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HYPERSURFACES OF EINSTEIN MANIFOLDS

BY NORIHITO KOISO ⁽¹⁾

0. Introduction and results

Let (\bar{M}, \bar{g}) be an Einstein manifold of dimension $n + 1$ ($n \geq 2$). We consider certain classes of hypersurfaces in (\bar{M}, \bar{g}) . First, let (M, g) be a totally umbilical hypersurface in (\bar{M}, \bar{g}) , i. e., we assume that the second fundamental form α satisfies $\alpha = fg$ for some function f on M . If we know completely the curvature tensor of (\bar{M}, \bar{g}) , we can get much information on (M, g) . For example, if (\bar{M}, \bar{g}) is a symmetric space, then (M, g) is also a locally symmetric space, and so the classification of such pairs $[(\bar{M}, \bar{g}), (M, g)]$ reduces to Lie group theory (see Chen [4] ⁽²⁾, Chen and Nagano [5], Naitoh [10]). But if we know nothing about (\bar{M}, \bar{g}) , we can only say that (M, g) has constant scalar curvature. In fact, we will prove the following.

THEOREM A. — *Let (M, g) be a real analytic riemannian manifold with constant scalar curvature. Then, there exists an Einstein manifold (\bar{M}, \bar{g}) (which may be non-complete) such that (M, g) is isometrically imbedded into (\bar{M}, \bar{g}) as a totally geodesic hypersurface.*

This Theorem means also that there exist many examples of totally geodesic Einstein hypersurfaces in Einstein manifolds. But, if we assume that (\bar{M}, \bar{g}) is complete (or compact), the situation changes drastically. In fact, we will show the following.

THEOREM B. — *Let (M, g) be a totally umbilical Einstein hypersurface in a complete Einstein manifold (\bar{M}, \bar{g}) . Then the only possible cases are:*

- (a) *g has positive Ricci curvature. Then g and \bar{g} have constant sectional curvature;*
- (b) *\bar{g} has negative Ricci curvature. If \bar{M} is compact or (\bar{M}, \bar{g}) is homogeneous, then g and \bar{g} have constant sectional curvature;*
- (c) *g and \bar{g} have zero Ricci curvature. If (\bar{M}, \bar{g}) is simply connected, then (\bar{M}, \bar{g}) decomposes as $(\tilde{M}, \tilde{g}) \times \mathbf{R}$, where (\tilde{M}, \tilde{g}) is a totally geodesic hypersurface in (\bar{M}, \bar{g}) which contains (M, g) .*

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⁽²⁾ Theorem 1 is not true as stated, but Theorem 2 is true. See Proof of Proposition 15 in Naitoh [10].

To prove this Theorem, we need essentially a result of D. M. DeTurck and J. L. Kazdan according to which all Einstein metrics are real analytic. In other words, the manifold (\bar{M}, \bar{g}) in Theorem A is uniquely defined by (M, g) (Prop. 4). If we apply Proposition 4 to a Kähler-Einstein manifold (\bar{M}, \bar{g}) , we can get much information on (M, g) and (\bar{M}, \bar{g}) , even without assuming anything on (M, g) , since in this situation, the Gauss-Codazzi equations imply many properties of (M, g) .

THEOREM C. — *Let (\bar{M}, \bar{g}) be a simply connected complete Kähler-Einstein manifold with Ricci curvature \bar{e} . If there exists a totally geodesic real hypersurface (M, g) in (\bar{M}, \bar{g}) , then there exists a totally geodesic complex hypersurface (\tilde{M}, \tilde{g}) in (\bar{M}, \bar{g}) , and (\bar{M}, \bar{g}) decomposes as $(\bar{M}, \bar{g}) = (\tilde{M}, \tilde{g}) \times (S, \bar{e})$, where (S, \bar{e}) means the simply connected and complete Riemann surface of constant Ricci curvature \bar{e} . In this decomposition, M is contained in $\tilde{M} \times \text{Im } \gamma$, where γ is a geodesic in S .*

Remark that Theorem C holds locally even if (\bar{M}, \bar{g}) is not complete. Next, let (M, g) be an orientable minimal hypersurface in an orientable manifold (\bar{M}, \bar{g}) . By Corollary 3.6.1 in Simons [11], if \bar{g} has positive Ricci curvature, then there is no orientable compact stable minimal hypersurface in (\bar{M}, \bar{g}) . By a similar method, we will show.

