ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 14, nº 4 (1981), p. 369-401
<http://www.numdam.org/item?id=ASENS_1981_4_14_4_369_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 14, 1981, p. 369 à 401.

SLOPES OF POWERS OF FROBENIUS ON CRYSTALLINE COHOMOLOGY

BY NIELS O. NYGAARD

Introduction

Recall that to an F-crystal (M, Φ) there are associated two polygons: The Newton polygon and the Hodge polygon (for an excellent exposition about these things see Katz's paper [6]). Assume that the F-crystal arises from a geometric situation, i. e. that there is a proper smooth variety X defined over a perfect field k of characteristic $p>0$ such that $M=H^i_{crys}(X/W)$/torsion and Φ is the canonical lifting of frobenius. In this case there is associated a third polygon: The Hodge polygon of the geometric Hodge numbers:

$$h^j=\dim_k H^{i-j}(X, \Omega^j_X),$$

of X. The Katz conjecture asserts that this geometric Hodge polygon lies below the Newton polygon. Katz's conjecture was proved first by Mazur under certain assumptions and later by Ogus in general ([8], [3]).

Mazur proved that for all F-crystals the Hodge polygon is always on or below the Newton polygon, and both polygons have the same end point ([8], [6]). Both Mazur and Ogus then prove that in the geometric situation the geometric Hodge polygon is below the Hodge polygon of the F-crystal.

Under Mazur's assumptions the two Hodge polygons actually coincide and Ogus shows that this continues to hold under considerably weaker conditions.

This paper grew out of an attempt to prove Mazur's and Ogus's Theorems using the de Rham-Witt complex of Bloch-Deligne-Illusie ([4], [5]).

It turns out that this method allows us to get estimates, not only for Φ, but also for the iterates Φ^n on $H^i_{crys}(X/W)$. The Hodge numbers of X are then replaced by the numbers:

$$h^j(n)=l(H^{i-j}(X, W_n\Omega^j_X)) \quad (l=\text{length}).$$

The Katz conjecture for $(H^i_{crys}(X/W)/\text{torsion}, \Phi^n)$ takes the following form: Let $(1/n)\,\mathrm{Hdg}(\Phi^n)$ denote the Hodge polygon of Φ^n except that all slopes have been divided by n, then $(1/n)\,\mathrm{Hdg}(\Phi^n)$is above the Hodge polygon of the numbers $\{h^0(n)/n, \ldots, h^i(n)/n\}$, in particular for $n=1$ we get Mazur's and Ogus's result.

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We also study the Hodge numbers of Φ^n and prove that under suitable conditions the F-crystal determines the two filtrations on $H^i_{crys}(X/W_n)$ coming from the two hypercohomology spectral sequences of the complex $\{W_n\Omega^{\cdot}_X\}$. In the case $n=1$ this determines the Hodge numbers of the crystal as being equal to the geometric Hodge numbers. This is no longer true if $n>1$; we give a formula for the Hodge numbers of Φ^n corresponding to slopes $<n$, this formula involves the iterates of F on the truncated Witt vector cohomology $H^i(X, W_n\mathcal{O}_X)$.

Finally we give a couple of applications to curves and surfaces: We show that a curve has Hasse-Witt matrix equal to zero if and only if its Jacobian is isomorphic to a product of supersingular elliptic curves, and that for a surface satisfying certain conditions the F-crystal $H^2_{crys}(X/W)$ determines the filtration coming from the slope spectral sequence. A corollary of this is that the formal Brauer group of the surface is uniquely determined by $H^2_{crys}(X/W)$, this should be compared with the corresponding result for $H^1_{crys}(X/W)$ which uniquely determines the completion of the Picard variety.

CONTENTS

1. The complexes $W\Omega^{\cdot}_X(r, n)$

In this section X is a smooth variety (not necessarily proper) over a perfect field k of characteristic $p>0$.

DEFINITION 1.1. — Let r, n be non-negative integers, we define a complex of $W(k)$-sheaves on X by:

$$W\Omega^{\cdot}_X(r, n)=0\to W\mathcal{O}_X\xrightarrow{d}\dots\xrightarrow{d}W\Omega_X^{r-1}\xrightarrow{dV^n}W\Omega^r_X\xrightarrow{d}\dots\xrightarrow{d}W\Omega^N_X\to 0.$$

($N=\dim X$), here $W\Omega^r_X$, $W\Omega^{r+1}_X$, ..., $W\Omega^N_X$ are viewed as $W(k)$-modules through σ^{-n}, where σ is the frobenius automorphism of $W(k)$. In the cases $r=0$ or $r\geqq N+1$ we have the ordinary de Rham-Witt complex [5].

DEFINITION 1.2. — Define maps of complexes:

$$\tilde{F}:\quad W\Omega^{\cdot}_X(r, n)\to W\Omega^{\cdot}_X(r+1, n)$$

and:

$$\tilde{V}:\quad W\Omega^{\cdot}_X(r, n)\to W\Omega^{\cdot}_X(r-1, n),$$

by the commutative diagrams:

$$\begin{array}{ccccccccccccc}
W\Omega^{\bullet}_X(r,n)=0 & \longrightarrow & W\mathcal{O}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^{r-1}_X & \xrightarrow{dV^n} & W\Omega^r_X & \xrightarrow{d} & W\Omega^{r+1}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^N_X & \to 0 \\
\downarrow{\scriptstyle\tilde{F}} & & \| & & \| & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & \\
W\Omega^{\bullet}_X(r+1,n)=0 & \to & W\mathcal{O}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^{r-1}_X & \xrightarrow{d} & W\Omega^r_X & \xrightarrow{dV^n} & W\Omega^{r+1}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^N_X & \to 0
\end{array}$$

and:

$$\begin{array}{ccccccccccccc}
W\Omega^{\bullet}_X(r,n)=0 & \longrightarrow & W\mathcal{O}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^{r-2}_X & \xrightarrow{d} & W\Omega^{r-1}_X & \xrightarrow{dV^n} & W\Omega^r_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^N_X & \to 0 \\
\downarrow{\scriptstyle\tilde{V}} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle V^n} & & \| & & \| & \\
W\Omega^{\bullet}_X(r-1,n)=0 & \to & W\mathcal{O}_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^{r-2}_X & \xrightarrow{dV^n} & W\Omega^{r-1}_X & \xrightarrow{d} & W\Omega^r_X & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega^N_X & \to 0
\end{array}$$

The commutativity of these diagrams follows from the identities:

$$F^n dV^n = d, \qquad dF^n = p^n F^n d, \qquad p^n dV^n = V^n d, \qquad F^n V^n = V^n F^n = p^n \quad [5].$$

It is clear from the definitions that $\tilde{F}\tilde{V} = \tilde{V}\tilde{F} = p^n$.

For any complex:

$$L^{\bullet} = 0 \to L^0 \xrightarrow{d} L^1 \xrightarrow{d} L^2 \xrightarrow{d} \dots \xrightarrow{d} L^i \xrightarrow{d} L^{i+1} \xrightarrow{d} \dots$$

let $t_{\leqq r} L^{\bullet}$ denote the complex:

$$0 \to L^0 \xrightarrow{d} \dots \xrightarrow{d} L^{r-1} \xrightarrow{d} ZL^r \to 0 \qquad (ZL^r = \ker d : L^r \to L^{r+1})$$

and $L^{\bullet \geqq r}$ the complex:

$$0 \to 0 \to \dots \to 0 \to L^r \xrightarrow{d} L^{r+1} \xrightarrow{d} L^{r+2} \xrightarrow{d} \dots$$

[this is the filtration canonique (resp. the filtration bête)].

DEFINITION 1.3. — Define maps of complexes:

$$\alpha(r,n):\quad W\Omega^{\bullet}_X(r,n)/\tilde{F}W\Omega^{\bullet}_X(r-1,n) \to W_n\Omega^{\bullet \geqq r}_X$$

and:

$$\beta(r,n):\quad W\Omega^{\bullet}_X(r,n)/\tilde{V}W\Omega^{\bullet}_X(r+1,n) \to t_{\leqq r} W_n\Omega^{\bullet}_X,$$

by the commutative diagrams:

$$\text{(i)}\qquad \begin{array}{ccccccc}
0\to 0\to 0 \dots & W\Omega^{r-1}_X/F^n & \xrightarrow{dV^n} & W\Omega^r_X/p^n & \xrightarrow{d} & W\Omega^{r+1}_X/p^n & \xrightarrow{d} \dots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to 0\to 0 \dots \longrightarrow & 0 & \longrightarrow & W_n\Omega^r & \xrightarrow{d} & W_n\Omega^{r+1} & \xrightarrow{d} \dots
\end{array}$$

where the vertical maps are defined by the projections $\mathrm{pr}_n^i : \mathrm{W}\Omega_X^i \to \mathrm{W}_n\Omega_X^i$, the diagram commutes since $\mathrm{Im}\, d\mathrm{V}^n \subset \ker \mathrm{pr}_n$ [5]:

$$\text{(ii)} \quad \begin{array}{ccccccccc} 0 \to \mathrm{W}\mathcal{O}_X/p^n & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathrm{W}\Omega_X^{r-1}/p^n & \xrightarrow{d\mathrm{V}^n} & \mathrm{W}\Omega_X^r/\mathrm{V}^n & \to 0 \dots \\ \downarrow & & & & \downarrow & & \downarrow \mathrm{F}^n & \\ 0 \to \mathrm{W}_n\mathcal{O}_X & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathrm{W}_n\Omega_X^{r-1} & \xrightarrow{d} & \mathrm{Z}\mathrm{W}_n\Omega_X^r & \to 0 \dots \end{array}$$

(here we use the fact that $\mathrm{pr}_n^i . \mathrm{F}^n$ maps $\mathrm{W}\Omega_X^i$ into $\mathrm{ZW}_n\Omega_X^i$ for all i [5], Prop. I 3.21). The diagram commutes since $\mathrm{F}^n d\mathrm{V}^n = d$.

Since $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{V}}$ commute we have induced maps:

$$\tilde{\mathrm{V}} : \quad \mathrm{W}\Omega_X^{\cdot}(r+1, n)/\tilde{\mathrm{F}}\,\mathrm{W}\Omega_X^{\cdot}(r, n) \to \mathrm{W}\Omega_X^{\cdot}(r, n)/\tilde{\mathrm{F}}\mathrm{W}\Omega_X^{\cdot}(r-1, n)$$

and:

$$\tilde{\mathrm{F}} : \quad \mathrm{W}\Omega_X^{\cdot}(r, n)/\tilde{\mathrm{V}}\,\mathrm{W}\Omega_X^{\cdot}(r+1, n) \to \mathrm{W}\Omega_X^{\cdot}(r+1, n)/\tilde{\mathrm{V}}\,\mathrm{W}\Omega_X^{\cdot}(r+2, n).$$

LEMMA 1.4. — *The following two diagrams commute:*

$$(a) \quad \begin{array}{ccc} \mathrm{W}\Omega_X^{\cdot}(r+1, n)/\tilde{\mathrm{F}}\mathrm{W}\Omega_X^{\cdot}(r, n) & \xrightarrow{\tilde{\mathrm{V}}} & \mathrm{W}\Omega_X^{\cdot}(r, n)/\tilde{\mathrm{F}}\mathrm{W}\Omega_X^{\cdot}(r-1, n) \\ \downarrow \alpha(r+1, n) & & \downarrow \alpha(r, n) \\ \mathrm{W}_n\Omega^{\cdot\, \geqq r+1} & \xrightarrow{\text{incl.}} & \mathrm{W}_n\Omega^{\cdot\, \geqq r} \end{array}$$

$$(b) \quad \begin{array}{ccc} \mathrm{W}\Omega_X^{\cdot}(r, n)/\tilde{\mathrm{V}}\mathrm{W}\Omega_X^{\cdot}(r+1, n) & \xrightarrow{\tilde{\mathrm{F}}} & \mathrm{W}\Omega_X^{\cdot}(r+1, n)/\tilde{\mathrm{V}}\mathrm{W}\Omega_X^{\cdot}(r+2, n) \\ \downarrow \beta(r, n) & & \downarrow \beta(r+1, n) \\ t_{\leqq r}\mathrm{W}_n\Omega_X^{\cdot} & \xrightarrow{\text{incl.}} & t_{\leqq r+1}\mathrm{W}_n\Omega_X^{\cdot} \end{array}$$

Proof: The only dimension where the commutativity is nontrivial is in dimension r. The commutativity of (a) amounts to the commutativity of:

$$\begin{array}{ccc} \mathrm{W}\Omega_X^r/\mathrm{F}^n & \xrightarrow{\mathrm{V}^n} & \mathrm{W}\Omega_X^r/p^n \\ \downarrow & & \downarrow \mathrm{pr}_n^r \\ 0 & \longrightarrow & \mathrm{W}_n\Omega_X^r \end{array}$$

which holds since $\mathrm{Im}\,\mathrm{V}^n \subset \ker \mathrm{pr}_n$.

