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COHOMOLOGY OF A GENERAL INSTANTON BUNDLE

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0. Introduction

In this paper we work over an algebraically closed field k of arbitrary characteristic. Let $\mathscr{F}$ be a coherent sheaf on a projective space $\mathbf{P}^m$. We say $\mathscr{F}$ has *natural cohomology* if for each $n \in \mathbf{Z}$, at most one of the cohomology groups $H^i(\mathbf{P}^m, \mathscr{F}(n))$, for $i=0, 1, \ldots, m$, is nonzero. For example, any line bundle on $\mathbf{P}^m$ has natural cohomology.

The purpose of this paper is to prove the following theorem.

THEOREM 0.1. — (*a*) *For $c_1=0$ and any $c_2>0$, there exists a stable rank 2 vector bundle $\mathscr{E}$ on $\mathbf{P}^3$ with Chern classes c_1 and c_2, having natural cohomology.*

(*b*) *For $c_1=-1$ and any even $c_2 \geqq 6$, there exists a stable rank 2 vector bundle on $\mathbf{P}^3$ with Chern classes c_1 and c_2, having natural cohomology.*

This result proves a conjecture made by one of us ([14], 5.2), and we refer to that paper for background and discussion of related questions. The corresponding result for rank 2 stable bundles on $\mathbf{P}^2$ was proved by Brun ([4], § 5) and Le Potier ([21], 6.1) (*see* also Lange [20], 1.4). One consequence of the theorem is that the conjectured bound ([14], 5.2), ([12], Problem 9), now proved in characteristic 0 [15], for the least integer t such that $H^0(\mathscr{E}(t))$ is nonzero for a rank 2 vector bundle $\mathscr{E}$ on $\mathbf{P}^3$ with Chern classes $c_1=0, c_2>0$, is the best possible. Another consequence of the theorem is the existence of nonsingular curves in $\mathbf{P}^3$, not contained in surfaces of low degree, for which the conjectural bound on the genus ([14], 3.4) is sharp (*see* [14], p. 99).

To explain the title of the paper, note that if $\mathscr{E}$ is a stable rank 2 vector bundle on $\mathbf{P}^3$ with $c_1=0$, $c_2>0$, and having natural cohomology, then the Euler characteristic $\chi(\mathscr{E}(-2))$ vanishes by the Riemann-Roch theorem. Hence $H^i(\mathscr{E}(-2))$ is zero for all i. In particular, $H^1(\mathscr{E}(-2))$ is zero, which characterizes (mathematical) instanton bundles ([11], 8.2.3). Let $M(c_1, c_2)$ denote the moduli space of stable rank 2 vector bundles on $\mathbf{P}^3$

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with Chern classes c_1 and c_2. At present one knows only one irreducible component of $M(0, c_2)$ containing instanton bundles, namely the one containing the bundles corresponding to skew lines (1.6.1). For $c_2 \leqq 4$ it is known that there is no other ([11], [6], [3]). The bundles we construct (2.3.1) lie in this irreducible component of $M(0, c_2)$. In this sense, then, we say "a general instanton bundle has natural cohomology".

In the case $k = \mathbf{C}$ we can deduce the existence of real instanton bundles with natural cohomology. A *real instanton bundle* is a holomorphic vector bundle on $\mathbf{P}^3_{\mathbf{C}}$ corresponding to an *instanton*, i.e. a self-dual solution of the Yang-Mills equation on S^4. These bundles can be characterized, in the rank 2 case, as stable rank 2 holomorphic vector bundles on $\mathbf{P}^3_{\mathbf{C}}$ with $c_1 = 0$, $c_2 > 0$, and having an extra real structure, described exactly in [12], 1.1. Because real instanton bundles are known to exist in the irreducible component of $M(0, c_2)$ mentioned above, and because the real manifold of moduli of real instanton bundles must intersect any non-empty Zariski-open subset of its complexification, which contains that irreducible component of $M(0, c_2)$, we obtain the following corollary.

COROLLARY 0.2. — *For any $c_2 > 0$, there exist rank 2 real instanton bundles on $\mathbf{P}^3_{\mathbf{C}}$ having natural cohomology.*

In the case $c_1 = -1$ of the theorem, the restriction that c_2 be even is imposed by the Riemann-Roch theorem ([11], 2.2), and stability implies $c_2 > 0$. For $c_2 = 2,4$, there are stable rank 2 vector bundles with $c_1 = -1$ on $\mathbf{P}^3$, but they are exceptions to our theorem (at least in characteristic zero): there are none with natural cohomology. For $c_2 = 2$, one has $h^0(\mathscr{E}(1)) = h^1(\mathscr{E}(1)) = 1$, which contradicts natural cohomology ([17], 2.2), and [22]. For the case $c_2 = 4$, *see* (1.6.3) below.

Note that the bundles we construct all have α-invariant 0 ([11], § 2). Indeed, a bundle with $c_1 = 0$, $\alpha = 1$ has $h^1(\mathscr{E}(-2)) \neq 0$, so cannot have natural cohomology.

The main idea of the proof will be to start from a rather common non locally free coherent sheaf $\mathscr{E}_0$ on $\mathbf{P}^3$, which is close enough to vector bundles in $M(c_1, c_2)$. More precisely, we will show that $\mathscr{E}_0$ has some deformations which are locally free. We will show that $\mathscr{E}_0$ has some (non locally free!) deformations with semi-natural cohomology (defined in § 1). Finally we show that $\mathscr{E}_0$ has an irreducible local deformation space.

Combining these results, we conclude that $\mathscr{E}_0$ has deformations which are both locally free and have natural cohomology, as required.

The major part of the proof is finding the deformations with semi-natural cohomology. This is reduced to a problem about lines and conics in general position in a certain geometric vector bundle over $\mathbf{P}^3$, whose solution uses methods similar to those in our earlier papers ([16], [18]).

The paper is organized as follows. Paragraph 1 contains preliminary material. Paragraph 2 gives the proof of the main theorem, modulo results proved in later sections. In paragraph 3 we prove the existence of a universal family of extensions of two given coherent sheaves. In paragraph 4 we review the **Quot** scheme and its infinitesimal study. In paragraph 5 we formulate the general position problems which are then solved in paragraphs 6-9.

1. Sheaves with natural cohomology. Examples

In this section we make a few elementary observations about sheaves with natural cohomology and give some examples.

LEMMA 1.1. — *Let $\mathscr{F}$ be a torsion-free coherent sheaf on $\mathbf{P}^m$ with natural cohomology. Then $\mathscr{F}$ is locally free.*

Proof. — Since $\mathscr{F}$ is torsion-free, $\operatorname{Hom}(\mathscr{F}(-n), \omega)$ is nonzero for $n \gg 0$. Then by duality, $H^m(\mathbf{P}^m, \mathscr{F}(-n))$ is nonzero for $n \gg 0$. Because $\mathscr{F}$ has natural cohomology, this implies $H^i(\mathbf{P}^m, \mathscr{F}(-n))=0$ for all $i<m$ and all $n \gg 0$. Again using duality, this implies $\operatorname{Ext}^j(\mathscr{F}(-n), \omega)=0$ for all $j>0$ and $n \gg 0$. But for $n \gg 0$,

$$\operatorname{Ext}^j(\mathscr{F}(-n), \omega)=H^0(\mathscr{E}xt^j(\mathscr{F}, \omega)(n)),$$

so in fact the sheaves $\mathscr{E}xt^j(\mathscr{F}, \omega)$ are zero for $j>0$, which implies $\mathscr{F}$ is locally free.

PROPOSITION 1.2. — *Let $\mathscr{E}$ be a locally free sheaf (vector bundle) of rank 2 on $\mathbf{P}^3$ with $c_1=0$, having natural cohomology. Then either: (1) $c_2=-1$ and $\mathscr{E} \cong \mathscr{O}(-1) \oplus \mathscr{O}(1)$, or (2) $c_2=0$ and $\mathscr{E} \cong \mathscr{O} \oplus \mathscr{O}$, or (3) $c_2>0$ and $\mathscr{E}$ is stable.*

Proof. — By the Riemann-Roch theorem ([11], 8.1), $\chi(\mathscr{E}(-2))=0$. Therefore, since $\mathscr{E}$ has natural cohomology, $H^i(\mathscr{E}(-2))=0$ for $i=0, 1, 2, 3$. It follows that $H^0(\mathscr{E}(n))=0$ for $n \leqq -2$.

Suppose $H^0(\mathscr{E}(-1)) \neq 0$. Then $H^0(\mathscr{E}(n)) \neq 0$ for all $n \geqq -1$, so using duality and natural cohomology, we see that $H^i(\mathscr{E}(n))=0$ for $i=1, 2$ and for all $n \in \mathbf{Z}$. This implies by a theorem of Horrocks [19], ([9], III, 6.3) that $\mathscr{E}$ is a direct sum of line bundles. Since $c_1=0$ and $H^0(\mathscr{E}(-1)) \neq 0$ but $H^0(\mathscr{E}(-2))=0$, $\mathscr{E} \cong \mathscr{O}(-1) \oplus \mathscr{O}(1)$, and $c_2=-1$. This is the first possibility.

Now suppose $H^0(\mathscr{E}(-1))=0$ but $H^0(\mathscr{E}) \neq 0$. By Riemann-Roch, $\chi(\mathscr{E})=2-2c_2$. On the other hand, since $\mathscr{E}$ has natural cohomology, $\chi(\mathscr{E})=h^0(\mathscr{E})>0$. Hence $c_2 \leqq 0$. Now let $s \in H^0(\mathscr{E})$ be a nonzero section. Then the zero set of s is a curve of degree c_2 in $\mathbf{P}^3$ ([11], § 1). So $c_2 \geqq 0$. Combining inequalities, $c_2=0$, the zero set of s is empty, so $\mathscr{E} \cong \mathscr{O} \oplus \mathscr{O}$. This is the second possibility.

In the remaining case, $H^0(\mathscr{E})=0$, so $\mathscr{E}$ is stable, which implies $c_2>0$ ([11], 8.4).

PROPOSITION 1.3. — *Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbf{P}^3$ with $c_1=-1$ having natural cohomology. Then either: (1) $c_2=-2$ and $\mathscr{E} \cong \mathscr{O}(1) \oplus \mathscr{O}(-2)$, or (2) $c_2=0$ and $\mathscr{E} \cong \mathscr{O} \oplus \mathscr{O}(-1)$, or (3) $c_2>0$ and $\mathscr{E}$ is stable.*

Proof. — The proof is almost identical to the proof of (1.2), so is omitted.

PROPOSITION 1.4. — *Let $\mathscr{E}$ be a rank 2 vector bundle on $\mathbf{P}^3$ with $c_1=0$ or -1, having natural cohomology. Then $H^0(\mathscr{E}(n))=H^1(\mathscr{E}(n))=0$ for $n \leqq -2$ and $H^2(\mathscr{E}(n))=H^3(\mathscr{E}(n))=0$ for $n \geqq -1$. Hence the dimensions $h^i(\mathscr{E}(n))$ of the cohomology groups are uniquely determined by the Chern classes, for all i, n.*

Proof. — This is clear for the direct sums of line bundles which occur in (1.2) and (1.3). So we need only consider the case $\mathscr{E}$ stable. Then $H^0(\mathscr{E}(n))=0$ for $n \leqq 0$, and by duality $H^3(\mathscr{E}(n))=0$ for $n \geqq -3$. Next we claim $H^1(\mathscr{E}(-2))=0$. If $c_1=0$, we have already seen this in the proof of (1.2). If $c_1=-1$, then $\chi(\mathscr{E}(-2))=(1/2)c_2>0$. So because of natural cohomology, $H^1(\mathscr{E}(-2))=0$. Now let H be a general plane in $\mathbf{P}^3$. Then $\mathscr{E}_H$ is semistable ([11], 3.3), [2]. The exact sequence:

$$0 \to \mathscr{E}(-1) \to \mathscr{E} \to \mathscr{E}_H \to 0,$$

gives a cohomology sequence:

$$H^0(\mathscr{E}_H(n)) \to H^1(\mathscr{E}(n-1)) \to H^1(\mathscr{E}(n)).$$

Now $H^0(\mathscr{E}_H(n))=0$ for $n<0$ by semistability, so by descending induction on n we find $H^1(\mathscr{E}(n))=0$ for all $n \leqq -2$.

The vanishing of H^2 and H^3 for $n \geqq -1$ follows by duality.

