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On the mixed Hodge structure associated to π_3 of a simply connected complex projective manifold

ON THE MIXED HODGE STRUCTURE
ASSOCIATED TO π_3
OF A SIMPLY CONNECTED COMPLEX
PROJECTIVE MANIFOLD

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1. Introduction

Let X be a simply connected complex projective manifold of (complex) dimension N. By the universal coefficient theorem for cohomology, $H^2(X; \mathbb{Z})$ is torsion-free. Thus:

$$(1.1) \qquad \mathcal{M} = H^{1,1}(X) \cap H^2(X; \mathbb{Z}) \subseteq H^2(X; \mathbb{C}),$$

makes sense and is free. $\mathcal{M}$ is identified with the *Neron-Severi group* of linear equivalence classes of divisors on X. Let S^r denote r-th symmetric product and form the natural map:

$$(1.2) \qquad \begin{cases} \alpha : \quad S^2 \mathcal{M} \to H_{2N-4}(X; \mathbb{Z}), \\ \sum a_{ij} D_i \otimes D_j \mapsto \{ \sum a_{ij} D_i . D_j \}, \end{cases}$$

obtained by intersecting divisors. If $\sum a_{ij} D_i \otimes D_j \in \ker \alpha$, then $\sum a_{ij} D_i . D_j$ is the boundary of a $(2N-3)$-chain Γ on X, and Γ is determined up to integral $(2N-3)$-cycles.

Next let F^* denote the Hodge filtration on $H^*(X; \mathbb{C})$. We recall that Griffiths has defined a set of complex tori, called *intermediate Jacobians*, associated to X, and that the intermediate Jacobian $J_{2N-3}(X)$ is defined as:

$$(1.3) \qquad (F^{N-1}(H^{2N-3}(X; \mathbb{C})))^* \Big/ \left\{ \int_\eta : \eta \in H_{2N-3}(X; \mathbb{Z}) \right\}.$$

Thus $\gamma \in \ker \alpha$ in (1.2) determines a well-defined element:

$$\int_\Gamma \in J_{2N-3}(X)$$

and so we obtain a homomorphism:

$$(1.4) \qquad \Phi : \ (\ker \alpha) \rightarrow J_{2N-3}(X).$$

The purpose of this paper is to show how the homomorphism (1.4) comes out of the *mixed Hodge structure* which Morgan has defined on $\pi_3(X)$ in [M]. We will also give examples of diffeomorphic algebraic manifolds X_1 and X_2 such that the induced isomorphism of cohomology rings:

$$H^*(X_1) \cong H^*(X_2),$$

preserves Hodge types but *not* the homomorphisms (1.4). Thus, although the rational homotopy type of X, and $\pi_3(X) \otimes \mathbb{Q}$ in particular, is determined by the cohomology ring $H^*(X; \mathbb{Z})$ (according a result of Deligne, Griffiths, Morgan and Sullivan), the mixed Hodge structures on $\pi_n(X)$ are not determined merely by the ring $H^*(X; \mathbb{Z})$ together with the Hodge structures on the groups of $H^*(X; \mathbb{C})$. Finally we will give some geometric applications which show the usefulness of the "extra information" contained in the mixed Hodge theory of the homotopy groups of X but not in the Hodge theory of the cohomology.

2. The group π_3^*

We define:

$$(2.1) \qquad \pi_n = \pi_n(X)/(\text{torsion subgroup}).$$

Furthermore, if no coefficients are specified for homology or cohomology, we will mean integral homology or cohomology *modulo torsion*. We let:

$$(2.2) \qquad X_2 = a \, K(\pi_2, 2)\text{-space}.$$

There is a continuous map:

$$(2.3) \qquad f : \ X \rightarrow X_2,$$

unique up to homotopy, such that $f* : \pi_2(X) \rightarrow \pi_2(X_2)$ is just the natural quotient map $\pi_2(X) \rightarrow \pi_2$.

Viewing (2.3) as a homotopy inclusion, the homotopy sequence for the pair (X_2, X) gives that:

$$\pi_3(X) \cong \pi_4(X_2, X)$$

The generalized relative Hurewicz theorem ([H], pp. 306-310), with $\mathscr{C}$ the category of one element groups, says that:

$$\pi_4(X_2, X) \rightarrow H_4(X_2, X; \mathbb{Z}),$$

is an epimorphism. Furthermore, since $\pi_3(X_2, X)$ is torsion, the generalized relative Hurewicz theorem also says that:

$$\pi_4(X_2, X) \to H_4(X_2, X)$$

is a $\mathscr{C}$-isomorphism, where $\mathscr{C}$ is the category of torsion groups. Thus $\pi_4(X_2, X)/(\text{torsion}) \to H_4(X_2, X)$ is onto and injective so that:

$$\pi_3 = \pi_4(X_2, X)/(\text{torsion}) = H_4(X_2, X),$$

and so by the universal coefficient theorem for cohomology:

(2.4) $$H^4(X_2, X) = \text{Hom}(\pi_3, \mathbb{Z}) \underset{(\text{def.})}{=} \pi_3^*.$$

Central to our results will be the examination of the mixed Hodge structure, defined over $\mathbb{Z}$, on π_3^*. As usual, this is a question of setting up the right complex in which to compute the relative cohomology group (2.4). The rest of this section will be devoted to doing just that.

