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ON THE SOBOLEV CONSTANT AND THE *p*-SPECTRUM OF A COMPACT RIEMANNIAN MANIFOLD

By Peter LI

On a m-dimensional compact C^∞ Riemannian manifold M^m, the Laplace-Beltrami operator Δ is defined by

$$\Delta = d\delta + \delta d, \tag{0.1}$$

where d is the exterior differential operator, and δ its adjoint operator with respect to the L^2 inner product. The Laplacian is a second-order self-adjoint elliptic operator acting on the space of L^2 p-forms. Its p-spectrum, $\mathrm{Spec}^p M$, consists of discrete eigenvalues $\{0=0=\ldots=0<\lambda_{p,1}\leqq\lambda_{p,2}\leqq\ldots\leqq\lambda_{p,n}\leqq\ldots\}$. The multiplicity, $n_{p,0}$, of the zero eigenvalue in $\mathrm{Spec}^p M$ are given by the p-th Betti number b_p.

Recent development shows that the non-zero eigenvalues also contain substantial geometric and analytic information. For instance, the first eigenvalue $\lambda_{0,1}$ for functions (0-forms) is given by the well-known Poincaré inequality

$$\lambda_{0,1} \inf_{a\in\mathbb{R}} \|f-a\|_2^2 \leqq \|\nabla f\|_2^2 \tag{0.2}$$

for all C^1 functions defined on M, where $\|\ \|_2$ denotes the L^2 norm. Equality holds iff f is the first eigenfunction satisfying

$$\Delta f = \lambda_{0,1} f. \tag{0.3}$$

Another important analytic inequality is the Sobolev inequality

$$C_0 \inf_{a\in\mathbb{R}} \|f-a\|_{m/(m-1)}^m \leqq \|\nabla f\|_1^m, \tag{0.4}$$

for all functions in $H_{1,1}$, the Sobolev space of functions which has L^1 derivatives. The best constant C_0 such that (0.4) holds is known as the Sobolev constant.

One of the important geometric inequalities is the isoperimetric inequality

$$C_1 (\min\{V(M_1), V(M_2)\})^{m-1} \leqq (A(N))^m, \tag{0.5}$$

where N is any codimension-1 submanifold which divides M into M_1 and M_2, with $A(N)$ = the (m-1)-dimensional measure of N and $V(M_i)$ = the m-dimensional measure of $M_i (i=1,2)$.

It is known that [1] the Sobolev inequality is equivalent to the isoperimetric inequality, namely,

$$2C_1 \geqq C_0 \geqq C_1. \tag{0.6}$$

Recently C. Croke [4] showed that C_1 has a lower bound depending on: lower bound of the Ricci curvature, $(m-1)K$; upper bound of diameter, d; lower bound of volume, V; and the dimension, m, of M. The purpose of this paper is to give estimates for the eigenvalues of $\mathrm{Spec}^p M$ from below in terms of C_0 and V. Hence combining with (0.6) and the result of Croke, one can estimate $\lambda_{p,n}$ from below in terms of d, K, m, and V. It is interesting to point out that Cheeger [2] gave a lower bound of $\lambda_{0,1}$ in terms of an isoperimetric constant h, which is different from C_1. Yau [8] later showed that this isoperimetric inequality is equivalent to an L^1-type Poincaré inequality. In the same paper he also gave a lower bound of h in terms of d, V, K and m, which provided a lower bound of $\lambda_{0,1}$.

For completeness sake, some known results concerning the Sobolev inequality will be proved. In particular, another version of the Sobolev inequality which is more suitable for our situation will be derived. We will also establish an elementary inequality between $\lambda_{0,1}$ and C_0.

Section 2 will be devoted to obtaining upper bounds for the multiplicities $n_p(\lambda)$ of all eigenvalues $\lambda \in \mathrm{Spec}^p M$. In the process, we will derive estimates on the supremum norms of the eigen-p-forms. As a corollary, upper bounds for the Betti numbers can be obtained in terms of the lower bound of the curvature operator, volume, and the Sobolev constant C_2 (*see* Lemma 1 for definition of C_2). In particular, when M is a manifold with constant curvature -1, b_p has an upper bound depending on V alone provided $m \geqq 4$.

Finally, we will estimate from above the sumpremum norm of any differential forms which are linear combinations of the first n-th eigen-p-forms. This enables us to obtain lower bounds for $\lambda_{p,n}$ in terms of V, d, lower bound of the curvature operator, and n.

Throughout this paper, all manifolds are assumed to be compact oriented C^∞ Riemannian manifolds, and $D(m)$ denotes a constant depending only on m. The results on functions can be generalized to non-oriented Riemannian manifolds by lifting to their two-folded coverings.

The author is greatly indebted to S. T. Yau for his interest in this work, without whose useful suggestions this paper would not be in the present form.

1. Sobolev inequalities

For our purposes, we will derive a weaker version of the Sobolev inequality for manifolds with dimension greater than or equal to 3.

LEMMA 1. — *Let f be a function in $H_{1,2}(M)$ satisfying $\int_M (\operatorname{sgn} f)\,|f|^{2/(m-2)}=0$. If $C_2=D(m)\,C_0^{2/m}$ and $m\geqq 3$, then*

$$C_2\,\|f\|^2_{2m/(m-2)}\leqq \|\nabla f\|_2^2.$$

Proof. — Consider the function

$$g=(\operatorname{sgn} f)\,|f|^{2(m-1)/(m-2)}.$$

The fact that $\int_M (\operatorname{sgn} f)\,|f|^{2/(m-2)}=0$ implies that g satisfied

$$\int (\operatorname{sgn} g)\,|g|^{1/(m-1)}=0. \tag{1.1}$$

This means

$$\|g\|_{m/(m-1)}=\inf_{a\in\mathbb{R}}\|g-a\|_{m/(m-1)}. \tag{1.2}$$

Hence by (0.4):

$$C_0\,\|f\|^{2m(m-1)/(m-2)}_{2m/(m-2)}=C_0\,\|g\|^m_{m/(m-1)}\leqq \|\nabla g\|_1^m$$
$$=\left\|\frac{2(m-1)}{m-2}f^{m/(m-2)}\nabla f\right\|_1^m\leqq D(m)\,\|f\|^{(m-2)/4}_{2m/(m-2)}\,\|\nabla f\|_2^m.$$

Dividing both sides by $\|f\|^{(m-2)/4}_{2m/(m-2)}$, we have

$$C_0\,\|f\|^m_{2m/(m-2)}\leqq D(m)\,\|\nabla f\|_2^m,$$

and the Lemma follows.

