ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 13, nº 4 (1980), p. 419-435
<http://www.numdam.org/item?id=ASENS_1980_4_13_4_419_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 13, 1980, p. 419 à 435.

SOME ISOPERIMETRIC INEQUALITIES AND EIGENVALUE ESTIMATES

BY CHRISTOPHER B. CROKE (1)

Introduction

In this paper we first find sharp isoperimetric inequalities

$$\text{I.}\qquad \frac{\mathrm{Vol}\,(\partial \mathrm{M})}{\mathrm{Vol}\,(\mathrm{M})} \geqq \frac{2\,\pi\alpha\,(n-1)}{\alpha\,(n).\,\mathrm{D}}\,\tilde{\omega},$$

$$\text{II.}\qquad \frac{\mathrm{Vol}\,(\partial \mathrm{M})^n}{\mathrm{Vol}\,(\mathrm{M})^{n-1}} \geqq \frac{2^{n-1}\,\alpha\,(n-1)^n}{\alpha\,(n)^{n-1}}\,\tilde{\omega}^{n+1},$$

where M^n is a compact Riemannian manifold with boundary $\partial \mathrm{M}$ and diameter D, $\alpha\,(n)$ is the volume of the unit n-dimensional sphere, and $\tilde{\omega}$ is a constant depending on M. For a history of isoperimetric inequalities see the survey article of Osserman [11].

In general the constant $\tilde{\omega}$ is hard to compute, but in some interesting cases it can be estimated.

For example, we consider the following case. Let N^n be a compact manifold without boundary. Define the isoperimetric type constants

$$\mathrm{I}\,(\mathrm{N}) = \inf_{\mathrm{S}} \frac{\mathrm{Vol}\,(\mathrm{S})}{\min\{\mathrm{Vol}\,(\mathrm{M}_1),\ \mathrm{Vol}\,(\mathrm{M}_2)\}},$$

$$\Phi\,(\mathrm{N}) = \inf_{\mathrm{S}} \frac{[\mathrm{Vol}\,(\mathrm{S})]^n}{[\min\{\mathrm{Vol}\,(\mathrm{M}_1),\ \mathrm{Vol}\,(\mathrm{M}_2)\}]^{n-1}}$$

where S runs over codimension one submanifolds of N which divide N into two pieces M_1, and M_2.

In [6], p. 196, Cheeger shows that the first eigenvalue of the Laplacian of N, $\lambda_1\,(\mathrm{N})$, can be bounded below in terms of I (N). In [13], p. 504, Yau shows that I (N) [and hence $\lambda_1\,(\mathrm{N})$] can be bounded below by the diameter, volume, and Ricci curvature of N. In this paper we

(1) This research was supported in part by Grant #MCS 76-01692.

reproduce Yau's result, with a slightly better constant, and show that in the two dimensional case $I(N)$ can be bounded below by the volume and injectivity radius of N.

In [10], Peter Li uses $\Phi(N)$ to get lower bounds for the higher eigenvalues of the Laplacian, for forms as well as functions, and upper bounds on their multiplicities. In this paper we show that $\Phi(N)$ can also be bounded below by the volume, diameter, and Ricci curvature of N, while in the two dimensional case it can be bounded by the volume and injectivity radius of N.

Another case where one can estimate $\tilde{\omega}$ is where M is contained in a compact manifold N without boundary, and the diameter of M is less than the injectivity radius of N. In this case $\tilde{\omega}=1$, so the isoperimetric inequality II is in terms only of the dimension of M. As a consequence we show that the volume of a metric ball of radius r in N, where r is less than or equal to one half the injectivity radius of N, is bounded below by a constant times r^n, where the constant depends only on the dimension of N.

We next turn our attention to universal (i.e., curvature independent) upper and lower bounds on the first eigenvalue, λ_1, of the Dirichlet problem for the Laplacian.

We prove a sharp lower bound for $\lambda_1(M)$ where M is a sufficiently nice compact manifold with boundary. In particular, if M is contained in a compact manifold N without boundary, and the diameter D of M is less than the injectivity radius of N, then $\lambda_1(M) \geqq \lambda_1(S_D^+)$ where S_D^+ is a hemisphere of the constant curvature sphere of diameter D. Further equality holds if and only if M is isometric to S_D^+. Cheng [8] has independently shown a universal bound for such M; however, his bound is not sharp.

We then show that there is a constant $\gamma(n)$ depending only on n such that for every compact manifold N^n without boundary of convexity radius $c(N)$, for every $m \in N$ and every $r < c(N)$ we have

$$\lambda_1(B(m, r)) \leqq \frac{\gamma(n)\,\mathrm{Vol}(N)^2}{r^{2n+2}},$$

where $B(m, r)$ is the metric ball of radius r about m. This allows us to show

$$\lambda_1(N) \leqq \frac{\gamma(n)\,\mathrm{Vol}(N)^2}{c(N)^{2n+2}}.$$

The proof of this result borrows much from the proof in [3]. In [3] Berger shows that there is a constant $\bar{\gamma}(n)$ depending only on the dimension n of N such that for every r less than the injectivety radius of N there is a point $m \in N$ such that

$$\lambda_1(B(m, r)) \leqq \frac{\bar{\gamma}(n)\,\mathrm{Vol}(N)}{r^{n+2}}.$$

Using this he gets an upper bound for $\lambda_1(N)$ under the assumption that N admits a fixed point free involutive isometry.

I would like to thank Peter Li for bringing the isoperimetric problem to my attention.
I would also like to thank Berger and Kazdan, whose work is used extensively throughout this paper.

Notation and Definitions

Let $(M, \partial M, g)$ be a smooth compact manifold M with smooth boundary ∂M and Riemannian metric g.

Let $UM \xrightarrow{\pi} M$ represent the unit sphere bundle with the canonical measure. For $v \in UM$ let γ_v be the geodesic with $\gamma'_v(0)=v$, let $\zeta^t(v)$ represent the geodesic flow, i.e. $\zeta^t(v)=\gamma'_v(t)$. Let $l(v)$ be the smallest value of $t>0$ (possibly ∞) such that $\gamma_v(t) \in \partial M$. Note $\zeta^t(v)$ is defined for $t \leq l(v)$. Let $\tilde{l}(v)=\sup\{t \mid \gamma_v \text{ minimizes up to } t \text{ and } t \leqq l(u)\}$.