Now $H^3(X_2; \mathbb{Z}) = 0$ and:

(2.5) $$H^4(X_2; \mathbb{Z}) \cong S^2 H^2(X_2; \mathbb{Z}) \doteq S^2 H^2(X; \mathbb{Z}),$$

where S^2 denotes second symmetric product. So we have the exact cohomology sequence:

$$0 \to H^3(X) \to H^4(X_2, X) \to S^2 H^2(X; \mathbb{Z}) \to H^4(X; \mathbb{Z}).$$

So if we let K denote the kernel of the natural cup-product mapping:

(2.6) $$\alpha : \quad S^2 H^2(X; \mathbb{Z}) \to H^4(X; \mathbb{Z}),$$

then we have the short exact sequence:

(2.7) $$0 \to H^3(X) \to \pi_3^* \to K \to 0.$$

Since α in (2.6) is a morphism of Hodge structures of weight four, K has naturally a Hodge structure of weight four. Also $H^3(X)$ has a Hodge structure of weight three. So the sequence (2.7) tells us that we may expect to find a natural mixed Hodge structure on π_3^* such that all morphisms in (2.7) are morphisms of mixed Hodge structures (*see* [G−S], §1).

Let:

(2.8) $$M_2 = S^* H^2(X; \mathbb{Z}),$$

where S^* denotes the graded symmetric algebra. Defining $d = 0$, (M_2, d) becomes a complex and:

$$H^*(M_2, d) \cong H^*(X_2; \mathbb{Z}).$$

Also $M_2 \otimes \mathbb{C} = H^*(X_2; \mathbb{C})$ carries a natural Hodge filtration:

$$F^p(M_2 \otimes \mathbb{C}),$$

induced formally from the Hodge filtration on $H^2(X; \mathbb{C})$. The cohomology $H^*(X_2, X; \mathbb{C})$ is computed from the complex:

$$(2.9) \qquad \begin{cases} A^*(X_2, X) = A^{*-1}(X) \oplus (M_2^* \otimes \mathbb{C}), \\ \quad d(\alpha, \beta) = (d\alpha - f^*\beta, -d\beta), \end{cases}$$

where A^* (space) means the $\mathbb{C}$-valued de Rham complex on the space, and f^* is obtained as follows. Pick a vector space V of closed forms in $A^2(X)$ such that the natural map:

$$(2.10) \qquad \{ \ \} : \ V \to H^2(X; \mathbb{C}),$$

is an isomorphism and such that, if $\{\omega\}$ is of type (p, q), then so is ω:

$$(2.11) \qquad f^* : \ S^* H^2(X; \mathbb{C}) \to A^*(X),$$

is then the homomorphism of differential graded algebras induced by the inverse of the mapping (2.10).

Morgan defines the Hodge filtration:

$$(2.12) \qquad F^p(A^*(X_2, X)) = \sum_{p_1 + p_2 = p} F^{p_1}(A^{*-1}(X)) \oplus F^{p_2}(M_2^* \otimes \mathbb{C}),$$

on the complex (2.9). This filtration, together with the natural integral structure on $H(X_2, X; \mathbb{C})$ gives the canonical mixed Hodge on $H^4(X_2, X) = \pi_3^*$. The exact sequence (2.7) is a sequence of morphisms of mixed Hodge structures. It is separated in the sense of [Ca] so that the mixed Hodge structure on π_3^* determines canonically an obstruction class to the splitting of sequence (2.7) over $\mathbb{Z}$. This obstruction is an element of:

$$(2.13) \qquad \frac{\mathrm{Hom}(K \otimes \mathbb{C}, H^3(X; \mathbb{C}))}{\mathrm{Hom}_F(K \otimes \mathbb{C}, H^3(X; \mathbb{C})) + \mathrm{Hom}(K, H^3(X))},$$

where Hom_F is the group of homomorphisms respecting the Hodge filtrations. To see what this element is, let:

$$s_{\mathbb{Z}} : \ K \to \pi_3^*,$$

be an integral splitting of (2.7) and let:

$$s_F : \ K \otimes \mathbb{C} \to \pi_3^* \otimes \mathbb{C},$$

be a splitting of the complexification of (2.7) respecting the Hodge filtrations. The element of (2.13) determined by the mixed Hodge structure on π_3^* is then simply the equivalence class of:

$$(2.14) \qquad -s_F.$$

We wish to get our hands on a portion of the information given by (2.14). As in the introduction, put:

$$\mathscr{M} = H^2(X; \mathbb{Z}) \cap H^{1,1}(X),$$

the *Neron-Severi group of* X. Let:

(2.15) $$K^{\text{alg.}} = S^2 \mathcal{M} \cap K.$$

Then, *via* restriction, the element (2.14) determines an element:

(2.16) $$u(\pi_3^*),$$

in the group:

$$\frac{\text{Hom}(K^{\text{alg.}} \otimes \mathbb{C}, H^3(X; \mathbb{C}))}{\text{Hom}_F(K^{\text{alg.}} \otimes \mathbb{C}, H^3(X; \mathbb{C})) + \text{Hom}(K^{\text{alg.}}, H^3(X))}.$$

As in [Ca] one checks that this last group is just:

(2.17) $$\text{Hom}\left(K^{\text{alg.}}, \frac{H^3(X; \mathbb{C})}{F^2 H^3(X; \mathbb{C}) + H^3(X)}\right) = \text{Hom}(K^{\text{alg.}}, J_{2N-3}(X)).$$

We can now rephrase our main result as the assertion that:

(2.18) $$u(\pi_3^*) = \Phi,$$

where Φ is the homomorphism (1.4). In the next section, we will prove this assertion.

3. Proof of the main assertion

To prove (2.18), we will first enlarge the complex $A^*(X_2, X)$ used to calculate π_3^*. We begin by enlarging the complex $A^*(X)$ to the complex of $\mathbb{C}$-valued integration currents of deRham (*see* [deR], §14) which we will call $B^*(X)$. $B^*(X)$ has a Hodge filtration such that the natural inclusion $A^*(X) \subseteq B^*(X)$ induces in cohomology an isomorphism of Hodge structures ([K], p. 169 ff.).

Next we want to enlarge $(M_2 \otimes \mathbb{C})^*$ as follows. Let ω_j, $j = 1, \ldots, s$ be the collection of $(1, 1)$ forms in the vector space V in (2.10) which correspond to a basis of $H^2(X; \mathbb{Z}) \cap H^{1,1}(X)$ under the isomorphism (2.10). Let D_j be a divisor on X whose Chern class is ω_j, $j = 1, \ldots, s$. Each $\{\omega_j\}$ is a basis element of M_2^2. For each j we make a new free closed generator δ_j to adjoin to the vector space $(M_2 \otimes \mathbb{C})^2$, and define:

(3.1) $$f^*(\delta_j) = [D_j],$$

where brackets denote integration currents. Next, for each j we adjoin a free generator σ_j in degree one and put:

$$d\sigma_j = \{\omega_j\} - \delta_j.$$

Define:

$$f^*(\sigma_j) = a\ (1, 0)\text{-current}\ \eta_j\ \text{such that}\ d\eta_j = \omega_j - \delta_j.$$

In $(M_2 \otimes \mathbb{C})^4$ we adjoin free generators:

$$\delta_i \wedge \delta_j,$$

for each $1 \leq i \leq j \leq s$. Finally for each of these we adjoin a free generator:

$$\sigma_i \wedge \omega_j + \delta_i \wedge \sigma_j,$$

of degree three, and define the coboundary of these new elements by the Leibniz rule and define f^* multiplicatively. So altogether we have a new complex:

$$N_\mathbb{C}^*,$$

such that:

$$(3.2) \begin{cases} \text{(i)} & N_\mathbb{C}^k = (M_2 \otimes \mathbb{C})^k \quad \text{for} \quad k = 0, 5, 6, \ldots, \\[2mm] \text{(ii)} & N_\mathbb{C}^1 = \sum_{j=1}^{s} \mathbb{C} \, \sigma_j, \\[2mm] \text{(iii)} & N_\mathbb{C}^2 = (M_2 \otimes \mathbb{C})^2 + \sum_{j=1}^{s} \mathbb{C}(\{\omega_j\} - \delta_j), \\[2mm] \text{(iv)} & N_\mathbb{C}^3 = \sum_{1 \leq i \leq j \leq s} \mathbb{C}(\sigma_i \wedge \{\omega_j\} + \delta_i \wedge \sigma_j), \\[2mm] \text{(v)} & N_\mathbb{C}^4 = (M_2 \otimes \mathbb{C})^4 + \sum_{1 \leq i \leq j \leq s} \mathbb{C}(\{\omega_i\} \wedge \{\omega_j\} - \delta_i \wedge \delta_j), \end{cases}$$

and it is immediate that the natural inclusion:

$$(3.3) \qquad\qquad (M_2 \otimes \mathbb{C})^* \subseteq N_\mathbb{C}^*$$

induces an isomorphism in cohomology.

Additionally, we can put a Hodge filtration on $N_\mathbb{C}^*$ by defining σ_j to be of type $(1, 0)$ and δ_j of type $(1, 1)$, and putting types on elements of (3.2) (iv) and (v) multiplicatively. Notice that the complex $N_\mathbb{C}^*$ is *not* defined over the real numbers since there is no conjugation operator in degrees 1 and 3. However it is clear that the cohomology isomorphism induced by (3.3) is an isomorphism of Hodge structures.