To prove the last statement, the Riemann-Roch theorem gives the Euler characteristic $\chi(\mathscr{E}(n))$ in terms of c_1 and c_2. Then, because of natural cohomology and the vanishing statements just proved, we have for $n \geqq -1$, if $\chi(\mathscr{E}(n)) \geqq 0$, then $h^0(\mathscr{E}(n))=\chi(\mathscr{E}(n))$ and $h^1(\mathscr{E}(n))=0$, and if $\chi(\mathscr{E}(n)) \leqq 0$, then $h^0(\mathscr{E}(n))=0$ and $h^1(\mathscr{E}(n))=-\chi(\mathscr{E}(n))$. Similarly h^2 and h^3 are determined for $n \leqq -2$.

DEFINITION. — A torsion-free coherent sheaf $\mathscr{F}$ on $\mathbf{P}^3$ has *semi-natural cohomology* if $c_1=0$ and for all $n \geqq -2$, or if $c_1=-1$ and for all $n \geqq -1$, at most one of the groups $H^i(\mathscr{F}(n))$, $i=0, 1, 2, 3$, is nonzero.

LEMMA 1.5. — *If $\mathscr{E}$ is locally free of rank 2 on $\mathbf{P}^3$, with $c_1=0$ or -1, then $\mathscr{E}$ has semi-natural cohomology if and only if $\mathscr{E}$ has natural cohomology.*

Proof. — This follows from Serre duality and the isomorphism $\check{\mathscr{E}} \cong \mathscr{E}(-c_1)$.

PROPOSITION 1.6. — *Let T be a scheme of finite type over k, and let $\mathscr{F}$ be a torsion-free coherent sheaf on $\mathbf{P}^3_T$, flat over* T. *Then the set of $t \in$ T for which the fibre $\mathscr{F}_t$ on $\mathbf{P}^3_{k(t)}$ is torsion-free and has semi-natural cohomology is an open subset of* T.

Proof. — The openess of the condition $\mathscr{F}_t$ torsion-free can be found in a paper of Maruyama ([23], 2.1). Next, using quasi-compacity of T and Serre's vanishing theorem, there is an n_0 such that for all $n \geqq n_0$ and all $t \in T$, $H^i(\mathbf{P}^3_{k(t)}, \mathscr{F}_t(n))=0$ for $i>0$. So the condition of semi-natural cohomology is verified for all $t \in T$ in the range $n \geqq n_0$. There remain only finitely many values of i and n to consider, so the openness of semi-naturality follows from the semi-continuity theorem applied to $H^i(\mathbf{P}^3, \mathscr{F}_t(n))$ for each i, n.

Example 1.6.1. — One way of constructing stable rank 2 bundles $\mathscr{E}$ on $\mathbf{P}^3$ with Chern classes $c_1=0$, $c_2>0$ is as follows ([11], 3.1.1, 4.3.1). Let Y be a disjoint union of $r=c_2+1$ lines in $\mathbf{P}^3$. Then there are extensions:

$$0 \to \mathcal{O} \to \mathscr{E}(1) \to \mathscr{I}_Y(2) \to 0,$$

where $\mathscr{E}$ is a stable vector bundle with $c_1=0$, c_2 given.

If $c_2=1$ these are the nullcorrelation bundles. It is easy to see that $h^1(\mathscr{E}(-1))=h^2(\mathscr{E}(-3))=1$, and otherwise all H^1 and H^2 groups are zero. Thus $\mathscr{E}$ has natural cohomology.

If $c_2=2$, these are the bundles studied in [11], paragraph 9. They have $h^1(\mathscr{E})=h^1(\mathscr{E}(-1))=h^2(\mathscr{E}(-3))=h^2(\mathscr{E}(-4))=2$ and otherwise all H^1 and H^2 groups are zero ([11], 9.4). So they also have natural cohomology.

If $c_2 \geqq 3$, then $\chi(\mathscr{E}(1))=8-3c_2<0$. Since $\mathrm{H}^0(\mathscr{E}(1)) \neq 0$ by construction, these bundles do not have natural cohomology.

To construct a bundle with $c_1=0$, $c_2=3$ having natural cohomology, one could use the construction of [11], 4.3.3 with a nonsingular elliptic curve Y of degree 7. The bundle $\mathscr{E}$ is obtained by an extension:

$$0 \rightarrow \mathcal{O} \rightarrow \mathscr{E}(2) \rightarrow \mathscr{I}_{\mathrm{Y}}(4) \rightarrow 0.$$

To show that $\mathscr{E}$ has natural cohomology it is sufficient to show $\mathrm{H}^0(\mathscr{E}(1))=0$ and $\mathrm{H}^1(\mathscr{E}(n))=0$ for $n \geqq 2$. This is equivalent to showing that Y is not contained in any cubic surface, and that for $n \geqq 4$, the natural map:

$$\mathrm{H}^0(\mathcal{O}_{\mathbf{P}^3}(n)) \rightarrow \mathrm{H}^0(\mathcal{O}_{\mathrm{Y}}(n)),$$

is surjective. The existence of an elliptic curve Y of degree 7 with those properties depends on a general position argument for elliptic curves which has recently been proved by Ballico and Ellia [1].

For higher values of c_2, the existence of bundles with natural cohomology can be similarly translated into questions of existence of curves of high degree and genus with suitable general position properties. To approach these questions of curves directly seems hopeless, and that is why we use an entirely different proof of our theorem in this paper.

Example 1.6.2. — One way of constructing stable rank 2 bundles on $\mathbf{P}^3$ with $c_1=-1$ and $c_2>0$ is as follows ([11], 3.1.2, 4.3.2). Let Y be a disjoint union of $r=1/2(c_2+2)$ conics in $\mathbf{P}^3$. Then one can obtain $\mathscr{E}$ by an extension:

$$0 \rightarrow \mathcal{O} \rightarrow \mathscr{E}(2) \rightarrow \mathscr{I}_{\mathrm{Y}}(3) \rightarrow 0.$$

If $c_2=2$, this construction gives all stable bundles with Chern classes $c_1=-1$, $c_2=2$, but none of them have natural cohomology [17], as already mentioned in the introduction.

If $c_2 \geqq 4$, then $\chi(\mathscr{E}(2))=14-(7/2)c_2 \leqq 0$. Since $\mathrm{H}^0(\mathscr{E}(2)) \neq 0$ by construction, these bundles do not have natural cohomology.

Even though the bundles obtained by this construction do not have natural cohomology, we will see as a consequence of our proof (2.3.1) that for $c_2 \geqq 6$ they do have deformations which have natural cohomology.

Example 1.6.3. — There is no rank 2 vector bundle on $\mathbf{P}^3$ with Chern classes $c_1=-1$, $c_2=4$ having natural cohomology, assuming char $k=0$ (we do not know if one exists in characteristic $p>0$). Indeed, suppose $\mathscr{E}$ were such a bundle. Then by (1.4), for $n \geqq -1$, $h^2(\mathscr{E}(n))=h^3(\mathscr{E}(n))=0$. The Riemann-Roch theorem gives:

$$\chi(\mathscr{E}(n))=\frac{1}{6}(n+1)(n+2)(2n+3)-2(2n+3).$$

In particular, $\chi(\mathscr{E}(2))=0$, so because of the hypothesis of natural cohomology, $h^0(\mathscr{E}(2))=h^1(\mathscr{E}(2))=0$. Now we can apply Castelnuovo's theorem ([25], p. 99) to conclude that $\mathscr{E}(3)$ is generated by global sections. In characteristic 0 it follows ([11], 1.4) that the zero set of a general section of $\mathscr{E}(3)$ is a nonsingular curve Y. The curve Y is related to $\mathscr{E}$ by the exact sequence:

$$0 \to \mathcal{O} \to \mathscr{E}(3) \to \mathscr{I}_Y(5) \to 0.$$

Since $h^1(\mathscr{E}(-2))=0$ by (1.4) again, we find $h^1(\mathscr{I}_Y)=0$, so $h^0(\mathcal{O}_Y)=1$, which implies that Y is connected, so Y is in fact an irreducible nonsingular curve.

The curve Y has degree 10 and genus 6, and $\omega_Y \cong \mathcal{O}_Y(1)$. Thus Y is a projection into $\mathbf{P}^3$ of a canonical curve of genus 6 in $\mathbf{P}^5$. A result of Gruson and Peskine ([8], p. 58) shows that any projection into $\mathbf{P}^3$ of a canonical curve of genus 6 is contained in a quartic surface. Thus $h^0(\mathscr{I}_Y(4))\neq 0$, which implies $h^0(\mathscr{E}(2))\neq 0$, contradicting the hypothesis of natural cohomology. Therefore $\mathscr{E}$ cannot exist.

See the paper of Ein [5] for a more detailed study of stable rank 2 bundles on $\mathbf{P}^3$ with Chern classes $c_1=-1$ and $c_2=4$.

2. Framework of the proof

In this section we present the proof of the main theorem, modulo various statements which will be proved in later sections. The basic idea is to study deformations of a certain torsion-free sheaf $\mathscr{E}_0$.

For each c_1, c_2, we define a sheaf $\mathscr{E}_0$ as follows. If $c_1=0$ and $c_2>0$, let Y_0 be a disjoint union of $r=c_2+1$ lines in $\mathbf{P}^3$, and let $\mathscr{E}_0=\mathcal{O}(-1)\oplus\mathscr{I}_{Y_0}(1)$. If $c_1=-1$ and $c_2>0$ is even, let Y_0 be a disjoint union of $r=1/2(c_2+2)$ conics in $\mathbf{P}^3$, and let $\mathscr{E}_0=\mathcal{O}(-2)\oplus\mathscr{I}_{Y_0}(1)$.

Since the Chern classes of a direct sum are the same as for a nontrivial extension, we see from the examples (1.6.1) and (1.6.2) that in each case $\mathscr{E}_0$ is a torsion-free coherent sheaf on $\mathbf{P}^3$ with the given Chern classes c_1, c_2, and with $c_3=0$. Note also, since Y_0 in each case is a Cohen-Macaulay scheme, that the *homological dimension* (namely the shortest length of a resolution by locally free sheaves) of $\mathscr{E}_0$ is 1.

PROPOSITION 2.1. — *There is an irreducible nonsingular scheme* T *and a torsion-free coherent sheaf* $\mathscr{F}$ *on* $\mathbf{P}^3_T$, *flat over* T, *such that for all sufficiently general* $t\in T$ *the fibre* $\mathscr{F}_t$ *is a locally free sheaf on* $\mathbf{P}^3_{k(t)}$, *and such that for one point* $t_0\in T$, *the fibre* $\mathscr{F}_{t_0}$ *is isomorphic to* $\mathscr{E}_0$.

Proof. — *See* paragraph 3. The idea is to construct a universal extension of $\mathscr{I}_{Y_0}(1)$ by $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$ so that the general fibre is one of the bundles $\mathscr{E}$ of examples (1.6.1), (1.6.2), and the special fibre is the trivial extension, namely $\mathscr{E}_0$.

PROPOSITION 2.2. — *For each* $c_1=0$, $c_2>0$ *and for each* $c_1=-1$, c_2 *even* $\geqq 6$, *there is an irreducible nonsingular scheme* T *and a torsion-free coherent sheaf* $\mathscr{F}$ *on* $\mathbf{P}^3$, *flat over* T, *such that for all sufficiently general points* $t\in T$, *the fibre* $\mathscr{F}_t$ *is torsion-free with semi-natural cohomology, and for one point* $t_0\in T$, *the fibre* $\mathscr{F}_{t_0}$ *is isomorphic to* $\mathscr{E}_0$.

Proof. – *See* paragraphs 5-9. This is the heart of the matter.

Next we construct a certain big enough family of deformations of $\mathscr{E}_0$ using the **Quot** scheme of Grothendieck. Having fixed c_1, c_2, choose N sufficiently large so that $\mathscr{E}_0(\mathrm{N})$ is generated by global sections and $\mathrm{H}^i(\mathscr{E}_0(\mathrm{N}))=0$ for $i>0$. Let $m=h^0(\mathscr{E}_0(\mathrm{N}))$. Then we can write $\mathscr{E}_0(\mathrm{N})$ as a quotient $\mathcal{O}^m \to \mathscr{E}_0(\mathrm{N}) \to 0$. Let P be the Hilbert polynomial of $\mathscr{E}_0(\mathrm{N})$. Consider the functor which to each scheme S over k assigns the set of quotients:

$$\mathcal{O}^m_{\mathbf{P}^3_\mathrm{S}} \to \mathscr{F} \to 0,$$

where $\mathscr{F}$ is coherent on $\mathbf{P}^3_\mathrm{S}$, flat over S, with fibres having Hilbert polynomial P, modulo isomorphisms $\mathscr{F} \cong \mathscr{F}'$ compatible with the maps from $\mathcal{O}^m_{\mathbf{P}^3_\mathrm{S}}$. This is the Quot functor of Grothendieck.