THEOREM D. — *Let (\bar{M}, \bar{g}) be an orientable Einstein manifold with zero Ricci curvature. Then all orientable compact stable minimal hypersurfaces without singularity are totally geodesic.*

Combining with Theorem C, we will get.

COROLLARY E. — *Let (\bar{M}, \bar{g}) be a Kähler-Einstein manifold with zero Ricci curvature and without local factor \mathbb{C} . Then there is no orientable compact stable minimal real hypersurface without singularity.*

Remark that we do not assume in Theorem A, B, C that (M, g) is complete. The paper is organized as follows: In 1, we derive some fundamental formulae and prove Theorem D. In 2, we consider the real case and prove Theorem A and Theorem B. In 3, we consider the Kähler case and prove Theorem C and Corollary E. The author would like to express his sincere gratitude to Professors J.-P. Bourguignon and R. Michel. Theorem A is an answer to a question of R. Michel and Corollary E is a generalization of a remark of J.-P. Bourguignon.

1. Preliminary and propositions

Let (\bar{M}, \bar{g}) be an Einstein manifold of dimension $n+1 \geq 3$ and M a hypersurface in (\bar{M}, \bar{g}) with induced metric g . In this paper, riemannian manifolds are not assumed to be complete, unless otherwise stated. The second fundamental form α is given by:

$$\alpha(X, Y)N = \bar{D}_X Y - D_X Y,$$

where N is the unit normal vector field, X and Y are vector fields on M , and D (resp. \bar{D}) is the

covariant derivative of (M, g) [resp. (\bar{M}, \bar{g})]. The following formulae are known as the Gauss-Godazzi equations:

$$\begin{aligned} \bar{R}(X, Y; Z, U) &= R(X, Y; Z, U) + \alpha(X, U)\alpha(Y, Z) - \alpha(X, Z)\alpha(Y, U), \\ \bar{R}(X, Y; Z, N) &= (D_Y \alpha)(X, Z) - (D_X \alpha)(Y, Z), \end{aligned}$$

where R (resp. \bar{R}) is the curvature tensor of (M, g) [resp. (\bar{M}, \bar{g})] and the sign convention is taken in such a way that $R(X, Y; X, Y) \geq 0$ for the standard sphere. Set:

$$\bar{R}(X, N; Y, N) = \beta(X, Y).$$

Then, the Ricci tensor \bar{r} of (\bar{M}, \bar{g}) is given by:

$$\begin{aligned} \bar{r}(X, Y) &= r(X, Y) + \alpha^2(X, Y) - \mu\alpha(X, Y) + \beta(X, Y), \\ \bar{r}(X, N) &= (d\mu)(X) + (\delta\alpha)(X), \\ \bar{r}(N, N) &= \text{tr } \beta, \end{aligned}$$

where r is the Ricci tensor of (M, g) , μ is the mean curvature defined by $\mu = \text{tr } \alpha$, and α^2 and $\delta\alpha$ are defined by:

$$\begin{aligned} (\alpha^2)_{ij} &= \alpha_i^k \alpha_{kj}, \\ (\delta\alpha)_i &= -D^k \alpha_{ki}. \end{aligned}$$

Since \bar{g} is an Einstein metric, i. e., $\bar{r} = \bar{e}g$ for some real number \bar{e} , we see that:

$$\begin{aligned} (1.1.a) \quad \bar{e}g &= r + \alpha^2 - \mu\alpha + \beta, \\ (1.1.b) \quad 0 &= d\mu + \delta\alpha, \\ (1.1.c) \quad \bar{e} &= \text{tr } \beta, \end{aligned}$$

and so:

$$(1.2) \quad (n-1)\bar{e} = u + \text{tr } \alpha^2 - \mu^2,$$

where u is the scalar curvature of (M, g) . Thus it is easy to check the following.

