}\,ds\,e^{\beta u}\,dW(u)\right)$$
$$=\frac{\lambda}{\lambda+\beta}\int_0^t e^{-\beta(t-u)}\,dW(u)-\frac{\lambda}{\lambda+\beta}\int_0^t (e^{-\beta t}-e^{-\beta u})\,e^{\beta u}\,dW(u)=\frac{\lambda}{\lambda+\beta}(W(t)-W(0).$$

We see immediately that $E(t)=N(t)=1$ for all $t>0$, and in fact $M_\lambda(t)$ has the form $f(\lambda)(W(t)-W(0))$ where $W(t)-W(0)$ provides the "white noise" for the moving average representation. Of course, by stationarity of $X(t)$ we can just as well extend the definition of $M_\lambda(t)$ to replace 0 by any $t_0<0$. Such stationarity always implies that, for fixed λ, $M_\lambda(t)$ is a process of stationary independent increments, hence it is always a Wiener process when $X(t)$ is stationary. The meaning of the factor of $f(\lambda)$ is provided in Section 2.

Example 1.14. — Let $T=[0,\infty)$, and consider a process $X(t)=\int_0^t (2t-u)\,dW(u)$, where again $W(u)$ is a Wiener process. This is an example, due to P. Lévy [12], of a "proper canonical representation" of $X(t)$. In such a case, it is easy to see that $E(t)=1$ for all t. In fact, by Theorem 1.6 of Hida [8], the necessary and sufficient condition that there exists a proper canonical representation of $X(t)$ is that $E(t)=1$ for all t, but since we are permitting degenerate Gaussian processes ($X(t)\equiv 0$, $0\leqq t\leqq t_0$) this conclusion should be restated as $E(t-)\leqq 1$ in our notation. Now we have:

$$E(X(t+s)\,|\,\mathscr{F}(t+))=\int_0^t (2(t+s)-u)\,dW(u).$$

Thus

$$M_\lambda(t)=\int_0^t\int_0^\infty \lambda e^{-\lambda s}(2(t+s)-u)\,ds\,dW(u)$$
$$+\lambda\int_0^t\left(\int_0^s (2s-u)\,dW(u)-\int_0^s (2s-u+2\lambda^{-1})\,dW(u)\right)ds$$
$$=\int_0^t (2t-u+2\lambda^{-1})\,dW(u)-2\int_0^t W(s)\,ds$$
$$=2(t+\lambda^{-1})\,W(t)-\int_0^t u\,dW(u)-2tW(t)+2\int_0^t u\,dW(u)$$
$$=(2\lambda^{-1}+t)\,W(t)-\int_0^t W(u)\,du.$$

We see in this case that $\{ M_\lambda(t), \lambda>0 \}$ generates a σ-field with respect to which both $W(t)$ and $\int_0^t W(u)\,du$ are measurable. Hence $N(t)=2$ for all $t>0$, and so $E(t)<N(t)$ in this example.

We will investigate the meaning of $N(t)$ more fully in Section 2. Here we give two more examples of $E(t)$, of a kind not found in Lévy [12] or Hida [8] (but they may be implicit in some of the many papers of Lévy on this subject).

Example 1.15. – Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be independent, standard normal random variables. The process

$$X(t)=\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!},$$

is well-defined for $0 \leqq t$, and has continuous covariance. It is readily seen that $E(t)=N(t)=\infty$ for all $t>0$ in this case.

Example 1.16. – Let $S \subset [1, \infty)$ be a set with the property that both S and $[1, \infty)-S$ have Lebesque density 1 on everywhere dense subsets of $[1, \infty)$. To obtain S, it suffices to choose any Borel set S such that both S and $[1, \infty)-S$ intersect every interval in sets of positive Lebesgue measure. By a Theorem of Lebesgue ([13], Chap. IV, Theorem (6.1)) S has density 1 at almost every point of S, and the same holds true of $[1, \infty)-S$, so these density sets are both dense in $[1, \infty)$. We now let $W_1(t)$ and $W_2(t)$ be two independent Wiener processes, and we set $X(t)=0$ for $0 \leqq t \leqq 1$, and (for $t>1$)

$$X(t)=\int_1^t (I_S(s)\,W_1(s-1)+(1-I_S(s))\,W_2(s-1))\,ds.$$

Then $d/dt\,X(t)=W_1(t-1)$ for t in the dense set where S has Lebesgue density 1 and $d/dt\,X(t)=W_2(t-1)$ for t in the dense set where $[1, \infty)-S$ has Lebesgue density 1. It therefore follows that, for $t>1$, the process $X(s)$, $1<s<t$, generates both $W_1(s-1)$ and $W_2(s-1)$ in this interval. On the other hand, at $t=1$ we known by the 0-1 Law for $(W_1(t), W_2(t))$ that $\mathscr{F}(1+)$ can contain only sets of probability 0 or 1. It follows that $E(t)=0$ for $0<t\leqq 1$, and $E(t)=2$ for all $t>1$. In particular, this shows that we do not obtain right-continuity of $E(t)$, in general, even if $X(t)$ is continuous. Similarly, since $M_\lambda(1)$ is $\mathscr{F}(1+)$-measurable, we must have $N(t)=0$ for $0<t<1$, and since $N(t)\geqq E(t)$ we see that $N(t)$ cannot be right-continuous in the present case. It is, of course, easy to give examples in which neither $E(t)$ or $N(t)$ is left-continuous by introducing discrete normal variables at a fixed time: for example

$$X(t)=\begin{cases} 0 & \text{for } 0\leqq t\leqq 1, \\ (t-1)\,\xi & \text{for } 1<t \quad (\xi \text{ standard normal}). \end{cases}$$

2. The Canonical Case, and Extensions Thereof

The object here is to examine briefly the relationship of $M_\lambda(t)$ and the representation of Theorem 1.12(a) in the case that $E(t)<\infty$. This includes, of course, the case of stationary $X(t)$. As before, we use a fixed but arbitrary choice of the Y_n in Theorem 1.12. The key result is the following [inwhich we do not require $E(t)<\infty$].

