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INFINITESIMAL BLASCHKE CONJECTURES ON PROJECTIVE SPACES

By Chiaki TSUKAMOTO

1. Let M be a closed smooth manifold. A Riemannian metric g on M is called a C_l-metric if all the geodesics on M are closed and have a common length l. Compact rank one symmetric spaces are the examples of manifolds of C_l-metrics. The standard $C_{2\pi}$-metric on the sphere S^n is non-trivially deformable (Zoll [11], Guillemin [6]). On the other hand, M. Berger proved that there exists no C_π-metric on the real projective space $P^n(R)$ $(n \geqq 2)$ other than the standard one (Besse [3], Appendix D). The purpose of this paper is to study a deformation of the standard C_π-metric on other projective spaces.

Let $g_t (t \in (-\varepsilon, \varepsilon), g_0 = g)$ be a smooth one-parameter family of C_π-metrics on M. We set $h = \partial g_t / \partial t \,|_{t=0}$. Then for any closed geodesic γ with respect to the metric g, we have:

$$(1.1) \qquad \int_0^\pi h(\dot{\gamma}(s), \dot{\gamma}(s))\, ds = 0,$$

where we parametrized γ by its arc-length s and denoted by $\dot{\gamma}(s)$ its tangent vector at $\gamma(s)$ (Michel [9], Besse [3], 5.86). If the family g_t is trivial, i.e., there exists a smooth one-parameter family φ_t of diffeomorphisms satisfying $g_t = \varphi_t^* g$, then h is a Lie derivative of the metric g by some vector field $X (h = L_X g)$.

We give the following definition according to Besse [3].

Definition 1.1. — A symmetric covariant 2-tensor h on a manifold M with a C_π-metric g is called an infinitesimal C_π-deformation if the condition (1.1) holds for any geodesic γ. We say the infinitesimal Blaschke conjecture (I.B.C.) holds for (M, g) when every infinitesimal C_π-deformation h is trivial, i.e., there exists some vector field X satisfying $h = L_X g$.

Let $(P^n, g_0) (n \geqq 2)$ be one of the projective spaces $P^n(R)$, $P^n(C)$, $P^n(H)$ and $P^2(Ca) (n=2)$ with the standard C_π-metric. We denote by P^1 the projective line over the same field of P^n [for $P^2(Ca)$, $P^1 = S^8$]. R. Michel gave in [9] a sufficient condition for an infinitesimal C_π-deformation of (P^n, g_0) to be trivial.

Theorem 1.2. — *Let h be an infinitesimal* C_π*-deformation of* (P^n, g_0). *Suppose that for any totally geodesic imbedding* $\iota : P^1 \to P^n$ *there exists a vector field* X *on* P^1 *satisfying:*

$$(1.2) \qquad \iota^* h = L_X \iota^* g_0.$$

Then there exists a vector field $\tilde{X}$ *on* P^n *satisfying* $h = L_{\tilde{X}} g_0$.

Especially, in case of $P^n(R)$ ($n \geqq 2$), the condition (1.1) implys the existence of a vector field X satisfying (1.2). Thus Michel proved:

THEOREM 1.3. — *The I.B.C. holds for* $(P^n(R), g_0)$ $(n \geqq 2)$.

See Besse [3] for another proof of Theorem 1.3. We notice that K. Kiyohara gave in the recent work [8] another sufficient condition. He replaced (1.2) by a conformality condition.

Now we state our main Theorem.

THEOREM 1.4. — *The I.B.C. holds for any* (P^n, g_0) $(n \geqq 2)$.

N. Tanaka comments in [8] that the I.B.C. for (P^n, g_0) implys the analytic non-deformability of the C_π-metric g_0. *See* also Michel [9]. Therefore we have:

THEOREM 1.5. — *Let* $g_t\,[t \in (-\varepsilon, \varepsilon)]$ *be a one-parameter family of* C_π*-metric on* P^n $(n \geqq 2)$ *around the standard* C_π*-metric which is analytic with respect to t. Then there exists a one-parameter family* φ_t *of diffeomorphisms of* P^n *satisfying* $g_t = \varphi_t^* g_0$.

It seems that $P^n(C)$, $P^n(H)$ $(n \geqq 2)$ and $P^2(Ca)$ admit few C_π-metrics. But the rigidity or the smooth non-deformability of the standard C_π-metric is still in question.

We can reduce Theorem 1.4 to the case $P^n = P^2(C)$, using Theorem 1.2. Our program is as follows: section 2 is devoted to the general theory on compact rank one symmetric spaces. In section 3, we prove that the I.B.C. holds for $(P^2(C), g_0)$ and we give the proof of Theorem 1.4 in the last section.

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2. We always assume the smoothness of class C^∞. The spaces of functions, vector fields and symmetric covariant 2-tensors on a manifold M are denoted by F(M), X(M) and $S^2(M)$, respectively.

Let a Riemannian manifold (M, g) be a C_l-manifold. Then the geodesic flow on the unit tangent bundle UM is a free S^1-action. Therefore Geod M, the set of oriented closed geodesics on M, naturally has a manifold structure.

For a C_π-manifold (M, g) we define linear mappings:

$$L:\ X(M) \to S^2(M) \qquad \text{and} \qquad A:\ S^2(M) \to F(\text{Geod}\,M)$$

by:

$$L(X) = L_X g \qquad [X \in X(M)],$$

$$A(h)(\gamma) = (1/\pi)\int_0^\pi h(\dot\gamma(s), \dot\gamma(s))\,ds \qquad [h \in S^2(M),\ \gamma \in \text{Geod}\,M].$$

In general Im L is included in Ker A, and the I.B.C. holds for (M, g) if Im L = Ker A.

Further we define linear mappings:

$$i:\ S^2(M) \to F(UM) \qquad \text{and} \qquad P:\ F(UM) \to F(\text{Geod}\,M)$$

by:

$$i(h)(x)=h(x, x) \qquad [h \in S^2(M),\ x \in UM],$$

$$P(f)(\gamma)=(1/\pi)\int_0^\pi f(\dot{\gamma}(s))\,ds \qquad [f \in F(UM),\ \gamma \in \operatorname{Geod} M].$$

Then the mapping i is injective and we have $A=P \circ i$. The I.B.C. for (M, g) holds if and only if $\operatorname{Im}(i \circ L)=\operatorname{Im} i \cap \operatorname{Ker} P$. Notice that this relation is unchanged as we complexify all the spaces and mappings. In the following we always assume that linear spaces and modules are over the complex number field C and that mappings are C-linear. For example $X(M)$ denotes the space of complex valued vector fields. Ker L is the complexification of the space of Killing vector fields with respect to the metric g.

Let (M, g) be a compact rank one symetric space. We can choose a compact connected Lie group G acting transitively on M as isometries and also transitively on UM. We denote the isotropy group at a point $o \in M$ by K and the isotropy group at a point $v_0 \in UM_o$ by H. The group G also acts transitively on Geod M. We denote the isotropy group at a geodesic $\gamma_0 \in \operatorname{Geod} M$ that is tangent to v_0 by L. We get $M \cong G/K$, $UM \cong G/H$, $\operatorname{Geod} M \cong G/L$, $L \cap K=H$ and $L/H \cong \gamma_0=S^1$.

LEMMA 2.1. — *Let $l(t)$ be a one-parameter subgroup of* L *such that the curve $l(t).o$ has a tangent vector v_0 at $t=0$. Then we have $\gamma_0(t)=l(t).o$, where t is the arc-length.*

Proof. — As a curve, $l(t).o$ coincides with γ_0, and its tangent vector $l(t)_* v_0$ is a unit vector.

Q.E.D.

The spaces $X(M)$, $S^2(M)$, $F(UM)$ and $F(\operatorname{Geod} M)$ are G-modules in the usual way, and it is easy to verify that the mappings L, A, i and P are G-homomorphisms. The G-modules $X(M)$ and $S^2(M)$ have natural G-invariant inner products induced by the Riemannian metric g. We regard $F(UM)$ and $F(\operatorname{Geod} M)$ as G-submodules of $F(G)$ as follows:

$$F(UM)=\{f \in F(G); f(gh)=f(g),\ g \in G,\ h \in H\},$$
$$F(\operatorname{Geod} M)=\{f \in F(G); f(gl)=f(g),\ g \in G,\ l \in L\}.$$

We define an inner product on $F(G)$, using a normalized Haar measure dg on G, by:

$$\langle f_1, f_2 \rangle=\int_G f_1(g)\overline{f_2(g)}\,dg \qquad [f_1, f_2 \in F(G)],$$

which induces G-invariant inner products on $F(UM)$ and $F(\operatorname{Geod} M)$. The following Lemma is easy to verify in view of Lemma 2.1.

LEMMA 2.2. — *Using a normalized Haar measure dl on* L, *the* G-*homomorphism* P *is expressed as follows:*

$$P(f)(g)=\int_L f(gl)\,dl \qquad [f \in F(UM),\ g \in G].$$

PROPOSITION 2.3. — *The* G-*homomorphism* P *is an orthogonal projection of* $F(UM)$ *onto* $F(\operatorname{Geod} M)$.

Proof. — It is easy to see that $P^2 = P$ and $\operatorname{Im} P = F(\operatorname{Geod} M)$. And $P^* = P$ is easily verified from Lemma 2.2.

Q.E.D.