Now let the subsets $\tilde{U}M \subset \overline{UM} \subset UM$ be defined by

$$\overline{UM}=\{v \in UM \mid l(-v)<\infty\}, \qquad \tilde{U}M=\{v \in UM \mid \tilde{l}(-v)=l(-v)\}.$$

Let $\overline{U}_p=\pi|_{\overline{UM}}^{-1}(p)$ and $\tilde{U}_p=\pi|_{\tilde{U}M}^{-1}(p)$. Define $\overline{\omega}_p=m(\overline{U}_p)/m(U_p)$ and $\tilde{\omega}_p=m(\tilde{U}_p)/m(U_p)$ where m represents the canonical measure on the unit sphere. Also let $\overline{\omega}=\inf_{p \in M} \overline{\omega}_p$, $\tilde{\omega}=\inf_{p \in M} \tilde{\omega}_p$.

For $p \in \partial M$ let N_p be the inwardly pointing unit normal vector. Let $U^+ \partial M \to \partial M$ be the bundle of inwardly pointing unit vectors. That is

$$U^+ \partial M=\{u \in UM|_{\partial M} \mid \langle u, N_{\pi(u)}\rangle \geqq 0\}.$$

Let $U^+ \partial M$ have the local product measure, where the measure on the fibre is the measure from the upper unit hemisphere.

We will let $\alpha(n)$ represent the volume of the unit n-sphere.

1. An isoperimetric inequality of type I and some consequences

PROPOSITION 1. — *For* $(M, \partial M, g)$ *we have:*

(i) $$\int_{\overline{UM}} f(v)\,dv=\int_{U^+ \partial M} \int_0^{l(u)} f(\zeta^r(u))\langle u, N_{\pi(u)}\rangle\, dr\, du;$$

(ii) $$\int_{\tilde{U}M} f(v)\,dv=\int_{U^+ \partial M} \int_0^{\tilde{l}(u)} f(\zeta^r(u))\langle u, N_{\pi(u)}\rangle\, dr\, du.$$

Where f is any integrable function. In particular for $f \equiv 1$ we have:

(iii) $$\operatorname{Vol}(\overline{UM})=\int_{U^+ \partial M} l(u)\langle u, N_{\pi(u)}\rangle\, du;$$

(iv) $$\operatorname{Vol}(\tilde{U}M)=\int_{U^+ \partial M} \tilde{l}(u)\langle u, N_{\pi(u)}\rangle\, du.$$

This formula occurs in [12], pp. 336-338, and [1], p. 286.

COROLLARY 2 :

(i) $$\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)} \geqq \frac{C_1 \overline{\omega}}{l};$$

(ii) $$\frac{\mathrm{Vol}\,(\partial M)}{\mathrm{Vol}\,(M)} \geqq \frac{C_1 \tilde{\omega}}{D},$$

where $C_1 = 2\pi\alpha(n-1)/\alpha(n)$, $l = \sup_{v \in U^+ \partial M} \{ l(v) \}$ *and* D *is the diameter of* M.

Note. — The inequalities are both sharp when M in the upper hemisphere of a constant curvature sphere. In this case $\overline{\omega} = \tilde{\omega} = 1$ and $l = D =$ diameter of the sphere.

Proof. — $\overline{\omega}.\alpha(n-1)$. Vol $(M) \leqq \mathrm{Vol}(\overline{U}\,\overline{M})$. From the Proposition we get

$$\mathrm{Vol}\,(\overline{U}\,\overline{M}) = \int_{U^+ \partial M} l(u) \langle u, N_{\pi(u)} \rangle \, du \leqq l \int_{U^+ \partial M} \langle u, N_{\pi(u)} \rangle \, du = l\,K\,\mathrm{Vol}(\partial M).$$

Where K is the constant achieved by integrating over the fibre. To finish the Corollary one can compute K directly or note that equality must hold everywhere for M the upper hemisphere of a constant curvature sphere. (ii) is proved similarly. □

Remark. — In general (ii) is more interesting than (i) as l may be infinite.

For M a compact manifold without boundary, and S a codimension one submanifold dividing M into two pieces M_1 and M_2, we let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ represent the $\tilde{\omega}$ corresponding to the manifolds with boundary M_1 and M_2 respectively. For $p \in M_i$ let

$$O_p = \{ q \in M \mid q = \exp_p tu, \ -u \in \tilde{U}_p, \ t \leqq C(u) \},$$

where $C(u)$ represents the distance along γ_u to its cut point in M. Since M is complete

$$M - O_p \subset \{ q \in M \mid q = \exp tu, \ -u \notin \tilde{U}_p, \ t \leqq C(u) = \tilde{l}(u) \} \subset M_i.$$

Therefore $M_j \subset O_p$ for $j \neq i$. Thus by a standard comparison Theorem we have:

LEMMA 3. — *Let* M *be a compact Riemannian manifold without boundary, such that the Ricci curvature is bounded below by* $(n-1)K$. *Then if* S *is any* $n-1$ *dimensional submanifold dividing* M *into two pieces* M_1 *and* M_2 *we have*

$$\tilde{\omega}_i \geqq \frac{\mathrm{Vol}(M_j)}{\alpha(n-1) \int_0^D (\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)^{n-1}\, dr} \qquad (i \neq j).$$

In particular if Vol $(M_i) \leqq \mathrm{Vol}(M_j)$ *then*

$$\tilde{\omega}_i \geqq \frac{\mathrm{Vol}(M)}{2\alpha(n-1) \int_0^D (\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)^{n-1}\, dr}$$

where we use the convention that $(\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)$ *is interpreted as* r *if* $K=0$ *and as* $\sqrt{1/K}\ \sin (K r)$ *if* $K>0$. D *represents the diameter of* M.