LEMMA 2.1. — *In the sense of quadratic mean, we have*

$$M_\lambda(t)=\sum_{n=1}^{E(t)}\int_0^t\left(\int_0^\infty \lambda e^{-\lambda s}F_n(u+s,\, u)\,ds\right)dY_n(u),$$

where the inner integrals converge absolutely for $d\mu_n$—*a. e. u., and are in* $L^2(d\mu_n)$ *on* $(0,\, t]$.

Proof. — We observe first the expression (1.1) for $M_\lambda(t)$ is linear in $X(.)$. Therefore, when we substitute into M_λ the expression of Theorem 1.12 *a*, the part involving $E(X(.)\,|\,\mathscr{F}(0+)$ separates out. Since it is independent of the rest and $\mathscr{F}(0+)$-measurable, it must already be a martingale and hence it must be identically 0 along with $M_\lambda(0)$. Hence we may assume here that $E(X(.)\,|\,\mathscr{F}(0+))=0$. Then $P_\lambda(t)$ becomes

$$P_\lambda(t)=\lambda\int_0^\infty e^{-\lambda s}E\left(\sum_{n=1}^{E(t+s)}\int_0^{t+s}F_n(t+s,\, u)\,dY_n(u)\,|\,\mathscr{F}(t+)\right)ds$$

$$=\lambda\sum_{n=1}^{E(t)}\int_0^\infty e^{-\lambda s}\left(\int_0^t F_n(t+s,\, u)\,dY_n(u)\right)ds,$$

because the summands are independent with independent increments, and $Y_n(t)=0$ for $E(t)<n\leqq E(t+s)$. It is necessary to interchange order of integration, which is justified as follows. We have

$$(2.1)\quad \infty>E^{1/2}\int_0^\infty e^{-\lambda s}X^2(t+s)\,ds$$

$$\geqq\int_0^\infty e^{-\lambda s}E^{1/2}\left(\int_0^t F_n(t+s,\, u)\,dY_n(u)\right)^2 ds$$

$$\geqq\lambda^{-1/2}\left(\int_0^\infty e^{-\lambda s}\int_0^t F_n^2(t+s,\, u)\,d\mu_n(u)\,ds\right)^{1/2}.$$

Since, analogously, we have

$$\left(\int_0^\infty e^{-\lambda s}F_n(t+s,\, u)\,ds\right)^2\leqq\lambda^{-1}\int_0^\infty e^{-\lambda s}F_n^2(t+s,\, u)\,ds,$$

this proves the existence of the L^2-integrals $\int_0^t\int_0^\infty e^{-\lambda s}F_n(t+s,\, u)\,ds\,dY_n(u)$. Since they are in the L^2-closure of $\{Y_n(u'),\, u'\leqq t\}$, to justify the interchange of integration it is enough to apply Fubini's Theorem to obtain

$$E(Y_n(u')\int_0^\infty e^{-\lambda s}\int_0^t F_n(t+s,\, u)\,dY_n(u)\,ds)$$

$$=\int_0^\infty e^{-\lambda s}\int_0^{u'}F_n(t+s,\, u)\,d\mu_n(u)\,ds$$

$$=E(Y_n(u')\int_0^t\int_0^\infty e^{-\lambda s}F_n(t+s,\, u)\,ds\,dY_n(u)),\qquad u'\leqq t.$$

Recalling now that $F_n(t, u)$ is continuous in t, and noting that $M_\lambda(t)$ is the sum of $E(t)$ independent terms corresponding to the terms of $P_\lambda(t)$, we may compute the n-th term as follows

$$M_{\lambda,n}(t)=\int_0^t\int_0^\infty \lambda e^{-\lambda s}F_n(t+s, u)\,ds\,dY_n(u)$$
$$+\lambda\int_0^t\left(\int_0^\infty e^{-\lambda s}\int_0^v (F_n(v, u)-F_n(v+s, u))\,dY_n(u)\,ds\right)dv$$
$$=\int_0^t\int_0^\infty \lambda e^{-\lambda s}F_n(t+s, u)\,ds\,dY_n(u)$$
$$+\lambda\int_0^t\left(\int_0^\infty \lambda e^{-\lambda s}\int_u^t (F_n(v, u)-F_n(v+s, u))\,dv\,ds\right)dY_n(u),$$

where the interchange of integration is not hard to justify using the same method as before. Therefore, our coefficient of $dY_n(u)$ becomes

$$\int_0^\infty \lambda e^{-\lambda s}(F_n(t+s, u)+\lambda\int_u^t F_n(v, u)\,dv)\,ds-\lambda\int_u^t F_n(v, u)\,dv$$
$$-\int_0^\infty \lambda e^{-\lambda s}\frac{d}{ds}\int_{u+s}^{t+s}F_n(w, u)\,dw\,ds=\int_0^\infty \lambda e^{-\lambda s}F_n(u+s, u)\,ds,$$

which completes the proof of the Lemma.

We turn now to characterizing $N(t)$ in the case $N(t)=1$.

THEOREM 2.2. — *(a) Suppose that* $E(t)=1$ *for a fixed* $t>0$, *and let the representation of Theorem* 1.12*(a) be*

$$X(t')=E(X(t')\,|\,\mathcal{F}(0+))+\int_0^{t'}F(t', u)\,dY(u), \tag{2.2}$$

plus additional terms which may appear only for $t'>t$. *Then* $N(t)=1$ *if and only if we may choose* $F(t', u)=G(t'-u)$, $0\leqq u\leqq t$, *for some function* G.