LEMMA 2. — *For all $f\in H_{1,2}(M)$ with $m\geqq 3$,*

$$\|\nabla f\|_2^2\geqq D(m)\,C_2\,[\,\|f\|^2_{2m/(m-2)}-V^{-2/m}\,\|f\|_2^2].$$

Proof. — Let $f\in H_{1,2}(M)$, consider $k\in\mathbb{R}$ such that

$$\int \operatorname{sgn}(f-k)\,|f-k|^{2/(m-2)}=0.$$

By Lemma 1,

$$(1.3)\quad \|\nabla f\|_2^2 \geqq C_2 \|f-k\|^2_{2m/(m-2)} \geqq C_2 [2^{-((m+2)/(m-2))} \|f\|^{2m/(m-2)}_{2m/(m-2)} - V k^{2m/(m-2)}]^{(m-2)/m}$$
$$\geqq C_2 [2^{-((m+2)/(m))} \|f\|^2_{2m/(m-2)} - V^{(m-2)/m} k^2].$$

However, if

$$M_+ = \{ x \in M : (f-k)(x) > 0 \}$$

and

$$M_- = \{ x \in M : (f-k)(x) < 0 \},$$

then

$$\int \operatorname{sgn}(f-k)\, |f-k|^{2/(m-2)} = 0$$

implies

$$(1.4)\quad \int_{M_+} (f-k)^{2/(m-2)} = \int_{M_-} (k-f)^{2/(m-2)}.$$

But

$$\int_{M_+} (f-k)^{2/(m-2)} \leqq 2^{\alpha} \int_{M_+} f^{2/(m-2)} - V_+ k^{2/(m-2)},$$

where

$$\alpha = \begin{cases} 0 & \text{if } m=3, \\ \dfrac{m-4}{m-2} & \text{if } m \geqq 4, \end{cases}$$

also

$$\int_{M_-} (k-f)^{2/(m-2)} \geqq 2^{\beta} V_- k^{2/(m-2)} - \int_{M_-} |f|^{2/(m-2)},$$

where

$$\beta = \begin{cases} \dfrac{m-4}{m-2} & \text{if } m=3, \\ 0 & \text{if } m \geqq 4. \end{cases}$$

Hence combining with (1.4):

$$2^{\alpha} \int_{M_+} f^{2/(m-2)} - V_+ k^{2/(m-2)} \geqq 2^{\beta} V_- k^{2/(m-2)} - \int_{M_-} |f|^{2/(m-2)}.$$

This gives

$$\int_{M_-} |f|^{2/(m-2)} + 2^{\alpha} \int_{M_+} |f|^{2/(m-2)} \geqq 2^{\beta} V_- k^{2/(m-2)} + V_+ k^{2/(m-2)} = (2^{\beta} V_- + V_+) k^{2/(m-2)}$$

Hence

$$D(m) \|f\|_{2/(m-2)}^{(m-2)/2} \geqq V k^{2/(m-2)}.$$

Applying Hölder inequality we have

$$D(m) \|f\|_2^2 \geqq V k^2. \tag{1.5}$$

Together with (1.3) yields the Lemma.

A lower bound of $\lambda_{0,1}$ can be obtained rather easily by substituting the first eigenfunction into the Sobolev inequality.

PROPOSITION 3. — *If* M *is a compact manifold, then*

$$\lambda_{0,1} \geqq D(m) (C_0 V^{-1})^{2/m}.$$

Proof. — Let f be the first eigenfunction satisfying (0.3), then $\int_M f = 0$. For the case $m=2$, 3, or 4, we consider the function

$$g = (\operatorname{sgn} f) |f|^{m-1}.$$

Clearly

$$\int (\operatorname{sgn} g) |g|^{1/(m-1)} = 0,$$

hence

$$(m-1)^m \|f^{m-2} \nabla f\|_1^m = \|\nabla g\|_1^m \geqq C_0 \|g\|_{m/(m-1)}^m = C_0 \|f\|_m^{m(m-1)}. \tag{1.6}$$

However

$$\|f^{m-2} \nabla f\|_1^m \leqq \|f\|_{2(m-2)}^{m(m-2)} \|\nabla f\|_2^m \leqq \|f\|_m^{m(m-2)} V^{(4-m)/2} \lambda_{0,1}^{m/2} \|f\|_2^m \leqq \lambda_{0,1}^{m/2} V \|f\|_m^{m(m-1)}.$$

Therefore $\lambda_{0,1} \geqq (C_0 V^{-1})^{2/m} (m-1)^{-2}$ as claimed. When $m \geqq 5$, let $g = (\operatorname{sgn} f) |f|^{(m-2)/2}$. Then g satisfies

$$\int (\operatorname{sgn} g) |g|^{2/(m-2)} = 0,$$

hence by Lemma 1:

$$\|\nabla g\|_2^2 \geqq C_2 \|g\|_{2m/(m-2)}^2 = C_2 \|f\|_m^{m-2}. \tag{1.7}$$

However

$$\| \nabla g \|_2^2 = \int_{M^+} |\nabla (f^{(m-2)/2})|^2 + \int_{M^-} |\nabla (|f|^{(m-2)/2})|^2, \tag{1.8}$$

where M^+ and M^- correspond to parts of M such that f takes positive and negative values, respectively. By regularity of ∂M^+ and ∂M^- [3], and the fact that $f^{(m-2)/2}|_{\partial M^+}=0$ and $|f^{(m-2)/2}|_{\partial M^-}=0$, we have

$$\| \nabla g \|^2 = \int_{M^+} f^{(m-2)/2} \Delta (f^{(m-2)/2}) + \int_{M^-} |f|^{(m-2)/2} \Delta (|f|^{(m-2)/2}).$$

