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On the point spectrum of Schrödinger operators


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OF SCHRODINGER OPERATORS

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1. Introduction

This paper is an extension of a work [2] on the spectral analysis of partial differential operators of Schrödinger type. The problem was the following: Let $A$ be a compact subset of $\mathbb{R}^n$, $\Sigma$ a finite interval in $\mathbb{R}$ and $H$ a self-adjoint elliptic differential operator in the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. We define $F(\Sigma)$ to be the spectral projection of $H$ associated with the interval $\Sigma$ and $E(A)$ the multiplication operator by the characteristic function $\chi_A$ of $A$. Do there exist vectors in $L^2(\mathbb{R}^n)$ which are contained both in the range $E(A)\mathcal{H}$ of $E(A)$ and in $F(\Sigma)\mathcal{H}$?

It turns out that the closed subspace $\mathcal{H}_p(H)$ generated by the set of eigenvectors of $H$ plays a different role from the subspace $\mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$ associated with the continuous spectrum of $H$. Notice that it is shown in [2], under regularity and integrability conditions on the coefficients of the differential operator, that there do not exist vectors of $\mathcal{H}_c(H)$ which belong both to $E(A)\mathcal{H}$ and to $F(\Sigma)\mathcal{H}$. On the other hand, to prove the non-existence of vectors in $\mathcal{H}_p(H)$ belonging to $E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H}$, we used an unique continuation theorem for solutions of the differential equation associated with $H$. Now, if for example $H = -\Delta + V$, where $V$ is the multiplication operator by a real function $v(x)$, the known results on unique continuation require a condition $L^s(\mathbb{R}^n)$ on $v$, where $N$ is a closed set of measure zero such that $\mathbb{R}^n \setminus N$ is connected ([3], [5]).

In the present paper, we propose to show that:

\begin{equation}
\mathcal{H}_p(H) \cap E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H} = \{0\},
\end{equation}

by imposing only an integrability condition on the function $v$. More precisely, we will prove (1) under the hypothesis that $v \in L^s_{\text{loc}}(\mathbb{R}^n)$ with $s = 2$ if $n = 1, 2, 3$ and $s > n - 2$ if $n \geq 4$.

This result shows that, under the above conditions on $v$, the operator $-\Delta + v$ has no eigenvector with compact support. This is essentially the content of our Theorem 1 in paragraph 2. (In the case $n = 1$, one obtains ordinary differential operators for which results of this type have been known for a long time [9]).
This result is also interesting from the point of view of “non-existence of positive eigenvalues of the operator $H$”. In the literature (for example [2], [12]) the non-existence of positive eigenvalues is obtained in two steps:

(i) under suitable decay conditions at infinity on the function $v$, it is shown that all eigenfunctions $f$ associated with a strictly positive eigenvalue of $H$ have compact support;

(ii) then one imposes suitable local conditions on $v$ (e.g. $v \in L^\infty_0(\mathbb{R}^n \setminus N)$ in order to apply the unique continuation theorem, which then leads to $f \equiv 0$. It turns out that the non-existence of positive eigenvalues is also obtained by assuming in (ii) as a local condition that $v \in L^s_0(\mathbb{R}^n)$ with $s = 2$ if $n = 1, 2, 3$ and $s > n - 2$ if $n \geq 4$ (Thm. 2).

Finally our method implies also the spectral continuity of a class of Schrödinger operators with periodic potentials $v(x)$.

The organization of the paper is as follows: first we give the principal results and deduce Theorems 1 and 2 from Theorem 3 in section 2, and we introduce a direct integral representation of Schrödinger operators in section 3. This representation will be used in section 4 for proving Theorem 3. The principal estimate of the proof is the subject of the last section 5.

2. Statements of the results

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We always suppose that:

\begin{equation}
(2) \quad v \in L^s_0(\mathbb{R}^n) \quad \text{with} \quad s = 2 \quad \text{if} \quad n = 1, 2, 3; \quad s > n - 2 \quad \text{if} \quad n \geq 4.
\end{equation}

Notice that $s > n - 2$ in all cases.

The function $v$ will be called periodic if there exist $n$ linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$ such that $v(x + \vec{a}_i) = v(x)$ for all $x \in \mathbb{R}^n$. A periodic function will be called ortho-periodic if:

\begin{equation}
(3) \quad \vec{a}_j, \vec{a}_k = L^2 \delta_{jk},
\end{equation}

with $L > 0$, i.e. if the vectors of the form $\sum_{i=1}^n \alpha_i \vec{a}_i$, $0 \leq \alpha_i < 1$, define a cube $C^n$ with side $L$.

We denote by $\hat{H}$ the symmetric operator:

\begin{equation}
(4) \quad \hat{H} = -\Delta + v(x),
\end{equation}

with domain $D(\hat{H}) = C^0_0(\mathbb{R}^n)$ and by $H_0$ the unique self-adjoint extension of $\hat{H}_0 = -\Delta$, $D(H_0) = C^0_0(\mathbb{R}^n)$. Let $H$ a self-adjoint extension of $\hat{H}$. We have the following lemma:

**Lemma 1.** — Assume that (2) and one of the following conditions are satisfied:

(i) $v$ is periodic;

(ii) $v \in L^\infty(\mathbb{R}^n)$ where $B_R = \{ x \in \mathbb{R}^n | |x| \leq R \}$ and $\mathbb{C} B_R$ denotes the complement of $B_R$.

Then:

(a) $v$ is $H_0$-bounded with $H_0$-bound $0$.
(b) \( \hat{H} \) is essentially self-adjoint;
(c) \( D(H) = D(H_0) \), where \( H \) is the unique self-adjoint extension of \( H \).

Proof. — (b) and (c) follow from (a) by using the Kato-Rellich Theorem ([7], Chapt. 5.4.1). Under hypothesis (i), (a) follows from Theorem XIII.96 of [11], whereas under the assumption (ii), (a) can be proved by the method used in the proof of Lemma 3 in [10]. Both cases are treated in [4].

We now state our principal results. In Theorem 2 we choose as conditions on the potential \( v \) at infinity those used in [4].