PROPOSITION 2.3. – *The functor described above is represented by a scheme* Q, *projective over* k. *Furthermore,* Q *is nonsingular at the point* $q_0 \in \mathrm{Q}$ *corresponding to* $\mathscr{E}_0(\mathrm{N})$.

Proof. – *See* paragraph 4. We use the differential study of Q described in ([7], exp 221).

Proof of Theorem 0.1. – Fix c_1, c_2 with either $c_1=0, c_2>0$, or $c_1=-1, c_2$ even $\geqq 6$. Let $\mathscr{E}_0$ be the torsion-free sheaf on $\mathbf{P}^3$ defined at the beginning of this section.

Let Q be the **Quot** scheme of (2.3) above. Let $\mathscr{G}$ denote the universal quotient sheaf on $\mathbf{P}^3_\mathrm{Q}$. Thus $\mathscr{G}$ comes with a natural map:

$$\mathcal{O}^m_{\mathbf{P}^3_\mathrm{Q}} \to \mathscr{G} \to 0,$$

$\mathscr{G}$ is flat over Q, and the fibre of $\mathscr{G}$ at the point q_0 is $\mathscr{E}_0(\mathrm{N})$. Since Q is nonsingular at q_0, the point q_0 is contained in a unique irreducible component of Q, which we call Q_0.

Now consider the flat family $\mathscr{F}$ on $\mathbf{P}^3_\mathrm{T}$ given by (2.1). Its fibre at the point t_0 is $\mathscr{E}_0$. Since $\mathscr{E}_0(\mathrm{N})$ is generated by global sections, and $\mathrm{H}^i(\mathscr{E}_0(\mathrm{N}))=0$ for $i>0$, it follows that the same is true for the fibres $\mathscr{F}_t$ for t in some neighborhood of t_0. In fact, in such a neighborhood, $\mathrm{R}^i f_*(\mathscr{F}(\mathrm{N}))=0$ and $f_*(\mathscr{F}(\mathrm{N}))$ is locally free of rank m and commutes with base change, by the semicontinuity theorems. The chosen mapping $\mathcal{O}^m \to \mathscr{E}_0(\mathrm{N}) \to 0$ gives a basis of $\mathrm{H}^0(\mathscr{E}_0(\mathrm{N}))$ which we can lift to a set of free generators of $f_*(\mathscr{F}(\mathrm{N}))$ in some neighborhood T_0 of t_0, thus obtaining an isomorphism $\mathcal{O}^m_{\mathrm{T}_0} \xrightarrow{\cong} f_*(\mathscr{F}(\mathrm{N}))$. This in turn gives a surjective map:

$$\mathcal{O}^m_{\mathbf{P}^3_{\mathrm{T}_0}} \to \mathscr{F}(\mathrm{N}) \to 0,$$

which restricts to the given map $\mathcal{O}^m \to \mathscr{E}_0(\mathrm{N}) \to 0$ at the point t_0.

We are now in a position to apply the universal property of the Quot scheme. It implies that there is a unique morphism $\varphi : \mathrm{T}_0 \to \mathrm{Q}$ such that the map $\mathcal{O}^m \to \mathscr{F}(\mathrm{N}) \to 0$ is obtained by applying φ^* to the universal quotient $\mathcal{O}^m \to \mathscr{G} \to 0$. In particular, $\varphi(t_0)=q_0$. Since T_0 is irreducible, $\varphi(\mathrm{T}_0) \subseteqq \mathrm{Q}_0$. Because of (2.1), the fibres $\mathscr{F}_t$ are locally free for all sufficiently general $t \in \mathrm{T}$, in particular, for points t in a non-empty Zariski open subset of T. It follows that $\mathscr{G}_q$ is locally free for q in an open subset of $\varphi(\mathrm{T}_0)$. In particular, since the property $\mathscr{G}_q$ locally free is an open condition on $q \in \mathrm{Q}$, it follows that there is a non-empty open subset U_1 of the irreducible component Q_0, such that $\mathscr{G}_q$ is locally free for all $q \in \mathrm{U}_1$.

Next we consider the flat family $\mathscr{F}$ on $\mathbf{P}^3_T$ of (2.2). The same argument shows that there is a neighborhood T_0 of t_0 in T and a morphism $\varphi : T_0 \to Q$ as before. We conclude, using (1.6), that there is a non-empty open subset $U_2 \subseteq Q_0$ such that $\mathscr{G}_q$ has semi-natural cohomology for all $q \in U_2$.

Now let $U = U_1 \cap U_2$. This is a non-empty open subset of the irreducible component Q_0 of Q. For $q \in U$, the fibre $\mathscr{G}_q$ is both locally free and has semi-natural cohomology. Hence by (1.5) it has natural cohomology. Let $\mathscr{E} = \mathscr{G}_q(-N)$. Then $\mathscr{E}$ is a rank 2 vector bundle with the given Chern classes, having natural cohomology. This proves the theorem, modulo (2.1), (2.2), and (2.3).

Remark 2.3.1. — The proof actually shows something slightly more precise. For each c_1, c_2 in the given range, the construction of (1.6.1) or (1.6.2) provides an irreducible family of bundles with the given Chern classes. This family is contained in a unique irreducible component M_0 of the moduli space $M(c_1, c_2)$ of stable bundles with Chern classes c_1, c_2. This component M_0 is reduced, and of the expected dimension $8c_2 - 3$ if $c_1 = 0$ or $8c_2 - 5$ if $c_1 = -1$ ([11], 4.3.1, 4.3.2). What our proof shows is that there is a non-empty open subset of the irreducible component M_0 whose points correspond to vector bundles with natural cohomology.

We do not know if the bundles with natural cohomology form an irreducible subset of the moduli space. We also do not know if the moduli space is nonsingular at every point corresponding to a bundle with natural cohomology.

3. Universal extensions

In this section we prove the existence of a universal family of extensions of two coherent sheaves on a projective scheme X. This should be well known, but we could not find a proof, so include one here.

PROPOSITION 3.1. — *Let* X *be a projective scheme over* k, *let* $\mathscr{F}, \mathscr{G}$ *be coherent sheaves on* X, *let* V *be the* k*-vector space* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$, *and let* $T = \mathrm{Spec}\, k[V^*]$. *Thus a closed point* $t \in T$ *corresponds to an element* $\xi_t \in V$. *Then there is an extension:*

$$0 \to p_1^* \mathscr{F} \to \mathscr{E} \to p_1^* \mathscr{G} \to 0,$$

on $X \times T$ *such that for each closed point* $t \in T$, *the induced extension:*

$$0 \to \mathscr{F} \to \mathscr{E}_t \to \mathscr{G} \to 0,$$

on X *is the extension given by* $\xi_t \in \mathrm{Ext}^1(\mathscr{G}, \mathscr{F})$.

LEMMA 3.2. — *Let* X *be a projective scheme over a noetherian ring* A, *let* $\mathscr{F}, \mathscr{G}$ *be coherent sheaves on* X, *and let* $A \to B$ *be a flat ring extension. Denote by* X_B, $\mathscr{F}_B$, $\mathscr{G}_B$ *the base extensions to* B. *Then for all* i,

$$\mathrm{Ext}^i_{X_B}(\mathscr{G}_B, \mathscr{F}_B) = \mathrm{Ext}^i_X(\mathscr{G}, \mathscr{F}) \otimes_A B.$$

Proof. — Take a locally free resolution of $\mathscr{G} : \mathscr{L}. \to \mathscr{G} \to 0$. Then $\mathrm{Ext}^i_X(\mathscr{G}, \mathscr{F})$ can be computed as the hypercohomology of the complex of sheaves $\mathscr{H}om(\mathscr{L}., \mathscr{F})$. As in the case of a single sheaf ([10], III, 9.3) the same proof using a Cech process shows that cohomology commutes with flat base extension. Hence:

$$H^i(X_B, \mathscr{H}om(\mathscr{L}., \mathscr{F})_B) = H^i(X, \mathscr{H}om(\mathscr{L}., \mathscr{F})) \underset{A}{\otimes} B.$$

Now:

$$\mathscr{H}om(\mathscr{L}., \mathscr{F})_B = \mathscr{H}om(\mathscr{L}._B, \mathscr{F}_B)$$

and since $A \to B$ is flat, $\mathscr{L}._B$ is a locally free resolution of $\mathscr{G}_B$, so this hypercohomology also computes Ext on X_B, which gives the result.

Proof of (3.1). — By the lemma,

$$\mathrm{Ext}^1_{X \times T}(p_1^* \mathscr{G}, p_1^* \mathscr{F}) = V \underset{k}{\otimes} k[V^*].$$

The identity map $V \to V$ gives an element $\eta \in V \otimes V^*$, hence an element $\eta \in V \otimes k[V^*]$. The corresponding extension of $p_1^* \mathscr{G}$ by $p_1^* \mathscr{F}$ on $X \times T$ has the required properties.

Remark 3.2.1. — It is easy to see in fact that T represents the functor which to each scheme S over k associates the set of extensions:

$$0 \to p_1^* \mathscr{F} \to \mathscr{E} \to p_1^* \mathscr{G} \to 0$$

on $X \times S$, modulo equivalence of extensions.

Proof of (2.1). — By (3.1) there is a universal extension of $\mathscr{I}_{Y_0}(1)$ by $\mathscr{O}(-1)$ or $\mathscr{O}(-2)$ on $\mathbf{P}^3_T$, where T is $\mathrm{Ext}^1(\mathscr{I}_{Y_0}(1), \mathscr{O}(-1))$ or $\mathrm{Ext}^1(\mathscr{I}_{Y_0}(1), \mathscr{O}(-2))$. For $t_0 \in T$ corresponding to 0, we get the direct sum, which is $\mathscr{E}_0$. For $t \in T$ sufficiently general, we get the vector bundles (1.6.1) or (1.6.2), which are locally free.

4. The Quot scheme

In this section we review the Quot scheme and its differential study, in order to prove (2.3).

Let X be a projective scheme over k, and let $\mathscr{G}$ be a coherent sheaf on X. Fix a polynomial $P \in \mathbf{Q}[z]$. Then the functor:

$$\mathrm{Quot}^P_{\mathscr{G}/X/k},$$

which to each scheme S over k assigns the set of quotients:

$$p_1^* \mathscr{G} \to \mathscr{F} \to 0$$

on $X \times S$, such that $\mathscr{F}$ is flat over S and the fibres have Hilbert polynomial P, is represented by a scheme:

$$Q = \mathbf{Quot}^P_{\mathscr{G}/X/k},$$

projective over k ([7], exp. 221, Theorem 3.2). In particular, this proves the existence of the scheme Q of (2.3), taking $X=\mathbf{P}^3$ and $\mathcal{G}=\mathcal{O}_X^m$.

To study the infinitesimal properties of Q, we use the differential study of Quot ([7], exp. 221, § 5). Let $q \in Q$ be a closed point corresponding to a quotient $\mathcal{F}$ of $\mathcal{G}$ on X, and let $\mathcal{H}$ be the kernel:

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{F} \to 0.$$

Assume that there are no local obstructions to deforming $\mathcal{F}$. This means given any surjective map $A' \to A \to 0$ of Artin rings over k, and given any extension of $\mathcal{F}$ to a quotient of $\mathcal{G}_A$ on X_A, then at least locally on X, this can be extended to a quotient of $\mathcal{G}'_A$ on X'_A.

Then the Zariski tangent space to Q at q is given by $H^0(X, \mathcal{H}om(\mathcal{H}, \mathcal{F}))$. Furthermore, the obstructions to global deformations lie in $H^1(X, \mathcal{H}om(\mathcal{H}, \mathcal{F}))$. In particular, if this H^1 is zero, then Q is nonsingular at q. All this is explained in the cited reference of Grothendieck.

To apply this to our situation we consider the special case $\mathcal{G}=\mathcal{O}_X^m$.

