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ERGODICITY OF TORAL LINKED TWIST MAPPINGS

By Feliks PRZYTYCKI [1]

0. Introduction, notations

Let $T^2=\mathbb{R}^2/\mathbb{Z}^2$ be the standard torus and let P, Q be closed annuli in T^2 defined by:

$$P=\{(x, y)\in T^2 : y_0\leqq y\leqq y_1,\ x \text{ arbitrary}\},\qquad |y_1-y_0|\leqq 1,$$
$$Q=\{(x, y)\in T^2 : x_0\leqq x\leqq x_1,\ y \text{ arbitrary}\},\qquad |x_1-x_0|\leqq 1\}.$$

Let $f:[y_0, y_1]\to\mathbb{R}$, $g:[x_0, x_1]\to\mathbb{R}$ be C^2-functions such that $f(y_0)=g(x_0)=0$, $f(y_1)=k$, $g(x_1)=l$, for some integers k and l.

Define $F=F_f$: $P\cup Q\circlearrowleft$ and $G=G_g$: $P\cup Q\circlearrowleft$ by:

$$F_f(x, y)=(x+f(y), y),\qquad G_g(x, y)=(x, y+g(x)),$$

on P and Q respectively and $F_f|_{Q\setminus P}=\mathrm{id}$, $G_g|_{P\setminus Q}=\mathrm{id}$.

Define $H=H_{f,g}=G_g\circ F_f$.

Observe that F, G and H preserve the Lebesgue measure μ on $P\cup Q$.

In the paper we shall consider f and g such that:

$$\frac{df}{dy}(y)\neq 0\qquad\text{and}\qquad\frac{dg}{dx}(x)\neq 0$$

for every $y\in[y_0, y_1]$ and $x\in[x_0, x_1]$.

If $k>0$ ($k<0$), what implies $df/dy>0(<0)$, define the slope of F by $\alpha=\inf\{df/dy(y) : y\in[y_0, y_1]\}$ (sup). Analogously define the slope β of G.

We call then $F|_P$ a (k, α)-twist, $G|_Q$ an (l, β)-twist and H a toral linked twist mapping, t. l. t. m.

In the presented paper we prove the following.

[1] The author gratefully acknowledges the financial support of the Stiftung Volkswagenwerk for a visit to the Institut des Hautes Études Scientifiques, during which this paper was written.

PROPOSITION. — *If a t. l. t. m.* H *is composed from* (k, α) – *and* (l, β) – *twists, where k and l have opposite signs,* $|k|, |l| \geqq 2$ *and* $|\alpha . \beta| \geqq C_0 \approx 17{,}244\,45$, *then* H *and all its powers are ergodic. In fact* H *satisfies Bernoulli property.*

Together with facts known before we obtain:

THEOREM. — *Let* H *be a t. l. t. m. composed from* (k, α) – *and* (l, β) – *twists.*

If $\alpha\beta > 0$ (i. e. $kl > 0$), *then* H *is Bernoulli.*

If $\alpha\beta < -4$, *then* H *is almost hyperbolic.*

If $\alpha\beta < -C_0 \approx 17{,}244\,45$ *and* $|k|, |l| \geqq 2$, *then* H *is Bernoulli.*

We add to the paper Appendix where we explain what we mean by "almost hyperbolic" and gather some facts from Pesin theory for mappings with singularities, which make background for this paper.

T. l. t. m. were introduced by Easton [4]. It seems however that the basic phenomena were observed earlier by Oseledec, *see* [5], Chap. 3.8.

The case $kl > 0$ has been done by Burton and Easton in [2] and by Wojtkowski in [7].

Almost hyperbolicity when $\alpha\beta < -4$ follows from Wojtkowski paper [7].

If $kl > 0$, global stable and unstable manifolds intersect each other since they are very long and go, roughly speaking, in different directions. This gives ergodicity.

If $\alpha\beta < -4$, global stable and unstable manifolds, although internally very long could have a very small diameter in $P \cup Q$. On Figure 1 we show what could happen with subsequent images of a local unstable manifold γ under iterations by H.