**THEOREM 1.** — Let \( v \in L^2_{\text{loc}}(\mathbb{R}^n) \) with \( s \) satisfying (2) and let \( H \) be a self-adjoint extension of \( H \):
(a) suppose that \( f \in L^2(\mathbb{R}^n) \) satisfies \( Hf = \lambda f \) for some \( \lambda \in \mathbb{R} \) and \( E(A)f \neq f \) for some compact subset \( A \) of \( \mathbb{R}^n \), (i.e., \( f \) is an eigenvector of \( H \) with compact support in \( \mathbb{R}^n \)). Then \( f = 0 \);
(b) for each compact subset \( A \) of \( \mathbb{R}^n \) and each bounded interval \( \Sigma \), one has:
\[
\mathcal{H}_p(H) \cap E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\}.
\]

**THEOREM 2.** — Suppose that:
(i) \( v \in L^4(B_R) \) with \( s \) satisfying (2) for some \( R < \infty \);
(ii) \( v = v_1 + v_2 \) such that:
\((\alpha) v_1, v_2 \in L^\infty(\mathbb{R}^n) \),
\((\beta) |\vec{x}| v_1(\vec{x}) \to 0 \quad \text{as} \quad |\vec{x}| \to \infty,
\((\gamma) r_2(\vec{x}) \to 0 \quad \text{as} \quad |\vec{x}| \to \infty,
\((\delta) r \mapsto v_2(r, \cdot))
\]
is differentiable as a function from \( (\mathbb{R}, \infty) \) to \( L^\infty(S^{n-1}) \), and \( \limsup_{r \to \infty} \partial_v v_2(\partial_r) \leq 0 \).

Then \( H = H_0 + V \) has no eigenvalues in \((0, \infty)\).

**THEOREM 3.** — Let \( v \) be ortho-periodic and \( v \in L^4_{\text{loc}}(\mathbb{R}^n) \) with \( s \) satisfying (2). Then the spectrum of \( H = H_0 + V \) is purely continuous.

Remark 1. — By following the proof of Theorem XIII.100 in [11], it is possible to show that the operator \( H \) in Theorem 3 is absolutely continuous. Other comments on Theorem 3 will be made at the end of this paper.

Remark 2. — Contrarily to [2], where the operator \( \hat{H} \) was defined by:
\[
\hat{H} = \sum_{j,k=1}^n a_{jk} \left( -i \frac{\partial}{\partial x_j} + b_j(\vec{x}) \right) \left( -i \frac{\partial}{\partial x_k} + b_k(\vec{x}) \right) + V(\vec{x}),
\]
we assume here that the vector potential \( \vec{b} = \{ b_k \} \) is equal to zero. It is possible to generalize Theorem 1 to the case where \( \vec{b} \neq 0 \).

Theorem 2 follows from results of [11] and [6], and from Theorem 1 as indicated in the introduction. (If \( Hf = \lambda f \) with \( \lambda > 0 \), then \( f \) has compact support by Theorem XIII.58 of ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPERIEURE

(on the point spectrum)
[11], and consequently \( f = 0 \) by our Theorem 1.) Theorem 1 (a) is deduced from Theorem 3: By the proof of Proposition 4 of [2], the vector \( f \) belongs to \( D(H_0) \cap D(V) \) and \( Hf = H_0 f + VE(A)f \). Let \( w \) be an ortho-periodic function such that \( w \in L^4_{loc}(\mathbb{R}^n) \) and \( w(\tilde{x}) = v(\tilde{x}) \) for \( \tilde{x} \in A \). If \( H_1 \) denotes the periodic Schrödinger operator \( H_1 = H_0 + W \) then \( H_1 f = Hf = \lambda f \). Therefore we deduce from Theorem 3 that \( f = 0 \).

To show Theorem 1 (b), let \( S = E(A) \cap F(\Sigma) \) (the orthogonal projection with range \( E(A) \cap F(\Sigma) \)) and suppose that \( f \in \mathcal{H}_p(H) \) satisfies \( Sf = f \). \( f \) is a linear combination of eigenvectors of \( H \), i.e. \( f = \sum \alpha_k g_k \), where \( Hg_k = \lambda_k g_k \) with \( \lambda_k \in \Sigma \). It follows that:

\[
Sf = f = \sum \alpha_k Sg_k.
\]

Now, by Proposition 2 of [2], \( S \) commutes with \( H \); in particular \( HSg_k = SHg_k = \lambda_k Sg_k \). This implies that each \( Sg_k \) is an eigenvector of \( H \) of compact support in \( A \), hence \( Sg_k = 0 \) by the part (a) of Theorem 1. We deduce from this that \( f = \sum \alpha_k Sg_k = 0 \). The condition "\( \Sigma \) bounded" is fundamental: we can choose a potential \( V \) such that \( \mathcal{H}_p(H) = \mathcal{H} \), i.e. such that the eigenvectors of \( \mathcal{H} \) generate \( \mathcal{H} \). In this case, we have:

\[
\mathcal{H}_p(H) \cap E(A) \mathcal{H} = E(A) \mathcal{H} \neq \{ 0 \}.
\]

### 3. Reduction of the translation group of the lattice

In this part, let \( v \) be an ortho-periodic potential. In a natural way, this implies a decomposition of the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n) \) and of the operators \( H \) and \( H_0 \) into direct integrals. This decomposition will be used in the next part for the proof of Theorem 3.

The potential \( v \) satisfies \( v(\tilde{x} + a_i) = v(\tilde{x}) \) where \( a_1, \ldots, a_n \) are as in (3). The points of the form \( \tilde{z} = \sum_{i=1}^n q_i \tilde{a}_i, \tilde{q} = \{ q_i \} \in \mathbb{Z}^n \), form a cubic lattice in \( \mathbb{R}^n \) which is invariant under the translations:

\[
\tilde{z} \mapsto \tilde{z} + \sum_{i} q_i \tilde{a}_i, \quad \tilde{q}' \in \mathbb{Z}^n.
\]

In \( L^2(\mathbb{R}^n) \), we consider the unitary representation \( U(\tilde{q}) \) of the additive group \( \mathbb{Z}^n \) given by:

\[
(5) \quad [U(\tilde{q}) f](\tilde{x}) = f(\tilde{x} - \sum q_i \tilde{a}_i) = f(\tilde{x} - L \tilde{q}),
\]

where we have written \( \sum q_i \tilde{a}_i = L \tilde{q} \), assuming that the directions of the \( \tilde{a}_i \) coincide with Cartesian coordinate system.