(b) If $T=(-\infty, \infty)$ *then* X *is stationary and nondeterministic* ($\notin\mathcal{F}(-\infty)$) *if and only if:*

(i) *for every* t_0, *the process* $X(t_0+t)$ *satisfies* $N(t)=1$ *for* $t>0$,

(ii) *when* $F(t', u)=G(t'-u)$, $-\infty<u\leqq t'<\infty$, *as in (a), we have* $dE(Y^2(u))=\sigma^2 du$ *for a constant* $\sigma^2>0$, *and*

(iii) *the process* $X(t)-\int_{-\infty}^t G(t-u)\,dY(u)$ *is stationary.*

Remark. — We note that in (2.2) $dY(u)$ is unique only up to some multiple $f(u)$ which may, in general, be absorbed in $F(t', u)$.

Proof. — By Lemma 2.1, $N(t)$ is the dimension of the Gaussian subspace generated by

$$\left\{\int_0^t \int_0^\infty e^{-\lambda s} F(u+s, u)\, ds\, dY(u);\ \lambda>0\right\}.$$

If $F=F(t'-u)$ for all $0<u\leqq t$, this becomes simply

$$\left\{\int_0^\infty e^{-\lambda s} F(s)\, ds\, Y(t);\ \lambda>0\right\},$$

so clearly $N(t)=1$.

Conversely, if $N(t)=1$ we may assume the entire subspace to be generated at a single λ, say $\lambda=1$. Thus

$$(2.3)\qquad \int_0^t \int_0^\infty e^{-\lambda s} F(u+s, u)\, ds\, dY(u)=c(\lambda)\int_0^t \int_0^\infty e^{-s} F(u+s, u)\, ds\, dY(u),$$

for constants $c(\lambda)$. Letting C_0 denote the continuous functions with limit 0 at ∞, we will obtain

$$(2.4)\qquad \int_0^t \int_0^\infty f(s) F(u+s, u)\, ds\, dY(u)=c(f)\int_0^t \int_0^\infty e^{-s} F(u+s, u)\, ds\, dY(u),$$

for all f with $e^{\lambda s} f\in C_0$ for some $\lambda>0$, where $c(f)$ depends only on f and equality is up to a P-null set. Indeed, by (2.1) with $t=0$ we have for any $\lambda>0$

$$(2.5)\qquad \int_0^t \left(\int_0^\infty e^{-\lambda s}\left|F(u+s, u)\right| ds\right)^2 d\mu(u)$$
$$=\int_0^t \left(e^{2\lambda u}\int_0^\infty e^{-\lambda(u+s)}\left|F(u+s, u)\right| ds\right)^2 d\mu(u)$$
$$\leqq e^{2\lambda t}\int_0^t \left(\int_u^\infty e^{-\lambda v}\left|F(v, u)\right| dv\right)^2 d\mu(u)<\infty.$$

Then if $e^{\lambda s} f(s)\in C_0$, for $\varepsilon>0$ there is by the Stone-Weierstrass Theorem a $g(s)$ with $\left|e^{\lambda s} f(s)-g(s)\right|<\varepsilon$ uniformly and $g(s)$ has the form $\sum_1^n c_i e^{-\lambda_i s}$. Thus applying (2.3) to $e^{-\lambda s} g(s)$, it follows by (2.5) that

$$\lim_{\varepsilon\to 0}\int_0^t \left(\int_0^\infty (f(s)-e^{-\lambda s} g(s))\, F(u+s, u)\, ds\right)^2 d\mu(u)$$
$$\leqq \lim_{\varepsilon\to 0} \varepsilon\int_0^t \left(\int_0^\infty e^{-\lambda s}\left|F(u+s, u)\right| ds\right)^2 d\mu(u)=0.$$

Hence the left side of (2.4) is well-defined as a limit in quadratic mean, and hence the constants $c(e^{-\lambda s} g)$ on the right side must converge to a limit $c(f)$. It is also clear that $c(f)$ is unique, hence linear and continuous in f for norms of the form $\sup|e^{\lambda s} f(s)|$. It follows by the Riesz Representation Theorem that there is a unique signed measure $\eta(ds)$ on $[0, \infty)$ such that

$$c(f)=\int_0^\infty f(s)\,\eta(ds) \quad \text{whenever } e^{\lambda s} f(s) \in C_0,$$

and $\eta(ds)$ is bounded in the sense that

$$\int_0^\infty e^{-\lambda s}|\eta(ds)|<\infty \quad \text{for all } \lambda>0$$

(this is an easy consequence of the usual statement given in [5], IV,6.3 for a compact space). Now it follows by (2.4), choosing f in a countable dense set with respect to any norm $\sup|e^{\lambda s} f(s)|$, that

$$0=\int_0^\infty f(s)\,F(u+s, u)\,ds-\left(\int_0^\infty f(s)\,\eta(ds)\right)\left(\int_0^\infty e^{-v} F(u+v, u)\,dv\right),$$

for $d\mu$ – a. e. $u \leqq t$, and for all f with $e^{\lambda s} f \in C_0$. Therefore we have for $d\mu$ – a. e. u the identity of measures

$$F(u+s, u)\,ds \equiv \left(\int_0^\infty e^{-v} F(u+v, u)\,dv\right)\eta(ds).$$