The commutativity of (b) is equivalent to the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{W}\Omega_X^r & \xrightarrow{\mathrm{F}^n} & \mathrm{W}\Omega_X^r/p^n \\ \downarrow \mathrm{F}^n & & \downarrow \\ \mathrm{ZW}_n\Omega_X^r & \to & \mathrm{W}_n\Omega_X^r \end{array}$$

which of course is trivial.

THEOREM 1.5. — *The maps $\alpha(r, n)$ and $\beta(r, n)$ are quasi-isomorphisms.*

For the proof of this Theorem we need two Lemmas both of which are essentially due to Illusie (private communication).

LEMMA 1.6. — *There is an exact sequence:*

$$0 \to W\Omega_X^{r-1}/F^n W\Omega_X^r \xrightarrow{dV^n} W\Omega_X^r/V^n W\Omega_X^r \to W_n\Omega_X^r \to 0.$$

Proof. — By [5], Proposition I 3.2 the kernel of pr_n^r is $dV^n W\Omega_X^{r-1} + V^n W\Omega_X^r$ so it suffices to show that:

$$dV^n: \quad W\Omega_X^{r-1}/F^n W\Omega_X^{r-1} \to W\Omega_X^r/V^n W\Omega_X^r,$$

is injective.

Let x be *a* local section of $W\Omega_X^{r-1}$ such that $dV^n x = V^n y$ for some local section y in $W\Omega_X^r$, then $dx = F^n dV^n x = F^n V^n y = p^n y$.. It follows that $x \in d^{-1}(p^n W\Omega_X^r)$ hence by [5], I 3.21.1.5, $x \in F^n W\Omega_X^{r-1}$ which proves the Lemma.

LEMMA 1.7. — *The map:*

$$pr_n^r F^n: \quad W\Omega_X^r \to W\Omega_X^r \to W_n\Omega_X^r,$$

induces an isomorphism:

$$C^{-n}: \quad W_n\Omega_X^r \xrightarrow{\sim} \mathscr{H}^r(W_n\Omega_X^\cdot).$$

Proof. — By [5], Proposition I 3.21, $pr_n^r . F^n$ maps $W\Omega_X^r$ onto $ZW_n\Omega_X^r$.

We have a commutative diagram:

$$\begin{array}{ccc} dV^n W\Omega_X^{r-1} + V^n W\Omega_X^r & \to & W\Omega_X^r \\ \downarrow F^n & & \downarrow pr_n^r . F^n \\ dW_n\Omega_X^r & \longrightarrow & ZW_n\Omega_X^r. \end{array}$$

which shows that $pr_n^r . F^n$ passes to the quotient and defines a surjective map:

$$C^{-n}: \quad W_n\Omega_X^r \to \mathscr{H}^r(W_n\Omega_X^\cdot).$$

Assume next that $\bar{x} \in W_n\Omega_X^r$ and that $C^{-n}\bar{x} = 0$ in $\mathscr{H}^r(W_n\Omega_X^\cdot)$. Let $x \in W\Omega_X^r$ such that $pr_n^r(x) = \bar{x}$. Since $dF^n x = p^n F^n dx$, $F^n x$ induces a section of $\mathscr{H}^r(W\Omega_X^\cdot/p^n)$ which maps to $C^{-n}\bar{x} = 0$ in $\mathscr{H}^r(W_n\Omega_X^\cdot)$ under the projection:

$$\mathscr{H}^r(W\Omega_X^\cdot/p^n) \to \mathscr{H}^r(W_n\Omega_X^\cdot).$$

By [5], I 3.17 the map:

$$W\Omega_X^\cdot/p^n \to W_n\Omega_X^\cdot,$$

is a quasi-isomorphism so $F^n x$ must be zero in $\mathscr{H}^r(W\Omega_X^\cdot/p^n)$, hence there are sections $y \in W\Omega_X^{r-1}$ and $z \in W\Omega_X^r$ such that $F^n x = dy + p^n z$. This gives:

$$F^n x = F^n(dV^n y + V^n z)$$

and since F is injective on $W\Omega_X^r$ we get:

$$x = dV^n y + V^n z,$$

hence $\bar{x}=\mathrm{pr}_n^r(x)=\mathrm{pr}_n^r(dV^n y+V^n z)=0$ which proves that C^{-n} is injective.

Proof of Theorem 1.5. — We use induction on r. For $r=0$ the statement that $\alpha(0, n)$ is a quasi-isomorphism is precisely [5], Corollaire 3.17.

Next we compute the cokernel of:

$$\tilde{V}: \quad W\Omega_X^{\cdot}(r+1, n)/\tilde{F}\, W\Omega_X^{\cdot}(r, n) \to W\Omega_X^{\cdot}(r, n)/\tilde{F}\, W\Omega_X^{\cdot}(r-1, n),$$

using the commutative diagram with exact columns:

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 & \to & 0 & - \dots \to & 0 & \longrightarrow & W\Omega_X^r/F^n & \xrightarrow{dV^n} & W\Omega_X^{r+1}/p^n & \xrightarrow{d} & \dots & \xrightarrow{d} & W\Omega_X^N/p^n \to 0 \\
 & & \downarrow{\scriptstyle \tilde{V}} & & \downarrow & & \downarrow & & \| & & & & \| \\
0 & \to & 0 & \to & W\Omega_X^{r-1}/F^n & \xrightarrow{dV^n} & W\Omega_X^r/p^n & \xrightarrow{d} & W\Omega_X^{r+1}/p^n & \xrightarrow{d} & \dots & \xrightarrow{d} & W\Omega_X^N/p^n \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & \to & W\Omega_X^{r-1}/F^n & \xrightarrow{dV^n} & W\Omega_X^r/V^n & \longrightarrow & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & 0 & & & &
\end{array}$$

To prove the induction step it is then enough to show that:

$$\begin{array}{ccccccc}
0 \to 0 & \to & W\Omega_X^{r-1}/F^n & \xrightarrow{dV^n} & W\Omega_X^r/V^n & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
0 \to 0 & \longrightarrow & 0 & \longrightarrow & W_n\Omega_X^r & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism, and this follows from 1.6.

To prove that $\beta(0, n)$ is a quasi-isomorphism we remark that:

$$W\Omega_X^{\cdot}(0, n)/\tilde{V}\, W\Omega_X^{\cdot}(1, n)=0 \to W_n\mathcal{O}_X \to 0 \to 0 \to 0 \to$$

and:

$$\beta(0, n)=\mathrm{pr}_n^0 . F^n: \quad W_n\mathcal{O}_X \to \mathcal{H}^0(W_n\Omega_X^{\cdot}),$$

which is an isomorphism by Lemma 1.5.

Consider the exact sequence of complexes:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
0 \to & W\mathcal{O}_X/p^n & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^{r-1}/p^n & \xrightarrow{dV^n} & W\Omega_X^r/V^n & \longrightarrow & 0 & \longrightarrow 0 \\
 & \| & & \| & & \downarrow{\scriptstyle F^n} & & \downarrow & \\
0 \to & W\mathcal{O}_X/p^n & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^{r-1}/p^n & \xrightarrow{d} & W\Omega_X^r/p^n & \xrightarrow{dV^n} & W\Omega_X^{r+1}/V^n & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \| & \\
0 \longrightarrow & 0 & \dots\dots\dots\dots & 0 & \longrightarrow & W\Omega_X^r/F^n & \xrightarrow{dV^n} & W\Omega_X^{r+1}/V^n & \to 0 \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 &
\end{array}$$

To prove the induction step it is then enough to show that:

$$\begin{array}{ccccc}
0 \to 0 \to \dots \to 0 \to & W\Omega_X^r/F^n & \xrightarrow{dV^n} & W\Omega_X^{r+1}/V^n & \to 0 \dots \\
 & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_n^{r+1}.F^n} & \\
0 \to & W_n\Omega_X^r/ZW_n\Omega_X^r & \xrightarrow{d} & ZW_n\Omega_X^{r+1} & \to 0
\end{array}$$

is a quasi-isomorphism and this follows from Lemmas 1.6 and 1.7.

2. The generalized Katz conjecture

In addition to the assumptions of section 1 we assume from now on that X is proper.

Let $h^j(n)$ be defined as the length of the $W(k)$-module $H^{i-j}(X, W_n\Omega_X^j)$, in this section we prove that $(1/n)\,\mathrm{Hdg}(\Phi^n)$ lies on or above the Hodge polygon of $\{h^0(n)/n, \dots, h^i(n)/n\}$. Since the Newton polygon of Φ is above $(1/n)\,\mathrm{Hdg}(\Phi^n)$ for all n ([6], Thm. 1.4.1) the Katz conjecture follows from the case $n=1$.

PROPOSITION 2.1. — *The composite map:*

$$\tilde{V}^r: \quad W\Omega_X^{\bullet}(r, n) \to W\Omega_X^{\bullet}$$

maps $\mathbb{H}^i(W\Omega_X^{\bullet}(r, n))$ *into* $\Phi^{-n}(p^{nr} H^i_{\mathrm{crys}}(X/W))$ *for all* r.

Proof. — Consider the commutative diagrams:

$$\begin{array}{cccccccccccc}
 & 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega_X^1 & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^{r-1} & \xrightarrow{dV^n} & W\Omega_X^r & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^N & \to 0 \\
\tilde{V}^r & & \downarrow{\scriptstyle p^{n(r-1)}V^n} & & \downarrow{\scriptstyle p^{n(r-2)}V^n} & & \downarrow{\scriptstyle V^n} & & \| & & \| & \\
 & 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega_X^1 & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^{r-1} & \xrightarrow{d} & W\Omega_X^r & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^N & \to 0 \\
\Phi^n & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle p^nF^n} & & \downarrow{\scriptstyle p^{n(r-1)}F^n} & & \downarrow{\scriptstyle p^{nr}F^n} & & \downarrow{\scriptstyle p^{nN}F^n} & \\
 & 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega_X^1 & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^{r-1} & \xrightarrow{d} & W\Omega_X^r & \xrightarrow{d} \dots \xrightarrow{d} & W\Omega_X^N & \to 0
\end{array}$$

and:

$$\tilde{F}^{N+1-r} \qquad \begin{array}{ccccccccc} 0 \to W\mathcal{O}_X & \xrightarrow{d} & \dots & W\Omega_X^{r-1} & \xrightarrow{dV^n} & W\Omega_X^r & \xrightarrow{d} \cdots\cdots\cdots \xrightarrow{d} & W\Omega_X^N \to 0 \\ \| & & & \| & & \downarrow F^n & & \downarrow p^{n(N-r)}F^n \\ 0 \to W\mathcal{O}_X & \xrightarrow{d} & \dots & W\Omega_X^{r-1} & \xrightarrow{d} & W\Omega_X^r & \xrightarrow{d} \cdots\cdots\cdots \xrightarrow{d} & W\Omega_X^N \to 0 \end{array}$$

comparing these diagrams and using $F^n V^n = p^n$ we get:

(2.1.1) $$\Phi^n \tilde{V}^r = p^{nr} \tilde{F}^{N+1-r},$$

which proves the Proposition.