PROPOSITION 1. — *If (M, g) is a minimal hypersurface (i. e., $\mu = 0$) of an Einstein manifold (\bar{M}, \bar{g}) , then $u \leq (n-1)\bar{e}$. Equality holds if and only if (M, g) is a totally geodesic hypersurface in (\bar{M}, \bar{g}) .*

PROPOSITION 2. — *If (M, g) is a totally umbilical hypersurface of an Einstein manifold (\bar{M}, \bar{g}) , i. e., $\alpha = fg$ for some $f \in C^\infty(M)$, then f is constant and $u \geq (n-1)\bar{e}$. Equality holds if and only if (M, g) is a totally geodesic hypersurface in (\bar{M}, \bar{g}) .*

Proof. — By (1.1.b), $0 = d \text{tr}(fg) + \delta(fg) = (n-1)df$, so f is constant. Since $\mu = nf$ and $\text{tr } \alpha^2 = nf^2$, the latter half is obvious by (1.2).

Q.E.D.

Without any further property of β , we cannot proceed any more. To answer the question "What is the meaning of β ?" we consider a one-parameter family of hypersurfaces in (\bar{M}, \bar{g}) . Denote by i and i_t the mappings: $M \times \mathbf{R} \rightarrow \bar{M}$ and $M \rightarrow \bar{M}$, defined by:

$$i(x, t) = \exp_x t N, \quad i_t(x) = i(x, t).$$

Then there is an open set R of $M \times \mathbf{R}$ containing $M \times \{0\}$ such that $g_t = i_t^* \bar{g}$ is a riemannian metric on $\{x \in M; (x, t) \in R\}$. We identify \bar{M} with its image R (locally) and we see that $g_t + dt^2$ coincides with \bar{g} . In fact, N extends as the vector field d/dt , whose integral curves are geodesics in (\bar{M}, \bar{g}) , and:

$$\frac{d}{dt} \bar{g}(X, N) = \bar{g}(\bar{D}_N X, N) + \bar{g}(X, \bar{D}_N N) = \bar{g}(\bar{D}_X N, N) = \frac{1}{2} X(\bar{g}(N, N)) = 0,$$

where we identify $X \in T_x M$ with the vector field along the geodesic $i_t(x)$ defined by $X(i_t(x)) = i_t^* X$. We derive the relation between g', g'' and α, β , where $'$ means the derivative with respect to t :

$$\begin{aligned} g'(X, Y) &= (\bar{g}(X, Y))' = \bar{g}(\bar{D}_N X, Y) + \bar{g}(X, \bar{D}_N Y) \\ &= X(\bar{g}(N, Y)) - \bar{g}(N, \bar{D}_X Y) + Y(\bar{g}(X, N)) - \bar{g}(\bar{D}_Y X, N) = -2\alpha(X, Y), \\ \beta(X, Y) &= \bar{g}(\bar{D}_X N, Y) = \bar{g}(\bar{D}_{[X, N]} Y - \bar{D}_X \bar{D}_N Y + \bar{D}_N \bar{D}_X Y, N) \\ &= -\bar{g}(\bar{D}_X \bar{D}_Y N, N) + (\bar{g}(\bar{D}_X Y, N))' - \bar{g}(\bar{D}_X Y, \bar{D}_N N) \\ &= -X(\bar{g}(\bar{D}_Y N, N)) + \bar{g}(\bar{D}_Y N, \bar{D}_X N) + (\alpha(X, Y))'. \end{aligned}$$

Here, $\bar{g}(\bar{D}_Y N, N) = 0$ and $\bar{g}(\bar{D}_Y N, X) = -\alpha(X, Y)$. Thus we get:

$$(1.3) \quad g' = -2\alpha,$$

$$(1.4) \quad \beta = \alpha^2 - (1/2)g''.$$

The Einstein equation becomes:

$$\begin{aligned} \bar{e}g &= r + (1/2)(g')^2 - (1/4)(\text{tr } g')g' - (1/2)g'', \\ 0 &= -(1/2)d \text{tr } g' - (1/2)\delta g', \\ \bar{e} &= -(1/2)\text{tr } g'' + (1/4)\text{tr } (g')^2. \end{aligned}$$

We conclude that:

$$(1.5.a) \quad g'' = -2\bar{e}g + 2r - (1/2)(\text{tr } g')g' + (g')^2,$$

$$(1.5.b) \quad d \text{tr } g' + \delta g' = 0,$$

$$(1.5.c) \quad \text{tr } (g')^2 - (\text{tr } g')^2 = 4(n-1)\bar{e} - 4u.$$

Remark that these equations hold on R , where $r, \text{tr } ()^2, \delta$ and u are defined by g_t . We shall solve this equation in 2.

