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EXISTENCE OF GLOBAL SOLUTIONS OF THE YANG-MILLS, HIGGS AND SPINOR FIELD EQUATIONS IN 3+1 DIMENSIONS

BY YVONNE CHOQUET-BRUHAT AND DEMETRIOS CHRISTODOULOU

Introduction

We prove in this paper the global existence on Minkowski space time of solutions of the Cauchy problem for the coupled Yang-Mills, Higgs and spinor classical field equations in $3+1$ dimensions.

Our proof relies on the transformation of the global Cauchy problem on Minkowski space time into a local Cauchy problem on the Einstein static universe, by a conformal transformation. The method applies to all conformally invariant systems (or more generally conformally regular, *cf.* § 8). It has been already introduced in the pure Yang-Mills case [9], and applied to the Maxwell-Dirac system [5]. Unlike the recent proofs of the global existence of solutions for Yang-Mills-Higgs equations of Ginibre and Velo [15] (case $n=2$) and of Eardley and Montcrief [11] (case $n=3$), it does not rely on *a priori* bounds or "no blow up" estimates: these estimates are not available in the presence of spinor fields for which there is no physically defined positive energy.

In paragraphs 1 and 2 we recall briefly the definitions of the classical Yang-Mills field equations on a hyperbolic manifold, coupled with gauge invariant equations for spinor and scalar multiplets. We give as examples the now classical Weinberg-Salam model, and the chromodynamics. In paragraph 3 we define the norms, and the functions spaces which we shall use. For the Cauchy data these spaces are the simple Sobolev spaces H_s or weighted Sobolev spaces $H_{s,\delta}$. In paragraph 4 we recall the local existence and uniqueness theorems for the solution of the Cauchy problem. We use as in ([17], [4], [9]) the Lorentz gauge (1), as an intermediate step. In paragraph 5 we state, in our notations, the conformal properties of the considered field equations, and in paragraph 6 we consider the case of the conformal mapping of Minkowski space time onto a bounded open subset of the Einstein, static universe.

(1) For a solution in the temporal gauge, in the case of Yang-Mills field on Minkowski space time *cf.* [24].

In paragraph 7 we prove the global existence of solutions of our Cauchy problem in Minkowski space-time for small data and study their asymptotic behavior, for systems conformally invariant in the sense of paragraph 5; we extend the results to conformally regular systems in paragraph 8. The global existence result also holds for space times conformal to Minkowski space time: we treat in paragraph 9 the case of De Sitter space time. Finally, in paragraph 10 we give some results about the solutions of the constraints equation on a compact initial 3-manifold, since the space of global solutions we construct on Minkowski space time originates from such data on the sphere S^3.

1. Space time and fields

A space time is a C^∞, $n+1$ dimensional manifold V endowed with a C^∞ hyperbolic metric g of signature $(+, -, -, \ldots, -)$. We shall suppose that (V, g) admits a spin structure (²), that is there exists a principal bundle SV with base V and structure group Spin $(n+1)$, which is the extension (*cf.* Lichnerowicz [19], § 10) of the bundle of orthonormal frames of (V, g).

We shall denote by γ^α, $\alpha = 0, \ldots, n$, a set of standard Dirac matrices, such that, in an orthonormal frame:

1.1 $$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\,\eta^{\alpha\beta} I, \qquad \eta^{\alpha\beta} = \mathrm{diag}(1, -1, \ldots, -1)$$

and denote by σ_λ the metric spin connection, which reads in a spin frame corresponding to a Lorentz frame where the coefficients of the metric connection are $\omega^\alpha_{\lambda\beta}$:

1.2 $$\sigma_\lambda = \frac{1}{4} \omega^\alpha_{\lambda\beta} \gamma_\alpha \gamma^\beta.$$

A Yang Mills connection is a 1-form ω on a principal (C^∞) fibre bundle P with base the space time V and structure group a Lie group G. The form ω has its values in the Lie algebra $\mathcal{G}$. We shall suppose in this paper that P is a trivial bundle (as is always the case when $V = \mathbb{R}^{n+1}$), though this hypothesis is not necessary for the local existence theorem of paragraph 4 to hold, if properly formulated.

We shall also suppose that G admits a non degenerate bi-invariant metric; a sufficient condition is that G be the product of abelian and semi-simple Lie groups. The more restrictive hypothesis of the compactness of G is required only in paragraph 10 on the constraint's equation.

Given a "gauge", that is a section (³) s of P, the connection ω is represented by the Yang Mills potential $A = s^* \omega$, a 1-form on V with values in $\mathcal{G}$. Under a change of gauge characterized by a mapping $u : V \to G$, A transforms by:

$$A \rightsquigarrow ad(u^{-1}) A + u^* \theta_{MC},$$

where θ_{MC} is the Maurer Cartan 1-form of G.

(²) A necessary and sufficient condition in the case $n+1=4$ is that the bundle of orthonormal frames admit a global, C^α, section.

(³) A section is in canonical correspondance with a trivialization of P.

This transformation is usually written:

1.3
$$A \rightsquigarrow u^{-1} A u + u^{-1} du$$

and is self explanatory when G is a subgroup of a linear group.

The Yang Mills field, representing the curvature Ω of the connection, is a 2-form $F = s^* \Omega$ on V, with values in $\mathcal{G}$, given by:

1.4
$$F \equiv DA \equiv dA + \frac{1}{2}[A, A].$$

If ∇ is the covariant derivative in the space time metric, and C^a_{bc} the structure constants for a basis (ε_a) of $\mathcal{G}$, 1-4 reads in a coordinate chart on V, and for the basis ε_a:

$$F^a_{\lambda\mu} \equiv \nabla_\lambda A^a_\mu - \nabla_\mu A^a_\lambda + C^a_{bc} A^b_\lambda A^c_\mu.$$

The scalar-multiplet field Φ is a section of a complex [or a real] vector bundle associated to P by a unitary [or an orthogonal] representation ρ of G, its change under a gauge transformation $u : V \to G$ is:

1.5
$$\Phi \rightsquigarrow \rho(u)\Phi, \qquad \text{with} \quad \rho(u(x)) \in U(m) \qquad [\text{or } O(m)].$$

The gauge covariant derivative of Φ in the connection represented by A is:

$$\hat{\nabla}\Phi = d\Phi + SA\Phi,$$

where $S = \rho'(\mathbf{1})$, $\mathbf{1}$ unit of G; the tangent space $T_1 G$ being identified with $\mathcal{G}$, S is a mapping $\mathcal{G} \to \mathcal{U}(m)$ [or $\mathcal{O}(m)$]. The covariant derivative is, in coordinates:

$$\hat{\nabla}_\mu \Phi^I = \nabla_\mu \Phi^I + S^I_{aJ} A^a_\mu \Phi^J; \qquad I, J = 1, \ldots, m,$$

with each $S_a = (S^I_{aJ})$ an antihermitian [a real antisymmetric] matrix, and ∇_μ the metric covariant derivative ($\nabla_\mu = \partial_\mu$ if Φ^I is scalar valued).

Remark. — We denote by Φ the hermitian conjugate of Φ [the transpose if Φ is real] that is $\tilde{\Phi}_I = \Phi^{*I}$ where Φ^* is the complex conjugate. It transforms under the change of gauge characterized by u as:

$$\tilde{\Phi} \rightsquigarrow \widetilde{\rho(u)\Phi} = \tilde{\Phi}\,\widetilde{\rho(u)} = \tilde{\Phi}(\rho(u))^{-1},$$

since ρ is unitary [orthogonal if Φ is real]. Thus $\tilde{\Phi}\Phi$ is invariant under gauge transformations.

The spinor-multiplet field Ψ is a section of a vector bundle associated to P, with typical fiber $\mathbb{C}^k \times \mathbb{C}^l$ where $\mathbb{C}^k$ is the fundamental representation space of the spinor group $\mathrm{Spin}(n+1)$ (therefore $k = 2^{[(n+1)/2]}$), and $\mathbb{C}^l$ is the space of a unitary representation r of G: under a change of section in the associated principal bundle characterized by a mapping (Λ, u):

$V \to \mathrm{Spin}(n+1) \times G$, ψ transforms by:

1.6
$$\Psi \rightsquigarrow (\Lambda^{-1}, r(u))\Psi,$$

(i. e. $\Psi^{A,L} \rightsquigarrow \Lambda_B^{-1A} r_L^M(u) \Psi^{B,L}$).

The metric-and-gauge covariant derivative of Ψ is:

$$\hat{\nabla}\Psi = d\Psi + (\sigma, r'(1)A)\Psi,$$

where σ is the spin connection corresponding to the metric g. If we denote by T the linear map $r'(\mathbf{1}): T_1 G \simeq \mathcal{G} \to \mathcal{U}(l)$, and by $\nabla = d + \sigma$ the metric covariant derivative we have:

$$\hat{\nabla}\Psi = \nabla\Psi + TA\Psi, \qquad \nabla = d + \sigma$$

(i. e. $\hat{\nabla}_\lambda \Psi^{A,L} = \partial_\lambda \Psi^{A,L} + \sigma^A_{\lambda B} \Psi^{B,L} + T^L_{aM} A^a_\lambda \Psi^{A,M}$).

The Dirac adjoint of the spinor-multiplet Ψ is the cospinor multiplet $\overline{\Psi}$ defined by:

$$\overline{\Psi} = \tilde{\Psi}\beta \qquad (\overline{\Psi}_{A,I} = \tilde{\Psi}^{B,I}\beta^B_A),$$

where $\tilde{\Psi}$ is the complex conjugate transpose of Ψ and β a $k \times k$ matrix such that:

$$\beta\gamma^\alpha\beta^{-1} = \tilde{\gamma}^\alpha.$$

We shall take a standard choice of Dirac matrices, that is we shall suppose:

$$\gamma^\alpha = \eta^{\alpha\alpha}\tilde{\gamma}^\alpha$$

and choose:

$$\beta = \gamma^0.$$

Note. — (1) Under the action of $(\Lambda^{-1}, u) \in \mathrm{Spin}(n+1) \times G$, $\overline{\Psi}$ transforms by:

$$\overline{\Psi} \rightsquigarrow \overline{\Psi}(\tilde{r}(u), \Lambda)$$

and $\overline{\Psi}\Psi$ is invariant.

(2) Recall that:

$$\hat{\nabla}\overline{\Psi} = d\overline{\Psi} - \overline{\Psi}\sigma - \overline{\Psi}\,TA,$$

because $\tilde{T} = -T$. Thus $\hat{\nabla}\overline{\Psi}$ is the Dirac adjoint of $\hat{\nabla}\Psi$:

$$\hat{\nabla}\overline{\Psi} = \widetilde{\hat{\nabla}\Psi}\beta,$$

since (*cf.* [19]) $\tilde{\sigma}\beta = -\beta\sigma$.