LEMMA 2.2. — *Let $\tilde{h}^0(n), \tilde{h}^1(n), \dots$ denote the Hodge numbers of Φ^n on $H^i_{\text{crys}}(X/W)$ then we have the following inequality for all $r \geqq 0$:*

$$nr.\tilde{h}^0(n) + (nr-1)\tilde{h}^1(n) + \dots + \tilde{h}^{nr-1}(n) \leqq r.h^0(n) + (r-1)h^1(n) + \dots + h^{r-1}(n).$$

Proof. — By definition of the Hodge numbers of Φ^n there are bases $\{e_1, e_2, \dots\}$ and $\{f_1, f_2, \dots\}$ of $M = H^i_{\text{crys}}(X/W)/\text{torsion}$ such that:

$$\begin{array}{ll} \Phi^n(e_t) = f_t, & t = 1, \dots, \tilde{h}^0(n), \\ \Phi^n(e_t) = pf_t, & t = \tilde{h}^0(n)+1, \dots, \tilde{h}^0(n)+\tilde{h}^1(n), \\ \vdots & \\ \Phi^n(e_t) = p^s f_t, & t = \sum_{j=0}^{s-1} \tilde{h}^j(n)+1, \dots, \sum_{j=0}^{s} \tilde{h}^j(n). \\ \vdots & \end{array}$$

Then we have the following basis of $\Phi^{-n}(p^{nr}M)$:

$$\{p^{nr}e_1, \dots, p^{nr}e_{u_0}, p^{nr-1}e_{u_0+1}, \dots, p^{nr-1}e_{u_1}, \dots, p^{nr-s}e_{u_{s-1}+1}, \dots\},$$

where:

$$u_s = \sum_{j=0}^{s} \tilde{h}^j(n), \qquad s = 0, 1, \dots$$

It follows that if l denotes the length then:

(2.2.1) $$l(M/\Phi^{-n}(p^{nr}M)) = nr.\tilde{h}^0(n) + (nr-1)\tilde{h}^1(n) + \dots + \tilde{h}^{nr-1}(n).$$

It is clear that:

(2.2.2) $$l(M/\Phi^{-n}(p^{nr}M)) \leqq l(H^i_{\text{crys}}(X/W)/\Phi^{-n}(p^{nr}H^i_{\text{crys}}(X/W)))$$

and by 2.1:

(2.2.3) $$l(H^i_{\text{crys}}(X/W)/\Phi^{-n}(p^{nr}H^i_{\text{crys}}(X/W))) \leqq l(H^i_{\text{crys}}(X/W)/\tilde{V}^r \mathbb{H}^i W\Omega^\bullet_X(r, n))).$$

Using the exact sequence:

$$\mathbb{H}^i(W\Omega^\bullet_X(r, n))/\tilde{V}\mathbb{H}^i(W\Omega^\bullet_X(r+1, n)) \xrightarrow{\tilde{V}^r} H^i_{\text{crys}}(X/W)/\tilde{V}^{r+1}\mathbb{H}^i(W\Omega^\bullet_X(r+1, n))$$
$$\to H^i_{\text{crys}}(X/W)/\tilde{V}^r\mathbb{H}^i(W\Omega^\bullet_X(r, n))$$

and induction it is now enough to prove that:

$$(2.2.4)\qquad l(\mathbb{H}^i(W\Omega^\bullet_X(r, n))/\tilde{V}\mathbb{H}^i(W\Omega^\bullet_X(r+1, n))) \leqq h^0(n)+h^1(n)+\ldots+h^r(n).$$

The exact sequence:

$$\mathbb{H}^i(W\Omega^\bullet_X(r+1, n)) \xrightarrow{\tilde{V}} \mathbb{H}^i(W\Omega^\bullet_X(r, n)) \to \mathbb{H}^i W\Omega^\bullet_X(r, n))/\tilde{V}\, W\Omega^\bullet_X(r+1, n)),$$

shows that:

$$l(\mathbb{H}^i(W\Omega^\bullet_X(r, n))/\tilde{V}\mathbb{H}^i(W\Omega^\bullet_X(r+1, n))) \leqq l(\mathbb{H}^i(W\Omega^\bullet_X(r, n)/\tilde{V}\, W\Omega^\bullet_X(r+1, n)))$$

and by Theorem 1.5:

$$\begin{aligned} l(\mathbb{H}^i(W\Omega^\bullet_X(r, n)/\tilde{V}\, W\Omega^\bullet_X(r+1, n)) &= l(\mathbb{H}^i(t_{\leq r} W_n\Omega^\bullet_X)) \\ &\leqq l(H^{i-r}(X, \mathscr{H}^r(W_n\Omega^\bullet_X))) + l(H^{i-(r-1)}(X, \mathscr{H}^{r-1}(W_n\Omega^\bullet_X)))\ldots \\ &\quad + l(H^i(X, \mathscr{H}^0(W_n\Omega^\bullet_X))) = h^r(n)+h^{r-1}(n)+\ldots+h^0(n), \end{aligned}$$

where the last equality follows from 1.7.

DEFINITION 2.3. — Let $C=\{c_0, c_1, c_2, \ldots\}$ be a finite set of real numbers, we define a function Hdg(C) by:

$$\text{Hdg}(C)(x) = c_1+2c_2+\ldots+jc_j+(j+1)(x-(c_0+c_1+c_2+\ldots+c_j))$$

for:

$$x \in [c_0+c_1+\ldots+c_j,\ c_0+c_1+\ldots+c_j+c_{j+1}].$$

The graph of Hdg(C) consists of line segments joining the points P_j, $j=0, 1, 2, \ldots$ with coordinates:

$$P_j:\quad (c_0+c_1+\ldots+c_j,\ c_1+2c_2+3c_3+\ldots+jc_j).$$

We call the graph of Hdg(C), the Hodge polygon of C.

LEMMA 2.4. — *Let* $A=\{a_0, a_1, \ldots\}$, $B=\{b_0, b_1, \ldots\}$ *be finite sets of non-negative real numbers and let* $n \geqq 1$ *be an integer. Assume that for all* $r \geqq 1$ *the following inequality holds:*

$$nr.a_0+(nr-1)a_1+\ldots+a_{nr-1} \leqq r.b_0+(r-1)b_1+\ldots+b_{r-1}$$

(*where we put* $a_t=b_t=0$ *for* $t \gg 0$). *Then the graph of* $(1/n)\,\text{Hdg}(A)$ *lies on or above the graph of* $\text{Hdg}(B/n)$, *where* $B/n=\{b_0/n, b_1/n, \ldots\}$.

Proof. — We follow the method of Ogus ([3], 8.37). Let s be any non-negative integer and choose $t \gg 0$ such that $b_j = 0$ for $j > t$.

For all $r > \max\{s, t\}$ we have:

$$nr.a_0 + (nr-1)a_1 + \ldots + (nr-s)a_s \leqq r.b_0 + (r-1)b_1 + \ldots + (r-t)b_t.$$

Dividing both sides of this inequality by nr and letting $r \to \infty$ we get:

$$(2.4.1) \qquad a_0 + a_1 + \ldots + a_s \leqq b_0/n + b_1/n + \ldots + b_t/n \quad \text{for all } s \geqq 0.$$

To prove that $(1/n)\,\mathrm{Hdg}(\mathrm{A})$ lies on or above $\mathrm{Hdg}(\mathrm{B}/n)$ it is enough to prove that all the break-points of $(1/n)\,\mathrm{Hdg}(\mathrm{A})$ lies above $\mathrm{Hdg}(\mathrm{B}/n)$. Let (x, y) be a break-point of $(1/n)\,\mathrm{Hdg}(\mathrm{A})$ so we have:

$$x = a_0 + a_1 + \ldots + a_s$$

and:

$$y = \frac{1}{n}a_1 + \frac{2}{n}a_2 + \ldots + \frac{s}{n}a_s,$$

for some s. By (2.4.1) we can find a j such that:

$$b_0/n + b_1/n + \ldots + b_{j-1}/n \geqq x \geqq b_0/n + b_1/n + \ldots + b_{j-1}/n + b_j/n,$$

so the value at x of $\mathrm{Hdg}(\mathrm{B}/n)$ is equal to:

$$y_1 = b_1/n + 2b_2/n + \ldots + (j-1)b_{j-1}/n + j(x - (b_0/n + \ldots + b_{j-1}/n))$$

and we must show that $y \geqq y_1$.

This is equivalent to proving that:

$$j.a_0 + \left(j - \frac{1}{n}\right)a_1 + \ldots + \left(j - \frac{s}{n}\right)a_s \leqq j.b_0/n + (j-1)b_1/n + \ldots + b_{j-1}/n,$$

but multiplying both sides by n we see that this is an immediate consequence of the assumptions.

THEOREM 2.5. — *$(1/n)\,\mathrm{Hdg}(\Phi^n)$ lies on or above the Hodge polygon of* $\{h^0(n)/n, h^1(n)/n, \ldots\}$.

Proof. — $\mathrm{Hdg}(\Phi^n)$ is the Hodge polygon of the numbers $\{\tilde{h}^0(n), \tilde{h}^1(n), \ldots\}$ and by 2.2 the two sets of numbers:

$$\{\tilde{h}^0(n), \tilde{h}^1(n), \ldots\}$$

and:

$$\{h^0(n), h^1(n), \ldots\},$$

satisfy the hypotheses of 2.4.

It would be interesting to know if 2.5 could be strengthened by replacing the "geometric Hodge numbers" $\{h^j(n)\}$ by the "reduced Hodge numbers" $\{\bar{h}^j(n) = l(\mathrm{E}_\infty^{i-j,\,j}(n))\}$ where $\mathrm{E}_\infty^{\cdot\,\cdot}(n)$ denotes the ∞-term in the spectral sequence:

$$\mathrm{E}_1^{p,\,q}(n) = \mathrm{H}^q(\mathrm{X}, \mathrm{W}_n\Omega_{\mathrm{X}}^p) \Rightarrow \mathrm{H}_{\mathrm{crys}}^{p+q}(\mathrm{X}/\mathrm{W}_n).$$

As a first step in this direction we have the following result:

PROPOSITION 2.6. — *For all* $n \geqq 1$ *we have* $\widetilde{h}^0(n) \leqq \overline{h}^0(n)/n$.

Proof. — For all $n \geqq 1$ we have by (2.2.1), (2.2.2) and (2.2.3):

$$n.\widetilde{h}^0(n) \leqq l(\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^n H^i_{\mathrm{crys}}(\mathrm{X/W}))) \leqq l(\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\tilde{\mathrm{V}}\mathbb{H}^i(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n))).$$

Consider the commutative diagram of complexes with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}\Omega^\bullet_{\mathrm{X}} & = & \mathrm{W}\Omega^\bullet_{\mathrm{X}} & & \\
 & & \downarrow{\scriptstyle \tilde{\mathrm{F}}} & & \downarrow{\scriptstyle p^n} & & \\
0 & \longrightarrow & \mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n) & \xrightarrow{\tilde{\mathrm{V}}} & W\Omega^\bullet_{\mathrm{X}} & \xrightarrow{\beta(0, n)} & t_{\leqq 0}\mathrm{W}_n\Omega^\bullet_{\mathrm{X}} \to 0 \\
 & & \downarrow{\scriptstyle \alpha(1, n)} & & \downarrow{\scriptstyle \mathrm{pr}_n} & & \uparrow{\scriptstyle \wr\, \mathrm{C}^{-n}} \\
0 & \longrightarrow & \mathrm{W}_n\Omega^{\bullet\, \geqq 1}_{\mathrm{X}} & \longrightarrow & \mathrm{W}_n\Omega^\bullet_{\mathrm{X}} & \longrightarrow & \mathrm{W}_n\mathcal{O}_{\mathrm{X}} \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

which gives a commutative diagram with exact rows and columns of the hypercohomology groups:

$$\begin{array}{ccccccc}
\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}) & = & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}) & & & & \\
\downarrow{\scriptstyle \tilde{\mathrm{F}}} & & \downarrow{\scriptstyle p^n} & & & & \\
\mathbb{H}^i(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n)) & \xrightarrow{\tilde{\mathrm{V}}} & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}) & \xrightarrow{\beta(0, n)} & \mathbb{H}^i(t_{\leqq 0}\mathrm{W}_n\Omega^\bullet_{\mathrm{X}}) & \to & \mathbb{H}^{i+1}(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n) \\
\downarrow{\scriptstyle \alpha(1, n)} & & \downarrow{\scriptstyle \mathrm{pr}_n} & & \uparrow{\scriptstyle \wr\, \mathrm{C}^{-n}} & & \downarrow \\
\mathbb{H}^i(\mathrm{W}_n\Omega^{\bullet\, \geqq 1}_{\mathrm{X}}) & \longrightarrow & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}_n) & \longrightarrow & \mathrm{H}^i(\mathrm{W}_n\mathcal{O}_{\mathrm{X}}) & \longrightarrow & \mathbb{H}^{i+1}(\mathrm{W}_n\Omega^{\bullet\, \geqq 1}_{\mathrm{X}})
\end{array}$$

It follows that we have an injection:

$$\ker(\mathbb{H}^i(t_{\leqq 0}\mathrm{W}_n\Omega^\bullet_{\mathrm{X}})) \to \mathbb{H}^{i+1}(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n))),$$
$$\subset \ker(\mathrm{H}^i(\mathrm{W}_n\mathcal{O}_{\mathrm{X}}) \to \mathbb{H}^{i+1}(\mathrm{W}_n\Omega^{\bullet\, \geqq 1}_{\mathrm{X}})),$$

but:

$$\ker(\mathbb{H}^i(t_{\leqq 0}\mathrm{W}_n\Omega^\bullet_{\mathrm{X}}) \to \mathbb{H}^{i+1}(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n))) = \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\mathrm{V}\mathbb{H}^i(\mathrm{W}\Omega^\bullet_{\mathrm{X}}(1, n))$$

and:

$$\ker(\mathrm{H}^i(\mathrm{W}_n\mathcal{O}_{\mathrm{X}}) \to \mathbb{H}^{i+1}(\mathrm{W}_n\Omega^{\bullet\, \geqq 1}_{\mathrm{X}})) = \mathrm{E}^{0,\, i}_{\infty}(n)$$

so the proposition follows.

