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A NOTE ON THE ISOPERIMETRIC CONSTANT

BY PETER BUSER

1. Introduction

The isoperimetric inequality for the standard sphere S^n implies that the volume ratio vol ∂A/vol A for open subsets A with vol A $\leqq 1/2$ vol S^n and sufficiently regular boundary ∂A is minimized for A equal a hemisphere. Similarly one defines an isoperimetric constant:

$$h(M^n) = \inf \frac{\text{vol } \partial A}{\text{vol } A},$$

for any compact n-dimensional Riemannian manifold M where A runs over all open subsets with not more than half of the total volume, and it is known from geometric measure theory that a minimizing set A always exists though in general ∂A need not be a *smooth* hypersurface (*see* below). We shall also deal with relatively compact (connected) subdomains D of a Riemannian manifold. In this case ∂A in the definition of h(D) is to be replaced by the part in the interior of D, i. e.:

$$h(D) = \inf \frac{\text{vol}(\partial A \cap \text{int } D)}{\text{vol } A}.$$

In 1970 Cheeger [10], proved the lower bound:

$$\lambda_1(M) \geqq \frac{1}{4} h^2(M),$$

where λ_1 is the smollest positive eigenvalue of the Laplace-Beltrami operator $\Delta = -\text{div}$ grad. If M has boundary then Cheeger's inequality still holds if λ_1 is meant subject to the *Neumann* boundary condition $\Delta u = \lambda u$, $*du \mid \partial M = 0$. Cheeger's inequality has found a number of applications e. g. ([5], [8], [11], [12], [16], [18], [21], [22], [24]), and for each compact manifold there exist Riemannian metrics for which the inequality becomes sharp [8]. This has suggested that λ_1 and the isoperimetric constant h ought to be *equivalent* in the following sense:

Given a family of Riemannian manifolds without boundary, then under suitable curvature conditions:

$$(1.1) \qquad \lambda_1 \to 0 \quad \textit{if and only if} \quad h \to 0.$$

From Cheeger's inequality (which gives the "only if") one might hope to dispense with additional assumptions on the manifolds. However it is possible to perturb the Riemannian structure of a manifold near any given subdividing hypersurface as to make h arbitrarily small with hardly affecting λ_1 (*cf.* the example in [6], *see* also [9]). Yet such procedures involve heavily negative curvature and small injectivity radii. Inspired by the article [16] by Gromov we prove here:

1.2 THEOREM. — *If the Ricci curvature of a compact unbordered Riemannian manifold* M^n *is bounded below by* $-(n-1)\delta^2\,(\delta \geqq 0)$ *then:*

$$\lambda_1(M) \leqq c_1(\delta h + h^2),$$

where c_1 *is a constant which depends only on the dimension.*

The two dimensional version of the theorem has already been proved in [6].

1.3 *Remarks.* — (*a*) In 3.2 we find more precisely $\lambda_1 \leqq 2\delta(n-1)h + 10\,h^2$. (*b*) Flat tori provide families of Riemannian manifolds where $h \to \infty$ such that $\lambda_1 = O(h^2)$. On the other hand Schoen-Wolpert-Yau [23] have shown that for compact Riemann surfaces of fixed genus $g \geqq 2$ (curvature $= -1$) one has $\lambda_1 \geqq \text{Const.}(g).h$, i. e. λ_1 is of the *same* order as h, as $h \to 0$, in this case. Hence to some extent the upper bound in Theorem 1.2 has the best possible form. (*c*) It would be interesting however to know whether 1.1 can also be proved under different circumstances, e. g. if there are no curvature conditions but if a lower bound on the injectivity radius is given instead. Such a possibility seems imaginable from the work of Berger ([2], [3]) and Croke [11].

1.4 *Example.* — Theorem 1.2 has no analogue if the manifold is bordered, at least not under the given circumstances. To obtain an illustrating example we consider the differential equation:

$$\frac{d^2 u}{dx^2} + m\frac{du}{dx} + u = 0 \qquad (x \geqq 0),$$

where m is an arbitrary large parameter, subject to the boundary-condition $u(0)=0$, $u'(0)=1$. Its unique solution is:

$$u(x) = \frac{1}{\lambda} e^{1/2\,(\lambda - m)\,x} - \frac{1}{\lambda} e^{-1/2\,(\lambda + m)\,x}, \qquad \lambda = {}^{+}\sqrt{m^2 - 4}.$$

We have:

$$u'(a) = 0 \qquad \text{for} \quad a = \frac{1}{\lambda}\log\frac{m+\lambda}{m-\lambda} \geqq \frac{1}{m}\log m^2,$$

and u is a solution of the eigenvalue equation:

$$u''+mu'+\eta\, u=0, \qquad x\in[0,\, a], \qquad u'(0)=0, \qquad u'(a)=0,$$

with the eigenvalue $\eta=1$.

Now consider the following *flat* domain (which imitates a cylindric surface of strongly negative curvature):

$$G(m,\,\varepsilon)=\{(x,\,y)\in\mathbb{R}^2;\ -a\leqq x\leqq a,\ 0\leqq y\leqq\varepsilon\, e^{m|x|}\}.$$

Standard techniques show that for small $\varepsilon>0$ the function $f(x,\,y)=u(|x|)$ is almost the first eigenfunction of the Laplacian $-(\partial^2/\partial x^2+\partial^2/\partial y^2)$ on $G(m,\,\varepsilon)$ subject to the Neumann boundary condition, and that $\lambda_1(G(m,\,\varepsilon))\to 1$ as $\varepsilon\to 0$. On the other hand obviously:

$$h(G(m,\,\varepsilon))=1\Big/\int_0^a e^{mx}\,dx\leqq m/(m^2-1).$$

The example also shows that additional curvature conditions for the boundary would still be insufficient to make 1.2 true: Glue two pieces of $G(m,\varepsilon)$ together along $|x|=a$ and smooth the "upper" boundary curve gently at $x=0$ and $x=a$. The flat annulus $Z(m,\,\varepsilon)$ thus obtained has $\lambda_1(Z(m,\,\varepsilon))>1/2$, $h(Z(m,\,\varepsilon))<2/m$ and the geodesic curvature of the boundary approaches zero as $\varepsilon\to 0$.

Sections 2, 3, 4 deal with the proof of Theorem 1.2. In section 6 we present another application of the used technique obtaining bounds for the higher eigenvalues which have the same growth rate as Weyl's asymptotic law. Section 7 provides a version of Theorem 1.2 for non compact manifolds and shows how to proceed if a separating hypersurface needs a haircut.

2. About the proof

We shall give two proofs of Theorem 1.2 (The proof given in [6] for the case of a surface does not generalize to higher dimensions.) The second proof (section 4) is elementary, based on standard comparison arguments. The first one (section 3) uses a result from geometric measure theory: It is shown in [1] or follows indirectly though more accessibly from [19] (see also remark 3.3 below) that $h(M)$, as mentioned in the introduction, is a minimum, obtained for an open submanifold of M whose boundary X is a rectifiable current ([13], p. 355) with the following.

2.1 Regularity property. – *If $p\in X$ is a point whose tangent cone is contained in a half space (regular point), then there exists a neighbourhood U of p in M such that $X\cap U$ is a smooth* submanifold of U.