We then replace $A^* (X_2, X)$ with the larger complex:

$$(3.4) \qquad\qquad B^*(X_2, X) = B^{*-1}(X) \oplus N_\mathbb{C}^*,$$

which, by the "five-lemma" is quasi-isomorphic to $A^*(X_2, X)$. The complex $B^*(X_2, X)$ has a Hodge filtration defined analogously to (2.12). What we must next check is that this filtration induces the same filtration on cohomology as does the filtration (2.12). To see this, suppose $(S, T) \in F^p (B^*(X_2, X))$ is closed. Then $T \in N_\mathbb{C}^*$ is closed. So, by construction, there exists $U \in F^p (N_\mathbb{C}^*)$ such that $T + dU = \varphi \in F^p (M_2^* \otimes \mathbb{C})$. Then:

$$(S, T) + d(0, U) = (S', \varphi) \in F^p,$$

and $d(\mathrm{S}', \varphi) = 0$. Thus:

$$d\mathrm{S}' = f^* \varphi$$

so that there exists $v \in \mathrm{F}^p (\mathrm{A}^* (\mathrm{X}))$ with $dv = f^* \varphi$. Therefore:

$$(\mathrm{S}', \varphi) + d(v, 0) = (\mathrm{S}'', \varphi) \in \mathrm{F}^p,$$

with $d\mathrm{S}'' = 0$. Since S'' is closed and of Hodge level p, there is a current V of Hodge level p such that in $\mathrm{B}^* (\mathrm{X})$:

$$\mathrm{S}'' + d\mathrm{V} = \psi \in \mathrm{F}^p (\mathrm{A}^* (\mathrm{X})).$$

So finally:

$$(\mathrm{S}, \mathrm{T}) \sim (\psi, \varphi) \in \mathrm{F}^p (\mathrm{A}^* (\mathrm{X}_2, \mathrm{X})),$$

which proves that the Hodge filtrations on $\mathrm{A}^* (\mathrm{X}_2, \mathrm{X})$ and $\mathrm{B}^* (\mathrm{X}_2, \mathrm{X})$ induce the same filtration in cohomology.

Next, let $\mathrm{K}^{\mathrm{alg.}}$ be as in (2.15), and for each $\{\gamma\}$ in some free basis for $\mathrm{K}^{\mathrm{alg.}}$ we write:

$$(3.5) \qquad\qquad \gamma = \sum a_{ij} \delta_i \wedge \delta_j$$

and:

$$f^* (\gamma) = \sum a_{ij} [\mathrm{D}_i] \wedge [\mathrm{D}_j],$$

the image current on X. Since $\sum a_{ij} \mathrm{D}_i . \mathrm{D}_j \sim 0$ there is a $(2\mathrm{N} - 3)$-chain $\Gamma(\gamma)$ such that:

$$\partial \Gamma = \sum a_{ij} \mathrm{D}_i . \mathrm{D}_j,$$

so that, as currents:

$$d[\Gamma] = f^* (\gamma).$$

Thus:

$$([\Gamma], \gamma) \in \mathrm{B}^4 (\mathrm{X}_2, \mathrm{X}),$$

is a cocycle. We wish to check that it is *integral*, that is, that it pairs with every element of $\mathrm{H}_4 (\mathrm{X}_2, \mathrm{X}; \mathbb{Z})$ to give an integer. For this we need only notice that this last group is computed from the complex:

$$(3.6) \qquad \begin{cases} \mathrm{C}_* (\mathrm{X}_2, \mathrm{X}) = (\mathrm{M}_2)_* \oplus \mathrm{C}_{*-1} (\mathrm{X}), \\ \partial (\mathrm{A}, \mathrm{B}) = (\partial \mathrm{A} + f_* \mathrm{B}, -\partial \mathrm{B}), \end{cases}$$

where:

$$(\mathrm{M}_2)_* = \text{dual of } \mathrm{M}_2^*,$$

$$\mathrm{C}_* (\mathrm{X}) = \text{complex of integral valued singular chains on } \mathrm{X}.$$

Since each class of $H_4(X_2, X; \mathbb{Z})$ has a representative (A, B) such that B is in general position with respect to each Γ defined after (3.5), the pairing becomes:

$$\langle (A, B), (\Gamma, \gamma) \rangle = (\Gamma . B) + \langle A, \gamma \rangle,$$

which clearly takes integral values.