Before developing this equation, we point out some facts related to Proposition 1. Assume that M is compact without boundary and that i_0 is a *stable* minimal immersion. (Here, stable means: the second derivative of volume is non-negative for any variation.) Then, if the unit normal vector field N is globally defined on M :

$$0 \leq \left(\int_M v_g \right)''_{t=0} = -\frac{1}{2} \int_M \text{tr}(g')^2 v_g + \frac{1}{2} \int_M \text{tr} g'' v_g + \frac{1}{4} \int_M (\text{tr} g')^2 v_g,$$

where v_g denotes the volume element of g . By (1.3) and (1.4), we see that:

$$0 \leq \int_M (-2(\alpha, \alpha) - (\text{tr} \beta - \text{tr} \alpha^2)) v_g = - \int_M (\text{tr} \alpha^2 + \bar{e}) v_g.$$

Here, $\text{tr} \alpha^2 + \bar{e} = n\bar{e} - u$ by (1.2), and we get:

PROPOSITION 3. — *If (M, g) is compact without boundary and immersed in an Einstein manifold (\bar{M}, \bar{g}) as a stable minimal hypersurface with trivial normal bundle then:*

$$\int_M uv_g \geq n\bar{e} \text{Vol}(M, g).$$

Moreover, if $\bar{e} = 0$, then $u = 0$ and (M, g) is totally geodesic.

Proof. — The integral inequality is obvious. If $\bar{e} = 0$, then $\int_M uv_g \geq 0$. But Proposition 1 implies $u \leq 0$, so $u = 0$. Then the equality in Proposition 1 holds, so (M, g) is totally geodesic.

Q.E.D.

Proof of Theorem D. — It is obtained as a corollary of Proposition 3.

Q.E.D.

Remark 4. — In Theorem D, if \bar{M} is simply connected, then the assumption that M is orientable is not necessary. In fact, Lemma 4.5 and Theorem 4.6 in Hirsch [8] says that all compact hypersurfaces in a simply connected manifold are orientable.

2. Solution of (1.5) — real case

Consider equation (1.5). Theorem 5.2 in DeTurck and Kazdan [6] says that all Einstein metrics are real analytic with respect to harmonic coordinates. This implies that the solution of (1.5) is unique for given initial data $g = g_0$ and $g' = h$, as long as g_t is positive definite. Moreover, we get the following global uniqueness property.

PROPOSITION 5. — *Let (M, g) be a real analytic hypersurface of a simply connected and complete Einstein manifold (\bar{M}, \bar{g}) with second fundamental form α . Assume that there is another simply connected and complete Einstein manifold (\bar{M}_1, \bar{g}_1) such that (M, g) is imbedded*

into $(\overline{M}_1, \overline{g}_1)$ as a real analytic hypersurface with the same second fundamental form α . Then $(\overline{M}, \overline{g})$ and $(\overline{M}_1, \overline{g}_1)$ are isometric with one another.

Proof. — By the uniqueness Theorem 5.4 in DeTurck and Kazdan [6].

Q.E.D.

Conversely, by Cauchy-Kovalevski's existence Theorem, we can solve (1.5.a) locally for any real analytic initial data, since the Ricci tensor r is expressed in terms of the derivatives up to the second order of the metric tensor g .

PROPOSITION 6. — *Let (M, g) be a real analytic riemannian manifold and α a real analytic symmetric bilinear form on M which satisfies $d \operatorname{tr} \alpha + \delta \alpha = 0$ and $\operatorname{tr} \alpha^2 - (\operatorname{tr} \alpha)^2 = (n-1)\overline{e} - u$. Then, there exists an Einstein manifold $(\overline{M}, \overline{g})$ with $\overline{r} = \overline{e}\overline{g}$ in which (M, g) is imbedded as a hypersurface with second fundamental form α .*

Proof. — There exists a unique real analytic solution g_t of (1.5.a) with initial data $g_0 = g$ and $g'_0 = -2\alpha$. We must check that this solution satisfies (1.5.b) and (1.5.c). By standard tensor calculus, we see using (1.5.a) that:

$$\begin{aligned} (\operatorname{tr} g')' &= -2n\overline{e} + 2u - (1/2)(\operatorname{tr} g')^2, \\ (\delta g')' &= (1/4)d \operatorname{tr} (g')^2 - (1/2)(\operatorname{tr} g')\delta g' - du, \\ (\operatorname{tr} (g')^2)' &= -4\overline{e} \operatorname{tr} g' - (\operatorname{tr} g') \operatorname{tr} (g')^2 + 4(r, g'), \\ u' &= \Delta \operatorname{tr} g' + \delta \delta g' - (r, g') \quad (\text{see Berger [1] (2.11)}). \end{aligned}$$