We also introduce the *reciprocal lattice* which is the set of points of the following form:

\[
\tilde{z} = \sum_{i=1}^n q_i e_i, \quad \tilde{q} \in \mathbb{Z}^n.
\]
where the vectors $\vec{e}_1, \ldots, \vec{e}_n$ are defined by:

(6) $\vec{e}_i, \vec{a}_k = 2\pi \delta_{ik}$.

We may write $\vec{z} = E \vec{q}$, with $E = 2\pi L^{-1}$. Let again:

$$\Gamma^n = \left\{ k \in \mathbb{R}^n | k = \sum_{i=1}^n \lambda_i \vec{e}_i, \ 0 \leq \lambda_i < 1 \right\}.$$ 

Consider the Hilbert space $\mathcal{G}$ of square-integrable functions $f : \Gamma^n \to l_2^n = l_2(\mathbb{Z}^n)$:

$$\mathcal{G} = L^2(\Gamma^n; l_2^n).$$

We write $f(\vec{k})_q$ for the component $f(\vec{q} \in \mathbb{Z}^n)$ of $f$ at the point $\vec{k} \in \mathbb{Z}^n$. Thus, we have:

$$\| f \|_2^2 = \int_{\mathbb{R}^n} dk \sum_{\vec{q} \in \mathbb{Z}^n} | f(\vec{k})_q |^2.$$ 

Now, let $\mathcal{U} : \mathcal{H} \to \mathcal{G}$ be the operator defined by:

(7) $$(\mathcal{U} f)(\vec{k})_q = f(\vec{k} + E \vec{q}),$$

where $\hat{f}$ is the Fourier transform of the function $f$:

$$\hat{f}(\vec{\xi}) = (2\pi)^{-n/2} \lim_{\mathbb{R}^n} \int dx \exp(-i\vec{\xi} \cdot \vec{x}) f(\vec{x}).$$

It follows from Plancherel's Theorem that the operator $\mathcal{U}$ is unitary, and its inverse is given by:

$$\mathcal{F} \{ \mathcal{U}^{-1} \{ f(\cdot) \} \}(\vec{\xi}) = f(\vec{k}),$$

where $\vec{q} \in \mathbb{Z}^n$ and $\vec{k} \in \Gamma^n$ are determined by $\vec{k} + E \vec{q} = \vec{\xi}$. If $m \in \mathbb{Z}^n$, one has:

(8) $$[\mathcal{U} U(m) f](\vec{k})_q = \exp(-i L \vec{k} \cdot \vec{m})(\mathcal{U} f)(\vec{k})_q,$$

i.e. $\mathcal{U} U(m) \mathcal{U}^{-1}$ is diagonalizable in $\mathcal{G}$ (i.e. a multiplication operator by a function of $\vec{k}$). As the functions $\{ \exp(iL \vec{k} \cdot \vec{m}) \}_{m \in \mathbb{Z}^n}$ form a basis of $L^2(\Gamma^n)$, each bounded diagonalizable operator is a function of $\{ \mathcal{U} U(m) \mathcal{U}^{-1} \}$. As $H_0$, $V$ and $H$ commute with every $U(m)$, these operators commute with each diagonalizable operator, i.e. $\mathcal{U} H_0 \mathcal{U}^{-1}$, $\mathcal{U} V \mathcal{U}^{-1}$ and $\mathcal{U} H \mathcal{U}^{-1}$ are decomposable in $L^2(\Gamma^n; l_2^n)$. Therefore there exist in $l_2^n$ measurable families of self-adjoint operators $H_0(\vec{k})$, $V(\vec{k})$ and $H(\vec{k})(\vec{k} \in \Gamma^n)$ such that, for $f \in D(H_0)$:

$$\begin{cases} (\mathcal{U} H_0 f)(\vec{k}) = H_0(\vec{k}) f(\vec{k}), \\
(\mathcal{U} V f)(\vec{k}) = V(\vec{k}) f(\vec{k}), \\
(\mathcal{U} H f)(\vec{k}) = H(\vec{k}) f(\vec{k}). \end{cases}$$

Now let us give the explicit form and the properties of these three families of operators.
Lemma 2. — (i) $H_0(k)$ is the self-adjoint multiplication operator in $l^2_n$ by $\varphi(q) = (k + Eq)^2$. If $g = \{g_q\} \in l^2_n$, then:

$$(H_0(k)g)_q = (k + Eq)^2g_q.$$ 

(ii) the domain of $D(H_0(k))$ is independent of $k$ and is given by:

$$D(H_0(k)) = D_0 = \{g \in l^2_n : \sum_{q \in \mathbb{Z}^n} |q^2g_q|^2 < \infty\}.$$ 

(iii) the resolvent $(H_0(k) - \mu)^{-1}$ of $H_0(k)$ is a compact operator for all $\mu \notin \sigma(H_0(k))$, where $\sigma(H_0(k))$ is the spectrum of $H_0(k)$.

Proof. — (i) and (ii) are obvious, since:

$$(H_0 f)(\xi) = \xi^2 f(\xi).$$ 

(iii) The resolvent $(H_0(k) - \mu)^{-1}$ is the multiplication operator by:

$$\psi(q) = [(k + Eq)^2 - \mu]^{-1}.$$ 

Let $\chi_M$ be the characteristic function of the set $\{q \in \mathbb{Z}^n : q^2 \leq M\}$ and $D_M$ the multiplication operator by $\psi(q)\chi_M(q)$. $D_M$ is a compact (even nuclear) operator, and:

$$(10) \quad ||(H_0(k) - \mu)^{-1} - D_M|| = \sup_{q > M} [(k + Eq)^2 - \mu]^{-1} \rightarrow 0,$$

as $M \to \infty$. Thus $(H_0(k) - \mu)^{-1}$ is compact as the uniform limit of the sequence $\{D_M\}$ of compact operators. □