Proof:

$$\operatorname{Vol}(M_j) \leqq \operatorname{Vol}(O_p) = \int_{-\tilde{U}_p} \int_0^{C(u)} F(u, r)\, dr\, du$$

$$\leqq \tilde{\omega}_p\, \alpha(n-1) \int_0^D (\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)^{n-1}\, dr.$$

(For the inequality $F(u,r) \leqq (\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)^{n-1}$, *see* [5], section 11.10) where

$$-\tilde{U}_p = \{ u \in U_p \mid -u \in \tilde{U}_p \} = \{ u \in U_p \mid l(u) = \tilde{l}(u) \},$$

and $F(u, r)$ is the volume form in normal polar coordinates. □

Now from Corollary 2 we have $\operatorname{Vol}(S)/\operatorname{Vol}(M_i) \geqq C_1 \tilde{\omega}_i / D$ (where C_1 is sharp). Thus using Lemma 3 we get

PROPOSITION 4:

$$I(M) \geqq \frac{\pi}{\alpha(n)} \frac{\operatorname{Vol}(M)}{D. \int_0^D (\sqrt{-1/K}\ \sinh \sqrt{-K}\, r)^{n-1}\, dr}$$

[I(M) was defined on page 1.]

Remark. — This Proposition was proved by Yau ([13], p. 504) with the constant $1/2\,\alpha(n-1)$, $\pi/\alpha(n) > 1/2\,\alpha(n-1)$. Neither constant is sharp as Lemma 3 (essentially used in both proofs) is not sharp.

THEOREM 5 (Yau). — *Let* M *be a compact n-dimensional Riemannian manifold whose Ricci curvature is bounded below by* $(n-1)K$. *Thus Proposition* 4 *holds. Since* $\lambda_1(M) \geqq I(M)^2/4$ *we find a lower bound of* λ_1 *in terms of* D, Vol(M), *and* K.

In some cases we are able to show that $\tilde{\omega}$ must be 1. For example let M be a compact manifold without boundary and let S be an $n-1$ dimensional submanifold dividing M into M_1, M_2 then we have:

LEMMA 6. — *If the maximum distance in* M *between any two points of* S *is less than the injectivity radius of* M, $i(M)$, *then* $\tilde{\omega}_i = 1$ *for* $i=1$ *or* 2.

Proof. — Let $p \in S$ then

$$S \subset B(p, i(M)) \equiv \{ q \in M \mid d(p, q) < i(M) \}.$$

Let M_i be the piece of M lying entirely inside $B(p, i(M))$. By continuity this choice is independent of the choice of p. Now for $x \in M_i$, $d(x, p) < i(M)$ for every $p \in S$, by the choice of M_i. Hence $S \subset B(x, i(M))$. Let M_j be the piece of M lying in $B(x, i(M))$. By continuity M_j is independent of x and hence must be M_i. Thus every geodesic from x minimizes up to S. Hence $\tilde{\omega}_i = 1$. □

If M is a compact manifold without boundary and $r < i(M)$, let

$$B(x, r) = \{ y \in M \mid d(x, y) \leqq r \} \qquad \text{and} \qquad S(x, r) = \partial B(x, r) = \{ y \in M \mid d(x, y) = r \}.$$

Then Lemma 6 and Corollary 2 give:

COROLLARY 7. — *For* $r < i(M)/2$:

$$\frac{\text{Vol}(S(x, r))}{\text{Vol}(B(x, r))} \geqq \frac{C_1}{2r} = \frac{\pi\alpha(n-1)}{r\alpha(n)}.$$

If M is a two dimensional compact manifold without boundary and S divides M into two pieces M_1, M_2 we can consider separately the cases where the length of $S \geqq 2i(M)$ and length of $S < 2i(M)$ to get:

COROLLARY 8. — *For* M *a compact 2-dimensional manifold*

$$I(M) \geqq \min\left\{ \frac{4i(M)}{\text{Vol}(M)}, \frac{C_1}{i(M)} \right\}.$$

Hence λ_1 can be bounded below by $i(M)$ *and* Vol (M).

2. An isoperimetric inequality of type II and consequences

To begin this section we introduce a Lemma essentially due to Berger and Kazdan ([4], Appendices D and E).

LEMMA 9. — *Let* M^n be a Riemannian manifold and $u \in UM$. *Then for every* $l \leqq C(u)$ (the distance to the cut locus in the direction u):

$$\int_{x=0}^{x=l} \int_{z=0}^{z=l-x} F(\zeta^x(u), z)\, dz\, dx \geqq C(n) \frac{l^{n+1}}{\pi^{n+1}},$$

where $C(n) = \pi\alpha(n)/2\alpha(n-1) = \pi^2/C_1$, *Further equality holds if and only if*

$$R(\gamma_u'(t), \,.\,)\gamma_u'(t) = (\pi/l)^2 \,\text{Id} \text{ for } 0 \leqq t \leqq l.$$

Here $F(v, z)$ *is the volume form in polar coordinates*

$$\left[\text{i.e.} \int_{U_p} \int_0^{C(v)} F(v, z)\, dz\, dv = \text{Vol}(M) \right],$$

R *is the curvature tensor and* γ_u *is the geodesic determined by* u.

This follows from a slight modification of the work of Berger ([4], Appendix D) (*see* Appendix).