2. Equations

The Dirac gauge covariant operator on Ψ is defined by:

2.1 $$\hat{\nabla\!\!\!\!/}\,\Psi \equiv \gamma^\alpha\hat{\nabla}_\alpha\Psi \equiv \gamma^\alpha(\nabla_\alpha\Psi + TA_\alpha\Psi).$$

While the wave operator on Φ is:

2.2 $$\hat{\square}\,\Phi \equiv \hat{\nabla}^\lambda\hat{\nabla}_\lambda\Phi \equiv g^{\lambda\mu}\hat{\nabla}_\lambda\hat{\nabla}_\mu\Phi.$$

The Yang-Mills operator on F is $D\star F$ where $\star F$ denotes the adjoint 2-form of F in the metric g. This operator may be written $\hat{\nabla}.F^{\#}$, where $F^{\#}$ is the contravariant 2-tensor associated with F in the metric g, and it reads, in an arbitrary frame on V and basis in $\mathcal{G}$:

2.3 $$(\hat{\nabla}.F^{\#})^{\mu,a}\equiv\hat{\nabla}_\lambda F^{\lambda\mu,a}\equiv\nabla_\lambda F^{\lambda\mu,a}+C^a_{bc}A^b_\lambda F^{\lambda\mu,c}.$$

As a consequence of the Bianchi identities we have:

2.4 $$\hat{\nabla}_\mu\hat{\nabla}_\lambda F^{\lambda\mu,a}\equiv 0.$$

The classical equations for the coupled Yang-Mills, scalar and spinor fields are usually obtained as the Euler equations of a Lagrangian, which must be invariant under isometries of g and gauge transformations of A.

The equations are, for the Yang-Mills field:

2.5 $$\hat{\nabla}_\lambda F^{\lambda\mu,a}=J^{\mu,a}.$$

With $J^{\mu,a}$ a vector on V, with values in $\mathcal{G}$ of type $\operatorname{ad}(u^{-1})$, under a change of gauge, called the Yang-Mills current, given by:

2.6 $$J^{\mu,a}=i\overline{\Psi}\gamma^\mu S^a\Psi+(\tilde{\Phi}\tilde{T}^a\hat{\nabla}^\mu\Phi+\hat{\nabla}^\mu\tilde{\Phi}T^a\Phi).$$

For the spinor and scalar multiplets the equations read:

2.7 $$\hat{\nabla}\!\!\!/\,\Psi=H,$$

2.8 $$\hat{\Box}\Phi=K.$$

With H and K some given smooth functions of the fields Φ, $\tilde{\Phi}$ and Ψ, $\overline{\Psi}$ which must be such that the equations 2.7 and 2.8 imply the generalized conservation law:

2.9 $$\hat{\nabla}_\mu J^{\mu,a}=0.$$

A possible choice, corresponding to the case where the fields Φ and Ψ do not interact and have no self interaction is:

2.10 $$H=0,\qquad K=0.$$

A more general, physical choice, is obtained by considering equations which derive from a gauge invariant lagrangian:

2.11 $$\mathcal{L}=\frac{1}{4}F^a_{\lambda\mu}F^{\lambda\mu}_a+\frac{1}{2}i(\overline{\Psi}\hat{\nabla}\!\!\!/\,\Psi-\hat{\nabla}\!\!\!/\,\overline{\Psi}\Psi)+\hat{\nabla}_\mu\tilde{\varphi}\nabla_\mu\varphi+\mathcal{L}_{\mathrm{Int}}.$$

The corresponding system 2.5, 2.7, 2.8 will satisfy 2.9 if the interaction lagrangian, $\mathcal{L}_{\mathrm{Int}}$, is invariant (like the first terms) by a gauge transformation. The minimal (Yukawa) coupling is:

2.12 $$\mathcal{L}_{\mathrm{Int}}=\overline{\Psi}C\Psi\Phi+\tilde{\Phi}\overline{\Psi}\tilde{C}\Psi+\mu(\tilde{\Phi}\Phi)^2.$$

where C is a given tri-linear map $\mathbb{C}^l \times \mathbb{C}^l \times \mathbb{C}^m \to S_1^1$, where S_1^1 is a 1-spinor, 1-cospinor.

In a given gauge and spin frame we have:

$$\mathscr{L}_{\mathrm{Int}} = \overline{\Psi}_{\mathrm{A, I}} \mathrm{C}^{\mathrm{IA}}_{\mathrm{JLB}} \Psi^{\mathrm{B, J}} \Phi^{\mathrm{L}} + \text{complex conjugate}.$$

We have then:

2.13 $$\mathrm{H} = i(\mathrm{C}\Phi + \tilde{\Phi}\tilde{\mathrm{C}})\Psi, \qquad \mathrm{K} = -\overline{\Psi}\tilde{\mathrm{C}}\Psi - \mu(\tilde{\Phi}\Phi)\Phi.$$

The interaction lagrangian is gauge invariant if:

2.14 $$\tilde{\mathrm{T}}\mathrm{C} + \mathrm{C}\mathrm{T} + \mathrm{C}\mathrm{S} = 0$$

(vanishing of a map $\mathscr{G} \times \mathbb{C}^l \times \mathbb{C}^l \times \mathbb{C}^m \to S_1^1$).

Physical models of such interactions are:

(1) The *Yukawa model* of the nuclear forces, before the adoption of the quark model. The equations reduce then to the wave equations 2.7 and 2.8 for the spinor field doublet [4] $\Psi = (\Psi_p, \Psi_n)$ and the scalar field triplet [5] $\Phi = (\Phi_0, \Phi_1, \Phi_2)$. The right hand sides H and K are given by 2-13, with:

$$\mathrm{C} = k\gamma_5\tau,$$

where $\tau = (\tau^a)$, $a = 1, 2, 3$ are the 2×2 Pauli matrices, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ a 1-spinor, 1-cospinor, and k a coupling constant. The corresponding interaction lagrangian is:

$$\mathscr{L}_{\mathrm{Int}} \equiv \overline{\Psi}\,\mathrm{C}\,\Psi\Phi + \text{Complex conjugate}$$
$$= k\{(\overline{\Psi}_p\gamma_5\Psi_n + \overline{\Psi}_n\gamma_5\Psi_p)\Phi_1 - i(\overline{\Psi}_p\gamma_5\Psi_n - \overline{\Psi}_n\gamma_5\Psi_p)\Phi_2 + (\overline{\Psi}_p\gamma_5\Psi_p - \overline{\Psi}_n\gamma_5\Psi_n)\Phi_3\}.$$

This lagrangian is invariant by "gauge" transformation associated with the group $\mathrm{G} = \mathrm{SU}(2)$ if the infinitesimal actions of SU(2) on the fields Ψ and Φ are respectively given by:

$$\mathrm{T}^a = i\tau^a \qquad (2 \times 2 \text{ matrices}),$$
$$\mathrm{S}^a = \varepsilon^a \qquad (3 \times 3 \text{ matrices } (\varepsilon^a)^b_c = \varepsilon^{ab}_c),$$

with ε^{ab}_c the totally antisymmetric structure constants of SU(2).

These equations can be coupled with an SU(2) Yang-Mills field.

(2) The *Weinberg-Salam model* for unification of weak and electromagnetic interactions.

The group G is here the product $\mathrm{SU}(2) \times \mathrm{U}(1)$.

The spinor field Ψ is a triplet, which is written [6] as a doublet L and *a* singlet R :

$$\Psi = (\mathrm{L}, \mathrm{R}), \qquad \mathrm{L} = (\mathrm{L}_1, \mathrm{L}_2).$$

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[4] p for proton and n for neutron.

[5] Associated to the 3 pions Π_0, Π_+, Π_-.

[6] Physically $\mathrm{L} = (1+\gamma_5)\binom{\nu_e}{e}$, $\mathrm{R} = (1-\gamma_5)e$. One takes R as a singlet because right handed neutrinos are not observed.

The scalar field Φ is a doublet; $\Phi=(\Phi_1, \Phi_2)$. The infinitesimal operators S^a, T^a are given by (k and k_1 are two independant coupling constants):

$$S^a \Psi = \left(-\frac{ik\tau^a}{2} L, 0\right), \qquad a=1, 2, 3,$$

$$S^a \Psi = \left(-\frac{ik_1 L}{2}, -ik_1 R\right), \qquad a=4$$

and:

$$T^a \Phi = -\frac{ik\tau^a}{2}, \qquad a=1, 2, 3,$$

$$T^a \Phi = \frac{ik_1}{2}\Phi, \qquad a=4.$$

The interaction lagrangian is the gauge invariant ([7]):

$$\mathscr{L}_{\mathrm{Int}} = (\overline{L}\,\Phi) R + \overline{R}(\tilde{\Phi} L) + \mu(\tilde{\Phi}\Phi)^2.$$

(3) *Chromodynamics.* A classical model for quarks dynamics mediated by the Yang-Mills field of the "gluons" is the Yang-Mills equation 2.5, with the group G=SU(3) [or some higher group to include the charmed quark, or more recently conjectured quarks] – the spinor field is a three or higher multiplet. There is no scalar field, and no self-interaction of quarks.

More generally we shall denote by $\mathscr{S}$ a system of the type 2.5, 2.7, 2.8, with J given by 2.6, and H and K some given smooth (for instance polynomial) functions of Φ and Ψ such that the system is invariant by an isometry of g (preserving also the choice of a spin structure) and by the gauge transformation:

$$A \rightsquigarrow u^{-1} A u + u^{-1} du, \qquad \Phi \rightsquigarrow \rho(u)\Phi, \qquad \Psi \rightsquigarrow r(u)\Psi,$$
$$\tilde{\rho}(u)\rho(u)=1, \qquad \tilde{r}(u) r(u)=1.$$

3. Norms and function spaces

(for analogous definitions *cf.* [8], [10])

(V, g) is now supposed to be globally hyperbolic, thus $V=\mathbb{R}\times S$. We suppose the curves $\mathbb{R}\times\{x\}$ uniformly time like [that is their tangent vector X such that $g(X, X)\geqq\alpha>0$], and the submanifolds $S_t=\{t\}\times S$ uniformly space like [that is their unit normal n such that $g(X, n)<\beta$]. The quadratic form:

$$\Gamma = X\otimes n + n\otimes X - g \qquad (\Gamma^{\lambda\mu}\equiv X^\lambda n^\mu + X^\mu n^\lambda - g^{\lambda\mu}).$$

is then positive definite and is used to define the norm at a point of V of vectors and tensors.

([7]) Weinberg-Salam also introduce a "mass" term $\nu\tilde{\Phi}\Phi$ which we do not write here. It has been argued (Coleman and Weinberg) that this term could be zero at the classical (non quantum) level.