We are finally ready to prove (2.18). By what we have done just above, we may evaluate the obstruction homomorphism $s_{\mathbb{Z}} - s_F$ of (2.14) by taking a generator $\{\gamma\} \in K^{alg}$:

$$\{\gamma\} = \{\textstyle\sum a_{ij} \delta_i \wedge \delta_j\} = \{\textstyle\sum a_{ij} \omega_i \wedge \omega_j\}$$

and putting:

$$s_{\mathbb{Z}}(\{\gamma\}) = ([\Gamma(\gamma)], \textstyle\sum a_{ij} \delta_i \wedge \delta_j),$$
$$s_F(\{\gamma\}) = (\rho, \textstyle\sum a_{ij} \{\omega_i\} \wedge \{\omega_j\}),$$

where $\rho \in F^2(A^3(X))$ and $d\rho = \sum a_{ij} \omega_i \wedge \omega_j$. To find a representative for $s_{\mathbb{Z}}(\{\gamma\}) - s_F(\{\gamma\})$ in $B^3(X)$, we simply add on:

$$d((0, \textstyle\sum a_{ij}(\sigma_i \wedge \{\omega_j\} + \delta_i \wedge \sigma_j)),$$

to obtain the representative:

(3.7) $$[\Gamma(\gamma)] - \rho - f^*(\textstyle\sum a_{ij}(\sigma_i \wedge \{\omega_j\} + \delta_i \wedge \sigma_j).$$

But since the last two terms in (3.7) are in $F^2 B^3(X)$, we see by (2.17) that:

$$u(\pi_3^*)(\{\gamma\}) = [\Gamma(\gamma)] = \int_{\Gamma(\gamma)} \in J_{2N-3}(X),$$

which is the assertion (2.18).

We next turn to some applications of (2.18) in the case in which X is a simply connected projective manifold of complex dimension three. For example, suppose we start with a smooth curve $C \subseteq \mathbb{P}^3$. Let us mark two distinct points $P, Q \in C$. First blow up P and Q in $\mathbb{P}^3$, then blow up the proper transform of C. We obtain a smooth threefold X with three irreducible exceptional divisors E_P, E_Q and E_C lying over P, Q, and C respectively. Then by (2.18):

$$u((E_P - E_Q) \otimes E_C) = \int_{L_Q}^{L_P} \in J_3(X),$$

where L_P is the fibre over P of the $\mathbb{P}^1$-bundle $E_C \to C$. But under the natural isomorphism $J_3(X) \cong J(C)$ (*see* [Cl − G], p. 294), $\int_{L_Q}^{L_P}$ becomes $\int_Q^P \in J(C)$.

This means that the subgroup $u(K^{alg})$ varies as we continuously vary P and Q on C (as long as C is not rational). However the cohomology ring with its Hodge structures defined over $\mathbb{Z}$ does *not* vary as P and Q are moved, showing that the mixed Hodge theory of $\pi_3^*(X)$ is not a formal consequence of the cohomology of X.

From these remarks it is clear that it will be important to understand what happens to the mixed Hodge structure on π_3^* under blowing up. We treat this in the next section.

4. Blowing up threefolds

From now on, X is a threefold so that the obstruction homomorphism associated to the mixed Hodge structure on π_3^* takes values in $J_3(X)$. We let

(4.1) $B^2(X) =$ Chow group of rational equivalence classes

of algebraic one-cycles on X which are homologous to 0.

There is a natural map, called the *Abel-Jacobi map:*

(4.2) $$\begin{cases} B^2(X) \to J_3(X), \\ Z \mapsto \int_\Gamma, \qquad \partial\Gamma = Z. \end{cases}$$

Then (2.18) simply tells us that the obstruction homomorphism:

(4.3) $$\sigma_X: \quad K_X^{\text{alg.}} \to J_3(X),$$

is the restriction of the Abel-Jacobi map (4.2) to the subgroup of $B^2(X)$ generated, *via* intersection, by the Neron-Severi group of X, which we denote $\mathcal{M}_X$.

We next analyze what happens when a threefold $\tilde{X}$ is obtained from X by blowing up a point. Then:

(4.4) $$K_{\tilde{X}}^{\text{alg.}} = K_X^{\text{alg.}} \oplus (\mathcal{M}_X \otimes \mathbb{Z}\{e\}),$$

where e is the cohomology class of the exceptional divisor, and:

(4.5) $$\begin{cases} u_{\tilde{X}}|_{K_X^{\text{alg.}}} = u_X, \\ u_{\tilde{X}}|_{\mathcal{M}_X \otimes \mathbb{Z}\{e\}} = 0. \end{cases}$$

So no new information is gained from $\mathcal{O}_{\tilde{X}}$.

The situation is quite different, however, when

(4.6) $$\pi: \quad \tilde{X} \to X,$$

is the monoidal transform with center C, a smooth irreducible curve on X. First of all ([Cl−G], p. 294):

(4.7) $$J_3(\tilde{X}) = J_3(X) \oplus J(C).$$

Again under the natural inclusion $S^{(2)}H^2(X) \subseteq S^{(2)}H^2(\tilde{X})$ we have $K_X^{\text{alg.}} \subseteq K_{\tilde{X}}^{\text{alg.}}$. To describe the rest of $K_{\tilde{X}}^{\text{alg.}}$, let e as before denote the cohomology class of the exceptional locus, let c denote the cohomology class of C in X, and let:

(4.8) $$\alpha_X: \quad S^{(2)}\mathcal{M}_X \to H^4(X; \mathbb{Z}),$$

be the wedge-product mapping.   Define:

(4.9)
$$\begin{cases} n_c = \min \{ n > 0 : nc \in \text{Image } \alpha_X \}, \\ m_c = \min \left\{ m \geq 0 : m = \int_X c \wedge \omega, \ \omega \in H^2(X) \right\}. \end{cases}$$

Finally choose:

(4.10)
$$\begin{cases} A \in S^{(2)} \mathcal{M}_X \text{ such that } \alpha_X(A) = n_c c, \\ B \in \mathcal{M}_X \text{ such that } \int c \wedge B = m_c. \end{cases}$$

Then:

(4.11)
$$K_{\tilde{X}}^{\text{alg.}} = K_X^{\text{alg.}} \oplus \mathbb{Z} \{ e \} \otimes \left\{ \omega \in H^2(X) : \int_C \omega = 0 \right\} \oplus \mathbb{Z} \{ \tau \},$$

where:

$$\tau = m_1 \, e \otimes e + m_2 \, e \otimes B + m_3 \, A,$$

and:

$$(m_1, m_2, m_3) \in \mathbb{Z}^3,$$

is the generator of the solution set of the system of equations:

(4.12)
$$n_c m_3 - m_1 = 0, \qquad \gamma m_1 + m_c m_2 = 0,$$

and:

(4.13)
$$\gamma = \text{degree of the normal bundle } N_C \text{ of } C \text{ in } X.$$

All this follows easily from the multiplication table for a monoidal transform of a threefold (*see*, for example, [I-M], p. 146).

The behavior of $u_{\tilde{X}}$ has been computed by M.   Letizia of the University of Trento, Italy, as follows:

(4.14)

(i) If $D.C = 0$ on X and ω_D is the Chern class of D, then:

$$u_{\tilde{X}}(e \otimes \omega_D) = (0, (D.C)) \in J_3(X) \oplus J(C).$$

(ii) If D is a divisor such that $\omega_D = B$ and:

$$L = (\Lambda^2 N_C)^{m_1} \otimes \mathcal{O}_C(m_2(D.C)),$$

then:

$$u_{\tilde{X}}(\tau) = \left(\int_\Gamma, \{ L \} \right) \in J_3(X) \oplus J(C),$$

where:

$$\partial \Gamma = m_3 (\text{image of } A \text{ under intersection map}) - m_1 C.$$

After getting accustomed to the geometric objects in (4.14) and the notation, the results contained therein are quite easy to apply. Some examples:

(4.15) Suppose we take two sets of smooth, mutally disjoint curves in $\mathbb{P}^3$:

$$\{C_1, \ldots, C_r\} \quad \text{and} \quad \{E_1, \ldots, E_s\},$$

and we blow up $\mathbb{P}^3$ along the C_j to obtain a rational threefold $\mathbb{P}(1)$, and then we start over with $\mathbb{P}^3$ and blow it up along the E_k to obtain a rational threefold $\mathbb{P}(2)$. We ask ourselves under what conditions it can be true that:

$$\mathbb{P}(1) \cong \mathbb{P}(2).$$

The required isomorphisms of integral cohomology H^2 and intermediate Jacobians tell us that $r = s$ and, reordering if necessary, there are abstract isomorphisms:

$$C_j \cong E_j, \quad j = 1, \ldots, r.$$

The ring structure on cohomology gives some additional numerical information which is very useful (as in [I-M]). Furthermore the required isomorphism:

$$\pi_3(\mathbb{P}(1)) \cong \pi_3(\mathbb{P}(2)),$$

gives:

$$u_{\mathbb{P}(1)} = u_{\mathbb{P}(2)}.$$

This means that, for example, if C_{j_0} and E_{j_0} have the same degree and:

$$C_{j_0} \neq C_j,$$

for any $j \neq j_0$, then up to a line bundle of specified finite order, sections of the *same* line bundle were used to embed C_{j_0} and E_{j_0} in $\mathbb{P}^3$.

(4.16) Let X be a smooth cubic hypersurface in $\mathbb{P}^4$. Let C be a smooth rational curve of degree 3 lying inside X. Then $K_X^{\text{alg.}}$ is generated by the single element:

$$\tau = -3\,e \otimes e + 4\,e \otimes h - 3\,h \otimes h,$$

where h is the Chern class of the hyperplane section of X. The image of τ in $J(\tilde{X}) = J(X)$ is then simply:

(4.17) $$3 \int_G^C,$$

where G is a cubic plane curve lying in X. Thus the mixed Hodge structure on $\pi_3(\tilde{X})^*$ determines the rational equivalence class of C modulo elements of order three. (It can be shown that the theta divisor of $J(X)$ exactly parametrizes rational equivalence classes of rational cubics.)

5. A "torelli-type" theorem
for projections of cubic threefolds

As a final illustration of the techniques described in this paper, we will give an example of a class of threefolds for which the polarized cohomology ring with its integral and Hodge structures is not sufficient to distinguish between the various threefolds in the class but the additional datum of a mixed Hodge structure on π_3^* does allow us to distinguish.