We denote the pre-Hilbert spaces $X(M)$, $S^2(M)$ and $F(UM)$ by H_1, H_2 and H_3. We will consider an irreducible decomposition of H_i as a G-module (for $i=1, 2, 3$).

For an irreducible G-module (ρ, V_ρ) we define a G-homomorphism

$$\iota_{\rho, i} : V_\rho \otimes \operatorname{Hom}_G(V_\rho, H_i) \to H_i \qquad (i=1, 2, 3)$$

by:

$$\iota_{\rho, i}(v \otimes \Phi) = \Phi(v) \qquad [v \in V_\rho,\ \Phi \in \operatorname{Hom}_G(V_\rho, H_i)].$$

Then $\iota_{\rho, i}$ is injective and $\operatorname{Im} \iota_{\rho, i}$, denoted by $\Gamma_{\rho, i}$, depends only on the equivalence class of (ρ, V_ρ).

We denote by V_1 the complexification of TM_o, the tangent space at o, considered as a K-module, and also by V_2 the K-module $S^2 V_1^*$.

For a K-module (ρ_K, V_K) we denote by $C^\infty(G, K; V_K)$ the G-module of V_K-valued functions f on G satisfying:

$$f(gk) = \rho_K(k^{-1}) f(g) \qquad [k \in K,\ g \in G].$$

Then the G-module H_i is isomorphic to $C^\infty(G, K; V_i)$ $(i=1, 2)$. By Frobenius' reciprocity law $\operatorname{Hom}_G(V_\rho, C^\infty(G, K; V_i))$ is canonically isomorphic to $\operatorname{Hom}_K(V_\rho, V_i)$. In the same way the G-module H_3 is isomorphic to $C^\infty(G, H; C)$, where C is considered as a trivial H-module, and $\operatorname{Hom}_G(V_\rho, C^\infty(G, H; C))$ is canonically isomorphic to $\operatorname{Hom}_H(V_\rho, C)$. We notice that $\operatorname{Hom}_K(V_\rho, V_i)$ $(i=1, 2)$ and $\operatorname{Hom}_H(V_\rho, C)$ are finite dimensional. Thus we get:

PROPOSITION 2.4. — *The* G-*module* $\Gamma_{\rho, i}$ *is finite dimensional* $(i=1, 2, 3)$, *and* $\Gamma_{\rho, 1}$ *is a direct sum of* $\dim \operatorname{Hom}_K(V_\rho, V_1)$-*copies of* V_ρ.

If two irreducible G-modules (ρ, V_ρ) and $(\rho', V_{\rho'})$ are not isomorphic, $\Gamma_{\rho, i}$ and $\Gamma_{\rho', i}$ are orthogonal. We denote by I_G the set of equivalence classes of irreducible G-modules.

PROPOSITION 2.5. — $\sum \Gamma_{\rho, i}([\rho] \in I_G)$ *is dense in* $H_i (i=1, 2, 3)$.

Proof. — Take a G-invariant elliptic differential operator $D_i : H_i \to H_i$. We denote by $E_{\lambda, i}$ the eigenspace of D_i with an eigenvalue λ. Then $\sum E_{\lambda, i}$ is dense in H_i. Since $E_{\lambda, i}$ is finite dimensional and G-invariant, $E_{\lambda, i}$ is a direct sum of irreducible G-submodules. Therefore $\sum E_{\lambda, i} \subset \sum \Gamma_{\rho, i}$.

Q.E.D.

We remark that we can take $L^* L$ as D_1. For the detail and the proof of the following Proposition we refer to Berger-Ebin [1].

PROPOSITION 2.6. — $\operatorname{Im} L$ *is closed in* H_2.

PROPOSITION 2.7. — $\sum L(\Gamma_{\rho, 1})$ $([\rho] \in I_G)$ *is dense in* $\operatorname{Im} L$.

Proof. — We set $S = \sum \Gamma_{\rho, 1}([\rho] \in I_G)$ and $K = \operatorname{Im} L$. We denote by $\tilde{H}_1$ and $\tilde{K}$ the completions of $(H_1, \langle\ ,\ \rangle)$ and $(K, \langle\ ,\ \rangle)$. We define an inner product $\langle\langle\ ,\ \rangle\rangle$ on K by

$\langle\langle x, y \rangle\rangle = \langle x, y \rangle + \langle L^* x, L^* y \rangle$ and denote by $\tilde{K}'$ the completion of $(K, \langle\langle\ ,\ \rangle\rangle)$. $\tilde{K}'$ is included in $\tilde{K}$ and L^* can be extended to a mapping from $\tilde{K}'$ to $\tilde{H}_1$.

It suffices to prove that $L(S)$ is dense in $(\tilde{K}', \langle\langle\ ,\ \rangle\rangle)$. Let $L(S)^\perp$ be the orthogonal complement of $L(S)$ in $(\tilde{K}', \langle\langle\ ,\ \rangle\rangle)$. Let $x \in L(S)^\perp$. We have for $\forall\ y \in S$:

$$0 = \langle\langle x, Ly \rangle\rangle = \langle x, Ly \rangle + \langle L^* x, L^* L y \rangle = \langle L^* x, y + L^* L y \rangle.$$

Since S is the direct sum of eigenspaces of $L^* L$, the set $\{ y + L^* L y;\ y \in S \}$ is also dense in H_1. Therefore $L^* x = 0$ and for $\forall\ z \in H_1$, we have:

$$\langle\langle Lz, x \rangle\rangle = \langle Lz, x \rangle + \langle L^* L z, L^* x \rangle = \langle z, L^* x \rangle = 0.$$

It means $x \perp K$, i.e., $x = 0$. Q.E.D.

The next Lemma and Proposition are easily seen.

LEMMA 2.8. — *The mapping i is a homeomorphism (into).*

PROPOSITION 2.9. — (*a*) $\mathrm{Im}\,(i \circ L)$ *is closed in* $\mathrm{Im}\, i$;
(*b*) $\sum (i \circ L)(\Gamma_{\rho, 1})([\rho] \in I_G)$ *is dense in* $\mathrm{Im}\,(i \circ L)$;
(*c*) $\sum i(\Gamma_{\rho, 2})([\rho] \in I_G)$ *is dense in* $\mathrm{Im}\, i$.

We notice that $L(\Gamma_{\rho, 1}) \subset \Gamma_{\rho, 2}$, $(i \circ L)(\Gamma_{\rho, 1}) \subset i(\Gamma_{\rho, 2}) \subset \Gamma_{\rho, 3}$ and $i(\Gamma_{\rho, 2}) = \mathrm{Im}\, i \cap \Gamma_{\rho, 3}$.

PROPOSITION 2.10. — $\sum i(\Gamma_{\rho, 2}) \cap \mathrm{Ker}\, P([\rho] \in I_G)$ *is dense in* $\mathrm{Im}\, i \cap \mathrm{Ker}\, P$.

Proof. — $i(\Gamma_{\rho, 2})$ is finite dimensional and hence we can define an orthogonal projection P_ρ of $\mathrm{Im}\, i$ onto $i(\Gamma_{\rho, 2})$. Since $f \in \mathrm{Im}\, i$ is approximated by a sum of $P_\rho f$, it suffices to show that if $f \in \mathrm{Im}\, i \cap \mathrm{Ker}\, P$, then $P_\rho f \in \mathrm{Ker}\, P$. But since P is continuous and $P(\Gamma_{\rho, 3}) \subset \Gamma_{\rho, 3}$ for $\forall\ [\rho] \in I_G$, $PP_\rho f$ and $P(f - P_\rho f)$ are orthogonal. Q.E.D.

PROPOSITION 2.11. — *The I.B.C. holds for a compact rank one symmetric space* (M, *g*), *if and only if for every* $[\rho] \in I_G$ *we have* $(i \circ L)(\Gamma_{\rho, 1}) = i(\Gamma_{\rho, 2}) \cap \mathrm{Ker}\, P\ (\subset \Gamma_{\rho, 3})$.

Proof. — Both $\mathrm{Im}\,(i \circ L)$ and $\mathrm{Im}\, i \cap \mathrm{Ker}\, P$ are closed in $\mathrm{Im}\, i$. If the above condition holds, then they include a dense subspace in common and hence they coincide.

A G-submodule W of $\Gamma_{\rho, 3}$ can be written as a direct sum of $\mathrm{Im}\, \Phi\,(\Phi \in \mathrm{Hom}_G(V_\rho, H_3))$. When m independent elements of $\mathrm{Hom}_G(V_\rho, H_3)$ are needed to express W, we say the G-module W has a multiplicity m. Thus we can verify the I.B.C. by computing the multiplicities of $(i \circ L)(\Gamma_{\rho, 1})$ and $i(\Gamma_{\rho, 2}) \cap \mathrm{Ker}\, P$. Since Ker L is a finite dimensional G-submodule of H_1, we have $\mathrm{Ker}\, L = \sum \mathrm{Ker}\, L \cap \Gamma_{\rho, 1}\,([\rho] \in I_G)$, and we can compute the multiplicity of $(i \circ L)(\Gamma_{\rho, 1})$ from Proposition 2.4. To compute the multiplicity of $i(\Gamma_{\rho, 2}) \cap \mathrm{Ker}\, P$ we will characterize the elements $\Phi \in \mathrm{Hom}_G(V_\rho, H_3)$ for which $\mathrm{Im}\, \Phi \subset i(\Gamma_{\rho, 2}) \cap \mathrm{Ker}\, P$.