3. Hodge numbers of Φ^n

THEOREM 3.1. — *Fix an integer $n \geqq 1$. Assume that the crystalline cohomology groups $H^i_{\mathrm{crys}}(X/W)$ are torsion free for all i and that the first hypercohomology spectral sequence:*

$$E_1^{p,q}(n) = H^q(X, W_n\Omega_X^p) \Rightarrow H^{p+q}_{\mathrm{crys}}(X/W_n),$$

degenerates at E_1, then the groups $\mathbb{H}^i(W\Omega_X^{\cdot}(r, n))$ are torsion free for all $r \geqq 0$ and the maps:

$$\tilde{F}: \mathbb{H}^i(W\Omega_X^{\cdot}(r, n)) \to \mathbb{H}^i(W\Omega_X^{\cdot}(r+1, n)),$$
$$\tilde{V}: \mathbb{H}^i(W\Omega_X^{\cdot}(r, n)) \to \mathbb{H}^i(W\Omega_X^{\cdot}(r-1, n))$$

are injective.

Proof. — Since kernels of $\tilde{F}$ and $\tilde{V}$ are killed by p^n (because $\tilde{F}\tilde{V} = \tilde{V}\tilde{F} = p^n$) it is clear that if $\mathbb{H}^i(W\Omega_X^{\cdot}(r, n))$ is torsion free then $\tilde{F}$ and $\tilde{V}$ are injective.

To prove that $\mathbb{H}^i(W\Omega_X^{\cdot}(r, n))$ is torsion free we use induction on r. For $r=0$ we have $W\Omega_X^{\cdot}(0, n) = W\Omega_X^{\cdot}$ hence $\mathbb{H}^i(W\Omega_X^{\cdot}(0, n)) = H^i_{\mathrm{crys}}(X/W)$ which is torsion free by assumption.

Assume now that the $\mathbb{H}^i(W\Omega_X^{\cdot}(s, n))$ are torsion free for $s \leqq r-1$. By 1.4 we have a commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{H}^i(W\Omega_X^{\cdot}(r-1, n)) & \xrightarrow{\tilde{V}} & \mathbb{H}^i(W\Omega_X^{\cdot}(r-2, n)) \\
\downarrow{\scriptstyle \tilde{F}} & & \downarrow \\
\mathbb{H}^i(W\Omega_X^{\cdot}(r, n)) & \xrightarrow{\tilde{V}} & \mathbb{H}^i(W\Omega_X^{\cdot}(r-1, n)) \\
\downarrow{\scriptstyle \alpha(r,n)} & & \downarrow{\scriptstyle \alpha(r-1,n)} \\
\mathbb{H}^i(W_n\Omega_X^{\cdot \geqq r}) & \longrightarrow & \mathbb{H}^i(W_n\Omega_X^{\cdot \geqq r-1}).
\end{array}$$

By the induction hypotheses and 1.5 the columns are exact and the $\tilde{V}$ on top is injective. The bottom horizontal map is injective by the degeneracy assumption, it follows that the $\tilde{V}$ in the middle is injective and hence $\mathbb{H}^i(W\Omega_X^{\cdot}(r, n))$ is torsion free.

For the remainder of this section we will assume that the conditions of 3.1 are satisfied.

LEMMA 3.2. — *The second hypercohomology spectral sequence:*

$$E_2^{p,q}(n) = H^p(X, \mathscr{H}^q(W_n\Omega_X^{\cdot})) \Rightarrow H^{p+q}_{\mathrm{crys}}(X/W_n),$$

degenerates at E_2.

Proof. — By 1.7 we have:

$$l(E_1^{p,q}(n)) = l(E_2^{q,p}(n))$$

and by assumption:

$$\sum_{p+q=i} l(E_1^{p,q}(n)) = l(H^i_{crys}(X/W_n)),$$

hence :

$$\sum_{p+q=i} l(E_2^{q,p}(n)) = l(H^i_{crys}(X/W_n)).$$

We also have:

$$\sum_{p+q=i} l(E_\infty^{q,p}(n)) = l(H^i_{crys}(X/W_n))$$

and since $l(E_\infty^{q,p}(n)) \leqq l(E_2^{q,p}(n))$ it follows that all the inequalities must be equalities hence $E_\infty^{q,p}(n) = E_2^{q,p}(n)$.

Corollary 3.3. — *For all* $r \geqq 0$ *we have:*

$$\Phi^{-n}(p^{nr} H^i_{crys}(X/W)) = \tilde{V}^r \mathbb{H}^i(W\Omega^\bullet_X(r, n))$$

and:

$$\operatorname{Im} \Phi^n \cap p^{nr} H^i_{crys}(X/W) = p^{nr} \tilde{F}^{N+1-r} \mathbb{H}^i(W\Omega^\bullet_X(r, n))$$

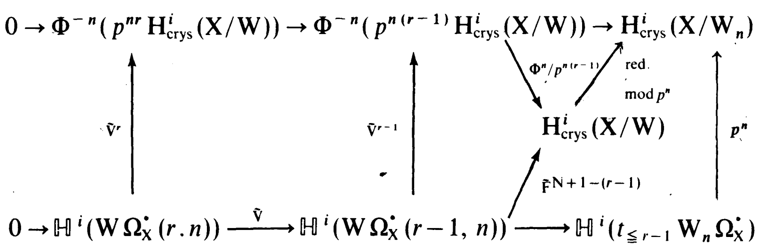
$(N = \dim X)$.

Proof. — We use induction on r. The statements are trivial for $r=0$ so assume they hold for $s=r-1$.

We have:

$$\Phi^{-n}(p^{nr} H^i_{crys}(X/W)) = \ker(\Phi^{-n}(p^{n(r-1)} H^i_{crys}(X/W)) \xrightarrow{\Phi^n/p^{n(r-1)}} H^i_{crys}(X/W) \xrightarrow[\text{mod. } p^n]{\text{red.}} H^i_{crys}(X/W_n)).$$

By 1.4; (*b*) and 2.1.1 we have a commutative diagram with exact rows:



The extreme right hand vertical map is injective by 3.2 and by the induction hypotheses $\tilde{V}^{r-1}$ is bijective hence:

$$\tilde{V}^r: \ \mathbb{H}^i(W\Omega^\bullet_X(r, n)) \to \Phi^{-n}(p^{nr} H^i_{crys}(X/W)),$$

is an isomorphism.

The second assertion follows using the commutative diagram (2.1.1):

$$
\begin{array}{ccc}
\Phi^{-n}(p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})) & \overset{\Phi^n}{\underset{\sim}{\rightarrow}} & \mathrm{Im}\,\Phi^n \cap p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \\
{\scriptstyle \tilde{\mathrm{V}}^r}\nwarrow\!\!\wr & & \nearrow{\scriptstyle p^{nr}\tilde{\mathrm{F}}^{\mathrm{N}+1-r}} \\
 & \mathbb{H}^i(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(r,\,n)) &
\end{array}
$$

COROLLARY 3.4 (Compare with Ogus [3], Thm. 8.26). — *Let:*

$$\pi:\quad \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \to \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n)$$

denote reduction mod p^n, *then* π *maps:*

$$\Phi^{-n}(p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})) \quad \textit{onto}\ \mathrm{Fil}^r\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n)$$

and π/p^{nr} *maps:*

$$\mathrm{Im}\,\Phi^n \cap p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \quad \textit{onto}\ \mathrm{Fil}_r\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n),$$

where $\mathrm{Fil}^{\bullet}$ *denotes the filtration induced from the spectral sequence:*

$$\mathrm{E}_1^{p,\,q}(n) = \mathrm{H}^q(\mathrm{X},\,\mathrm{W}_n\,\Omega^p_{\mathrm{X}}) \quad \Rightarrow \quad \mathrm{H}^{p+q}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n),$$

and $\mathrm{Fil}_{\bullet}$ *denotes the filtration induced from the spectral sequence:*

$$\mathrm{E}_2^{p,\,q}(n) = \mathrm{H}^p(\mathrm{X},\,\mathscr{H}^q(\mathrm{W}_n\,\Omega^{\bullet}_{\mathrm{X}})) \quad \Rightarrow \quad \mathrm{H}^{p+q}_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n).$$

Furthermore we have a commutative diagram:

$$
\begin{array}{ccc}
\Phi^{-n}(p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})) & \overset{\Phi^n}{\underset{\sim}{\rightarrow}} & \mathrm{Im}\,\Phi^n \cap p^{nr}\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \\
\downarrow & & \downarrow{\scriptstyle \pi/p^{nr}} \\
\mathrm{Fil}^r\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n) & & \mathrm{Fil}_r\,\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n) \\
\downarrow & & \downarrow \\
\mathrm{H}^{i-r}(\mathrm{X},\,\mathrm{W}_n\,\Omega^r_{\mathrm{X}}) & \xrightarrow[\sim]{\mathrm{C}^{-n}} & \mathrm{H}^{i-r}(\mathrm{X},\,\mathscr{H}^r(\mathrm{W}_n\,\Omega^{\bullet}_{\mathrm{X}})).
\end{array}
$$

Proof. — By 1.4 we have commutative diagrams:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{H}^i(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(r-1,\,n)) & \xrightarrow{\tilde{\mathrm{V}}^{r-1}} & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \\
\downarrow{\scriptstyle \tilde{\mathrm{F}}} & & \downarrow{\scriptstyle p^n} \\
\mathbb{H}^i(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(r,\,n)) & \xrightarrow{\tilde{\mathrm{V}}^{r}} & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \\
\downarrow{\scriptstyle \alpha(r,\,n)'} & & \downarrow{\scriptstyle \pi} \\
\mathbb{H}^i(\mathrm{W}_n\,\Omega^{\bullet\,\geq r}_{\mathrm{X}}) & \longrightarrow & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}_n) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{H}^i(W\Omega_X^{\cdot}(r+1,\, n)) & \xrightarrow{\tilde{F}^{N+1-(r-1)}} & H^i_{\mathrm{crys}}(X/W) \\
\downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle p^n} \\
\mathbb{H}^i(W\Omega_X^{\cdot}(r,\, n)) & \xrightarrow{\tilde{F}^{N+1-r}} & H^i_{\mathrm{crys}}(X/W) \\
\downarrow{\scriptstyle \beta(r,\, n)} & & \downarrow{\scriptstyle \pi} \\
\mathbb{H}^i(t_{\leqq r}\, W_n\Omega_X^{\cdot}) & \longrightarrow & H^i_{\mathrm{crys}}(X/W_n) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Under our assumptions the columns are exact and since:

$$\mathrm{Fil}^r\, H^i_{\mathrm{crys}}(X/W_n) = \mathrm{Im}(\mathbb{H}^i(W_n\Omega_X^{\cdot\, \geqq r}) \to H^i_{\mathrm{crys}}(X/W_n))$$

and:

$$\mathrm{Fil}_r\, H^i_{\mathrm{crys}}(X/W_n) = \mathrm{Im}(\mathbb{H}^i(t_{\leqq r}\, W_n\Omega_X^{\cdot}) \to H^i_{\mathrm{crys}}(X/W_n)),$$

the first two parts of the Corollary follow from 3.3.

The third statement is an immediate consequence of these and of the diagram of complexes:

$$
\begin{array}{ccccc}
W\Omega_X^{\cdot}(r,\, n) \to W\Omega_X^{\cdot}(r,\, n)/\tilde{F}W\Omega_X^{\cdot}(r-1,\, n) & \xrightarrow{\alpha(r,\, n)} & W_n\Omega_X^{\cdot\, \geqq r} & \to & W_n\Omega_X^r \\
\downarrow & & & & \downarrow{\scriptstyle C^{-n}} \\
W\Omega_X^{\cdot}(r,\, n)/\tilde{V}W\Omega_X^{\cdot}(r+1,\, n) & \xrightarrow{\beta(r,\, n)} & t_{\leqq r}\, W_n\Omega_X^{\cdot} & \to & \mathscr{H}^r(W_n\Omega_X^{\cdot})
\end{array}
$$

The verification that this diagram commutes is left to the reader.

COROLLARY 3.5. — *The Hodge numbers* $\{\tilde{h}^j(n)\}$ *of* Φ^n *on* $H^i_{\mathrm{crys}}(X/W)$ *satisfy the equations:*

$$nr.\tilde{h}^0(n) + (nr-1)\tilde{h}^1(n) + \ldots + \tilde{h}^{nr-1}(n) = r.h^0(n) + (r-1)h^1(n) + \ldots + h^{r-1}(n),$$

for all $r \geqq 0$.

