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On the point spectrum of Schrödinger operators


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1. Introduction

This paper is an extension of a work [2] on the spectral analysis of partial differential operators of Schrödinger type. The problem was the following: Let $A$ be a compact subset of $\mathbb{R}^n$, $\Sigma$ a finite interval in $\mathbb{R}$ and $H$ a self-adjoint elliptic differential operator in the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. We define $F(\Sigma)$ to be the spectral projection of $H$ associated with the interval $\Sigma$ and $E(A)$ the multiplication operator by the characteristic function $\chi_A$ of $A$. Do there exist vectors in $L^2(\mathbb{R}^n)$ which are contained both in the range $E(A)\mathcal{H}$ of $E(A)$ and in $F(\Sigma)\mathcal{H}$?

It turns out that the closed subspace $\mathcal{H}_p(H)$ generated by the set of eigenvectors of $H$ plays a different role from the subspace $\mathcal{H}_c(H) = \mathcal{H}_p(H) \upharpoonright$ associated with the continuous spectrum of $H$. Notice that it is shown in [2], under regularity and integrability conditions on the coefficients of the differential operator, that there do not exist vectors of $\mathcal{H}_c(H)$ which belong both to $E(A)\mathcal{H}$ and to $F(\Sigma)\mathcal{H}$. On the other hand, to prove the non-existence of vectors in $\mathcal{H}_p(H)$ belonging to $E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H}$, we used an unique continuation theorem for solutions of the differential equation associated with $H$. Now, if for example $H = -\Delta + V$, where $V$ is the multiplication operator by a real function $v(x)$, the known results on unique continuation require a condition $L^s(\mathbb{R}^n \setminus N)$ on $v$, where $N$ is a closed set of measure zero such that $\mathbb{R}^n \setminus N$ is connected ([3], [5]).

In the present paper, we propose to show that:

\begin{equation}
\mathcal{H}_p(H) \cap E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H} = \{0\},
\end{equation}

by imposing only an integrability condition on the function $v$. More precisely, we will prove (1) under the hypothesis that $v \in L^s_{\text{loc}}(\mathbb{R}^n)$ with $s = 2$ if $n = 1, 2, 3$ and $s > n - 2$ if $n \geq 4$.

This result shows that, under the above conditions on $v$, the operator $-\Delta + v$ has no eigenvector with compact support. This is essentially the content of our Theorem 1 in paragraph 2. (In the case $n = 1$, one obtains ordinary differential operators for which results of this type have been known for a long time [9]).
This result is also interesting from the point of view of "non-existence of positive eigenvalues of the operator $H". In the literature (for example [2], [12]) the non-existence of positive eigenvalues is obtained in two steps:

(i) under suitable decay conditions at infinity on the function $v$, it is shown that all eigenfunctions $f$ associated with a strictly positive eigenvalue of $H$ have compact support;

(ii) then one imposes suitable local conditions on $v$ (e.g., $v \in L^\infty_\text{loc}(\mathbb{R}^n \setminus N)$ in order to apply the unique continuation theorem, which then leads to $f \equiv 0$. It turns out that the non-existence of positive eigenvalues is also obtained by assuming in (ii) as a local condition that $v \in L^s_\text{loc}(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s>n-2$ if $n \geq 4$ (Thm. 2).

Finally our method implies also the spectral continuity of a class of Schrödinger operators with periodic potentials $v(x)$.

The organization of the paper is as follows: first we give the principal results and deduce Theorems 1 and 2 from Theorem 3 in section 2, and we introduce a direct integral representation of Schrödinger operators in section 3. This representation will be used in section 4 for proving Theorem 3. The principal estimate of the proof is the subject of the last section 5.

2. Statements of the results

Let $v: \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We always suppose that:

(2) $v \in L^s_\text{loc}(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$; $s>n-2$ if $n \geq 4$.

Notice that $s>n-2$ in all cases.

The function $v$ will be called periodic if there exist $n$ linearly independent vectors $\tilde{a}_1, \ldots, \tilde{a}_n \in \mathbb{R}^n$ such that $v(x+\tilde{a}_j)=v(x)$ for all $x \in \mathbb{R}^n$. A periodic function will be called ortho-periodic if:

(3) $\tilde{a}_j \cdot \tilde{a}_k = L^2 \delta_{jk},$

with $L>0$, i.e. if the vectors of the form $\sum_{i=1}^n \alpha_i \tilde{a}_i$, $0 \leq \alpha_i < 1$, define a cube $C^n$ with side $L$.

We denote by $\hat{H}$ the symmetric operator:

(4) $\hat{H} = -\Delta + v(\vec{x}),$

with domain $D(\hat{H})=C^\infty_0(\mathbb{R}^n)$ and by $H_0$ the unique self-adjoint extension of $\hat{H}_0 = -\Delta$, $D(\hat{H}_0)=C^\infty_0(\mathbb{R}^n)$. Let $H$ a self-adjoint extension of $\hat{H}$. We have the following lemma:

LEMMA 1. — Assume that (2) and one of the following conditions are satisfied:

(i) $v$ is periodic;
(ii) $v \in L^\infty(\mathbb{C}B_R)$ where $B_R = \{ \vec{x} \in \mathbb{R}^n | |x| \leq R \}$ and $\mathbb{C}B_R$ denotes the complement of $B_R$.

Then:

(a) $v$ is $H_0$-bounded with $H_0$-bound 0;
(b) $\hat{H}$ is essentially self-adjoint;
(c) $D(H) = D(H_0)$, where $H$ is the unique self-adjoint extension of $H$.

**Proof.** — (b) and (c) follow from (a) by using the Kato-Rellich Theorem ([7], Chapt. 5.4.1). Under hypothesis (i), (a) follows from Theorem XIII. 96 of [11], whereas under the assumption (ii), (a) can be proved by the method used in the proof of Lemma 3 in [10]. Both cases are treated in [4].

We now state our principal results. In Theorem 2 we choose as conditions on the potential $v$ at infinity those used in [4].