We fix a G-invariant inner product on V_ρ. Then $\mathrm{Hom}_H(V_\rho, C)$ is isomorphic to V_ρ^H, the space of H-invariant vectors in V_ρ, by:

$$V_\rho^H \ni w \mapsto \Psi_w \in \mathrm{Hom}_H(V_\rho, C); \qquad \Psi_w(v) = \langle v, w \rangle \quad [v \in V_\rho].$$

As we have mentioned, $\mathrm{Hom}_G(V_\rho, H_3)$ is canonically isomorphic to $\mathrm{Hom}_H(V_\rho, C)$, so is to V_ρ^H. We have explicitly:

$$V_\rho^H \ni w \mapsto \Phi_w \in \mathrm{Hom}_G(V_\rho, H_3);$$
$$\Phi_w(v)(g) = \Psi_w(\rho(g^{-1})v) = \langle \rho(g^{-1})v, w \rangle = \langle v, \rho(g)w \rangle \qquad [v \in V_\rho, g \in G].$$

First we seek the condition for $\mathrm{Im}\,\Phi_w \subset i(\Gamma_{\rho, 2})(w \in V_\rho^H)$.

DEFINITION 2.12. — A function on a standard sphere $S^n = \{ x \in R^{n+1}; |x| = 1 \}$ is called of degree 2 if and only if it is expressed as the restriction of a homogeneous polynomial of degree 2. A function f on UM is called of degree 2 at $x \in M$ if and only if $f|_{UM_x}$ is of degree 2.

Obviously $f \in F(UM)$ is contained in Im i if and only if f is of degree 2 at $\forall\ x \in M$.

The "of degree 2" property has an intrinsic meaning. Let Δ be the Laplacian on a standard sphere. Then $f \in F(S^n)$ is of degree 2 if and only if f is contained in the sum of eigenspaces of Δ with the eigenvalues 0 and $2n+2$.

For the standard sphere UM_0, we have a group theoretical characterization, too. Since UM_0 is a homogeneous Riemannian manifold K/H, eigenspaces of Δ are K-modules and so is the space of functions of degree 2. As we have done for $F(UM) = F(G/H)$, a finite dimensional K-submodule of $F(K/H)$ can be written as a direct sum of Im φ ($\varphi \in \mathrm{Hom}_K(U_\sigma, F(K/H))$, $[(\sigma, U_\sigma)] \in I_K$). Thus there exist irreducible K-modules $(\sigma_i, U_i)(1 \leqq i \leqq \mu)$ and H-invariant vectors $w_{i,j}(\neq 0)$ in $U_i (1 \leqq j \leqq \nu_i)$ by which the space of functions of degree 2 can be written as $\sum \varphi_{w_{i,j}}(U_i)$, where $\varphi_{w_{i,j}}$ is an element of $\mathrm{Hom}_K(U_i, F(K/H))$ determined by $w_{i,j}$ and some fixed invariant inner product on U_i:

$$\varphi_{w_{i,j}}(u)(k) = \langle u, \sigma_i(k) w_{i,j} \rangle \qquad [u \in U_i, k \in K].$$

PROPOSITION 2.13. — *Assume that an irreducible* G*-module* V_ρ *has a* K*-irreducible orthogonal decomposition* $V_\rho = \sum_{a=1}^{t} \tilde{U}_a$ *where* $\tilde{U}_a$ *is a* K*-module isomorphic to some* U_i *for* $1 \leqq a \leqq s$ *and not isomorphic to any* U_i *for* $s+1 \leqq a \leqq t$. *When* $\tilde{U}_a$ *is isomorphic to* U_i, *we denote by* $w_{a,j} (1 \leqq j \leqq \nu_i)$ *the* H*-invariant vector in* $\tilde{U}_a$ *identified with* $w_{i,j}$ *in* U_i. *Then* $\mathrm{Im}\,\Phi_w$ $(w \in V_\rho^H)$ *is included in* Im i, *if and only if* w *is a linear combination of* $w_{a,j}$.

Proof. — If $\mathrm{Im}\,\Phi_w \subset \mathrm{Im}\,i$, then $\Phi_w(v)$ is of degree 2 at $o = [K]$ for $\forall\ v \in V_\rho$. On the other hand, since $g \in G$ induces an isometry between UM_o and $UM_{g.o}$, $\Phi_w(v)$ is of degree 2 at $g.o$ if and only if $g^{-1}.\Phi_w(v) = \Phi_w(\rho(g^{-1})v)$ is of degree 2 at o. Thus $\mathrm{Im}\,\Phi_w \subset \mathrm{Im}\,i$ if and only if $\Phi_w(v)$ is of degree 2 at o for $\forall v \in V$.

Let w_a be the orthogonal projection of w on $\tilde{U}_a$. For $u \in \tilde{U}_a$ and $k \in K$ we have:

$$\Phi_w(u)(k) = \langle \rho(k^{-1})u, w \rangle = \langle \rho(k^{-1})u, w_a \rangle = \varphi_{w_a}(u)(k).$$

Therefore the restriction of $\Phi_w(u)$ on UM_o is equal to $\varphi_{w_a}(u)$. When w_a and u do not vanish, $\varphi_{w_a}(u)$ is of degree 2 if and only if U_a is isomorphic to some U_i and w_a is a linear combination of $w_{a,j}$.

Q.E.D.

We denote by V_ρ^I the subspace of V_ρ^H spanned by $w_{a,j}$.
Next we compute how P acts to $\Phi_w(v)$ $(w \in V_\rho^H, v \in V_\rho)$:

$$P\Phi_w(v)(g) = \int_L \Phi_w(v)(gl)\,dl = \int_L \langle v, \rho(gl)w\rangle\,dl = \langle v, \rho(g).\int_L \rho(l)w\,dl\rangle.$$

We set $pw = \int_L \rho(l)w\,dl$. Let V_ρ^L be the space of L-invariant vectors in V_ρ. The next Lemma is easily verified.

LEMMA 2.14. – *The mapping p is an orthogonal projection of* V_ρ^H *onto* $V_\rho^L (\subset V_\rho^H)$.

Since $\rho(g^{-1})v$ $(g \in G)$ span V_ρ for $v \neq 0$, $P\Phi_w(v)$ vanishes if and only if $w \in \operatorname{Ker} p$. Thus $\operatorname{Im}\Phi_w$ is included in Ker P if and only if w is contained in Ker p.

PROPOSITION 2.15. – *The multiplicity of* $i(\Gamma_{\rho,2}) \cap \operatorname{Ker} P$ *is equal to* $\dim(V_\rho^I \cap \operatorname{Ker} p)$.

3. In this section we prove that the I.B.C. holds for $(P^2(C), g_0)$ by means given in section 2. Let $G = U(n+1)$ be the group of $(n+1)\times(n+1)$ unitary matrices, which acts on C^{n+1} as linear mappings and on $(P^n(C), g_0)$ transitively as isometries. The isotropy group K at $o = [1:0:\ldots:0]$ is $U(1)\times U(n)$. We set $H = \Delta(U(1)\times U(1))\times U(n-1)$ and $L = T \times U(n-1)$, where $\Delta(U(1)\times U(1))$ is the diagonal subgroup of $U(1)\times U(1)$ and $T \subset U(2)$ is a total group given by:

$$T = \left\{ \begin{bmatrix} s\cos t & s\sin t \\ s\sin t & s\cos t \end{bmatrix};\ s \in U(1),\ t \in R \right\}.$$

Then we have $U(P^n(C)) \cong G/H$ and $\operatorname{Geod}(P^n(C)) \cong G/L$. We take a maximal abelian subalgebra A of $u(n+1)$, the Lie algebra of $U(n+1)$, as follows:

$$A = \{\operatorname{diag}(\mu_0, \mu_1, \ldots, \mu_n);\ \mu_i \in \sqrt{-1}\,R\}.$$

We define $\lambda_i \in A^*$ $(i = 0, 1, \ldots, n)$ by $\lambda_i(\operatorname{diag}(\mu_0, \mu_1, \ldots, \mu_n)) = \mu_i$ and take $\lambda_0 - \lambda_1$, $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n$ as the simple roots of $U(n+1)$ [1]. The highest weight of an irreducible $U(n+1)$-module is written as $\sum_{i=0}^{n} f_i \lambda_i$, where f_i are integers satisfying $f_0 \geqq f_1 \geqq \ldots \geqq f_n$. Thus we can identify $I_{U(n+1)}$ with $\{(f_i) \in Z^{n+1}; f_0 \geqq f_1 \geqq \ldots \geqq f_n\}$. The Lie algebra of $U(1)\times U(n)$ also includes A as its maximal abelian subalgebra. We take $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n$ as the simple roots of $U(1)\times U(n)$. The highest weight of an irreducible $U(1)\times U(n)$-module is written as $\sum_{i=0}^{n} g_i \lambda_i$, where g_i are integers satisfying $g_1 \geqq \ldots \geqq g_n$. We can identify $I_{U(1)\times U(n)}$ with $\{(g_i) \in Z^{n+1};\ g_1 \geqq \ldots \geqq g_n\}$. We cite the following branching law from Boerner [4].

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[1] We do not include the center part in simple roots.