PROPOSITION 10. — *For* (M, ∂M, *g*) *we have*

$$\operatorname{Vol}(M)^2 \geqq C_2 \int_{U^+\partial M} (\tilde{l}(v))^{n+1} \langle v, N_{\pi(v)} \rangle \, dv,$$

with $C_2 = \alpha(n)/2\pi^n \alpha(n-1)$. *Equality holds for the upper hemisphere of a constant curvature sphere.*

Proof:

$$\operatorname{Vol}(M)^2 \geqq \int_M \int_{U_p} \int_0^{\tilde{l}(u)} F(u, t)\, dt\, du\, dp = \int_{UM} \int_0^{\tilde{l}(u)} F(u, t)\, dt\, du$$

$$\geqq \int_{\tilde{U}M} \int_0^{\tilde{l}(u)} F(u, t)\, dt\, du \tag{8.1}$$

$$= \int_{U^+\partial M} \int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(\zeta^s(v))} F(\zeta^s(v), t) \langle v, N_{\pi(v)} \rangle \, dt\, ds\, dv$$

$$\geqq \int_{U^+\partial M} \left[\int_0^{\tilde{l}(v)} \int_0^{\tilde{l}(v)-s} F(\zeta^s(v), t)\, dt\, ds \right] \langle v, N_{\pi(v)} \rangle \, dv$$

$$\geqq \frac{C(n)}{\pi^{n+1}} \int_{U^+\partial M} (\tilde{l}(v))^{n+1} \langle v, N_{\pi(v)} \rangle \, dv. \tag{8.2}$$

The above follows from Proposition 1, Lemma 9, and the fact that $\tilde{l}(\zeta^s(v)) \geqq \tilde{l}(v) - s$. Equality holds for the upper hemisphere of a sphere at each stage. □

THEOREM 11. — *For* (M, ∂M, *g*) *we have the isoperimetric inequality:*

$$\frac{\operatorname{Vol}(\partial M)^n}{\operatorname{Vol}(M)^{n-1}} \geqq C_3 \tilde{\omega}^{n+1},$$

where $C_3 = 2^{n-1} \alpha(n-1)^n / \alpha(n)^{n-1}$.

Equality holds iff $\tilde{\omega} = 1$ *and* M *is the upper hemisphere of a constant curvature sphere.*

Proof. — From Proposition 10 and a Hölder inequality we have

$$\operatorname{Vol}(M)^2 \geqq C_2 \int_{U^+\partial M} (\tilde{l}(u))^{n+1} \langle u, N_{\pi(u)} \rangle \, du \geqq C_2 \frac{\left\{ \int_{U^+\partial M} \tilde{l}(u) \langle u, N_{\pi(u)} \rangle \, du \right\}^{n+1}}{\left\{ \int_{U^+\partial M} \langle u, N_{\pi(u)} \rangle \, du \right\}^n}, \tag{9.1}$$

using Proposition 1 we have

$$\operatorname{Vol}(M)^2 . \left\{ \int_{U^+\partial M} \langle u, N_{\pi(u)} \rangle \, du \right\}^n \geqq C_2 \operatorname{Vol}(\tilde{U}M)^{n+1} \geqq C_2 [\tilde{\omega}\, \alpha(n-1) \operatorname{Vol}(M)]^{n+1}$$

giving

$$\frac{\mathrm{Vol}(\partial \mathrm{M})^n}{\mathrm{Vol}(\mathrm{M})^{n-1}} \geqq \mathrm{C}_3 \,\tilde{\omega}^{n+1}.$$

To compute C_3 one need only note that equality holds everywhere for upper hemisphere.

To order for equality to hold we must have equality in (8.1), (8.2) and (9.1). Equality in (9.1) implies $\tilde{l}(v)$ is a constant l almost everywhere in $U^+ \partial M$. Equality in (8.1) implies $\tilde{\omega}=1$. Equality in (8.2) implies equality in Lemma 9. Thus we see that M must have constant curvature equal to $(\pi/l)^2$.

For p an interior point of M, $x \in S^n$, the sphere of curvature $(\pi/l)^2$, and $I: T_p M \to T_x S^n$ an isometry, we see that $\mathrm{Exp}_x \circ I \circ \mathrm{Exp}_p^{-1}: M \to S^n$ must be an isometry by the Cartan-Ambrose-Hicks Theorem ([7], p. 37). To see that the image is a hemisphere one need only look at $q \in \partial M$ and note that $\tilde{l}(q) = l(q) = l$. □

Note. — The equality condition only says that the upper hemisphere minimizes $\mathrm{Vol}(\partial M)^n/\mathrm{Vol}(M)^{n-1}$ over spaces $(M, \partial M, g)$ with $\tilde{\omega}=1$.

Remark. — If $(M, \partial M, g)$ has non-positive sectional curvature one can bypass Lemma 9 and use $F(u, t) \geqq t^{n-1}$ in Proposition 10 to get a better constant $(\overline{C}_2 = 1/n(n+1))$, and thus a better constant in Theorem 11 $(\overline{C}_3 = (2\pi)^n \alpha(n-1)^{n+1}/n(n+1)\alpha(n)^n)$.

Consider M a compact Riemannian manifold without boundary, and S a codimension one submanifold dividing M into two pieces M_1 and M_2. If the maximum distance in M between any two points of S is less than the injectivity radius, then we can combine Lemma 6 and Theorem 11 to get

$$\frac{\mathrm{Vol}(S)^n}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\}^{n-1}} \geqq C_3 = \frac{2^{n-1}\alpha(n-1)^n}{\alpha(n)^{n-1}}.$$

Using this in the case that M is two dimensional we see:

PROPOSITION 12. — *Let* M *be a compact* 2-*dimensional Riemannian manifold then:* $\Phi(M) \geqq 8\, i(M)^2/\mathrm{Vol}(M)$, *which is sharp for a constant curvature sphere.*

Proof. — Since $n=2$ we can assume that S is a smooth closed curve of length l. If $l \geqq 2\, i(M)$ then

$$\frac{\mathrm{Vol}(S)^2}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\}} \geqq \frac{4(i(M))^2}{\mathrm{Vol}(M)/2} = \frac{8\, i(M)^2}{\mathrm{Vol}(M)}.$$

If $l < 2\, i(M)$ then by the above

$$\frac{\mathrm{Vol}(S)^2}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)} \geqq \frac{2(2\pi)^2}{4\pi} = 2\pi.$$

Now in [2], p. 36, and [9], p. 296, Berger and L. Green show $\mathrm{Vol}(M) \geqq 4\, i(M)^2/\pi$. Thus $2\pi \geqq 8\, i(M)^2/\mathrm{Vol}(M)$. □

For $n \geqq 2$ we need only combine Theorem 11 with Lemma 3 to get:

THEOREM 13:

$$\Phi(M) \geqq C_4 \left(\frac{\mathrm{Vol}(M)}{\int_0^D (\sqrt{-1/K}) \sinh \sqrt{-K}\, r)^{n-1}\, dr} \right)^{n+1},$$

with the same convention as Lemma 3 *for* $K \geqq 0$. $C_4 = 1/4\alpha(n-1)\alpha(n)^{n-1}$.