Proposition 4.1. — *Let* X *be a projective scheme over* k, P *a polynomial, and let* Q *be the Quot scheme of quotients of* $\mathcal{O}_X^m$ *with Hilbert polynomial* P *on* X. *Let* $q \in Q$ *correspond to a quotient* $\mathcal{O}_X^m \to \mathcal{F} \to 0$ *on* X. *Assume:*

(1) $hd\,\mathcal{F} \leqq 1$;

(2) $H^1(X, \mathcal{F})=0$;

(3) $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})=0$.

Then Q *is nonsingular at* q.

Proof. — (*See* also [24], 6.6, 6.7). Since $\mathcal{F}$ has homological dimension 1, the kernel $\mathcal{H}$ of $\mathcal{O}_X^m \to \mathcal{F}$ will be locally free. Thus locally $\mathcal{F}$ is a cokernel of a map $\alpha: \mathcal{O}_X^n \to \mathcal{O}_X^m$ of free sheaves, given by a matrix of maximal rank. One can always lift a matrix over a larger Artin ring simply by lifting its entries. So there are no local obstructions to lifting $\mathcal{F}$. Thus the previous discussion applies, and to show Q nonsingular at q it is sufficient to show $H^1(X, \mathcal{H}om(\mathcal{H}, \mathcal{F}))=0$.

Apply the functor Hom (., $\mathcal{F}$) to the exact sequence:

$$0 \to \mathcal{H} \to \mathcal{O}^m \to \mathcal{F} \to 0.$$

This gives:

$$\mathrm{Ext}^1(\mathcal{O}^m, \mathcal{F}) \to \mathrm{Ext}^1(\mathcal{H}, \mathcal{F}) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}).$$

The first term is m copies of $H^1(\mathcal{F})$, which is zero by hypothesis. The last term is also zero by hypothesis. Therefore $\mathrm{Ext}^1(\mathcal{H}, \mathcal{F})=0$. But $\mathcal{H}$ is locally free, so this is equal to $H^1(\mathcal{H}^\vee \otimes \mathcal{F})=H^1(\mathcal{H}om(\mathcal{H}, \mathcal{F}))$. Hence Q is nonsingular at q.

Proposition 4.2. — *Let* Y *be a locally complete intersection curve in* $\mathbf{P}^3$, *let* $a \in \mathbf{Z}$, *and let* $\mathcal{F}=\mathcal{O}(-a) \oplus \mathcal{I}_Y$. *Assume:*

(1) $H^1(\mathcal{O}_Y(a))=0$;

(2) $H^1(\mathscr{I}_Y(a-4))=0$;

(3) $H^1(\mathscr{N}_Y)=0$.

Then $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F})=0$.

Proof. — Since $\mathscr{F}$ is a direct sum of two sheaves, the Ext group is a direct sum of four pieces. We treat them individually.

(*a*) $\mathrm{Ext}^2(\mathcal{O}(-a), \mathcal{O}(-a))=H^2(\mathcal{O})=0$ since we are working on $\mathbf{P}^3$.

(*b*) $\mathrm{Ext}^2(\mathcal{O}(-a), \mathscr{I}_Y)=H^2(\mathscr{I}_Y(a))\cong H^1(\mathcal{O}_Y(a))$ which is zero by hypothesis.

(*c*) $\mathrm{Ext}^2(\mathscr{I}_Y, \mathcal{O}(-a))$ is dual to $H^1(\mathscr{I}_Y(a-4))$ by Serre duality. It is also zero by hypothesis.

(*d*) To compute $\mathrm{Ext}^2(\mathscr{I}_Y, \mathscr{I}_Y)$, we use the spectral sequence of local and global Ext. First note $\mathscr{H}om(\mathscr{I}_Y, \mathscr{I}_Y)=\mathcal{O}$. Next we claim $\mathscr{E}xt^1(\mathscr{I}_Y, \mathscr{I}_Y)\sim\mathscr{N}_Y$, the normal sheaf of Y in $\mathbf{P}^3$. Indeed, apply $\mathscr{H}om(\mathscr{I}_Y, \;)$ to the exact sequence $0\to\mathscr{I}_Y\to\mathcal{O}\to\mathcal{O}_Y\to 0$. This gives:

$$0\to\mathscr{H}om(\mathscr{I}_Y, \mathscr{I}_Y)\xrightarrow{\alpha}\mathscr{H}om(\mathscr{I}_Y, \mathcal{O})\to\mathscr{H}om(\mathscr{I}_Y, \mathcal{O}_Y)$$
$$\to\mathscr{E}xt^1(\mathscr{I}_Y, \mathscr{I}_Y)\to\mathscr{E}xt^1(\mathscr{I}_Y, \mathcal{O})\xrightarrow{\beta}\mathscr{E}xt^1(\mathscr{I}_Y, \mathcal{O}_Y).$$

Now $\mathscr{H}om(\mathscr{I}_Y, \mathcal{O}_Y)\cong\mathscr{N}_Y$ by definition of the normal bundle. So it is sufficient to show that α and β are isomorphisms. This is an easy local calculation using a resolution:

$$0\to\mathcal{O}\xrightarrow{(y,-x)}\mathcal{O}\oplus\mathcal{O}\xrightarrow{(x,y)}\mathscr{I}_Y\to 0$$

of $\mathscr{I}_Y$.

Since $hd\,\mathscr{I}_Y=1$, $\mathscr{E}xt^i(\mathscr{I}_Y, \mathscr{I}_Y)=0$ for $i\geqq 2$. So the spectral sequence of local and global Ext is very simple. In particular, it gives an isomorphism:

$$\mathrm{Ext}^2(\mathscr{I}_Y, \mathscr{I}_Y)\sim H^1(\mathscr{E}xt^1(\mathscr{I}_Y, \mathscr{I}_Y))=H^1(\mathscr{N}_Y).$$

This is also zero by hypothesis, which proves the proposition.

Proof of (2.3). — The existence of Q follows directly from Grothendieck's theorem, as noted above. To show that Q is nonsingular at the point q_0 corresponding to $\mathscr{E}_0(N)$, we apply the two previous results. We have seen that $hd\,\mathscr{E}_0=1$. Furthermore $H^1(\mathscr{E}_0(N))=0$ by choice of N. So we can apply (4.1), and it remains to verify $\mathrm{Ext}^2(\mathscr{E}_0(N), \mathscr{E}_0(N))=0$. The twist by N is irrelevant, so we can apply (4.2) with $\mathscr{F}=\mathscr{E}_0(-1)=\mathcal{O}(-a)\oplus\mathscr{I}_Y$, where $a=2$ or 3, and Y is a union of lines or conics in $\mathbf{P}^3$. In these cases the assumptions of (4.2) are immediately verified so we conclude $\mathrm{Ext}^2(\mathscr{E}_0(N), \mathscr{E}_0(N))=0$ as required.

5. General position statements

In this section we begin the proof of (2.2). First we construct the family $\mathscr{F}$ of (2.2). Then we formulate the general position statements (5.1) and (5.2) which are needed to prove that the general fibre $\mathscr{F}_t$ has semi-natural cohomology. We show that (5.2)$\Rightarrow$(5.1)$\Rightarrow$(2.2). The proof of (5.2) will be carried out in sections 6-9.

Recall from paragraph 2 that we defined the sheaf $\mathscr{E}_0$ to be:

$$\mathscr{E}_0=\mathcal{O}\,(-a+1)\oplus\mathscr{I}_{Y_0}(1),$$

depending on c_1 and c_2 as follows. If $c_1=0, c_2>0$, take Y_0 to be a disjoint union of $r=c_2+1$ lines in $\mathbf{P}^3$, and take $a=2$. If $c_1=-1$, $c_2>0$, c_2 even, take Y_0 to be a disjoint union of $r=1/2\,(c_2+2)$ conics in $\mathbf{P}^3$, and take $a=3$. Note that $\mathscr{E}_0$ can be written as a kernel:

$$0\to\mathscr{E}_0\to\mathcal{O}(-a+1)\oplus\mathcal{O}(1)\xrightarrow{\alpha_0(1)}\mathcal{O}_{Y_0}(1)\to 0,$$

where the map $\alpha_0\,(1)$ is zero on the first factor, and the natural restriction map on the second factor. We write $\alpha_0\,(1)=(0,\,1)$.

To construct the family $\mathscr{F}$ we first allow Y to range over all disjoint unions of r lines or conics, as the case may be. We allow α to be any map of the form $(\beta,\,1)$ where β: $\mathcal{O}\,(-a)\to\mathcal{O}_Y$ is an arbitrary map. Then we consider all sheaves $\mathscr{E}$ which are kernels of the maps $\alpha\,(1)$:

$$0\to\mathscr{E}\to\mathcal{O}(-a+1)\oplus\mathcal{O}(1)\xrightarrow{\alpha(1)}\mathcal{O}_Y(1)\to 0.$$

We construct a family having all these sheaves as its fibres as follows. Let H be the open subset of the Hilbert scheme corresponding to disjoint unions of r lines or conics. Then H is an irreducible nonsingular quasiprojective variety, and there is a universal closed subscheme $\mathscr{Y}\subseteqq\mathbf{P}^3_H$. Next, note that the map β factors through the natural projection $\mathcal{O}(-a)\to\mathcal{O}_Y(-a)$, so to give β is equivalent to giving the map $\mathcal{O}_Y(-a)\to\mathcal{O}_Y$, which corresponds to a section of $\mathcal{O}_Y(a)$. So we consider the locally free sheaf $\pi_*\mathcal{O}_{\mathscr{Y}}(a)$ on H, where $\pi:\mathbf{P}^3_H\to H$ is the projection, and let T be the geometric vector bundle $T=\mathbf{V}((\pi_*\mathcal{O}_{\mathscr{Y}}(a))^{\check{}})$ ([10], II, Ex. 5.18). Make the base change $T\to H$ and let $\mathscr{Y}_T\subseteqq\mathbf{P}^3_T$, with projection $\pi':\mathbf{P}^3_T\to T$, be the extended situation. By construction (*see* also §6) T comes equipped with a natural section of $\pi'_*(\mathcal{O}_{\mathscr{Y}_T}(a))$ which lifts to a natural section of $\mathcal{O}_{\mathscr{Y}_T}(a)$ which in turn defines a natural map $\beta:\mathcal{O}_{\mathbf{P}^3_T}(-a)\to\mathcal{O}_{\mathscr{Y}_T}$. Now we define the family $\mathscr{F}$ on $\mathbf{P}^3_T$ by:

$$0\to\mathscr{F}\to\mathcal{O}_{\mathbf{P}^3_T}(-a+1)\oplus\mathcal{O}_{\mathbf{P}^3_T}(1)\xrightarrow{\alpha(1)}\mathcal{O}_{\mathscr{Y}_T}(1)\to 0,$$

where $\alpha=(\beta,\,1)$.

This is the family we will use to prove (2.2). $\mathscr{F}$ is clearly torsion-free. It is flat over T because $\mathscr{Y}_T$ is. Furthermore, this exact sequence commutes with passage to fibres for $t\in T$, so the fibres $\mathscr{F}_t$ are exactly the sheaves $\mathscr{E}$ described above, and the points $t\in T$ correspond to all possible choices of Y and α. It remains to show that for t sufficiently general, $\mathscr{F}_t$ has semi-natural cohomology.

PROPOSITION 5.1. — *Let* $Y\subseteqq\mathbf{P}^3$ *be a disjoint union of* $r\geqq 1$ *lines* (*resp.* $r\geqq 4$ *conics*) *and let* $a=2$ (*resp.* $a=3$). *If* Y *is sufficiently general, and if* $\beta:\mathcal{O}\,(-a)\to\mathcal{O}_Y$ *is sufficiently general, then for all* $n\in\mathbf{Z}$, *taking* $\alpha=(\beta,\,1)$, *the induced map:*

$$H^0(\alpha(n)):\quad H^0(\mathbf{P}^3,\,\mathcal{O}(n-a)\oplus\mathcal{O}(n))\to H^0(Y,\,\mathcal{O}_Y(n))$$

is of maximal rank (*i. e., either injective or surjective or both*).