**Theorem 1.** — Let $v \in L^4_\infty(\mathbb{R}^n)$ with $s$ satisfying (2) and let $H$ be a self-adjoint extension of $H$:
(a) suppose that $f \in L^2(\mathbb{R}^n)$ satisfies $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $E(A) f = f$ for some compact subset $A$ of $\mathbb{R}^n$, (i.e. $f$ is an eigenvector of $H$ with compact support in $\mathbb{R}^n$). Then $f = 0$;
(b) for each compact subset $A$ of $\mathbb{R}^n$ and each bounded interval $\Sigma$, one has:

$$\mathcal{H}_\Sigma(\mathcal{H}) \cap E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\}.$$

**Theorem 2.** — Suppose that:
(i) $v \in L^4(B_R)$ with $s$ satisfying (2) for some $R < \infty$;
(ii) $v = v_1 + v_2$ such that:

(a) $v_1, v_2 \in L^\infty(\mathbb{R}^n)$,

(b) $|x|v_1(x) \to 0$ as $|x| \to \infty$,

(γ) $r \to v_2(r, \cdot)$

is differentiable as a function from $(\mathbb{R}, \infty)$ to $L^{\infty}(S^{n-1})$, and $\limsup_{r \to \infty} \partial v_2/\partial r \leq 0$. ($S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$.)

Then $H = H_0 + V$ has no eigenvalues in $(0, \infty)$.

**Theorem 3.** — Let $v$ be ortho-periodic and $v \in L^4_\infty(\mathbb{R}^n)$ with $s$ satisfying (2). Then the spectrum of $H = H_0 + V$ is purely continuous.

**Remark 1.** — By following the proof of Theorem XIII. 100 in [11], it is possible to show that the operator $H$ in Theorem 3 is absolutely continuous. Other comments on Theorem 3 will be made at the end of this paper.

**Remark 2.** — Contrarily to [2], where the operator $\hat{H}$ was defined by:

$$\hat{H} = \sum_{j, k=1}^{n} a_{jk} \left(-i \frac{\partial}{\partial x_j} + b_j(\vec{x})\right) \left(-i \frac{\partial}{\partial x_k} + b_k(\vec{x})\right) + V(\vec{x}),$$

we assume here that the vector potential $\vec{b} = \{b_k\}$ is equal to zero. It is possible to generalize Theorem 1 to the case where $\vec{b} \neq 0$.

Theorem 2 follows from results of [11] and [6], and from Theorem 1 as indicated in the introduction. (If $Hf = \lambda f$ with $\lambda > 0$, then $f$ has compact support by Theorem XIII. 58 of...
Theorem 1 (a) is deduced from Theorem 3: By the proof of Proposition 4 of [2], the vector \( f \) belongs to \( D(H_0) \cap D(V) \) and \( Hf = H_0 f + VE(A) f \). Let \( w \) be an ortho-periodic function such that \( w \in L^1_{loc}(\mathbb{R}^n) \) and \( w(x) = v(x) \) for \( x \in A \). If \( H_1 \) denotes the periodic Schrödinger operator \( H_1 = H_0 + W \) then \( H_1 f = Hf = \lambda f \). Therefore we deduce from Theorem 3 that \( \lambda f = 0 \).

To show Theorem 1 (b), let \( S = E(A) \cap F(\Sigma) \) (the orthogonal projection with range \( E(A) \cap F(\Sigma) \)) and suppose that \( f \in \mathcal{H}(H) \) satisfies \( Sf = f \). \( f \) is a linear combination of eigenvectors of \( H \), i.e. \( f = \sum \lambda_k g_k \), where \( Hg_k = \lambda_k g_k \) with \( \lambda_k \in \Sigma \). It follows that:

\[
Sf = f = \sum \lambda_k g_k = \sum \lambda_k Sg_k.
\]

Now, by Proposition 2 of [2], \( S \) commutes with \( H \); in particular \( HSg_k = SHg_k = \lambda_k Sg_k \). This implies that each \( Sg_k \) is an eigenvector of \( H \) of compact support in \( A \), hence \( Sg_k = 0 \) by the part (a) of Theorem 1. We deduce from this that \( f = \sum \lambda_k Sg_k = 0 \). The condition "\( \Sigma \) bounded" is fundamental: we can choose a potential \( V \) such that \( H_\Sigma(H) = \mathcal{H} \), i.e. such that the eigenvectors of \( \mathcal{H} \) generate \( \mathcal{H} \). In this case, we have:

\[
\mathcal{H}_\Sigma(H) \cap E(A) = E(A) \mathcal{H} \neq \{0\}.
\]

### 3. Reduction of the translation group of the lattice

In this part, let \( v \) be an ortho-periodic potential. In a natural way, this implies a decomposition of the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n) \) and of the operators \( H \) and \( H_0 \) into direct integrals. This decomposition will be used in the next part for the proof of Theorem 3.

The potential \( v \) satisfies \( v(x + \alpha_i) = v(x) \) where \( \alpha_1, \ldots, \alpha_n \) are as in (3). The points of the form \( \tilde{z} = \sum_{i=1}^{n} q_i \alpha_i, \tilde{q} = \{q_i\} \in \mathbb{Z}^n \), form a cubic lattice in \( \mathbb{R}^n \) which is invariant under the translations:

\[
\tilde{z} \mapsto \tilde{z} + \sum_{i} q_i \tilde{\alpha}_i, \quad \tilde{q} \in \mathbb{Z}^n.
\]

In \( L^2(\mathbb{R}^n) \), we consider the unitary representation \( U(\tilde{q}) \) of the additive group \( \mathbb{Z}^n \) given by:

\[
(U(\tilde{q}) f)(\tilde{x}) = f(\tilde{x} - \sum_{i} q_i \tilde{\alpha}_i) = f(\tilde{x} - L \tilde{q}),
\]

where we have written \( \sum_{i} q_i \tilde{\alpha}_i = L \tilde{q} \), assuming that the directions of the \( \tilde{\alpha}_i \) coincide with Cartesian coordinate system.