Now Proposition 12 and Theorem 13 can be applied to the results of Peter Li [10]. Thus we get a lower bound on the higher eigenvalues of M as well as upper bounds on their multiplicities in terms of the volume of M, the diameter of M, and a lower bound on the Ricci curvature of M.

Remark — For $(M, \partial M, g)$ we can consider

$$\Phi(M) = \inf_S \frac{\mathrm{Vol}(S)^n}{(\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\})^{n-1}},$$

where S moves over submanifolds dividing M into two pieces M_1 and M_2 ($S \cap \partial M$ not necessarily empty). If for given S we let $\tilde{\tilde{U}}M$ be the set of vectors whose geodesics minimize up to the point they intersect S, and define $\tilde{\tilde{\omega}}$ analogously, then the same method will give an isoperimetric inequality. If M is geodesically convex, then an argument similar to Lemma 3 will put a lower bound on $\tilde{\tilde{\omega}}$. This will give a lower bound on Φ (M).

Let M be a compact Riemannian manifold without boundary. Define

$$r_p(M) = \inf\{0 < r \mid (B(p, r), S(p, r), g) \text{ has } \tilde{\omega} < 1\}.$$

Since $\tilde{\omega} = 1$ is equivalent to the statement that the cut locus to any interior point of $B(p, r)$ lies outside $B(p, r)$, we see that $r_p(M) \geqq i(M)/2$ for all $p \in M$.

PROPOSITION 14. — *For* $r \leqq r_p$ *(or in particular* $r \leqq i(M)/2$*) we have*

$$\mathrm{Vol}(B(p, r)) \geqq \frac{C_3}{n^n} r^n,$$

$$\mathrm{Vol}(S(p, r)) \geqq \frac{C_3}{n^{n-1}} r^{n-1},$$

in particular

$$\mathrm{Vol}\left(B\left(p, \frac{i(M)}{2}\right)\right) \geqq \frac{\alpha(n-1)^n}{2 n^n \alpha(n)^{n-1}} i(M)^n.$$

Proof. — By Theorem 11 for $0 < t \leqq r$:

$$\frac{\mathrm{Vol}(S(p, t))}{\mathrm{Vol}(B(p, t))^{(n-1)/n}} \geqq C_3^{1/n}$$

integrating both sides with respect to t yields

$$n.\mathrm{Vol}\,(\mathrm{B}\,(p,\,r))^{1/n} \geqq \mathrm{C}_3^{1/n}\,r.$$

This gives the first statement. The second follows from Theorem 11 and the first statement. □

This relates to a question of Berger. Berger is interested in bounding the volume of a compact manifold from below in terms of the injectivity radius. In [4], p. 242, he proves that $\mathrm{Vol}\,(\mathrm{M}) \geqq (1/2)(\alpha(n)/\pi^n)\,i(\mathrm{M})^n$. Proposition 14 can be considered as a local version of this result (although not as good). One has from Proposition 14 that

$$\mathrm{Vol}(\mathrm{M}) \geqq \mathrm{Cat}(\mathrm{M}).\frac{\mathrm{C}_3}{2^n n^n}\,i(\mathrm{M})^n,$$

where Cat (M) is the topological category of M (i. e., the number of topological n-balls needed to cover M). To see this one need only note that for every $x \in \mathrm{M}$, $\mathrm{B}(x,\,i(\mathrm{M}))$ (open) is a topological n-ball, then choose $x_1 \in \mathrm{M}$, choose $x_2 \in \mathrm{M} - \mathrm{B}(x_1,\,i(\mathrm{M}))$, in general choose $x_i \in \mathrm{M} - \bigcup_{j=1}^{i-1} \mathrm{B}(x_j,\,i(\mathrm{M}))$; by the definition of Cat(M) we can choose at least Cat(M) such x_i. Now for $j \neq i$, $d(x_i, x_j) > i(\mathrm{M})$ hence $\mathrm{B}(x_i, i(\mathrm{M})/2) \bigcap \mathrm{B}(x_j, i(\mathrm{M})/2) = \varnothing$. Hence Proposition 14 gives the result.

Proposition 14 also allows us to get good lower bounds on Vol (M) when $r_p(\mathrm{M})$ is large for some p even though the injectivity radius may be small. Another consequence is:

Corollary 15. — *Let* M *be a compact Riemannian manifold then*

$$\frac{\mathrm{Vol}(\mathrm{M})}{\mathrm{D}} > \frac{\alpha(n-1)^n}{2\,n^n\,\alpha(n)^{n-1}}\,i(\mathrm{M})^{n-1}.$$

Proof. — Let I be the integer such that $\mathrm{I}+1 > \mathrm{D}/i(\mathrm{M}) \geqq \mathrm{I} \geqq 1$. Let γ be a minimizing geodesic from p to q in M of length D. Choose points $p = x_0, x_1, x_2, \ldots, x_\mathrm{I} = q$ along γ such that $d(x_i,\,x_{i+1}) \geqq i(\mathrm{M})$. Then the geodesic balls $\mathrm{B}(x_i,\,i(\mathrm{M})/2)$ will be disjoint and have volume $\geqq (\alpha(n-1)^n/2\,n^n\,\alpha(n)^{n-1})\,i(\mathrm{M})^n$. Thus

$$\mathrm{Vol}\,(\mathrm{M}) \geqq (\mathrm{I}+1)\frac{\alpha(n-1)^n\,i(\mathrm{M})^n}{2\,n^n\,\alpha(n)^{n-1}} \geqq \frac{\mathrm{D}}{i(\mathrm{M})}\,\frac{\alpha(n-1)^n\,i(\mathrm{M})^n}{2\,n^n\,\alpha(n)^{n-1}}.\quad \square$$

3. A universal lower bound for the first eigenvlaue of the Laplacian

In this section we prove the following lower bound for the first eigenvalue of the Dirichlet problem for the Laplacian.