The set X^0 of all regular points in X will be called the regular part. It is known [14] that the complement $X-X^0$ has Hausdorff dimension $\leqq n-8$ (in particular $X=X^0$ if $\dim M\leqq 7$) but we shall not need this fact here. Note that since X is an area minimizing current, X^0 is a hypersurface of constant mean curvature.

For the sake of simplicity the metric of M is scaled such that $\delta=1$ i. e. such that the Ricci curvature is bounded below by $-(n-1)$ in the sequel.

3. Proof using a minimizing current

Consider a hypersurface X which satisfies the regularity property 2.1 and which subdivides M into two open submanifolds A, B such that (*cf.* remark 3.3) $\partial A=\partial B=X$, $A\cap B=\emptyset$ and such that:

$$h(M)=\frac{\mathrm{vol}\, X}{\min\{\mathrm{vol}\, A,\ \mathrm{vol}\, B\}}.$$

By Courant's minimax principle we have $\lambda_1(M)\leqq\max\{\lambda_1(A), \lambda_1(B)\}$, where $\lambda_1(A)$, $\lambda_1(B)$ are the smallest non trivial eigenvalues of A, B for $\Delta u=\lambda u$ with respect to the *Dirichlet* boundary condition $u\,|\,X=0$. Hence it suffices to estimate the Rayleigh quotient $\int\|\mathrm{grad}\, f\|^2\Big/\int f^2$ for a suitable test function, say on A. To this end we put:

$$A(t)=\{p\in A;\ \mathrm{dist}\,(p, X)\leqq t\}$$

and define for sufficiently small $t>0$:

$$f(p)=\begin{cases}\dfrac{1}{t}\,\mathrm{dist}\,(p, X) & \text{if } p\in A(t),\\ 1 & \text{if } p\in A-A(t).\end{cases}$$

The function f is Lipschitzian satisfying $\|\mathrm{grad}\ f\|^2\leqq t^{-2}$ on $A(t)$ and $\|\mathrm{grad}\, f\|=0$ on $A-A(t)$. Therefore we have only to estimate the volume of $A(t)$:

Let $C=C(X)$ denote the cut locus of X, i. e. the closure of the set of all points $p\in M$ to which more than one distance minimizing geodesic from p to X exists. It is known that C has zero measure. Now consider $p\in M-C$ and let $p_X\in X$ be the endpoint of the unique distance minimizing geodesic from p to X. Then X and the open metric ball U of radius dist (p, p_X) around p are disjoint. Moreover p and p_X are not conjugate points ($p\notin C(X)$), hence U has a well defined tangent hyperplane T at p_X and the tangent cone of X at p_X is contained in one of the half spaces defined by T, i. e.:

$$p_X\in X^0\quad\text{for all } p\in M-C.$$

Therefore we can apply the comparison theorem of Heintze-Karcher [17]: Since the regular part X^0 has constant mean curvature, say η (with respect to the normal vector which points towards A) and since the Ricci curvature of M is bounded below by $-(n-1)$, it follows:

$$\mathrm{vol}\ A(t)=\mathrm{vol}\,(A(t)-C)\leqq\mathrm{vol}\ X^0\int_0^t J_\eta(\tau)\,d\tau\leqq h(M)\ \mathrm{vol}\ A\int_0^t J_\eta(\tau)\,d\tau,$$

where $J_\eta(\tau)=(\cosh\tau-\eta\ \sinh\tau)^{n-1}$ as long as the term in the bracket is positive and $J_\eta(\tau)=0$ otherwise. Now:

$$\int_A \|\operatorname{grad} f\|^2\, dM \leqq \frac{1}{t^2}\ \mathrm{vol}\ A(t) \qquad \text{and} \qquad \int_A f^2\, dM \geqq \mathrm{vol}\ A - \mathrm{vol}\ A(t).$$

Hence the Rayleigh principle yields:

$$\lambda_1 \leqq \frac{h(M)}{t^2}\ \frac{\displaystyle\int_0^t (\cosh\tau+\eta\sinh\tau)^{n-1}\, d\tau}{1-h\displaystyle\int_0^t (\cosh\tau+\eta\sinh\tau)^{n-1}\, d\tau}, \tag{3.1}$$

for all t for which the denominator becomes positive. The inequality 3.1 holds for $\lambda_1(A)$ as well as $\lambda_1(B)$ and *a fortiori* for $\lambda_1(M)$ if η is now interpreted as the *absolute value* of the mean curvature of X^0.

In order to eliminate η we estimate η in terms of h: Nothing has to be done if $0\leqq\eta\leqq 1$. If $\eta=1+\varepsilon$ with $\varepsilon>0$ then $\cosh\tau-\eta\ \sinh\tau=e^{-\tau}-\varepsilon\sinh\tau\leqq 1-\varepsilon\tau$, from which:

$$\int_0^\infty J_\eta(\tau)\, d\tau \leqq \int_0^{1/\varepsilon} (1-\varepsilon\tau)^{n-1}\, d\tau = \frac{1}{\varepsilon n},$$

$$\int_0^\infty J_\eta(\tau)\, d\tau \leqq \int_0^\infty e^{-(n-1)\tau}\, d\tau = \frac{1}{n-1}.$$

Since $h\geqq \mathrm{vol}\ X^0/\min\{\mathrm{vol}\ A,\ \mathrm{vol}\ B\}\geqq 1\Big/\displaystyle\int_0^\infty J_\eta(\tau)\, d\tau$, the first inequality shows that:

$$\eta\leqq 1+\frac{h}{n}$$

(which is trivially true if $\eta\leqq 1$) and the second inequality implies:

$$h\geqq(n-1) \qquad \text{if} \quad \eta\geqq 1.$$

Now an elementary calculation yields from 3.1:

$$\lambda_1(M)\leqq 2(n-1)\,h(M)+10h^2(M). \tag{3.2}$$

[Take $t=3/(4n-4)$ if $0<h\leqq(n-1)/2$ resp. $t=1/(2n-2)$ if $(n-1)/2\leqq h\leqq n-1$ and use that $\eta\leqq 1$ for these two cases. If $h\geqq n-1$ use the fact $\eta\leqq 1+h/n$ and take $t=1/(5h)$.]

3.3 *Remark.* – The precise result from geometric measure theory is that the ratio vol X/vol A is minimized by a current X with regularity property 2.1 *if* A runs over all open subsets of M *with a fixed volume* vol $A=v$, whereas in the definition of $h(M)$ this v also varies in the interval (0, 1/2 vol M). Hence in order to find a minimizing current, one needs an additional argument. One way to proceed is as follows:

3.4 Lemma. – *There exists* $v>0$ *and a series of open submanifolds* $A_k\subset M$ *such that* vol $A_k=v$ *for all* k *and such that* $h(M)=\lim_{k\to\infty}\ \mathrm{vol}\ \partial A_k/\mathrm{vol}\ A_k$.