Computation shows that

$$\begin{aligned} \Delta (f^{(m-2)/2}) &= -\left(\frac{m-2}{2}\right)\left(\frac{m-4}{2}\right) f^{(m-6)/2} |\nabla f|^2 + \left(\frac{m-2}{2}\right) f^{(m-4)/2} \Delta f \\ &= \left(\frac{m-2}{2}\right)\left[-\left(\frac{m-4}{2}\right) f^{(m-6)/2} |\nabla f|^2 + \lambda_{0,1} f^{(m-2)/2}\right]. \end{aligned} \tag{1.9}$$

Hence (1.8) and (1.9) together yield

$$\| \nabla g \|^2 = \left(\frac{m-2}{2}\right)^2 \left(\frac{1}{m-3}\right) \lambda_{0,1} \| f \|_{m-2}^{m-2}.$$

Substituting into (1.7), we have

$$\lambda_{0,1} D(m) \| f \|_{m-2}^{m-2} \geqq C_2 \| f \|_m^{m-2}.$$

Hölder inequality implies

$$\lambda_{0,1} D(m) V^{2/m} \geqq C_2.$$

The Proposition follows from the definition of C_2.

In the case when $m=2$, Lemmas 1 and 2 are not valid, hence we need the following:

LEMMA 4. — *Let* M *be a compact surface. If* $f \in H_{1,2}(M)$, *then*

$$\| \nabla f \|_2^2 \geqq D(2) C_0 [V^{-1/2} \| f \|_4^2 - V^{-1} \| f \|_2^2].$$

Proof. — Let g be defined by

$$g(x) = |f| f(x).$$

Then (0.4) gives

$$\| \nabla g \|_1^2 \geqq C_0 \inf_{a \in \mathbb{R}} \| g - a \|_2^2 = C_0 \| g - \alpha \|_2^2, \tag{1.10}$$

where $\alpha = V^{-1} \cdot \int g$. By definition of g, we have

$$(1.11)\quad \| g-\alpha \|_2^2 = \int (f|f|-\alpha)^2 = \| f \|_4^4 - 2\alpha \int f|f| + \alpha^2 V$$
$$= \| f \|_4^4 - \alpha^2 V \geqq \| f \|_4^4 - \| f \|_2^4 V^{-1}.$$

On the other hand

$$(1.12)\qquad \| \nabla g \|_1^2 = 4 \| f \nabla f \|_1^2 \leqq 4 \| f \|_2^2 \| \nabla f \|_2^2.$$

(1.10), (1.11), and (1.12) then give

$$(1.13)\qquad D(2) \| f \|_2^2 \| \nabla f \|_2^2 + C_0 \| f \|_2^4 V^{-1} \geqq C_0 \| f \|_4^4.$$

But

$$\| f \|_2^2 \leqq \| f \|_4^2 V^{1/2},$$

therefore

$$D(2) V^{1/2} \| \nabla f \|_2^2 + C_0 V^{-1/2} \| f \|_2^2 \geqq C_0 \| f \|_4^2$$

and the Lemma follows as claimed.

When M is a compact manifold with boundary the Sobolev inequality

$$(1.14)\qquad \| \nabla f \|_1^m \geqq \tilde{C}_0 \| f \|_{m/(m-1)}^m$$

is valid for all $f \in H_{1,1}$ satisfying $f|_{\partial M} = 0$. As in the case of manifolds without boundary (1.14) is equivalent to the isoperimetric inequality ([1], [7]):

$$(1.15)\qquad (A(N))^m \geqq \tilde{C}_1 (V(M_1))^{m-1}$$

for all codimension-1 submanifolds N dividing M, and M_1 is the part of M such that $M_1 \cap \partial M = \emptyset$. In fact, $\tilde{C}_0 = \tilde{C}_1$. Following the proofs of Lemmas 1, 4, and Proposition 3, one can prove the following Lemmas:

LEMMA 5. — *Let* M *be a compact manifold with boundary. If* $f \in H_{1,2}(M)$ *satisfying* $f|_{\partial M} = 0$ *then*

$$\| \nabla f \|_2^2 \geqq \tilde{C}_2 \| f \|_{2m/(m-2)}^2$$

when $m \geqq 3$, *with* $\tilde{C}_2 = \tilde{C}_0^{2/m} D(m)$, *and*

$$\| \nabla f \|_2^2 \geqq D(2) \tilde{C}_0 [V^{-1/2} \| f \|_4^2]$$

when $m = 2$.

PROPOSITION 6. — *Let $\lambda_{0,1}$ be the first eigenvalue for functions with the Dirichlet boundary condition. Then*

$$\lambda_{0,1} \geqq D(m)\left(\frac{C_0}{V}\right)^{2/m}.$$

Remark. — Sobolev inequalities similar to those of Lemmas 2 and 4 can be derived for functions that do not necessarily satisfy $f|_{\partial M}=0$.

2. L^∞ estimate and multiplicities of $\lambda_{p,n}$

Let M^m be a compact oriented Riemannian manifold. By duality $\text{Spec}^p M = \text{Spec}^{m-p} M$, moreover if w is an eigen-p-form satisfying

$$(2.1) \qquad \Delta w = \lambda w,$$

then $\star w$ is an eigen-$(m-p)$-form satisfying

$$(2.2) \qquad \Delta(\star w) = \lambda(\star w).$$

Hence the task of studying the spectrum is reduced to the study of $\text{Spec}^p M$, for $0 \leqq p \leqq [m/2]$ ($[m/2]$ = largest integer less than $m/2$). From here on p is assumed to be in the above range.