Therefore:

$$\begin{aligned} (d \operatorname{tr} g' + \delta g')' &= -(1/2)(\operatorname{tr} g')(d \operatorname{tr} g' + \delta g') + (1/4)d(\operatorname{tr} (g')^2 - (\operatorname{tr} g')^2 + 4u), \\ (\operatorname{tr} (g')^2 - (\operatorname{tr} g')^2 + 4u)' &= 4\delta(d \operatorname{tr} g' + \delta g') - (\operatorname{tr} g')(\operatorname{tr} (g')^2 - (\operatorname{tr} g')^2 + 4u - 4(n-1)\overline{e}). \end{aligned}$$

Thus analyticity implies that (1.5.b) and (1.5.c) hold for all t .

Q.E.D.

Proof of Theorem A. — In the above Proposition, set $\alpha = 0$ and $\overline{e} = u/(n-1)$.

Q.E.D.

Remark 7. — In the situation of Theorem A, the change $t \rightarrow -t$ of the parameter t preserves the solution. Therefore there is an isometry of $(\overline{M}, \overline{g})$ of order 2 such that all points of M are fixed.

Let g_t be an analytic solution of (1.5) with initial data $g_0 = g$ and $g'_0 = h$. If the metric $g_t + dt^2$ on \mathbb{R} does not extend to a complete metric, for example, if the sectional curvature of $g_t + dt^2$ diverges for $t \rightarrow t_0$, then we see that (M, g) cannot be immersed in any complete Einstein manifold as a hypersurface with second fundamental form $\alpha = -(1/2)h$. We apply this method to a family $g_t = f(t)^2 g_0$ where g_0 is an Einstein metric and $f(t)$ is a positive function of t such that $f(0) = 1$. Let this family g_t be a solution of (1.5). Then:

$$\begin{aligned} g'_t &= 2(f'(t)/f(t))g_t, \\ g''_t &= 2((f'(t)/f(t))^2 + f''(t)/f(t))g_t. \end{aligned}$$

From now on, we will omit t for simplicity. Since the Ricci tensor is invariant under multiplication by a scalar factor:

$$r = r_0 = e_0 g_0 = e_0 f^{-2} g,$$

where e_0 is the Ricci curvature of g_0 . As a result, (1.5.c) becomes:

$$(2.1) \quad \begin{aligned} 4n(f'/f)^2 - 4n^2(f'/f)^2 &= 4(n-1)\bar{e} - 4ne_0 f^{-2}, \\ (f')^2 &= e_0/(n-1) - (\bar{e}/n) f^2. \end{aligned}$$

Further (1.5.a) becomes:

$$(2.2) \quad \begin{aligned} ff'' &= -\bar{e}f^2 + e_0 - (n-1)(f')^2 = -(\bar{e}/n)f^2 \quad [\text{using (2.1)}], \\ f'' &= -(\bar{e}/n)f. \end{aligned}$$

Equation (2.2) reduces to (2.1), except in the case where f is constant. We get the following solutions.

(2.3.a) If $\bar{e} > 0$, then $e_0 > 0$ and:

$$f(t) = (\sqrt{e_0/(n-1)}/2\sqrt{\bar{e}/n}) \sin(\pm\sqrt{\bar{e}/n}(t+C)).$$

(2.3.b) If $\bar{e} = 0$, then $e_0 \geq 0$ and:

$$f(t) = \pm\sqrt{e_0/(n-1)}t + C.$$

(2.3.c) If $\bar{e} < 0$, then:

$$f(t) = |(n/4\bar{e}) \exp(\pm\sqrt{-\bar{e}/n}(t+C)) + (e_0/(n-1)) \exp(\mp\sqrt{-\bar{e}/n}(t+C))|.$$

Therefore, if (\bar{M}, \bar{g}) is an Einstein manifold and if (M, g_0) is an Einstein manifold which is isometrically immersed into (\bar{M}, \bar{g}) as a totally umbilical hypersurface, then \bar{g} is locally isometric with $f(t)^2 g_0 + dt^2$, where $f(t)$ is one of the solutions (2.3). In fact, since the equation expressing that a hypersurface is totally umbilical is elliptic, (M, g_0) is analytically immersed into (\bar{M}, \bar{g}) . Now, we check completeness of the metric $\bar{g} = f(t)^2 g_0 + dt^2$.