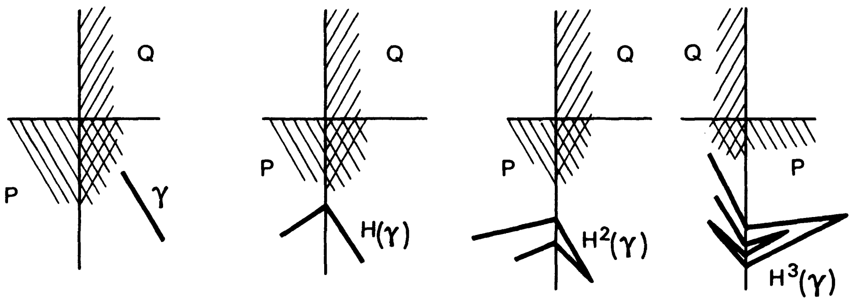

Fig. 1.

We prove however that if $\alpha\beta < -C_0$, this is not so, that piecewise differentiable global stable and unstable manifolds contain pieces winding along the whole annulus P(Q). (A similar phenomenon appears in an example studied by Wojtkowski in [8].), § 1, will be devoted to this aim.

In section 2 we briefly list some generalizations of Theorem for larger classes of linked twist mapping. The reader can find details in [11].

1. Proof of proposition

To simplify notation assume that the functions f and g are linear.

We may assume $|\alpha|=|\beta|$. Otherwise we may change the coordinates on T^2, taking the coordinates $(x, \sqrt{|\alpha/\beta|}.y)$. Then we consider the torus $\mathbb{R}^2/\mathbb{Z}\times\sqrt{|\alpha/\beta|}.\mathbb{Z}$. We may assume that $\alpha>0$. Let us repeat according to [2] and [14] the proof of almost hyperbolicity of H. The matrix:

$$\mathrm{DG}\circ\mathrm{DF}=\begin{pmatrix}1 & 0\\ -\alpha & 1\end{pmatrix}\circ\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \alpha\\ -\alpha & -\alpha^2+1\end{pmatrix}\quad \text{for}\quad \alpha>2,$$

is hyperbolic. Its eigenvalues are:

$$\lambda_{\pm}=\frac{-\alpha^2+2\pm\sqrt{\alpha^4-4\alpha^2}}{2}$$

and the expanding eigenvector (ξ_1, ξ_2) satisfies:

$$\xi_1/\xi_2=-\left(\frac{\alpha}{2}\right)+\sqrt{\left(\frac{\alpha}{2}\right)^2-1}.$$

Let us denote this number by L.

In $\mathbb{R}^2$ let us take the cone $C=\{(x, y) : L\leqq x/y\leqq 0\}$.

It is easy to see that for every positive integers s, r, $\mathrm{DG}^s\circ\mathrm{DF}^r(C)\subset C$ and for every $v\in C$, $\|D(G^s\circ F^r)(v)\|\geqq\lambda\|v\|$ where $\lambda=|\lambda_-|>1$.

Denote $P\cap Q=S$ and denote the first return mapping, to S, under F, G, H, by F_s, G_s, H_s respectively. It is easy to see that $H_s=G_s\circ F_s$.

For almost every $x\in S$ there exists $D(H_s)$ and $D(H_s^{-1})$ and:

$$\int_s \log^+\|DH_s^{(-1)}(x)\|\,d\mu(x)\leqq\int_{P\cup Q}\log^+\|DH^{(-1)}(x)\|\,d\mu(x)<+\infty.$$

Hence for all vectors tangent to s at x there exist Lyapunov exponents for DH_s. One of them, corresponding to the vectors from C is positive (not less than $\log\lambda$), the other one is negative.

For almost every $x\in S$ its H-orbit returns to S with positive frequency (this is a corollary from Birkhoff Ergodic Theorem, *see* [2], Lemma 4.4). Moreover $\mu\left((P\cup Q)\setminus\left(\bigcup_{n=-\infty}^{+\infty}H^n(S)\right)\right)=0$. So Lyapunov exponents for H itself are nonzero almost everywhere and in view of Pesin Theory (*see* Appendix) for almost every $x\in P\cup Q$ there exist local stable and unstable manifolds $\gamma^s(x)$, $\gamma^u(x)$ [Under the linearity assumption $\gamma^{s\,(u)}(x)$, are linear segments.]

To prove, ergodicity of H and all its powers it is enough to show that for almost every x, $y\in S$, $H^m(\gamma^u(x))$ intersects $H^{-n}(\gamma^s(y))$ for all m, n large enough.