We define K_p by

$$K_p = \begin{cases} \text{lower bound of the curvature operator on M, for } p>1, \\ (m-1)^{-1} \times (\text{lower bound of the Ricci curvature}), \text{ for } p=1, \\ 0, \text{ when } p=0. \end{cases}$$

THEOREM 7. — *Let M be a compact manifold with $m \geqq 3$. Suppose w satisfies* (2.1), *and if $\lambda - p(m-p)K_p \neq 0$, then*

$$D(m)[C_2^{-1}(\lambda - p(m-p)K_p)]^{m/2} \exp\left[\frac{D(m)}{C_2^{-1}V^{2/m}(\lambda - p(m-p)K_p)}\right] \| w \|_2^2 \geqq \| w \|_\infty^2.$$

Remark. — By [5], p. 270, $\lambda = p(m-p)K_p$ only when $\lambda = 0 = K_p$. In which case, w has constant length, hence

$$\| w \|_2^2 = V \| w \|_\infty^2.$$

When $K_p < 0$ and $\lambda = 0$, Theorem 7 gives supremum norm estimates for harmonic p-forms.

Before we attempt to prove the Theorem, we will show the following Lemma:

LEMMA 8. — *Suppose w is an eigen-p-form satisfying* (2.1). *Then*

$$|w| \Delta |w| \leqq (\lambda - p(m-p)K_p)|w|^2.$$

Proof. — By Bochner's formula

$$\frac{1}{2}\Delta|w|^2=(\Delta w, w)-|\nabla w|^2-F(w), \tag{2.3}$$

where $F(w)$ is a function defined on M involving w and the curvature tensor. It is known that [5], p. 264:

$$F(w)\geqq p(m-p)K_p|w|^2. \tag{2.4}$$

Since

$$\Delta|w|^2=2|w|\Delta|w|-2|\nabla|w||^2,$$

(2.3) becomes

$$|w|\Delta|w|-|\nabla|w||^2\leqq(\Delta w, w)-|\nabla w|^2-p(m-p)K_p|w|^2. \tag{2.5}$$

Now we claim that $|\nabla|w||^2\leqq|\nabla w|^2$. Indeed, if one chooses orthonormal coframe $w_1, \ldots, w_m$ at a point with $w=\sum_{|I|=p} a_I w_I$, then

$$|w|=(\sum_{|I|=p} a_I^2)^{1/2}.$$

If $a_{I,j}$'s denote the covariant derivatives of a_I, then

$$|\nabla|w||^2=\sum_j\left(\sum_{|I|=p}\frac{a_{I,j}a_I}{|w|}\right)^2\leqq\frac{1}{|w|^2}\sum_j(\sum_{|I|=p}a_{I,j}^2)(\sum_{|I|=p}a_I^2)=\sum_j\sum_{|I|=p}a_{I,j}^2=|\nabla w|^2.$$

The Lemma follows by substituting this inequality into (2.5).

Proof of Theorem 7. — Define $f=|w|$. For $k>1/2$, Lemma 9 shows

$$\int f^{2k-1}\Delta f\leqq(\lambda-p(m-p)K_p)\|f\|_{2k}^{2k}. \tag{2.6}$$

On the other hand

$$\int f^{2k-1}\Delta f=(2k-1)\int(f^{2k-2}\nabla f, \nabla f)=\frac{2k-1}{k^2}\|\nabla f^k\|_2^2. \tag{2.7}$$

Applying Lemma 2 to the function f^k, (2.6) and (2.7) yield

$$D(m)\left[C_2^{-1}(\lambda-p(m-p)K_p)\left(\frac{k^2}{2k-1}\right)+V^{2/m}\right]\|f\|_{2k}^{2k}\geqq\|f\|_{2mk/(m-2)}^{2k}. \tag{2.8}$$

Denote $\beta=m/(m-2)$ and let $k=\beta^i$, $i=0, 1, 2, \ldots$ Equation (2.8) can be written in the form

$$(2.9) \quad \left\{D(m)\left[C_2^{-1} V^{2/m}(\lambda-p(m-p)K_p)\frac{\beta^{2i}}{2\beta^i-1}+1\right]\right\}^{1/2\beta^i} \times \| f \|_{2\beta^i} V^{-1/2\beta^i} \geqq \| f \|_{2\beta^{i+1}} V^{-1/2\beta^{i+1}}.$$

Since $\lim_{i\to\infty} \| f \|_{2\beta^{i+1}} V^{-1/2\beta^{i+1}} = \| f \|_\infty$, by iteration of (2.9), we have

$$(2.10) \quad \prod_{i=0}^{\infty}\left\{D(m)\left[C_2^{-1} V^{2/m}(\lambda-p(m-p)K_p)\frac{\beta^{2i}}{2\beta^i-1}+1\right]\right\}^{1/\beta^i} \| f \|_2^2 V^{-1} \geqq \| f \|_\infty^2.$$

The Theorem follows from the appendix.

By utilizing Lemmas 4 and 5 instead, the above proof gives the following Theorems.

THEOREM 9. — *Let M be a 2-dimensional compact surface. Suppose f is a function defined on M satisfying*

$$(2.11) \quad \Delta f=\lambda f, \qquad \lambda\neq 0,$$

then

$$D(2)\left(\frac{\lambda}{C_0}\right)^2 V \| f \|_2^2 \geqq \| f \|_\infty^2.$$

THEOREM 10. — *Let M be a compact manifold with boundary. Suppose f is a function on M satisfying*

$$(2.12) \quad \Delta f=\lambda f$$

and

$$f|_{\partial M}=0$$

then

$$D(m)(\lambda \tilde{C}_2^{-1})^{m/2} \| f \|_2^2 \geqq \| f \|_\infty^2 \quad \textit{if} \quad m\geqq 3$$

and

$$D(2)V(\lambda \tilde{C}_0^{-1})^2 \| f \|_2^2 \geqq \| f \|_\infty^2 \quad \textit{if} \quad m=2.$$

Remark. — Similar estimates for eigen-p-forms on a compact manifold with boundary satisfying absolute or relative boundary conditions can be obtained. In which case, we utilize the Sobolev inequalities for non-compact support which was mentioned at the end of section 1.