We also introduce the reciprocal lattice which is the set of points of the following form:

\[
\tilde{z} = \sum_{i=1}^{n} q_i \tilde{e}_i, \quad \tilde{q} \in \mathbb{Z}^n,
\]
where the vectors $\vec{e}_1, \ldots, \vec{e}_n$ are defined by:

$$
\vec{e}_i, \vec{a}_k = 2\pi \delta_{ik}.
$$

We may write $\vec{z} = E \vec{q}$, with $E = 2\pi L^{-1}$. Let again:

$$
\Gamma^n = \left\{ k \in \mathbb{R}^n \mid k = \sum_{i=1}^n \lambda_i e_i, \ 0 \leq \lambda_i < 1 \right\}.
$$

Consider the Hilbert space $\mathcal{H}$ of square-integrable functions $f : \Gamma^n \to l^2_{\mathbb{Z}^n}$:

$$
\mathcal{H} = L^2(\Gamma^n; l^2_{\mathbb{Z}^n}).
$$

We write $f(\vec{k})$ for the component $q(\vec{k} \in \mathbb{Z}^n)$ of $f$ at the point $\vec{k} \in \mathbb{Z}^n$. Thus, we have:

$$
\|f\|_2^2 = \int_{\Gamma^n} dk \sum_{\vec{k} \in \mathbb{Z}^n} |f(\vec{k})|^2.
$$

Now, let $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ be the operator defined by:

$$
(\mathcal{U} f)(\vec{k}) = f(\vec{k} + E \vec{q}),
$$

where $f$ is the Fourier transform of the function $f$:

$$
\hat{f}(\xi) = (2\pi)^{-n/2} \lim_{R \to \infty} \int_{\mathbb{R}^n} dx \exp(-ix \cdot \xi) f(x).
$$

It follows from Plancherel's Theorem that the operator $\mathcal{U}$ is unitary, and its inverse is given by:

$$
\mathcal{F} [\mathcal{U}^{-1} \{ f(\cdot) \}](\xi) = f(\vec{k}_q),
$$

where $\vec{q} \in \mathbb{Z}^n$ and $\vec{k} \in \Gamma^n$ are determined by $\vec{k} + E \vec{q} = \vec{\xi}$. If $\vec{m} \in \mathbb{Z}^n$, one has:

$$
[\mathcal{U} \mathcal{U}(\vec{m}) f](\vec{k}) = \exp(-iL \vec{k} \cdot \vec{m})(\mathcal{U} f)(\vec{k}),
$$

i.e. $\mathcal{U} \mathcal{U}(\vec{m}) \mathcal{U}^{-1}$ is diagonalizable in $\mathcal{H}$ (i.e. a multiplication operator by a function of $\vec{k}$). As the functions $\{ \exp(i \vec{k} \cdot \vec{m}) \}_{\vec{m} \in \mathbb{Z}^n}$ form a basis of $L^2(\Gamma^n)$, each bounded diagonalizable operator is a function of $\{ \mathcal{U} \mathcal{U}(\vec{m}) \mathcal{U}^{-1} \}$. As $H_0$, $V$ and $H$ commute with every $\mathcal{U}(\vec{m})$, these operators commute with each diagonalizable operator, i.e. $\mathcal{U} H_0 \mathcal{U}^{-1}$, $\mathcal{U} V \mathcal{U}^{-1}$ and $\mathcal{U} H \mathcal{U}^{-1}$ are decomposable in $L^2(\Gamma^n; l^2_{\mathbb{Z}^n})$. Therefore there exist in $l^2_{\mathbb{Z}^n}$ measurable families of self-adjoint operators $H_0(\vec{k})$, $V(\vec{k})$ and $H(\vec{k})(\vec{k} \in \Gamma^n)$ such that, for $f \in D(H_0)$:

$$
\begin{align*}
(\mathcal{U} H_0 f)(\vec{k}) &= H_0(\vec{k}) f(\vec{k}), \\
(\mathcal{U} V f)(\vec{k}) &= V(\vec{k}) f(\vec{k}), \\
(\mathcal{U} H f)(\vec{k}) &= H(\vec{k}) f(\vec{k}).
\end{align*}
$$

Now let us give the explicit form and the properties of these three families of operators.
LEMMA 2. — (i) $H_0(k)$ is the self-adjoint multiplication operator in $l^2_n$ by $\varphi_\xi(q) = (k + E \bar{q})^2$ : If $g = \{g_\xi\} \in l^2_n$, then:

$$ (H_0(k)g)_\xi = (k + E \bar{q})^2 g_\xi. $$

(ii) the domain of $D(H_0(k))$ is independent of $k$ and is given by:

$$ D(H_0(k)) = D_0 = \{ g \in l^2_n \mid \sum_{\xi \in \mathbb{Z}^n} |\bar{q}^2 g_\xi|^2 < \infty \}; $$

(iii) the resolvent $(H_0(k) - \mu)^{-1}$ of $H_0(k)$ is a compact operator for all $\mu \notin \sigma(H_0(k))$, where $\sigma(H_0(k))$ is the spectrum of $H_0(k)$.

Proof. — (i) and (ii) are obvious, since:

$$ (H_0 f)(\xi) = \xi^2 f(\xi). $$

(iii) The resolvent $(H_0(k) - \mu)^{-1}$ is the multiplication operator by:

$$ \psi(q) = [(k + E \bar{q})^2 - \mu]^{-1}. $$

Let $\chi_M$ be the characteristic function of the set $\{ q \in \mathbb{Z}^n \mid |\bar{q}|^2 \leq M \}$ and $D_M$ the multiplication operator by $\psi(q) \chi_M(q)$. $D_M$ is a compact (even nuclear) operator, and:

$$ \| (H_0(k) - \mu)^{-1} - D_M \| = \sup_{\bar{q} > M} [(k + E \bar{q})^2 - \mu]^{-1} \to 0, $$

as $M \to \infty$. Thus $(H_0(k) - \mu)^{-1}$ is compact as the uniform limit of the sequence $\{ D_M \}$ of compact operators. ■

Let us denote by $\{ \tilde{v}_q \}_{q \in \mathbb{Z}^n}$ the Fourier coefficients of the periodic function $v$:

$$ \tilde{v}_q = L^{-n/2} \int_\mathbb{C} \exp (-i E \cdot \bar{q} \cdot \bar{x}) v(x). $$

Notice that $v \in L^p(\mathbb{C}^n)$ for all $p \in [1, s]$. To establish the relation between the Fourier coefficients of $v$ and the operator $V(k)$ we need the following result:

LEMMA 3. — Given $\varphi, \psi : \mathbb{Z}^n \to \mathbb{C}$, we define an operator $A_{\varphi \psi} : l^2_n \to l^2_n$ as follows:

$$ (A_{\varphi \psi} g)_\xi = \sum_{\tilde{m} \in \mathbb{Z}^n} \varphi(\tilde{m}) \psi(q - \tilde{m}) g_{\tilde{m}}. $$

Assume that $2 \leq p < \infty$, $\psi \in l^p(\mathbb{Z}^n)$ and let $\{ \varphi(q) \}$ be the Fourier coefficients of a function $\Phi$ belonging to $L^p(\mathbb{C}^n)$. Then $A_{\varphi \psi}$ is a compact operator and one has:

$$ \| A_{\varphi \psi} \| \leq L^{-(n/2) - (n/p)} \| \Phi \|_{L^p(\mathbb{C}^n)} \| \eta \|_{L^r(\mathbb{C}^n)}. $$

Proof. — For $g = \{g_\xi\} \in l^p_n$, define $\psi g = \{ \psi(q) g_\xi \}$. By the Hölder inequality, $\psi g \in l^r_n$ with $r^{-1} = (1/2) + p^{-1}$, i.e. $1 \leq r < 2$, and:

$$ \| \psi g \|_r \leq \| \psi \|_p \| g \|_2. $$
Let:

\[ \gamma(x) = L^{-n/2} \sum_{q \in \mathbb{Z}^n} \exp \left( i \mathbf{E} \cdot \mathbf{x} \right) \psi(q) g_q, \quad x \in \mathbb{C}^n. \]

By the Hausdorff-Young inequality [8], \( \gamma \in L'(\mathbb{C}^n) \) with \( (r')^{-1} = 1 - r^{-1} = 1/2 - p^{-1} \) and:

\[ \| \gamma \|_r \leq L^{(n/r')-(n/2)} \| \psi \|_r \leq L^{(n/r')-(n/2)} \| \psi \|_p \| g \|_2. \]

Since \( 1/2 = p^{-1} + (r')^{-1} \) and \( \Phi \in L^p(\mathbb{C}^n) \), the Hölder inequality implies that \( \Phi \gamma \in L^2(\mathbb{C}^n) \) and:

\[ \| \Phi \gamma \|_2 \leq \| \Phi \|_p \| \gamma \|_r \leq L^{(n/r')-(n/2)} \| \Phi \|_p \| \psi \|_p \| g \|_2. \]

Now:

\[ (A_{\psi \Phi} g)_q = \int_{\mathbb{C}^n} dx \exp \left( -i \mathbf{E} \cdot \mathbf{x} \right) \Phi(\mathbf{x}) \gamma(\mathbf{x}), \]

and by Plancherel's theorem we have:

\[ \| A_{\psi \Phi} g \|_2 = L^{n/2} \| \Phi \gamma \|_2 \leq L^{n/r'} \| \Phi \|_p \| \psi \|_p \| g \|_2. \]

This shows that \( A_{\psi \Phi} \) is defined everywhere with the bound (12):

(b) Let \( D_M \) be the multiplication operator by \( \psi_M(q) = \psi(q) \chi_M(q) \) (see the proof of Lemma 2). By (a), \( A_{\psi \psi_M} \) is bounded, and \( A_{\psi \psi_M} \) is non-zero only on a subspace of finite dimension. Therefore \( A_{\psi \psi_M} \) is nuclear. By using (12) we obtain:

\[ \| A_{\psi \Phi} - A_{\psi \psi_M} \| \leq L^{n/2} \| \Phi \gamma \|_2 \leq L^{(n/2)-(n/p)} \| \Phi \|_p \| (1-\chi_M) \psi \|_p. \]

Since \( \psi \in l^p_n, \| (1-\chi_M) \psi \|_p \to 0 \) as \( M \to \infty \). This proves the compactness of \( A_{\psi \Phi} \).

**Lemma 4.** Let \( Y \) be the operator in \( l^2_n \) defined by:

\[ (Y g)_q = L^{-n/2} \sum_{m \in \mathbb{Z}^n} \hat{v}_m g_{\mathbf{q}+\mathbf{m}}. \]

Then:

(i) \( D_0 \subseteq D(Y) \) and \( Y \) is symmetric on \( D_0 \);

(ii) \( Y \) is relatively compact with respect to \( H_0(\mathbf{k}) \);

(iii) \( V(\mathbf{k}) = Y \) on \( D_0 \), for all \( \mathbf{k} \in \Gamma_n \) (in particular \( V(\mathbf{k}) \) is independent of \( k \));

(iv) \( H(\mathbf{k}) = H_0(\mathbf{k}) + Y \) and \( D(H(\mathbf{k})) = D_0 \).

**Proof.** (i) If \( g \in D_0 \), then \( g = [H(\mathbf{k}) + 1]^{-1} \) for some \( h \in l^2_n \). (15) shows that \( \| Y g \|_2 < \infty \), therefore \( D_0 \subseteq D(Y) \). By using \( v_{\mathbf{q}} = v_{\mathbf{q}+\mathbf{m}} \), one obtains easily that \( (f, Y g) = (f, Y g) \) for \( f, g \in D_0 \);

(ii) \( Y(H_0(\mathbf{k}) + 1)^{-1} \) is of the form \( A_{\psi \Phi} \), with \( \Phi(x) = L^{-n/2} \psi(x) \) and \( \psi(q) = [k + E q]^2 + 1]^{-1} \). Notice that \( \psi \in l^p_n \) for each \( p > n/2 \). As \( v \in L^2(\mathbb{C}^n) \) for \( s = 2 \) if \( n = 2, 3 \) and \( s > n/2 \) if \( n \geq 4 \), Lemma 3 implies that \( Y(H_0(\mathbf{k}) + 1)^{-1} \) is compact;
(iii) this can be verified by calculating the Fourier transform of $Vf$;
(iv) by (i) and (ii), $H_0(\hat{k})$ is self-adjoint. $H(\hat{k}) = H_0(\hat{k}) + Y$ follows from (iii) and Lemmas 1 and 2.