The norm at a point of the derivative of order k, $\nabla^k \Phi$, of the scalar-multiplet Φ (in a chosen gauge) is the coordinate invariant positive number:

$$|\nabla^k \Phi| = \{ \Gamma^{\lambda_1\mu_1} \dots \Gamma^{\lambda_1\mu_1} \nabla_{\lambda_1 \dots \lambda_k} \tilde{\Phi} \nabla_{\mu_1 \dots \mu_k} \Phi \}^{1/2}.$$

Remark. — The number $\Gamma^{\lambda_1\mu_1} \dots \Gamma^{\lambda_k\mu_k} \hat{\nabla}_{\lambda_1 \dots \lambda_k} \tilde{\Phi} \hat{\nabla}_{\mu_1 \dots \mu_k} \Phi$ is also positive since we have:

$$\hat{\nabla}_{\lambda_1 \dots \lambda_k} \tilde{\Phi} = \widetilde{\hat{\nabla}_{\lambda_1 \dots \lambda_k} \Phi}$$

and is gauge invariant.

The norm of a spinor at a point is the positive, frame independent, scalar, linked to the choice of submanifolds S_t, given by:

$$|\Psi| = (\overline{\Psi} \gamma^\lambda n_\lambda \Psi)^{1/2},$$

In a spin frame such that the corresponding lorentzian frame has axis $e_0 = n$, we have:

$$|\Psi| = (\tilde{\Psi} \Psi)^{1/2},$$

The norm of the derivative $\nabla^k \Psi$ is given (in a chosen gauge and spin frame) by the positive number:

$$|\nabla^k \Psi| = \Gamma^{\lambda_1\mu_1} \dots \Gamma^{\lambda_k\mu_1} \widetilde{\nabla_{\lambda_1 \dots \lambda_k} \Psi} \nabla_{\mu_1 \dots \mu_k} \Psi,$$

where the $\sim$ denotes the hermitian conjugate in $C^k \times C^l$.

Remark. — We have the identity (*cf.* [18]):

$$(\widetilde{\nabla \Psi}) \beta = \nabla \overline{\Psi}.$$

Thus:

$$|\nabla \Psi| = (\Gamma^{\lambda\mu} \nabla_\lambda \overline{\Psi} \gamma^\alpha n_\alpha \nabla_\mu \Psi)^{1/2}$$

and analogous formulas for the higher order derivatives: $|\nabla^k \psi|$ do not depend on the spin frame.

If G is a compact Lie group it admits an Ad-invariant positive definite metric. This metric can be used together with Γ to define at a point of V, for $\mathcal{G}$ valued tensors of type Ad, a norm invariant by coordinates and gauge transformations. However, since the potential A is not of type Ad we will work in a specific gauge. We choose a basis $\mathcal{E}_a$ of $\mathcal{G}$ and we define the coordinate independent norms of the (real valued) derivatives $\nabla^k A$ by:

$$|\nabla^k A| = \left\{ \sum_{a=1}^{N} \Gamma^{\lambda_1\mu_1} \dots \Gamma^{\lambda_k\mu_k} \Gamma^{\alpha\beta} \nabla_{\lambda_1 \dots \lambda_k} A^a_\alpha \nabla_{\mu_1 \dots \mu_k} A^a_\beta \right\}^{1/2}.$$

This definition does not require that the group has a positive definite invariant metric ([8]).

([8]) Such an hypothesis will be useful in the study of the constraint problem (and seems necessary for the construction of Hilbert spaces in the quantum context).

FUNCTION SPACES. — Let h be a measurable section of a vector bundle over V (a measurable field of tensor — spinor multiplets whose norms at a point we have previously defined). We suppose that h and its distribution — derivatives of order $\leqq s$ are also measurable and admit for each t a restriction to S_t which is square integrable in the metric $\bar{g}_t$ induced by g on S_t. We set:

$$\| h \|_s^{S_t} = \left\{ \int_{S_t} \sum_{k=0}^{s} | \nabla^k h |^2 \, d\mu(\bar{g}_t) \right\}^{1/2}.$$

We call [9] $E_s(I \times S)$ a Banach space which is the closure of the space of restrictions to $I \times S$ of $C_0^\infty(\mathbb{R} \times S)$ tensor-spinor multiplets in the norm:

$$\| h \|_{E_s(I \times S)} = \operatorname*{Sup}_{t \in I} \| h \|_s^{S_t}.$$

Remark. — If $g=(1, \bar{g})$ with $\bar{g}$ a positive definite metric on S (i.e. $g_{\lambda\mu} dx^\lambda dx^\mu = dt^2 - \bar{g}_{ij} dx^i dx^j$, $\bar{g}_{ij}$ independent of t) then:

$$E_s(I \times S) = \left\{ h \,\middle|\, \frac{\partial^k h}{\partial t^k} \in C^0(\bar{I}, H_{s-k}(S)), \, k=0, \ldots s \right\},$$

if $\bar{g}$ is (s, S) Sobolev-regular, that is if $C_0^\infty(S)$ is dense in $H_s(S)$.

4. Cauchy problem, local existence

The Cauchy problem for a system $\mathscr{S}$ (*cf.* § 2) is the data, on the submanifold $S_0 \equiv \{ t=0 \}$, of the potential A and the fields F, Φ, $n^k \nabla_\lambda \Phi$, Ψ. The data cannot be independent, they must satisfy the constraint:

4.1 $$\hat{\nabla}_\lambda F^{\lambda 0, a} = J^{0, a}.$$

The electric Yang Mills field, relative to the slicing $S \times I$ is the vector, with values in $\mathscr{G}$, $F.n = (F^{\alpha\beta} n_\beta)$. In particular we set:

4.2 $$E = F.n_{|S_0}.$$

In a lorentzian frame with time axis $e_0 = n$ we have:

$$E^{i, a} = F^{i0, a}$$

and the equations 4.1 reads:

4.3 $$\widehat{\operatorname{div}}\, E = J.n,$$

that is on S_0:

4.4 $$\hat{\bar{\nabla}}_i E^{i, a} \equiv \bar{\nabla}_i E^{i, a} + C^a_{bc} A^b_i E^{i, c} = J^{0, a},$$

where $\bar{\nabla}_i$ denotes the covariant derivative in the metric $\bar{g}_0$ induced by g on S_0.

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[9] The definition is more restrictive than in previous papers ([4], [10]) where quasi-linear (not semi-linear) equations were treated.

The data of A and F on S_0 is equivalent to the data of A and E. The data of A on S_0 is equivalent to the data of a $\mathscr{G}$ valued covariant vector a on S_0, projection of A:

$$a = \Pi\, A_{|S_0} \qquad \text{(in a Lorentz frame } a_\lambda = (\delta^\mu_\lambda - n_\lambda n^\mu) A_{\mu|S_0})$$

and a $\mathscr{G}$ valued scalar [10]:

$$\alpha = n^\lambda A_{\lambda|S_0}.$$

We set:

4.5 $$\Phi_{|S_0} = \varphi, \qquad n.\nabla\Phi_{|S_0} = \dot{\varphi}, \qquad \Psi_{|S_0} = \psi.$$

The system $(\mathscr{S})$ being invariant by the gauge transformation 1.3, 1.5, 1.6, is not well posed, we turn it into an hyperbolic system by imposing on A the lorentz-gauge condition:

4.6 $$\nabla_\lambda A^{\lambda, a} = 0.$$

If 4.6 is satisfied, 2.5 becomes:

4.7 $$\nabla_\lambda \nabla^\lambda A^{\mu, a} - R^\mu_\lambda A^{\lambda, a} + C^a_{bc} A^b_\lambda (F^{\lambda\mu, c} + \nabla^\lambda A^{\mu, c}) = J^{\mu, \alpha},$$

with $R_{\lambda\mu}$ the Ricci tensor of g.

The system does not appear as strictly hyperbolic, due to multiple characteristics in the Dirac operator, but will be proved to be equivalent (in the relevant spaces) to a quasi-diagonal semi-linear, second order, strictly hyperbolic system, by replacing equation 2.7 and Cauchy data $\Psi_{|S_0} = \psi$ by (*cf.* [5], $\nabla\!\!\!/ = \gamma^\alpha \nabla_\alpha$):

4.8 $$\nabla\!\!\!/\, \hat{\nabla\!\!\!/}\, \Psi \equiv \nabla^\lambda \nabla_\lambda \Psi + \frac{1}{4} R \Psi + e T \nabla (\gamma^\alpha A_\alpha \Psi) = \nabla\!\!\!/\, H.$$

and $\Psi_{|S_0} = \psi$ while $\dot{\psi} = \partial_0 \Psi_{|S_0}$ is determined by the original equation 2.7.

It is also possible to write directly for equation 2.7 the energy estimates which lead to the local existence theorem (for such a theorem for pure Yang-Mills fields (*cf.* [17], [24]), for Yang-Mills field coupled with gravity (*cf.* [4]).

THEOREM (local existence). — *Suppose given on* S_0 *the Cauchy data* $\varphi, a, \alpha, \psi \in H_s(S_0)$ *and* E, $\dot{\varphi} \in H_{s-1}(S_0)$, *with* $s > n/2$, $s \geq 2$ *satisfying the constraint* 4.3. *Then there exists an interval* $I_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbb{R}$, *and* $\Psi, \Phi, A \in E_s(I_\varepsilon \times S)$ *satisfying the system* $\mathscr{S}$ *and:*

$$\Psi_{|S_0} = \psi, \qquad \Pi\, A_{|S_0} = a, \qquad n.A_{|S_0} = \alpha, \qquad n.F_{|S_0} = E,$$
$$\Phi_{|S_0} = \varphi, \qquad \nabla_0 \Phi_{|S_0} = \dot{\varphi}.$$

The number ε *depends continuously on:*

$$M = \|\varphi\|_{H_s} + \|\psi\|_{H_s} + \|a\|_{H_s} + \|\dot{\varphi}\|_{H_{s-1}} + \|E\|_{H_{s-1}} + \|\alpha\|_{H_s}$$

and tends to zero when M *tends to zero* [11].

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[10] Note that α is not a "dynamical variable": it can always be taken zero without restricting the generality of solutions (*cf.* below, uniqueness theorem). It is known (Segal [24]) that on Minkowski space every solution of Yang-Mills equations is gauge equivalent to a solution in temporal gauge.

[11] The same result holds, when M is fixed and the coupling constants **C**, T, S, C, μ ... tend to zero.

Proof. — (1) Suppose that A satisfies the Lorentz condition:

4.9 $$\nabla_\lambda A^{\lambda, a} = 0.$$

It implies $\nabla_\lambda A^{\lambda, a}{}_{|S_0} = 0$, which determines $\dot{\alpha} = \partial_0 A^{0, a}{}_{|S_0} \in H_{s-1}$ once a, α are given. The Dirac equation 2.7 determines $\dot{\psi} = \partial_0 \Psi_{|S_0} \in H_{s-1}$, in terms of the other data.