The next Lemma enables us to derive upper bounds for the multiplicities of eigenvalues.

LEMMA 11. — *Let E be a finite dimensional subspace of the space of L^2 p-forms on M. Then there exists $w\in E$ such that*

$$\frac{\dim E}{V} \| w \|_2^2 \leqq \| w \|_\infty^2 \min\left\{\binom{m}{p}, \dim E\right\}.$$

Proof. — Let $\{w_i\}_{i=1}^r$, $r=\dim E$, be an orthonormal basis of E. Define the function

$$F(x)=\sum_{i=1}^{r}|w_i(x)|^2, \qquad x\in M. \tag{2.13}$$

Clearly $F(x)$ is well-defined under orthogonal change of basis. Since $E\neq\{0\}$, $\|F\|_\infty\neq 0$. Let $x_0\in M$ such that $F(x_0)=\|F\|_\infty$. Define the subspace E_0 of E by

$$E_0=\{w\in E \mid w(x_0)=0\}.$$

By the choice of x_0, $E_0\neq E$. We claim that the orthogonal complement $E_0^\perp$ of E_0 is of at most dimension $\binom{m}{p}$. In fact, if $\{w_\alpha\}_{\alpha=1}^s$ form an orthonormal basis for $E_0^\perp$ with $s>\binom{m}{p}$, then there exists $(a_\alpha)_{\alpha=1}^s\in R^s$ with $(a_\alpha)\neq 0$ such that $\sum_{\alpha=1}^{s}a_\alpha w_\alpha(x_0)=0$. This is true because the dimension of the vector space of antisymmetric p-tensors on an m-dimensional vector space is $\binom{m}{p}$. However this implies $\sum_{\alpha=1}^{s}a_\alpha w_\alpha\in E_0$, which is a contradition.

Now we choose orthonormal basis for E such that $\{w\}_{\alpha=1}^s$ form an orthonormal basis for $E_0^\perp$ and $\{w_i\}_{i=s+1}^r$ an orthonormal basis for E_0. Then

$$\dim E=\int F(x)\leqq\|F\|_\infty V=F(x_0)V \;=V\sum_{\alpha=1}^{s}|w_\alpha(x_0)|^2\leqq\binom{m}{p}V\max_\alpha\|w_\alpha\|_\infty^2.$$

Since $\|w\|_2^2\leqq V\|w\|_\infty^2$ for all $w\in E$, the Lemma follows.

THEOREM 12. — *Let λ be an eigenvalue for p-forms on a compact Riemannian manifold M^m with $m\geqq 3$. Suppose n_λ denotes the multiplicity of λ, and if $\lambda-p(m-p)K_p\neq 0$, then either*

$$n_\lambda\leqq\binom{m}{p}$$

or

$$n_\lambda\leqq\binom{m}{p}D(m)V(C_2^{-1}(\lambda-p(m-p)K_p))^{m/2}\exp\left[\frac{D(m)}{C_2^{-1}V^{2/m}(\lambda-p(m-p)K_p)}\right].$$

When $p=0$, then

$$n_\lambda\leqq D(m)V(C_2^{-1}\lambda)^{m/2}.$$

Proof. — This follows from Proposition 3, Theorem 7 and Corollary 11.

Similarly, we obtain the following Corollaries.

COROLLARY 13. — *Let λ be an eigenvalue for functions on a compact surface M. If n_λ is the multiplicity of λ, then $n_\lambda\leqq D(2)(\lambda C_0^{-1}V)^2$.*

COROLLARY 14. — *Let* λ *be an eigenvalue for functions satisfying* $f|_{\partial M}=0$ *on a compact manifold with boundary. If* n_λ *is the multiplicity of* λ, *then:*

(i) $$n_\lambda \leqq D(m)\,V(\lambda\,\tilde{C}_2^{-1})^{m/2} \quad \text{if} \quad m \geqq 3,$$

(ii) $$n_\lambda \leqq D(2)(\lambda\,V\tilde{C}_0^{-1})^2 \quad \text{if} \quad m=2.$$

If $K_p \geqq 0$, and there exists a point on M such that the curvature operator (Ricci curvature, if $p=1$) is positive, then it is known that [5] $b_p=0$. When the curvature operator is identically zero, the manifold is flat, hence is covered by a flat torus. Therefore

$$b_p \leqq \text{Betti number of the torus} = \binom{m}{p}.$$

Suppose the Ricci curvature of M is identically zero then by Bochner's formula, any harmonic 1-form is parallel, hence with constant length. This means

$$\|w\|_2^2 = V\,\|w\|_\infty^2.$$

Comining with Lemma 11, we get

$$b_p \leqq \min\{m,\, b_p\}.$$

Therefore

$$b_p \leqq m.$$

THEOREM 15. — *Let* M *be a compact manifold. Assume* $K_p<0$, *then*

$$b_p \leqq \binom{m}{p} D(m)\,V(-C_2^{-1}\,p(m-p)\,K_p)^{m/2} \exp\left[\frac{D(m)\,C_2}{-p(m-p)\,K_p\,V^{2/m}}\right].$$

COROLLARY 16. — *Let* M *be a compact manifold of dimension* $m \geqq 4$. *Suppose the curvature of* M *is identically* -1, *then*

$$b_p \leqq Const.(m,\, p,\, V),$$

where $Const.(m, p, V)$ *is a constant depending only on* m, p, *and* V.