Remark 8. – If (M, g_0) is a complete Einstein manifold with negative Ricci curvature, then (2.3c) gives a complete Einstein metric. This metric is not homogeneous by Theorem B, if (M, g_0) does not have constant sectional curvature.

Let $f(t)$ be one of the solutions (2.3) and set $g_t = f(t)^2 g_0$ and $\bar{g} = g_t + dt^2$ on $\bar{M} = M \times I$. Denote by $\bar{K}(V, W)$ [resp. $K_0(X, Y)$] the sectional curvature of (\bar{M}, \bar{g}) [resp. (M, g_0)] of the plane spanned by V and W [resp. X and Y]. Suppose that X and Y are unit orthogonal vectors on (M, g_0) . Then, by the identification $\bar{M} = M \times I$ and the formulae in 1, we see that:

$$(2.4) \quad \begin{aligned} \bar{K}_t(X, Y) &= \bar{R}(X, Y; X, Y)/(g(X, X)g(Y, Y)) \\ &= f^{-4}(R(X, Y; X, Y) + \alpha(X, Y)^2 - \alpha(X, X)\alpha(Y, Y)) \\ &= f^{-4}(g(R(X, Y)X, Y) - (1/4)g'(X, X)g'(Y, Y)) \\ &= f^{-4}(f^2 K_0(X, Y) - f^2 (f')^2) = f^{-2}(K_0(X, Y) + (\bar{e}/n)f^2 - e_0/(n-1)) \\ &= \bar{e}/n + f^{-2}(K_0(X, Y) - e_0/(n-1)), \end{aligned}$$

$$\begin{aligned}
(2.5) \quad \bar{K}_t(X, N) &= \bar{R}(X, N; X, N)/g(X, X) \\
&= f^{-2}((1/4)(g'(X, X))^2 - (1/2)g''(X, X)) \\
&= f^{-2}((f'/f)^2 g(X, X) - ((f'/f)^2 + f''/f)g(X, X)) = -f''/f = \bar{e}/n, \\
(2.6) \quad \bar{K}_t(X, N+aY) &= \bar{R}(X, N+aY; X, N+aY)/g(X, X)\bar{g}(N+aY, N+aY) \\
&= f^{-2}(1+a^2 f^2)^{-1}(\bar{R}(X, N; X, N) + 2a\bar{R}(X, N; X, Y) + a^2\bar{R}(X, Y; X, Y)) \\
&= f^{-2}(1+a^2 f^2)^{-1}(f^2\bar{K}(X, N) + a^2 f^4\bar{K}(X, Y)) \\
&= (1+a^2 f^2)^{-1}(\bar{K}(X, N) + a^2 f^2\bar{K}(X, Y)).
\end{aligned}$$

By these formulae, we see that \bar{g} has constant sectional curvature if and only if g_0 has constant sectional curvature. From now on, we assume that (\bar{M}, \bar{g}) extends to a complete Einstein manifold, which we denote by the same symbol (\bar{M}, \bar{g}) .

LEMMA 9. — Assume that g_0 does not have constant sectional curvature. Then, (a) $f(t) \neq 0$ for all real number t . (b) If $f(t)$ converges to 0 for $t \rightarrow \infty$ or $-\infty$, then the sectional curvature of (\bar{M}, \bar{g}) is not bounded.

Proof. — Easy, by (2.4).

Q.E.D.

Denote by G the isometry group of (\bar{M}, \bar{g}) and by d the metric on \bar{M} induced by \bar{g} .

LEMMA 10. — Assume that there is a positive number D such that $d(p, G(q)) < D$ for all $p, q \in \bar{M}$. If $f(t)$ converges to ∞ for $t \rightarrow \infty$ or $-\infty$, then g_0 has constant sectional curvature.