PROPOSITION 3.1. — *An irreducible* $U(n+1)$*-module with the highest weight* $\sum_{i=0}^{n} f_i \lambda_i$ *includes an irreducible* $U(1) \times U(n)$*-submodule with the highest weight* $\sum_{i=0}^{n} g_i \lambda_i$ *if and only if* $\sum_{i=0}^{n} f_i = \sum_{i=0}^{n} g_i$ *and* $f_{i-1} \geqq g_i \geqq f_i (1 \leqq i \leqq n)$. *The irreducible* $U(1) \times U(n)$*-submodule with the highest weight* $\sum_{i=0}^{n} g_i \lambda_i$ *is unique, if it exists.*

Using this Proposition, we can compute dim $\mathrm{Hom}_K(V_\rho, V_1)$ in case of $P^n(C)$. The K-module V_1 is a sum of two irreducible $U(1) \times U(n)$-module with the highest weight $\lambda_0 - \lambda_n$ and $-\lambda_0 + \lambda_1$. By Schur's Lemma dim $\mathrm{Hom}_K(V_\rho, V_1)$ is equal to how many times either of these irreducible K-module appears in the K-irreducible decomposition of a G-module V_ρ. We denote the highest weight of an irreducible G-module V_ρ by H. W. (V_ρ).

PROPOSITION 3.2. — $\mathrm{Hom}_K(V_\rho, V_1)$ *has the following dimension.*

H.W. (V_ρ)	dim $\mathrm{Hom}_K(V_\rho, V_1)$
$h\lambda_0 - h\lambda_n$, $h \geqq 1$	2
$h\lambda_0 + \lambda_1 - (h+1)\lambda_n$, $h \geqq 1$	1
$(h+1)\lambda_0 - \lambda_{n-1} - h\lambda_n$, $h \geqq 1$	1
Otherwise	0

The space of Killing vectors on $(P^n(C), g_0)$ is isomorphic to $su(n+1)$, the semisimple part of $u(n+1)$, and Ker L is isomorphic to its complexification $sl(n+1, C)$, which is an irreducible $U(n+1)$-module with the highest weight $\lambda_0 - \lambda_n$.

PROPOSITION 3.3. — $(i \circ L)(\Gamma_{\rho, 1})$ *has the following multiplicity.*

H.W. (V_ρ)	Multiplicity
$h\lambda_0 - h\lambda_n$, $h \geqq 2$	2
$h\lambda_0 - h\lambda_n$, $h = 1$	1
$h\lambda_0 + \lambda_1 - (h+1)\lambda_n$, $h \geqq 1$	1
$(h+1)\lambda_0 - \lambda_{n-1} - h\lambda_n$, $h \geqq 1$	1
Otherwise	0

Next we investigate the subspace V_ρ^I and the operator p. We identify the hermitian vector space TM_o with $C^n = \{z = (z_1, \ldots, z_n);\ z_i \in C\}$, where we assume that the hermitian inner product on C^n is given by $\langle a, b \rangle = \sum_{i=1}^{n} a_i \overline{b}_i$ $(a, b \in C^n)$. Then $UM_o \cong K/L = U(1) \times U(n)/\Delta(U(1) \times U(1)) \times U(n-1)$ is identified with $S^{2n-1} = \{z \in C^n;\ \sum_{i=1}^{n} |z_i|^2 = 1\}$. We notice that K acts unitarily on C^n and therefore isometically on S^{2n-1} by:

$$(e^{\sqrt{-1}\theta}, U_0).z = e^{-\sqrt{-1}\theta} U_0 z \qquad [(e^{\sqrt{-1}\theta}, U_0) \in U(1) \times U(n),\ z \in C^n].$$

The space of homogeneous polynomials of degree 2 on C^n is a K-module, which consists of four irreducible K-modules U_i $(i=1, 2, 3, 4)$:

$$U_1=\left\{a\sum_{i=1}^{n}|z_i|^2;\ a\in C\right\},$$

$$U_2=\left\{\sum_{i,\,j=1}^{n} a_{ij}z_i\bar{z}_j;\ a_{ij}\in C,\ \sum_{i=1}^{n} a_{ii}=0\right\},$$

$$U_3=\left\{\sum_{i,\,j=1}^{n} a_{ij}z_i z_j;\ a_{ij}\in C,\ a_{ij}=a_{ji}\right\},$$

$$U_4=\left\{\sum_{i,\,j=1}^{n} a_{ij}\bar{z}_i\bar{z}_j;\ a_{ij}\in C,\ a_{ij}=a_{ji}\right\}.$$

Their highest weights are 0, $\lambda_1-\lambda_n$, $2\lambda_0-2\lambda_n$ and $-2\lambda_0+2\lambda_1$, respectively. We notice that $U_0\cong U_0^*$, $U_1\cong U_1^*$, $U_3\cong U_4^*$ and $U_4\cong U_3^*$ as K-modules.

Since each U_i has a unique $\Delta(U(1)\times U(1))\times U(n-1)$-invariant vector up to a constant factor (, which can be verified using Proposition 3.1 again), the submodule of $C^\infty(S^{2n-1})$ isomorphic to some U_i consists only of functions of degree 2. Using Proposition 3.1, we can determine the irreducible $U(n+1)$-module V_ρ for which $V_\rho^I\neq\{0\}$.

PROPOSITION 3.4. — *An irreducible* $U(n+1)$-*module* $V_\rho (n\geqq 2)$ *which includes a* $U(1)\times U(n)$-*submodule isomorphic to some* U_i *has the following highest weight.*

H.W. (V_ρ)		U_i included	dim V_ρ^I
	$h=0$	U_1	1
$h\lambda_0-h\lambda_n$,	$h=1$	U_1, U_2	2
	$h\geqq 2$	U_1, U_2, U_3, U_4	4
$(h+1)\lambda_0-\lambda_{n-1}-h\lambda_n$,	$h=1$	U_2	1
	$h\geqq 2$	U_2, U_3	2
$h\lambda_0+\lambda_1-(h+1)\lambda_n$,	$h=1$	U_2	1
	$h\geqq 2$	U_2, U_4	2
$(h+2)\lambda_0-2\lambda_{n-1}-h\lambda_n$,	$h\geqq 2$	U_3	1
$h\lambda_0+2\lambda_1-(h+2)\lambda_n$,	$h\geqq 2$	U_4	1

And if $n\geqq 3$, *we have further the following.*

$h\lambda_0+\lambda_1-\lambda_{n-1}-h\lambda_n$,	$h\geqq 1$	U_2	1

The I.B.C. holds for $(P^n(C), g_0)$ if and only if $\dim(V_\rho^I\cap \operatorname{Ker} p)$ is equal to $\operatorname{mult}.((i\circ L)(\Gamma_{\rho,1}))$, the multiplicity of $(i\circ L)(\Gamma_{\rho,1})$, for every $[\rho]\in I_{U(n+1)}$. But, since $\dim(V_\rho^I\cap \operatorname{Ker} p)\geqq \operatorname{mult}.((i\circ L)(\Gamma_{\rho,1}))$, it is enough to show that $\dim(p(V_\rho^I))\geqq \dim V_\rho^I-\operatorname{mult}.((i\circ L)(\Gamma_{\rho,1}))$. We will check this for $n=2$. We will freely use the representation theory, especially, the theorems on the structure of irreducible modules. Consult, for example, Humphreys [7].

The linear mapping p is an orthogonal projection of V_ρ^H onto V_ρ^L. We first study the T-invariant vectors in an irreducible $U(2)$-module (V, ρ). We choose the following elements in $gl(2, \mathbf{C})$, the complexification of $u(2)$:

$$X=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then we have $[X, Y]=H$, $[\operatorname{diag}(\lambda_0, \lambda_1), X]=(\lambda_0-\lambda_1)X$ and $[\operatorname{diag}(\lambda_0, \lambda_1), Y]=-(\lambda_0-\lambda_1)Y$. We denote the action of the Lie algebras $u(2)$ and $gl(2, \mathbf{C})$ on V by ρ, too.

A maximal vector v_0 in (V, ρ) is a non-zero vector satisfying $\rho(X)v_0=0$. When the highest weight of V is $h_0\lambda_0+h_1\lambda_1$, the vectors $v_i=\rho(Y)^i v_0$ $(0\leqq i\leqq h_0-h_1)$ form a basis of V and $\rho(Y)^{h_0-h_1+1}v_0$ vanishes. Each v_i is a vector of weight $(h_0-i)\lambda_0+(h_1+i)\lambda_1$. Since $\rho(\operatorname{diag}(\lambda, \lambda))v_i=(h_0+h_1)\lambda v_i$ for each i, we have $\rho(\operatorname{diag}(\lambda, \lambda))v=(h_0+h_1)\lambda v$ for $\forall\ v\in V$.