Proof. — Consider a sequence $\tilde{A}_k$ of open submanifolds of M such that:

$$\lim_{k\to\infty} \frac{\text{vol } \partial\tilde{A}_k}{\text{vol } \tilde{A}_k} = h(M)$$

and such that the limit $v := \lim_{k\to\infty} \text{vol } \tilde{A}_k \leqq 1/2$ exists. We first show that $v \neq 0$. To this end we represent M as a sum $M = M_1 \cup \ldots \cup M_l$, where each M_i is a cell which is mapped homeomorphically onto the euclidean n-ball B^n of radius 1 by some fixed quasi isometry Φ_i, and where $\text{int } M_i \cap \text{int } M_j = \emptyset$ $(i \neq j)$. (This can be achieved by a suitable triangulation of M, or with the Dirichlet regions of section 4. The length distortion of the Φ_i is irrelevant.) Now if the volume of $\tilde{A} = \tilde{A}_k$ is sufficiently small (if this occurs at all) then $\text{vol } \Phi_i(\tilde{A} \cap M_i) \leqq 1/2 \text{ vol } B^n$, $i = 1, \ldots, l$. It follows from the classical isoperimetric inequality in $B^n \subset \mathbb{R}^n$ that (if $\tilde{A} \cap M_i \neq \emptyset$):

$$\frac{\text{vol } \Phi_i (\text{int } M_i \cap \partial\tilde{A})}{\text{vol } \Phi_i(\tilde{A} \cap M_i)} \geqq c_1 (\text{vol } \Phi_i(\tilde{A} \cap M_i))^{-1/n},$$

where c_1 is a dimension constant. Therefore:

$$\frac{\text{vol}(\text{int } M_i \cap \partial\tilde{A})}{\text{vol}(\tilde{A} \cap M_i)} \geqq c_2 (\text{vol}(\tilde{A} \cap M_i))^{-1/n} \geqq c_2 (\text{vol } \tilde{A})^{1/n},$$

with a constant c_2 which depends on the length distortion of the quasiisometry Φ_i, but which is independent of the subset $\tilde{A} = \tilde{A}_k$. It follows:

$$\frac{\text{vol } \partial\tilde{A}_k}{\text{vol } \tilde{A}_k} \geqq c_2 (\text{vol } \tilde{A}_k)^{-1/n}.$$

Since $\text{vol } \partial\tilde{A}_k/\text{vol } \tilde{A}_k$ converges to $h(M) < \infty$, this shows that $\text{vol } \tilde{A}_k$ cannot approach zero.

In a second step we replace the $\tilde{A}_k$ whose volumes converge to $v > 0$ by subsets A_k of M whose volumes equal v. We restrict ourseleves to the case that $\text{vol } \tilde{A}_k < v$. The remaining case is similar. We try to attach a little ball $U(p, r)$ of radius r and center p to $\tilde{A} = \tilde{A}_k$. We may assume that $|v - \text{vol } \tilde{A}|$ is very small. Then the formula:

$$\int_M \text{vol } (\tilde{A} \cap U(p, r))\, dp = \int_{\tilde{A}} \text{vol } U(p, r)\, dp$$

and the fact that $\text{vol } \tilde{A} \leqq 1/2 \text{ vol } M$ help us to find a point p and a radius r such that:

$$\text{vol}(U(p, r) \cap \tilde{A}) \leqq \frac{3}{4} \text{ vol } U(p, r), \qquad \text{vol } U(p, r) \geqq 4\,|v - \text{vol } \tilde{A}|.$$

Note that $|v - \text{vol } \tilde{A}|$ is arbitrarily small, so we can assume that:

$$0.99\, \omega\, r^n \leqq \text{vol } U(p, r) \leqq 1.01\, \omega\, r^n,$$

whenever the radius r is so small, that at *some* point $q \in \mathrm{M}$ we have:

$$\operatorname{vol} \mathrm{U}(q, r) \leqq 10 \, | v - \operatorname{vol} \tilde{\mathrm{A}} \, |,$$

(ω = volume of the euclidean unit ball).

We now take $r_k \in [0, r]$ such that the volume of $\mathrm{A}_k := \tilde{\mathrm{A}}_k \cup \mathrm{U}(p, r_k)$ equals v and obtain the lemma with this new sequence of domains.

4. Elementary proof

We give a second proof of Theorem 1.2 which avoids area minimizing currents, since the regularity Theorem 2.1 is not very accessible. For the sake of simplicity we do not hesitate to loose large factors here.

Let this time X be a smooth hypersurface which subdivides M into $\mathrm{M} - \mathrm{X} = \mathrm{A} \cup \mathrm{B}$, $\mathrm{A} \cap \mathrm{B} = \emptyset$ with:

$$\mathscr{H} : = \frac{\operatorname{vol} \mathrm{X}}{\min \{ \operatorname{vol} \mathrm{A}, \operatorname{vol} \mathrm{B} \}}, \tag{4.1}$$

arbitrarily close to $h(\mathrm{M})$. The difficulty which arises in dimension $n \geqq 3$ is well known: The maximal distance $\operatorname{dist}(p, \mathrm{X})$, $p \in \mathrm{M}$ might be smaller than ε for any $\varepsilon > 0$ (the problem of "hairs", see *fig.* 2) and therefore no *a priori* bound exists for the volume of $\mathrm{A}(t)$. Hence in order to define a test function f as in section 3 we shall first replace X by a more "bald headed" subset $\tilde{\mathrm{X}}$ and then define f in terms of the distance to $\tilde{\mathrm{X}}$.

The comparison argument will be the usual one: Let $p \in \mathrm{M}$ and describe the inside of the cut locus $\mathrm{C}(p)$ with polar coordinates (ρ, φ), $0 < \rho < \infty$, $\varphi \in \mathrm{S}^{n-1}$ centered at p. Then the volume element of the given Riemannian metric takes the form:

$$d\mathrm{M} = g(\rho, \varphi) \, d\rho \, d\mathrm{S}^{n-1},$$

where $d\mathrm{S}^{n-1}$ is the volume of the standard S^{n-1}; e.g. if M^n has constant sectional curvature -1, then $g(\rho, \varphi) = \sinh^{n-1} \rho$. In general, i.e. if the Ricci curvature of M^n is bounded below by $-(n-1)$, standard arguments on Jacobi fields (e.g. [4], p. 256 or [17]) yield $d/d\rho \, [g(\rho) . \sinh^{1-n} \rho] \leqq 0$ or:

$$\frac{g(\rho_1, \varphi)}{g(\rho_2, \varphi)} \geqq \frac{\sinh^{n-1} \rho_1}{\sinh^{n-1} \rho_2} = \frac{\sigma(\rho_1)}{\sigma(\rho_2)} \qquad (0 < \rho_1 \leqq \rho_2). \tag{4.2}$$

For quicker reference we list the following immediate consequences:

$$\frac{g(r, \varphi)}{\displaystyle\int_r^{\mathrm{R}} g(\rho, \varphi) \, d\rho} \geqq \frac{\sigma(r)}{\beta(\mathrm{R}) - \beta(r)} \qquad (0 < r < \mathrm{R}), \tag{4.3.1}$$

$$(4.3.2)\qquad \frac{\displaystyle\int_{r_1}^{r_2} g(\rho,\varphi)\,d\rho}{\displaystyle\int_{r_0}^{r_1} g(\rho,\varphi)\,d\rho} \leqq \frac{\beta(r_2)-\beta(r_1)}{\beta(r_1)-\beta(r_0)} \qquad (0 \leqq r_0 < r_1 < r_2),$$

$$(4.3.3)\qquad \int_0^R g(\rho,\varphi)\,d\rho \leqq \frac{\beta(R)}{\beta(r)} \cdot \int_0^r g(\rho,\varphi)\,d\rho \qquad (0 < r \leqq R),$$

where $\beta(\rho)$ is the n-volume of the hyperbolic ρ-ball (curvature -1) and $\sigma(\rho) = d/d\rho\,\beta(\rho) = \text{Const.}\,(n)\sinh^{n-1}\rho$ is the $(n-1)$-volume of the corresponding boundary sphere. The inequalities 4.2 and 4.3 hold as long as the considered geodesic segment $\rho \mapsto (\rho, \varphi)$ does not meet the cut locus.