Proof. — From Theorem 15, we have

$$b_p \leqq D(m,\, p)\,VC_2^{-m/2} \exp\left[\frac{D(m,\, p)\,C_2}{V^{2/m}} + m - 1\right].$$

However, since C_2 can be estimated from below in terms of Ricci curvature, volume, and upper bound of diameter (*see* [4]), clearly the Corollary follows if one can estimate d from above by V. In fact, a Theorem of Gromov [6] showed

$$w(m)(1+d) \leqq V.$$

3. Lower bounds for $\lambda_{p,n}$

In view of Theorem 11, if one takes E to be the vector space spanned by the set of eigen-p-forms with eigenvalue $\lambda_{p,i} \leqq \lambda_{p,n}$, then in order to get a lower bound for $\lambda_{p,n}$ one needs a similar estimate as in Theorem 7, for any $w \in E$. The main theme of this section is to deal with this question. Since in general we cannot conclude that

$$\int |w|^{2k-2}(w, \Delta w) \leqq \lambda_{p,n} \| w \|_{2k}^{2k}$$

for $w \in E$, the proof of Theorem 7 will not carry through directly. To get around this difficulty we need the following Lemma:

LEMMA 17. — *Let $\{w_i\}_{i=1}^{n}$ be a set of linearly independent p-forms defined on* M. *If $q \geqq 2$ and $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ are positive real numbers, then there exists a subset $\{\alpha\} \subset \{1, 2, \ldots, n\}$ such that*

$$\left\| \sum_{i=1}^{n} \lambda_i w_i \right\|_q \leqq \left\| \sum_{\alpha} \lambda_n w_\alpha \right\|_q.$$

Proof. — Define the function

$$F(\lambda_j) = \left\| \sum_{i=1}^{n} \lambda_i w_i \right\|_q^q \qquad \text{for} \quad 1 \leqq j \leqq n.$$

Differentiating F twice with respect to λ_j, we have

$$F''(\lambda_j) = q(q-2) \int_M \left| \sum_{i=1}^{n} \lambda_i w_i \right|^{q-4} \left(w_j, \sum_{i=1}^{n} \lambda_i w_i \right)^2 + q \int_M \left| \sum_{i=1}^{n} \lambda_i w_i \right|^{q-2} |w_j|^2.$$

For $q \geqq 2$, this is a positive function. Hence F is a concave function of λ_j, which attains its maximum on the interval $[0, \lambda_n]$ at either 0 or λ_n. Inductively, one replaces the maximum point (0 or λ_n) for each λ_j, $1 \leqq j \leqq n$, and we obtain

$$\left\| \sum_{i=1}^{n} \lambda_i w_i \right\|_q^q \leqq \left\| \sum_{\alpha} \lambda_n w_\alpha \right\|_q^q.$$

Clearly, the set $\{\alpha\}$ is non empty.

THEOREM 18. — *Let* E *be the finite dimensional space spanned by the first n-th orthonormal eigenfunctions on a compact manifold* M *of dimension $m \geqq 3$. If $f \in E = \langle \varphi_i \rangle_{i=1}^{n}$, then*

$$\|f\|_\infty^2 \leqq D(m) \, [C_2^{-1} \lambda_{0,n}]^{m-1} \, V^{(m-2)/2} \exp\left[\frac{D(m) \, C_2}{V^{2/m} \lambda_{0,n}} \right] \|f\|_2^2.$$

Proof. — Let $f=\sum_{i=1}^{n} a_i \varphi_i$, $a_i \in \mathbb{R}$. Then as in Theorem 7, if $\beta=m/(m-2)$:

$$(3.1)\quad \int |f|^{2\beta^i-2} f\Delta f=\frac{2\beta^i-1}{\beta^{2i}}\, \|\nabla |f|^{\beta^i}\|_2^2 \geqq C_2\, \frac{2\beta^i-1}{\beta^{2i}}\, D(m)\, \|f\|_{2\beta^{i+1}}^{2\beta^i} - V^{-2/m}\|f\|_{2\beta^i}^{2\beta^i}.$$

Hence

$$(3.2)\quad \|f\|_{2\beta^{i+1}}^{2\beta^i} \leqq D(m)\left[C_2^{-1}\left(\frac{\beta^{2i}}{2\beta^{i}-1}\right)\int |f|^{2\beta^i-2} f\Delta f + V^{-2/m}\|f\|_{2\beta^i}^{2\beta^i}\right].$$

When $i=0$, we have

$$(3.3)\quad \|f\|_{2\beta}^2 \leqq D(m)\left[C_2^{-1}\int f\Delta f + V^{-2/m}\|f\|_2^2\right]$$
$$= D(m)\left[C_2^{-1}\sum_{i=1}^{n}\lambda_{0,i}a_i^2 + V^{-2/m}\|f\|_2^2\right] \leqq D(m)\,[C_2^{-1}\lambda_{0,n}+V^{-2/m}]\,\|f\|_2^2.$$

Therefore

$$(3.4)\quad \|f\|_{2\beta} V^{-1/2\beta} \leqq D(m)\,[C_2^{-1}\lambda_{0,n} V^{2/m}+1]^{1/2}\, \|f\|_2^2\, V^{-1/2}.$$

We claim that for $1 \leqq i < \infty$,

$$(3.5)\quad \|f\|_{2\beta^{i+1}} V^{-1/2\beta^{i+1}} \leqq D(m)\,[C_2^{-1}\lambda_{0,n}V^{2/m}+1]^{1/2}$$
$$\times \prod_{j=1}^{i}\left\{D(m)\left[C_2^{-1}\lambda_{0,n}V^{2/m}\,\frac{\beta^{2j}}{2\beta^j-1}+1\right]\right\}^{1/(2\beta^j-1)} \|f\|_2\, V^{-1/2}.$$

Assuming this is true for $i-1$, by induction, we need to show (3.5) for i. First consider the function $g \in E$ with the property that

$$(3.6)\quad \frac{\|g\|_{2\beta^{i+1}}}{\|g|_2} \geqq \frac{\|f\|_{2\beta^{i+1}}}{\|f\|_2} \quad \text{for all } f\in E.$$

By scaling, we may assume $\|g\|_2=1$. (3.2) then gives

$$(3.7)\quad \|g\|_{2\beta^{i+1}}^{2\beta^i} \leqq D(m)\left[C_2^{-1}\left(\frac{\beta^{2i}}{2\beta^i-1}\right)\int |g|^{2\beta^i-2} g\,\Delta g + V^{-2/m}\|g\|_{2\beta^i}^{2\beta^i}\right]$$
$$\leqq D(m)\left[C_2^{-1}\left(\frac{\beta^{2i}}{2\beta^i-1}\right)\|g\|_{2\beta^i}^{2\beta^i-1}\,\|\Delta g\|_{2\beta^i} + V^{-2/m}\|g\|_{2\beta^i}^{2\beta^i}\right].$$