4. Proof of Theorem 3

Let $f$ be an eigenvector of $H$, i.e. $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$. By defining $v'(x) = v(x) - \lambda$ and $H' = H_0 + V'$, we have $H'f = 0$. Since $V'$ satisfies also the hypothesis (2), it is possible to assume without loss of generality that $\lambda = 0$.

Let $\Gamma_0 = \{ \hat{k} \in \Gamma \mid (\mathcal{U}f)(\hat{k}) \neq 0 \text{ in } l^2 \}$. $\Gamma_0$ is measurable. Since $H(\hat{k})(\mathcal{U}f)(\hat{k}) = 0$, $H(\hat{k})$ must have the eigenvalue 0 for almost all the $\hat{k} \in \Gamma_0$. We will show that, for all $p \in (k, \ldots, k_{n-1}, 0) \in \mathbb{R}^{n-1}$ the set $\theta(p)$ of the points $k \in (0, E)$ such that $0 \in \sigma(H(p + k_0^{-1}e_n))$ is a set of measure zero. Thus the measure of $\Gamma_0$ is zero, i.e. $(\mathcal{U}f)(\hat{k}) = 0$ a.e., i.e. $f = 0$. Therefore $H$ cannot have any eigenvalues.

Fix $\bar{p} = (\bar{k}_1, \ldots, \bar{k}_{n-1})$. To show that the measure of $\theta(\bar{p})$ is zero, we shall use the Fredholm theory of holomorphic families of operators of type (A), [7]. Let $\Omega$ be the following complex domain:

$$\Omega = \{ \mathcal{X} + ir \mid \mathcal{X} \in (0, 1), r \in \mathbb{R} \}.$$  

For $z \in \Omega$, we define $H_0(\bar{p}, z\bar{e}_n)$ to be the multiplication operator in $l^2_\mathcal{X}$ by $(\bar{p} + z\bar{e}_n + E\bar{q})^2$ and:

$$H(\bar{p}, z\bar{e}_n) = H_0(\bar{p}, z\bar{e}_n) + Y.$$  

We shall see that:

(I) $\{H(\bar{p}, z\bar{e}_n)\}$ is a holomorphic family of type (A) with respect to $z$. (See the terminology in [7]);

(II) the resolvent of $H(\bar{p}, z\bar{e}_n)$ is compact;

(III) the resolvent set of $H(\bar{p}, z\bar{e}_n)$ is not empty.

Under these conditions, Theorem VII.1.10 of [7] says that we have the following alternative:

- either $0 \in \sigma(H(\bar{p}, z\bar{e}_n))$ for each $z \in \Omega$;

- or every compact $\Omega_0$ in $\Omega$ contains only a finite number of points $z$ such that $0 \in \sigma(H(\bar{p}, z\bar{e}_n))$.

We shall show that:

(IV) 0 belongs to the resolvent set of $H(\bar{p}, z\bar{e}_n)$ for Im $z$ sufficiently large. Hence the first alternative is excluded, so that the measure of $\theta(\bar{p})$ is zero.

The remainder of the paper is devoted to the verification of the properties I to IV of $H(\bar{p}, z\bar{e}_n)$. To simplify the notations we write $H(\bar{p}, \bar{z})$ for $H(\bar{p}, z\bar{e}_n)$.

**Lemma 5.** — (i) $H_0(\bar{p}, \bar{z})$ is a self-adjoint holomorphic family of type (A) in $\Omega$ with domain $D(H_0(\bar{p}, \bar{z})) = D_0$;
(ii) \( \forall z \in \Omega \), the resolvent of \( H_0(\vec{p}, z) \) is compact;
(iii) \( 0 \) belongs to the resolvent set \( \rho(H_0(\vec{p}, z)) \) of \( H_0(\vec{p}, z) \) for all \( z \) with \( \text{Im } z \neq 0 \).

**Proof.** — (i) Let \( P_j (j = 1, \ldots, n) \) be the following operator in \( l_n^2 \):

\[
P_j g_q = \theta_j g_q.
\]

One has:

\[
H_0(\vec{p}, z) = (\vec{p} + \vec{q} + z e_n)^2 = (\vec{p} + \vec{q})^2 + E^2 z^2 + 2 E^2 z P_n.
\]

and the result is immediate:

(ii) the proof is the same as in Lemma 2 (iv).
(iii) for \( z = \mathcal{X} + ir \), we have:

\[
\text{Im}(\vec{p} + E \vec{q} + z e_n)^2 = 2 E^2 r (\mathcal{X} + q_n),
\]

which is different from zero if \( r \neq 0 \). Since \( q_n \in \mathbb{Z} \) and \( \mathcal{X} \in (0, 1) \) it follows that:

\[
\| [H_0(\vec{p}, z)]^{-1} \| = \sup_{q \in \mathbb{Z}} |(\vec{p} + E \vec{q} + z e_n)^2|^{-1} < \infty,
\]

i.e. \( 0 \in \rho(H_0(\vec{p}, z)) \). 

**Lemma 6.** — (i) \( H(\vec{p}, z) \) is a self-adjoint holomorphic family of type (A) in \( \Omega \) with domain \( D_0 \);
(ii) \( \forall z \in \Omega \) the resolvent of \( H(\vec{p}, z) \) is compact;
(iii) for all \( \vec{p} \in \Gamma^{-1} \) and \( z \in \Omega \), \( \rho(H(\vec{p}, z)) \) is not empty.

**Proof.** — (i) this follows from Lemmas 5 (i) and 4 (ii);
(ii) it suffices to show:

\[
\lim_{\lambda \to +\infty} \| Y[H_0(\vec{p}, z) - i \lambda]^{-1} \| = 0,
\]

since then the Neumann series for \( [H(\vec{p}, z) - i \lambda]^{-1} \), i.e.:

\[
[H(\vec{p}, z) - i \lambda]^{-1} = [H_0(\vec{p}, z) - i \lambda]^{-1} \sum_{n=0}^{\infty} \{ - Y[H_0(\vec{p}, z) - i \lambda]^{-1} \}^n,
\]

is convergent if \( \lambda \) is sufficiently large. Now, by (12):

\[
\| Y[H_0(\vec{p}, z) - i \lambda]^{-1} \| \leq L^{-n/4} \| v \| \sum_{q \in \mathbb{Z}} |(\vec{p} + E \vec{q} + z e_n)^2 - i \lambda |^{-1/4}.
\]

We have with the notations \( z = \mathcal{X} + ir, \vec{k} = (\vec{p}, \vec{X} e_n) \in \Gamma^n \):

\[
|(\vec{p} + E \vec{q} + z e_n)^2 - i \lambda |^{-2} \leq |(\vec{k} + E \vec{q})^2 - E^2 r^2 |^{1/2}
\]

\[
+ 4 E^4 r^2 [\mathcal{X} + q_n - \lambda (2 E^2 r)^{-1}]^{1/2} \leq |(\vec{k} + E \vec{q})^2 - E^2 r^2 |^{-2}.
\]
This shows that each term of the sum in (26) converges to zero as \( \lambda \to +\infty \), and that the series in (26) is uniformly majorized in \( \lambda \) by a convergent serie (since \( s > n/2 \)). Therefore (23) is proven.