The geometric meaning, say of 4.3.1 is that of an isoperimetric inequality:

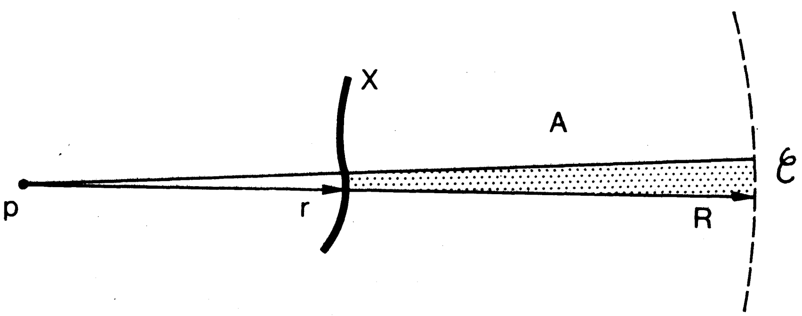

Fig. 1

Consider for instance a point $p \in M - A$ and an infinitesimal cone $\mathscr{C}$ of geodesics of length $R \leqq \text{dist}(p, \text{cut locus of } p)$ issuing from p, and assume that $X = \partial A$ cuts $\mathscr{C}$ into two pieces like in figure 1. Then 4.3.1 says:

$$\frac{\text{vol}(X \cap \mathscr{C})}{\text{vol}(A \cap \mathscr{C})} \geqq \frac{\sigma(r)}{\beta(R)-\beta(r)},$$

where r is the distance along $\mathscr{C}$ from p to the intersection of $\mathscr{C}$ with X. The inequality is sharp for constant negative curvature, if X meets $\mathscr{C}$ perpendicularly.

In order to define the set $\tilde{X}$ we consider a collection of points $p_1, \ldots, p_k \in M$ with pairwise distances $\geqq 2r$ such that the *open metric balls* $U(p_i, 2r)$ with center p_i and radius $2r$ cover M, where r is an adjustible sufficiently small parameter. ($U(p_i, 2r)$ need not be homeomorphic to a euclidean ball). Such a collection of points will be called a *complete r-package*.

It can be obtained in the following way: First let $p_1 \in M$ be just any point. If there exists a point in M whose distance to p_1 is greater or equal $2r$, let p_2 be such a point. Now assume by induction that we have points $p_1, \ldots, p_i$ with pairwise distances $\geqq 2r$. If the open metric balls $U(p_j, 2r), j=1, \ldots, i$ still don't cover M, we find again a point p_{i+1} with $\operatorname{dist}(p_{i+1}, p_j) \geqq 2r, j=1, \ldots, i$. And so on. Since M is compact, there is a positive lower bound $\varepsilon(r)$ for the volume of $U(p, r), p \in M$. Since the balls $U(p_j, r)$ are pairwise disjoint, we can have at most $k \leqq \operatorname{vol} M/\varepsilon(r)$ such points. So eventually the package will be complete.

The complete r-package $p_1, \ldots, p_k$ gives rise to *Dirichlet regions:*

$$D_i: \ =\{q \in M; \operatorname{dist}(q, p_i) \leqq \operatorname{dist}(q, p_j) \text{ for all } j=1, \ldots, k\}.$$

Clearly each Dirichlet region satisfies:

$$U(p_i, r) \subset D_i \subset U(p_i, 2r), \tag{4.4}$$

The main tool is the following lower bound for the isoperimetric constant of a Dirichlet region D (satisfying 4.4):

$$h(D) \geqq j(r) := \frac{\sigma(r/4)\ \beta(r/2)}{4\beta(2r)\beta(r)} \geqq \frac{1}{r} c_2^{1+r}. \tag{4.5}$$

Here $c_2 < 1$ is a constant which depends only on the dimension of M. The *proof* of 4.5 is postponed to section 5. We now assume r at least so small that:

$$\mathscr{H} \cdot \frac{\beta(4r)}{\beta(r) j(r)} \leqq \frac{1}{8}, \tag{4.6}$$

(where $\mathscr{H}$ is from 4.1) and enumerate the collection $p_1, \ldots, p_k$ in such a way that:

$$\operatorname{vol}(A \cap U(p_i, r)) \leqq \frac{1}{2} \operatorname{vol} U(p_i, r) \quad \text{for} \quad i=1, \ldots, m$$

$(m \in \{0, \ldots, k\})$ and:

$$\operatorname{vol}(A \cap U(p_i, r)) > \frac{1}{2} \operatorname{vol} U(p_i, r) \quad \text{for} \quad i=m+1, \ldots, k.$$

It follows from 4.3.3 and 4.4 that $\operatorname{vol} D_i \leqq \operatorname{vol} U(p_i, r)\beta(2r)/\beta(r)$. Therefore by 4.5:

$$\operatorname{vol}(A \cap D_i) \leqq \frac{2\ \beta(2r)}{j(r)\beta(r)} . \operatorname{vol}(X \cap \operatorname{int} D_i), \qquad i=1, \ldots, m.$$

Since the Dirichlet regions do not overlap we obtain from 4.6 because of $\operatorname{vol} X/\operatorname{vol} A \leqq \mathscr{H}$ (4.1).

$$\sum_{i=1}^{m} \operatorname{vol}(A \cap D_i) \leqq \frac{2\beta(2r)}{j(r)\beta(r)} . \mathscr{H} . \operatorname{vol} A < \frac{1}{4} \operatorname{vol} A. \tag{4.7}$$

Hence there exists at least one ball $U(p_i, r)$ where B contributes to more than half of the total volume and by the same reason there exists another ball $U(p_j, r)$ where A is predominating, i. e. the following sets are non empty:

$$\tilde{A}: \quad =\left\{p\in M;\ \mathrm{vol}(A\cap U(p, r))>\frac{1}{2}\ \mathrm{vol}\ U(p, r)\right\},$$

$$\tilde{B}: \quad =\left\{p\in M;\ \mathrm{vol}(B\cap U(p, r))>\frac{1}{2}\ \mathrm{vol}\ U(p, r)\right\}.$$

By the continuity of the function $p\mapsto \mathrm{vol}(A\cap U(p, r))-\mathrm{vol}(B\cap U(p, r))$ we see that the open submanifolds $\tilde{A}$, $\tilde{B}$ are separated by the closed subset:

$$\tilde{X}: \quad =\{p\in M;\ \mathrm{vol}(A\cap U(p, r))=\mathrm{vol}(B\cap U(p, r))\},$$

which need not be a null set, let alone a hypersurface.