However, if $g=\sum b_i \varphi_i$, then $\Delta g=\sum \lambda_{0,i} b_i \varphi_i$ and by Lemma 17, we have

$$(3.8)\quad \| \Delta g \|_{2\beta^i} \leqq \| \Delta g \|_{2\beta^{i+1}} V^{(\beta-1)/2\beta^{i+1}} = \left\| \sum_i \lambda_{0,i} b_i \varphi_i \right\|_{2\beta^{i+1}} V^{(\beta-1)/2\beta^{i+1}}$$

$$\leqq \left\| \sum_\alpha \lambda_{0,n} b_\alpha \varphi_\alpha \right\|_{2\beta^{i+1}} V^{(\beta-1)/2\beta^{i+1}} \leqq \lambda_{0,n} \| g \|_{2\beta^{i+1}} \left\| \sum_\alpha b_\alpha \varphi_\alpha \right\|_2 V^{(\beta-1)/2\beta^{i+1}} \quad \text{[by (3.6)]}$$

$$\leqq \lambda_{0,n} \| g \|_{2\beta^{i+1}}.$$

Therefore

$$(3.9)\quad \| g \|_{2\beta^{i+1}}^{2\beta^i} \leqq D(m) \left[C_2^{-1} \left(\frac{\beta^{2i}}{2\beta^i - 1} \right) \| g \|_{2\beta^{i-1}}^{2\beta^i} \lambda_{0,n} \right.$$

$$\left. \times \| g \|_{2\beta^{i+1}} V^{(\beta-1)/2\beta^{i+1}} + V^{-2/m} \| g \|_{2\beta^i}^{2\beta^i} \right]$$

$$\leqq D(m) \left[C_2^{-1} \lambda_{0,n} \left(\frac{\beta^{2i}}{2\beta^i - 1} \right) + V^{-2/m} \right] V^{(\beta-1)/2\beta^{i+1}} \| g \|_{2\beta^i}^{2\beta^{i-1}} \| g \|_{2\beta^{i+1}}.$$

This implies

$$(3.10)\quad \| g \|_{2\beta^{i+1}} V^{-1/2\beta^{i+1}} \leqq \left\{ D(m) \left[C_2^{-1} \lambda_{0,n} V^{2/m} \times \left(\frac{\beta^{2i}}{2\beta^i - 1} \right) + 1 \right] \right\}^{1/(2\beta^i - 1)} \| g \|_{2\beta^i} V^{-1/2\beta^i}.$$

Together with our hypothesis, we obtain

$$\| g \|_{2\beta^{i+1}} V^{-1/2\beta^{i+1}} \leqq D(m) \left[C_2^{-1} \lambda_{0,n} V^{2/m} + 1 \right]^{1/2}$$

$$\times \prod_{j=1}^{i} \left\{ D(m) \left[C_2^{-1} \lambda_{0,n} V^{2/m} \left(\frac{\beta^{2j}}{2\beta^j - 1} \right) + 1 \right] \right\}^{1/(2\beta^j - 1)} \| g \|_2 V^{-1/2}.$$

However, by (3.6), one can replace f in terms of g, and the claim is proved. By iteration of (3.5) and applying the appendix, the Theorem follows:

THEOREM 19. — *Let* E *be the finite dimensional space spanned by the first n-th orthonormal eigen-p-forms on* M *with* $m \geqq 3$. *If* $w \in E$, *and* $\lambda_{p,n} > \lambda_{p,0}$ *then*

$$\| w \|_\infty^2 \leqq D(m) [C_2^{-1} (\lambda_{p,n} - p(m-p) K_p)]^{m-1} V^{(m-2)/m}$$

$$\times \exp \left[\frac{D(m) C_2}{V^{2/m} (\lambda_{p,n} - p(m-p) K_p)} \right] \| w \|_2^2.$$

THEOREM 20. — *Let* M *be a compact surface, and* E *the finite dimensional space spanned by the first n-th orthonormal eigenfunctions with* $n > 1$. *if* $f \in E$, *then*

$$\| f \|_\infty^2 \leqq D(2)\, V (C_0^{-1} \lambda_{0,n})^2 \| f \|_2^2.$$

THEOREM 21. — *Let* M *be a compact manifold with boundary. Suppose* E *is the finite dimensional space spanned by the first n-th orthonormal eigenfunctions satisfying the Dirichlet boundary condition with* $n>1$. *If* $f \in E$, *then*

$$\|f\|_\infty^2 \leqq (\tilde{C}_2^{-1} \lambda_{0,n})^{m-1} V^{(m-2)/m} D(m) \|f\|_2^2$$

for $m>3$, *and*

$$\|f\|_\infty^2 \leqq D(2) (\tilde{C}_0^{-1} \lambda_{0,n})^2 V \|f\|_2^2 \qquad \textit{for} \quad m=2.$$

The proofs of Theorems 19-21 are essentially the same as in Theorem 18 and will be omitted.