Proof. — Without loss of generality, we may assume that $f(t)$ converges to ∞ for $t \rightarrow \infty$. Let B be the closed ball with center $x \in \bar{M}$ and radius r in (\bar{M}, g_0) , where r is sufficiently small so that B is compact. By assumption, there exists t_0 such that $f(t)r > D$ for all $t \geq t_0$. Then for all $t > t_0 + D$, $B \times (t_0, \infty) (\subset \bar{M})$ contains the closed ball \bar{B}_t with the center $(x, t) \in \bar{M}$ and the radius D in (\bar{M}, \bar{g}) . By (2.4), (2.5) and (2.6), the sectional curvature of (\bar{M}, \bar{g}) at the point (y, t) converges uniformly in B to \bar{e}/n for $t \rightarrow \infty$. Thus the sectional curvature of (\bar{M}, \bar{g}) is constant, since:

$$\bigcap_{t > t_0 + D} G(\bar{B}_t) = \bar{M}.$$

Q.E.D.

Proof of Theorem B. — Remark that $f'(a) = 0$ if and only if $i_a : (M, g_a) \rightarrow (\bar{M}, \bar{g})$ is totally geodesic.

(a) $e_0 > 0$. There is a real number a such that $f(a) = 0$. By Lemma 8 (a), g_0 and \bar{g} have constant sectional curvature.

(b) $e_0 = \bar{e} = 0$. $f' \equiv 0$. Then (\bar{M}, \bar{g}) is the riemannian product $(M, g_0) \times \mathbf{R}$ locally. If (\bar{M}, \bar{g}) is simply connected, then (\bar{M}, \bar{g}) decomposes globally as $(\tilde{M}, \tilde{g}) \times \mathbf{R}$, since \bar{g} is real analytic. Here (\tilde{M}, \tilde{g}) is a complete totally geodesic hypersurface of (\bar{M}, \bar{g}) which contains M .

(c) $e_0=0, \bar{e}<0$. $f(t) \rightarrow 0$ for $t \rightarrow \infty$ or $-\infty$. By Lemma 8 (b), if the sectional curvature of (\bar{M}, \bar{g}) is bounded, then g_0 and \bar{g} have constant sectional curvature.

(d) $e_0, \bar{e}<0$. There is a real number a such that $f(a)>0$ and $f'(a)=0$. So i_a is totally geodesic. Moreover, $f(t)$ converges to ∞ for $t \rightarrow \infty$. If (\bar{M}, \bar{g}) satisfies the condition in Lemma 9, then g_0 and \bar{g} have constant sectional curvature.

By Proposition 2, these are the only possible cases.

Q.E.D.

3. Real hypersurfaces of a Kähler-Einstein manifold

In the general situation, we saw in Theorem A that we cannot get much information on (M, g) , even if (M, g) is a totally geodesic hypersurface in an Einstein manifold (\bar{M}, \bar{g}) . But if (\bar{M}, \bar{g}) is a Kähler-Einstein manifold, the Gauss-Codazzi equations give more information on (M, g) . Let (M, g) be a totally umbilical real hypersurface in a Kähler-Einstein manifold (\bar{M}, \bar{g}) . By Proposition 2, the second fundamental form α is expressed as $\alpha=ag$ for some real number a . Then, the Codazzi equation and formula (1.1.a) become:

$$(3.1) \quad \bar{R}(X, Y; Z, N)=0,$$

$$(3.2) \quad r=(\bar{e}+(n-1)a^2)g-\beta.$$

Denote by J the almost complex structure of (\bar{M}, \bar{g}) and set $H=JN$. In equation (3.1), if X is orthogonal to H , then JX is tangent to M , and we see that:

$$(3.3) \quad \beta(X, Y)=\bar{R}(X, N; Y, N)=-\bar{R}(JX, H; Y, N)=0.$$

Then equation (1.1.c) implies:

$$(3.4) \quad \beta(H, H)=\bar{e}.$$

PROPOSITION 11. — *Let (\bar{M}, \bar{g}) be a complete Kähler-Einstein manifold with zero Ricci curvature. Assume that there exists a totally umbilical but not totally geodesic real hypersurface (M, g) in (\bar{M}, \bar{g}) (i.e., $a \neq 0$). Then both (\bar{M}, \bar{g}) and (M, g) have constant sectional curvature.*

Proof. — By equations (3.2), (3.3) and (3.4), g is an Einstein metric with positive Ricci curvature. Thus the proof reduces to Theorem B(a).