PROPOSITION 3.5. — *An irreducible* U(2)-*module* (V, ρ) *contains a non-zero* T-*invariant vector if and only if the highest weight of* V *is of the form* $h(\lambda_0-\lambda_1)(h\geqq 0)$. *The* T-*invariant vector is unique up to a constant factor and is a linear combination of vectors of weight* $(h-2k)(\lambda_0-\lambda_1)$ $(k=0, 1, \ldots, h)$ *with non-zero coefficients.*

Proof. — A vector $v\in V$ is T-invariant if and only if $\rho(\operatorname{diag}(\lambda, \lambda))v$ and $\rho(X-Y)v$ vanish. For a non-zero vector v, $\rho(\mathit{diag}(\lambda, \lambda))v$ vanishes if and only if the highest weight of V is of the form $h(\lambda_0-\lambda_1)(h\geqq 0)$. Now assume that V has the highest weight $h(\lambda_0-\lambda_1)$. We set $v=\sum_{i=0}^{2h} a_i v_i$, where $\{v_i;\ 0\leqq i\leqq 2h\}$ is a basis of V given ahead. From the formula:

$$\begin{aligned}\rho(X)v_i=\rho(X)\rho(Y)v_{i-1}&=\rho(Y)\rho(X)v_{i-1}+\rho([X, Y])v_{i-1}\\ &=\rho(Y)\rho(X)v_{i-1}+2(h-i+1)v_{i-1},\end{aligned}$$

one can easily deduce:

$$\begin{aligned}&\rho(X)v_i=i(2h-i+1)v_{i-1}:\\ &\rho(X-Y)v=\sum_{i=0}^{2h} a_i\{i(2h-i+1)v_{i-1}-v_{i+1}\}\\ &\qquad=2ha_1v_0+\sum_{i=1}^{2h-1}\{(i+1)(2h-i)a_{i+1}-a_{i-1}\}v_i-a_{2h-1}v_{2h}.\end{aligned}$$

Thus $\rho(X-Y)v$ vanishes if and only if $a_1=a_{2h-1}=0$ and $a_{i-1}=(i+1)(2h-i)a_{i+1}$ $(1\leqq i\leqq 2h-1)$, where the coefficients of a_{i+1} do not vanish.

Q.E.D.

We shall describe the structure of an irreducible U(3)-module (V_ρ, ρ). We choose in $gl(3, \mathbf{C})$ elements X_i, Y_i and $H_i (i=1, 2, 3)$, as follows:

$$X_1=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y_i={}^tX_i \quad (i=1, 2, 3) \quad \text{(the transpose)},$$

$$H_i=[X_i, Y_i] \qquad (i=1, 2, 3).$$

We denote the action of the Lie algebra $gl(3, \mathrm{C})$ and the universal enveloping algebra $\mathcal{U}(gl(3, \mathrm{C}))$ on V_ρ by ρ, too. We conventionally set $\mathrm{U}^0=1$ and $\mathrm{U}^t=0$ $(t<0)$ for $\mathrm{U}\in gl(3, \mathrm{C})$.

We denote the highest weight of V_ρ by Λ and fix a maximal vector v_Λ, which is a non-zero vector satisfying $\rho(\mathrm{X}_i)v_\Lambda=0$ $(i=1, 2, 3)$. When $\Lambda(\mathrm{H}_1)=r$ and $\Lambda(\mathrm{H}_2)=s$, $\rho(\mathrm{Y}_1^{r+1})v_\Lambda$ and $\rho(\mathrm{Y}_2^{s+1})v_\Lambda$ vanish. We set $v_{i,j,k}=\rho(\mathrm{Y}_1^i\mathrm{Y}_2^j\mathrm{Y}_3^k)v_\Lambda$, which is a vector of wieght $\Lambda-i(\lambda_0-\lambda_1)-j(\lambda_1-\lambda_2)-k(\lambda_0-\lambda_2)$ if it does not vanish. The module V_ρ is spanned by the vectors $v_{i,j,k}$ for non-negative integers i, j and k.

LEMMA 3.6. — $[\mathrm{Y}_2, \mathrm{Y}_1]=\mathrm{Y}_3$, $[\mathrm{Y}_1, \mathrm{Y}_3]=[\mathrm{Y}_2, \mathrm{Y}_3]=0$.

LEMMA 3.7. — $\mathrm{Y}_2^n\mathrm{Y}_1^m=\sum {}_n\mathrm{C}_i\, m!/(m-n+i)!\,\mathrm{Y}_1^{m-n+i}\mathrm{Y}_2^i\mathrm{Y}_3^{n-i}$, *where the summation is taken over the integers i for which* $\mathrm{Y}_1^{m-n+i}\mathrm{Y}_2^i\mathrm{Y}_3^{n-i}$ *does not vanish.*

Proof. — We first prove $[\mathrm{Y}_2, \mathrm{Y}_1^m]=m\mathrm{Y}_1^{m-1}\mathrm{Y}_3$ by induction:

$$[\mathrm{Y}_2, \mathrm{Y}_1^{m+1}]=[\mathrm{Y}_2, \mathrm{Y}_1^m]\mathrm{Y}_1+\mathrm{Y}_1^m[\mathrm{Y}_2, \mathrm{Y}_1]=m\mathrm{Y}_1^{m-1}\mathrm{Y}_3\mathrm{Y}_1+\mathrm{Y}_1^m\mathrm{Y}_3=(m+1)\mathrm{Y}_1^n\mathrm{Y}_3.$$

Therefore we have $\mathrm{Y}_2\mathrm{Y}_1^m=m\mathrm{Y}_1^{m-1}\mathrm{Y}_3+\mathrm{Y}_1^m\mathrm{Y}_2$. We prove the Lemma by induction on n:

$$\begin{aligned}
\mathrm{Y}_2^{n+1}\mathrm{Y}_1^m&=\mathrm{Y}_2.\sum {}_n\mathrm{C}_i.\frac{m!}{(m-n+i)!}\mathrm{Y}_1^{m-n+i}\mathrm{Y}_2^i\mathrm{Y}_3^{n-i}\\
&=\sum {}_n\mathrm{C}_i.\frac{m!}{(m-n+i)!}\{(m-n+i)\mathrm{Y}_1^{m-n+i-1}\mathrm{Y}_3+\mathrm{Y}_1^{m-n+i}\mathrm{Y}_2\}\mathrm{Y}_2^i\mathrm{Y}_3^{m-i}\\
&=\sum_{i\leqq n} {}_n\mathrm{C}_i.\frac{m!}{(m-n+i-1)!}\mathrm{Y}_1^{m-n+i-1}\mathrm{Y}_2^i\mathrm{Y}_3^{n-i+1}\\
&\quad+\sum_{0\leqq i} {}_n\mathrm{C}_i.\frac{m!}{(m-n+i)!}\mathrm{Y}_1^{m-n+i}\mathrm{Y}_2^{i+1}\mathrm{Y}_3^{n-i}\\
&=\sum_{i\leqq n} {}_n\mathrm{C}_i.\frac{m!}{(m-(n+1)+i)!}\mathrm{Y}_1^{m-(n+1)+i}\mathrm{Y}_2^i\mathrm{Y}_3^{(n+1)-i}\\
&\quad+\sum_{1\leqq i} {}_n\mathrm{C}_{i-1}.\frac{m!}{(m-(n+1)+i)!}\mathrm{Y}_1^{m-(n+1)+i}\mathrm{Y}_2^i\mathrm{Y}_3^{(n+1)-i}\\
&=\sum {}_{n+1}\mathrm{C}_i.\frac{m!}{(m-(n+1)+i)!}\mathrm{Y}_1^{m-(n+1)+i}\mathrm{Y}_2^i\mathrm{Y}_3^{(n+1)-i}.
\end{aligned}$$

Q.E.D.

LEMMA 3.8. — *We have the following identities:*

$$\begin{aligned}
&[\mathrm{X}_1, \mathrm{Y}_1^n]=\{n\mathrm{H}_1+n(n-1)\}\mathrm{Y}_1^{n-1},\\
&[\mathrm{X}_1, \mathrm{Y}_2^n]=0,\\
&[\mathrm{X}_1, \mathrm{Y}_3^n]=-n\mathrm{Y}_2\mathrm{Y}_3^{n-1},\\
&[\mathrm{X}_2, \mathrm{Y}_1^n]=0,\\
&[\mathrm{X}_2, \mathrm{Y}_2^n]=\{n\mathrm{H}_2+n(n-1)\}\mathrm{Y}_2^{n-1},\\
&[\mathrm{X}_2, \mathrm{Y}_3^n]=n\mathrm{Y}_1\mathrm{Y}_3^{n-1}.
\end{aligned}$$

Proof. — Use the induction on n.

LEMMA 3.9. — *We have the following identities:*

$$\rho(X_1)v_{i,j,k}=i(\Lambda(H_1)+j-k-i+1)v_{i-1,j,k}-kv_{i,j+1,k-1},$$
$$\rho(X_2)v_{i,j,k}=j(\Lambda(H_2)-j+1)v_{i,j-1,k}+kv_{i+1,j,k-1}.$$

Proof. — Using Lemma 3.8 and the fact $\rho(X_1)v_\Lambda=0$, we get:

$$\begin{aligned}\rho(X_1)v_{i,j,k}&=\rho(X_1)\rho(Y_1^i Y_2^j Y_3^k)v_\Lambda\\&=\rho([X_1, Y_1^i]Y_2^j Y_3^k)v_\Lambda+\rho(Y_1^i[X_1, Y_2^j]Y_3^k)v_\Lambda\\&\quad+\rho(Y_1^i Y_2^j[X_1, Y_3^k])v_\Lambda+\rho(Y_1^i Y_2^j Y_3^k)\rho(X_1)v_\Lambda\\&=i\rho(H_1)v_{i-1,j,k}+i(i-1)v_{i-1,j,k}-kv_{i,j+1,k-1}.\end{aligned}$$

Since $v_{i-1,j,k}$ is a vector of weight $\Lambda-(i-1)(\lambda_0-\lambda_1)-j(\lambda_1-\lambda_2)-k(\lambda_0-\lambda_2)$ (if $v_{i-1,j,k}$ does not vanish), we have $\rho(H_1)v_{i-1,j,k}=\{\Lambda(H_1)-2(i-1)+j-k\}v_{i-1,j,k}$. Thus follows the first identity; the second one can be shown similarly.