For any segment γ we denote by $l_h(\gamma)$ and $l_v(\gamma)$ the lengths of the orthogonal projections of γ to the horizontal, respect. vertical axes. We shall prove that for any linear segment $\gamma \subset H_s^r(\gamma^u(x))$, the image $F_s(\gamma)$, which is a union of linear segments, contains a segment γ' such that either $l_h(\gamma') > \delta . l_v(\gamma)$, for a constant $\delta > 1$ independent of γ, or γ' joins the left and right sides of S. In the latter case, due to $|l| \geqq 2$, $G(\gamma')$ contains a segment joining the upper and lower sides of S (*Fig.* 2). (We shall call any segment in S joining the uper and lower sides of S a v-segment, and joining the left and right sides of S and h-segment.)

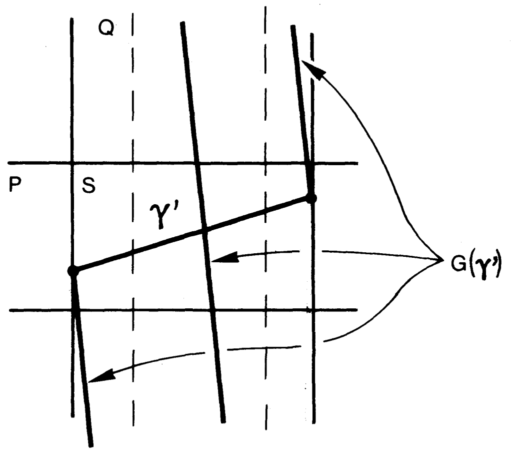

Fig. 2.

In the first case we act on γ' with G_s and find a segment $\gamma'' \subset G_s(\gamma')$ such that either $l_v(\gamma'') \geqq \delta . l_h(\gamma')$ or γ'' is a v-segment. In the first case we continue the process.

We get a sequence of segments of exponentially growing length. So it must finish with a v-segment or an h-segment. Then, if we continue iterating with F and G alternately we find an h-segment or v-segment alternately at each step (since $|k|, |l| \geqq 2$). The same happens for all sufficiently high iterations of H^{-1} on $\gamma^s(y)$. Concluding: $H^m(\gamma^u(x))$ contain v-segments and H^{-n} contain h-segments for all m, n sufficiently large. But the h-segments intersect v-segments.

So let us fix a segment $\gamma \subset H_s^r(\gamma^u(x))$. Let $m_1 > 0$ be the first time when $F^{m_1}(\gamma)$ intersects S. Then we have four possibilities:

1) $F^{m_1}(\gamma)$ contains an h-segment. This case has just been discussed.

2) The right side $F^{m_1}(\gamma)$ intersects S (*Fig.* 3).

3) The left side of $F^{m_1}(\gamma)$ intersects S [this case if fully analogous to the case 2)].

4) Both sides of $F^{m_1}(\gamma)$ intersect S (*Fig.* 4).

We study the case 2) and make the *assumption* ($\bigstar$) that $F_s(\gamma)$ does not contain any h-segment. We divide $F^{m_1}(\gamma) \setminus S$ into three intervals I_1, I_2, I_3. Let us denote $F^{m_1}(\gamma) \cap S = I_4$.

Along I_2 the rotation number $f(y)$ changes by $\alpha . l_v(I_2)$. So, for any integer $n > 0$ such that $1/n < \alpha . l_v(I_2)$ there exists a horizontal circle in P, intersecting I_2 on which the rotation number is m/n for an integer m. So there exists an F-periodic point $p \in I_2$ with the period $[(1/\alpha . l_v(I_2))+1]$. It divides I_2 into I'_2 and I''_2 (*Fig.* 3). The distance d between the different points of the F-orbit of p, $\mathrm{Orb}_F(p)$, is at least:

$$\frac{1}{[(1/\alpha . l_v(I_2))+1]}$$

Let us denote the last (to the right) point of $\mathrm{Orb}_F(p)$ in S by p_1 and the next one (to the right) by p_2 (*Fig.* 3). Let $m_2>0$ be the first time when $F^{m_2}(p)$ is between p and RS – the right side of S. [Inclusing p and p_1. Observe however that $F^{m_2}(p) \neq p$. Otherwise p would have its F-orbit disjoint with S. But $F^{-m_1}(p) \in \gamma \subset S$, a contradiction].

