ANNALES SCIENTIFIQUES DE L'É.N.S.

DAN BARBASCH

Filtrations on Verma modules

Annales scientifiques de l'É.N.S. 4e série, tome 16, nº 3 (1983), p. 489-494
<http://www.numdam.org/item?id=ASENS_1983_4_16_3_489_0>

Ann. scient. Éc. Norm. Sup.
4e série, t. 16, 1983, p. 489 à 494.

FILTRATIONS ON VERMA MODULES

BY Dan BARBASCH (1)

1. Introduction

The purpose of this note is to make several remarks on the Jantzen filtration that follow from the paper of O. Gabber and A. Joseph [G-J]. The first is to show that their techniques show that in fact the filtration has to coincide with the Socle filtration for Verma modules. The second is to apply their result to calculate Ext^1 between two irreducible Verma modules.

Since the publication of [G-J], the conjecture that the Jantzen filtration is hereditary has been proved by A. Beilinson and J. Bernstein (unpublished).

I would like to thank D. Vogan for valuable discussions.

2. Notation and preliminary results

We adopt the notation and conventions of [G-J] paeticularly section 4. We assume that the Verma modules all have fixed regular infinitesimal character ρ (to keep the notation to a minimum). We write $M(w)$ for $M(-w\rho)$ if $w \in W$, the Weyl group. Let α be a simple root, $s = s_\alpha$ the corresponding simple reflection and assume $ws > w$ in the Bruhat ordering. Fix an inclusion $M(w) \to M(ws)$. If $\{M^j\}$ is the Jantzen filtration, then the Jantzen conjecture states that

(2.1) $$M(w)^j = M(ws)^{j+1} \cap M(w).$$

Set $M_j = M^j/M^{j+1}$. Let $\theta_\alpha = \varphi_\alpha \psi_\alpha$ be the coherent continuation functor. Write $X = M(ws)$, $Z = M(w)$ and $Y = \theta_\alpha X = \theta_\alpha Z$.

(1) Supported in part by an NSF grant and a Rutgers Research Council Grant.

Then there is an exact sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0. \tag{2.2}$$

Y also has a canonical filtration satisfying the following properties ([G-J], section 4)

$$\begin{cases} Y^j = \theta_\alpha Z^j = \theta_\alpha X^{j+1}, \\ Y_j = \theta_\alpha Z_j = \theta_\alpha X_{j+1}, \end{cases} \tag{2.3}$$

$$\pi(Y^j) \subset Z^j, \tag{2.4}$$

$$Y^{j+1} \cap X \subset X^{j+1} \subset Y^j, \tag{2.5}$$

$$Z^{j+1} \subset \pi(Y^j).$$

Then (*cf.* [G-J], section 4) one can set up the following exact sequences. Let

$$X^a_{j+1} = X^{j+1}/(Y^{j+1} \cap X),$$
$$X^b_{j+1} = (Y^{j+1} \cap X)/X^{j+2}.$$

Then

$$0 \to X^b_{j+1} \to X_{j+1} \to X^a_{j+1} \to 0. \tag{2.7}$$

The sequence splits, X_{j+1} is completely reducible and X^b_{j+1} is the largest submodule so that $\theta_\alpha(X^b_{j+1}) = 0$.

Let

$$Z^a_j = \pi(Y^j)/Z^{j+1}$$
$$Z^b_{j+1} = Z^{j+1}/\pi(Y^{j+1}).$$

Then

$$0 \to Z^a_{j+1} \to Z_{j+1} \to Z^b_{j+1} \to 0. \tag{2.8}$$

Z_{j+1} is completely reducible and Z^a_{j+1} is the smallest submodule so that $\theta_\alpha(Z_{j+1}/Z^a_{j+1}) = 0$.

Let

$$Y^x_j = (Y^j \cap X)/(Y^{j+1} \cap X),$$
$$Y^z_j = \pi(Y^j)/\pi(Y^{j+1}).$$

Then

$$0 \to Y^x_j \to Y_j \to Y^z_j \to 0, \tag{2.9.1}$$

$$0 \to X^a_{j+1} \to Y^x_j \to X^b_j \to 0, \tag{2.9.2}$$

$$0 \to Z^b_{j+1} \to Y^z_j \to Z^a_j \to 0, \tag{2.9.3}$$

$$X^a_{j+1} \simeq Z^a_j. \tag{2.10}$$

Let now $v \in W$ be such that $vs > v$ and $L = L(v)$, the corresponding irreducible quotient of $M(v)$. Then there are exact sequences

$$0 \to L \to \theta_\alpha L \to Q_\alpha \to 0, \tag{2.10.1}$$

$$0 \to U_\alpha L \to Q_\alpha \to L \to 0, \tag{2.10.2}$$

which do not split.