Theorem 16. — *Let* $(\mathrm{M}, \partial\mathrm{M}, g)$ *be a compact Riemannian manifold with boundary such that every geodesic ray in* M *intersects* $\partial\mathrm{M}$. *(i.e.,* $\bar{\omega}=1$*). Let* l *be the maximum length of any geodesic (from boundary point to boundary point). Then we have* $\lambda_1(\mathrm{M}) \geqq \lambda_1(\mathrm{S}_l^+)$. *If* *further every geodesic ray minimizes distance up to the point that it intersects the boundary* (*i.e.*, $\tilde{\omega}=1$), *then equality holds if and only if* M *is isometric to* S_l^+.

Remark. — One suspects that the equality condition is also true for $\bar{\omega}=1$ without assuming $\tilde{\omega}=1$.

COROLLARY 17. — *Let* N *be a complete Riemannian manifold of injectivity radius* $i(N)$. *Then for every* $m \in N$ *and every* $r \leqq i(N)/2$ *we have* $\lambda_1(B(m,r)) \geqq \lambda_1(S_{2r}^+)$, *with equality holding if and only if* B (m, r) *is isometric to* S_{2r}^+ [*in which case* $r=i(N)/2$].

Proof (Thm. 16). — By the minimum principle we need only show that

$$\frac{\int_M |\nabla f|^2 \, dm}{\int_M f^2 \, dm} \geqq \lambda_1(S_l^+)$$

for all f such that $f|_{\partial M}=0$.

We first note that

$$|\nabla f(p)|^2 = \frac{n}{\alpha(n-1)} \int_{U_p} (vf)^2 \, dv,$$

where vf represents differentiation.

Using this, Proposition 1 (with $\overline{UM}=UM$) and the one dimensional version:

$$\int_0^a f'(t)^2 \, dt \geqq \frac{\pi^2}{a^2} \int_0^a f(t)^2 \, dt, \qquad f(0)=0, \quad f(a)=0,$$

with equality if and only if $f(t)=A \sin((\pi/a)\,t)$, we see

$$\begin{aligned}
\int_M |\nabla f|^2 \, dm &= \frac{n}{\alpha(n-1)} \int_{UM} (vf)^2 \, dv \\
&= \frac{n}{\alpha(n-1)} \int_{U^+\partial M} \int_0^{l(u)} [(\zeta^t(u))f]^2 \, dt \langle u, N_{\pi(u)} \rangle \, du \\
&\geqq \frac{n}{\alpha(n-1)} \int_{U^+\partial M} \frac{\pi^2}{l(u)^2} \int_0^{l(u)} [f(\pi(\zeta^t(u))]^2 \, dt \langle u, N_{\pi(u)} \rangle \, du \\
&\geqq \frac{n\pi^2}{\alpha(n-1)l^2} \int_{U^+\partial M} \int_0^{l(u)} [f(\pi(\zeta^t(u)))]^2 \, dt \langle u, N_{\pi(u)} \rangle \, du \\
&= \frac{n\pi^2}{\alpha(n-1)l^2} \int_{UM} [f(\pi(v))]^2 \, dv = \frac{n\pi^2}{l^2} \int_M f^2 \, dm = \lambda_1(S_l^+) \int_M f^2 \, dm.
\end{aligned}$$

Now we assume that equality holds. Equality holds if and only if:

(*a*) $l=l(u)$ for every $u \in U^+ \partial M$ and

(*b*) $f(\gamma_u(t))=A(u) \sin(\pi/l)\, t$ for all $u \in U^+ \partial M$, where $\gamma_u(t)$ represents the geodesic with initial tangent vector u and $A(u)$ is a constant depending on u.

By scaling we may assume that $\sup(f)=1$. Let $m\in M$ be such that $f(m)=1$. Then if γ is any geodesic through m (parameterized from boundary point to boundary point), m will take on the maximum value of f hence $m=\gamma(l/2)$. Thus it is not hard to see:

(1) M is the metric ball of radius $l/2$ around m and $\partial M=\{q\in M\mid d(m,q)=l/2\}$.

(2) $f(q)=\cos[\pi\,(d(p,q))/l]$ for all $q\in M$.

(3) $A(u)=\langle u, N_{\pi(u)}\rangle$ for all $u\in U^+\partial M$.

Let $u\in T_q\partial M$, $q\in\partial M$. By continuity $\gamma_u(t)$ is defined (i. e. lies in M) for $0\leqq t\leqq l$. Since $A(u)=\langle u, N_{\pi(u)}\rangle=0$ we see that $f(\gamma_u(t))=0$ for all $t\leqq l$. Hence $\gamma_u(t)\in\partial M$ for $0\leqq t\leqq l$. Thus ∂M is totally geodesic.

For $q\in\partial M$ we let $\tilde{q}$ represent the (antipodal) point $\gamma_{N_q}(l)\in\partial M$. We now assume (as in the statement of the Theorem) that every geodesic minimizes length up to the point it intersects ∂M. As M is the metric ball of radius $l/2$ around m the unique point of distance l from q is $\tilde{q}$. Hence if γ is any geodesic from q we have $\gamma(l)=\tilde{q}$. Hence this holds for geodesics in ∂M. Hence the metric on ∂M is that of a Blaske structure on a sphere. Hence by Berger's Theorem ([4], p. 236) ∂M is isometric to the constant curvature sphere ∂S_l^+. In particular $\mathrm{Vol}(\partial M)=\mathrm{Vol}(\partial S_l^+)$. Now using the assumptions of the Theorem, the fact that $l(u)=l$, and the proof of Corollary 2 we see that

$$\frac{\mathrm{Vol}(\partial M)}{\mathrm{Vol}(M)}=\frac{\mathrm{Vol}(\partial S_l^+)}{\mathrm{Vol}(S_l^+)}.$$

Thus

$$\frac{\mathrm{Vol}(\partial M)^n}{\mathrm{Vol}(M)^{n-1}}=\frac{\mathrm{Vol}(\partial S_l^+)^n}{\mathrm{Vol}(S_l^+)^{n-1}}=C_3.$$