Proof. — Since $H^i_{\mathrm{crys}}(X/W)$ is torsion free (by assumption), (2.2.1) implies that the left hand side is equal to:

$$l(H^i_{\mathrm{crys}}(X/W)/\Phi^{-n}(p^{nr}\, H^i_{\mathrm{crys}}(X/W))),$$

which by 3.3 is equal to:

$$l(H^i_{\mathrm{crys}}(X/W)/\tilde{V}^r\mathbb{H}^i(W\Omega_X^{\cdot}(r,\, n))).$$

We use induction on r to show that $l(H^i_{\mathrm{crys}}(X/W)/\tilde{V}^r\mathbb{H}^i(W\Omega_X^{\cdot}(r,\, n)))$ is equal to the right hand side.

For $r=0$ the equality is trivial so assume it holds for $s=r-1$. Since:

$$\tilde{V}^{r-1}: \quad \mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n)) \to H^i_{\text{crys}}(X/W),$$

is injective by 3.1, we have an exact sequence:

$$0 \to \mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n))/\tilde{V}\mathbb{H}^i(W\Omega_X^{\bullet}(r, n)) \xrightarrow{\tilde{V}^{r-1}} H^i_{\text{crys}}(X/W)/\tilde{V}^r\mathbb{H}^i(W\Omega_X^{\bullet}(r, n)) \to H^i_{\text{crys}}(X/W)/\tilde{V}^{r-1}\mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n)) \to 0,$$

hence it suffices to show that:

$$l(\mathbb{H}^i(W\Omega_X^{\bullet}(r-1))/\tilde{V}\mathbb{H}^i(W\Omega_X^{\bullet}(r, n))) = h^0(n)+h^1(n)+\ldots+h^{r-1}(n).$$

Since $\mathbb{H}^{i+1}(W\Omega_X^{\bullet}(r, n))$ is torsion free by 3.1 we have:

$$\mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n))/\tilde{V}\mathbb{H}^i(W\Omega_X^{\bullet}(r, n)) = \mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n)/\tilde{V}W\Omega_X^{\bullet}(r, n))$$

and by 1.5:

$$\mathbb{H}^i(W\Omega_X^{\bullet}(r-1, n)/\tilde{V}W\Omega_X^{\bullet}(r, n)) = \mathbb{H}^i(t_{\leqq r-1}W_n\Omega_X^{\bullet}).$$

Now 3.2 implies that:

$$\begin{aligned} l(\mathbb{H}^i(t_{\leqq r-1}W_n\Omega_X^{\bullet})) &= l(H^{i-(r-1)}(X, \mathscr{H}^{r-1}(W_n\Omega_X^{\bullet}))) \\ &\quad + l(H^{i-(r-1)+1}(X, \mathscr{H}^{(r-1)+1}(W_n\Omega_X^{\bullet}))) + \ldots \\ &\quad + l(H^i(X, \mathscr{H}^0(W_n\Omega_X^{\bullet}))) = h^{r-1}(n)+h^{r-2}(n)+\ldots+h^0(n), \end{aligned}$$

where the last equality follows from 1.7.

For $n=1$ these equations show that $\tilde{h}^j(1)=h^j(1)$, but for $n>1$ they do not suffice to determine the Hodge numbers of Φ^n. In the next Theorem we give a formula for $\tilde{h}^j(n)$, $j \leqq n-1$.

THEOREM 3.6. — *Assume that* $H^i(X, W\mathcal{O}_X) \to H^i(X, W_s\mathcal{O}_X)$ *is surjective for* $s \leqq n$ *and all* i.

Let $v^j(n)=l(H^i(X, W_j\mathcal{O}_X)/F^{n-j})$ *then we have:*

$$\tilde{h}^0(n)=h^0(1)-v^1(n),$$

$$\tilde{h}^i(n)=2v^i(n)-v^{i+1}(n)-v^{i-1}(n), \qquad 0<i<n-1,$$

$$\tilde{h}^{n-1}(n)=2v^{n-1}(n)-v^{n-2}(n).$$

Proof. — Consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \Phi^{-n}(p^{n-s}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) & \to & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}) & \to & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n-s}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) & \to 0 \\
& \downarrow{\scriptstyle p^s} & & {\scriptstyle p^s}\downarrow & & \downarrow{\scriptstyle \underline{p}^s} & \\
0 \to & \Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) & \to & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}) & \to & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}_s) & \longrightarrow & \mathrm{Coker}\, p^s & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

The map:

$$\underline{p}^s:\ \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n-s}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) \to \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})),$$

is clearly injective so we have:

$$l(\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n-s}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}))) = l(\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W}))) - l(\mathrm{Coker}\,\underline{p}^s).$$

By 3.3:

$$\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) = \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\tilde{\mathrm{V}}\mathbb{H}^i(\mathrm{W}\Omega^{\bullet}_{\mathrm{X}}(1, n)) = \mathrm{H}^i(\mathrm{X}, \mathrm{W}_n\mathcal{O}_{\mathrm{X}}).$$

The surjectivity assumption implies that we have exact sequences:

$$0 \to \mathrm{H}^i(\mathrm{X}, \mathrm{W}_s\mathcal{O}_{\mathrm{X}}) \xrightarrow{\mathrm{V}^t} \mathrm{H}^i(\mathrm{X}, \mathrm{W}_{s+t}\mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^i(\mathrm{X}, \mathrm{W}_t\mathcal{O}_{\mathrm{X}}) \to 0,$$

for $s+t \leqq n$, it follows that:

$$l(\mathrm{H}^i(\mathrm{X}, \mathrm{W}_n\mathcal{O}_{\mathrm{X}})) = n\,.\,h^0(1).$$

The commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) & & \\
{\scriptstyle p^s}\downarrow & \searrow & \mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n-s}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})) \\
& \swarrow{\scriptstyle \underline{p}^s} & \\
\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})/\Phi^{-n}(p^{n}\mathrm{H}^i_{\mathrm{crys}}(\mathrm{X/W})), & &
\end{array}$$

shows that:

$$\mathrm{Coker}\,\underline{p}^s = \mathrm{Coker}\, p^s : \mathrm{H}^i(\mathrm{X}, \mathrm{W}_n\mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^i(\mathrm{X}, \mathrm{W}_n\mathcal{O}_{\mathrm{X}}).$$

Now p^s factors:

$$H^i(X, W_n\mathcal{O}_X) \to H^i(X, W_{n-s}\mathcal{O}_X) \xrightarrow{V^s} H^i(X, W_n\mathcal{O}_X) \xrightarrow{F^s} H^i(X, W_n\mathcal{O}_X),$$

so since $H^i(X, W_n\mathcal{O}_X) \to H^i(X, W_{n-s}\mathcal{O}_X)$ is surjective we get:

$$\text{Coker } p^s = \text{Coker}(H^i(X, W_{n-s}\mathcal{O}_X) \xrightarrow{V^s} H^i(X, W_n\mathcal{O}_X)).$$

From the commutative diagram:

$$\begin{array}{ccccccc}
 & H^i(X, W_{n-s}\mathcal{O}_X) & = & H^i(X, W_{n-s}\mathcal{O}_X) & \longrightarrow & & 0 \\
 & \downarrow F^s & & \downarrow F^s V^s & & & \\
0 \to & H^i(X, W_{n-s}\mathcal{O}_X) & \xrightarrow{V^s} & H^i(X, W_n\mathcal{O}_X) & \to & H^i(X, W_s\mathcal{O}_X) & \to 0 \\
 & \downarrow & & \downarrow & & & \\
0 \to & H^i(X, W_{n-s}\mathcal{O}_X)/F^s & \to & \text{Coker } p^s & \to & H^i(X, W_s\mathcal{O}_X) & \longrightarrow 0
\end{array}$$

we get:

$$l(\text{Coker } p^s) = s.h^0(1) + v^{n-s}(n).$$

It follows that:

$$l(H^i_{\text{crys}}(X/W)/\Phi^{-n}(p^{n-s} H^i_{\text{crys}}(X/W))) = n.h^0(1) - (s.h^0(1) + v^{n-s}(n)).$$

Now:

$$l(H^i_{\text{crys}}(X/W)/\Phi^{-n}(p^{n-s} H^i_{\text{crys}}(X/W))) = (n-s)\tilde{h}^0(n) + \ldots + \tilde{h}^{n-s-1}(n),$$

so we get the following set of equations:

$$\begin{aligned}
\tilde{h}^0(n) &= n.h^0(1) - ((n-1)h^0(1) + v^1(n)) = h^0(1) - v^1(n), \\
2\tilde{h}^0(n) + \tilde{h}^1(n) &= n.h^0(1) - ((n-2)h^0(1) + v^2(n)) = 2h^0(1) - v^2(n), \\
&\vdots \\
s.\tilde{h}^0(n) + (s-1)\tilde{h}^1 n) + \ldots + \tilde{h}^{s-1}(n) &= n.h^0(1) - ((n-s)h^0(1) + v^s(n)) = s.h^0(1) + v^s(n), \\
&\vdots \\
n.\tilde{h}^0(n) + (n-1)\tilde{h}^1(n) + \ldots + \tilde{h}^{n-1}(n) &= n.h^0(1).
\end{aligned}$$

Solving these equations for $\tilde{h}^0(n), \tilde{h}^1(n), \ldots, \tilde{h}^{n-1}(n)$ gives the formula of the Theorem.

Example. — Let X be a supersingular K3 surface with Artin invariant σ_0, then it can be shown (e.g. by the methods of [10]) that the spectral sequences:

$$E_1^{p,q}(n) = H^q(X, W_n\Omega_X^p) \Rightarrow H^{p+q}_{\text{crys}}(X/W_n),$$

degenerate for $n \leqq \sigma_0$. Frobenius is zero on all the $H^2(X, W_s\mathcal{O}_X)$ and $H^1(X, W_s\mathcal{O}_X)=0$, the crystalline cohomology groups are torsion free so the conditions of 3.1 and 3.6 are satisfied. We have:

$$\nu^s(n)=l(H^2(X, W_s\mathcal{O}_X)/F^{n-s})=l(H^2(X, W_s\mathcal{O}_X))=s.h^0(1)=s.$$

By 3.6 we get:

$$\tilde{h}^i(n)=0 \qquad \text{for} \quad i<n-1,$$

$$\tilde{h}^{n-1}(n)=n \qquad \text{for} \quad n \leqq \sigma_0.$$

Let us show that $\tilde{h}^i(n)=0$ for $i>n+1$. If not, there will be an element x which is part of a basis of $H^2_{\text{crys}}(X/W)$ such that:

$$\Phi^n(x)=p^i y, \qquad i>n+1.$$

We know that $\Phi(x) \in H^1(X, W\Omega^1_X)$ and $\Phi \mid H^1(X, W\Omega^1_X)=pF$ [5] where F is σ-linear automorphism of $H^1(X, W\Omega^1_X)$.

It follows that:

$$\Phi^n(x)=p^{n-1}F^{n-1}(\Phi(x))$$

and so:

$$F^{n-1}(\Phi(x))=p^{i-(n-1)}y=p^{i-n}(py).$$

Since $py \in H^1(X, W\Omega^1_X)$ we have:

$$\Phi(x)=p^{i-n}F^{-(n-1)}(py).$$

Now $i-n \geqq 2$ so we can write:

$$\Phi(x)=p^2 z \qquad \text{with} \quad z \in H^1(X, W\Omega^1_X),$$

or equivalently:

$$F(px)=p(pz),$$

so:

$$px=pF^{-1}(pz),$$

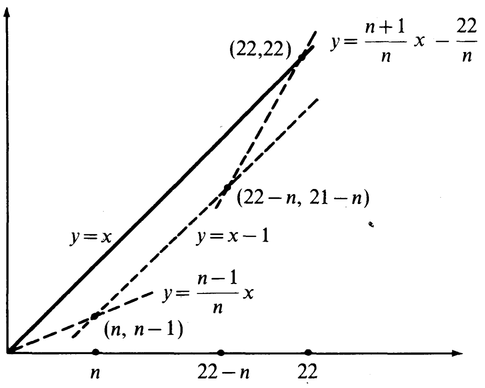
i.e.

$$x=F^{-1}(pz)=pF^{-1}(z),$$

but this shows that x is divisible by p which is a contradiction since x is part of a basis of

$H^2_{crys}(X/W)$, so $\tilde{h}^i(n)$ must be 0. From the Figure below we get the Hodge rumbers of Φ^n on $H^2_{crys}(X/W)$:

$$\tilde{h}^{n-1}(n)=n,$$
$$\tilde{h}^{n}(n)=22-2n,$$
$$\tilde{h}^{n-1}(n)=n, \qquad n \leqq \sigma_0, \quad \text{all others are 0.}$$



4. Applications

If X is a curve the conditions of 3.1 and 3.5 are satisfied for all n (this follows since the slope spectral sequence of a curve degenerates at E_1 and the E_1 term maps onto the E_1 term of the spectral sequence:

$$H^q(X, W_n\Omega^p_X) \Rightarrow H^{p+q}_{crys}(X/W_n) \text{ [5]).}$$

We shall apply the results of section 3 to prove the following Theorem:

THEOREM 4.1. — *Let* X *be a curve. Then the Jacobian* J(X) *is isomorphic to a product of supersingular elliptic curves if and only if the Cartier operator* $C : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ *vanishes.*

Proof. — Assume that $C=0$. By Serre duality this is equivalent to the vanishing of $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$. We show first that J(X) is a supersingular abelian variety (this generalizes a theorem of Yui for hyperelliptic curves [13]).