The system 4.7, 4.8, 2.8 is a second order, semi-linear, quasidiagonal strictly hyperbolic system. The local existence, that is on a manifold $I_\varepsilon \times S$, of a solution of the Cauchy problem results from the general theorem of Leray [18], refined by Dionne in the space $H_s(I_\varepsilon \times S)$, $s > n/2 + 2$. The existence in $E_s(I_\varepsilon \times S)$, $s > n/2 + 1$ for second order, hyperquasilinear, quasi-diagonal systems has been proved by Hughes, Kato, Marsden [16] ($S = \mathbb{R}^n$) and Choquet-Bruhat, Christodoulou, Francaviglia [7]. The existence in $E_s(I_\varepsilon \times S)$, $s > n/2$, is a consequence of the semi-linearity and the particular form of the right hand side (linear in the first derivatives). It can be proved directly on 4.7, 2.7, 2.8 by writing energy type estimates and using the multiplication lemma:

$$H_{s_1} \times H_{s_2} \to H_s \quad \text{if} \quad s \leqq s_1, \quad s \leqq s_2, \quad s < s_1 + s_2 - \frac{n}{2},$$

which gives:

$$H_s \times H_{s-1} \to H_{s-1} \quad \text{if} \quad s > n/2.$$

(2) Let $(\Psi, \Phi, A) \in E_s(I_\varepsilon \times S)$, $s > n/2$, $s \geqq 2$, be a solution of 4.7, 4.8 and 2.8. Suppose the corresponding Cauchy data for $t = 0$ satisfy 2.7, 4.3 and 4.9.

(*a*) We deduce from 4.8 that $f \equiv \hat{\nabla}\!\!\!\!\!\diagdown\, \Psi - H \in E_{s-1}(I_\varepsilon \times S)$ satisfies the linear hyperbolic system:

4.10 $$\left(\nabla^\lambda \nabla_\lambda + \frac{1}{4} R\right) f = 0.$$

The corresponding Cauchy data $f_{t=0}$ and $\partial f / \partial t|_{t=0}$ are well defined if $s \geqq 2$; they vanish by hypothesis, and by the fact that 4.8 is satisfied. The distributions $Y_+ f$ [resp. $Y_- f$], with $Y_+(t, x) = 1$ if $t > 0$, $Y_+(t, x) = 0$ if $t < 0$ [resp. 1 if $t < 0$, 0 if $t > 0$] satisfy 4.10 and have support compact toward the past [resp. the future] they are therefore identically zero. Thus the equation 2.7 is satisfied on $I_\varepsilon \times S$.

(*b*) We know, that if (Ψ, Φ, A) satisfies 2.7, 2.8 it also satisfies the conservation equation:

$$\hat{\nabla}_\beta J^{\beta, a} = 0.$$

We deduce therefore from the identity $\hat{\nabla}_\alpha \hat{\nabla}_\beta F^{\alpha\beta, a} = 0$ that every solution $(\Psi, A, \Phi) \in E_s(I_\varepsilon \times S)$ of 3.6, 2.7, 2.8 satisfies:

$$\nabla^\beta \nabla_\beta (\nabla_\alpha A^{\alpha, a}) = 0, \qquad \nabla_\alpha A^{\alpha, a} \in E_{s-1}(I_\varepsilon \times S).$$

The same argument as in a) on hyperbolic systems implies $\nabla_\alpha A^{\alpha, a} = 0$ on $I_\varepsilon \times S$ if $\nabla_\alpha A^\alpha{}_{|S_0} = 0$ (satisfied by hypothesis) and $\nabla_0 (\nabla_\alpha A^\alpha)_{|S_0} = 0$ (satisfied as a consequence of the constraint 4.3 and the equation 3.6 of index $\beta = 0$).

The considered solution (Ψ, Φ, A) of 3.6, 2.7, 2.8 satisfies therefore the original system $\mathscr{S}$. ■

We shall now give a theorem of uniqueness up to gauge transformations.$\mathscr{S}$ A mapping $u: S_0 \to \omega$ [resp. $U: S_0 \times I \to \omega$] with $\omega \subset G$ the domain of a chart of G can be identified with a mapping $S_0 \to \mathbb{R}^N$ [resp. $S_0 \times I \to \mathbb{R}^N$], and the space $H_s(S_0)$ [resp. $E_s(S_0 \times I)$] for such mappings can be defined without the help of an invariant, positive, metric on G. On the other hand if G admits such a metric (for instance G is compact) these spaces have been defined, as Banach-Lie groups for mappings $u: S_0 \to G$ [resp. $U: S_0 \times I \to G$] by I. Segal [24], who has proved a global uniqueness theorem for Yang-Mills field equations in Minkowski space time, using the temporal gauge. We enunciate in our context the uniqueness theorem and sketch a proof using the Lorentz gauge.

THEOREM (uniqueness). — *Let* (A, Φ, Ψ) *and* (A_1, Φ_1, Ψ_1) *be two solutions of an* $\mathscr{S}$ *system in* $E_s(I \times S)$, *with Cauchy data on* S_0 *differing by a gauge transformation: that is there exists two mappings* $u: S_0 \to G$ *and* $\dot{u}: S_0 \to T_u G$, $u \in H_s$, $\dot{u} \in H_{s-1}$ *such that:*

4.11
$$\begin{cases} a_1 = u^{-1} a u + u^{-1} du, \qquad \alpha_1 = u^{-1} \alpha u + u^{-1} \dot{u}, \\ \varphi_1 = r(u)\varphi, \qquad \dot{\varphi}_1 = r(u)\dot{\varphi} + r'(u)\dot{u}\varphi, \\ \psi_1 = \rho(u)\psi. \end{cases}$$

We suppose, if G *is not compact, that u takes its values in a compact subset* K *of the domain* ω *of a chart at* **1**, *unit of* G.

Then there exists, if $s > 1 + n/2$ *an interval* $I_\varepsilon \subset I$ *and a gauge transformation* $U: I_\varepsilon \times S \to G$, $U \in E_s(I_\varepsilon \times S)$ *such that* $U_{|S_0} = u$, $(n.\nabla U)_{|S_0} = \dot{u}$ *and, on* $I_\varepsilon \times S$:

4.12
$$\begin{cases} A_1 = U^{-1} A U + U^{-1} dU, \\ \Phi_1 = r(U)\Phi, \qquad \Psi_1 = \rho(U)\Psi. \end{cases}$$

COROLLARY. — *If* G *is compact* U *exists on* $I \times S$.

Proof. — (1) We prove the local uniqueness by bringing the two solutions in the Lorentz gauge:

We look for a gauge transformation $V: S_0 \to G$ such that the potentiel **A**, the transform of A by V satisfies:

$$\nabla_\lambda \mathbf{A}^\lambda = 0,$$

A necessary and sufficient condition on V is:

$$\nabla_\lambda (V^{-1} A^\lambda V) + \nabla^\lambda (V^{-1} \partial_\lambda V) = 0.$$

There exists an interval $I_\eta = (-\eta, \eta) \subset I$ such that this second order semi linear hyperbolic system has a solution $V \in E_s(I_\eta \times S)$, with values in ω, if v has values in K, if $s > 1 + n/2$ (the equation is not linear in V), with Cauchy data $v \in H_s(S)$, with values in $K \subset \Omega$, $\dot{v} \in H_{s-1}(S)$. The potential **A** is then in $E_{s-1}(I_\eta \times S)$, the transforms $\mathbf{\Phi}$ and $\mathbf{\Psi}$ of Φ and Ψ are in $E_s(I_\eta \times S)$.

We apply the same reasoning to (A_1, Φ_1, Ψ_1). The transforms by V_1, $(\mathbf{A}_1, \mathbf{\Phi}_1, \mathbf{\Psi}_1)$ are in $E_{s-1}(I_{\eta_1} \times S) \times E_s(I_{\eta_1} \times S) \times E_s(I_{\eta_1} \times S)$. They satisfy the same hyperbolic system on $I_\varepsilon \times S$, $\varepsilon = \min(\eta, \eta_1)$, therefore they coincide if their Cauchy data coincide, which we obtain by choosing appropriate Cauchy data for V and V_1, for instance $v=u$ and $\dot{v}=\dot{u}$, $v_1=1$, $\dot{v}_1=0$. The required U is defined on $I_\varepsilon \times S$ by $U=V\circ V_1^{-1}$.

Remark. — The given proof shows that one does not restrict the class of solutions by choosing Cauchy data such that $A_{0|S_0}=0$: we can always make $\alpha_1=0$ by choice of $\dot{u}$.

(2) To prove the global uniqueness we consider the set of numbers $\varepsilon>0$ such that $U\in E_s(I_\varepsilon\times S)$ exists, satisfying 4.12 and taking Cauchy data u and $\dot{u}$, with $I_\varepsilon=(0, \varepsilon)$ [resp. $I_\varepsilon=(-\varepsilon, 0)$]. This set is both open (by the local uniqueness) and closed (if G is compact, because then, by 4.12, U is uniformly continuous on $\bigcup I_\varepsilon$) in the interval $I\cap\{t>0\}$ [resp. $I\cap\{t<0\}$]. ■

The methods used to prove global existence on Minkowski space time in the case of electromagnetic or Yang Mills field coupled to scalar fields (*cf.* Montcrief [20], Ginibre et Velo [15] for $n=2$, Eardley and Montcrief [11], for $n=3$) do not apply here since they rely on the non blow up of the H_2 norm, proved through the property of conservation of the physical energy. This energy is not a positive quantity in the presence of spinor fields, and cannot be used to limit the H_1 norms of the fields. We shall use an entirely different method in the case $n=3$, the conformal mapping of Minkowski space time onto a bounded set of the Einstein cylinder.

5. Conformal transformation of Dirac, Yang-Mills and Higgs operators

Let g and $\underline{g}$ be two conformal hyperbolic metrics on the manifold V :

5.1 $$g=\Omega^2\underline{g},$$

where Ω is a C^∞ positive function on V.

To a lorentzian frame on (V, g) corresponds a lorentzian frame $O_{\underline{g}}$ on (V, $\underline{g}$). The associated coframes are such that:

5.2 $$g=(\theta^0)^2-\sum_{i=1}^{n}(\theta^i)^2,\qquad \underline{g}=(\underline{\theta}^0)^2-\sum_{i=1}^{n}(\underline{\theta}^i)^2,$$

5.3 $$\theta^\alpha=\Omega\underline{\theta}^\alpha.$$

We shall underline all operations relative to $\underline{g}$ and all indexes relative to the frame $O_{\underline{g}}$.