(If \( z \) is such that \((\vec{q} + E, \vec{q}) - E^2 r^2 = 0\) for certain \( \vec{q} \in \mathbb{Z}^n \), then there exist \( c > 0 \) and \( \lambda_0 < \infty \) such that \( 4E^2 r^2 [\xi + q_n - \lambda (2E^2 r^{-1})]^2 \geq c \) for all these \( \vec{q} \) and for each \( \lambda \geq \lambda_0 \). For these values of \( \vec{q} \) we can take as majorization in (26) the number \( c^{-1} \)).

(ii) Now we use the first and the second resolvent equation:

\[
(H(p, z) - \xi)^{-1} = (H(p, z) - \mu)^{-1} + (\xi - \mu) \left( (H(p, z) - \mu)^{-1} \right)^{-1} [H(p, z) - \xi]^{-1} \left( (H(p, z) - \mu)^{-1} \right)^{-1}
\]

\[
(H(p, z) - \mu)^{-1} = (H_0(p, z) - \mu)^{-1} - (H(p, z) - \mu)^{-1} Y[H_0(p, z) - \mu]^{-1}.
\]

(27) shows that if \( (H(p, z) - \mu)^{-1} \) is compact for \( \mu \in \rho(H(p, z)) \), then \( (H(p, z) - \xi)^{-1} \) is compact for each \( \xi \in \rho(H(p, z)) \). Since \( (H_0(p, z) - \mu)^{-1} \) and \( Y[H_0(p, z) - \mu]^{-1} \) are compact if \( \mu \in \rho(H_0(p, z)) \), by (28) it suffices to show that:

\[
\rho(H_0(p, z)) \cap \rho(H(p, z)) \neq \emptyset.
\]

We know from (iii) that there exists a point \( \mu_0 \in \rho(H(p, z)) \). If \( \mu_0 \notin \rho(H_0(p, z)) \), there exists a point close to \( \mu \in \rho(H_0(p, z)) \cap \rho(H(p, z)) \), since:

(\( \tau \)) \( \rho(H(p, z)) \) is open;

(\( \beta \)) \( \sigma(H_0(p, z)) \) consists of isolated eigenvalues only, because the resolvent of \( H_0(p, z) \) is compact ([7], Thm. III 6.29).

By Lemma 6 we have verified the properties (I) to (III) of the family \( \{H(p, z)\} \). It now remains to prove (IV) i.e. \( 0 \in \rho(H(p, z)) \) for some \( z = \xi + ir \) in \( \Omega \). We have seen that \( 0 \in \rho(H_0(p, z)) \) if \( r \neq 0 \). We shall show that:

\[
\lim_{r \to \infty} \|Y[H_0(p, \xi + ir)]^{-1}\| = 0.
\]

By using the Neumann series (24) with \( \lambda = 0 \) and \( r \) sufficiently large, (29) implies \( 0 \in \rho(H(p, z)) \) if \( r = \text{Im} z \) is sufficiently large.

To obtain (29), we use the inequality (25). By virtue of the first inequality in (26), it suffices to show that:

\[
\lim_{r \to \infty} \sum_{q \in \mathbb{Z}^n} \left\{ \left( \left( \frac{q + \vec{k}}{E} \right)^2 - r^2 \right)^2 + 4r^2 |q_n + \xi|^2 \right\}^{-s/2} = 0,
\]

which will be done in the next section.

5. Estimation of the series (30)

We now show that (30) holds if \( s = 2 \) for \( n = 2, 3, s > n - 2 \) for \( n \geq 4 \) and \( \xi \in (0, 1) \). We use the following notations:

\[
a = 2r |q_n + \xi|, \quad b = (q_n + \xi)^2 - r^2.
\]
We set \( \overline{p} = E^{-1}(k_1, \ldots, k_{n-1}) \in \Gamma_1^{n-1} \), where \( \Gamma_1^{n-1} = \{ \overline{p} \in \mathbb{R}^{n-1} \mid 0 \leq p_j < 1 \} \), and:

\[
S(q_n, r) = \sum_{m \in \mathbb{Z}^{n-1}} \left\{ (m+p)^2 + b^2 + a^2 \right\}^{s/2}.
\]

(30) is then equivalent to:

\[
\lim_{r \to \infty} \sum_{q_n \in \mathbb{Z}^{n-1}} S(q_n, r) = 0.
\]

To prove (33), we first give a preliminary estimate in Lemma 7.