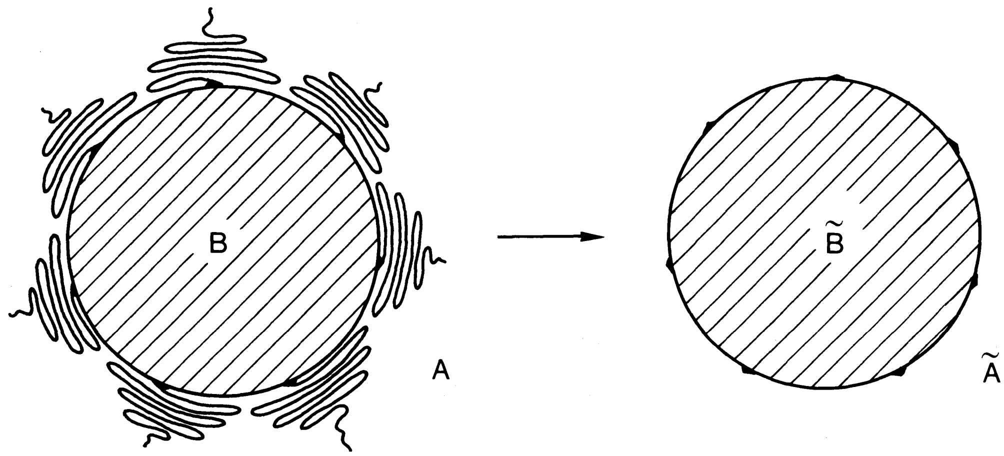

Fig. 2

Note that passing form A to $\tilde{A}$ we eliminate all "lower dimensional looking" parts of A (the problem of hairs).

Let us see whether it is now possible to estimate the volume of the t-hull:

$$\tilde{X}^t: \quad =\{p\in M;\ \mathrm{dist}(p, \tilde{X})\leqq t\}.$$

We take a *new* complete r-package $p_1, \ldots, p_k$ which satisfies the following conditions: 1) $p_1, \ldots, p_s\in\tilde{X}$ and $\tilde{X}$ is covered by the balls $U(p_i, 2r)$, $i=1, \ldots, s$. 2) $p_{s+1}, \ldots, p_m\in\tilde{B}$ and $p_{m+1}, \ldots, p_k\in\tilde{A}$. Now $\tilde{X}^t$ is covered by the balls $U(p_i, 2r+t)$, $i=1, \ldots, s$, and we obtain from 4.3.3, since $p_1, \ldots, p_s\in\tilde{X}$:

$$\mathrm{vol}\ \tilde{X}^t\leqq\frac{\beta(2r+t)}{\beta(r)}\sum_{i=1}^{s}\mathrm{vol}\ U(p_i, r)=\frac{2\beta(2r+t)}{\beta(r)}\sum_{i=1}^{s}\mathrm{vol}(A\cap U(p_i, r)).$$

Therefore by Lemma 5.1:

$$(4.8)\quad \operatorname{vol} \tilde{X}^t \leqq \frac{2\beta(2r+t)}{j(r)\beta(r)} \sum_{i=1}^{s} \operatorname{vol}(X \cap U(p_i, r)) \leqq \frac{2\beta(2r+t)}{j(r)\beta(r)} . \mathscr{H} . \min.\{\operatorname{vol} A, \operatorname{vol} B\}.$$

Now fix $t=2r$. If $p \in \tilde{B}-\tilde{X}^{2r}$, then $U(p, 2r)$ is contained in $\tilde{B}$ since it is impossible to get from p to $\tilde{A}$ without crossing $\tilde{X}$. Hence p is contained in a Dirichlet region D_i with $s+1 \leqq i \leqq m$, and we have $\tilde{B}-\tilde{X}^{2r} \subset D_{s+1} \cup \ldots \cup D_m$. Therefore together with 4.7 and 4.8:

$$(4.9)\quad \operatorname{vol}(\tilde{A}-\tilde{X}^{2r}) \geqq \operatorname{vol}(A \cap (\tilde{A}-\tilde{X}^{2r})) = \operatorname{vol} A - \operatorname{vol}(A \cap (\tilde{B} \cup \tilde{X}^{2r}))$$

$$\geqq \operatorname{vol} A - \operatorname{vol} \tilde{X}^{2r} - \sum_{i=s+1}^{m} \operatorname{vol}(A \cap D_i) \geqq \left(1 - \frac{4\mathscr{H}\beta(4r)}{j(r)\beta(r)}\right) \operatorname{vol} A.$$

With 4.6, 4.8 and 4.9 we are now in a position to proceed like in section 3 by defining $f(p) = \operatorname{dist}(p, \tilde{X})/2r$ if $p \in \tilde{A} \cap \tilde{X}^{2r}$ and $f(p)=1$ on $\tilde{A}-\tilde{X}^{2r}$ and ditto on $\tilde{B}$ to obtain:

$$\lambda_1(M) \leqq \frac{\mathscr{H}}{r^2} \frac{\beta(4r)}{j(r)\beta(r)} \leqq \frac{\mathscr{H}}{r} c_3^{1+r},$$

where r is assumed to satisfy 4.6 and $c_3 > 1$ is another constant depending on the dimension such that the inequality on the right hand side is true for arbitrary $r>0$. Taking $r=1/8$ in case $\mathscr{H} \leqq 1/c_3^2$ and $r=(8\mathscr{H} c_3^2)^{-1}$ if $\mathscr{H} \geqq 1/c_3^2$ we obtain:

$$\lambda_1(M) \leqq 8 c_3^4 (\mathscr{H} + \mathscr{H}^2),$$

for $\mathscr{H}$ arbitrarily close to $h(M)$.

Q.E.D.

5. Starlike domains

Let $p_0 \in M$ and D be a domain such that each distance minimizing geodesic from p_0 to $q \in D$ is contained in D. The Dirichlet regions of the preceeding section, e. g. are *starlike* in this sense.

5.1 LEMMA. — *If* D *is as above and if* $U(p_0, r) \subset D \subset U(p_0, R)$ *then:*

$$h(D) \geqq \max_{0<t<r/2} \frac{\sigma(t)(\beta(r/2)-\beta(t))}{2\beta(R)\beta(r)} \geqq c_4^{1+R} \frac{r^{n-1}}{R^n},$$

where $c_4 < 1$ *is a constant which depends on the dimension.*

Proof. — Recall that the Ricci curvature is bounded below by $-(n-1)$, $\sigma(\rho)$ is the surface area of the ρ sphere in hyperbolic space. We shall write U_ρ instead of $U(p_0, \rho)$. For $q \in D$ we let $C(q)$ denote the cut locus of q with respect to M.

Let χ be a smooth hypersurface of D which divides the interior of D into two open disjoint subsets $\mathscr{A}$ and $\mathscr{B}$, satisfying $\operatorname{vol}\chi = \operatorname{vol}(\chi \cap \operatorname{int} D)$, assume that $\mathscr{A}$ is the part for which:

$$\operatorname{vol}(\mathscr{A} \cap U_{r/2}) \leqq \frac{1}{2} \operatorname{vol} U_{r/2}.$$

In order to obtain a lower bound for $\operatorname{vol}\chi/\operatorname{vol}\mathscr{A}$ we fix t, $0<t<r/2$. It is then necessary to distinguish two cases, 1) where the major part of $\mathscr{A}$ is contained in $D-U_{r/2}$ and 2) where a considerable part of $\mathscr{A}$ lies inside $U_{r/2}$. To this end we introduce a further parameter x, $0<x<1$ which will be suitably chosen at the end of the proof.