Let us denote by $\{\hat{v}_q\}_{q \in \mathbb{Z}^n}$ the Fourier coefficients of the periodic function $v$:

$$(11) \quad \hat{v}_q = L^{-n/2} \int_{\mathbb{C}^n} dx \exp(-iE \cdot \tilde{q} \cdot \tilde{x}) v(x).$$

Notice that $v \in L^p(\mathbb{C}^n)$ for all $p \in [1, s]$. To establish the relation between the Fourier coefficients of $v$ and the operator $V(k)$ we need the following result:

Lemma 3. — Given $\varphi, \psi : \mathbb{Z}^n \to \mathbb{C}$, we define an operator $A_{\varphi\psi} : l^2_n \to l^2_n$ as follows:

$$(A_{\varphi\psi} g)_q = \sum_{\tilde{m} \in \mathbb{Z}^n} \varphi(\tilde{m}) \psi(q - \tilde{m}) g_{\tilde{q} - \tilde{m}}.$$ 

Assume that $2 \leq p < \infty, \psi \in L^p(\mathbb{Z}^n)$ and let $\{\varphi(q)\}$ be the Fourier coefficients of a function $\Phi$ belonging to $L^p(\mathbb{C}^n)$. Then $A_{\varphi\psi}$ is a compact operator and one has:

$$(12) \quad ||A_{\varphi\psi}|| \leq L^{-\frac{n}{2} - \frac{n}{p}} ||\Phi||_{L^p(\mathbb{C}^n)} ||\psi||_{l^p(\mathbb{Z}^n)}.$$ 

Proof. — For $g = \{g_q\} \in l^2_n$, define $\psi g = \{\psi(\tilde{q}) g_{\tilde{q}}\}$. By the Hölder inequality, $\psi g \in l^r_n$ with $r^{-1} = (1/2) + p^{-1}$, i.e. $1 \leq r < 2$, and:

$$(13) \quad ||\psi g||_r \leq ||\psi||_p ||g||_2.$$
Let:
\[ \gamma(x) = L^{-n/2} \sum_{\vec{q} \in \mathbb{Z}^n} \exp\left(i \vec{E} \cdot \vec{x}\right) \psi(\vec{q}) g_{\vec{q}}, \quad x \in \mathbb{C}^n. \]

By the Hausdorff-Young inequality [8], \( \gamma \in L' (\mathbb{C}^n) \) with \( (r')^{-1} = 1 - r^{-1} = 1/2 - p^{-1} \) and:

\[
\| \gamma \|_r \leq L^{(n/r)-(n/2)} \| \psi g \|_r \leq L^{(n/r)-(n/2)} \| \psi \|_p \| g \|_2.
\]

Since \( 1/2 = p^{-1} + (r')^{-1} \) and \( \Phi \in L^p(\mathbb{C}^n) \), the Hölder inequality implies that \( \Phi \gamma \in L^2(\mathbb{C}^n) \) and:

\[
\| \Phi \gamma \|_2 \leq \| \Phi \|_p \| \gamma \|_r \leq L^{(n/r)-(n/2)} \| \Phi \|_p \| \psi \|_p \| g \|_2.
\]

Now:

\[
(A_{\psi \psi} g)_{\vec{q}} = \int_{\mathbb{C}^n} dx \exp\left(-i \vec{E} \cdot \vec{x}\right) \Phi(\vec{x}) \gamma(\vec{x}),
\]

and by Plancherel's theorem we have:

\[
\| A_{\psi \psi} g \|_2 = L^{n/2} \| \Phi \gamma \|_2 \leq L^{(n/2)-(n/p)} \| \Phi \|_p \| \psi \|_p \| g \|_2.
\]

This shows that \( A_{\psi \psi} \) is defined everywhere with the bound (12):

(b) Let \( D_M \) be the multiplication operator by \( \psi_M(\vec{q}) = \psi(\vec{q}) \chi_M(\vec{q}) \) (see the proof of Lemma 2). By (a), \( A_{\psi \psi} M \) is bounded, and \( A_{\psi \psi} M \) is non-zero only on a subspace of finite dimension. Therefore \( A_{\psi \psi} M \) is nuclear. By using (12) we obtain:

\[
\| A_{\psi \psi} - A_{\psi \psi} M \| \leq L^{(n/2)-(n/p)} \| \Phi \|_p \| (1-\chi_M) \|_p.
\]

Since \( \psi \in l^n_p \), \( \| (1-\chi_M) \|_p \to 0 \) as \( M \to \infty \). This proves the compactness of \( A_{\psi \psi} \).

**Lemma 4.** Let \( Y \) be the operator in \( l^n_2 \) defined by:

\[
(Yg)_{\vec{q}} = L^{-n/2} \sum_{\vec{m} \in \mathbb{Z}^n} v_{\vec{m}} g_{\vec{q}+\vec{m}}.
\]

Then:

(i) \( D_0 \subseteq D(Y) \) and \( Y \) is symmetric on \( D_0 \);

(ii) \( Y \) is relatively compact with respect to \( H_0(\vec{k}) \);

(iii) \( V(\vec{k}) = Y \) on \( D_0 \), for all \( \vec{k} \in \Gamma_n \) (in particular \( V(\vec{k}) \) is independent of \( \vec{k} \));

(iv) \( H(\vec{k}) = H_0(\vec{k}) + Y \) and \( D(H(\vec{k})) = D_0 \).

**Proof.** (i) If \( g \in D_0 \), then \( g = [H(\vec{0})+1]^{-1} \) for some \( h \in l^n_2 \). (15) shows that \( \| Yg \|_2 < \infty \), therefore \( D_0 \subseteq D(Y) \). By using \( \vec{v}_{\vec{q}} = v_{\vec{q}} \), one obtains easily that \( (f, Yg) = (Yf, g) \) for \( f, g \in D_0 \);

(ii) \( Y(H_0(\vec{k})+1)^{-1} \) is of the form \( A_{\psi \psi} \), with \( \Phi(x) = L^{-n/2} v(\vec{x}) \) and \( \psi(\vec{q}) = [(k+E \vec{q})^2 + 1]^{-1} \). Notice that \( \psi \in l^n_2 \) for each \( p > n/2 \). As \( v \in L^s(\mathbb{C}^n) \) for \( s = 2 \) if \( n = 2, 3 \) and \( s > n/2 \) if \( n \geq 4 \), Lemma 3 implies that \( Y(H_0(\vec{k})+1)^{-1} \) is compact;
(iii) this can be verified by calculating the Fourier transform of $Vf$;
(iv) by (i) and (ii), $H_o(k)$ is self-adjoint. $H(k) = H_o(k) + Y$ follows from (iii) and
Lemmas 1 and 2.