We denote by $\omega^\alpha_{\lambda\beta}$ [resp $\underline{\omega}^{\underline{\alpha}}_{\underline{\lambda\beta}}$] the connection coefficients of g [resp. $\underline{g}$] in the frame O_g [resp. $O_{\underline{g}}$], by ∂_α [resp. $\underline{\partial}_{\underline{\alpha}}$] the Pfaff derivative with respect to θ^α [resp. $\underline{\theta}^{\underline{\alpha}}$].

A straight forward and well known computation, gives:

5.4 $$\begin{cases}\omega_{\lambda\beta,\,\alpha}=\Omega^{-1}\,\underline{\omega}_{\underline{\lambda\beta},\underline{\alpha}}+\Omega^{-2}\,(\eta_{\alpha\lambda}\,\underline{\partial}_\beta\,\Omega-\eta_{\beta\lambda}\,\underline{\partial}_\alpha\,\Omega),\\ \omega^\mu_{\lambda\beta}=\eta^{\alpha\mu}\,\omega_{\lambda\beta\alpha},\qquad \underline{\omega}^{\underline{\mu}}_{\underline{\lambda\beta}}=\underline{\eta}^{\underline{\alpha\mu}}\,\underline{\omega}_{\underline{\lambda\beta},\underline{\alpha}}\end{cases}$$

(recall: $\eta_{\alpha\lambda}=\eta^{\alpha\lambda}=\underline{\eta}_{\underline{\alpha\lambda}}=\underline{\eta}^{\underline{\alpha\lambda}}$, $(\eta_{\alpha\lambda})=\operatorname{diag}(1, -1, \ldots, -1)$).

The bundles of Lorentz frames on (V, g) and $(V, \underline{g})$ are isomorphic. We identify them by the bijective map $O_g \to O_{\underline{g}}$. Then, if SV is a principal spin bundle for (V, g), it is also a principal spin bundle over $(V, \underline{g})$, and conversely. A spinor field Ψ on (V, g) is then equal to a spinor field $\underline{\Psi}$ on $(V, \underline{g})$ if Ψ and $\underline{\Psi}$ are the same $\mathbb{C}^k$ vector in a given (arbitrary) spin-frame ([12]).

We choose for g and $\underline{g}$ the same Dirac matrices:

5.5 $$\gamma^\alpha = \underline{\gamma}^\alpha.$$

The spin connections σ and $\underline{\sigma}$ are then such that:

5.6 $$\sigma_\lambda = \Omega^{-1} \underline{\sigma_\lambda} + \frac{1}{4} \Omega^{-2} (\gamma_\lambda \gamma^\beta - \gamma^\beta \gamma_\lambda) \underline{\partial_\beta} \Omega.$$

DIRAC OPERATOR. — We deduce from 5.6:

$$\nabla_\lambda \Psi \equiv \Omega^{-1} \underline{\nabla_\lambda} \Psi + \frac{1}{4} \Omega^{-2} \underline{\partial_\beta} \Omega (\gamma_\lambda \gamma^\beta - \gamma^\beta \gamma_\lambda) \Psi;$$

thus, since $\gamma^\alpha = \underline{\gamma}^\alpha$:

5.7 $$\not\nabla \Psi \equiv \gamma^\alpha \nabla_\alpha \Psi \equiv \Omega^{-(1+n/2)} \underline{\not\nabla} \underline{\Psi},$$

if Ψ is the spinor-multiplet given by:

5.8 $$\underline{\Psi} = \Omega^{n/2} \Psi.$$

If the potential A is unchanged on V we have also:

5.9 $$\gamma^\mu T A_\mu \Psi \equiv \Omega^{-(1+n/2)} \underline{\gamma}^\mu T \underline{A_\mu} \underline{\Psi},$$

since T is linear and $\underline{A_\mu} = \Omega^{-1} A_\mu$. Therefore:

5.10 $$\hat{\not\nabla} \Psi \equiv \Omega^{-(1+n/2)} \underline{\hat{\not\nabla}} \underline{\Psi}.$$

Note that if $\underline{\Psi} = \Omega^{n/2} \Psi$ the Dirac adjoint is also such that:

5.11 $$\underline{\overline{\Psi}} = \Omega^{n/2} \overline{\Psi},$$

HIGGS OPERATOR. — We have set:

5.12 $$\hat{\Box} \Phi \equiv (\nabla^\alpha + S A^\alpha)(\nabla_\alpha + S A_\alpha) \Phi,$$

([12]) We could also choose (as Penrose [22]) to say that $\Psi = \underline{\Psi}$ on V if vectors of $\mathbb{C}^k$ representing them in a spin-frame are related by $\underline{\Psi}^A = \Omega^{1/2} \Psi^A$ (which we can also write $\Psi^{\underline{A}} = \Omega^{1/2} \Psi^A$). This identification is natural if one wants to identify vectors (in an appropriate sense) with tensor products of spinors.

that is:

5.13 $$\Box\hat{\Phi} \equiv \nabla^\alpha \nabla_\alpha \Phi + 2\,\mathrm{S}A^\alpha \nabla_\alpha \Phi + \mathrm{S}\,\nabla_\alpha A^\alpha \Phi + (\mathrm{S}A^\alpha)(\mathrm{S}A_\alpha)\Phi,$$

For conformal manifolds of dimension $n+1$ the following identity holds for each component of the scalar multiplet Φ, and therefore for Φ itself:

$$\nabla^\alpha \nabla_\alpha \Phi + \frac{n-1}{4n} R\,\Phi \equiv \Omega^{-(3+n/2)}\left(\underline{\nabla}^\alpha \underline{\nabla}_\alpha \underline{\Phi} + \frac{n-1}{4n} \underline{R}\,\underline{\Phi}\right),$$

with R and $\underline{R}$ the scalar curvature of g and $\underline{g}$ and:

5.14 $$\underline{\Phi} = \Omega^{(n-1)/2}\,\Phi.$$

Using 5.4, and since $A^\alpha = \Omega^{-1}\underline{A}^\alpha$, $A_\alpha = \Omega^{-1}\underline{A}_\alpha$, we obtain:

$$\nabla_\alpha A^\alpha \equiv \Omega^{-2}\,\underline{\nabla}_\alpha \underline{A}^\alpha + (n-1)\,\Omega^{-3}\,\underline{A}^\beta\,\underline{\partial}_\beta\,\Omega,$$

while:

$$A^\alpha \nabla_\alpha \Phi \equiv \Omega^{-(3+n)/2}\,\underline{A}^\alpha\,\underline{\nabla}_\alpha\,\underline{\Phi} + \frac{3-n}{2}\,\Omega^{-(5+n)/2}\,\underline{A}^\alpha\,\underline{\partial}_\alpha\,\Omega$$

and:

$$(\mathrm{S}A^\alpha)(\mathrm{S}A_\alpha) = \Omega^{-2}(\mathrm{S}\underline{A}^\alpha)(\mathrm{S}\underline{A}_\alpha).$$

Reassembling these identities we obtain:

5.15 $$\hat{\Box}\,\Phi + \frac{n-1}{4n}\,R\,\Phi \equiv \Omega^{-(3+n)/2}\left(\underline{\hat{\Box}}\,\underline{\Phi} + \frac{n-1}{4n}\,\underline{R}\,\underline{\Phi}\right).$$

YANG-MILLS OPERATOR. — Let F be a 2-form on V, with values in $\mathcal{G}$, whose components are $F_{\lambda\mu}$ in the frame dual of O_g. Let $F^{\#}$ be the contravariant tensor associated to F in the metric g. Its components in the frame O_g are the elements of $\mathcal{G}$ given by:

$$F^{\lambda\mu} = \eta^{\alpha\lambda}\,\eta^{\beta\mu}\,F_{\alpha\beta}.$$

By using 5.4 one finds:

5.16 $$\nabla_\lambda F^{\lambda\mu} \equiv \Omega^{-n}\,\underline{\nabla}_\lambda\,\underline{F}^{\lambda\mu}, \qquad \text{with} \quad \underline{F}^{\lambda\mu} = \Omega^{n-1}\,F^{\lambda\mu}.$$

The contravariant tensor with components $\underline{F}^{\lambda\mu}$ in $O_{\underline{g}}$ is associated through the metric $\underline{g}$ to the 2-form $\underline{F}$ with components in the dual frame:

$$\underline{F}_{\alpha\beta} = \eta_{\alpha\lambda}\,\eta_{\beta\mu}\,\underline{F}^{\lambda\mu} = \Omega^{n-1}\,F_{\alpha\beta};$$

the 2-forms F and $\underline{F}$ are therefore linked on V by the relation:

5.17 $$\underline{F} = \Omega^{n-3}\,F.$$

In particular, if $n+1=4$:

5.18 $$\underline{F} = F.$$

In this case, we can set:

5.19 $$A = \underline{A}$$

and we shall have both:

5.20 $$F = DA, \qquad \underline{F} = D\underline{A}$$

and:

$$\nabla . F^{\#} \equiv \Omega^{-4} \underline{\nabla} . \underline{F}^{\#},$$

where $F^{\#}$ and $\underline{F}^{\#}$ denote respectively the 2-contravariant tensors associated with F in the metrics g and $\underline{g}$.

When $n+1 \neq 4$ one cannot choose $\underline{A}$ such that the relations 5.17 and 5.20 both hold.

In the case $n+1=4$, $A=\underline{A}$, we deduce from 2.3:

$$\hat{\nabla} . F^{\#} \equiv \Omega^{-4} \underline{\hat{\nabla}} . \underline{F}^{\#}.$$

We deduce from the definitions 5.8, 5.14, with $n=3$ that:

$$\overline{\Psi} \gamma^\mu S^a \Psi = \Omega^{-3} \underline{\overline{\Psi}} \underline{\gamma}^\mu S^a \underline{\Psi}$$

and:

$$\tilde{\Phi} T^a \hat{\nabla}^\mu \Phi + \hat{\nabla}^\mu \tilde{\Phi} T^a \Phi = \Omega^{-3} (\underline{\tilde{\Phi}} T^a \hat{\nabla}^\mu \underline{\Phi} + \underline{\hat{\nabla}}^\mu \underline{\tilde{\Phi}} T^a \underline{\Phi})$$

(because T^a is a skew-hermitian linear operator, the term in $\partial^\mu \Omega$ vanishes). Thus, on V, with the definition 2.6 of J:

$$J = \Omega^{-4} \underline{J}$$

and

$$\hat{\nabla} . F^{\#} - J = \Omega^{-4} (\underline{\nabla} . \underline{F}^{\#} - \underline{J}).$$

If we take as source terms in the Dirac and Klein-Gordon operators the quantities 2.13 we obtain ($n=3$):

$$H = \Omega^{-5/2} \underline{H},$$

and:

$$K = \Omega^{-3} \underline{K}.$$

the equations are conformally invariant in the following sense:

$$\underline{\hat{\nabla}}\,\underline{\Psi} - \underline{H} \equiv \Omega^{5/2} (\hat{\nabla} \Psi - H),$$

$$\underline{\hat{\Box}}\,\underline{\Phi} + \frac{1}{6} \underline{R}\,\underline{\Phi} - \underline{K} \equiv \Omega^3 \left(\hat{\Box} \Phi + \frac{1}{6} R \Phi - K \right),$$

$$\underline{\Phi} = \Omega \Phi, \qquad \underline{\Psi} = \Omega^{3/2} \Psi.$$

We deduce from all these transformation properties the theorem:

THEOREM. — *If* (A, Ψ, Φ) *is a solution of the Yang Mills, Dirac, Higgs system, with* H *and* K *given by* 2.13, *on a four dimensional hyperbolic manifold* (V, g) *then* $(A, \underline{\Psi}, \underline{\Phi})$, *with* $\underline{\Psi} = \Omega^{3/2}\Psi$, $\underline{\Phi} = \Omega\Phi$ *is a solution of this system on* $(V, \underline{g})$, *with* $\underline{g} = \Omega^{-2} g$.