Now the fact that every geodesic minimizes up to ∂M combined with Theorem 11 gives M is isometric to S_l^+. □

4. A universal upper bound for the first eigenvalue of the Laplacian

THEOREM 18. — *Let N^n be a compact Riemannian manifold without boundary. There exists a constant $\gamma(n)$ depending only on the dimension n of* N such that for every $m\in N$ *and every* $r\leqq c(N)$, *the convexity radius of* N, *we have*

$$\lambda_1(B(m,r))+\frac{9\pi^2}{4r^2}<\frac{\gamma(n)\,\mathrm{Vol}(B(m,r))^2}{r^{2n+2}}.$$

COROLLARY 19. — *For N^n a compact Riemannian manifold we have*

$$\lambda_1(N)<\frac{\gamma(n)\,\mathrm{Vol}(N)^2}{c(N)^{2n+2}}.$$

Remark. — Let M be a compact Riemannian manifold with boundary. Let R be the supremum of all r such that there is an $m \in M$ with:

(1) $B(m, r) \cap \partial M = \emptyset$;

(2) $B(m, r/3)$ is convex;

(3) for $p \in B(m, 2r/3)$, $\operatorname{Exp}|_p : D(0, r/3) \to B(p, r/3)$ is a diffeomorphism.

$$(D(0, r/3) = \{ V \in T_p M \mid \|V\| \leqq r/3 \}).$$

The proof of Theorem 18 allows us to conclude

$$\lambda_1(M) < \frac{\gamma(n) \operatorname{Vol}(M)^2}{R^{2n+2}}.$$

Proof. — Theorem 18 is also proved using the minimum principle. For r less than the injectivity radius (or for our purposes the convexity radius) and for $m \in N$ we define the function $K_{(m, r)}$ on $B(m, r)$ as follows

$$K_{(m, r)}(p) = \cos \frac{\pi d(m, p)}{2r}.$$

By direct computation, or by Berger ([3], p. 6) one has:

$$\text{(i)} \qquad \left\{ \begin{aligned} \int_{B(m, r)} |\nabla K_{(m, r)}|^2 dp &= \frac{\pi^2}{4r^2} \int_{U_m} \int_0^r \sin^2\left(\frac{\pi t}{2r}\right) F(u, t)\, dt\, du, \\ \int_{B(m, r)} K^2_{(m, r)} dp &= \int_{U_m} \int_0^r \cos^2\left(\frac{\pi t}{2r}\right) F(u, t)\, dt\, du, \end{aligned} \right.$$

where $F(u, t)$ is the volume form in polar coordinates.

We will need two lemmas.

LEMMA 1 (Berger). — *There is a constant $c(n)$ depending only on the dimension n of* N *such that for all real r less than the injectivity radius of* N we have

$$\int_{x=0}^{r} \int_{t=0}^{r} \cos^2\left(\frac{\pi t}{2r}\right) F(\zeta^x(u), t)\, dt\, dx > c(n) r^{n+1}.$$

Proof. — See [3], p. 7.

LEMMA 2. — *For $m \in N$ and r less than or equal to the convexity radius of* N *we have*

$$\int_{UB(m, 2r)} g(u)\, du \geqq \int_{U^+ \partial(B(m, r))} \int_0^r g(\zeta^t u)\, dt \langle u, N_{\pi(u)} \rangle du$$

for all non-negative integrable functions g on $UB(m, 2r)$.

Remark. — This is the only point in the proof where the convexity radius (rather than the injectivity radius) is needed.

Proof. — Consider the geodesic flow

$$\zeta: \quad U^{+}\partial(B(m, r))\times[0, r]\to UB(m, 2r).$$

The fact that the image lies in $UB(m, 2r)$ is a simple consequence of the triangle inequality. The Jacobian is computed to be $\langle u, N_{\pi(u)}\rangle$ as in Proposition 1. Thus the Lemma will follow if we show that ζ is one to one.

Assume ζ is not one to one. Then there exists $0\leqq t_1<t_2\leqq r$ and $u_1, u_2\in U^{+}\partial(B(m, r))$ such that $\zeta^{t_1}(u_1)=\zeta^{t_2}(u_2)$. Thus $\zeta^{t_2-t_1}(u_2)=u_1$. Thus if γ is the geodesic with initial tangent u_2 we have $\gamma'(t_2-t_1)=u_1$. As $t_2-t_1\leqq r$ we see that γ minimizes length from $\pi(u_2)$ to $\pi(u_1)$. Since $B(m, r)$ is convex $\gamma(t)\subset B(m, r)$ for all $0\leqq t\leqq t_2-t_1$ but this contradicts $\gamma'(t_2-t_1)=u_1\in U^{+}\partial(B(m, r))$. □

We now fix $m\in N$ and r less than or equal to the convexity radius of N. We let λ_1 represent $\lambda_1(B(m, r))$. For every $q\in B(m, 2r/3)$ we have $B(q, r/3)\subset B(m, r)$ and hence $\lambda_1 B(q, r/3)\geqq\lambda_1$. Since $K_{(q, r/3)}$ is 0 on $\partial B(q, r/3)$ the minimum principle gives

$$\int_{B(q, r/3)}|\nabla K_{(q, r/3)}|^2\,dp\ \geqq\lambda_1(B(q, r/3))\int_{B(q, r/3)}K^2_{(q, r/3)}\,dp\ \geqq\lambda_1\int_{B(q, r/3)}K^2_{(q, r/3)}\,dp$$

substituting (i) in we have

$$\frac{9\pi^2}{4r^2}\int_{U_q}\int_0^{r/3}\sin^2\left(\frac{3\pi t}{2r}\right)F(u, t)\,dt\,du\ \geqq\lambda_1\int_{U_q}\int_0^{r/3}\cos^2\left(\frac{3\pi t}{2r}\right)F(u, t)\,dt\,du,$$

using $\sin^2=1-\cos^2$ we get

$$\frac{9\pi^2}{4r^2}\mathrm{Vol}(B(q, r/3))=\frac{9\pi^2}{4r^2}\int_{U_q}\int_0^{r/3}F(u, t)\,dt\,du$$

$$\geqq\left(\lambda_1+\frac{9\pi^2}{4r^2}\right)\int_{U_q}\int_0^{r/3}\cos^2\left(\frac{3\pi t}{2r}\right)F(u, t)\,dt\,du.$$