COROLLARY 22. — *Let* M *be a compact manifold of dimension* $m \geqq 3$. *Suppose* $\lambda_{p,n}$ *is the n-th eigenvalue for p-forms. If* $\lambda_{p,n}>0$, *then*

$$n \leqq \binom{m}{p} [C_2^{-1} (\lambda_{p,n} - p(m-p) K_p)]^{m-1} \; V^{(2m-2)/m} D(m) \exp\left(\frac{C_2 D(m)}{V^{2/m} (\lambda_{p,n} - p(m-p) K_p)}\right).$$

When $p=0$, *we have*

$$n \leqq \lambda_{0,n}^{m-1} V^{2(m-1)/m} C_2^{1-m} D(m).$$

The proof follows from Proposition 3, and Theorems 11, 18, and 19. Similarly one can show the following Corollaries:

COROLLARY 23. — *Let* M *be a compact surface, then*

$$n \leqq D(2) (C_0^{-1} V \lambda_{0,n})^2.$$

COROLLARY 24. — *Let* M *be a compact manifold with boundary. Then*

$$n \leqq D(m) V^{2(m-1)/m} (\tilde{C}_2^{-1} \lambda_{0,n})^{m-1} \qquad \textit{if} \quad m \geqq 3$$

and

$$n \leqq D(2) (\tilde{C}_0^{-1} V \lambda_{0,n})^2 \qquad \textit{if} \quad m=2.$$

Remark 1. — In the case of differential p-forms, if $K_p>0$, it was shown ([5], p. 270) that

$$\lambda_{p,n} \geqq K_p \times \min \{ p(m-p+1), (p+1)(m-p) \}.$$

This implies

$$\lambda_{p,n} - p(m-p) K_p \geqq K_p \times \min \{ m-p, p \} = p K_p \qquad \text{since} \quad p \leqq \left[\frac{m}{2}\right].$$

Hence the first part of Corollary 22 takes the form

$$n \leqq \binom{m}{p} V^{2(m-1)/m} (C_2^{-1} \lambda_{p,n})^{m-1} \exp\left[\frac{C_2 D(m)}{V^{2/m} p K_p}\right] D(m).$$

When $K_p < 0$, the estimate of $\lambda_{p,n}$ becomes ineffective when n is small. It can be written as

$$C_2 n^{1/(m-1)} \exp\left[\frac{C_2 D(m)}{V^{2/m} p(m-p) K_p}\right] D(m, p) V^{-2/m} + p(m-p) K_p \leqq \lambda_{p,n}.$$

Remark 2. — In [4], C. Croke showed that

$$C_1 \geqq \frac{2^{m-1} \alpha(m-1)^m}{\alpha(m)^{m-1}} \left(\frac{V}{\int_0^d \sqrt{-K^{-1}} \sinh \sqrt{-K}\, r dr^{m-1}}\right)^{m+1}$$

Hence C_0 and $C_2 = C_0^{2/m} D(m)$ can be bounded from below by the following quantities: upper bound of diameter, lower bound of volume, lower bound of Ricci curvature and the dimension of M. When $m=2$, he also showed that $C_1 \geqq 8\delta^2/V$, where δ = injectivity radius of M.

APPENDIX

Let $\beta > 1$ and $\alpha > 0$, then

$$\text{(A)} \qquad \prod_{i=0}^{\infty} \left[\frac{\alpha\beta^{2i}}{2\beta^i - 1} + \gamma\right]^{1/\beta^i} \leqq \alpha^{\beta/(\beta-1)} \exp\left(\frac{\gamma\beta}{\alpha(\beta-1)} + \frac{1}{\beta^{1/2}-1}\right)$$

and

$$\text{(B)} \qquad \prod_{i=1}^{\infty} \left[\frac{\alpha\beta^{2i}}{2\beta^i - 1} + \gamma\right]^{1/(2\beta^i-1)} \leqq \alpha^{1/(\beta-1)} \exp\left(\frac{1}{\alpha(\beta-1)} + \frac{1}{\beta^{1/2}-1}\right)$$

Proof. — (A) Since for $i \geqq 1$:

$$\log\left[\frac{\alpha\beta^{2i}}{2\beta^i-1} + \gamma\right] \leqq \log \alpha + \log \frac{\beta^{2i}}{2\beta^i - 1} + \frac{\gamma}{\alpha} \leqq \log \alpha + \log \beta^i + \frac{\gamma}{\alpha} \leqq \log \alpha + \beta^{i/2} + \frac{\gamma}{\alpha},$$

hence

$$\prod_{i=0}^{\infty} \left[\frac{\alpha\beta^{2i}}{2\beta^i-1} + \gamma\right]^{1/\beta^i} = \exp\left[\sum_{i=0}^{\infty} \frac{1}{\beta^i} \log\left(\frac{\alpha\beta^{2i}}{2\beta^i-1} + \gamma\right)\right]$$

$$\leqq \exp\left[\left(\log \alpha + \frac{\gamma}{\alpha}\right) \sum_{i=0}^{\infty} \frac{1}{\beta^i} + \sum_{i=1}^{\infty} \frac{1}{\beta^{i/2}}\right]$$

$$= \exp\left[\frac{\beta}{\beta-1}\left(\log \alpha + \frac{\gamma}{\alpha}\right) + \frac{1}{\beta^{1/2}-1}\right] = \alpha^{\beta/(\beta-1)} \exp\left(\frac{\gamma\beta}{\alpha(\beta-1)} + \frac{1}{\beta^{1/2}-1}\right).$$

If $\beta=m/(m-2)$, we have

$$\prod_{i=0}^{\infty}\left[\frac{\alpha\beta^{2i}}{2\beta^i-1}+\gamma\right]^{1/\beta^i}\leqq\alpha^{m/2}\exp\left(\frac{\gamma m}{2\alpha}+m-1\right).$$

(B) First note that

$$\prod_{i=1}^{\infty}\alpha^{1/(2\beta^i-1)}=\exp\left(\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\log\alpha\right)$$

and

$$\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\leqq\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\qquad\text{since}\quad\beta>1=\frac{1}{\beta}\left(\sum_{i=1}^{\infty}\frac{1}{2\beta^{i-1}-1}\right)=\frac{1}{\beta}\left(1+\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\right).$$

Hence

$$\left(1-\frac{1}{\beta}\right)\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\leqq\frac{1}{\beta}.$$

This implies

$$\sum_{i=1}^{\infty}\frac{1}{2\beta^i-1}\leqq\frac{1}{\beta-1},$$

therefore

$$\prod_{i=1}^{\infty}\alpha^{1/(2\beta^i-1)}\leqq\alpha^{1/(\beta-1)}.$$

The proof of (B) follows similarly as in (A).

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