From this we see that $\eta(ds)$ is absolutely continuous, and setting $\eta(ds)=h(s)\,ds$,

$$F(u+s, u)=h(s)\int_0^\infty e^{-v} F(u+v, u)\,dv,$$

except for a set of (s, u) which is $ds \times d\mu$-null on $(0, \infty) \times (0, t]$. Equivalently, we have outside a $dt \times d\mu$-null set

$$F(t, u)=h(t-u)\int_0^\infty e^{-v} F(u+v, u)\,dv. \tag{2.6}$$

But since $F(t, u)$ is continuous in t for fixed u, it follows by Fubini's Theorem that the essential right limits $\operatorname*{esslim}_{\tau \to s+} h(\tau-u)$, exist for all $s \geqq u$ and $u \leqq t$, except on a set of $u \leqq t$ which is either $d\mu$-null or on which $\int_0^\infty e^{-v} F(u+v, u)\,dv=0$ (for a discussion of the "essential topology", *see* [14]). However a set of the latter type must also be $d\mu$-null, since the representation (2.2) in conjunction with (2.6) implies that we can replace $dY(u)$ by 0 on such a set a.e. t, and hence for all t since $X(t)$ is continuous in quadratic mean, implying that $dY(u)$ on such a set would not be $\mathscr{F}(t+)$-measurable. Thus $\operatorname*{esslim}_{\tau \to s+} h(\tau-u)$ exists for all $s \geqq u$ and $d\mu$-a. e. $u \leqq t$. Clearly these limits are consistent in u whenever they exist for all $s \geqq u$, hence we can define $\hat{F}(t-u) = \operatorname{esslim}_{\tau \to t+} h(\tau - u)$ for all u outside a $d\mu$-null set, and obtain from (2.6):

$$(2.7) \qquad F(t, u) = \hat{F}(t-u) \int_0^\infty e^{-v} F(u+v, u)\, dv \quad \text{for all } t \geqq u,$$

except for u in a $d\mu$-null set. Finally, setting

$$dZ(u) = \left(\int_0^\infty e^{-v} F(u+v, u)\, dv \right) dY(u),$$

we obtain in place of (2.2) the representation

$$(2.8) \qquad X(t') = E(X(t') \mid \mathscr{F}(0+)) + \int_{0+}^{t'} \hat{F}(t'-u)\, dZ(u), \qquad 0 < t' \leqq t,$$

which completes the proof of (*a*).

Before proving (*b*), we insert another

Remark. — In the general case $E(t) = 1$, we showed in Lemma 2.1 that

$$\lambda \int_0^t \left(\int_0^\infty e^{-\lambda s} F(u+s, u)\, ds \right) dY(u) = M_\lambda(t).$$

Thus it is plausible that observation of $M_\lambda(s)$, $s \leqq t$, should lead, via inversion of the transform, to $F(t', u)$, $0 < u \leqq t' \leqq t$, and to $dY(u)$, $0 < u \leqq t$, up to equivalence. By (2.10) below, this is easily implemented whenever $N(t) = 1$.

Turning to the proof of (*b*), we require one fact from the general theory of wide-sense stationary processes in one dimension, namely that if $X(t)$ is stationary then $E(t) \leqq 1$. This is obvious in the discrete parameter case, and the present case may be reduced to this by use of a transform, as in [4], p. 583. We next show that $N(t) = E(t)$. Indeed, since $M_\lambda(t_1 + t) - M_\lambda(t_1)$ is $M_\lambda(t)$ for $X(t_1 + t)$ (Proposition 1.10) we see that for fixed $t > 0$, $M_\lambda(t_1 + t) - M_\lambda(t_1)$ is stationary in t_1. But if $N(t_0) > E(t_0)$ for some t_0, then there are *orthonormal*

$$Y_i(t_0) = \Sigma_n c_{i,n} M_{\lambda_{i,n}}(t_0), \qquad 1 \leqq i \leqq E(t_0) + 1,$$

and it follows easily that $t_0^{1/2} Y_i(t)$ are independent Wiener processes. Hence $E(t) \geqq E(t_0) + 1$, which is a contradiction.

The proof of (*a*), applied in the present case, now shows that $F = F(t-u)$ may be chosen the same for all t [where $F = Y = 0$ if $N(t) = 0$; we note that, by stationarity, $N(t)$ does not depend on t]. Thus we obtain a representation

$$X(t_0 + t) = E(X(t_0 + t) \mid \mathscr{F}(t_0+)) + \int_{t_0+t}^{t_0} F(t_0 + t - u)\, dY(u).$$

If we set $t_1 = t_0 + t$ and let $t_0 \to -\infty$ then this becomes

$$\text{(2.9)} \qquad X(t_1) = E(X(t_1) \mid \mathscr{F}(-\infty)) + \int_{-\infty}^{t_1} F(t_1 - u)\, dY(u),$$

and in particular

$$\int_{-\infty}^{t_1} F^2(t_1 - u)\, d\mu(u) < \infty \qquad \text{where} \quad d\mu(u) = dEY^2(u).$$

On the other hand, whenever $N(t) \equiv 1$ and $F = F(t-u)$, Lemma 2.1 implies

$$\text{(2.10)} \qquad M_\lambda(t_1 + t) - M_\lambda(t_1) = \lambda \int_0^\infty e^{-\lambda s} F(s)\, ds (Y(t_1 + t) - Y(t_1)).$$

In our case this is stationary in t_1, hence either $Y \equiv 0$ or else $d\mu(u) = \sigma^2\, du$ for a $\sigma^2 > 0$, and then $\sigma^{-1}\, dY$ is an incremental Wiener process. Since this implies the stationarity of the last term in (2.9), which is orthogonal to the first term on the right, this is also stationary. The converse in (*b*) is immediate, completing the proof of Theorem 2.2.