6. Conformal mapping from M_4 into Σ_4

We recall that the Einstein cylinder is the manifold $\mathbb{R} \times S^3$ endowed with the metric $g = (\mathbf{1}, \bar{g})$, with $\bar{g}$ the canonical metric of the S^3 sphere. In canonical coordinates $(T, \alpha, \theta, \varphi)$ on $\mathbb{R} \times S^3$ this metric reads:

6.1 $$g = dT^2 - (d\alpha^2 + \sin^2\alpha\,(d\theta^2 + \sin^2\theta\, d\varphi^2)).$$

LEMMA (Penrose [22]). — *The Minkowski space time* $M_4 \equiv (V, \eta)$, V *diffeomorphic to* $\mathbb{R}^4$, *is conformal to a bounded open set* (V, g) *of the Einstein cylinder* Σ_4. *One has:*

6.2 $$g = \Omega^2 \eta \quad \text{on } V.$$

In the canonical coordinates of M_4 *the metric* η *reads:*

6.3 $$\eta = dt^2 - \sum_{i=1}^{3} (dx^i)^2$$

and V *is represented by* $(t, x^i) \in \mathbb{R} \times \mathbb{R}^3$.

In the canonical coordinates of Σ_4, V is represented by:

6.4 $$\begin{cases} V: \; 0 \leqq \alpha < \Pi, \quad 0 \leqq \theta < \Pi, \quad 0 \leqq \varphi < 2\Pi, \\ \qquad \alpha - \Pi < T < \Pi - \alpha. \end{cases}$$

The correspondance between the two coordinates systems is:

6.5 $$x^1 = r \sin\theta \sin\varphi, \qquad x^2 = r \sin\theta \cos\varphi, \qquad x^3 = r\cos\theta,$$

with:

6.6 $$r = \frac{1}{2}\left(\operatorname{tg}\frac{T+\alpha}{2} - \operatorname{tg}\frac{T-\alpha}{2}\right),$$

6.7 $$t = \frac{1}{2}\left(\operatorname{tg}\frac{T+\alpha}{2} + \operatorname{tg}\frac{T-\alpha}{2}\right).$$

The conformal factor Ω^2 is expressed in coordinates (t, x^i) by:

6.8 $$\Omega^2 \equiv 4(1+u^2)^{-1}(1+v^2)^{-1},$$

where $u = t + r$, $v = t - r$, $r^2 = \sum_{i=1}^{3} (x^i)^2$, and in the coordinate $(T, \alpha, \theta, \varphi)$ by:

6.9 $$\Omega^2 \equiv (\cos\alpha + \cos T)^2.$$

The submanifold S_0 of M_4, with equation $t=0$, is diffeomorphic to $\mathbb{R}^3$. It is represented on Σ_4 by the submanifold $T=0$, minus the point I_0 of coordinates $T=0$, $\alpha=\Pi$.

The metrics η and g induce respectively on S_0 the euclidean metric e and the metric $\bar{g}$ of S^3, they are linked on S_0 by:

$$6.10 \qquad \bar{g}=\Omega^2|_{S_0} e=\sigma^{-4} e, \qquad \sigma=\left(\frac{1+r^2}{2}\right)^{1/2}.$$

We denote by D | resp. $\bar{\nabla}$ | the covariant derivation in the metric e | resp. $\bar{g}$ | of a tensor field on S. If ψ is the restriction to S of a spinor field Ψ on V we define $\bar{\nabla}\psi$ to be the covariant vector on S with spinor values, projection of $\nabla\Psi$ on S, $\bar{\nabla}\psi=\Pi\nabla\Psi|_S$; $\bar{\nabla}\psi$ depends only on ψ. In an orthonormal frame with time axis n it has components:

$$\bar{\nabla}_i\psi=\nabla_i\Psi|_S=\partial_i\psi+\sigma_i\psi.$$

The H_s spaces on $(S, \bar{g})$ [resp. S, e] are defined as usual. For instance:

$$\|\psi\|_{H_s(S,\bar{g})}=\left\{\sum_{k=0}^{s}\int_S |\bar{\nabla}^k\psi|_{\bar{g}}^2\, d\mu(\bar{g})\right\}^{1/2},$$

with:

$$|\bar{\nabla}^k\psi|_{\bar{g}}^2=\sum_{i_1,\ldots,i_k=1}^{3} \bar{\nabla}_{i_1\ldots i_k}\bar{\psi}\gamma^\lambda n_\lambda \bar{\nabla}^{i_1\ldots i_k}\psi$$

(quantity invariant under a change of associated spin and Lorentz frame if the Lorentz frame keeps n as time axis).

We also extends trivially the usual definition of $H_{s,\delta}$ spaces on $(\mathbb{R}^3, e)$:

DEFINITION. — A tensor [spinor] field (distribution) h on $\mathbb{R}^3$ with values in a vector space $\mathbb{R}^N$ or C^N is in $\underline{H}_{s,\delta}\equiv H_{s,\delta}(\mathbb{R}^3, e)$ if, for $0\leqq k\leqq s$, $D^k h$ is measurable and:

$$\sigma^{\delta+k}|D^k h|_e\equiv\sigma^{\delta+k}(\Sigma D_{i_1\ldots i_k} h D^{i_1\ldots i_k} h)^{1/2}\in L^2(\mathbb{R}^3, e).$$

LEMMA 1. — *If $\underline{h}$ is a p-covariant tensor field on $\mathbb{R}^3$, then $h=\underline{h}\sigma^\alpha\in H_s(S^3, \bar{g})$ if $\underline{h}\in\underline{H}_{s,\delta}$*, $\delta=s+2p-3+\alpha$.

The same is true, with $p=0$, if ψ is a section $S\to DV$ *of a bundle of spinors on* (V, g).

Proof. — (1) If $\underline{h}$ is a p-covariant tensor field on $\mathbb{R}^3$ it induces a p-covariant, almost everywhere defined tensor field on S^3 (correspondance between canonical coordinates given by 6.5 and 6.6 with $T=0$). The norms in the metric $\bar{g}$ and e are linked by the relation:

$$|\underline{h}|_{\bar{g}}=\sigma^{2p}|\underline{h}|_e.$$

The derivatives of h in the connexions $\bar{\nabla}$ of $\bar{g}$ and D of e are linked by:

$$\bar{\nabla} h=D h+\Sigma S.h,$$

where $\Sigma\, S.h$ is a bilinear expression in h and the tensor S, difference of the connexions of $\bar{g}$ and e, which is C^∞ on $S^3 \setminus I_0 \simeq \mathbb{R}^3$, and has components in the orthonormal frame of e (i. e. in cartesian coordinates on $\mathbb{R}^3$):

$$S^k_{ji} = \delta^k_j \partial_i \log \sigma - \delta_{ij} \partial_k \log \sigma,$$

with:

$$\partial_i \sigma = x^i/(1+r^2)^{1/2}.$$

There is therefore a constant C such that on $\mathbb{R}^3$:

$$|D^l S| \leqq C \sigma^{-(l+1)}$$

and a constant such that:

$$|\bar{\nabla}^k h|_{\bar{g}} = \frac{1}{2^{p+k}} \sigma^{2p+2k} |\bar{\nabla}^k h|_e \leqq C \sum_{l=0}^{k} \sigma^{2p+k+l} |D^l h|_e.$$

The conclusion follows from the fact that:

$$d\mu(\bar{g}) = \sigma^{-6} d\mu(e),$$

and that $\underline{h}\sigma^\alpha \in \underline{H}_{s,\delta}$ if $\underline{h} \in H_{s,\delta+\alpha}$.

(2) If ψ is a section $S \to DV$ of a bundle of spinors on (V, g) (which we have identified with a bundle of spinors on (V, η) through identification of the spin bundles), we have, by the definitions ([13]):

$$|\psi|_e = |\psi|_{\bar{g}}$$

and:

$$|\bar{\nabla}^k \psi|_{\bar{g}} \leqq C \sum_{l=0}^{k} \sigma^{k+l} |D^l \psi|_e.$$

The conclusion follows as before.

Remark. — The condition $\underline{h} \in H_{s,\delta}$, $\delta = s+2p-3+\alpha$ is not necessary for $h = \underline{h}\sigma^\alpha$ to be in $H_s(S^3, \bar{g})$. The tensor [spinor] field $k = \underline{h}\sigma^\alpha + f$ with f a C^s tensor [spinor] field on S^3 is in $H_s(S^3, \bar{g})$ if $\underline{h} \in H_{s,\delta}$; the corresponding $\underline{k}$ are of the form $\underline{h} + f\sigma^{-\alpha}$, $\underline{h} \in H_{s,\delta}$. For instance if we take $a \in C^2(S^3, \bar{g})$ then $a \in H_2(S^3, \bar{g})$ but $\underline{a} = a \in H_{2,1}(\mathbb{R}^3)$ if a does not vanish at I_0, since $|\underline{a}|_e = 2/(1+r^2)\,|a|_{\bar{g}}$.