Proof of (2.2). — Admitting this result, we can complete the proof of (2.2). Take $\mathscr{F}$ on $\mathbf{P}^3_T$ to be the family just constructed. At the point $t_0 \in T$ corresponding to $Y = Y_0$, $\beta = 0$, the fibre of $\mathscr{F}$ is just $\mathscr{E}_0$. For sufficiently general $t \in T$, by (5.1), for any $n \in \mathbf{Z}$, at most one of $H^0(\mathscr{F}_t(n))$ and $H^1(\mathscr{F}_t(n))$ is nonzero. Furthermore, taking into account the known cohomology of the sheaves $\mathcal{O}(n)$ on lines, conics, and on $\mathbf{P}^3$, we see easily that for $n \geqq -2$ (resp. $n \geqq -1$), the groups $H^2(\mathscr{F}_t(n))$ and $H^3(\mathscr{F}_t(n))$ are zero. Hence $\mathscr{F}_t$ has semi-natural cohomology.

Remark 5.1.1. — Note that the statement of (5.1) is false for 2 or 3 conics, although it is trivially true for one conic. Indeed, if $r=2$, $n=2$, then both vector spaces have dimension 10, but $H^0(\alpha(2))$ is not injective because the pair of conics is contained in a quadric surface, namely the union of the planes of the two conics. Similarly, for $r=n=3$, both spaces have dimension 21, but $H^0(\alpha(3))$ is not injective, because Y is contained in a union of 3 planes.

Proof of (5.1). — First we note, as in the proof of (1.6), that $H^0(\alpha(n))$ is automatically surjective for $n \gg 0$ (depending on r), and both sides are 0 for $n<0$, so there are only finitely many values of n to consider. Hence the conclusion of the proposition is an open condition on the parameter space T, and it is sufficient to verify it for each r, n individually. Furthermore, if we can find a single choice of Y, β for which $H^0(\alpha(n))$ has maximal rank, then it is also true for all sufficiently general choices of Y, β.

Given n, suppose one can choose r so that:

$$h^0(\mathcal{O}(n-a) \oplus \mathcal{O}(n)) = h^0(\mathcal{O}_Y(n))$$

and suppose one can find Y, β so that $H^0(\alpha(n))$ is bijective. Then if one removes some lines (resp. conics) from Y, $H^0(\alpha(n))$ will still be surjective, and if one adds some lines (resp. conics) to Y, $H^0(\alpha(n))$ will still be injective. In other words, the result for that given pair (n, r) implies the result for the same n and any r whatsoever.

Unfortunately for given n one cannot always find such an r. Therefore we will make adjustments by adding some isolated points to Y. Suppose we add q points. In the case of lines, with $a=2$, the desired equality of h^0's above says:

$$\binom{n+1}{3} + \binom{n+3}{3} = r(n+1) + q.$$

So we take:

$$r = \left[\frac{n^2+2n+3}{3}\right]$$

and:

$$q = (n+1)\left(\frac{n^2+2n+3}{3} - r\right).$$

For each $n \geqq 0$ consider the statement:

(H_n) *Taking r and q as above, if* Y *is a disjoint union of r lines and q collinear points in* $\mathbf{P}^3$ *in sufficiently general position, and if* $\beta : \mathcal{O}(-2) \to \mathcal{O}_Y$ *is sufficiently general, then taking* $\alpha = (\beta, 1)$, *the induced map:*

$$H^0(\alpha(n)) : \quad H^0(\mathcal{O}_{\mathbf{P}^3}(n-2) \oplus \mathcal{O}_{\mathbf{P}^3}(n)) \to H^0(\mathcal{O}_Y(n)),$$

is bijective.

In the case of conics, with $a=3$, the desired equality of h^0's says:

$$\binom{n}{3} + \binom{n+3}{3} = r(2n+1) + q.$$

So take:

$$r = \left[\frac{n^2+n+6}{6}\right]$$

and:

$$q = (2n+1)\left(\frac{n^2+n+6}{6} - r\right).$$

For each $n \geqq 0$ consider the statement:

(H'_n) *Taking r and q as above, if* Y *is a disjoint union of r conics and q points lying on a conic in* $\mathbf{P}^3$ *in sufficiently general position, and if* $\beta : \mathcal{O}(-3) \to \mathcal{O}_Y$ *is sufficiently general, then taking* $\alpha = (\beta, 1)$, *the induced map:*

$$H^0(\alpha(n)) : \quad H^0(\mathcal{O}_{\mathbf{P}^3}(n-3) \oplus \mathcal{O}_{\mathbf{P}^3}(n)) \to H^0(\mathcal{O}_Y(n)),$$

is bijective.

PROPOSITION 5.2. — *The statement* H_n *is true for all* $n \geqq 0$. *The statement* H'_n *is true for* $n=0$, 1, *and all* $n \geqq 4$.

Proof. — See (8.1) and (9.1).

Proof of (5.1), *continued.* — Using (5.2), we can complete the proof of (5.1). First we consider the case of lines. For $n<0$ there is nothing to prove. For each $n \geqq 0$ there is a union Y of r lines and q points for which the corresponding map $H^0(\alpha(n))$ is bijective. To prove (5.1) for that n and any $r' \leqq r$, simply remove the q points and $r-r'$ lines. Then $H^0(\alpha(n))$ will be surjective. To prove it for $r''>r$, first add a line passing through the q collinear points, then add $r''-r-1$ disjoint lines. Then the corresponding $H^0(\alpha(n))$ will be injective. This proves the statement for lines for all $r \geqq 1$ and all $n \in \mathbf{Z}$.

Now consider the case of conics. Again for $n<0$ there is nothing to prove. For $n=0$, 1 or for $n \geqq 4$, the same argument shows the statement of (5.1) is true for all $r \geqq 1$. It remains to verify the cases $n=2$, 3 and $r \geqq 4$. If $n=2$, the map in question is:

$$H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(\mathcal{O}_Y(2)).$$

This is clearly injective for $r \geqq 3$ (and surjective for $r=1$). If $n=3$, the map is:

$$H^0(\alpha(3)) : \quad H^0(\mathcal{O}_{\mathbf{P}^3}) \oplus H^0(\mathcal{O}_{\mathbf{P}^3}(3)) \to H^0(\mathcal{O}_Y(3)).$$

If $r \geqq 4$, then this map is clearly injective on the $H^0(\mathcal{O}_{\mathbf{P}^3}(3))$ part. Furthermore, $h^0(\mathcal{O}_{\mathbf{P}^3}(3))=20$, and $h^0(\mathcal{O}_Y(3))=7r$, so for $r \geqq 3$, $h^0(\mathcal{O}_Y(3)) > h^0(\mathcal{O}_{\mathbf{P}^3}(3))$. Since we can choose the map β to send the generator of $H^0(\mathcal{O}_{\mathbf{P}^3})=k$ to any element of $H^0(\mathcal{O}_Y(3))$ we like, for $r \geqq 4$ we can make $H^0(\alpha(3))$ injective. This completes the proof of (5.1) modulo (5.2). (It is not hard to check also that $H^0(\alpha(3))$ is surjective for $r=1, 2$).

Remark 5.2.1. — In fact, this argument shows that the statement of (5.1) in the case of conics is true for all $r \geqq 1$, $n \in \mathbf{Z}$ with the exception of only the two cases $r=n=2$ and $r=n=3$ mentioned in (5.1.1), and in those two cases, the map $H^0(\alpha(n))$ fails to be injective by a 1-dimensional subspace. The consequence of this for vector bundles is that in the two excluded cases of theorem (0.1), namely $c_1=-1$, $c_2=2$ (resp. $c_2=4$) there are stable vector bundles having natural cohomology with the exception of $h^0(E(1))=h^1(E(1))=1$ (resp. $h^0(E(2))=h^1(E(2))=1$), and the exception for h^2 and h^3 implied by duality.

6. Reformulations and inductive procedure of proof

In this section we explain a reformulation of the statements (H_n), (H'_n) of (5.2) in terms of certain geometric vector bundles over $\mathbf{P}^3$. We also describe the inductive procedure involving analogous statements in lower dimensions which we will use to prove these results.

DEFINITION. — Let X be a quasiprojective variety over k with a given line bundle $\mathcal{O}_X(1)$. A *line* in X is a closed subscheme $L \subseteqq X$ such that $L \cong \mathbf{P}^1$ and $\mathcal{O}_X(1) \otimes \mathcal{O}_L \cong \mathcal{O}_{\mathbf{P}^1}(1)$. A *conic* in X is a closed subscheme $C \subseteqq X$ such that $C \cong \mathbf{P}^1$ and $\mathcal{O}_X(1) \otimes \mathcal{O}_C \cong \mathcal{O}_{\mathbf{P}^1}(2)$. A set of points of X is *collinear* if there exists a line containing all of them.

If $X=\mathbf{P}^n$, this is of course the usual notion of line and conic. We include the definition so that we can speak without ambiguity of lines and conics in the geometric vector bundles defined below.

Let $(X, \mathcal{O}_X(1))$ be a projective variety (which in our applications will be $\mathbf{P}^1$, $\mathbf{P}^2$, a nonsingular quadric surface Q, or $\mathbf{P}^3$) and let $a \geqq 0$ be an integer (which in our case will be 2, 3, 4, or 6). Let Z be the geometric vector bundle $Z=\mathbf{V}(\mathcal{O}_X(-a))$. Recall ([10], II, Ex. 5.18) that this is defined as follows. Let $\mathcal{S}$ be the symmetric algebra on $\mathcal{O}_X(-a)$, namely $\mathcal{S}=\bigoplus_{n \geqq 0} \mathcal{O}_X(-na)$. Then $\mathcal{S}$ is a sheaf of $\mathcal{O}_X$-algebras, and we define $\mathbf{V}(\mathcal{O}_X(-a))=\mathbf{Spec}\,\mathcal{S}$. The natural map $\mathcal{O}_X(-a) \to \mathcal{S}$ of $\mathcal{O}_X$-modules induces a map $\mathcal{O}_X(-a) \otimes \mathcal{S} \to \mathcal{S}$ of $\mathcal{S}$-modules, and hence a natural map $\varphi : \mathcal{O}_Z(-a) \to \mathcal{O}_Z$ on $Z=\mathbf{V}(\mathcal{O}_X(-a))$. Here if $p : Z \to X$ is the projection, we denote by $\mathcal{O}_Z(1)$ the sheaf $p^* \mathcal{O}_X(1)$ on Z.

Now let $f : Y \to X$ be a morphism, and let $\mathcal{O}_Y(1)=f^* \mathcal{O}_X(1)$. If we give a lifting $g : Y \to Z$ of f (i. e., a morphism g such that $f=p \circ g$), then $g^*(\varphi)=\psi$ is a map $\psi : \mathcal{O}_Y(-a) \to \mathcal{O}_Y$. Conversely, given $f : Y \to X$ and given any map $\psi : \mathcal{O}_Y(-a) \to \mathcal{O}_Y$, then ψ induces a morphism of $\mathcal{O}_Y$-algebras $f^*(\mathcal{S}) \to \mathcal{O}_Y$, hence a morphism of schemes $g : Y \to Z = \mathbf{Spec}\ \mathcal{S}$, lifting f, such that $\psi = g^*(\varphi)$. Thus we see that (Z, φ) represents the functor on X-schemes:

$$Y \mapsto \{\psi : \mathcal{O}_Y(-a) \to \mathcal{O}_Y\}.$$

To apply this to our situation, think of (H_n) for example. Let Y be a disjoint union of lines in $\mathbf{P}^3$, and let $Z = \mathbf{V}(\mathcal{O}_{\mathbf{P}^3}(-2))$. Then to give a map $\beta : \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_Y$ is equivalent to giving its restriction to Y, $\mathcal{O}_Y(-2) \to \mathcal{O}_Y$, which in turn is equivalent to giving a lifting $g : Y \to Z$ of the inclusion $i : Y \subsetneq \mathbf{P}^3$. The image $Y' = g(Y)$ will be a disjoint union of lines in Z, in the sense defined above. Conversely, any sufficiently general disjoint union of lines in Z arises in this way. Thus in the statement of (H_n), the choice of $Y \subseteqq \mathbf{P}^3$ and $\beta : \mathcal{O}_Y(-2) \to \mathcal{O}_Y$ is equivalent to the choice of $Y' \subseteqq Z$. Note however that it is possible to have disjoint lines in Z whose projections to $\mathbf{P}^3$ are not disjoint. In fact, we will make essential use of such sets of lines in Z, which do not correspond to any sheaves in the family $\mathcal{F}$. A similar discussion applies to (H'_n). In this case, and henceforth, we consider only those conics in Z whose projections to X are also conics, excluding those whose projection to X is a line.