Q.E.D.

LEMMA 12. — *Let (\bar{M}, \bar{g}) be a Kähler-Einstein manifold. Assume that there exists a totally geodesic real hypersurface (M, g) in (\bar{M}, \bar{g}) . Then there exists a totally geodesic complex hypersurface (\tilde{M}, \tilde{g}) in (\bar{M}, \bar{g}) which is contained in (M, g) . Moreover, (\tilde{M}, \tilde{g}) is a Kähler-Einstein manifold and (M, g) decomposes locally as $(M, g)=(\tilde{M}, \tilde{g}) \times \mathbb{R}$.*

Proof. — Since (M, g) is totally geodesic, $\overline{D}_X N = 0$ holds for any tangent vector X of M . Then we see that:

$$(3.5) \quad D_X H = \overline{D}_X H = \overline{D}_X (JN) = J(\overline{D}_X N) = 0,$$

which implies that there is a hypersurface (\tilde{M}, \tilde{g}) in (M, g) and (M, g) decomposes locally as $(M, g) = (\tilde{M}, \tilde{g}) \times \mathbb{R}$. Here J preserves the tangent space of \tilde{M} . This implies that \tilde{M} is a complex submanifold of \overline{M} . Moreover, equations (3.2) and (3.3) imply that \tilde{g} is an Einstein metric.

Q.E.D.

Proof of Theorem C. — Let γ be a geodesic in (S, \bar{e}) . By Lemma 12, (M, g) may be immersed into $(\tilde{M}, \tilde{g}) \times \text{Im } \gamma$. On the other hand, since \tilde{g} is an Einstein metric with Ricci curvature \bar{e} , $(\tilde{M}, \tilde{g}) \times (S, \bar{e})$ becomes an Einstein manifold and $(\tilde{M}, \tilde{g}) \times \text{Im } \gamma$ is totally geodesic in (\overline{M}, \bar{g}) . Then Proposition 4 implies that $(\tilde{M}, \tilde{g}) \times (S, \bar{e})$ is an open set of (\overline{M}, \bar{g}) . Remark that this identification preserves the complex structure. Since (\overline{M}, \bar{g}) is real analytic, this decomposition extends globally. That is, (\tilde{M}, \tilde{g}) extends to a complete complex hypersurface of (\overline{M}, \bar{g}) and we get a global decomposition.

Q.E.D.

Remark 13. — Even if (\overline{M}, \bar{g}) is not complete, the above decomposition holds locally.

Proof of Corollary E. — Assume that there is a compact stable minimal real hypersurface (M, g) in (\overline{M}, \bar{g}) . Then by Theorem D, (M, g) is totally geodesic. Therefore we can apply Theorem C to the universal riemannian covering of (\overline{M}, \bar{g}) and get a global decomposition. This contradicts the assumption.

Q.E.D.

Remark 14. — In Corollary E, if \overline{M} is simply connected, the assumption that M is orientable is not necessary. See Remark 4.

Remark 15. — In particular, there is no compact stable minimal hypersurface in the K3-surfaces \overline{M} with zero Ricci curvature. By Theorem 2.9 in Bourguignon [2], there is no stable closed geodesic in \overline{M} . We may say that these results are dual with one another.

COROLLARY 16. — *Let (\overline{M}, \bar{g}) be a compact Kähler-Einstein manifold with zero Ricci curvature of complex dimension ≤ 3 . If $\pi_1(\overline{M})$ is not finite, then (M, g) has a local factor C .*

Proof. — Since $\pi_1(\overline{M})$ is not finite, $H_n(\overline{M}, \mathbb{Z})$ is not trivial by Poincaré duality. For $\dim_{\mathbb{R}} \overline{M} \leq 6$, a non-trivial homology class in $H_n(\overline{M}, \mathbb{Z})$ can be represented by stable minimal real hypersurfaces M without singularity (Federer [7], Thm. 5.4.15, Lawson Jr. [9], Remark 3.4). Then by Corollary E, (\overline{M}, \bar{g}) decomposes locally with a factor C .

Q.E.D.

Remark 17. — We can get Corollary 16 in more general situation by Theorem 3 in Cheeger and Gromoll [3]. But the proof is different.

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