Now $v^1(2)=l(H^1(X, \mathcal{O}_X)/F)=h^0(1)$ hence $\tilde{h}^0(2)=h^0(1)-h^0(1)=0$. It follows that the first slope in $(1/2)\,\mathrm{Hdg}(\Phi^2)$ is $1/2$ but since the Newton polygon lies between the straight line with slope $1/2$ and $(1/2)\,\mathrm{Hdg}(\Phi^2)$ it follows that the first slope and hence all the slopes in the Newton polygon are equal to $1/2$. This shows that the Newton polygon of $(H^1_{crys}(X/W), \Phi)=(\mathbb{D}(J(X)), F)$ [$\mathbb{D}(J(X))$ = Dieudonne module of the p-divisible group of J(X)] is a straight line with slope $1/2$ so J(X) is supersingular.

By Ogus's Torelli Theorem for supersingular abelian varieties ([11], Cor. 6.7) it is now enough to show that:

$$\mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) = \mathrm{Fil}^{\mathrm{con}}_1 \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k).$$

We have:

$$\mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) = \mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}/k})$$

and:

$$\mathrm{Fil}^{\mathrm{con}}_1 \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) = \mathrm{H}^1(\mathrm{X}, \mathscr{H}^0(\Omega^{\bullet}_{\mathrm{X}/k})).$$

This fits into a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(\mathrm{X}, \mathscr{H}^0(\Omega^{\bullet}_{\mathrm{X}/k})) & \rightarrow & \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) \\ \wr \uparrow \mathrm{F} & & \downarrow \\ \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) & \xrightarrow{\mathrm{F}} & \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \end{array}$$

Since $\mathrm{F} : \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \rightarrow \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ is zero it follows that:

$$\mathrm{Fil}^{\mathrm{con}}_1 \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) = \mathrm{H}^1(\mathrm{X}, \mathscr{H}^0(\Omega^{\bullet}_{\mathrm{X}/k})) \subset \ker(\mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) \rightarrow \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}})) = \mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k)$$

and since rank:

$$\mathrm{H}^1(\mathrm{X}, \mathscr{H}^0(\Omega^{\bullet}_{\mathrm{X}/k})) = \mathrm{rank}\ \mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}/k})$$

these two subspaces must be equal.

Conversely if $\mathrm{J}(\mathrm{X}) = \mathrm{E}_1 \times \ldots \times \mathrm{E}_g$ where the E_i's are super-singular elliptic curves, we have:

$$\mathrm{H}^1_{\mathrm{DR}}(\mathrm{X}/k) = \mathrm{H}^1_{\mathrm{DR}}(\mathrm{J}(\mathrm{X})/k) = \mathrm{H}^1_{\mathrm{DR}}(\mathrm{E}_1/k) \times \ldots \times \mathrm{H}^1_{\mathrm{DR}}(\mathrm{E}_g/k)$$

and:

$$\mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}/k}) = \mathrm{H}^0(E_1, \Omega^1_{\mathrm{E}_1/k}) \times \ldots \times \mathrm{H}^0(\mathrm{E}_g, \Omega^1_{\mathrm{E}_g/k}),$$

but C vanishes on $\mathrm{H}^0(\mathrm{E}_i, \Omega^1_{\mathrm{E}_i/k})$, $i = 1, 2, \ldots, g$.

Remark 4.2. — One can use the formula from Theorem 3.6 to give a necessary and sufficient condition for J(X) to be supersingular in terms of vanishing of certain powers of F on a finite number of the Witt vector cohomology groups $\mathrm{H}^1(\mathrm{X}, \mathrm{W}_i(\mathcal{O}_{\mathrm{X}}))$.

Assume now that X is a surface with $\mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ torsion free and assume that the Hodge to de Rham spectral sequence:

$$\mathrm{E}_1^{p,\,q} = \mathrm{H}^q(\mathrm{X}, \Omega^p_{\mathrm{X}/k}) \quad \Rightarrow \quad \mathrm{H}^{p+q}_{\mathrm{DR}}(\mathrm{X}/k),$$

degenerates at E_1.

The following Theorem was proved by Illusie [5] under stronger assumptions.

THEOREM 4.3. — *Let* $P^i H^2_{crys}(X/W)$ *denote the filtration induced from the slope spectral sequence, and let* M^i *denote the largest subcrystal of* $H^2_{crys}(X/W)$ *such that* $\Phi \mid M^i$ *is divisible by* p^i, *then we have:*

$$P^i H^2_{crys}(X/W) = M^i, \qquad i=1, 2.$$

Proof. — Define:

$$v_1 : \quad W\Omega^\cdot_X(1, n) \to W\Omega^\cdot_X(1, n-1),$$

by the commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & W\mathcal{O}_X & \xrightarrow{dV^n} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \\ v_1 & \downarrow & & \| & & \| & \\ 0 \to & W\mathcal{O}_X & \xrightarrow{dV^{n-1}} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0. \end{array}$$

Remark that $v_1^n : W\Omega^\cdot_X(1, n) \to W\Omega^\cdot_X$ is just $\tilde{V}$ hence we have a commutative diagram of complexes:

$$\begin{array}{ccc} W\Omega^\cdot_X(1, n) & \xrightarrow{\tilde{V}} & \\ {\scriptstyle v_1}\downarrow & & W\Omega^\cdot_X \\ W\Omega^\cdot_X(1, n-1) & \xrightarrow{\tilde{V}} & \end{array}$$

It follows that:

$$\tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n)) \subset \tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n-1)),$$

for all $n \geqq 1$, hence by 2.1:

$$\begin{aligned} \tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n)) &\subset \Phi^{-n}(p^n H^2_{crys}(X/W)) \cap \tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n-1)) \\ &\subset \Phi^{-n}(p^n H^2_{crys}(X/W)) \cap \Phi^{-(n-1)}(p^{n-1} H^2_{crys}(X/W)) \cap \tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n-2)) \end{aligned}$$

and by induction:

$$\tilde{V}\mathbb{H}^2(W\Omega^\cdot_X(1, n)) \subset \bigcap_{i=0}^{n} \Phi^{-i}(p^i H^2_{crys}(X/W)).$$

Next define:

$$v_2 : \quad W\Omega^\cdot_X(2, n) \to W\Omega^\cdot_X(2, n-1),$$

by the commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_X & \xrightarrow{dV^n} & W\Omega^2_X & \to 0 \\ v_2 & \downarrow{\scriptstyle pV} & & \downarrow{\scriptstyle V} & & \| & \\ 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_X & \xrightarrow{dV^{n-1}} & W\Omega^2_X & \to 0 \end{array}$$

then $v_2^n : W\Omega^{\bullet}_X(2, n) \to W\Omega^{\bullet}_X$ is just $\tilde{V}^2$ and we have a commutative diagram:

$$\begin{array}{ccc} W\Omega^{\bullet}_X(2, n) & \xrightarrow{\tilde{V}^2} & \\ {\scriptstyle v_2}\downarrow & & W\Omega^{\bullet}_X \\ W\Omega^{\bullet}_X(2, n-1) & \xrightarrow{\tilde{V}^2} & \end{array}$$

so $\tilde{V}^2 \mathbb{H}^2(W\Omega^{\bullet}_X(2, n)) \subset \tilde{V}^2 \mathbb{H}^2(W\Omega^{\bullet}_X(2, n-1))$ and the same argument as before shows that:

$$\tilde{V}^2 \mathbb{H}^2(W\Omega^{\bullet}_X(2, n)) \subset \bigcap_{i=0}^{n} \Phi^{-i}(p^{2i} H^2_{crys}(X/W)).$$

We claim now that it is enough to show that:

(4.3.1) $$\tilde{V} : \mathbb{H}^2(W\Omega^{\bullet}_X(1, n)) \to \bigcap_{i=0}^{n} \Phi^{-i}(p^i H^2_{crys}(X/W))$$

and:

(4.3.2) $$\tilde{V}^2 : \mathbb{H}^2(W\Omega^{\bullet}_X(2, n)) \to \bigcap_{i=0}^{n} \Phi^{-i}(p^{2i} H^2_{crys}(X/W)),$$

are isomorphisms.

To see this consider the commutative diagram with exact columns:

$$\begin{array}{ccccccc} 0 & & 0 & & & & \\ \downarrow & & \downarrow & & & & \\ H^2_{crys}(X/W) & = & H^2_{crys}(X/W) & & & & \\ \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle p^n} & & & & \\ \mathbb{H}^2(W\Omega^{\bullet}_X(1, n)) & \xrightarrow{\tilde{V}} & H^2_{crys}(X/W) & \to & H^2_{crys}(X/W)/\tilde{V}\mathbb{H}^2(W\Omega^{\bullet}_X(1, n)) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{H}^2(W_n\Omega^{\bullet \geqq 1}_X) & \longrightarrow & H^2_{crys}(X/W_n) & \longrightarrow & E^{0,2}_{\infty}(n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

(the two vertical maps are surjective since $H^2_{crys}(X/W)$ torsionfree implies $H^3_{crys}(X/W)$ torsionfree by Poincaré duality), which shows that:

$$H^2_{crys}(X/W)/\tilde{V}\, \mathbb{H}^2(W\Omega^{\bullet}_X(1, n)) = E^{0,2}_{\infty}(n).$$

So assuming (4.3.1) we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & \bigcap_{i=0}^{n} \Phi^{-i}(p^i H^2_{crys}(X/W)) & \to & H^2_{crys}(X/W) & \to & E^{0,2}_{\infty}(n) & \longrightarrow 0 \\ & \cap & & \| & & \downarrow & \\ 0 \to & \bigcap_{i=0}^{n-1} \Phi^{-i}(p^i H^2_{crys}(X/W)) & \to & H^2_{crys}(X/W) & \to & E^{0,2}_{\infty}(n-1) & \to 0 \end{array}$$

Passing to the limit we get an exact sequence [since $l(E_\infty^{0,2}(n))<\infty$]:

$$0 \to \bigcap_{i=0}^{\infty} \Phi^{-i}(p^i H^2_{\text{crys}}(X/W)) \to H^2_{\text{crys}}(X/W) \to E_\infty^{0,2} \to 0,$$

which shows that:

$$P^1 H^2_{\text{crys}}(X/W) = \bigcap_{i=0}^{\infty} \Phi^{-i}(p^i H^2_{\text{crys}}(X/W))$$

on the other hand it is clear that:

$$M^1 = \bigcap_{i=0}^{\infty} \Phi^{-i}(p^i H^2_{\text{crys}}(X/W)),$$

so:

$$M^1 = P^1 H^2_{\text{crys}}(X/W).$$

Next consider the commutative diagram with exact columns:

$$\begin{array}{ccc} \mathbb{H}^2(W\Omega_X^{\bullet}(1, n)) & \xrightarrow{\tilde{V}} & H^2_{\text{crys}}(X/W) \\ \downarrow{\scriptstyle \tilde{F}} & & \downarrow{\scriptstyle p^n} \\ \mathbb{H}^2(W\Omega_X^{\bullet}(2, n)) & \xrightarrow{\tilde{V}^2} & H^2_{\text{crys}}(X/W) \\ \downarrow{\scriptstyle \alpha(2, n)} & & \downarrow \\ H^0(X, W_n\Omega_X^2) = \mathbb{H}^2(W_n\Omega_X^{\bullet \geq 2}) & \longrightarrow & H^2_{\text{crys}}(X/W_n) \end{array}$$

This shows that under the projection:

$$H^2_{\text{crys}}(X/W) \xrightarrow{\pi} H^2_{\text{crys}}(X/W_n),$$

$$\bigcap_{i=0}^{n} \Phi^{-i}(p^{2i} H^2_{\text{crys}}(X/W)) = \tilde{V}^2 \mathbb{H}^2(W\Omega_X^{\bullet}(2, n))$$

maps into $H^0(X, W_n\Omega_X^2)$ hence passing to the limit we get:

$$M^2 = \bigcap_{i=0}^{\infty} \Phi^{-i}(p^{2i} H^2_{\text{crys}}(X/W)) \subset H^0(X, W\Omega_X^2) = P^2 H^2_{\text{crys}}(X/W).$$

Since Φ is divisible by p^2 on $H^0(X, W\Omega_X^2)$ the other inclusion is trivial.