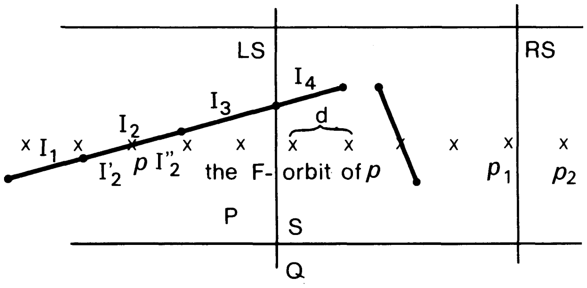

Fig. 3.

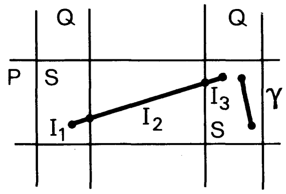

Fig. 4.

We denote $\mathscr{J}_0 = I''_2 \cup I_3$ and $\mathscr{J}_m = F(\mathscr{J}_{m-1} \setminus S)$ for $m=1, 2, \ldots, m_2$.

Then:

$$l_h(\mathscr{J}_m) \geqq \min(d + l_h(I''_2 \cup I_3),\ l_h(I''_2 \cup I_3) + \alpha . l_v(I''_2 \cup I_3))$$

for $m=1, 2, \ldots, m_2$.

If $F^{m_2}(p)$ is between p and LS (the left side of S), then $l_h(\mathscr{J}_{m_2} \cap S) \geqq d$. If $F^{m_2}(p) \in S \setminus \{p_1\}$, then:

$$(2) \qquad l_h(\mathscr{J}_{m_2} \cap S) \geqq \min(d,\ l_h(I''_2 \cup I_3) + \alpha . l_v(I''_2 \cup I_3)).$$

Assume $F^{m_2}(p) = p_1$. Let $\mathrm{dist}(p_1, \mathrm{RS}) = \tau . d$.

Then $\mathrm{dist}(\mathrm{RS}, p_2) = (1-\tau) . d$. We have either (2) satisfied or $l_h(\mathscr{J}_{m_2} \cap S) = \tau . d$ and $\mathscr{J}_{m_2} \cap S$ touches RS with its right end.

Define $\tilde{\mathscr{J}}_0 = I_1 \cup I'_2$ and $\tilde{\mathscr{J}}_m = F(\tilde{\mathscr{J}}_{m-1} \setminus S)$ for $m=1, \ldots, m_2$: We have either:

$$l_h(\tilde{\mathscr{J}}_{m_2} \cap S) \geqq \min((1-\tau)\, d,\ \alpha . l_v(I_1 \cup I'_2) + l_h(I_1 \cup I'_2)$$

or $\tilde{\mathscr{J}}_{m_2} \cap S$ touches LS with its left end [the number $(1-\tau)\, d$ appears because there can exist $m : 0 < m < m_2$ for which $F^m(p) = p_2$]. So due to assumption $(\star)$:

$$l_h(\mathscr{J}_{m_2} \cup \tilde{\mathscr{J}}_{m_2}) \cap S) \geqq \min(d,\ \alpha . l_v(I_3) + l_h(I_3)),$$
$$\alpha . l_v(I_1) + l_h(I_1)).$$

Thus in order to have $l_h((\mathcal{I}_{m_2} \cup \tilde{\mathcal{I}}_{m_2}) \cap S) \geqq \delta . l_v(\gamma)$ it is enough that the following inequalities hold:

$$d \geqq \delta . l_v(\gamma), \tag{3}$$

$$\alpha . l_v(I_3) + l_h(I_3) \geqq \delta . l_v(\gamma), \tag{4}$$

$$\alpha . l_v(I_1) + l_h(I_1) \geqq \delta . l_v(\gamma). \tag{5}$$

For (3) it is enough that:

$$\frac{\alpha . l_v(I_2)}{1 + \alpha . l_v(I_2)} \geqq \delta . l_v(\gamma)$$

or:

$$l_v(I_2) \geqq \frac{\delta . l_v(\gamma)}{\alpha(1 - \delta . l_v(\gamma))}. \tag{6}$$