Our class will be formed from a fixed smooth cubic threefold:

$$(5.1) \qquad\qquad\qquad Y \subseteq \mathbb{P}^4.$$

Let $U \subseteq Y$ denote the Zariski open subset of Y consisting of those points $y \in Y$ such that:

$\left\{\begin{array}{l} \text{(i) if } H_y \subseteq \mathbb{P}^4 \text{ is the hyperplane tangent to Y at } y, \text{ then } H_y . Y \text{ has an ordinary node} \\ \text{at } y; \\ \text{(ii) } (H_y . Y) \text{ contains six distinct lines through } y. \quad (See [\text{Cl} - \text{G}]). \end{array}\right.$

For each $y \in U$, we build a threefold:

$$(5.3) \qquad\qquad\qquad X_y,$$

as follows. Beginning with Y, we first blow up y and then blow up the proper transforms of each of the six lines through y. It is this family (5.3), over the parameter space U, that we wish to study.

We begin by laying out the cohomology ring for X_y in even degrees. This is simply the Chow ring modulo numerical equivalence, which we denote:

$$A^*(X_y) \quad H^{2*}(X_y).$$

Let us denote by H_2 the irreducible rational surface in X_y which lies over $y \in Y$, and let H_1 denote the proper transform of the tangent hyperplane section $(H_y . Y)$ of Y at y. Let:

$$E_1, \ldots, E_6,$$

stand for the six irreducible surfaces obtained by blowing up the (proper transforms of the) six lines through y. Then the group of divisor classes on X_y is given by:

$$(5.4) \qquad A^1(X_y) = \mathbb{Z}\{H_1\} \oplus \mathbb{Z}\{H_2\} \oplus \overset{6}{\underset{j=1}{\oplus}} \mathbb{Z}\{E_j\}.$$

In this situation we will have to distinguish two divisor classes:

$$(5.5) \qquad\qquad D^+ = H_1 + H_2, \qquad D^- = H^{1\cdot} - H^2,$$

to serve as "polarizing classes" for X_y. D^+ allows us to define a symmetric bilinear form on (5.4) by the formula:

$$(5.6) \qquad\qquad\qquad \langle A, B \rangle = \frac{1}{2}(A.B.D^+).$$

D^- has norm one with respect to the form (5.6) and we will be interested in the subspace:

$$(5.7) \qquad\qquad (D^-)^\perp \cong A^1(X_y).$$

To recognize the subspace (5.7) and compute the form (5.6) on this subspace, recall that the projection map:

$$\pi_y: \quad Y \to \mathbb{P}^3,$$

centered at y can be resolved so as to realize X_y as a finite branched double cover:

$$(5.8) \qquad\qquad \pi: \quad X_y \to \tilde{\mathbb{P}},$$

where $\tilde{\mathbb{P}}$ is obtained from $\mathbb{P}^3$ by blowing up the six image points $P_1, \ldots, P_6$ of the six lines through y. [By (5.2) (i), the P_j lie on a non-degenerate conic.] Then:

$$\pi^{-1}(H) = H_1 + H_2,$$

where $H \subseteq \tilde{\mathbb{P}}$ is the proper transform of the plane through the P_j.

From the discussion just above, we can conclude first of all that:

$$(5.9) \qquad\qquad \{D^-\}^\perp = \pi^*(A^1(\tilde{\mathbb{P}})).$$

Also, since $\pi^*(H) = D^+$ and the intersection map:

$$(5.10) \qquad\qquad A^1(\tilde{\mathbb{P}}) \to A^1(H),$$

is an isomorphism, we conclude that the bilinear form $\langle \ , \ \rangle$ on $\{D^-\}^\perp$ is simply given by the usual intersection pairing on $A^1(H)$ via the isomorphism (5.9).

Next, H is simply $\mathbb{P}^2$ blown up at six points on a conic. Let h be the divisor class on H given by the proper transform of a plane cubic through the six points. It is easy to check that:

$$\{h\}^\perp \subseteq A^1(H),$$

corresponds under the isomorphisms (5.9) and (5.10) to:

$$(5.11) \qquad\qquad \mathscr{E} = \{B \in \{D^-\}^\perp : (D^- . D^- . B) = 0\}.$$

However, it is well-known ([B], p. 260) that the quadratic space $\{h\}^\perp$ is simply the negative definitive quadratic space E_6. So the same is true of the quadratic space $\mathscr{E}$. The important thing is that, $\mathscr{E}$ is completly determined as a quadratic space by the ring structure of $H^*(X_y)$ and the choice of D^+ and D^{-1} ([1]).

([1]) A similar situation obtains for the union X of a smooth cubic surface and a plane: the polarized one-motif attached to $H^2(X)$ maps a lattice of type E_6 to an elliptic curve, and determines X up to isomorphism. *See* J. CARLSON, *Extensions of Mixed Hodge Structures (Journées de Géométrie Algébrique d'Angers*, A. BEAUVILLE, Ed., Sijthoff and Noordhoff, 1980).