This result gives direct access to the moving average representation whenever the prediction problem for $X(t)$ is solved. For instance, in Example 1.13 of the Ornstein-Uhlenbeck process, it is just as simple to begin with the transition function of this (Markov) process, write the solution of the prediction problem, and then deduce the moving average representation. There is a variety of means available to solve the prediction problem, and we do not enter into them here. To give another illustration of how they lead directly to the moving average, we apply our method to the final example of Yaglom [17 *b*], Example 4.

Example 2.3. – Let $X(t)$ be the process with spectral density $(\xi^2 + \alpha^2)(\xi^4 + \alpha^4)^{-1}$, where $\alpha > 0$ is fixed and ξ is the spectral variable. A classical criterion of Szego shows that the term (iii) of Theorem 2.2 (*b*) is zero. The prediction problem may be solved by "Yaglom's method" and yields [17], (6.68)

$$E(X(t+s) \mid \mathscr{F}(t+)) = A(s) X(t) - \alpha (A(s) - B(s)) \int_0^\infty e^{-\alpha\tau} X(t - \tau)\, d\tau,$$

where

$$A(s) = \exp(-\alpha s/\sqrt{2})(\cos \alpha s/\sqrt{2} + (\sqrt{2} - 1) \sin \alpha s/\sqrt{2}),$$
$$B(s) = \exp(-\alpha s/\sqrt{2})(\cos \alpha s/\sqrt{2} + (1 - \sqrt{2}) \sin \alpha s/\sqrt{2}).$$

Then it is straightforward to compute our

$$P_\lambda(t) = f_1(\lambda) X(t) + f_2(\lambda) \int_0^\infty e^{-\alpha\tau} X(t - \tau)\, d\tau,$$

where:

$$f_1(\lambda) = 2\lambda(\lambda + \alpha)((\sqrt{2}\lambda + \alpha)^2 + \alpha^2)^{-1},$$

and:

$$f_2(\lambda)=2(2-\sqrt{2})\lambda\alpha^2((\sqrt{2}\lambda+\alpha)^2+\alpha^2)^{-1}.$$

From this it is again straightforward, but somewhat tedious, to compute our

$$\text{(2.11)}\quad M_\lambda(t)=f_1(\lambda)\left[X(t)-X(0)+(2-\sqrt{2})\alpha\left(e^{-\alpha t}\int_{-\infty}^{t} e^{\alpha u}X(u)\,du - \int_{-\infty}^{0} e^{\alpha u}X(u)\,du\right)+\alpha\int_0^t X(u)\,du\right].$$

Therefore, the factor in brackets on the right of (2.11) is just $c\,W(t)$ for a normalizing constant c (depending on α), and we have

$$\lambda\int_0^\infty e^{-\lambda v}h(v)\,dv=c^{-1}f_1(\lambda).$$

This is easily inverted by reversing the calculation used to obtain $f_1(\lambda)$, and we get $h(v)=c^{-1}A(v)$. Therefore, denoting the process in brackets at the right of (2.11) by $B(t)$, our moving average representation is

$$\text{(2.12)}\qquad X(t)=\int_{-\infty}^{t} A(t-u)\,dB(u).$$

We emphasize that this expression involves only observable quantities, unlike the spectral representation of $X(t)$ which involves quantities dependent on the future.

In the most general case of arbitrary $E(t)\leqq N(t)$ the picture is much the same as in Theorem 2.2*a*. We have

THEOREM 2.4. — *If* $E(t)=K<\infty$ *for given* $t>0$, *then* $N(t)=k<\infty$ *if and only if, in the representation of Theorem* 1.12*a*

$$\text{(2.13)}\qquad X(t)=E(X(t)\mid\mathscr{F}(0+))+\sum_{n=1}^{K}\int_0^t F_n(t,u)\,dY_n(u),$$

there are $c_j(v)$ *and* $g_{n,j}(u)$, $1\leqq j\leqq k$, *such that*

$$\text{(2.14)}\qquad F_n(t',u)=\sum_{j=1}^{k}c_j(t'-u)\,g_{n,j}(u),\qquad 1\leqq n\leqq K,$$

outside a set which is $dt'\times d\mu_n$*-null in* $\{t'\geqq u,\ u\leqq t\}$, *where the signed measures* $c_j(v)\,dv$ *are linearly independent in* $(0,\infty)$, *and the random variables*

$$Z_j=\sum_{n=1}^{K}\int_0^t g_{n,j}(u)\,dY_n(u),\qquad 1\leqq j\leqq k,$$

are also linearly independent.

Proof. — The same argument which led to (2.4) [based on (2.5)] applies here to yield from Lemma 2.1 that $N(t)$ is the dimension of the Gaussian space generated by

$$\bigcup_{\lambda>0}\left\{\sum_{n=1}^{K}\int_0^t\left(\int_0^\infty f(s)\,F_n(u+s,u)\,ds\right)dY_n(u) : e^{\lambda s}f(s)\in C_0\right\}.$$

Thus, if $N(t)=k$, there are $f_1, \ldots, f_k$ such that these k sums generate the entire space. Consequently, if $e^{\lambda s}f\in C_0$, there exist $c_1(f), \ldots, c_k(f)$ such that

$$(2.15)\quad 0=\left(\sum_{n=1}^{K}\int_0^t\int_0^\infty f(s)\,F_n(u+s,u)\,dY_n(u)\right) - \sum_{j=1}^{k}\left(c_j(f)\sum_{n=1}^{K}\int_0^t\int_0^\infty f_j(s)\,F_n(u+s,u)\,ds\,dY_n(u)\right).$$