7. Global solutions of the Cauchy problem on M^4

THEOREM. — *The Cauchy problem for the Yang-Mills Higgs Dirac conformally invariant system* 2.5, 2.7, 2.8 *with* 2.6, 2.13 *and* 2.14 *with data* $\underline{\psi}, \underline{a}, \underline{\alpha}, \underline{\varphi}, \underline{E}, \underline{\dot{\varphi}}$ *on* $\mathbb{R}^3$ *such that:*

$$\sigma^2 \alpha,\ \sigma^3 \underline{\psi},\ \underline{a},\ \sigma^2 \underline{\varphi} \in H_2(S^3); \qquad \sigma^2 \underline{E},\ \sigma^4 \underline{\dot{\varphi}} \in H_1(S^3),$$

([13]) With Penrose identification (*cf.* Note, § 5) we would have:

$$|\psi|^2_e = \bar{\psi}_{\underline{A}} \gamma^{\underline{\lambda}} n_\lambda \psi^{\underline{A}} = \Omega \bar{\psi}_A \gamma^\lambda n_\lambda \psi^A = \Omega |\psi|^2_{\bar{g}}.$$

satisfying the constraint $\widehat{\operatorname{div}}\,\underline{\mathrm{E}}=\mathscr{J}^{\underline{0}}(\underline{\alpha}, \underline{\varphi}, \underline{\dot{\varphi}}, \underline{\psi})$ admits on M^4 a global solution A, Φ, $\Psi \in \mathrm{H}_2^{\mathrm{loc}}(\mathbb{R}^4)$ if the number:

$$d \equiv \| \sigma^3 \underline{\psi} \|_{\mathrm{H}_2(\mathrm{S}^3)} + \| \underline{a} \|_{\mathrm{H}_2(\mathrm{S}^3)} + \| \sigma^2 \underline{\alpha} \|_{\mathrm{H}_2(\mathrm{S}^3)} + \| \sigma^2 \underline{\varphi} \|_{\mathrm{H}_2(\mathrm{S}^3)} + \| \sigma^2 \underline{\mathrm{E}} \|_{\mathrm{H}_1(\mathrm{S}^3)} + \| \sigma^4 \underline{\dot{\varphi}} \|_{\mathrm{H}_1(\mathrm{S}^3)},$$

is sufficiently small.

Remark 1. — Recall by the lemma of paragraph 6, a sufficient condition for $\sigma^3 \underline{\psi}$, $\underline{a}$, $\sigma^2 \underline{\alpha}$, $\sigma^2 \underline{\varphi} \in \mathrm{H}_2(\mathrm{S}^3)$ is $\underline{\psi} \in \mathrm{H}_{2,2}(\mathbb{R}^3)$, $\underline{a}$, $\underline{\alpha}$, $\underline{\varphi} \in \mathrm{H}_{2,1}(\mathbb{R}^3)$. Also $\sigma^2 \underline{\mathrm{E}}$, $\sigma^4 \underline{\dot{\varphi}} \in \mathrm{H}_1(\mathrm{S}^3)$ if $\underline{\mathrm{E}}$, $\underline{\dot{\varphi}} \in \mathrm{H}_{1,2}(\mathbb{R}^3)$.

Remark 2. — If morover the data $\underline{\psi}$, $\underline{a}$, $\underline{\alpha}$, $\underline{\varphi} \in \mathrm{H}_s^{\mathrm{loc}}(\mathbb{R}^3)$, $\underline{\mathrm{E}}$, $\underline{\dot{\varphi}} \in \mathrm{H}_{s-1}^{\mathrm{loc}}(\mathbb{R}^3)$, $s \geqq 2$, then the solution A, Φ, $\Psi \in \mathrm{H}_s^{\mathrm{loc}}(\mathbb{R}^4)$.

Proof. — We consider the Cauchy problem for the system on Σ^4. The Cauchy data are defined almost everywhere on S^3 by:

$$\psi = \sigma^3 \underline{\psi}, \qquad a = \underline{a}, \qquad \alpha = \sigma^2 \underline{\alpha}, \qquad \varphi = \sigma^2 \underline{\varphi}, \qquad \dot{\varphi} = (n.\nabla\Phi)_{|\mathrm{S}_0} = \sigma^4 \underline{\dot{\varphi}}$$

and $\mathrm{E} = \sigma^2 \underline{\mathrm{E}}$ (covariant vectors).

By the hypothesis of the theorem we have, on S^3, a, α, φ, $\psi \in \mathrm{H}_2$ and $\dot{\varphi}$, $\mathrm{E} \in \mathrm{H}_1$. The problem satisfies the hypotheses of the local theorem; there exists therefore a number $\mathrm{M} > 0$ such that if the norm of the Cauchy data is less than M a solution Ψ, Φ, A exists in $\mathrm{E}_2(\mathrm{S} \times \mathrm{I})$ with $\mathrm{I} =]-\Pi, \Pi[$. Since $\mathrm{S} \times \mathrm{I} \supset \mathrm{V}$, the fields $\underline{\Psi} = \Omega^{3/2} \Psi$, $\underline{\Phi} = \Omega\Phi$, $\underline{\mathrm{A}} = \mathrm{A}$ satisfy the system on Minkowski space $\mathrm{M}^4 = (\mathrm{V}, \eta)$ and take on $t = 0$ the given Cauchy data (note that $(\partial\Omega/\partial\mathrm{T})_{|\mathrm{S}_0} = (n.\nabla\Omega)_{|\mathrm{S}_0} = (\sin \mathrm{T})_{|\mathrm{T}=0} = 0$). Since Ω is a C^∞ function on $\mathrm{V} \simeq \mathbb{R}^4$ the solution $\underline{\Psi}$, $\underline{\Phi}$, $\underline{\mathrm{A}}$ is in $\mathrm{H}_2^{\mathrm{loc}}(\mathbb{R}^4)$.

Decay properties. — By the continuous inclusion property:

7.1 $$\mathrm{E}_2(\mathrm{S} \times \mathrm{I}) \subset \mathrm{C}^0(\overline{\mathrm{S} \times \mathrm{I}}).$$

Φ, $\Psi \in \mathrm{E}_2(\mathrm{S} \times \mathrm{I})$ imply that Φ and Ψ are uniformly continuous on $\mathrm{S} \times \mathrm{I}$. Hence there is a constant c such that:

7.2 $$(\underline{\tilde{\Phi}\Phi})^{1/2} \leqq c\,\Omega$$

and

7.3 $$(\underline{\overline{\Psi}}\, \gamma^\lambda n_\lambda \underline{\Psi})^{1/2} \leqq c\,\Omega^{3/2},$$

7.2 shows that $(\underline{\tilde{\Phi}\Phi})^{1/2}$ decays like $|s|^{-2}$ along a spacelike and timelike geodesics and like $|s|^{-1}$ along null geodesics, where s is an affine parameter on the geodesic. Expressing $n = \partial/\partial\mathrm{T}$ in the form:

$$n = f\frac{\partial}{\partial t} + k,$$

where k is a future directed null vectorfield, and taking into account the fact that:

$$\underline{\overline{\Psi}}\, \gamma^\lambda k_\lambda \underline{\Psi} \geqq 0,$$

we conclude from 7.3 that $(\tilde{\underline{\Psi}}\underline{\Psi})^{1/2}$ decays like $|s|^{-3}$ along spacelike and timelike geodesics and like $|s|^{-1}$ along null geodesics.

If in the proof of the theorem we assume ψ, a, α, $\varphi \in H_s(S^3)$, E, $\dot{\varphi} \in H_{s-1}(S^3)$, $s \geqq 2$, we obtain Ψ, Φ, $A \in E_s(S \times I)$. Hence taking $s=3$ we conclude by property 7.1 that F is uniformly continuous on $S \times I$. We shall deduce from this the decay properties of $\underline{F}$ in Minkowski space.

We introduce on M_4 the null tetrad:

$$\underline{l} = \frac{1}{\sqrt{2}}\, du, \qquad \underline{m} = \frac{1}{\sqrt{2}}\, dv, \qquad \underline{\zeta} = r\, d\theta, \qquad \underline{\xi} = r \sin\theta\, d\varphi.$$

The corresponding null tetrad on Σ_4, given by $l = 1/\sqrt{2}\, dU$, $m = 1/\sqrt{2}\, dV$, $\zeta = \sin\alpha\, d\theta$, $\xi = \sin\alpha \sin\theta\, d\varphi$, $U = T - \alpha$, $V = T + \alpha$, is orthogonal with respect to the positive definite metric Γ of paragraph 3:

$$\Gamma^{\mu\nu} \equiv g^{\mu\nu} + 2\, n^\mu n^\nu.$$

The relation between the two tetrads is:

$$\underline{l} = \frac{(1+u^2)}{2}\, l, \qquad \underline{m} = \frac{(1+v^2)}{2}\, m, \qquad \underline{\zeta} = \Omega^{-1} \zeta, \qquad \underline{\xi} = \Omega^{-1} \xi.$$

Therefore we have:

$$7.4 \qquad \left\{ \begin{aligned} (\underline{F}_{\underline{l}\underline{\zeta}}, \underline{F}_{\underline{l}\underline{\xi}}) &= (2\Omega/(1+u^2))(F_{l\zeta}, F_{l\xi}) \\ (\underline{F}_{\underline{\zeta}\underline{\xi}}, \underline{F}_{\underline{l}\underline{m}}) &= \Omega^2 (F_{\zeta\xi}, F_{lm}) \\ (F_{m,\zeta}, \underline{F}_{\underline{m}\underline{\xi}}) &= (2\Omega/(1+v^2))(F_{m\zeta}, F_{m\xi}). \end{aligned} \right.$$

By the uniform continuity of F on $S \times I$ the components of F in the Γ-orthonormal tetrad (l, m, ζ, ξ) are uniformly bounded on V. Then the decay properties of $\underline{F}$ in Minkowski space follow from relations 7.4.

UNIQUENESS. — The local uniqueness theorem (§ 4) applied to Σ^4 gives the following corollary, which can be translated into a global uniqueness theorem for small data in Minkowski space.

COROLLARY (uniqueness). — *If the data* $(a, \alpha, \varphi, \psi) \in H_s(S^3)$, $(\dot{\varphi}, E) \in H_{s-1}(S^3)$, $s \geqq 3$ *are small enough in the* $H_s(S^3) \oplus H_{s-1}(S^3)$ *norms [or if the group* G *is compact], then every solution* $(A, \Phi, \Psi) \in E_s((-\mathbb{T}, \mathbb{T}), S)$ *taking these Cauchy data on* S_0 *coincides by a gauge transformation with the one previously constructed in the Lorentz gauge of* Σ^4.