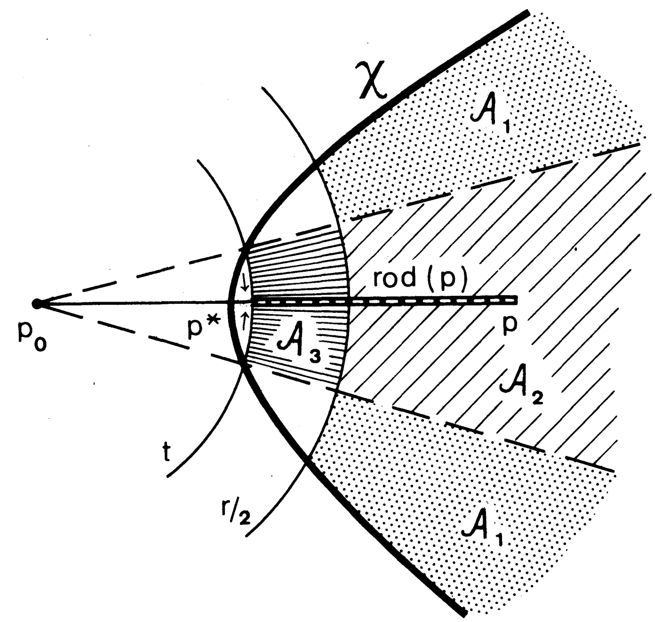

Fig. 3

1. *Case*. — $\operatorname{vol}(\mathscr{A} \cap U_{r/2}) \leqq x \operatorname{vol}\mathscr{A}$. For $p \in \mathscr{A} - C(p_0)$ we define p^* to be the *first* point along the distance minimizing geodesic pp_0 from p to p_0 which meets χ, resp. $p^* = p_0$ if the whole segment pp_0 is contained in $\mathscr{A}$. Hence if we introduce polar coordinates $p = (\rho, \varphi)$, $\rho \geqq 0$, $\varphi \in S^{n-1}$ with center p_0 [for $M - C(p_0)$] then $p^* = (\rho^*, \varphi)$, $\rho^* < \rho$, and the geodesic segment $\{(\rho', \varphi);\ \rho^* \leqq \rho' \leqq \rho\}$ is contained in $\mathscr{A}$. For $p = (\rho, \varphi) \in \mathscr{A} - C(p_0) - U_{r/2}$ we further define the rod of p as the geodesic segment:

$$\operatorname{rod}(p) = \{(\rho', \varphi);\ t \leqq \rho' \leqq \rho\}.$$

Finally we let (the bar denotes closure):

$$\mathscr{A}_1 = \{\, p \in \mathscr{A} - C(p_0) - \overline{U}_{r/2};\ p^* \notin U_t \},$$

$$\mathscr{A}_2 = \{\, p \in \mathscr{A} - C(p_0) - \overline{U}_{r/2};\ \operatorname{rod}(p) \subset \mathscr{A} \},$$

$$\mathscr{A}_3 = \{\, q \in U_{r/2} - \overline{U}_t;\ q \in \operatorname{rod}(p) \text{ for some } p \in \mathscr{A}_2 \}.$$

$\mathscr{A}_2$ can also be described as the set of points in $\mathscr{A} - \overline{\mathrm{U}}_{r/2} - \mathrm{C}(p_0)$ with $p^* \in \mathrm{U}_t$, and $\mathscr{A}_3$ is the intersection of the shell $\mathrm{U}_{r/2} - \overline{\mathrm{U}}_t$ with the union of all rod(p), $p \in \mathscr{A}_2$. Note that $\mathscr{A}_3 \subset \mathscr{A}$. From 4.3.2 follows:

$$\frac{\mathrm{vol}\,\mathscr{A}_2}{\mathrm{vol}\,\mathscr{A}_3} \leqq \frac{\beta(\mathrm{R}) - \beta(r/2)}{\beta(r/2) - \beta(t)} =: \gamma.$$

By the assumption in this first case we have $\mathrm{vol}(\mathscr{A} - \mathrm{U}_{r/2}) \geqq (1-x)\,\mathrm{vol}\,\mathscr{A}$, on the other hand since $\mathscr{A} - \overline{\mathrm{U}}_{r/2} - \mathrm{C}(p_0)$ is contained in $\mathscr{A}_2 \cup \mathscr{A}_1$ we have $\mathrm{vol}(\mathscr{A} - \mathrm{U}_{r/2}) \leqq \mathrm{vol}\,\mathscr{A}_2 + \mathrm{vol}\,\mathscr{A}_1$. Finally since $\mathrm{vol}\,\mathscr{A}_3 \leqq \mathrm{vol}(\mathscr{A} \cap \mathrm{U}_{r/2}) \leqq x\,\mathrm{vol}\,\mathscr{A}$ we obtain:

$$\mathrm{vol}\,\mathscr{A}_1 \geqq (1 - x - \gamma x)\,\mathrm{vol}\,\mathscr{A}.$$

Now we are ready to use 4.3.1: $\mathrm{vol}\,\chi / \mathrm{vol}\,\mathscr{A}_1 \geqq \sigma(t)(\beta(\mathrm{R}) - \beta(t))^{-1}$ by the definition of $\mathscr{A}_1$, and therefore:

$$\frac{\mathrm{vol}\,\chi}{\mathrm{vol}\,\mathscr{A}} \geqq \frac{\sigma(t)}{\beta(\mathrm{R}) - \beta(t)} - \frac{x\,\sigma(t)}{\beta(r/2) - \beta(t)}. \tag{5.2}$$

2. *Case.* — $\mathrm{vol}(\mathscr{A} \cap \mathrm{U}_{r/2}) \geqq x\,\mathrm{vol}\,\mathscr{A}$. We use an argument due to Gromov ([16], lemma (C)). Let $\mathrm{W}_0 = \mathscr{A} \cap \mathrm{U}_{r/2}$ and $\mathrm{W}_1 = \mathscr{B} \cap \mathrm{U}_{r/2}$ or vice versa. Then for one of the two possible choices of W_0, W_1 we have the following.

CLAIM. — *There exists* $w_0 \in \mathrm{W}_0$ *and a measurable subset* $\widetilde{\mathrm{W}}_1$ *of* W_1 *such that:*

(*a*) $\mathrm{vol}\,\widetilde{\mathrm{W}}_1 \geqq 1/2\,\mathrm{vol}\,\mathrm{W}_1$.

(*b*) *Each distance minimizing geodesic from* $q \in \widetilde{\mathrm{W}}_1$ *to* w_0 *intersects* χ *in a first point* q^* *for which* $\mathrm{dist}(q, q^*) \leqq \mathrm{dist}(q^*, w_0)$.