A maximal vector of an irreducible $U(2)\times U(1)$-submodule [resp. $U(1)\times U(2)$-submodule] of V_ρ is a non-zero vector v which satisfies $\rho(X_1)v=0$ [resp. $\rho(X_2)v=0$].

When the highest weight of the submodule is λ, $\rho(Y_1^i)v$ [resp. $\rho(Y_2^i)v$] is a vector of weight $\lambda-i(\lambda_0-\lambda_1)$ [resp. $\lambda-i(\lambda_1-\lambda_2)$] $(i=0, 1, \ldots, \lambda(H_1)$ [resp. $\lambda(H_2)$]) and they form a basis of the irreducible $U(2)\times U(1)$-module [resp. $U(1)\times U(2)$-module].

We now study V_ρ^I and p when the highest weight of V_ρ is $h\lambda_0-h\lambda_2$, $(h+1)\lambda_0-\lambda_1-h\lambda_2$ and $(h+2)\lambda_0-2\lambda_1-h\lambda_2$, separately.

The case $\Lambda=h\lambda_0-h\lambda_2$ $(h\geqq 0)$. — We have $\Lambda(H_1)=\Lambda(H_2)=h$. Since $\rho(Y_2^{h+1})v_\Lambda$ vanishes, $v_{i,j,k}$ vanishes for $j\geqq h+1$.

LEMMA 3.10. — *For non-negative integers i, j and k satisfying $i+k\leqq h$ and $j\leqq h$, the vectors $v_{i,j,k}$ are linearly independent.*

Proof. — If $v_{i,j,k}$ does not vanish, it is a vector of weight $(h-i-k)\lambda_0+(i-j)\lambda_1+(j+k-h)\lambda_2$. The sum of weight spaces of weight $p\lambda_0+q\lambda_1+r\lambda_2$ satisfying $p\geqq 0$, which we denote by V_ρ^+, is spanned by $v_{i,j,k}$ satisfying $i+k\leqq h$ and $j\leqq h$. The highest weight of $U(1)\times U(2)$-submodules appearing in the $U(1)\times U(2)$-irreducible decomposition of V_ρ are $(t_2-t_1)\lambda_0+t_1\lambda_1-t_2\lambda_2$ $(0\leqq t_1,\ t_2\leqq h)$ (Lemma 3.1), and the dimension of each irreducible $U(1)\times U(2)$-submodule is t_1+t_2+1. Therefore the dimension of V_ρ^+ is $\sum_{0\leqq t_1\leqq t_2\leqq h}(t_1+t_2+1)=(h+1)^2(h+2)/2$, which agrees with the number of sets of non-negative integers (i, j, k) satisfying $i+k\leqq h$ and $j\leqq h$.

Q.E.D.

In order to determine V_ρ^L $(L=T\times U(1))$, it is enough to know maximal vectors of irreducible $U(2)\times U(1)$-submodules with the highest weights $t\lambda_0-t\lambda_1$ $(t\geqq 0)$, in view of Proposition 3.5. From Lemma 3.10, the vectors $v_{i,i+t,h-i-t}$ $(0\leqq i\leqq h-t)$ form a basis of the weight space of weight $t\lambda_0-t\lambda_1$ $(t\geqq 0)$.

LEMMA 3.11. — *Let w be a maximal vector in the irreducible* $\mathrm{U}(2)\times\mathrm{U}(1)$*-submodule of* V_ρ *with the highest weight* $t\lambda_0 - t\lambda_1$ $(t \geqq 0)$. *Then w is written as:*

$$w=\sum_{i=0}^{h-t} a_i v_{i,\, i+t,\, h-i-t},$$

where $a_0 \neq 0$ *and:*

$$a_i=\frac{(h-t-i+1)}{i(2t+i+1)} a_{i-1} \qquad (1 \leqq i \leqq h-t).$$

Proof. — Write down the condition that $\rho(\mathrm{X}_1).\sum_{i=0}^{h-t} a_i v_{i,\, i+t,\, h-i-t}$ vanishes, using Lemma 3.9, and we have the Lemma.

LEMMA 3.12. — *A maximal vector w in the irreducible* $\mathrm{U}(2)\times\mathrm{U}(1)$*-submodule of* V_ρ *with the highest weight* $t\lambda_0 - t\lambda_1$ $(t \geqq 0)$ *is perpendicular to the subspace spanned by* $v_{i,\, i+t,\, h-i-t}$ $(1 \leqq i \leqq h-t)$ *with respect to the invariant inner product on* V_ρ.

Proof. — A vector in the above subspace is written as $\rho(\mathrm{Y}_1)v'$, where v' is a vector of weight $(t+1)\lambda_0-(t+1)\lambda_1$. Since an irreducible $\mathrm{U}(2)\times\mathrm{U}(1)$-module with the highest weight $t\lambda_0 - t\lambda_1$ does not have the weight space of weight $(t+1)\lambda_0-(t+1)\lambda_1$, v' is perpendicular to the irreducible $\mathrm{U}(2)\times\mathrm{U}(1)$-submodule containing w, and so is $\rho(\mathrm{Y}_1)v'$.

Q.E.D.

The irreducible $\mathrm{U}(1)\times\mathrm{U}(2)$-module U_0 is included in V_ρ for every h $(\geqq 0)$. U_0 is one-dimensional and each vector in U_0 is $\mathrm{U}(1)\times\mathrm{U}(2)$-invariant and therefore $\Delta(\mathrm{U}(1)\times\mathrm{U}(1))\times\mathrm{U}(1)$-invariant. A vector u_0 is contained in U_0 if and only if u_0 is contained in the weight space of weight 0 and $\rho(\mathrm{X}_2)u_0$ vanishes.

LEMMA 3.13. — *A vector* $u_0 \in \mathrm{U}_0$ *is written as:*

$$\sum_{i=0}^{h} b_i v_{i,\, i,\, h-i} \quad \textit{where} \quad b_i=\frac{(-1)^i}{i!} b_0 \quad (0 \leqq i \leqq h).$$

The irreducible $\mathrm{U}(2)\times\mathrm{U}(1)$-submodule W_0 of V_ρ with the highest weight 0 is one-dimensional and each vector in W_0 is $\mathrm{U}(2)\times\mathrm{U}(1)$-invariant and therefore $\mathrm{T}\times\mathrm{U}(1)$-invariant.

PROPOSITION 3.14. — *Let u_0 be a non-zero vector in* U_0 *and w_0 a non-zero vector in* W_0. *Then* $\langle u_0, w_0 \rangle$ *does not vanish.*

Proof. — We have $w_0 = cu_0 +$ a linear combination of $v_{i,\, i,\, h-i}$ $(1 \leqq i \leqq h)$, $c \neq 0$, by Lemmas 3.11 and 3.13. Thus the proposition follows from Lemma 3.12.

The irreducible $\mathrm{U}(1)\times\mathrm{U}(2)$-module U_3 is included in V_ρ for $h \geqq 2$. A maximal vector u_3 in U_3 is a vector of weight $2\lambda_0 - 2\lambda_2$ in V_ρ which satisfies $\rho(\mathrm{X}_2)u_3 = 0$. We notice that the vectors $v_{i,\, i,\, h-i-2}$ $(0 \leqq i \leqq h-2)$ form a basis of the weight space of weight $2\lambda_0 - 2\lambda_2$.

LEMMA 3.15. — *A maximal vector* $u_3 \in \mathrm{U}_3$ *is written as:*

$$\sum_{i=0}^{h-2} b_i v_{i,\, i,\, h-i-2} \quad \textit{where} \quad b_i=\frac{(-1)^i(h-i)(h-i-1)}{i!\,h(h-1)} b_0 \quad \textit{and} \quad b_0 \neq 0.$$

A non-zero $\Delta(\mathrm{U}(1)\times\mathrm{U}(1))\times\mathrm{U}(1)$-invariant vector in U_3 is a vector of weight $2\lambda_0-2\lambda_1=2\lambda_0-2\lambda_2-2(\lambda_1-\lambda_2)$ in U_3, which is given by $\rho(\mathrm{Y}_2^2)u_3$.

LEMMA 3.16:

$$\rho(\mathrm{Y}_2^2)u_3=\sum_{i=0}^{h-2}\frac{2(-1)^i}{i!\,h(h-1)}b_0 v_{i,\,i+2,\,h-i-2}.$$

PROPOSITION 3.17. — *Let w_3 be a non-zero $\mathrm{T}\times\mathrm{U}(1)$-invariant vector in the irreducible $\mathrm{U}(2)\times\mathrm{U}(1)$-submodule of V_ρ with the highest weight $2\lambda_0-2\lambda_1$. Then $\langle\rho(\mathrm{Y}_2^2)u_3, w_3\rangle$ does not vanish and $\langle\rho(\mathrm{Y}_2^2)u_3, w_0\rangle$ vanishes.*

Proof. — We have $w_3=c\,\rho(\mathrm{Y}_2^2)u_3+$a linear combination of $v_{i,\,i+2,\,h-i-2}$ $(1\leqq i\leqq h-2)+$ a linear combination of vectors of weight other than $2\lambda_0-2\lambda_1, c\neq 0$. Hence $\langle\rho(\mathrm{Y}_2^2)u_3, w_3\rangle$ $(=c)$ does not vanish. Since $\rho(\mathrm{Y}_2^2)u_3$ and w_0 are vectors of weights $2\lambda_0-2\lambda_1$ and 0, respectively, $\langle\rho(\mathrm{Y}_2^2)u_3, w_0\rangle$ vanishes.