4. Proof of Theorem 3

Let $f$ be an eigenvector of $H$, i.e. $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$. By defining $v'(x) = v(x) - \lambda$ and $H' = H_o + V'$, we have $H'f = 0$. Since $V'$ satisfies also the hypothesis (2), it is possible to assume without loss of generality that $\lambda = 0$.

Let $\Gamma_0 = \{ k \in \Gamma | (\mathcal{U}f)(k) \neq 0 \text{ in } l^2_n \}$. $\Gamma_0$ is measurable. Since $H(k)(\mathcal{U}f)(k) = 0$, $H(k)$ must have the eigenvalue 0 for almost all the $k \in \Gamma_0$. We will show that, for all $p \in (k_1, \ldots, k_{n-1}, 0) \in \mathbb{R}^{n-1}$ the set $\theta(p)$ of the points $k_n \in (0, E)$ such that $0 \in \sigma(H(k + k_n E^{-1} e_n))$ is a set of measure zero. Thus the measure of $\Gamma_0$ is zero, i.e. $(\mathcal{U}f)(k) = 0$ a.e., i.e. $f = 0$. Therefore $H$ cannot have any eigenvalues.

Fix $p = (k_1, \ldots, k_{n-1})$. To show that the measure of $\theta(p)$ is zero, we shall use the Fredholm theory of holomorphic families of operators of type (A), [7]. Let $\Omega$ be the following complex domain:

$$(18) \quad \Omega = \{ \mathcal{F} + ir | \mathcal{F} \in (0, 1), r \in \mathbb{R} \}.$$  

For $z \in \Omega$, we define $H_o(p, z \mathcal{E}_n)$ to be the multiplication operator in $l^2_n$ by $(p + z \mathcal{E}_n + E \mathcal{Q})^2$ and:

$$(19) \quad H(p, z \mathcal{E}_n) = H_o(p, z \mathcal{E}_n) + Y.$$  

We shall see that:

(I) $\{ H(p, z \mathcal{E}_n) \}$ is a holomorphic family of type (A) with respect to $z$. (See the terminology in [7]);

(II) the resolvent of $H(p, z \mathcal{E}_n)$ is compact;

(III) the resolvent set of $H(p, z \mathcal{E}_n)$ is not empty.

Under these conditions, Theorem VII.1.10 of [7] says that we have the following alternative:

- either $0 \in \sigma(H(p, z \mathcal{E}_n))$ for each $z \in \Omega$;
- or every compact $\Omega_0$ in $\Omega$ contains only a finite number of points $z$ such that $0 \in \sigma(H(p, z \mathcal{E}_n))$.

We shall show that:

(IV) $0$ belongs to the resolvent set of $H(p, z \mathcal{E}_n)$ for $\text{Im } z$ sufficiently large. Hence the first alternative is excluded, so that the measure of $\theta(p)$ is zero.

The remainder of the paper is devoted to the verification of the properties I to IV of $H(p, z \mathcal{E}_n)$. To simplify the notations we write $H(p, z \mathcal{E}_n)$ for $H(p, z \mathcal{E}_n)$.

**Lemma 5.** - (i) $H_o(p, z)$ is a self-adjoint holomorphic family of type (A) in $\Omega$ with domain $D(H_o(p, z)) = D_0$;
(ii) \( \forall z \in \Omega \), the resolvent of \( H_0(\vec{p}, z) \) is compact;
(iii) 0 belongs to the resolvent set \( \rho(H_0(\vec{p}, z)) \) of \( H_0(\vec{p}, z) \) for all \( z \) with \( \text{Im} \ z \neq 0 \).

**Proof.** — (i) Let \( P_j(j = 1, \ldots, n) \) be the following operator in \( l_n^2 \):

\[
P_j \bar{q}_i = \theta_j \bar{q}_i.
\]

One has:

\[
H_0(\vec{p}, z) = (\vec{p} + E\vec{q} + z\vec{e}_n)^2 = (\vec{p} + E\vec{q})^2 + E^2 z^2 + 2 E^2 z \vec{p}.
\]

and the result is immediate:

(ii) the proof is the same as in Lemma 2 (iv).
(iii) for \( z = \xi + ir \), we have:

\[
\text{Im} (\vec{p} + E\vec{q} + z\vec{e}_n)^2 = 2 E^2 r (\xi + q_n),
\]

which is different from zero if \( r \neq 0 \). Since \( q_n \in \mathbb{Z} \) and \( \xi \in (0, 1) \) it follows that:

\[
\|[(H_0(\vec{p}, z)]^{-1} = \sup_{\xi \in \mathbb{Z}} \|((\vec{p} + E\vec{q} + z\vec{e}_n)^2)^{-1} < \infty,
\]

i.e. \( 0 \in \rho(H_0(\vec{p}, z)) \). \( \blacksquare \)

**Lemma 6.** — (i) \( H(\vec{p}, z) \) is a self-adjoint holomorphic family of type (A) in \( \Omega \) with domain \( D_0 \);
(ii) \( \forall z \in \Omega \) the resolvent of \( H(\vec{p}, z) \) is compact;
(iii) for all \( \vec{p} \in \Gamma^{n-1} \) and \( z \in \Omega \), \( \rho(H(\vec{p}, z)) \) is not empty.