For the proof of (4.3.1) and (4.3.2) remark first that:

$$\tilde{F}: \quad W\Omega_X^{\bullet}(2, n) \to W\Omega_X^{\bullet},$$

is an isomorphism, indeed this follows from the fact that $F: W\Omega_X^2 \to W\Omega_X^2$ is an automorphism ([5], I 3.7 *b*). This shows in particular that $\mathbb{H}^2(W\Omega_X^{\bullet}(2, n))$ and $\mathbb{H}^3(W\Omega_X^{\bullet}(2, n))$ are torsion free for all $n \geq 0$, hence:

$$\tilde{V}: \quad \mathbb{H}^i(W\Omega_X^{\bullet}(2, n)) \to \mathbb{H}^i(W\Omega_X^{\bullet}(1, n)),$$

is injective for all $n \geq 0$, $i=2, 3$.

It follows now from 1.5 that we have an exact sequence:

$$0 \to \mathbb{H}^2(W\Omega_X^\bullet(2, n)) \xrightarrow{\tilde{V}} \mathbb{H}^2(W\Omega_X^\bullet(1, n)) \xrightarrow{\beta(1,n)} \mathbb{H}^2(t_{\leq 1} W_n \Omega_X^\bullet) \to 0,$$

which fits into a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to \Phi^{-n}(p^{2n} H^2_{crys}(X/W)) & \to & \Phi^{-n}(p^n H^2_{crys}(X/W)) & \longrightarrow & H^2_{crys}(X/W_n) & & \\
\uparrow \tilde{V}^2 & & \uparrow \tilde{V} \quad \searrow^{\Phi^n/p^n} & H^2_{crys}(X/W) & \nearrow^{\pi} \quad \uparrow & & \\
0 \to \mathbb{H}^2(W\Omega_X^\bullet(2, n)) & \xrightarrow{\tilde{V}} & \mathbb{H}^2(W\Omega_X^\bullet(1, n)) & \to & \mathbb{H}^2(t_{\leq 1} W_n \Omega_X^\bullet) & \to & 0.
\end{array}$$

Since $\mathbb{H}^2(W\Omega_X^\bullet(2, n))$ is torsion free:

$$\tilde{V}^2: \quad \mathbb{H}^2(W\Omega_X^\bullet(2, n)) \to \Phi^{-n}(p^{2n} H^2_{crys}(X/W)),$$

is injective, and:

$$\mathbb{H}^2(t_{\leq 1} W_n \Omega_X^\bullet) \to H^2_{crys}(X/W_n),$$

is injective since:

$$\mathbb{H}^1(W_n \Omega_X^\bullet / t_{\leq 1} W_n \Omega_X^\bullet) = \mathbb{H}^1(X, \mathscr{H}^2(W_n \Omega_X^\bullet)[-2]) = 0.$$

This shows that:

$$\tilde{V}: \quad \mathbb{H}^2(W\Omega_X^\bullet(1, n)) \to \Phi^{-n}(p^n H^2_{crys}(X/W)) \subset H^2_{crys}(X/W),$$

is injective so $\mathbb{H}^2(W\Omega_X^\bullet(1, n))$ is torsion free for all $n \geqq 0$.

Remark that we have an exact sequence of complexes:

$$0 \to W\Omega_X^\bullet(1, n) \xrightarrow{v_1} W\Omega_X^\bullet(1, n-1) \to \mathcal{O}_X \to 0,$$

and since $\mathbb{H}^2(W\Omega_X^\bullet(1, n))$ is torsion free we get a commutative diagram with exact columns:

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
0 \to \mathbb{H}^2(W\Omega_X^\bullet(1, n)) & \xrightarrow{\tilde{V}} & \bigcap_{i=0}^{n} \Phi^{-i}(p^i H^2_{crys}(X/W)) & \to & \mathrm{Coker}_n & \longrightarrow & 0 \\
\downarrow v_1 & & \downarrow & & \downarrow & & \\
0 \to \mathbb{H}^2(W\Omega_X^\bullet(1, n-1)) & \xrightarrow{\tilde{V}} & \bigcap_{i=0}^{n-1} \Phi^{-i}(p^i H^2_{crys}(X/W)) & \to & \mathrm{Coker}_{n-1} & - & 0 \\
\downarrow \quad \searrow^{\Phi\tilde{F}} & & \swarrow_{\Phi^n/p^{n-1}} \quad \downarrow & & & & \\
 & H^2_{crys}(X/W) & & & & & \\
\downarrow & & \searrow^{\pi} \quad \downarrow & & & & \\
H^2(X, \mathcal{O}_X) & \xrightarrow{F} & H^2_{DR}(X/k) & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

Since the Hodge to de Rham spectral sequence degenerates at E_1 it follows that the conjugate spectral sequence:

$$E_2^{p,q}=H^p(X, \mathscr{H}^q(\Omega^{\cdot}_{X/k})) \Rightarrow H^{p+q}_{DR}(X/k),$$

degenerates at E_2, hence:

$$(H^2(X, \mathcal{O}_X) \xrightarrow{F} H^2_{DR}(X/k)) = (H^2(X, \mathcal{O}_X) \overset{F}{\underset{\sim}{\rightarrow}} H^2(X, \mathscr{H}^0(\Omega^{\cdot}_X)) \rightarrow H^2_{DR}(X/k)),$$

is injective and so the diagram shows that $\mathrm{Coker}_n \rightarrow \mathrm{Coker}_{n-1}$ is injective for all $n \geqq 1$, hence to show $\mathrm{Coker}_n = 0$ for all $n \geqq 0$ it suffices to show that $\mathrm{Coker}_0 = 0$.

But $W\Omega^{\cdot}_X(1, 0) = W\Omega^{\cdot}_X$ and $\tilde{V}$ is the identity so it is clear that $\mathrm{Coker}_0 = 0$. This shows that:

$$\tilde{V}: \quad \mathbb{H}^2(W\Omega^{\cdot}_X(1, n)) \rightarrow \bigcap_{i=0}^{n} \Phi^{-i}(p^i H^2_{crys}(X/W)),$$

is surjective for all $n \geqq 0$ and the injectivity has already been proven so we get (4.3.1).

Next consider the commutative diagram with exact columns:

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 \rightarrow & \mathbb{H}^2(W\Omega^{\cdot}_X(2, n)) & \xrightarrow{\tilde{V}^2} & \bigcap_{i=0}^{n} \Phi^{-i}(p^{2i} H^2_{crys}(X/W)) & \subset & \Phi^{-n}(p^{2n} H^2_{crys}(X/W)) & \\
& \downarrow & & \downarrow & & \cap & \\
0 \rightarrow & \mathbb{H}^2(W\Omega^{\cdot}_X(1, n)) & \xrightarrow[\sim]{\tilde{V}} & \bigcap_{i=0}^{n} \Phi^{-i}(p^{i} H^2_{crys}(X/W)) & \subset & \Phi^{-n}(p^{n} H^2_{crys}(X/W)) & \\
& \downarrow & & \downarrow & \swarrow{\scriptstyle \Phi^n/p^n} & \downarrow & \\
& & & & H^2_{crys}(X/W) & & \\
& & & & \searrow & & \\
& \mathbb{H}^2(t_{\leqq 1} W_n\Omega^{\cdot}_X) & \longrightarrow & \mathrm{Coker} & \longrightarrow & H^2_{crys}(X/W_n). &
\end{array}$$

Since the map:

$$\mathbb{H}^2(t_{\leqq 1} W_n\Omega^{\cdot}_X) \rightarrow H^2_{crys}(X/W_n),$$

is injective, the map:

$$\mathbb{H}^2(t_{\leqq 1} W_n\Omega^{\cdot}_X) \rightarrow \mathrm{Coker},$$

is injective and it follows that:

$$\tilde{V}^2: \quad \mathbb{H}^2(W\Omega^{\cdot}_X(2, n)) \rightarrow \bigcap_{i=0}^{n} \Phi^{-i}(p^{2i} H^2_{crys}(X/W)),$$

is an isomorphism which proves (4.3.2).

The following Corollary was suggested by Raynaud.

COROLLARY 4.4. — *Under the assumptions of* 4.3 *the formal Brauer group*, $\widehat{\mathrm{Br}}_X$, *is uniquely determined by the* F-*crystal* $(H^2_{crys}(X/W), \Phi)$.

Proof. — The assumption that $H^2_{crys}(X/W)$ is torsion free implies that $\widehat{\mathrm{Br}}_X$ is a smooth formal group ([5], II 5.16, 5.21) hence is uniquely determined by its covariant Dieudonné module $H^2(X, W\mathcal{O}_X)$. So it suffices to show that we can construct $H^2(X, W\mathcal{O}_X)$ with its frobenius F and its Verschiebung V, from the F-crystal $(H^2_{crys}(X/W), \Phi)$.

Define the complex $W\Omega^\bullet_X(1)$ by:

$$W\Omega^\bullet_X(1) = 0 \to W\mathcal{O}_X \xrightarrow{Fd} W\Omega^1_X \xrightarrow{d} W\Omega^2_X \to 0$$

and define maps:

$$\underline{V}, \underline{F}: \quad W\Omega^\bullet_X \to W\Omega^\bullet_X(1)$$

and:

$$\mathscr{F}: \quad W\Omega^\bullet_X(1) \to W\Omega^\bullet_X,$$

by the commutative diagrams:

$$\underline{V}\qquad \begin{array}{ccccccccc} 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \\ & \downarrow{\scriptstyle V} & & \| & & \| & \\ 0 \to & W\mathcal{O}_X & \xrightarrow{Fd} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \end{array}$$

$$\underline{F}\qquad \begin{array}{ccccccccc} 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \\ & \| & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & \\ 0 \to & W\mathcal{O}_X & \xrightarrow{Fd} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \end{array}$$

and:

$$\mathscr{F}\qquad \begin{array}{ccccccccc} 0 \to & W\mathcal{O}_X & \xrightarrow{Fd} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \\ & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & \\ 0 \to & W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_X & \xrightarrow{d} & W\Omega^2_X & \to 0 \end{array}$$

We show first that:

$$\mathrm{Im}(F^n d: H^2(X, W\mathcal{O}_X) \to H^2(X, W\Omega^1_X)) = \mathrm{Im}(d: H^2(X, W\mathcal{O}_X) \to H^2(X, W\Omega^1_X)),$$

for all $n \geqq 0$.

Using the degeneration properties of the slope spectral sequence of a surface [9] we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to H^1(X, W\Omega^2_X) & \to & \mathbb{H}^3(W\Omega^\bullet_X(1)) & \to & H^2(X, W\Omega^1_X)/\mathrm{Im}\, Fd & \to 0 \\ \| & & \uparrow{\scriptstyle \underline{V}} & & \uparrow & \\ 0 \to H^1(X, W\Omega^2_X) & \to & H^3_{crys}(X/W) & \longrightarrow & H^2(X, W\Omega^1_X)/\mathrm{Im}\, d & \to 0 \end{array}$$

Since $H^3_{crys}(X/W)$ is torsion free:

$$\underline{V}: \quad H^3_{crys}(X/W) \to \mathbb{H}^3(W\Omega^\bullet_X(1)),$$

is injective hence:

$$\mathrm{Im}\, F d = \mathrm{Im}\, d.$$

This immediately implies that $\operatorname{Im} F^n d = \operatorname{Im} F^{n+1} d$ for all $n \geqq 0$ since F maps $\operatorname{Im} F^n d/\operatorname{Im} F^{n-1} d$ onto $\operatorname{Im} F^{n+1} d/\operatorname{Im} F^n d$ so by induction we get:

$$\operatorname{Im} d = \operatorname{Im} F d = \operatorname{Im} F^2 d = \ldots = \operatorname{Im} F^n d = \ldots$$

The commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to \ker F^m d & \longrightarrow & H^2(X, W\mathcal{O}_X) & \xrightarrow{F^m d} & \operatorname{Im} F^m d & \longrightarrow & 0 \\
\downarrow{\scriptstyle V^n} & & \downarrow{\scriptstyle V^n} & & \| & & \\
0 \to \ker F^{m+n} d & - & H^2(X, W\mathcal{O}_X) & \xrightarrow{F^{m+n} d} & \operatorname{Im} F^{m+n} d & - & 0 \\
& \searrow{\scriptstyle \rho_{m,n}} & \downarrow & & & & \\
& & H^2(X, W_n\mathcal{O}_X) & & & &
\end{array}$$

shows that we have isomorphisms:

$$\rho_{m,n}: \quad \ker F^{m+n} d/V^n \ker F^m d \to H^2(X, W_n\mathcal{O}_X),$$

and commutative diagrams with exact rows:

$$\begin{array}{ccccc}
0 \to \ker F^{m+1} d & \xrightarrow{V^n} & \ker F^{m+n+1} d & \xrightarrow{\rho_{m,n}} & H^2(X, W_n\mathcal{O}_X) \to 0, \\
\downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}} & & \| \\
0 \to \ker F^m d & \xrightarrow{V^n} & \ker F^{m+n} d & \xrightarrow{\rho_{m+1,n}} & H^2(X, W_n\mathcal{O}_X) \to 0
\end{array}$$

so:

$$\begin{array}{ccc}
\ker F^m d & \xrightarrow{V^n} & \ker F^{m+n} d \\
\downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}} \\
\ker F^{m+1} d & \xrightarrow{V^n} & \ker F^{m+n+1} d,
\end{array}$$

is a cocartesian diagram.