Extending the definition of U_α to semisimple modules, we get an exact sequence

$$0 \to X_j^b \to U_\alpha Z_j^a \to Z_{j+1}^b \to 0. \tag{2.11}$$

Since $Y_j = \theta_\alpha Z_j^a$ and θ_α is exact, we can write

$$Y_j = \bigoplus_{vs > v} [L(v) : Z_j^a]\, \theta_\alpha L(v). \tag{2.12}$$

LEMMA. — *For $y \in W$ we have*

$$\operatorname{Hom}[L(y) : Y_j^x] = \begin{cases} \operatorname{Hom}[L(y) : X_{j+1}^a] & \text{if } ys > y, \\ 0 & \text{if } ys < y. \end{cases}$$

Proof. — Suppose $ys > y$. Then $\operatorname{Hom}[L(y) : X_j^b] = 0$. The long exact sequence

$$0 \to \operatorname{Hom}[L(y) : X_{j+1}^a] \to \operatorname{Hom}[L(y) : Y_j^x] \to \operatorname{Hom}[L(y) : X_j^b] \to$$

gives the assertion.

Suppose $ys < y$ but $\operatorname{Hom}[L(y) : Y_j^x] \neq 0$. Then $\operatorname{Hom}[L(y) : Y_j] \neq 0$. By (2.12) there must be v such that $\operatorname{Hom}[L(y) : \theta_\alpha L(v)] \neq 0$. But the only submodule of $\theta_\alpha L(v)$ is $L(v)$ itself and v must be such that $vs > v$, a contradiction. This proves the lemma.

We now proceed to define the Socle filtration. Given $M = M(w)$, it is well known that the largest semisimple submodule of M is $M(\mathrm{id})$. We label $M^{[l(w)]} = M(\mathrm{id})$. Suppose we have defined $M^{[j+1]}$. Then we define $M^{[j]}$ as the largest submodule so that $M_{[j]} = M^{[j]}/M^{[j+1]}$ is semisimple. This is a well-defined construction and $\{M^{[j]}\}$ is called the Socle filtration of M.

3. The main theorem

THEOREM. — $M^{[j]} = M^j$.

Proof. — The proof goes by ascending induction in $l(w)$ and descending induction in j. It is enough to assume the statement to be true for Z and for X up to $j+3$ and to show it is true for $j+2$.

It is clear that $X^{j+2} \subseteqq X^{[j+2]}$. Suppose $X^{j+2} \neq X^{[j+2]}$. Let $k \leqq j+1$ be the largest integer such that

$$X^{k,\, j+2} = X^{[j+2]} \cap X^k / X^{[j+2]} \cap X^{k+1} \neq 0.$$

Let $y \in W$ be such that $[L(y) : X^{k,\,j+2}] \neq 0$.

Then

$$(3.1) \qquad 0 \to X^{[j+2]} \cap X^k / X^{[j+2]} \cap X^{k+2} \to X^k / X^{k+2}$$

and its image under the map π

$$(3.2) \qquad 0 \to X^{k+1}/X^{k+2} \to X^k/X^{k+2} \xrightarrow{\pi} X^k/X^{k+1} \to 0$$

is nonzero.

Then

$$(3.3) \qquad \operatorname{Hom}[L(y) : X^k/X^{k+2}] > \operatorname{Hom}[L(y) : X^{k+1}/X^{k+2}].$$

We show that this cannot be.

Case 1. — $y < ys$.

$$\operatorname{Hom}[L(y) : X^k] = \operatorname{Hom}[L(y) : Z^{k-1}] \qquad \text{since} \quad [L(y) : X^k/Z^{k-1}] = 0.$$

By the induction hypothesis, the socle of Z/Z^{k+1} is Z^k/Z^{k+1}, which implies

$$(3.4) \qquad \operatorname{Hom}[L(y) : Z^{k-1}/Z^{k+1}] = \operatorname{Hom}[L(y) : Z^k/Z^{k+1}].$$

This is a contradiction.

Case 2. — $y > ys$.

Since $X^{k+2} \subset Y^{k+1} \subset Y^k$, we get $Y^k \cap X^{k+2} = X^{k+2}$.

Then we have an exact sequence

$$(3.5) \qquad 0 \to (Y^k \cap X)/X^{k+2} \to X^k/X^{k+2} \to X^k/Y^k \cap X \to 0$$

But $X^k/Y^k \cap X = X_k^a$ and $[L(y) : X_k^a] = 0$.