We can assume here $1 - \delta . l_v(\gamma) > 0$ since the assumption $(\star)$ implies:

$$l_v(\gamma) . (L + \alpha) < 2 \tag{7}$$

and $L + \alpha \geqq -1 + \sqrt{17} > 3$. We can replace (4), (5) by

(8) [and (9)]

$$l_v(I_{3\,(1)}) \geqq \frac{\delta . l_v(\gamma)}{L + 2\alpha}.$$

(due to the fact that $\gamma \in C$ — the cone $L \leqq x/y \leqq 0$). The work would be also done if:

$$l_h(I_4) \geqq \delta . l_v(\gamma)$$

which follows from:

$$l_v(I_4) \geqq \frac{\delta . l_v(\gamma)}{L + \alpha}. \tag{10}$$

There exists $\delta > 1$ and a partition of $F^{m_i}(\gamma) \setminus S$ into I_1, I_2, I_3 satisfying (6), (8), (9), or the inequality (10) is satisfied if:

$$l_v(\gamma) = \sum_{i=1}^{4} l_4(I_i) > l_v(\gamma) \left(\frac{1}{\alpha(1 - l_v(\gamma))} + \frac{2}{2\alpha + L} + \frac{1}{\alpha + L} \right). \tag{11}$$

We divide both sides by $l_v(\gamma)$ and due to (7) we obtain the condition:

$$1 > \frac{1}{\alpha(1 - (2/(L + \alpha)))} + \frac{2}{2\alpha + L} + \frac{1}{\alpha + L} \tag{12}$$

[recall that $L = -(\alpha/2) + \sqrt{(\alpha/2)^2 - 1}$].

In the case 4) the situation is simpler. $F^{m_i}(\gamma)$ divides into I_1, I_2, I_3 as on Figure 4. We need either $l_h(I_1) \geqq \delta . l_v(\gamma)$, or $l_h(I_3) \geqq \delta . l_v(\gamma)$, or $l_h F(I_2) - l_h(I_2) \geqq \delta . l_v(\gamma)$. [The sufficiency of the last inequality follows from the following: Lift everything to $\mathbb{R}^2$, denote two consecutive components of the lift of Q by Q_1 and Q_2. Then, assumed ($\star$), $F(I_2)$ has a component $\tilde{I}$ of its lift between left sides of Q_1 and Q_2 or between right sides of Q_1 and Q_2. Otherwise $\tilde{I}$ would intersect a component of the lift of I_2, which would imply the existence of an F-fixed point q S. But $F^{-m_1}(q) \in \gamma \subset S$ – a contradiction].

For this it suffices that:

$$(L+\alpha).l_v(I_1) \geqq \delta.l_v(\gamma),$$

or:

$$(L+\alpha).l_v(I_3) \geqq \delta.l_v(\gamma),$$

or:

$$\alpha.l_v(I_2) \geqq \delta.l_v(\gamma).$$

For that it is enough if:

$$l_v(\gamma) = \sum_{i=1}^{3} l_v(I_i) \geqq l_v(\gamma)\left(\frac{2}{L+\alpha}+\frac{1}{\alpha}\right)$$

i. e. :

(13) $$1 > \frac{2}{L+\alpha}+\frac{1}{\alpha}$$

(12) is satisfied for $\alpha > \alpha_0 \approx 4{,}152\,643$;

(13) is satisfied for $\alpha > \alpha_1 \approx 3{,}239$.

This gives the constant $C_0 = \alpha_0^2 \approx 17{,}244\,45$ in the statement of Proposition.

Remark. – If $l_v(\gamma)$ is small, then in (13) we can write $(L+m_1\alpha)$ instead of $L+\alpha$, where m_1 is large. So (13) can be replaced by:

$$1 > \frac{1}{\alpha}.$$

Also (12) can be replaced by:

(14) $$1 > \frac{1}{\alpha}+\frac{2}{2\alpha+L}+\frac{1}{\alpha+L},$$

since we can omit $l_v(\gamma)$ in the denominator of the ratio $1/(1-l_v(\gamma))$ of (11):

(14) holds for $\alpha > \alpha_2 \approx 3{,}183\,590$.