**Proof.** — (i) this follows from Lemmas 5 (i) and 4 (ii);
(iii) it suffices to show:

\[
\lim_{\lambda \to +\infty} \|Y[H_0(\vec{p}, z) - i\lambda]^{-1} \|=0
\]

since then the Neumann series for \( [H(\vec{p}, z) - i\lambda]^{-1} \), i.e.:

\[
[H(\vec{p}, z) - i\lambda]^{-1} = [H_0(\vec{p}, z) - i\lambda]^{-1} \sum_{n=0}^{\infty} \{-Y[H_0(\vec{p}, z) - i\lambda]^{-1}\}^n,
\]

is convergent if \( \lambda \) is sufficiently large. Now, by (12):

\[
\|Y[H_0(\vec{p}, z) - i\lambda]^{-1} \| \leq L^{-n/\alpha} \|v\|_n \sum_{\xi \in \mathbb{Z}} \|((\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda)^{-1}\|^{1/\alpha}.
\]

We have with the notations \( z = \xi + ir, \vec{k} = (\vec{p}, \xi \vec{e}_n) \in \Gamma^{n} \):

\[
|(\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda|^{-2} \leq [(\vec{k} + E\vec{q})^2 - E^2 r^2]^2
\]

\[
+ 4 E^4 r^2 [\xi + q_n - \lambda(2 E^2 r)^{-1}]^2 \leq [(\vec{k} + E\vec{q})^2 - E^2 r^2]^{-2}.
\]
This shows that each term of the sum in (26) converges to zero as $\lambda \to +\infty$, and that the series in (26) is uniformly majorized in $\lambda$ by a convergent series (since $s > n/2$). Therefore (23) is proven.

(If $z$ is such that $(\tilde{E} + q - z)^2 - E^2 r^2 = 0$ for certain $q \in \mathbb{Z}^n$, then there exist $c > 0$ and $\lambda_0 < \infty$ such that $4 E^4 r^2 [x + a_n - \lambda (2 E^2 r)^{-1}]^2 \geq c$ for all these $q$ and for each $\lambda \geq \lambda_0$. For these values of $q$ we can take as majorization in (26) the number $c^{-1}$).

(ii) Now we use the first and the second resolvent equation:

(27) \[ (H(p, z) - \xi)^{-1} = (H(p, z) - \mu)^{-1} + (\xi - \mu) [H(p, z) - \xi]^{-1} (H(p, z) - \mu)^{-1}. \]

(28) \[ (H(p, z) - \mu)^{-1} = (H_0(p, z) - \mu)^{-1} - (H(p, z) - \mu)^{-1} Y [H_0(p, z) - \mu]^{-1}. \]

(27) shows that if $(H(p, z) - \mu)^{-1}$ is compact for $\mu \in \rho(H(p, z))$ then $(H(p, z) - \xi)^{-1}$ is compact for each $\xi \in \rho(H(p, z))$. Since $(H_0(p, z) - \mu)^{-1}$ and $Y [H_0(p, z) - \mu]^{-1}$ are compact if $\mu \in \rho(H_0(p, z))$, by (28) it suffices to show that:

\[ \rho(H_0(p, z)) \cap \rho(H(p, z)) \neq \emptyset. \]

We know from (iii) that there exists a point $\mu_0 \in \rho(H(p, z))$. If $\mu_0 \notin \rho(H_0(p, z))$, there exists a point close to $\mu \in \rho(H_0(p, z)) \cap \rho(H(p, z))$, since:

(\sigma) $\rho(H(p, z))$ is open;

(\beta) $\rho(H_0(p, z))$ consists of isolated eigenvalues only, because the resolvent of $H_0(p, z)$ is compact ([7], Thm. III 6.29).

By Lemma 6 we have verified the properties (I) to (III) of the family $H(p, z)$. It now remains to prove (IV) i.e. $0 \in \rho(H(p, z))$ for some $z = \mathcal{A} + ir$ in $\Omega$. We have seen that $0 \in \rho(H_0(p, z))$ if $r \neq 0$. We shall show that:

(29) \[ \lim_{r \to \infty} \| Y [H_0(p, \mathcal{A} + ir)]^{-1} \| = 0. \]

By using the Neumann series (24) with $\lambda = 0$ and $r$ sufficiently large, (29) implies $0 \in \rho(H(p, z))$ if $r = \text{Im} z$ is sufficiently large.

To obtain (29), we use the inequality (25). By virtue of the first inequality in (26), it suffices to show that:

(30) \[ \lim_{r \to \infty} \sum_{q \in \mathbb{Z}^n} \left\{ \left[ \left( \frac{q + \tilde{E}}{E} \right)^2 - r^2 \right]^2 + 4 r^2 |q_n + \mathcal{A}|^2 \right\}^{-s/2} = 0, \]

which will be done in the next section.

5. Estimation of the series (30)

We now show that (30) holds if $s = 2$ for $n = 2, 3$, $s > n - 2$ for $n \geq 4$ and $\mathcal{A} \in (0, 1)$. We use the following notations:

(31) \[ a = 2r |q_n + \mathcal{A}|, \quad b = (q_n + \mathcal{A})^2 - r^2. \]
We set \( p = E^{-1}(k_1, \ldots, k_{n-1}) \in \Gamma_1^{n-1} \), where \( \Gamma_1^{n-1} = \{ p \in \mathbb{R}^{n-1} \mid 0 \leq p_j < 1 \} \), and:

\[
S(q_n, r) = \sum_{m \in \mathbb{Z}^{n-1}} \{((m + p)^2 + b^2 + a^2)^{-1/2}.
\]

(30) is then equivalent to:

\[
\lim_{r \to \infty} \sum_{q_n \in \mathbb{Z}} S(q_n, r) = 0.
\]

To prove (33), we first give a preliminary estimate in Lemma 7.