To explain the analogue of $H^0(\alpha(n))$ in this new interpretation, we return to the general situation. Let $(X, \mathcal{O}_X(1))$ be a projective variety, let $a \geqq 0$, and let $Z = \mathbf{V}(\mathcal{O}_X(-a))$. Let $Y' \subseteqq Z$ be a closed subscheme. Then the induced map $\varphi : \mathcal{O}_{Y'}(-a) \to \mathcal{O}_{Y'}$ and the projection $p : Y' \to X$ allow us to define for any $n \in \mathbf{Z}$ an induced map:

$$\rho(n) : \quad H^0(\mathcal{O}_X(n-a)) \oplus H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_{Y'}(n)),$$

by composing $p^* : H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_{Y'}(n))$ with $H^0((\varphi \oplus \mathrm{id})(n))$. If Y' is obtained by lifting a closed subscheme $Y \subseteqq X$ by means of a map $\beta : \mathcal{O}_Y(-a) \to \mathcal{O}_Y$, then $\rho(n)$ can be identified with $H^0(\alpha(n))$ *via* the isomorphism $g : Y \to Y'$. Thus we have proved:

PROPOSITION 6.1. — *In the statement* (H_n) (*resp.* (H'_n)) *of* (5.2) *we obtain an equivalent statement if we replace* (Y, β) *by a disjoint union of lines and collinear points in* $Z = \mathbf{V}(\mathcal{O}_{\mathbf{P}^3}(-2))$ (*resp. conics and points lying on a conic in* $Z = \mathbf{V}(\mathcal{O}_{\mathbf{P}^3}(-3))$), *and replace* $H^0(\alpha(n))$ *by* $\rho(n)$.

Our strategy for proving the modified statements (H_n) and (H'_n), which we still denote (H_n) and (H'_n) for simplicity, is to use induction on n and on the dimension of $\mathbf{P}^3$. This will involve analogous statements on lower-dimensional varieties $X = \mathbf{P}^1$, $\mathbf{P}^2$, or a nonsingular quadric surface Q.

A typical statement will be like this. For a given choice of X a projective variety and $a \geqq 0$, let $Z = \mathbf{V}(\mathcal{O}_X(-a))$. Then we will consider closed subschemes Y of Z consisting of a certain number of lines, conics, points, etc. We also consider an integer n which is so chosen in relation to the schemes Y being considered that:

$$h^0(\mathcal{O}_X(n-a)) + h^0(\mathcal{O}_X(n)) = h^0(\mathcal{O}_Y(n)).$$

The statement will then say that *in general the induced map* $\rho(n)$ *is bijective*. We use this phrase subject to the following:

CONVENTION. — The phrase *in general* $\rho(n)$ *is bijective* will mean that the set of subschemes Y of Z under consideration form an *irreducible* subset of the Hilbert scheme of closed subschemes of Z proper over X, and that $\rho(n)$ is bijective for all choices of Y in a non-empty Zariski open subset of this family.

Suppose given X, a, n, as above and a certain irreducible subset T of closed subschemes of Z, and we wish to prove for $Y \in T$ in general $\rho(n)$ is bijective. First we consider the closure $\overline{T}$ of T in the Hilbert scheme of closed subschemes of Z, proper over X. Since the property $\rho(n)$ bijective is open, it will be sufficient to find one $Y \in \overline{T}$, or a small family $T_1 \subseteq \overline{T}$ of such Y's, for which $\rho(n)$ is bijective. To make an induction, let $X' \subseteq X$ be a hypersurface of degree d. [In practice, d will be 1 or 2, or if X is the nonsingular quadric surface Q, sometimes d is the bidegree (0, 1) or (1, 0).] Denote by Z' the base extension $Z \times_X X'$, which is also equal to $\mathbf{V}(\mathcal{O}_{X'}(-a))$. Let Y' be the scheme-theoretic intersection $Y \cap Z'$, and let Y'' be the residual intersection $\operatorname{res}_{Z'} Y$ defined by the sheaf of ideals:

$$\mathcal{I}_{Y''} = f^{-1}.\ker(\mathcal{I}_{Y,Z} \to \mathcal{I}_{Y',Z'}),$$

where f is the equation of Z' in Z. Then there is an exact sequence:

$$0 \to \mathcal{O}_{Y''}(-d) \xrightarrow{f} \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0$$

([16], § 2, Def. and Ex. 2.1.1). Let $\rho'(n)$ and $\rho''(n-d)$ be the corresponding maps ρ associated to Y' and Y''.

LEMMA 6.2. — *With the hypotheses above, let* $Y \subseteq Z$ *be one such closed subscheme. Assume that* X *is* $\mathbf{P}^2$ *or* $\mathbf{P}^3$, *and that* $\rho'(n)$ *and* $\rho''(n-d)$ *are bijective. Then* $\rho(n)$ *is bijective.*

Proof. — This is an easy application of the 5-lemma. The exact sequence for Y, Y', Y'' above, coupled with the exact sequence:

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

and the fact that $H^1(\mathcal{O}_X(m)) = 0$ for all $m \in \mathbf{Z}$ give us an exact commutative diagram:

$$\begin{array}{ccccccc}
0 \to H^0(\mathcal{O}_X(n-a-d)) \oplus H^0(\mathcal{O}_X(n-d)) & \to & H^0(\mathcal{O}_X(n-a)) \oplus H^0(\mathcal{O}_X(n)) & \to & H^0(\mathcal{O}_{X'}(n-a)) \oplus H^0(\mathcal{O}_{X'}(n)) \to 0 \\
\downarrow{\scriptstyle \rho''(n-d)} & & \downarrow{\scriptstyle \rho(n)} & & \downarrow{\scriptstyle \rho'(n)} \\
0 \longrightarrow H^0(\mathcal{O}_{Y''}(n-d)) & \longrightarrow & H^0(\mathcal{O}_Y(n)) & \longrightarrow & H^0(\mathcal{O}_{Y'}(n)).
\end{array}$$

Since $\rho''(n-d)$ and $\rho'(n)$ are bijective by hypothesis, and the top line is exact on the right, $\rho(n)$ is also bijective.

If X is a nonsingular quadric surface Q, we must modify this statement to take into account the bidegree of a divisor. If $Y \subseteq Z = \mathbf{V}(\mathcal{O}_X(-a))$, and if (n_1, n_2) is a bidegree, then we have an associated map:

$$\rho(n_1, n_2): \quad H^0(\mathcal{O}_Q(n_1-a, n_2-a)) \oplus H^0(\mathcal{O}_Q(n_1, n_2)) \to H^0(\mathcal{O}_Y(n_1, n_2)),$$

where $\mathcal{O}_Y(n_1, n_2)=\mathcal{O}_Q(n_1, n_2)\otimes\mathcal{O}_Y$. If $X'\subseteqq X$ is a divisor of bidegree (d_1, d_2), we consider $Y'=Y\cap Z$ and $Y''=\mathrm{res}_{Z'}\, Y$ as above.

LEMMA 6.3. — *Let X be the quadric surface Q, let $Y\subseteqq Z$ as above, and assume that $\rho'(n_1, n_2)$ and $\rho''(n_1-d_1, n_2-d_2)$ are bijective. Assume furthermore that $n_1-d_1-a\geqq -1$ and $n_2-d_2-a\geqq -1$. Then $\rho(n_1, n_2)$ is bijective.*

Proof. — The same as the proof of (6.2) except for one point: to know that the upper row of the diagram is exact at the right, we need $H^1(\mathcal{O}_Q(n_1-d_1-a, n_2-d_2-a))=0$ and $H^1(\mathcal{O}_Q(n_1-d_1, n_2-d_2))=0$. This follows from our hypothesis and the fact that $H^1(\mathcal{O}_Q(m_1, m_2))=0$ if $m_1\geqq -1$ and $m_2\geqq -1$ (use Serre duality and [10], III, Ex. 5.6 *a*).

Remark 6.3.1. — To make use of these lemmas, we need to know something about Y' and Y''. In practice, we will not know about specific choices of Y', Y'', but we will know something if Y' and Y'' are sufficiently general. So we will always invoke these lemmas in the following situation. Suppose given X, *a*, *n* as above, and an irreducible family T of closed subschemes of Z, for which we wish to prove in general if $Y\in T$, then $\rho(n)$ is bijective. We will define an irreducible subfamily $T_1\subseteqq\overline{T}$ of the closure of T in the Hilbert scheme. We will fix a divisor $X'\subseteqq X$ as above. For each $Y\in T_1$ we consider $Y'=Y\cap Z'$ and $Y''=\mathrm{res}_{Z'}\, Y$. The schemes Y', Y'' thus obtained will form irreducible families T', T''. We will then refer to earlier results to show that in general for $Y'\in T'$, $\rho'(n)$ is bijective, and in general for $Y''\in T''$, $\rho''(n-d)$ is bijective. It will then follow by the lemmas that in general for $Y\in T$, $\rho(n)$ is bijective. Each time we use this technique, we must verify that the irreducible families T', T'' obtained are the same (or at least have open subsets the same) as the families considered in the earlier results alluded to. It will be understood that this is so each time this technique is used.

7. Lower-dimensional results

In this section we prove the general position results over varieties X of dimensions 1 and 2 which will be used in the proof of (5.2). Notations and terminology are those of paragraph 6.

PROPOSITION 7.1. — *Let $X=\mathbf{P}^1$, let $a\geqq 0$, let $Z=\mathbf{V}(\mathcal{O}_X(-a))$, and let $n\geqq a-1$. Let $Y\subseteqq Z$ be the union of one line and $q=n-a+1$ points. Then in general:*

$$\rho(n):\quad H^0(\mathcal{O}_X(n-a))\oplus H^0(\mathcal{O}_X(n))\to H^0(\mathcal{O}_Y(n))$$

is bijective.

Proof. — According to the conventions of paragraph 6, we must verify that the set of schemes Y we are considering is irreducible, which is obvious, and that for an open set of choices of Y, $\rho(n)$ is bijective. In this case $\rho(n)$ will be bijective provided the points have distinct projections to X and do not lie on the line L of Y. Indeed, by counting dimensions, it is enough to show $\rho(n)$ is injective. So suppose (f, g) is in the kernel of $\rho(n)$, with $f\in H^0(\mathcal{O}_X(n-a))$ and $g\in H^0(\mathcal{O}_X(n))$.

As we saw in paragraph 6, the line $L \subseteq Z$ is a lifting of X to Z, and so corresponds to a map $\beta : \mathcal{O}_X(-a) \to \mathcal{O}_X$. Since $\rho(f, g)=0$ along L, we conclude that $\beta f+g=0$ on X, i.e., $g=-\beta f$ in $H^0(\mathcal{O}_X(n))$.

Now let P_i be the points of Y, and Q_i their projections to X. Each P_i is determined by a map $\beta_i : \mathcal{O}_X(-a) \to \mathcal{O}_{Q_i}$. Since $P_i \notin L$, β_i is not equal to β restricted to $\mathcal{O}_{Q_i}$. Since $\rho(f, g)$ is 0 at P_i, we have $g(Q_i)=-\beta_i f(Q_i)$. But $g=-\beta f$, and $\beta(Q_i) \neq \beta_i$, so this implies $f(Q_i)=0$. Since f is a polynomial of degree $n-a$ which is zero at the $n-a+1$ points Q_i, it must be identically 0. So $g=-\beta f$ is also 0, which proves that $\rho(n)$ is injective, and hence bijective.

LEMMA 7.2. — *Let* $X=\mathbf{P}^2$, *and* $Z=\mathbf{V}(\mathcal{O}_X(-2))$, *Let* $n \geqq 0$ *be an integer. Let* $Y \subseteq Z$ *be a disjoint union of n lines and one point. Then in general:*

$$\rho(n): \quad H^0(\mathcal{O}_X(n-2)) \oplus H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_Y(n))$$

is bijective.

Proof. — By induction on n, the case $n=0$ being trivial. If $n \geqq 1$, fix a line $L \subseteq X$ and consider the subfamily consisting of those Y having one line lying over L. This is an irreducible subfamily. For this family we take L to be the divisor X′ in the notation of paragraph 6, and apply (6.2) and (6.3.1). Then $L \cong \mathbf{P}^1$, $Z'=\mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(-2))$, and $Y'=Y \cap Z'$ consists of one line and the $n-1$ points of intersection of the other lines with Z′. The family of such Y′ is general for (7.1), so we conclude that in general $\rho'(n)$ is bijective. On the other hand, the residual intersection Y″ consists of $n-1$ lines and one point. So by the induction hypothesis $\rho''(n-1)$ is bijective. It follows from (6.3.1) that $\rho(n)$ is bijective.