Thus

$$(3.6) \qquad \operatorname{Hom}[L(y) : X^k/X^{k+2}] = \operatorname{Hom}[L(y) : (Y^k \cap X)/X^{k+2}].$$

Also

$$(3.7) \qquad 0 \to (Y^{k+1} \cap X)/X^{k+2} \to (Y^k \cap X)/X^{k+2} \to (Y^k \cap X)/(Y^{k+1} \cap X) \to 0.$$

But $(Y^k \cap X)/(Y^{k+1} \cap X) = Y_k^x$ so by Lemma 2.13

$$(3.8) \quad \operatorname{Hom}[L(y) : (Y^k \cap X)/X^{k+2}] = \operatorname{Hom}[L(y) : (Y^{k+1} \cap X)/X^{k+2}] \leqq \operatorname{Hom}[L(y) : X_{k+1}].$$

Thus, combining (3.6) and (3.8) we get

$$\operatorname{Hom}[L(y) : X^k/X^{k+2}] = \operatorname{Hom}[L(y) : X_{k+1}].$$

This contradicts (3.3) so

$$X^{[j+2]}=X^{j+2}.$$

The proof is now complete.

4. Computation of Ext^1

Let $y<w$. We compute $\mathrm{Ext}^1[L(y), L(w)]$. Consider the exact sequences

$$0 \to M(w)^1 \to M(w) \to L(w) \to 0, \tag{4.1}$$

$$0 \to M(y)^1 \to M(y) \to L(y) \to 0 \tag{4.2}$$

(4.1) gives rise to a long exact sequence

$$0 \to \mathrm{Hom}[L(w), L(y)] \to \mathrm{Hom}[M(w), L(y)] \to \mathrm{Hom}[M^1(w), L(y)] \to$$
$$\to \mathrm{Ext}^1[L(w), L(y)] \to \mathrm{Ext}^1[M(w), L(y)] \to$$

But $\mathrm{Hom}[M(w), L(y)]=0$ because $L(w)$ is the unique irreducible quotient of $M(w)$ and $\mathrm{Ext}^1[M(w), L(y)]=0$ because it is equal to $H^1(n, L(y))^{w\rho-\rho}$ which is zero since $y<w$. Thus

$$\mathrm{Ext}^1[L(w), L(y)] \simeq \mathrm{Hom}[M^1(w), L(y)]. \tag{4.3}$$

Next, consider the long exact sequence coming from (4.2)

$$0 \to \mathrm{Hom}[L(y), L(w)] \to \mathrm{Hom}[M(y), L(w)] \to \mathrm{Hom}[M(y)^1, L(w)] \to$$
$$\to \mathrm{Ext}^1[L(y), L(w)] \to \mathrm{Ext}^1[M(y), L(y)] \to .$$

Since $\mathrm{Hom}[M(y)^1, L(w)]=0$

$$\dim \mathrm{Ext}^1[L(y), L(w)] \leqq \dim \mathrm{Ext}^1[M(y), L(w)]=\mu(y, w) \tag{4.4}$$

(from the Kazhdan-Lusztig conjecture as phrased in [V], conjecture 3.4 and the fact that $\mathrm{Ext}^1[M(y), L(w)] \simeq H^1[n, L(w)]^{y\rho-\rho}$).

Putting (4.3) and (4.4) together, we get

$$\dim \mathrm{Hom}[M(w)^1, L(y)] \leqq \dim \mathrm{Ext}^1[L(w), L(y)] = \dim \mathrm{Ext}^1[L(y), L(w)] \leqq \mu(y, w). \tag{4.5}$$

By [G-J], Corollary 4.9, $\dim \mathrm{Hom}[M^1(w), L(y)] \geqq \mu(ww_0, yw_0)$.

By [K-L], Corollary 3.2, $\mu(y, w)=\mu(ww_0, yw_0)$. We summarize our result.

PROPOSITION. — *Let $y<w$. Then*

$$\dim \mathrm{Ext}^1[L(y), L(w)]=\mu(y, w).$$

REFERENCES

[G-J] O. GABBER and A. JOSEPH, *Towards the Kazhdan-Lusztig Conjecture* (*Ann. scient. Éc. Norm. Sup.*, 4e série, T. 14, 1981, pp. 261-302).

[K-L] D. KAZHDAN and G. LUSZTIG, *Representations of Coxeter Groups and Hecke Algebras* (*Invent. Math.*, Vol. 53, 1979, pp. 127-149).

[V] D. VOGAN, *Irreducible Characters of Semisimple Lie Groups II, The Kazhdan-Lusztig Conjectures* (*Duke Math. J.*, Vol. 46, No. 4, 1979, pp. 805-859).

(Manuscrit reçu le 8 septembre 1982,
révisé le 10 décembre 1982).

D. BARBASCH,
Rutgers,
The State University of New Jersey,
New Brunswick,
New Jersey 08903, U.S.A.