(The geodesic segment qw_0 is not assumed to be contained in $\mathrm{U}_{r/2}$.) For the *proof* of the claim we consider the cartesian product $\mathrm{W}_1 \times \mathrm{W}_0$ with the product measure. Since cut loci are nullsets, it follows from Fubini's theorem that, apart from a nullset $\mathrm{N} \subset \mathrm{W}_1 \times \mathrm{W}_0$, each pair (q, w) of points $q \in \mathrm{W}_1$, $w \in \mathrm{W}_0$ is connected by a *unique* distance minimizing geodesic qw. This geodesic is contained in $\mathrm{U}_r \subset \mathrm{D}$ since its length is less than r, *so it intersects* χ. Now let V_0 (resp. V_1) in $\mathrm{W}_1 \times \mathrm{W}_0 - \mathrm{N}$ be the set of pairs (q, w) which satisfy $\mathrm{dist}(q, q^*) \leqq \mathrm{dist}(q^*, w)$ [resp. $\mathrm{dist}(q, w^*) \geqq \mathrm{dist}(w^*, w)$] where q^* (resp. w^*) is the intersection point of qw with χ next to q (resp. w). Since $\mathrm{V}_0 \cup \mathrm{V}_1 = \mathrm{W}_1 \times \mathrm{W}_0 - \mathrm{N}$ now $\mathrm{V}_0 \geqq 1/2\,\mathrm{vol}(\mathrm{W}_1 \times \mathrm{W}_0)$ for one of the two choices of W_0, W_1. The claim is now a consequence of Fubini's theorem.

In order to estimate $\mathrm{vol}\,\chi / \mathrm{vol}\,\widetilde{\mathrm{W}}_1$ we introduce (new) polar coordinates (ρ, φ) centered at w_0 [inside the cut locus $\mathrm{C}(w_0)$]. If $q \in \widetilde{\mathrm{W}}_1$, $q = (\rho, \varphi)$, then the corresponding point $q^* = (\rho^*, \varphi) \in \chi$ satisfies $\rho^* \geqq 1/2\,\rho$. Let ρ^{**} be maximal such that the geodesic segment $\{(\rho', \varphi);\ \rho^* \leqq \rho' \leqq \rho^{**}\}$ is contained in $\widetilde{\mathrm{W}}_1 - \mathrm{C}(w_0)$. We also have $\rho^* \geqq 1/2\,\rho^{**}$ and

$\rho^{**} \leqq r$. Now 4.8.1 implies [the volume element is $dM = g(\rho, \varphi)\, d\rho\, dS^{n-1}$ inside the cut locus $C(w_0)$]:

$$\frac{g(\rho^*, \varphi)}{\int_{\rho^*}^{\rho^{**}} g(\rho, \varphi)\, d\rho} \geqq \frac{\sigma(\rho^*)}{\beta(\rho^{**}) - \beta(\rho^*)} \geqq \frac{\sigma(\rho^{**}/2)}{\beta(\rho^{**}) - \beta(\rho^{**}/2)} \geqq \frac{\sigma(r/2)}{\beta(r) - \beta(r/2)}$$

[observe that $d/d\tau(\sigma(\tau)(\beta(2\tau) - \beta(\tau))^{-1}) < 0$ for $\tau > 0$]. This is at the same time a lower bound for $\operatorname{vol}\chi/\operatorname{vol}\tilde{W}_1$ and since we assumed $x \operatorname{vol}\mathscr{A} \leqq \operatorname{vol}(\mathscr{A} \cap U_{r/2}) \leqq 2 \operatorname{vol}\tilde{W}_1$ [recall that $\operatorname{vol}\mathscr{A} \cap U_{r/2} \leqq \operatorname{vol}(\mathscr{B} \cap U_{r/2})$] we obtain:

$$\frac{\operatorname{vol}\chi}{\operatorname{vol}\mathscr{A}} \geqq \frac{x\,\sigma(r/2)}{2(\beta(r) - \beta(r/2))}. \tag{5.3}$$

In order to obtain the best possible bound from 5.2 and 5.3 we take x such that both bounds become equal. Thus after some elementary simplifications we end up with:

$$h(D) \geqq \max_{0<t<r/2} \frac{\sigma(t)(\beta(r/2) - \beta(t))}{2\beta(R)\beta(r)}.$$

Q.E.D.

6. An application

The above estimate of $h(D)$ for Dirichlet regions provides a simplified proof of Gromov's bound [16]:

$$\lambda_m \geqq \varepsilon^{-2} c_5^{1+\varepsilon}, \tag{6.1}$$

for the m-th eigenvalue of the Laplacian on M, where $m = m(\varepsilon)$ is the minimal cardinality of a complete ε-package of M, and c_5 is a dimension constant.

Proof. — The simplification lies in the possibility of applying Cheeger's inequality directly. By Courant's minimax principle $\lambda_m \geqq \min\{\lambda_1(D_i);\ i = 1, \ldots, m\}$ where D_i are the Dirichlet regions due to an optimal complete ε-package $p_1, \ldots, p_m$, and $\lambda_1(D_i)$ is the smallest positive eigenvalue of $\Delta u = \lambda u$ with respect to the *Neumann* condition $*du \mid \partial D_i = 0$. By Cheeger's inequality $\lambda_1(D_i) \geqq 1/4 h^2(D_i)$, and by 4.5 $h(D_i) \geqq 1/\varepsilon c_2^{1+\varepsilon}$.

Q.E.D.

We use this place to answer a question of Gromov in [16] concerning bounds for the higher eigenvalues λ_k which have growth rate as given by Weyl's asymptotic law [20]:

$$\lambda_k(M) \sim \text{Const.}(n) \left(\frac{k}{\operatorname{vol} M}\right)^{2/n}, \qquad k \to \infty.$$

The bound will be obtained as a simple application of a result of C. Croke. In [11], proposition 14, Croke proves the inequality:

$$\operatorname{vol} U_r \geqq c_6\, r^n, \qquad c_6 = \frac{n}{2\pi}\left(\frac{2}{n\pi}\right)^{n/2} . b_n.$$

(b_n = volume of the unit ball in $\mathbb{R}^n$) for a distance ball U_r in M^n of radius r less or equal half the injectivity radius ρ_{inj} of M. The result holds in fact without any assumption on the curvature of M and generalizes (up to a constant) Berger's inequality [3]:

$$\operatorname{vol} M \geqq n . b_n . \left(\frac{\rho_{\text{inj}}}{\pi}\right)^n.$$

6.2 THEOREM. — *Let* M^n *be a compact unbordered Riemannian manifold with* $\operatorname{vol} M = V$, *injectivity radius* $\rho_{\text{inj}} = \rho$ *and Ricci curvature bounded below by* $-(n-1)\delta^2$, $\delta \geqq 0$. *Then:*

(a) $$\lambda_k \geqq \left(\frac{k\,\rho^{n-1}}{V}\right)^2 c_7^{1+\delta_\rho^{1-n} V/k} \qquad \text{if} \quad k \leqq \frac{4V}{c_6 (\rho/2)^n},$$

(b) $$\lambda_k \geqq \left(\frac{k}{V}\right)^{2/n} . c_7^{1+\delta (V/k)^{1/n}} \qquad \text{if} \quad k \geqq \frac{4V}{c_6 (\rho/2)^n},$$

(c) $$\lambda_k \leqq \left(\frac{k}{V}\right)^{2/n} . c_8 + \frac{(n-1)^2 \delta^2}{4},$$

where $c_7 < 1$ *and* $c_8 > 1$ *are dimension constants.*

Proof. — Scale again $\delta = 1$. The upper bound (*c*) is from [7] for the sake of completeness (*see* also [15] for particularly sharp bounds).