PROPOSITION 7.3. — *Let* $X=\mathbf{P}^2$, *let* $Z=\mathbf{V}(\mathcal{O}_X(-2))$, *and let* $n \geqq 0$ *be an integer. Let* $Y \subseteq Z$ *be a disjoint union of r lines, q points, and q′ collinear points, such that:*

(1) $r(n+1)+q+q'=n^2+n+1$,

(2) $q' \leqq n+1$.

Then in general $\rho(n)$ is bijective.

Proof. — We may arrange the $q+q'$ points into $n-r$ sets of $n+1$ collinear points each, with one point left over. If L′ is a line containing a set Y′ of $n+1$ collinear points, then $h^0(\mathcal{O}_{L'}(n))=h^0(\mathcal{O}_{Y'}(n))$. So the result follows by applying (7.2) to the r original lines plus the $n-r$ lines containing the sets of collinear points, plus the one extra point.

LEMMA 7.4. — *Let* $X=\mathbf{P}^2$, *Let* $Z=\mathbf{V}(\mathcal{O}_X(-3))$, *and let* $k>0$.

(*a*) *Let* $Y \subseteq Z$ *be a union of* $k-1$ *conics,* $3k+2$ *points on a conic, and one further point. Then in general* $\rho(2k)$ *is bijective.*

(*b*) *Let* $Y \subseteq Z$ *be a union of* $k-1$ *conics and* $k+2$ *points on a conic. Then in general* $\rho(2k-1)$ *is bijective.*

Proof. — (*a*) By induction on k, the case $k=1$ being straightforward (it says in general 6 points of $\mathbf{P}^2$ do not lie on a conic). If $k \geqq 2$, fix a conic $C \subseteq \mathbf{P}^2$, which we take to be the divisor X′ of paragraph 6. Then $C \cong \mathbf{P}^1$ and $Z' \cong \mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(-6))$. We consider the irreducible subfamily of those Y for which one of the conics lies over C, and 3 of the $3k+2$ points on a conic lie over C, and apply (6.2) and (6.3.1). Then $Y' = Y \cap Z'$ consists of one conic plus the $4(k-2)$ points of intersection of the other conics, plus 3 more points. So (7.1) applies to show that $\rho'(4k)$ over $\mathbf{P}^1$ is bijective. On the other hand, the residual intersection Y'' consists of $k-2$ conics and $3k-1$ points on a conic and one further point. So the induction hypothesis implies $\rho''(2k-2)$ is bijective. We conclude that $\rho(2k)$ is bijective.

(*b*) The proof is entirely analogous. The case $k=1$ is trivial. For $k \geqq 2$ we fix a conic C as above, and consider the subfamily of those Y for which one conic lies over C, and one of the $k+2$ points lies over C. Then Y' is one conic plus $4k-7$ points. By (7.1), $\rho'(4k-2)$ is bijective. The residual intersection is $k-2$ conics plus $k+1$ points, so the induction hypothesis implies $\rho''(2k-3)$ is bijective. As above we conclude $\rho(2k-1)$ is bijective.

PROPOSITION 7.5. — *Let* $X = \mathbf{P}^2$, *let* $Z = \mathbf{V}(\mathcal{O}_X(-3))$, *and let* $n > 0$. *Let* $Y \subseteqq Z$ *be a disjoint union of r conics, q points, and q' points lying on a conic, such that:*

(1) $r(2n+1) + q + q' = n^2 + 2$;

(2) $q' \leqq 2n+1$.

Assume furthermore $q > 0$ *or n odd. Then in general:*

$$\rho(n): \quad H^0(\mathcal{O}_X(n-3)) \oplus H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_Y(n))$$

is bijective.

Proof. — If n is even and $q > 0$, we can arrange the $q+q'$ points into $(n/2)-r-1$ sets of $2n+1$ points on a conic each, plus $(3n/2)+2$ points on a conic and one extra point. If C' is a conic containing a set Y' of $2n+1$ points, then $h^0(\mathcal{O}_{C'}(n)) = h^0(\mathcal{O}_{Y'}(n))$. So we can apply (7.4 *a*) with $k = n/2$ to the original r conics plus the $(n/2)-r-1$ conics containing sets of $2n+1$ points to obtain the result.

If n is odd, we can arrange the $q+q'$ points into $((n+1)/2)-r-1$ sets of $2n+1$ points on a conic, plus a set of $((n+1)/2)+2$ points on a conic. Then (7.4 *b*) applies with $k = ((n+1)/2)$ to give the result.

PROPOSITION 7.6. — *Let X be a nonsingular quadric surface Q, and let* $Z = \mathbf{V}(\mathcal{O}_X(-2))$. *Let* $Y \subseteqq Z$ *be a disjoint union of* r_1 *lines in the first family,* r_2 *lines in the second family, and q points. (We fix the convention that if L is a line in the first family, its ideal sheaf* $\mathcal{I}_L$ *is* $\mathcal{O}_Q(-1, 0)$.) *Let* $n_1, n_2 \geqq 1$ *be integers. Assume:*

(1) $r_1(n_2+1) + r_2(n_1+1) + q = 2n_1 n_2 + 2$;

(2) $r_1 \leqq n_1 - 1$ *and* $r_2 \leqq n_2 - 1$.

Then in general:

$$\rho(n_1, n_2): \quad H^0(\mathcal{O}_Q(n_1-2, n_2-2)) \oplus H^0(\mathcal{O}_Q(n_1, n_2)) \to H^0(\mathcal{O}_Y(n_1, n_2)),$$

is bijective.

Proof. — By induction on $n_1 + n_2$. If $n_1 + n_2 = 2$, then $n_1 = n_2 = 1$, $r_1 = r_2 = 0$, so Y consists of 4 points, and the result simply says in general 4 points of Q are not contained in a hyperplane section.

For the general case $n_1+n_2 \geqq 3$, we first arrange the q points into n_1-r_1-1 sets of n_2+1 points lying on lines of the first family, plus n_2-r_2-1 sets of n_1+1 points lying on lines of the second family, with 4 points left over. For a line L_1 of the first family, $\mathcal{O}_Q(n_1, n_2) \otimes \mathcal{O}_{L_1} \cong \mathcal{O}_{\mathbf{P}^1}(n_2)$, and similarly for the second family. Thus we can replace these sets of collinear points by the lines containing them, and so reduce to the special case of the proposition for which $r_1=n_1-1$, $r_2=n_2-1$ and $q=4$.

Since $n_1+n_2 \geqq 3$, we may assume $n_1 \geqq 2$, and so $r_1 \geqq 1$. Fix a line L_1 in the first family, and consider the irreducible subfamily of those Y for which one line lies over L_1. We take L_1 to be the divisor X′ of paragraph 6, and apply (6.3). Then $L_1 \cong \mathbf{P}^1$, and $Z' \cong \mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(-2))$. The intersection $Y'=Y \cap Z'$ consists of one line and the n_2-1 points of intersection of the lines of the second family with Z′. Since $\mathcal{O}_Q(n_1, n_2) \otimes \mathcal{O}_{L_1} \cong \mathcal{O}_{\mathbf{P}^1}(n_2)$, we can apply (7.1) to conclude that $\rho'(n_1, n_2)$ is bijective. The residual intersection Y″ consists of n_1-2 lines of the first family, n_2-1 lines of the second family, and 4 points. So the induction hypothesis implies $\rho''(n_1-1, n_2)$ is bijective. Finally note that in the notation of (6.3), $d_1=1$, $a=2$, so $n_1 \geqq 2$ implies $n_1-d_1-a \geqq -1$, so the hypotheses of (6.3) are verified, and we conclude in general $\rho(n_1, n_2)$ is bijective.

DEFINITION. — Let X be $\mathbf{P}^3$ or a quadric surface Q. A set of points in $Z=\mathbf{V}(\mathcal{O}_X(-a))$ will be called *coplanar* if its projection to X lies in a plane (of $\mathbf{P}^3$) or a plane section (of Q).

LEMMA 7.7. — *Let* X *be a nonsingular quadric surface* Q, *let* $Z=\mathbf{V}(\mathcal{O}_X(-3))$, *and let* $n \geqq 2$. *Let* $Y \subseteq Z$ *be a disjoint union of* $n-2$ *conics*, $[(n+7)/4]$ *sets of* 4 *coplanar points and* $n+7-4[(n+7)/4]$ *further points. Then in general* $\rho(n)$ *is bijective.*

Proof. — By induction on n. If $n=2$, Y consists of 2 sets of 4 coplanar points and one further point. In this case the results say that in general their projections to Q do not lie on any other quadric surface, which can be seen easily.

If $n \geqq 3$, we fix a conic C in Q, which we take to the divisor X′ of paragraph 6, and consider the irreducible subfamily of those Y having one conic lying over C, and one point (chosen from among the $n+7-4[(n+7)/4]$ further points if there are any) lying over C. Then we apply (6.2) and (6.3.1). In this case $C \cong \mathbf{P}^1$, $Z' \cong \mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(-6))$, and the intersection $Y'=Y \cap Z'$ consists of one conic plus $2n-5$ points. So (7.1) applies to show in general $\rho'(2n)$ is bijective. On the other hand the residual intersection satisfies the conditions of the induction hypothesis, so $\rho''(n-1)$ is bijective. We conclude $\rho(n)$ is bijective.

PROPOSITION 7.8. — *Let* X *be a nonsingular quadric surface* Q, *let* $Z=\mathbf{V}(\mathcal{O}_X(-3))$, *and let* $n \geqq 2$. *Let* $Y \subseteq Z$ *be a disjoint union of* r *conics*, q *points, and* $q'=4k$ *points in* k *sets of* 4 *coplanar points each. Assume:*

(1) $r(2n+1)+q+q'=2n^2-2n+5$;

(2) $r \leqq n-2$ [*which follows from* (1) *if* $n \geqq 7$].

Then in general:

$$\rho(n): \quad H^0(\mathcal{O}_Q(n-3)) \oplus H^0(\mathcal{O}_Q(n)) \to H^0(\mathcal{O}_Y(n)),$$

is bijective.

Proof. — By induction on $n-2-r$, the case $n-2-r=0$ being (7.7), after we arrange the q free points into as many sets of 4 coplanar points as possible.

If $n-2-r>0$, we claim it is possible to move exactly $2n+1$ of the $q+q'$ points so as to lie on a conic. The problem is that a set of 4 coplanar points can be moved so that 1, 2, or 4 of them lie on a given conic. So the only difficulty is in case $2n+1 \equiv 3 \pmod 4$. Note that $2n^2-2n+5-(n-2)(2n+1)=n+7$. So, at the moment we need to move 3 points onto the conic, there are at least $n+10 \geqq 12$ points available. So we can use 2 from one set of 4, and one from another set of 4.

Now adjoin to Y the conic containing the $2n+1$ points. This increases r by 1. Furthermore, the points not on this conic, other than those still in sets of 4 coplanar points, are subject to no additional restrictions. So the induction hypothesis applies to give the result.

8. Proof of (H_n)

In this section we will prove that the statement (H_n) of (5.2) is true for all $n \geqq 0$. We will prove it in the reformulation of (6.1), which for convenience we state again.

PROPOSITION 8.1. — *Let $X=\mathbf{P}^3$ and let $Z=V(\mathcal{O}_X(-2))$. For any $n \geqq 0$, let r and q be defined as in (H_n) of paragraph 5. Let $Y \subseteq Z$ be a disjoint union of r lines and q collinear points. Then in general:*

$$\rho(n): \quad H^0(\mathcal{O}_X(n-2)) \oplus H^0(\mathcal{O}_X(n)) \to H^0(\mathcal{O}_Y(n)),$$

is bijective.

Proof. — We denote this statement still by (H_n). Note that (H_0) is trivial. In that case $r=1$, $q=0$, and $\rho(0)$ is the obvious isomorphism $H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_Y)$. The rest of the proof will be by induction on n. We will show that $(H_{n-1}) \Rightarrow (H_n)$ for $n \equiv 1, 2 \pmod 3$, and $(H_{n-2}) \Rightarrow (H_n)$ for $n \equiv 0 \pmod 3$, $n \geqq 3$.