**Lemma 7.** Let \( \delta > 0 \), \( c > 0 \) and \( R > 0 \). Then:

\[
\varepsilon = \inf_{r \geq R} \inf_{a \geq b > r} \inf_{b \geq r} \inf_{t, z \geq 0} \frac{(z^2 + b^2 + a^2)}{(t^2 + b^2 + a^2)} > 0.
\]

**Proof.** Setting \( \alpha = a/r \), \( \beta = br^{-2} \), \( \sigma = z/r \), \( \tau = t/r \) and \( \Omega_r = \{ (\alpha, \beta, \sigma, \tau) \mid \alpha \geq \delta, \beta \geq -1, \sigma \geq 0, \tau \geq 0, |\sigma - \tau| \leq c r^{-1} \} \), we see that (34) is equivalent to:

\[
\varepsilon = \inf_{r \geq R} \inf_{a \geq b > r} \frac{(\sigma^2 + \beta^2) + (\alpha/r)^2}{(\tau^2 + \beta^2) + (\alpha/r)^2} > 0.
\]

The quotient on the r.h.s. of (35) is \( \geq 1 \) if \( |\tau^2 + \beta| \leq |\sigma^2 + \beta| \). Hence the infimum is obtained by taking \( |\tau^2 + \beta| \leq |\sigma^2 + \beta| \). Under this restriction we have:

\[
\frac{(\sigma^2 + \beta^2) + (\alpha/r)^2}{(\tau^2 + \beta^2) + (\alpha/r)^2} \geq \max \left\{ \frac{(\sigma^2 + \beta)^2}{(\tau^2 + \beta)^2}, \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \right\}.
\]

Also notice the following inequalities, valid on each \( \Omega_r \), with \( r \geq R \):

\[
\tau^2 + \beta = |(\tau - \sigma) + \alpha|^2 + \beta \leq 2(\tau - \sigma)^2 + 2\sigma^2 + \beta
\]

\[
= 2(\sigma^2 + \beta) - \beta + 2(\tau - \sigma)^2 \leq 2(\sigma^2 + \beta) + 1 + 2c^2 R^{-2}.
\]

(38) implies that:

\[
(\tau^2 + \beta)^2 \leq 2(\sigma^2 + \beta)^2 + 2(\sigma + \tau)^2 c^2 R^{-2}.
\]

We denote by \( \varepsilon_+ \) and \( \varepsilon_- \) the infimum in (35) under the restriction \( \sigma^2 + \beta \geq 1 \) and \( \sigma^2 + \beta \in [-1, +1] \) respectively. It suffices to show that \( \varepsilon_+ > 0 \) and \( \varepsilon_- > 0 \). In the first case (i.e. for \( \sigma^2 + \beta \geq 1 \)), we use the first expression on the r.h.s. of (36) and the inequality (37). Setting \( x = \sigma^2 + \beta \), we see that:

\[
\varepsilon_+ = \inf_{x \geq 1} \frac{x^2}{(2x + 1 + 2c^2 R^{-2})^2} > 0.
\]
In the second case (i.e. for $\sigma^2 + \beta \in [-1, +1]$), we have $\sigma^2 \leq 2$, hence $\sigma + \gamma \leq 2 \sqrt{2} + c R^{-2} \equiv \eta$. After inserting this into (39) and using the second expression on the r.h.s. of (36), one obtains by setting $y = (\sigma^2 + \beta)^2$:

$$
\varepsilon_+ = \inf_{r \geq R} \inf_{0 \leq y \leq 1} \frac{y + (\sigma/r)^2}{2y + 2 \eta^2 c^2 r^{-2} + 2(\sigma/r)^2}
$$

$$
= \inf_{r \geq R} \inf_{\eta \geq \delta} \frac{(\sigma/r)^2}{2 \eta^2 c^2 r^{-2} + 2(\sigma/r)^2} = \frac{\delta^2}{2 \eta^2 c^2 + 2 \delta^2} > 0.
$$

**Proof of (33).** — Let $\vec{m} \in \mathbb{Z}^{n-1}$ and $\Gamma(\vec{m})$ be the cube:

$$
\Gamma(\vec{m}) = \{ \vec{x} \in \mathbb{R}^{n-1} \mid \vec{x} = \vec{p} + \vec{m} + \vec{y}, \ y \in \Gamma_1^r \}.
$$

We have $\Gamma(\vec{m}) \cap \Gamma(\vec{m}') = \emptyset$ if $\vec{m} \neq \vec{m}'$ and:

$$
\mathbb{R}^{n-1} = \bigcup_{\vec{m} \in \mathbb{Z}^{n-1}} \Gamma(\vec{m}).
$$

Let $c = \sqrt{n-1}$. Then for each $\vec{x} \in \Gamma(\vec{m})$ and each $\vec{m} \in \mathbb{Z}^{n-1}$:

$$
||\vec{m} + \vec{p}|| - ||\vec{x}|| \leq c.
$$

Let $\delta = 1/2 \min (\delta', 1 - \delta')$. By assumption $\delta > 0$; since $a \geq \delta r$ and $b \geq -r^2$, Lemma 7 implies the existence of a number $\varepsilon > 0$ such that, for each $\vec{m} \in \mathbb{Z}^{n-1}$, each $\vec{x} \in \Gamma(\vec{m})$, each $a \geq \delta r$ and $b \geq -r^2$ and all $r \geq R$:

$$
[(\vec{m} + \vec{p})^2 + b^2] + a^2 \geq \varepsilon [(\vec{x}^2 + b)^2 + a^2].
$$

Thus:

$$
S(q, r) = \sum_{\vec{m} \in \mathbb{Z}^{n-1}} \left\{ [(\vec{m} + \vec{p})^2 + b^2] + a^2 \right\}^{-s/2}
$$

$$
= \sum_{\vec{m} \in \mathbb{Z}^{n-1}} \int_{\Gamma(\vec{m})} dx \left\{ [(\vec{m} + \vec{p})^2 + b^2 + a^2]^{-s/2} \right\}
$$

$$
\leq \varepsilon^{-1} \sum_{\vec{m}} \int_{\Gamma(\vec{m})} dx \left\{ [(\vec{x}^2 + b)^2 + a^2]^{-s/2} \right\}
$$

$$
= \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} dx \left\{ [(\vec{x}^2 + b)^2 + a^2]^{-s/2} \right\}
$$

$$
= \frac{1}{2} \varepsilon^{-1} w_{n-1} \int_0^\infty y^{n-3/2} \left\{ (y + b)^2 + a^2 \right\}^{-s/2} dy,
$$

where we have introduced spherical polar coordinates, $y = |\vec{x}|^2$ and $w_{n-1}$ denotes the area of the unit sphere in $\mathbb{R}^{n-1}$. 