8. Higher couplings

If the right hand sides $\underline{H}$ and $\underline{K}$ of the Dirac and Higgs equations of an $\mathscr{S}$ system on $(V, \underline{g})$ are polynomials in $\underline{\Psi}$, $\underline{\overline{\Psi}}$, Φ, $\underline{\tilde{\Phi}}$ such that when $\underline{\Psi} = \Omega^{3/2} \Psi$, $\underline{\Phi} = \Omega \Phi$ we have $\underline{H} = \Omega^{5/2+\alpha} H$, $\underline{K} = \Omega^{3+\beta} K$, then $(A, \underline{\Psi}, \underline{\Phi})$ will satisfy the system on $(V, \underline{g})$ if and only if (A, Ψ, Φ) satisfy on (V, g), $g = \Omega^2 \underline{g}$, the Yang Mills equations 2.5 with J given by 2.6 and:

$$\hat{\nabla} \Psi = \Omega^\alpha H, \qquad \hat{\square} \Phi + \frac{1}{6} R\, \Phi = \Omega^\beta K.$$

If we have $\alpha \geqq 0$, $\beta \geqq 0$ then Ω^α and Ω^β are C^∞ functions on Σ^4, and the global existence theorem on Minkowski space time proved for conformally invariant systems in paragraph 7 is valid without change.

An example of such a coupling is the "four fermions coupling", combined or not with a higher degree interaction of the scalar multiplet, that is:

$$\underline{H} = \lambda \{ (\overline{\underline{\Psi}}\, C_1 \underline{\Psi})\, C_2 \Psi + (\overline{\underline{\Psi}}\, C_2 \underline{\Psi})\, C_1 \Psi \},$$
$$K = \mu (\tilde{\underline{\Phi}}\, \underline{\Phi})^p\, \underline{\Phi},$$

where $p \geqq 1$ and C_1, C_2 are linear hermitian maps $\mathbb{C}^l \to \mathbb{C}^l$ such that, to insure $\underline{\hat{\nabla}} \,.\, \underline{J} = 0$:

$$SC_1 - C_1 S = 0, \qquad SC_2 - C_2 S = 0,$$

where S is the map defined paragraph 2 for the spinor representation of G.

More generally the coupling of an $\mathscr{S}$ system on M^4 will be said to be "conformally regular" if the conformally transformed equations from M^4 to Σ^4 are again an $\mathscr{S}$ system. The global existence theorem applies to conformally regular $\mathscr{S}$ systems.

9. Existence in other conformally flat space times, for instance De Sitter space time

The method obviously applies to prove existence in various conformally flat space-times. We give the details in the case of De Sitter space time.

De Sitter space time is the manifold $\mathbb{R} \times S^3$, with the metric where a is some positive constant:

$$\underline{g} = \underline{ds^2} = dt^2 - a^2\, ch^2 (a^{-1} t)(d\alpha^2 + \sin^2 \alpha\, (d\theta^2 + \sin^2 \theta\, d\varphi^2)),$$
$$-\infty < t < \infty, \qquad 0 \leqq \alpha < \Pi, \qquad 0 \leqq \theta < \Pi, \qquad 0 \leqq \varphi < 2\Pi.$$

It satisfies the Einstein equations with cosmological constant a:

$$R_{\lambda\mu} + a g_{\lambda\mu} = 0.$$

De Sitter space time is conformal to M^4 and to a bounded open set (V, g) of the Einstein cylinder Σ^4. The correspondance between the canonical coordinates $(t, \alpha, \theta, \varphi)$ of De Sitter, and $(T, \alpha, \theta, \varphi)$ of Σ^4 is, in V:

$$T = 2 \operatorname{Arctg}(\exp(a^{-1} t)) - \frac{\Pi}{2}, \qquad -\frac{\Pi}{2} < T < \frac{\Pi}{2},$$

that is:

$$t = a \log tg\left(\frac{T}{2} + \frac{\Pi}{4}\right), \qquad -\infty < t < \infty.$$

We have, on V:

$$g = \Omega^2 \underline{g},$$

with:

$$\Omega^2 = a^{-2}\,\mathrm{ch}^{-2}(a^{-1}\,t) = a^{-2}\sin^2\left(\mathrm{T}+\frac{\Pi}{2}\right).$$

The image of $t=0$ is $\mathrm{T}=0$ and the image of $t=$constant is also a submanifold $\mathrm{T}=$Const. of Σ^4.

On $\mathrm{S}_0 \equiv \{t=\mathrm{T}=0\}$ we have the relation between induced metrics:

$$\bar{g} \equiv g_{|\mathrm{S}_0} = a^{-2}\,\bar{\underline{g}} \equiv (\Omega^2\,\underline{g})_{|\mathrm{S}_0},$$

the norms in the metrics $\bar{g}$ and $\bar{\underline{g}}$ are therefore equivalent. Moreover:

$$\partial_\lambda\,\Omega\,|_{\mathrm{S}_0} = 0,$$

the connections ∇ and $\underline{\nabla}$ are therefore identical on S_0.

The results of the previous paragraphs give the following theorem:

THEOREM. — *The Cauchy problem for the Yang-Mills equations, coupled with scalar and spinor fields*, 2.5, 2.7, 2.8 *with* 2.6, 2.13 *and* 2.14, *with data on* S^3:

$$\underline{\psi},\ \underline{a},\ \underline{\alpha},\ \underline{\varphi} \in \mathrm{H}_2; \qquad \underline{\mathrm{E}},\ \dot{\underline{\varphi}} \in \mathrm{H}_1,$$

satisfying the constraint $\widehat{\mathrm{div}}\,\underline{\mathrm{E}} = n\,.\,\mathrm{J}(\underline{\varphi},\ \dot{\underline{\varphi}},\ \underline{\psi}^\alpha)$, *admits a global solution on the De Sitter space-time if the norms of the Cauchy data, or the coupling constants are small enough.*

10. Initial value constraint on a compact manifold S

The Cauchy data a, α, E, φ, $\dot{\varphi}$, ψ must satisfy on the initial manifold S the constraint (we set $n\,.\,\mathrm{J}_{|\mathrm{S}} = q$):

10.1 $$\widehat{\mathrm{div}}\ \mathrm{E} = q.$$

q is a given smooth function of α, φ, $d\varphi$, $\dot{\varphi}$, ψ and their hermitian conjugates, while $\widehat{\mathrm{div}}$ E is a linear operator on E depending on a:

10.2 $$\widehat{\mathrm{div}}:\quad \mathrm{E} \mapsto \mathrm{div}\ \mathrm{E} + [a,\ \underline{\mathrm{E}}].$$

where div is the divergence in the metric $\bar{g}$ induced by g on S.

We recall that, for a $\mathcal{G}$-valued fonction of type Ad we have defined:

10.3 $$\widehat{\mathrm{grad}}\,f = df + [a, f].$$

The operators $\widehat{\mathrm{grad}}$ and $\widehat{\mathrm{div}}$ are formal adjoints.

The operator $\hat{\Delta} \equiv \widehat{\mathrm{div}}\ \widehat{\mathrm{grad}}$ is self adjoint, elliptic, quasi-diagonal with principal part the classical laplacian $\Delta \equiv \nabla^i\,\nabla_i$.

The kernels of $\hat{\Delta}$ and $\widehat{\mathrm{grad}}$ coincide if the Ad-invariant metric on $\mathcal{G}$ is positive definite, therefore if G is the product of an abelian Lie group by a compact semi-simple Lie group. The Fredholm alternative leads then, as in the classical Berger-Ebin case, to the continuous decomposition:

$$\mathrm{H}_s = \text{range}\ \widehat{\mathrm{div}} \oplus \ker\ \mathrm{grad},$$

$$\mathrm{H}_s = \text{range}\ \widehat{\mathrm{grad}} \oplus \ker\ \widehat{\mathrm{div}}.$$

the sumands are L^2 orthogonal in the sense of the integral over $(S, \bar{g})$ of the scalar product in both the metric $\bar{g}$ and the metric of $\mathscr{G}$; these decompositions can be formulated as a lemma.

LEMMA. — *If* G *is a product of an abelian Lie group by a compact semi-simple Lie group the constraint on a compact manifold* S:

$$\widehat{\operatorname{div}}\ E = q,$$

admits solutions E, *for a given* q, *if and only if this* q *is orthogonal to the space* $\ker \widehat{\operatorname{grad}}$.

If q is L_2-orthogonal to ker grad, then the space of solutions of 10.1 is the affine space:

$$E = \widehat{\operatorname{grad}\, f} \oplus \mathbf{E},$$

where f is the unique solution orthogonal to $\ker \widehat{\operatorname{grad}}$ of:

$$\hat{\Delta} f = q,$$

and $\widehat{\operatorname{div}}\ \mathbf{E} = 0$.

A potential a on S is called generic if is such that $\ker \widehat{\operatorname{grad}} = 0$. By the lemma the equation div $E = q$ have solutions for arbitrarily given q if and only if a is generic.

We know that $f \in \ker \widehat{\operatorname{grad}}$, if and only if it generates a 1-parameter group of automorphisms of a since, by the gauge transformation law of a connection:

$$\delta_f a = [a, f] + df = \widehat{\operatorname{grad}}\, f.$$

As a consequence, if $f \in \ker \widehat{\operatorname{grad}}$ then:

10.4 $$[f, H] = 0,$$

where H is the Ad-invariant magnetic Yang Mills vector field:

$$H = \star \left(da + \frac{1}{2} [a, a] \right)$$

with $\star$ the duality Hodge operator in the metric $\bar{g}$. We have moreover, if $f \in \ker \widehat{\operatorname{grad}}$:

10.5 $$f.f = (\text{Const.}) \quad \text{on S.}$$

If the structure constants are totally antisymmetric (which can be always supposed if G is compact and semi-simple) the equation (1) implies that H must be of the form:

10.6 $$H = \mu \otimes \mathscr{H},$$

where μ is a vector field on S and $\mathscr{H}$ a $\mathscr{G}$ valued function. The solutions are then given by, λ being a constant:

$$f = \lambda \frac{\mathscr{H}}{(\mathscr{H} . \mathscr{H})^{1/2}},$$

The kernel of $\widehat{\operatorname{grad}}$ is non empty (and coincides with the f's of the above type) if and only if $(\mathscr{H} . \mathscr{H})\ \widehat{\operatorname{grad}}\ \mathscr{H} - \mathscr{H} (\mathscr{H} . \widehat{\operatorname{grad}}\ \mathscr{H}) = 0$.

AN EXAMPLE OF A GENERIC POTENTIAL (14). — Take as a potential a on the 3-dimensional manifold S:

$$a = \sum_{A=1}^{3} \mu_{(A)} \mathcal{A}_{(A)},$$

where $\mathcal{A}_{(A)}$ are three linearly independant elements of $\mathcal{G}$, and $\mu_{(A)}$ three linearly independent 1-forms on S which we suppose to be closed. The corresponding magnetic field is:

$$H = \star\left(da + \frac{1}{2}[a, a]\right) = \star\frac{1}{2}[a, a],$$

of the form, non compatible with 10.6:

$$H = \sum_{A=1}^{3} \nu_{(A)} \otimes \mathcal{A}_{(A)},$$

where the $\nu_{(A)}$ are non zero 2-forms, if G is semi-simple.

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