Let for the moment $\varepsilon > 0$ be given and consider a complete ε-package $p_1, \ldots, p_m$ (*see* above 4.4) with minimal cardinality $m = m(\varepsilon)$. If $\varepsilon \leqq \rho/2$ then $\operatorname{vol} U(p_i, \varepsilon) \geqq c_6 \varepsilon^n$ by Crokes inequality and $m(\varepsilon)$ has the upper bound $V \varepsilon^{-n} c_6^{-1}$. If $\varepsilon \geqq \rho/2$ then each $U(p_i, \varepsilon)$ contains at least $\operatorname{int}(1/2 + \varepsilon/\rho)$ disjoint balls of radius $\rho/2$ (as long as ε does not exceed max dist (p_i, q), $q \in M$, which is satisfied if $m(\varepsilon) > 1$). Hence vol $U(p_i, \varepsilon) \geqq c_6\, \varepsilon (\rho/2)^n / 2\rho$ and $m(\varepsilon) \leqq 2^{n+1} V \varepsilon^{-1} \rho^{1-n} c_6^{-1}$.

Now let $k \in \mathbb{N}$ be given and define $\varepsilon = \varepsilon(k) = (4V/c_6 k)^{1/n}$ if $k \geqq 4V c_6^{-1} (\rho/2)^{-n}$ resp. $\varepsilon(k) = 4V c_6^{-1} k^{-1} (\rho/2)^{1-n}$ if $k \leqq 4V c_6^{-1} (\rho/2)^{-n}$. Clearly $k \geqq m(\varepsilon(k))$, $\lambda_k \geqq \lambda_{m(\varepsilon)}$ and (*a*), (*b*) follow from 6.1.

Remarks. — (*a*) The growth rate of the lower bound has a sudden change from order k^2 to order $k^{2/r}$ if $\delta = 0$. Such a behaviour can indeed be observed for λ_k, e. g. on the product $S_1^1 \times S_r^{n-1}$ of the unit circle with a very small r-sphere. (*b*) Gromov's bound in [16] (based on 4.3) is in terms of the diameter d of M^n:

$$\lambda_k \geqq \frac{1}{d^2} k^{2/n} c_9^{1 + d/\delta}.$$

For the first n^n eigenvalues or so, this bound is better than the one given by 6.2 since $V/\rho^{n-1} \geqq \text{Const.}(n) . d$ as follows from Croke's inequality.

7. Non compact manifolds

We now assume M is a *complete non compact* Riemannian manifold of dimension n. Here one defines:

$$\lambda(\mathrm{M}) = \inf \frac{\displaystyle\int_{\mathrm{M}} \| \operatorname{grad} f \|^2 \, d\mathrm{M}}{\displaystyle\int_{\mathrm{M}} f^2 \, d\mathrm{M}},$$

where f runs over all sufficiently smooth functions; if M has finite volume, we require the mean value $\int_{\mathrm{M}} f \, d\mathrm{M} = 0$; we require f to have compact support if the volume of M is infinite. $\lambda(\mathrm{M})$ is the greatest lower bound for the spectrum of the Laplacian, except that on manifolds with finite volume, one also has zero as a trivial eigenvalue corresponding to the constant functions.

Cheeger's inequality is still true for $\lambda(\mathrm{M})$ if $h(\mathrm{M})$ is defined as in paragraph 1 but with the additional condition that $\mathrm{A} \cup \partial \mathrm{A}$ be compact. Upper bounds for $\lambda(\mathrm{M})$ in terms of $h(\mathrm{M})$ have recently become of interest on foliations and on universal coverings of compact manifolds [5]. We are now going to check that in fact Theorem 1.2 holds without restriction in the non compact case as well. Since foliations are often assumed of differentiability class C^1 only, we give the theorem a curvature free formulation introducing the condition δ [which is for example satisfied if M has Ricci curvature bounded below by $-\delta^2(n-1)$, c.f. 4.2]. To emphasize our point of view in this paragraph, we assume that M is of *differentiability class piecewise* C^1, though this is not the weakest possible assumption to make 7.1 and 7.2 true.

CONDITION δ ($\delta > 0$). — *In polar coordinates—up to the cut locus of the coordinate center—the volume element* $d\mathrm{M} = g(\rho, \varphi)\, d\rho \, d\mathrm{S}^{n-1}$ *always satisfies:*

$$\frac{g(\rho_1, \varphi)}{g(\rho_2, \varphi)} \geqq \frac{\sinh^{n-1} \delta\rho_1}{\sinh^{n-1} \delta\rho_2}, \qquad (0 < \rho_1 < \rho_2).$$

7.1 THEOREM. — *If* M *is complete, non compact and satisfies condition* δ, *then:*

$$\lambda(\mathrm{M}) \leqq c . \delta . h(\mathrm{M}),$$

where c is a constant depending only on the dimension.

Proof. — The term $h^2(\mathrm{M})$ of 1.2 can be suppressed here by choosing c properly, for in the non compact case we cannot have large $h(\mathrm{M})$ [take two arbitrarily large disjoint distance balls to prove $h(\mathrm{M}) \leqq \delta(n-1)$].

The method of paragraph 3 cannot be applied here, since minimizing currents need not exist, even if M is C^∞. However the procedure of paragraph 4 carries over. Assume $\delta = 1$. Take A relatively compact with $\mathrm{vol}_{n-1} \partial \mathrm{A} / \mathrm{vol}_n \mathrm{A}$ close to $h(\mathrm{M})$ and restrict consideration to a sufficiently large distance ball U which contains A. Now observe that paragraphs 4 and 5 use the curvature assumption only via 4.2, 4.3, and carry out the remainder of paragraph 4 on U.

The main point is of course that a "hairy" ∂A can be replaced by a hypersurface which allows us to estimate the volume of its tubular neighbourhoods. Since we consider this an appropriate substitute for the non-existence of minimizing currents, we formulate it here as a lemma, omitting the proof which follows from 4.8 and 4.9 by handling constants.

7.2 LEMMA (Cutting off hairs). — *Assume* M *satisfies condition* δ. *Consider an arbitrary relatively compact domain* $A \subset M$ *with* $\mathrm{vol}_{n-1} \partial A / \mathrm{vol}_n A := \mathscr{H} \geqq h(M)$, *and let* $0 < r \leqq 1/2\, c \min\{1, 1/\mathscr{H}\}$. *Then there exists a domain* $\tilde{A} = \tilde{A}(r)$ *with boundary* $\tilde{X} = \partial \tilde{A}$ *with the following properties:*

1. $\tilde{A}$ *is contained in* A^r *and* $\mathrm{vol}\, \tilde{A} \geqq (1 - r \mathscr{H}/c) \mathrm{vol}\, A$.
2. $\mathrm{vol}_n \tilde{X}^t / \mathrm{vol}_n \tilde{A} \leqq c''/r^n . \mathscr{H} . \int_0^t (\cosh \delta\tau)^{n-1} d\tau$ *for all* $t \geqq r$.

A^r *is the tubular neighbourhood of radius r, the constants* c', c'' *depend only on the dimension.*

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