Notice too that if $B \in \{ D^- \}^\perp$, then B is symmetric with respect to the involution associated to (2.8), so that $(B . D^-)$ is skew. But the skew elements of $A^1(X_y)$ form a rank one submodule, as do the skew elements of $A^2(X_y)$. Consequently $(B . D^- . D^-) = 0$ if and only if $(B . D^-)$ is itself zero in $A^2(X_y)$. Thus:

$$(5.12) \qquad\qquad \mathscr{E} = \{ B \in \pi^*(A^1(\widetilde{\mathbb{P}})) : B . D^- = 0 \}.$$

Now for each $y \in U$ [see (5.1) and (5.2)] we can identify $H^3(X_y)$ and $J_3(X_y)$ with $H^3(Y)$ and $J_3(Y)$ respectively. So we have short exact sequences:

$$(5.13) \qquad\qquad 0 \to H^3(Y) \to \pi_3(X_y)^* \to K_y \to 0,$$

for each $y \in U$, and also obstruction homomorphisms:

$$(5.14) \qquad\qquad u_y : \quad K_y \to J_3(Y).$$

As we have seen in (5.12), the designation of the polarizing classes D^+ and D^- allows us to distinguish the subspace:

$$D^- \otimes \mathscr{E} \subseteq K_y,$$

as well as to determine an E_6 quadratic structure on this space.

So the restriction of (5.14) to the subspace $D^- \otimes \mathscr{E}$ can be considered, via (5.9)-(5.11), as a map:

$$(5.15) \qquad\qquad u_y : \quad \{ h \}^\perp \to J_3(Y).$$

Using our main result (2.18), we have a very simple way to describe (5.15). Let:

$$\psi : \quad H \to (H_y . Y),$$

be the natural birational mapping which blows down the proper transform of the conic through $P_1, \ldots, P_6$. (Recall that H_y is the tangent hyperplane to Y at y.) Then for any element:

$$\gamma \in \{ h \}^\perp,$$

$\psi_*(\gamma)$ bounds in Y. Choose Γ such that:

$$\partial \Gamma = \psi_*(\gamma).$$

Then:

$$(5.16) \qquad\qquad u_y(\gamma) = \int_\Gamma \in J_3(Y).$$

Now it is a classical fact ([B], p. 260) that if E_6 is considered to be the module of finite cycles on a smooth cubic surface, then the only elements of E_6 with square norm -2 are differences of skew lines. From this one concludes immediately that if $\gamma \in \{ h \}^\perp$ has square norm -2,

then $\psi_*(\gamma)$ is either a difference of skew lines on $(H_j . Y)$ or a difference of lines both of which pass through y, or zero.

Summing up, the ring structure on $H^*(X_y)$, the choice of the cohomology classes D^+ and D^-, and the mixed Hodge structure on $\pi_3(X_y)^*$ determine the principally polarized abelian variety:

$$J_3(X_y) = J_3(Y)$$

and so, by [Cl−G], the cubic Y. These data also determine a certain finite set $\mathscr{S}$ in $J_3(Y)$, namely the points given by:

(5.17)
$$\begin{cases} \text{(i)} & l - l' \text{ where } l \text{ and } l' \text{ are skew lines in } (H_y . Y); \\ \text{(ii)} & l - l' \text{ where } l \text{ and } l' \text{ are lines through } y. \end{cases}$$

So we will establish the result claimed at the beginning of this section by showing that the subset:

$$\mathscr{S} \subseteq J_3(Y),$$

uniquely determines the point y.

It is shown in [Cl−G] that, if S is the Fano surface of lines on Y, then the Abel-Jacobi mapping:

(5.18)
$$\begin{cases} \Phi: & S \times S \to J_3(Y), \\ & (l, l') \mapsto \int_l^{l'}, \end{cases}$$

has as image the theta divisor Θ of $J_3(Y)$. Furthermore the Gauss map:

(5.19)
$$g: \quad \Theta \to \mathbb{P} \text{ (tangent space to } J_3(Y) \text{ at } 0)^*,$$

assigns to each $\Phi(l, l')$ the hyperplane spanned by l and l' as long as they are skew. Finally the branch locus of g is the dual variety to Y:

$$Y^* \subseteq (\mathbb{P}^4)^*.$$

Now our restrictions (5.2) assure that H_y is a smooth point of the dual variety Y^*. So g is defined at (l, l') if the pair is as in (5.17), and the image:

$$\mathscr{G}(\mathscr{S}) = \{ H_y \}$$

and the dual map:

$$\mathscr{D}^*: \quad Y^* \to (Y^*)^* = Y,$$

is defined at $\{ H_y \}$ since it is a smooth point of Y^*. Finally $\mathscr{D}^*(\{ H_y \}) = y$ and our claim is established.

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