Case $n \equiv 1$ (mod 3). — We will show $(H_{n-1}) \Rightarrow (H_n)$. Write $n=3k+1$, $k \geqq 0$. Then $r=3k^2+4k+2$ and $q=0$. Fix a plane $H \subseteq \mathbf{P}^3$, and consider the subfamily T_1 of schemes Y for which $2k+1$ lines lie over H. Then $H \cong \mathbf{P}^2$ and $Z' \cong V(\mathcal{O}_{\mathbf{P}^2}(-2))$. The intersection $Y'=Y \cap Z'$ consists of $r'=2k+1$ lines and $q'=3k^2+2k+1$ points where the remaining lines intersect Z'. With $n'=n$, Y' satisfies the hypotheses of (7.3), so in general $\rho'(n)$ is bijective.

The residual intersection Y'' consists of $3k^2+2k+1$ lines and no points, so according to the induction hypothesis (H_{n-1}), in general $\rho''(n-1)$ is bijective. We conclude (6.3.1) that in general $\rho(n)$ is bijective, i. e. (H_n) holds.

Case $n \equiv 2$ (mod 3). — We will show $(H_{n-1}) \Rightarrow (H_n)$. Write $n=3k+2$, $k \geqq 0$. Then $r=3k^2+6k+3$ and $q=2k+2$. Fix a plane $H \subseteq \mathbf{P}^3$, and move $2k+1$ lines and the collinear points over H. Then $Y'=Y \cap Z'$ consists of $r'=2k+1$ lines, $q'=3k^2+4k+2$ points, and $q=2k+2$ collinear points. Thus Y' satisfies the hypotheses of (7.3) (with notation q and q' interchanged), so in general $\rho'(n)$ is bijective.

The residual intersection Y'' consists of $3k^2+4k+2$ lines and no points, so (H_{n-1}) implies that in general $\rho''(n-1)$ is bijective. It follows (6.3.1) that in general $\rho(n)$ is bijective.

Case $n\equiv 0$ (*mod* 3). — In this case we prove $(H_{n-2})\Rightarrow(H_n)$ for $n\geqq 3$. Write $n=3k$ with $k\geqq 1$. Then $r=3k^2+2k+1$ and $q=0$. Fix a nonsingular quadric surface $Q\subseteq \mathbf{P}^3$ and take Q to be the divisor X' of paragraph 6. Then $Z'=\mathbf{V}(\mathcal{O}_Q(-2))$. Put $4k$ lines over Q, with $2k$ in each family. Then Y' has $r_1=2k$ lines in the first family and $r_2=2k$ lines in the second family, and $2(3k^2-2k+1)$ points of intersection of the remaining lines with Z'. This satisfies the hypotheses of (7.6) with $n_1=n_2=n$, so we conclude in general $\rho'(n)$ is bijective.

The residual intersection Y'' consists of $3k^3-2k+1$ lines and no points so by (H_{n-2}) we conclude $\rho''(n-2)$ in general is bijective. It follows (6.3.1) that $\rho(n)$ is bijective.

9. Proof of (H'_n)

In this section we will prove that (H'_n) is true for $n=0, 1$ and all $n\geqq 4$, in the reformulation of (6.1).

PROPOSITION 9.1. — *Let* $X=\mathbf{P}^3$ *and let* $Z=\mathbf{V}(\mathcal{O}_X(-3))$. *For* $n=0, 1$ *or* $n\geqq 4$, *let r and q be defined as in* (H'_n) *of paragraph* 5. *Let* $Y\subseteq Z$ *be a disjoint union of r conics and q points lying on a conic. Then in general:*

$$\rho(n):\quad H^0(\mathcal{O}_X(n-3))\oplus H^0(\mathcal{O}_X(n))\to H^0(\mathcal{O}_Y(n)),$$

is bijective.

Proof. — We denote this statement still by (H'_n). If $n=0$, then $r=1$, $q=0$, and $H^0(\mathcal{O}_X)\to H^0(\mathcal{O}_Y)$ is bijective, so (H'_0) is trivial. If $n=1$, then $r=q=1$, and the map $H^0(\mathcal{O}_X(1))\to H^0(\mathcal{O}_Y(1))$ is bijective provided the point does not lie in the plane of the conic. So (H'_1) is true.

The rest of the proof is by induction on n. We will prove (H'_4) and (H'_5) individually. Also we will prove that $(H'_{n-1})\Rightarrow(H'_n)$ for $n\equiv 0$, 1 (*mod* 3), $n\geqq 1$ and $(H'_{n-2})\Rightarrow(H'_n)$ for $n\equiv 2$ (mod 3), $n\geqq 5$.

Case $n\equiv 0$ (*mod* 3), $n\geqq 3$. — We will prove $(H'_{n-1})\Rightarrow(H'_n)$. Write $n=3k$, $k\geqq 1$. Then $r=1/2k(3k+1)+1$ and $q=0$. Fix a plane $H\subseteq \mathbf{P}^3$ and move k conics over H. Then $Y'=Y\cap Z'$ consists of $r'=k$ conics and $q'=k(3k-1)+2$ points of intersection of the remaining conics with Z'. Thus Y' satisfies the hypotheses of (7.5), so in general $\rho'(n)$ is bijective. The residual intersection Y'' consists of $1/2k(3k-1)+1$ conics and no points, so (H'_{n-1}) implies $\rho''(n-1)$ in general is bijective. We conclude (6.3.1) that $\rho(n)$ is bijective.

Case $n\equiv 1$ (*mod* 3), $n\geqq 1$. — We will prove $(H'_{n-1})\Rightarrow(H'_n)$. Write $n=3k+1$, $k\geqq 0$ Then $r=1/2(3k^2+3k+2)$, and $q=2k+1$. Fix a plane $H\subseteq \mathbf{P}^3$, and move k conics and the $2k+1$ points lying on a conic over H. Then $Y'=Y\cap Z'$ consists of $r'=k$ conics and $q_H=3k^2+k+2$ points and $q'=2k+1$ points on a conic. Thus Y' satisfies the hypotheses of (7.5) so we conclude that $\rho'(n)$ in general is bijective.

The residual intersection Y'' consists of $1/2(3k^2+k+2)$ conics and no points, so (H'_{n-1}) implies in general $\rho''(n-1)$ is bijective. We conclude (6.3.1) in general $\rho(n)$ is bijective.

Case $n\equiv 2$ (*mod* 3), $n\geqq 5$. — We will prove $(H'_{n-2})\Rightarrow(H'_n)$. Write $n=3k+2$, $k\geqq 1$. Then $r=1/2(3k^2+5k+4)$ and $q=0$. Fix a nonsingular quadric surface $Q\subseteq\mathbf{P}^3$, and take this to be the divisor X' of paragraph 6. Then $Z'=\mathbf{V}(\mathcal{O}_Q(-3))$. Move $2k+1$ conics over Q. Then Y' consists of $r'=2k+1$ conics and $q'=2(3k^2+k+2)$ points, in sets of 4 coplanar points (*see* definition in § 7). So Y' satisfies the hypotheses of (7.8) for $n\geqq 5$, and we conclude in general $\rho'(n)$ is bijective.

The residual intersection consists of $1/2(3k^2+k+2)$ conics, so by (H'_{n-2}), in general $\rho''(n-2)$ is bijective. We conclude (6.3.1) that in general $\rho(n)$ is bijective.

Case $n=4$. — We will prove (H'_4) directly. In this case Y consists of 4 conics and 3 points. We need to consider some degenerations of Y. So let T be the family of all disjoint unions of 4 conics and 3 points in Z, and let $\overline{T}$ be the closure of T in the Hilbert scheme of closed subschemes of Z, proper over X. In particular, we will consider schemes $Y\in\overline{T}$ where three of the conics degenerate into a pair of distinct lines meeting at a point. We call these *degenerate conics*. Fix a plane $H\subseteq\mathbf{P}^3$, and let $T_1\subseteq\overline{T}$ be the subfamily of schemes Y consisting of one conic, three degenerate conics, and 3 points, such that one line of each degenerate conic lies over H, and one of the three points lies over H. Take the divisor X' of paragraph 6 to be H. Then Y' consists of 3 lines and 3 points, and the residual intersection Y'' consists of 1 conic, 3 lines, and 2 points. So using (6.2) and (6.3.1) it suffices to solve the following two problems:

(*a*) $X=\mathbf{P}^3$, $n=3$, $Z=\mathbf{V}(\mathcal{O}_X(-3))$, and $Y\subseteq Z$ is a union of 1 conic, 3 lines, and 2 points. Then in general $\rho(3)$ is bijective.

(*b*) $X=\mathbf{P}^2$, $n=4$, $Z=\mathbf{V}(\mathcal{O}_X(-3))$, and $Y\subseteq Z$ is a union of 3 lines and 3 points. Then in general $\rho(4)$ is bijective.

Neither of these fits exactly any of our earlier results, but we can handle them by exactly the same techniques.

To prove (*a*), fix a plane H, and move 2 lines over H. Then restricting to H, we get the problem:

(*c*) $X=\mathbf{P}^2$, $n=3$, $Z=\mathbf{V}(\mathcal{O}_X(-3))$, and Y is 2 lines and 3 points. Then in general $\rho(3)$ is bijective.

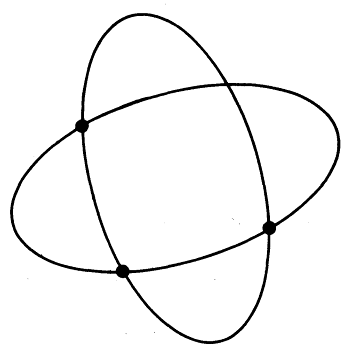
Assuming (*c*) for the moment, the residual intersection Y'' is 1 conic, 1 line, and 2 points, and we must show $\rho''(2)$ in general in bijective. But $\rho''(2)$ is the map $H^0(\mathcal{O}_X(2))\to H^0(\mathcal{O}_Y(2))$, so we need only show in general Y'' is not contained in a quadric surface, which is clear. This proves (*a*), modulo (*c*).

To prove (*c*), fix a line L in $\mathbf{P}^2$, and move one line of Y over L. Then restricting to L we have 1 line and 1 point over L, $n=3$, $a=3$, so by (7.1) in general $\rho'(3)$ is bijective. The residual intersection Y'' is 1 line and 3 points, and we must show $\rho''(2)$ is bijective. For this it is sufficient to show in general Y'' is not contained in a conic, which is obvious. This proves (*c*).

To prove (*b*), fix a line $L \subseteq \mathbf{P}^2$, and move one line of Y over L. Then restricting to L we have Y′ is 1 line and 2 points, $n=4$, $a=3$, so by (7.1) in general $\rho'(4)$ is bijective. The residual intersection Y″ is 2 lines and 3 points. We must show $\rho''(3)$ in general is bijective, which is just assertion (*c*).

This completes the proof of (H'_4).

Case $n=5$. — We will show $(H'_4) \Rightarrow (H'_5)$. In this case Y consists of 6 conics and no points. Again we consider the closure $\overline{T}$ of the family of Y's. We will consider the family $T_1 \subseteq \overline{T}$ of specializations of Y where two of the conics lie over a fixed plane $H \subseteq \mathbf{P}^3$ and meet in 3 points. Of course the projections of those two conics to H meet in 4 points, but the lifting of a conic C to Z is given by a section of $H^0(\mathcal{O}_C(3)) \cong H^0(\mathcal{O}_{\mathbf{P}^1}(6))$, so we can determine the lifting arbitrarily at 7 points. In particular, we can make the lifted conics in Z meet at exactly 3 points. At each of these three intersection points, the scheme Y will have an embedded point ([16], 2.1.1), which in general will not be contained in $Z' = Z \times_X H$.



Now we apply the technique of paragraph 6. The residual scheme Y″ will consist of the four remaining conics and the three points where Y had embedded components. Thus $\rho''(Y)$ will be bijective in general by (H'_4).

Restricting to the plane H we must show for a scheme consisting of two conics meeting in 3 points (without embedded points) and 8 points, in general $\rho(5)$ is bijective. We fix a conic C in H, and move one of the conics and 4 points over C. We get one further point of intersection with the other conic. Since $C \cong \mathbf{P}^1$ and $Z_C \cong \mathbf{V}(\mathcal{O}_{\mathbf{P}^1}(-6))$, and $n' = 2n = 10$, we see from (7.1) that $\rho'(5)$ is bijective over C. The residual intersection Y″ is one conic and 4 points. We must show $\rho''(3)$ is bijective in general, which follows from (7.5).

This completes the proof of (H'_5), hence also of (5.2) and (0.1).

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Added in proof. — In a recent paper (Universal families of extensions, preprint, Erlangen 1982) H. Lange proves results which generalize those of our Section 3.