In this case the H^m-images of any unstable segment $\gamma^u(x)$, for m sufficiently large, contain segments of length larger than a constant. The same concerns H^{-n}-images of $\gamma^s(y)$, for n large. Does it imply that there exists a decomposition into *finite* number of Bernoulli components?

2. Generalizations

Denote by (♮) the places where we are not exact in statements.

1. P and Q can be replaced by two finite families of annuli embedded (♮) into a surface, $\{P_i\}_{i=1}^p$ and $\{Q_i\}_{i=1}^q$, such that $P_i \cap P_j = Q_i \cap Q_j = \emptyset$ for $i \neq j$ and P_i intersect Q_j transversally (♮). Assume that $M = \bigcup_{i=1}^{p} P_i \cup \bigcup_{j=1}^{q} Q_j$ is connected. Consider H which is a composition of two mappings: (α_i, k_i)-twists on $\bigcup_{i=1}^{p} P_i$ with (β_j, l_j)-twists on $\bigcup_{j=1}^{q} Q_j$.

If H preserves a probability measure on M which restriction to each P_i and Q_j is equivalent to Lebesgue measure with bounded density, if $|\alpha_i|$, $|\beta_j|$ are large enough and $|k_i|$, $|l_j| \geqq 2$, then H is Bernoulli.

In particular assume that changes of coordinates on M from P_i to Q_j are only translations composed with multiplication by -1. Then:

If for every pair i, j such that $P_i \cap Q_j \neq \emptyset$, $|\alpha_i \beta_j| \geqq 4$, then H is almost hyperbolic.

If $|\alpha_i \beta_j| > X(i).Y(j)$ where $X(i)$, $Y(j)$ are respectively the largest solutions of the equations:

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X - 3q(i)} + \frac{2}{X-1},$$

$$1 = \frac{2p(j)}{Y} + \frac{p(j)}{Y - 3p(j)} + \frac{2}{Y-1},$$

where $q(i)$, respectively $p(j)$, is the number of components of $P_i \cap \bigcup_{s=1}^{q} Q_s$, respectively of $Q_j \cap \bigcup_{s=1}^{p} P_s$, then H is Bernoulli.

For the sufficient estimates on α_j, β_j, to have almost hyperbolicity and Bernoulli property, in the general case *see* [11], § 2.

This generalization includes the Bowen example [1], Chap. 10, F. Devaney generalized t. l. t. m. [3] and Thurston pseudo-Anosov examples built from Dehn twists on two thickened simple closed curves (families of curves) "filling up" a surface, *see* [10], § 6.

2. We can consider a composition of a finite number of families of twists alternately on the annuli $\{P_i\}$ and $\{Q_j\}$ (rather than to compose two family only), so that every annulus is twisted at least once and all twists on it go in the same direction.

Then also as in the generalization 1, if slopes of twists are large enough, H is Bernoulli.

3. H is Bernoulli if the above annuli P_i, Q_i are sufficiently narrow collars for the circles A_i, B_j respectively, embedded into a compact orientable surface N, such that A_i intersect B_j transversally.

Indeed, then, since $|k_i|$, $|l_j| \neq 0$ (as $\geqq 2$), slopes of the twists are large (if we consider linear twists, i. e. powers of Dehn twists).

Due to this we can find in any isotopy class of orientation preserving homeomorphism on N a piecewise linear homeomorphism which is Bernoulli (for a probabilistic measure which support is N) with arbitrarily small measure entropy.

4. We can prove Bernoulli property in the generalization 1 with the assumptions $|k_i|$, $|l_j| \geqq 2$ weakened in various ways (*see* [11], the section "graphs of linkage").

For example this assumption can be completely dropped if for every $i, j : p(i), q(j) \geqq 3$.

APPENDIX

We refer in this paper to the so-called Pesin Theory [9] for maps with singularities, recently developped by Katok and Strelcyn [6]. For the comfort of the reader we list below the results.

Katok-Strelcyn, (K-S)-conditions

Let X be a complete metric space with a metric ρ. Let $N \subset X$ be an open subset which is a Riemannian manifolds with a Riemannian metric inducing $\rho|_N$. Assume that there exists a number $r>0$ such that for each $x \in N$, $\exp_x$ restricted to the ball $B(x) = B(x, \min(r, \mathrm{dist}_\rho(x, X \setminus N)))$ is injective.