**Lemma 7.** Let \( \delta > 0 \), \( c > 0 \) and \( R > 0 \). Then:

\[
\varepsilon = \inf_{r \geq R} \inf_{a \geq br} \frac{(z^2 + b)^2 + a^2}{(t^2 + b)^2 + a^2} > 0.
\]

**Proof.** Setting \( \alpha = a/r \), \( \beta = br^{-2} \), \( \sigma = z/r \), \( \tau = t/r \) and \( \Omega_\sigma = \{ (\alpha, \beta, \sigma, \tau) \mid \alpha \geq \delta, \beta \geq -1, \sigma \geq 0, \tau \geq 0, |\sigma - \tau| \leq cr^{-1} \} \), we see that (34) is equivalent to:

\[
\varepsilon = \inf_{r \geq R} \inf_{\alpha \geq b/r} \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} > 0.
\]

The quotient on the r.h.s. of (35) is \( \geq 1 \) if \( |\tau^2 + \beta| \leq |\sigma^2 + \beta| \). Hence the infimum is obtained by taking \( |\tau^2 + \beta| \geq |\sigma^2 + \beta| \). Under this restriction we have:

\[
\frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \geq \max \left[ \frac{(\sigma^2 + \beta)^2}{(\tau^2 + \beta)^2}, \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + 2(\alpha/r)^2} \right].
\]

Also notice the following inequalities, valid on each \( \Omega_\sigma \), with \( r \geq R \):

\[
\tau^2 + \beta = (|\tau - \sigma| + \sigma)^2 + \beta \leq 2(\tau - \sigma)^2 + 2 \sigma^2 + \beta = 2(\sigma^2 + \beta) - \beta + 2(\tau - \sigma)^2 \leq 2(\sigma^2 + \beta) + 1 + 2c^2 R^{-2}.
\]

(38) implies that:

\[
(\tau^2 + \beta)^2 \leq 2(\sigma^2 + \beta)^2 + 2(\sigma + \tau)^2 c^2 R^{-2}.
\]

We denote by \( \varepsilon_+ \) and \( \varepsilon_- \) the infimum in (35) under the restriction \( \sigma^2 + \beta \geq 1 \) and \( \sigma^2 + \beta \in [-1, +1] \) respectively. It suffices to show that \( \varepsilon_+ > 0 \) and \( \varepsilon_- > 0 \). In the first case (i.e. for \( \sigma^2 + \beta \geq 1 \)), we use the first expression on the r.h.s. of (36) and the inequality (37).

Setting \( x = \sigma^2 + \beta \), we see that:

\[
\varepsilon_+ = \inf_{x \geq 1} \frac{x^2}{(2x + 1 + 2c^2 R^{-2})^2} > 0.
\]
In the second case (i.e., for $\sigma^2 + \beta \in \{-1, +1\}$), we have $\sigma^2 \leq 2$, hence $\sigma + \tau \leq 2 \sqrt{2} + c R^{-2} \equiv \eta$. After inserting this into (39) and using the second expression on the r.h.s. of (36), one obtains by setting $y = (\sigma^2 + \beta)^2$:

$$\varepsilon_- = \inf_{r \geq R} \inf_{0 \leq y \leq 1} \inf_{\eta \geq \delta} \frac{y + (\alpha/r)^2}{(2 y + 2 \eta^2 c^2 r^{-2} + 2(\alpha/r)^2}$$

$$= \inf_{r \geq R} \inf_{\eta \geq \delta} \frac{(\eta/r)^2}{(2 \eta^2 c^2 r^{-2} + 2(\alpha/r)^2} = \frac{\delta^2}{2 \eta^2 c^2 + 2 \delta^2} > 0. \ \Box$$

**Proof of (33).** Let $\tilde{m} \in \mathbb{Z}^{n-1}$ and $\Gamma(\tilde{m})$ be the cube:

$$\Gamma(\tilde{m}) = \{ x \in \mathbb{R}^{n-1} | x = \tilde{p} + \tilde{m} + y, y \in \Gamma^1 \}.$$  

We have $\Gamma(\tilde{m}) \cap \Gamma(\tilde{m}') = \emptyset$ if $\tilde{m} \neq \tilde{m}'$ and:

$$\mathbb{R}^{n-1} = \bigcup_{\tilde{m} \in \mathbb{Z}^{n-1}} \Gamma(\tilde{m}).$$

Let $c = \sqrt{n-1}$. Then for each $\tilde{x} \in \Gamma(\tilde{m})$ and each $\tilde{m} \in \mathbb{Z}^{n-1}$:

$$||\tilde{m} + \tilde{p}|| - ||\tilde{x}|| \leq c.$$

Let $\delta = 1/2 \min (\tilde{x}', 1 - \tilde{x}')$. By assumption $\delta > 0$; since $a \geq \delta r$ and $b \geq -r^2$, Lemma 7 implies the existence of a number $\varepsilon > 0$ such that, for each $\tilde{m} \in \mathbb{Z}^{n-1}$, each $x \in \Gamma(\tilde{m})$, each $a \geq \delta r$ and $b \geq -r^2$ and all $r \geq R$:

$$[(\tilde{m} + \tilde{p})^2 + b^2] + a^2 \geq \varepsilon [(\tilde{x}^2 + b^2) + a^2].$$

Thus:

$$S(q_m, r) = \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} [(\tilde{m} + \tilde{p})^2 + b^2] + a^2 \geq \varepsilon [(\tilde{x}^2 + b^2) + a^2].$$

Thus:

$$S(q_m, r) = \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} \int_{\Gamma(\tilde{m})} dx \left([(\tilde{m} + \tilde{p})^2 + b^2] + a^2 \right]^{-s/2}$$

$$\leq \varepsilon^{-1} \sum_{\tilde{m} \in \mathbb{Z}^{n-1}} \int_{\Gamma(\tilde{m})} dx \left([(\tilde{x}^2 + b^2) + a^2 \right]^{-s/2}$$

$$= \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} dx \left[(\tilde{x}^2 + b^2) + a^2 \right]^{-s/2}$$

$$= \frac{1}{2} \varepsilon^{-1} w_{n-1} \int_0^\infty y^{n-3/2} \left[(y + b)^2 + a^2 \right]^{-s/2} dy,$$

where we have introduced spherical polar coordinates, $y = ||\tilde{x}||^2$ and $w_{n-1}$ denotes the area of the unit sphere in $\mathbb{R}^{n-1}$.  