Assume now that we can construct $\ker d$, $\ker F d$, incl. : $\ker d \to \ker F d$:

$$F: \quad \ker d \to \ker d,$$

$$F: \quad \ker F d \to \ker F d \qquad \text{and} \qquad V: \quad \ker d \to \ker F d,$$

such that $FV = p.\text{incl.} : \ker d \to \ker F d$. We claim that we can then construct the Cartier module $(H^2(X, W\mathcal{O}_X), F, V)$.

First of all we can construct $\ker F^n d$ for all $n \geqq 0$ inductively by the cocartesian diagrams:

$$\begin{array}{ccc}
\ker F^{n-2} d & \xrightarrow{V} & \ker F^{n-1} d \\
\downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}} \\
\ker F^{n-1} d & \xrightarrow{V} & \ker F^n d
\end{array}$$

and then construct $H^2(X, W_n\mathcal{O}_X)$ as the quotient $\ker F^n d/V^n \ker d$.

The restriction maps:

$$H^2(X, W_n\mathcal{O}_X) \to H^2(X, W_{n-1}\mathcal{O}_X),$$

are constructed as the composition:

$$H^2(X, W_n\mathcal{O}_X) \xrightarrow{\rho_{n,0}^{-1}} \ker F^n d/V^n \ker d \to \ker F^n d/V^{n-1} \ker F d \xrightarrow{\rho_{n-1,1}} H^2(X, W_{n-1}\mathcal{O}_X).$$

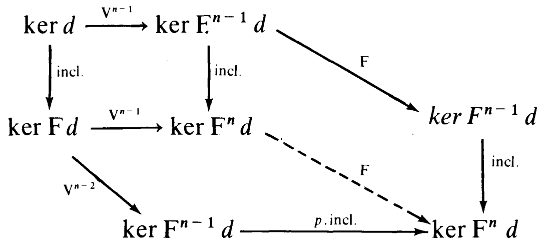
To construct the frobenius:

$$F: \quad H^2(X, W_n\mathcal{O}_X) \to H^2(X, W_n\mathcal{O}_X),$$

we first construct:

$$F: \quad \ker F^n d \to \ker F^n d,$$

inductively using the commutative diagram:



frobenius is then the composite:

$$H^2(X, W_n\mathcal{O}_X) \underset{\sim}{\xrightarrow{\rho_{n,0}^{-1}}} \ker F^n d/V^n \ker d \xrightarrow{F} \ker F^n d/V^n \ker d \underset{\sim}{\xrightarrow{\rho_{n,0}}} H^2(X, W_n\mathcal{O}_X).$$

The Verschiebung:

$$V: \quad H^2(X, W_n\mathcal{O}_X) \to H^2(X, W_{n+1}\mathcal{O}_X),$$

is the composition:

$$H^2(X, W_n\mathcal{O}_X) \underset{\sim}{\xrightarrow{\rho_{n,0}^{-1}}} \ker F^n d/V^n \ker d \xrightarrow{V} \ker F^{n+1} d/V^{n+1} \ker d \underset{\sim}{\xrightarrow{\rho_{n+1,0}}} H^2(X, W_{n+1}\mathcal{O}_X).$$

We first construct $\ker d$:

By [9] the slope spectral sequence degenerates at E_2 hence $\ker d = E_2^{0,2} = E_\infty^{0,2}$.

By 4.3. We have $P^1 H^2_{crys}(X/W) = M^1$ and M^1 can be constructed from the F-crystal structure on $H^2_{crys}(X/W)$ so the exact sequence:

$$0 \to P^1 H^2_{crys}(X/W) \to H^2_{crys}(X/W) \to E_\infty^{0,2} \to 0,$$

shows that $\ker d$ can be constructed.

Now for the construction of $\ker F d$:

Consider the frobenius:

$$\varphi: \quad H^2_{DR}(X/k) \to H^2_{DR}(X/k).$$

By Mazur's and Ogus's Theorem (3.4) we have $\ker \varphi = \mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k)$ and:

$$\varphi = (\mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})/p\,\mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \xrightarrow{\Phi} \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})/p\,\mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})),$$

so we can construct $\mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k)$ and hence also:

$$\mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k)/\mathrm{Fil}^1_{\mathrm{Hdg}} \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k),$$

can be constructed.

The map:

$$\mathrm{F}: \quad \mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \overset{\sim}{\rightarrow} \mathrm{H}^2(\mathrm{X}, \mathcal{H}^0(\Omega^{\bullet}_{\mathrm{X}/k})) \rightarrow \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k),$$

can be constructed as the factorization:

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k) & \xrightarrow{\varphi} & \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k) \\ \downarrow & & \uparrow \mathrm{F} \\ \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k)/\ker\varphi & = & \mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \end{array}$$

It follows from the definition of $\underline{\mathrm{V}}$ and the absence of torsion in $\mathrm{H}^3_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$ that we have an exact sequence:

$$0 \rightarrow \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \xrightarrow{\underline{\mathrm{V}}} \mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1)) \rightarrow \mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \rightarrow 0,$$

which fits into a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) & \xrightarrow{p} & \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) & \rightarrow & \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k) & \rightarrow & 0 \\ & & \| & & \uparrow \mathscr{F} & & \uparrow \mathrm{F} & & \\ 0 & \rightarrow & \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) & \xrightarrow{\underline{\mathrm{V}}} & \mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1)) & \rightarrow & \mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) & \rightarrow & 0 \end{array}$$

This shows that:

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) & \longrightarrow & \mathrm{H}^2_{\mathrm{DR}}(\mathrm{X}/k) \\ \uparrow \mathscr{F} & & \uparrow \mathrm{F} \\ \mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1)) & \rightarrow & \mathrm{H}^2(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \end{array}$$

is a cartesian diagram hence we can construct $\mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1))$ and the maps:

$$\mathscr{F}: \quad \mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1)) \rightarrow \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W})$$

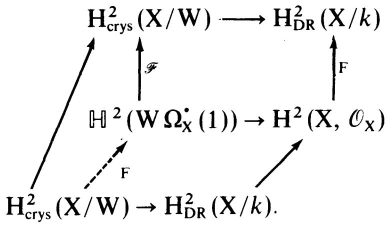
and:

$$\underline{\mathrm{V}}: \quad \mathrm{H}^2_{\mathrm{crys}}(\mathrm{X}/\mathrm{W}) \rightarrow \mathbb{H}^2(\mathrm{W}\,\Omega^{\bullet}_{\mathrm{X}}(1)).$$

The map:

$$\underline{F}:\quad H^2_{crys}(X/W)\to \mathbb{H}^2(W\Omega^{\bullet}_X(1)),$$

is constructed from the commutative diagram:



The exact sequence of complexes:

$$0\to W\Omega_X^{\bullet\geqq 1}\to W\Omega^{\bullet}_X(1)\to W\mathcal{O}_X\to 0,$$

gives a long exact sequence:

$$\mathbb{H}^1(W\Omega_X^{\bullet\geqq 1})\to\mathbb{H}^1(W\Omega^{\bullet}_X(1))\to H^1(X, W\mathcal{O}_X)$$
$$\to\mathbb{H}^2(W\Omega_X^{\bullet\geqq 1})\to\mathbb{H}^2(W\Omega^{\bullet}_X(1))\to \ker F\,d\to 0.$$

Since $H^1(X, W\mathcal{O}_X)$ is finitely generated all differentials out of $H^1(X, W\mathcal{O}_X)$ vanish and so:

$$\mathbb{H}^1(W\Omega^{\bullet}_X(1))\to H^1(X, W\mathcal{O}_X),$$

is surjective.

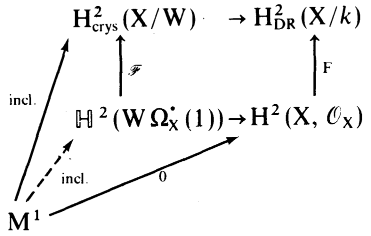
It follows that we have a short exact sequence:

$$0\to\mathbb{H}^2(W\Omega_X^{\bullet\geqq 1})\xrightarrow{\text{incl.}}\mathbb{H}^2(W\Omega^{\bullet}_X(1))\to \ker F\,d\to 0.$$

Now $\mathbb{H}^2(W\Omega_X^{\bullet\geqq 1})=P^1\,H^2_{crys}(X/W)=M^1$ so this module can be constructed. The inclusion:

$$M^1\to\mathbb{H}^2(W\Omega^{\bullet}_X(1));$$

is constructed using the commutative diagram:



hence the exact sequence:

$$0\to M^1\to\mathbb{H}^2(W\Omega^{\bullet}_X(1))\to \ker F\,d\to 0,$$

shows that $\ker F\,d$ can be constructed.

The commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \to & M^1 & \to & H^2_{crys}(X/W) & \longrightarrow & \ker d & \to & 0 \\
 & & \downarrow{\scriptstyle \Phi/p} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle incl.} & & \\
0 & \to & M^1 & \to & \mathbb{H}^2(W\Omega^{\bullet}_X(1)) & \to & \ker F d & \to & 0
\end{array}$$

and:

$$\begin{array}{ccccccccc}
0 & \to & M^1 & \to & H^2_{crys}(X/W) & \longrightarrow & \ker d & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & & \\
0 & \to & M^1 & \to & \mathbb{H}^2(W\Omega^{\bullet}_X(1)) & \to & \ker F d & \to & 0,
\end{array}$$

show that we can construct:

$$\text{incl.}: \quad \ker d \to \ker F d$$

and:

$$V: \quad \ker d \to \ker F d.$$

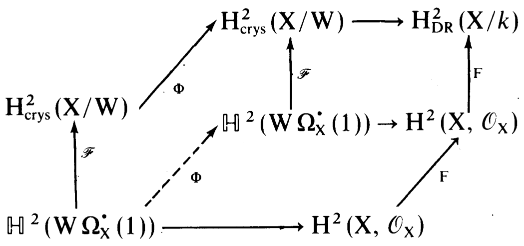
To construct:

$$F: \quad \ker F d \to \ker F d,$$

we construct:

$$\Phi: \quad \mathbb{H}^2(W\Omega^{\bullet}_X(1)) \to \mathbb{H}^2(W\Omega^{\bullet}_X(1)),$$

using the commutative diagram:



and then pass to the quotient:

$$\ker F d = \mathbb{H}^2(W\Omega^{\bullet}_X(1))/M^1 \xrightarrow{\Phi} \mathbb{H}^2(W\Omega^{\bullet}_X(1))/M^1 = \ker F d.$$

The relation $FV = p.\text{incl.}$ is proved by checking the commutativity of the diagram:

$$\begin{array}{ccc}
H^2_{crys}(X/W) & \xrightarrow{F} & \mathbb{H}^2(W\Omega^{\bullet}_X(1)) \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle p} \\
\mathbb{H}^2(W\Omega^{\bullet}_X(1)) & \xrightarrow{\Phi} & \mathbb{H}^2(W\Omega^{\bullet}_X(1))
\end{array}$$

Remark 4.6. — The above construction is clearly functorial so any map of F-crystals $\gamma : H^2_{crys}(X/W) \to H^2_{crys}(X/W)$ induces a map $\hat{\gamma} : \hat{Br}_X \to \hat{Br}_X$.

Remark 4.7. — The construction applies to an abstract F-crystal (M, F) and associates in a functorial way a Cartier module (H_M, F, V). It would be interesting to find conditions under which this Cartier module actually is the covariant Dieudonné module of a smooth formal group, and in this case study this formal group.

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(Manuscrit reçu le 27 janvier 1981,
accepté le 28 juin 1981.)

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