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To estimate the integral in (43), we distinguish the two cases \( b \geq 0 \) and \( b < 0 \). For \( b \geq 0 \), we have \( \{(y + b)^2 + a^2\}^{-s/2} \leq \{y^2 + a^2 + b^2\}^{-s/2} \), and (43) leads to:

\[
S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} (a^2 + b^2)^{-s/2} \int_0^{+\infty} z^{(n-3)/2} (z^2 + 1)^{-s/2} \, dz.
\]

Notice that the integral in this expression is convergent since \( s > n/2 \). By observing that:

\[
a^2 + b^2 = [(q + \mathcal{A})^2 + r^2].
\]

we obtain:

\[
\sum_{|q, +\mathcal{A}| \geq r} S(q_n, r) \leq C \sum_{|q, +\mathcal{A}| \geq r} |q_n + \mathcal{A}|^{-2s + n - 1}.
\]

The hypothesis \( s > n/2 \) implies that the last series is convergent so that this term tends to zero as \( r \to \infty \).

We now turn to the case \( b < 0 \). We set \( z = (y + b)/a \). (43) then gives:

\[
S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} a^{-s+1} \int_{b/a}^{+\infty} (az - b)^{n-3/2} \{1 + z^2\}^{-s/2} \, dz.
\]

If \( n \geq 3 \), this leads to:

\[
S(q_n, r) \leq c_1 a^{-s+1} \int_{b/a}^{+\infty} \{|az|^{(n-3)/2} + |b|^{(n-3)/2}\} \{1 + z^2\}^{-s/2} \, dz
\]

\[
\leq c_2 a^{-s+1} \{|a|^{(n-3)/2} + |b|^{(n-3)/2}\} \leq c_3 a^{-s+1} (a^2 + b^2)^{(n-3)/4}.
\]

Using (47), (44) and (31), we obtain in this case that:

\[
\sum_{|q, +\mathcal{A}| < r} S(q_n, r) \leq c_4 r^{-s+1} r^{n-3} \sum_{|q, +\mathcal{A}| < r} |q_n + \mathcal{A}|^{-s+1} = O(r^{-s+n-2} \log r),
\]

since \( s \geq 2 \). Under the hypothesis \( s > n - 2 \), this converges to zero as \( r \to \infty \).

Finally, if \( n = 2 \), one may bound the integral in (46) by a constant which is independent of \( a \) and \( b \) on the set \( \{a \geq a_0 > 0, b < 0\} \); this is easily achieved by splitting the domain of integration into \( \{z | az - b \leq 1\} \cup \{z | az - b > 1\} \). Thus:

\[
S(q_n, r) \leq c_5 r^{-s+1} |q_n + \mathcal{A}|^{-s+1}, \quad \forall q_n, \forall r \geq r_0.
\]

For any \( s > 3/2 \), this implies that:

\[
\lim_{r \to \infty} \sum_{|q, +\mathcal{A}| < r} S(q_n, r) = 0. \quad \blacksquare
\]

Remark 3. — One sees from the preceding proof that, for \( n = 3 \), the limit in (33) is zero under the weaker hypothesis that \( s > 3/2 \). By using a modified resolvent equation, one obtains the result of Theorem 1 for \( s > 3/2 \). The case \( s = 2, n = 3 \) was first treated by Thomas.
in [12]. Similarly, for \( n = 2 \), a more careful estimate of the integral in (46) shows that it suffices to require \( s > 1 \).

**Remark 4.** — Theorem 3 remains true if the condition of ortho-periodicity of \( v \) is replaced by the weaker condition of periodicity. Indeed, the estimation of the series given in section 5, may be applied if, instead of \( \tilde{a}_i, \tilde{a}_j = \delta_{ij} \), one requires only that \( \tilde{a}_i, \tilde{a}_n = \delta_{in} \) (i.e. the vector \( \tilde{a}_n \) is orthogonal to the hyperplane spanned by \( \tilde{a}_1, \ldots, \tilde{a}_{n-1} \)). Clearly the direction \( \tilde{a}_n \) is distinguished in our estimation. A similar result for an arbitrary periodic lattice is given in Theorem XIII.100 of [11], under a more restrictive assumption on the local behaviour of the function \( v(x) \) than that of Theorem 3.

**Remark 5.** — We also have the following result which generalizes Theorem 1:

**Theorem V.** — Let \( v \in L^s_\text{loc} (\mathbb{R}^n \setminus N) \), where \( s \) satisfies \( s = 2 \) if \( n = 1, 2, 3 \) and \( s > n - 2 \) if \( n \geq 4 \), and where \( N \) is a closed set of measure zero. Let \( H \) be a self-adjoint extension of \( \hat{H} \), \( D(\hat{H}) = C_0^\infty (\mathbb{R}^n \setminus N) \). Suppose that \( f \in L^2 (\mathbb{R}^n) \) satisfies \( Hf = \lambda f \) for some \( \lambda \in \mathbb{R} \) and \( E(A) f = f \) for some compact subset of \( \mathbb{R}^n \setminus N \) (i.e. \( f \) is an eigenvector of \( H \) having compact support in \( \mathbb{R}^n \setminus N \)). Then \( f = 0 \).

**Proof.** — One has \( \chi_A(.)(v(.)) \in L^s (\mathbb{R}^n) \). Let \( C \) be a cube in \( \mathbb{R}^n \) such that \( A \subseteq C \). Define \( w \) by:

\[
 w(x + \sum q_i \tilde{a}_i) = \chi_A(x) v(x), \quad x \in C,
\]

\( w \) is ortho-periodic and in \( L^s_\text{loc} (\mathbb{R}^n) \). Since \( (H_0 + w) f = \lambda f \), one has \( f = 0 \) by Theorem 3. \( \blacksquare \)

**Remark 6.** — The hypothesis "\( \Sigma \) bounded" in Theorem 1(b) is essential. Assume for example that \( v \) is such that \( H_0 + v \) has pure point spectrum (e.g. \( v(x) \to + \infty \) as \( |x| \to \infty \)). Take \( \Sigma = \mathbb{R} \). Then:

\[
 F(\Sigma) \mathcal{H} = \mathcal{H} \quad \text{and} \quad E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = E(A) \mathcal{H}.
\]

Since \( E(A) \mathcal{H} \neq \{0\} \) if \( A \) has positive measure, it is clear that one cannot have \( E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = \{0\} \) in this case.

**Remark 7.** — By combining our Theorem 1 with Proposition 4 of [2], one may also prove that \( E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\} \) under assumptions of Theorem 1(b).

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ON THE POINT SPECTRUM


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