PROPOSITION 3.18. — *For an irreducible $\mathrm{U}(3)$-module V_ρ with the highest weight $h\lambda_0-h\lambda_2$, we have:*

$$\dim(p(\mathrm{V}_\rho^{\mathrm{I}}))\begin{cases}\geqq 1, & h=0,\ 1,\\ \geqq 2, & h\geqq 2.\end{cases}$$

The case $\Lambda=(h+1)\lambda_0-\lambda_1-h\lambda_2$ $(h\geqq 2)$. — Since $\Lambda(\mathrm{H}_2)=h-1$, $\rho(\mathrm{Y}_2^h)v_\Lambda$ vanishes and therefore $v_{i,\,j,\,k}$ vanishes for $j\geqq h$. When $v_{i,\,j,\,k}$ does not vanish, it is a vector of weight $(h-i-k+1)\lambda_0+(i-j-1)\lambda_1+(j+k-h)\lambda_2$.

LEMMA 3.19. — *For non-negative integers i, j and k satisfying $i+k\leqq h+2$ and $j\leqq h-1$, the vectors $v_{i,\,j,\,k}$ are linearly independent.*

In fact, they form a basis of the sum of weight spaces of weight $p\lambda_0+q\lambda_1+r\lambda_2$ satisfying $p\geqq -1$. In particular, the weight space of weight $t\lambda_0-t\lambda_1$ $(t\geqq 1)$ has as its basis the vectors $v_{i,\,i+t-1,\,h-i-t+1}$ $(0\leqq i\leqq h-t)$.

LEMMA 3.20. — *Let w be a maximal vector in the irreducible $\mathrm{U}(2)\times\mathrm{U}(1)$-submodule of V_ρ with the highest weight $t\lambda_0-t\lambda_1$ $(t\geqq 1)$. Then w is written as:*

$$w=\sum_{i=0}^{h-t}a_i v_{i,\,i+t-1,\,h-i-t+1},$$

where:

$$a_i=\frac{h-i}{i(i+5)}a_{i-1}\qquad(1\leqq i\leqq h-t)\quad\textit{and}\quad a_0\neq 0,$$

and w is perpendicular to the subspace spanned by $v_{i,\,i+t-1,\,h-i-t+1}$ $(1\leqq i\leqq h-t)$.

LEMMA 3.21. — *A maximal vector u_3 in the irreducible $\mathrm{U}(1)\times\mathrm{U}(2)$-submodule U_3 of V_ρ is written as:*

$$u_3=\sum_{i=0}^{h-2}b_i v_{i+1,\,i,\,h-i-2}$$

where:

$$b_i = \frac{(-1)^i (h-i-1)}{i!(h-1)} b_0 \quad \textit{and} \quad b_0 \neq 0.$$

LEMMA 3.22. — *A non-zero* $\Delta(\mathrm{U}(1)\times\mathrm{U}(1))\times\mathrm{U}(1)$-*invariant vector in* U_3 *is a vector of weight* $2\lambda_0 - 2\lambda_1$ *in* U_3, *that is:*

$$\rho(\mathrm{Y}_2^2)u_3 = \sum_{i=0}^{h-2} \frac{2(-1)^i}{i!(h-1)} b_0 v_{i,\, i+1,\, h-i-1}.$$

PROPOSITION 3.23. — *Let* w_3 *be a non-zero* $\mathrm{T}\times\mathrm{U}(1)$-*invariant vector in the irreducible* $\mathrm{U}(2)\times\mathrm{U}(1)$-*submodule of* V_ρ *with the highest weight* $2\lambda_0 - 2\lambda_1$. *Then* $\langle \rho(\mathrm{Y}_2^2)u_3, w_3 \rangle$ *does not vanish.*

PROPOSITION 3.24. — *For an irreducible* $\mathrm{U}(3)$-*module* V_ρ *with the highest weight* $(h+1)\lambda_0 - \lambda_1 - h\lambda_2$ $(h \geqq 2)$, *we have* $\dim(p(\mathrm{V}_\rho^1)) \geqq 1$.

The case $\Lambda = (h+2)\lambda_0 - 2\lambda_1 - h\lambda_2$ $(h \geqq 2)$. — Since $\Lambda(\mathrm{H}_2) = h-2$, $\rho(\mathrm{Y}_2^{h-1})v_\Lambda$ vanishes and therefore $v_{i,j,k}$ vanishes for $j \geqq h-1$. When $v_{i,j,k}$ does not vanish, it is a vector of weight $(h-i-k+2)\lambda_0 + (i-j-2)\lambda_1 + (j+k-h)\lambda_2$.

LEMMA 3.25. — *For non-negative integers* i, j *and* k *satisfying* $i+k \leqq h+4$ *and* $j \leqq h-2$, *the vectors* $v_{i,j,k}$ *are linearly independent.*

In fact, they form a basis of the sum of weight spaces of weight $p\lambda_0 + q\lambda_1 + r\lambda_2$ satisfying $p \geqq -2$. In particular, the weight space of weight $t\lambda_0 - t\lambda_1$ $(t \geqq 2)$ has as its basis the vectors $v_{i,\, i+t-2,\, h-i-t+2}$ $(0 \leqq i \leqq h-t)$.

LEMMA 3.26. — *Let* w *be a maximal vector in the irreducible* $\mathrm{U}(2)\times\mathrm{U}(1)$-*submodule of* V_ρ *with the highest weight* $t\lambda_0 - t\lambda_1$ $(t \geqq 2)$. *Then* w *is written as:*

$$w = \sum_{i=0}^{h-t} a_i v_{i,\, i+t-2,\, h-i-t+2}$$

where:

$$a_i = \frac{h-i+1}{i(i+5)} a_{i-1} \quad (1 \leqq i \leqq h-t) \qquad \textit{and} \qquad a_0 \neq 0,$$

and w *is perpendicular to the subspace spanned by the vectors* $v_{i,\, i+t-2,\, h-i-t+2}$ $(1 \leqq i \leqq h-t)$.

LEMMA 3.27. — *A maximal vector* u_3 *in the irreducible* $\mathrm{U}(1)\times\mathrm{U}(2)$-*submodule* U_3 *of* V_ρ *is written as:*

$$u_3 = \sum_{i=0}^{h-2} b_i v_{i+2,\, i,\, h-i-2} \qquad \textit{where} \quad b_i = \frac{(-1)^i}{i!} b_0 \quad \textit{and} \quad b_0 \neq 0.$$

A non-zero $\Delta(\mathrm{U}(1)\times\mathrm{U}(1))\times\mathrm{U}(1)$-*invariant vector in* U_3 *is a vector of weight* $2\lambda_0 - 2\lambda_1$ *in* U_3, *that is:*

$$\rho(\mathrm{Y}_2^2)u_3 = \sum_{i=0}^{h-2} \frac{2(-1)^i}{i!} b_0 v_{i,\, i,\, h-i}.$$

PROPOSITION 3.28. — *Let* w_3 *be a non-zero* $T \times U(1)$*-invariant vector in the irreducible* $U(2) \times U(1)$*-submodule of* V_ρ *with the highest weight* $2\lambda_0 - 2\lambda_1$. *Then* $\langle \rho(Y_2^2) u_3, w_3 \rangle$ *does not vanish.*

PROPOSITION 3.29. — *For an irreducible* $U(3)$*-module* V_ρ *with the highest weight* $(h+2)\lambda_0 - 2\lambda_1 - h\lambda_2$ $(h \geqq 2)$, *we have* $\dim(p(V_\rho^I)) \geqq 1$.

We now study the remaining cases. For an irreducible $U(3)$-module V_ρ, the dual vector space V_ρ^* becomes canonically an irreducible $U(3)$-module and an invariant hermitian inner product on V_ρ gives an anti-linear isomorphism Ic between V_ρ and V_ρ^*. When V_ρ includes some $U(1) \times U(2)$-module U_i listed before, Ic maps U_i to a $U(1) \times U(2)$-submodule U_i^*, which is again isomorphic to some U_j. Since $\mathrm{Ic}(V_\rho^H) = (V_\rho^*)^H$ and $\mathrm{Ic}(V_\rho^L) = (V_\rho^*)^L$, we have $\mathrm{Ic}(V_\rho^I) = (V_\rho^*)^I$ and $\dim(p(V_\rho^I)) = \dim(p((V_\rho^*)^I))$. When V_ρ is an irreducible $U(3)$-module with the highest weight $p\lambda_0 + q\lambda_1 + r\lambda_2$, V_ρ^* is an irreducible $U(3)$-module with the highest weight $-r\lambda_0 - q\lambda_1 - p\lambda_2$. Thus we have:

PROPOSITION 3.30. — *For an irreducible* $U(3)$*-module with the highest weight* $h\lambda_0 + \lambda_1 - (h+1)\lambda_2$ $(h \geqq 2)$ *or* $h\lambda_0 + 2\lambda_1 - (h+2)\lambda_2$ $(h \geqq 2)$, *we have* $\dim(p(V_\rho^I)) \geqq 1$.

Comparing Proposition 3.4 with Propositions 3.18, 3.24, 3.29 and 3.30, we get:

THEOREM 3.31. — *The I.B.C. holds for* $(P^2(C), g_0)$.