We will show that, in fact, the $c_j(f)$ are continuous linear functionals on $\{f : e^{\lambda s}f\in C_0, \|f\|=\sup|e^{\lambda s}f|\}$ for each $\lambda>0$. It is clear, first, that the $c_j(f)$ are unique, since a difference of two such representations would contradict the choice of f_j. The linearity then follows trivially. To prove continuity, it suffices to consider a sequence

$$g_m : \lim_{m\to\infty}\sup_s|e^{\lambda s}g_m(s)|=0 \text{ and show that } \lim_m c_j(g_m)=0, \qquad 1\leqq j\leqq k.$$

In the contrary case, either there is a subsequence (also denoted $m\to\infty$) such that $\max_j|c_j(g_m)|\to\infty$, or there is a subsequence and two bounds $0<\varepsilon<M$ such that, for all m, $\varepsilon<\max_j|c_j(g_m)|<M$. In the former circumstance, if we divide (2.15) with $f=g_m$ by the constant $c_j(g_m) : |c_j(g_m)|=\max_i|c_i(g_m)|$, and choose a further subsequence to make each new coefficient on the right of (2.15) converge as $m\to\infty$, we obtain a nontrivial relation among the generating set contradicting the choice of the f_j. In the latter case, the same contradiction is obtained without the preliminary division, since the first terms on the right in (2.15) with $f=g_m$ tend to zero in quadratic mean by (2.5).

It follows that there are unique signed measures $\eta_1, \ldots, \eta_k$ such that $c_j(f)=\int f(s)\,\eta_j(ds)$, $1\leqq j\leqq k$, and by independence of the terms in (2.15) we have, for $1\leqq n\leqq K$

$$(2.16)\quad 0=\int_0^t\int_0^\infty f(s)\,F_n(u+s,u)\,ds\,dY_n(u) - \sum_{j=1}^{k}\int_0^\infty f(s)\,\eta_j(ds)\int_0^\infty f_j(v)\,F_n(u+v,u)\,dv\,dY_n(u).$$

Choosing, as before, f in a countable dense subset (for some fixed $\lambda > 0$) it follows that:

$$\mathrm{F}_n(u+s,\, u)\, ds = \sum_{j=1}^{k}\left(\eta_j(ds)\int_0^\infty f_j(v)\,\mathrm{F}_n(u+v,\, u)\, dv\right),$$

holds, as an identity of measures, for $d\mu_n$-a. e. $u \leqq t$, $1 \leqq n \leqq \mathrm{K}$. Obviously this will also hold if we replace each $\eta_j(ds)$ by its absolutely continuous component (if there is any singular part) and we henceforth assume this replacement. Setting

$$\eta(ds) = c_j(s)\, ds \qquad \text{and} \qquad g_{n,\, j}(u) = \int_0^\infty f_j(v)\,\mathrm{F}_n(u+v,\, u)\, dv,$$

we have:

$$\mathrm{F}_n(u+s,\, u) = \sum_{j=1}^{k} c_j(u+s,\, u)\, g_{n,\, j}(u),$$

except on a set which is $ds \times d\mu_n$-null in $(0, \infty) \times (0, t]$, as asserted in the Theorem.

Substituting this expression into the generating set of (2.15), we find that the random variables

$$\sum_{n=1}^{\mathrm{K}} \int_0^t \sum_{j=1}^{k}\left(\int_0^\infty f_m(s)\, c_j(s)\, ds\right) g_{n,\, j}(u)\, d\mathrm{Y}_n(u)$$

must be linearly independent, $1 \leqq m \leqq k$. Hence, in particular, $\sum_{n=1}^{\mathrm{K}} \int_0^t g_{n,\, j}(u)\, d\mathrm{Y}_n(u) = \mathrm{Z}_j$ must be linearly independent, $1 \leqq j \leqq k$. Moreover, the measures $c_j(s)\, ds$ must be linearly independent on $(0, \infty)$, or else we could write the above sums using fewer than k linear combinations of the Z_j, which would clearly not generate a space of dimension k.

Suppose, conversely, that such functions c_j and $g_{n,\, j}$ exist

$$\left(\text{with} \int_0^t g_{n,\, j}^2(u)\, d\mu(u) < \infty \text{ and } \int_0^\infty e^{-\lambda s}\,|c_j(s)|\, ds < \infty, \text{ by hypothesis}\right).$$

Then we may write, if $e^{\lambda s} f \in \mathrm{C}_0$

$$\sum_{n=1}^{\mathrm{K}} \int_0^t \int_0^\infty f(s)\,\mathrm{F}_n(u+s,\, u)\, ds\, d\mathrm{Y}_n(u) = \sum_{j=1}^{k}\left(\int_0^\infty f(s)\, c_j(s)\, ds \sum_{n=1}^{\mathrm{K}} \int_0^t g_{n,\, j}(u)\, d\mathrm{Y}_n(\mathrm{u})\right).$$

Since the measures $c_j(s)\, ds$ are linearly independent, we may choose functions $h_1, \ldots, h_k$ such that $\left(\int_0^\infty h_i(s)\, c_j(s)\, ds\right)$ has non-zero determinant. Then the dimension of the Gaussian space generated by the above expressions with $f = h_i$, $1 \leqq i \leqq k$, is the same as that generated by $\mathrm{Z}_1, \ldots, \mathrm{Z}_k$, which by hypothesis is k, completing the proof.