Let μ be a probability measure on X and Φ be a μ-preserving, C^2-, 1-1 mapping defined on an open set $V \subset N$, into N. Denote $\mathrm{sing}\,\Phi = X \setminus V$.

(K-S, 1) There exist positive constants a, C_1 such that for every $\varepsilon > 0$:

$$\mu(B(\mathrm{sing}\,\Phi, \varepsilon)) \leqq C_1 \varepsilon^a.$$

[$B(\mathrm{sing}\,\Phi, \varepsilon)$ means the neighbourhood of $\mathrm{sing}\,\Phi$ with radius ε.]:

$$(\mathrm{K-S}, 2) \quad \begin{cases} \displaystyle\int \log^+ \| D\Phi(x) \| \, d\mu(x) < \infty, \\ \displaystyle\int \log^+ \| D\Phi^{-1}(x) \| \, d\mu(x) < \infty \end{cases}$$

(K-S, 3) There exist positive constants b, C_2 such that for every $x \in X \setminus \mathrm{sing}\,\Phi$:

$$\| D^2 \Phi(x) \| \leqq C_2 (\mathrm{dist}(x, \mathrm{sing}\,\Phi))^{-b}$$

[By $\| D^2 \Phi(x) \|$ we mean $\sup \{ \| D^2 (\exp_z^{-1} \circ \Phi \circ \exp_y) \| : x \in B(y), \Phi(x) \in B(z) \}$].

Remark. — If $\Phi = \Phi_n \circ \ldots \circ \Phi_1$ and Φ preserves μ we can replace the (K-S)-conditions for Φ by the analogous conditions for each Φ_i separately, with $\mathrm{Sing}\,\Phi_i$ and $\mu_i = (\Phi_{i-1} \circ \ldots \circ \Phi_1)_* (\mu)$ respectively.

Theorem. — (*a*) *If Φ satisfies the* (K-S)-*conditions then for almost every $x \in X$, for all vectors tangent at x, there exist Lyapunov exponents and there exist local stable and unstable* *manifolds* $\gamma^s(x)$, $\gamma^u(x)$, *corresponding to negative and positive exponents respectively. Denote by* $\Lambda^{s(u)}(k)$ *the set of points where the number of negative (positive) Lyapunov exponents computed with multiplicities is equal to* k [*i.e.* $\dim \gamma^{s(u)}(x)=k$]. *Consider* $\Lambda^{s(u)}(k)$ *if* $\mu(\Lambda^{s(u)}(k))>0$. *Then for a sequence of sets* $\Lambda^{s(u)}(k, m)$ *increasing with m which exhaust almost all of* $\Lambda^{s(u)}(k)$, *the families* $\{\gamma^{s(u)}(x) : x \in \Lambda^{s(u)}(k, m)\}$ *are absolutely continuous.*

(*b*) *If we assume additionally that the measure* μ *is equivalent to the Riemannian measure on* N *and all Lyapunov exponents are almost everywhere different from* 0, *then* X *decomposes into a countable family of positive measure* μ, Φ-*invariant pairwise disjoint sets* $X=\bigcup_{i=1}^{\infty\,(\text{or } N)} \Lambda_i$ *such that for every i*, $\Phi|_{\Lambda_i}$ *is ergodic and* $\Lambda_i=\bigcup_{j=1}^{j(i)} \Lambda_i^j$ *where* $\Lambda_i^j \cap \Lambda_i^{j'}=\emptyset$ *for* $j\neq j'$, $\Phi|_{\Lambda_i}$ *permutes* Λ_i^j *and for each j* $\Phi^{j(i)}|\Lambda_i^j$ *is Bernoulli.* (In the above situation we sometimes call the system almost hyperbolic and say that it decomposes into a countable family of Bernoulli components.)

(*c*) *If additionally for almost every* $z, z' \in X$ *there exist integers m, n such that* $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$ *then in the decomposition of* X *we have only one set* $\Lambda_i=\Lambda_1$ (i. e. $N=1$). *In particular* Φ *is ergodic.*

(*d*) *If additionally for almost all points* z, $z' \in X$ *and every pair of integers m, n large enough* $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$ *then all powers of* Φ *are ergodic. This implies* $j(1)=1$ *so* Φ *is a Bernoulli system.*

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