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To estimate the integral in (43), we distinguish the two cases $b \geq 0$ and $b < 0$. For $b \geq 0$, we have $\{(y+b)^2 + a^2\}^{-s/2} \leq \{y^2 + a^2 + b^2\}^{-s/2}$, and (43) leads to:

$$S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} (a^2 + b^2)^{-s/2} \int_0^\infty z^{(n-3)/2} (z^2 + 1)^{-(s/2)} dz.$$  

Notice that the integral in this expression is convergent since $s > n/2$. By observing that:

$$(44) \quad a^2 + b^2 = [(q + \mathcal{A})^2 + r^2]^2,$$

we obtain:

$$(45) \quad \sum_{|q_n + \mathcal{A}| \geq r} S(q_n, r) \leq \text{Cte} \sum_{|q_n + \mathcal{A}| \geq r} |q_n + \mathcal{A}|^{-2s+n-1}.$$  

The hypothesis $s > n/2$ implies that the last series is convergent so that this term tends to zero as $r \to \infty$.

We now turn to the case $b < 0$. We set $z = (y+b)/a$. (43) then gives:

$$(46) \quad S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} a^{-s+1} \int_{b/a}^{+\infty} (az-b)^{n-3/2} (1+z^2)^{-s/2} dz.$$  

If $n \geq 3$, this leads to:

$$S(q_n, r) \leq c_1 a^{-s+1} \int_{-\infty}^{+\infty} |az|^{|n-3)/2|} |b|^{(n-3)/2} (1+z^2)^{-s/2} dz \leq c_2 a^{-s+1} (a^2 + b^2)^{(n-3)/4}.$$  

Using (47), (44) and (31), we obtain in this case that:

$$\sum_{|q_n + \mathcal{A}| < r} S(q_n, r) \leq c_4 r^{-s+1} r^{n-3} \sum_{|q_n + \mathcal{A}| < r} |q_n + \mathcal{A}|^{-s+1} = \mathcal{O} (r^{-s+n-2} \log r),$$  

since $s \geq 2$. Under the hypothesis $s > n-2$, this converges to zero as $r \to \infty$.

Finally, if $n=2$, one may bound the integral in (46) by a constant which is independent of $a$ and $b$ on the set $\{a \geq a_0 > 0, b < 0\}$; this is easily achieved by splitting the domain of integration into $\{z \mid az - b \leq 1\} \cup \{z \mid az - b > 1\}$. Thus:

$$S(q_n, r) \leq c_5 r^{-s+1} |q_n + \mathcal{A}|^{-s+1}, \quad \forall q_n, \forall r \geq r_0.$$  

For any $s > 3/2$, this implies that:

$$\lim_{r \to \infty} \sum_{|q_n + \mathcal{A}| < r} S(q_n, r) = 0.$$  

**Remark 3.** One sees from the preceding proof that, for $n=3$, the limit in (33) is zero under the weaker hypothesis that $s > 3/2$. By using a modified resolvent equation, one obtains the result of Theorem 1 for $s > 3/2$. The case $s=2$, $n=3$ was first treated by Thomas...
Similarly, for \( n=2 \), a more careful estimate of the integral in (46) shows that it suffices to require \( s > 1 \).

**Remark 4.** — Theorem 3 remains true if the condition of ortho-periodicity of \( v \) is replaced by the weaker condition of periodicity. Indeed, the estimation of the series given in section 5, may be applied if, instead of \( \tilde{\alpha}_i, \tilde{\alpha}_j = \delta_{ij} \), one requires only that \( \tilde{\alpha}_i, \tilde{\alpha}_n = \delta_{im} \) (i.e. the vector \( \tilde{\alpha}_n \) is orthogonal to the hyperplane spanned by \( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1} \)). Clearly the direction \( \tilde{\alpha}_n \) is distinguished in our estimation. A similar result for an arbitrary periodic lattice is given in Theorem XIII.100 of [11], under a more restrictive assumption on the local behaviour of the function \( v(x) \) than that of Theorem 3.

**Remark 5.** — We also have the following result which generalizes Theorem 1:

**THEOREM V.** — Let \( v \in L^s_{\text{loc}}(\mathbb{R}^n \setminus N) \), where \( s \) satisfies \( s = 2 \) if \( n = 1, 2, 3 \) and \( s > n - 2 \) if \( n \geq 4 \), and where \( N \) is a closed set of measure zero. Let \( H \) be a self-adjoint extension of \( \tilde{H} \), \( \text{D}(\tilde{H}) = C^0_c(\mathbb{R}^n \setminus N) \). Suppose that \( f \in L^2(\mathbb{R}^n) \) satisfies \( \tilde{H}f = \gamma f \) for some \( \gamma \in \mathbb{R} \) and \( \text{E}(A) f = f \) for some compact subset of \( \mathbb{R}^n \setminus N \) (i.e. \( f \) is an eigenvector of \( H \) having compact support in \( \mathbb{R}^n \setminus N \)). Then \( f = 0 \).

**Proof.** — One has \( \chi_A(.) \ v(.) \in L^s(\mathbb{R}^n) \). Let \( C \) be a cube in \( \mathbb{R}^n \) such that \( A \subseteq C \). Define \( w \) by:

\[
w(x + \sum q_i \tilde{\alpha}_i) = \chi_A(x) v(x), \quad x \in C,
\]

\( w \) is ortho-periodic and in \( L^s_{\text{loc}}(\mathbb{R}^n) \). Since \( (H_0 + w) f = \chi f \), one has \( f = 0 \) by Theorem 3.

**Remark 6.** — The hypothesis "\( \Sigma \) bounded" in Theorem 1(b) is essential. Assume for example that \( v \) is such that \( H_0 + v \) has pure point spectrum (e.g. \( v(x) \to +\infty \) as \( |x| \to \infty \)). Take \( \Sigma = \mathbb{R} \). Then:

\[
\text{F}(\Sigma) = \mathcal{H} \quad \text{and} \quad \text{E}(A) \mathcal{H} \cap \text{F}(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = \text{E}(A) \mathcal{H}.
\]

Since \( \text{E}(A) \mathcal{H} \neq \{0\} \) if \( A \) has positive measure, it is clear that one cannot have \( \text{E}(A) \mathcal{H} \cap \text{F}(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = \{0\} \) in this case.

**Remark 7.** — By combining our Theorem 1 with Proposition 4 of [2], one may also prove that \( \text{E}(A) \mathcal{H} \cap \text{F}(\Sigma) \mathcal{H} = \{0\} \) under assumptions of Theorem 1(b).

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