The general expression (2.14) explains our use of the term "index of stationarity" for $\mathrm{N}(t)$. Of course, real stationarity is obtained only if the $g_{n,\, j}$ are constants and the $d\mathrm{Y}_n$ are homogeneous, in which case the dY_n are combined into a single Wiener process so that $E(t)=N(t)=1$, in accordance with Theorem 2.2. We do not know if the exceptional $dt \times d\mu_n$- null sets in Theorem 2.4 can be avoided [relying on the known continuity of $F_n(t, u)$ in $t \geqq u$, as in the proof of Theorem 2.2 *a*]. As examples of cases in which $N(t) < \infty$, we may extend Example 1.14 to any case in which $E(t) < \infty$ and each of the $F_n(t, u)$, $1 \leqq n \leqq E(t)$, is a polynomial in (t, u). In fact, if the polynomial is of degree K_n in t, then this term of (2.13) generates a space of degree at most K_n+1, and consequently

$$N(t) \leqq \sum_{n=1}^{E(t)} (K_n+1).$$

3. The covariance

In Theorem 1.4 we showed that, for a given Gaussian process $X(t)$, the martingales $M_k(s)$, $k \geqq K > 0$, determine the same σ-fields as the projections of $X(s)$, $0 < s \leqq t$, on the orthogonal complement of the Gaussian subspace corresponding to $\mathscr{F}(0+)$ (i. e. orthogonal to $E(X(t) \mid \mathscr{F}(0+))$ for all t). In fact, $M_k(s)$ together with $P_k(0)$, $k \geqq K$, determine $X(s)$ almost uniquely in $0 \leqq s \leqq t$. Then it follows that the determination of $X(s)$ from $M_k(s)$ and $P_k(0)$ can be made for any two processes X by the same Borel function, and hence the joint distributions of $P_k(0)$ and $M_k(s)$ determine uniquely those of $X(s)$, $0 \leqq s \leqq t$. Unfortunately, it does not seem easy to write the covariance Γ of X explicity in terms of that of $P_k(0)$ and $M_k(s)$. On the other hand, it is an interesting and nontrivial exercise to write the covariance of $M_k(s)$ in terms of that of $P_k(s)$ (hence, ultimately, in terms of Γ). This exercise may indicate how to estimate the covariance of $P_k(0)$ and M_k in the non-stationary case, thus leading perhaps to estimates of the elements of the canonical representation of Theorem 1.12 *a* once the prediction problem is solved.

To obtain the covariance of M_λ, since the M_λ have mutually independent increments in time (i. e. $M_{\lambda_1}(s_2)-M_{\lambda_1}(s_1)$ and $M_{\lambda_2}(t_2)-M_{\lambda_2}(t_1)$ are independent for $s_1 < s_2 \leqq t_1 < t_2$ and all λ_1, λ_2), it suffices by writing

$$4\,E(M_{k_1}(t)\,M_{k_2}(t)) = E(M_{k_1}(t)+M_{k_2}(t))^2 - E(M_{k_1}(t)-M_{k_2}(t))^2,$$

to obtain only the second moments of $M_{k_1}+M_{k_2}$ and of $M_{k_1}-M_{k_2}$, for each t. We shall write the proof for $EM_\lambda^2(t)$, and leave it to the reader to check that it extends easily to $E(M_{\lambda_1} \pm M_{\lambda_2})^2$. This last result is therefore simply stated as a Corollary.

THEOREM 3.1. — *With the notation of Definition* 1.3

$$EM_\lambda^2(t) = EP_\lambda^2(t) - EP_\lambda^2(0) + 2\lambda \int_0^t E(X(u)\,P_\lambda(u) - P_\lambda^2(u))\,du.$$

Proof. — We make repeated use of the device of writing, for $\varepsilon > 0$

$$(3.1)\quad \int_0^\infty e^{-\lambda s} X(t+\varepsilon+s)\,ds - \int_0^\infty e^{-\lambda s} X(t+s)\,ds$$
$$= (e^{\lambda\varepsilon}-1) \int_\varepsilon^\infty e^{-\lambda s} X(t+s)\,ds - \int_0^\varepsilon e^{-\lambda s} X(t+s)\,ds.$$

As a first consequence, by "projecting" this identity onto $\mathscr{F}(t+)$ it follows immediately that $E(P_\lambda(t+\varepsilon)\,|\,\mathscr{F}(t+)) - P_\lambda(t) = \varepsilon\lambda P_\lambda(t) - \varepsilon\lambda X(t) + o(\varepsilon)$ in the sense of quadratic mean, i. e. that $\lim\limits_{\varepsilon\to 0+}$

$$\varepsilon^{-1} E(P_\lambda(t+\varepsilon)-P_\lambda(t)\,|\,\mathscr{F}(t+) = \lambda P_\lambda(t) - X(t)$$

in quadratic mean, uniformly in finite time intervals.

Setting for convenience $t=1$, we now write

$$\begin{aligned}
EM_\lambda^2(1) &= \lim_{n\to\infty} E\sum_{i=1}^{n}\left(M_\lambda\left(\frac{i}{n}\right)-M_\lambda\left(\frac{i-1}{n}\right)\right)^2\\
&= \lim_{n\to\infty} E\left(\sum_{i=1}^{n}\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)\right)+\lambda\int_{(i-1/n}^{i/n}(X(u)-P_\lambda(u))\,du\right)^2\\
&= \lim_{n\to\infty} E\left(\sum_{i=1}^{n}\left(\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)-E\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)|\,\mathscr{F}\left(\frac{i-1}{n}+\right)\right)\right)\right)\right)^2\\
&= \lim_{n\to\infty} E\sum_{i=1}^{n}\left(\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)\right)^2-E\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)|\,\mathscr{F}\left(\frac{i-1}{n}+\right)\right)^2\right)\\
&= \lim_{n\to\infty} E\sum_{i=1}^{n}\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)\right)^2,
\end{aligned}$$

where the third and fifth equalities are justified by the above estimate (and right-continuity of $X(u)-P_\lambda(u)$ in quadratic mean) while the fourth is by squaring the sum and introducing conditional expectations given $\mathscr{F}((i-1)/n+)$ term by term. We remark that this calculation is also valid in the non-Gaussian case, where projection onto $H((i-1)/n+)$ replaces conditional expectation.