4. We now prove Theorem 1.4. It suffices to show:

THEOREM 4.1. — *For* $(P^n(C), g_0)$, $(P^n(H), g_0)$ *and* $(P^2(Ca), g_0)$ *the I.B.C. holds.*

We start with a preparatory Lemma.

LEMMA 4.2. — *Let* (M, g) *and* (N, g') *be Riemannian manifolds and* $\iota : N \to M$ *be a totally geodesic immersion. If there exists* $X \in \mathfrak{X}(M)$ *for* $h \in S^2(M)$ *such that* $L_X g = h$, *then there exists* $X' \in \mathfrak{X}(N)$ *such that* $L_{X'} g' = \iota^* h$.

Proof. — The pull back $\iota^* TM$ have an inner product defined by g and includes TN as a subbundle. Let $\tilde{X}$ be a section of $\iota^* TM$ defined by the restriction of X and let X′ be a section of TN given by the orthogonal projection of $\tilde{X}$ to TN. Since ι is a totally geodesic immersion, X′ satisfies the required condition of the Lemma.

Q.E.D.

By this Lemma and Theorem 1.2, we can prove that the I.B.C. holds for $(P^n(C), g_0)$ $(n \geqq 3)$ from Theorem 3.31. Let $\iota : P^1(C) \to P^n(C)$ $(n \geqq 3)$ be any totally geodesic imbedding. Then there exist totally geodesic imbeddings $\iota_1 : P^1(C) \to P^2(C)$ and $\iota_2 : P^2(C) \to P^n(C)$ satisfying $\iota = \iota_2 \circ \iota_1$. (There exists a projective plane including a given projective line.)

When $h \in S^2(P^n(C))$ is an infinitesimal C_π-deformation of $(P^n(C), g_0)$, $\iota_2^* h$ is an infinitesimal C_π-deformation of $(P^2(C), g_0)$, for ι_2 is totally geodesic. Because the I.B.C. holds for $(P^2(C), g_0)$, there exists $X \in \mathfrak{X}(P^2(C))$ such that $\iota_2^* h = L_X g_0$. By Lemma 4.2, there exists $X' \in \mathfrak{X}(P^1(C))$ such that $L_{X'} g_0 = \iota_1^*(\iota_2^* h) = \iota^* h$. Thus from Theorem 1.2, there exists $\tilde{X} \in \mathfrak{X}(P^n(C))$ such that $h = L_{\tilde{X}} g_0$, which implys that the I.B.C. holds. We get:

PROPOSITION 4.3. — *The I.B.C. holds for* $(P^n(C), g_0)$ $(n \geqq 2)$.

For $(P^n(H), g_0)$ $(n \geqq 2)$ and $(P^2(Ca), g_0)$ we need a further consideration, which was indicated by K. Sugahara. We first quote a result known in the projective geometry.

PROPOSITION 4.4. — *Let* P^n *be* $(P^n(H), g_0)$ $(n \geqq 2)$ *or* $(P^2(Ca), g_0)$:

(*a*) *let* $\iota_1, \iota_2 : S^2 \to P^n$ *be totally geodesic imbeddings. Then there exists an isometry* σ *of* P^n *satisfying* $\iota_2 = \sigma \circ \iota_1$;

(*b*) *there exists a totally geodesic imbedding* $\iota : P^2(C) \to P^n$;

(*c*) *for a totally geodesic imbedding* $\iota : S^2 \to P^n$, *there exist totally geodesic imbeddings* $\iota_1 : S^2 \to P^2(C)$ *and* $\iota_2 : P^2(C) \to P^n$ *satisfying* $\iota = \iota_2 \circ \iota_1$.

Proof. — For (*a*) and (*b*), *see* Wolf [10]. There exist some totally geodesic imbeddings $\bar{\iota}_1 : S^2 = P^1(C) \to P^2(C)$ and $\bar{\iota}_2 : P^2(C) \to P^n$ by (*b*). For a given totally geodesic imbedding $\iota : S^2 \to P^n$, there exists an isometry σ satisfying $\iota = \sigma \circ (\bar{\iota}_2 \circ \bar{\iota}_1)$. Setting $\iota_2 = \sigma \circ \bar{\iota}_2$ and $\iota_1 = \bar{\iota}_1$, we get the imbeddings needed in (*c*).

Q.E.D.

Next we quote an integrability condition of the equation $L_X g = h$ on a space form obtained by E. Calabi. Let (X, g) be a space of constant curvature K. For $h \in S^2(X)$ we define a 4-tensor r_h by:

$$r_h(x, y, z, w) = (\nabla_x \nabla_z h)(y, w) - (\nabla_y \nabla_z h)(x, w) - (\nabla_x \nabla_w h)(y, z) + (\nabla_y \nabla_w h)(x, z)$$
$$+ K\{g(x, z)h(y, w) - g(y, z)h(x, w) - g(x, w)h(y, z) + g(y, w)h(x, z)\},$$
$$[x, y, z, w \in TX_p].$$

One can verify:

LEMMA 4.5. — *The tensor* r_h *is curvature-like:*

$$r_h(x, y, z, w) = -r_h(y, x, z, w) = -r_h(x, y, w, z),$$
$$r_h(x, y, z, w) + r_h(y, z, x, w) + r_h(z, x, y, w) = 0.$$

The next Theorem is stated in Calabi [5], but for a strict proof, *see* Bérard Bergery-Bourguignon-Lafontaine [2].

THEOREM 4.6. — *Let* (S^n, g_0) *be a sphere of constant curvature. For* $h \in S^2(S^n)$, *there exists* $X \in \mathfrak{X}(S^n)$ *satisfying* $L_X g_0 = h$, *if and only if* r_h *vanishes on* S^n.

Proof of Theorem 4.1. — Let (P^n, g_0) be any of $(P^n(H), g_0)$ $(n \geqq 2)$ and $(P^2(Ca), g_0)$. Let $h \in S^2(P^n)$ be any infinitesimal C_π-deformation of (P^n, g_0) and let $\iota : P^1 \to P^n$ be any totally geodesic imbedding of a projective line. If we can prove that $\iota^* h$ is trivial on P^1, we have done in view of Theorem 1.2. We will prove $r_{\iota^* h}$ vanishes on P^1 (= a sphere of constant curvature). Since $r_{\iota^* h}$ is a curvature-like tensor, $r_{\iota^* h}$ vanishes if and only if $r_{\iota^* h}(x, y, x, y)$ vanishes for any $p \in P^1$ and $x, y \in TP^1_p$. Let $\iota' : S^2 \to P^1$ be a totally geodesic imbedding of a sphere of constant curvature such that the image is tangent to x and y at p. Since ι' is totally geodesic, $\iota'^* r_{\iota^* h}$ coincides with $r_{\iota'^*\iota^* h}$, a 4-tensor on S^2 constructed from $(\iota\circ\iota')^* h$. For the totally geodesic imbedding $\iota\circ\iota' : S^2 \to P^n$, there exist totally geodesic imbeddings $\iota_1 : S^2 \to P^2(C)$ and $\iota_2 : P^2(C) \to P^n$ satisfying $\iota\circ\iota' = \iota_2\circ\iota_1$. By the same reasoning of Proposition 4.2, $(\iota_2\circ\iota_1)^* h = (\iota\circ\iota')^* h$ is trivial on S^2. From Theorem 4.6, $r_{\iota'^*\iota^* h}$ vanishes on S^2 and $r_{\iota^* h}(x, y, x, y)$ vanishes.

Q.E.D.

REFERENCES

[1] M. Berger and D. G. Ebin, *Some Decomposition of the Space of Symmetric Tensors on a Riemannian Manifold* (*J. Differential Geometry*, Vol. 3, 1969, pp. 379-392).

[2] L. Bérard Bergery, J.-P. Bourguignon and J. Lafontaine, *Déformations localement triviales des variétés riemanniennes* (*Proc. Symp. Pure Math.*, Vol. 27, Part 1, 1975, pp. 3-32).

[3] A. Besse, *Manifolds all of Whose Geodesics are Closed*, Springer-Verlag, Berlin-Heiderberg-New York, 1978.

[4] H. Boerner, *Representations of Groups*, North-Holland, Amsterdam, 1963.

[5] E. Calabi, *On Compact Riemannian Manifolds with Constant Curvature I* (*Proc. Symp. Pure Math.*, Vol. 3, 1961, pp. 155-180).

[6] V. Guillemin, *The Radon Transform on Zoll Surfaces* (*Advances in Math.*, Vol. 22, 1976, pp. 85-119).

[7] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (*Graduate Texts in Mathematics*, Vol. 9, Springer-Verlag, New York-Heiderberg-Berlin, 1972).

[8] K. Kiyohara, *On the Infinitesimal Blaschke Conjecture* (*Hokkaido Math. J.*, Vol. 10, 1981, pp. 124-142).

[9] R. Michel, *Problèmes d'analyse géométrique liés à la conjecture de Blaschke* (*Bull. Soc. Math. Fr.*, T. 101, 1973, pp. 17-69).

[10] J. A. Wolf, *Elliptic Spaces in Grassmann Manifolds* (*Illinois J. Math.*, Vol. 7, 1963, pp. 447-462).

[11] O. Zoll, *Über Flächen mit Scharen geschlossener geodätischer Linien* (*Math. Ann.*, Vol. 57, 1903, pp. 108-133).

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