On the other hand, (3.1) also shows that

$$\begin{aligned}
&\lim_{n\to\infty} E\sum_{i=1}^{n}\left(P_\lambda\left(\frac{i}{n}\right)-P_\lambda\left(\frac{i-1}{n}\right)\right)^2\\
&\quad= \lim_{n\to\infty}\lambda^2 E\sum_{i=1}^{n}\left(E\left(\int_0^\infty e^{-\lambda s}X\left(\frac{i}{n}+s\right)ds\,|\,\mathscr{F}\left(\frac{i}{n}+\right)\right)\right.\\
&\qquad\left.-E\left(\int_0^\infty e^{-\lambda s}X\left(\frac{i}{n}+s\right)ds\,|\,\mathscr{F}\left(\frac{i-1}{n}+\right)\right)\right)^2\\
&\quad= \lim_{n\to\infty}\lambda^2 E\sum_{i=1}^{n}\left(E\int_0^\infty e^{-\lambda s}X\left(\frac{i}{n}+s\right)ds\,\middle|\,\mathscr{F}\left(\frac{i}{n}+\right)\right)^2\\
&\qquad-\left(E\int_0^\infty e^{-\lambda s}X\left(\frac{i}{n}+s\right)ds\,|\,\mathscr{F}\left(\frac{i-1}{n}+\right)\right)^2.
\end{aligned}$$

Reordering the terms of the sum by a familiar device due to Abel, and then applying a simple algebraic identity term by term, this becomes

$$E(P_\lambda^2(1)-P_\lambda^2(0))$$

$$-\lim_{n\to\infty}\lambda^2 E\sum_{i=1}^{n}\left(E\int_0^\infty e^{-\lambda s}\left(X\left(\frac{i+1}{n}+s\right)-X\left(\frac{i}{n}+s\right)\right)ds\,\Big|\,\mathscr{F}\left(\frac{i}{n}+\right)\right)$$
$$\times\left(E\int_0^\infty e^{-\lambda s}\left(X\left(\frac{i+1}{n}+s\right)+X\left(\frac{i}{n}+s\right)\right)ds\,|\,\mathscr{F}\left(\frac{i}{n}+\right)\right).$$

Now two more applications of (3.1) show that this is the same as

$$EP_\lambda^2(1)-P_\lambda^2(0)-\lim_{n\to\infty}E\lambda\sum_{i=1}^{n}\left(n^{-1}\left(P_\lambda\left(\frac{i}{n}\right)-X\left(\frac{i}{n}\right)\right)\left(2P_\lambda\left(\frac{i}{n}\right)+0\left(\frac{1}{n}\right)\right)\right)$$
$$=EP_\lambda^2(1)-EP_\lambda^2(0)+2\lambda\int_0^1 E(X(u)P_\lambda(u)-P_\lambda^2(u))\,du,$$

as asserted.

As remarked before the theorem, the same reasoning may be applied to linear combinations $M_{\lambda_1}(t)\pm M_{\lambda_2}(t)$, and leads without more difficulty to

COROLLARY 3.2.

$$E(M_{\lambda_1}(t)M_{\lambda_2}(t))=E(P_{\lambda_1}(t)P_{\lambda_2}(t))-E(P_{\lambda_1}(0)P_{\lambda_2}(0))$$
$$+\int_0^t E(X(u)(\lambda_2 P_{\lambda_1}(u)+\lambda_1 P_{\lambda_2}(u))-(\lambda_1+\lambda_2)P_{\lambda_1}(u)P_{\lambda_2}(u))\,du.$$

We note that these expressions may be written out explicitly in terms of the covariance $E(E(X(u+s_1)\,|\,\mathscr{F}(u+))\,E(X(u+s_2)\,|\,\mathscr{F}(u+)))$ of the assumed solution to the prediction problem, and hence in terms of the covariance Γ. However, while we know that $\{E(M_{\lambda_1}(t)M_{\lambda_2}(t)),\ E(P_{\lambda_1}(0)P_{\lambda_2}(0)),\ 0<t,\ \lambda_1,\lambda_2\}$ determines $\Gamma(s,t)$, it does not seem possible to solve the equations explicitly for $E(P_{\lambda_1}(t)P_{\lambda_2}(t))$ (which would lead to Γ, using the fact that $E(X(t)P_\lambda(t))=\lim_{\lambda_1\to\infty}E(P_{\lambda_1}(t)P_\lambda(t))$). On the other hand, we should again call attention to the fact that when $E(t)=1$, which is by all odds the most prevalent case, the proof of Lemma 2.1 gave us

$$M_\lambda(t)=\int_0^t\left(\int_0^\infty e^{-\lambda s}F(u+s,u)\,ds\right)dY(u),$$

from which it would not seem too difficult to obtain the function $F(s,t)$ and corresponding variance $\sigma^2(t)=EY^2(t)$ by making observations of $M_\lambda(t)$. With these it is, of course, a trivial matter to write both the covariance of P_λ and the covariance of X. Finally, we recall that the problem of obtaining $F(s,t)$ from the covariance of X (in the case $E(t)=1$) has been extensively studied by P. Lévy (*see* for example [11]; 2.3). However, even under strict differentiability assumptions on Γ, it leads to a Fredholm integral equation which is not explicitly solvable in general. This reinforces our viewpoint that it is often necessary to use the predictor in obtaining the canonical representation of $X(t)$.

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