Integrating both sides over $q\in B(m, 2r/3)$ and using

$$\mathrm{Vol}\,B(m, r)\geqq\mathrm{Vol}\,B(q, r/3)\qquad\text{and}\qquad\mathrm{Vol}(B(m, r))\geqq\mathrm{Vol}(B(m, 2r/3))$$

we get

$$\frac{9\pi^2}{4r^2}[\mathrm{Vol}\,B(m, r)]^2\geqq\left(\lambda_1+\frac{9\pi^2}{4r^2}\right)\int_{UB(m, 2r/3)}\int_0^{r/3}\cos^2\left(\frac{3\pi t}{2r}\right)F(u, t)\,dt\,du.$$

Using Lemma 2:

$$\frac{9\pi^2}{4r^2}[\mathrm{Vol}(B(m, r))]^2\geqq\left(\lambda_1+\frac{9\pi^2}{4r^2}\right)$$

$$\times\int_{U^{+}\partial B(m, r/3)}\left[\int_{x=0}^{r/3}\int_{t=0}^{r/3}\cos^2\frac{3\pi t}{2r}\,F(\zeta^x u, t)\,dt\,dx\right]\langle u, N_{\pi(u)}\rangle\,du.$$

Using Lemma 1:

$$\frac{9\pi^2}{4r^2}[\mathrm{Vol}(\mathrm{B}(m, r))]^2 \geqq \left(\lambda_1 + \frac{9\pi^2}{4r^2}\right) \int_{\mathrm{U}^+\partial\mathrm{B}(m, r/3)} c(n)\frac{r^{n+1}}{3^{n+1}}\langle u, \mathrm{N}_{\pi(u)}\rangle\, du$$

$$= \left(\lambda_1 + \frac{9\pi^2}{4r^2}\right) c(n)\frac{r^{n+1}}{3^{n+1}} k(n)\,\mathrm{Vol}(\partial\mathrm{B}(m, r/3)),$$

where $k(n)$ is the constant $\int_{\mathrm{U}_q^+}\langle u, \mathrm{N}_{\pi(u)}\rangle\, du$, for $q \in \partial\mathrm{B}(m, r/3)$.

By Proposition 14 we have

$$\mathrm{Vol}\left(\partial\mathrm{B}\left(m, \frac{r}{3}\right)\right) \geqq \frac{2^{n-1}\alpha(n-1)^n}{n^{n-1}\alpha(n)^{n-1}}\,\frac{r^{n-1}}{3^{n-1}}.$$

Thus we have

$$\frac{9\pi^2}{4r^2}[\mathrm{Vol}(\mathrm{B}(m, r))]^2 \geqq \left(\lambda_1 + \frac{9\pi^2}{4r^2}\right) c(n)\, k(n)\frac{2^{n-1}\alpha(n-1)^n}{n^{n-1}\alpha(n)^{n-1}}\,\frac{r^{2n}}{3^{2n}}.$$

Combining the constants together and rearranging we get the result

$$\frac{\gamma(n)[\mathrm{Vol}(\mathrm{B}(m, r))]^2}{r^{2n+2}} \geqq \left(\lambda_1 + \frac{9\pi^2}{4r^2}\right). \quad \square$$

Proof of the Corollary. — Choose points $m_1, m_2 \in \mathrm{N}$ such that $\mathrm{B}(m_1, c(\mathrm{N})) \cap \mathrm{B}(m_2, c(\mathrm{N}))$ has measure 0. This is possible by the definition of $c(\mathrm{N})$ (the convexity radius of N).

Let f_1 and f_2 be corresponding first eigenfunctions of the Laplacian. Let

$$c_1 = \int_{\mathrm{B}(m_1, c(\mathrm{N}))} f_1 \quad \text{and} \quad c_2 = \int_{\mathrm{B}(m_2, c(\mathrm{N}))} f_2.$$

Define f on N by

$$f(m) = \begin{cases} f_1(m) & \text{if } m \in \mathrm{B}(m_1, c(\mathrm{N})), \\ -\dfrac{c_1}{c_2} f_2(m) & \text{if } m \in \mathrm{B}(m_2, c(\mathrm{N})), \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\int_{\mathrm{N}} f = \int_{\mathrm{B}(m, c(\mathrm{N}))} f_1 - \frac{c_1}{c_2}\int_{\mathrm{B}(m_2, c(\mathrm{N}))} f_2 = 0.$$

Hence we can apply the minimum principle to f:

$$\int_{\mathrm{N}} |\nabla f|^2 = \int_{\mathrm{B}(m_1, c(\mathrm{N}))} |\nabla f_1|^2 + \left(\frac{c_1}{c_2}\right)^2 \int_{\mathrm{B}(m_2, c(\mathrm{N}))} |\nabla f_2|^2$$

since f_1 and f_2 are eigenfunctions we get

$$=\lambda_1(B(m_1, c(N))\int_{B(m_1, c(N))} f_1^2$$
$$+\lambda_1(B(m_2, c(N)))\int_{B(m_2, c(N))}\left(\frac{c_1}{c_2}\right)^2 f_2^2 \leqq \frac{\gamma(n)[\mathrm{Vol}(N)]^2}{c(N)^{2n+2}}\left[\int_N f^2\right],$$

hence

$$\lambda_1(N) \leqq \frac{\gamma(n)[\mathrm{Vol}(N)]^2}{c(N)^{2n+1}}. \quad \square$$

APPENDIX

Berger and Kazdan show in [4], Appendices D and E, that for $\pi < C(u)$:

$$\int_{x=0}^{x=\pi}\int_{z=0}^{z=\pi-x} F(\zeta^x(u), z)\,dz\,dx \geqq C(n),$$

with equality holding if and only if

(A.1) $$A^*(t)A(t)=\varphi^2(u, t)\,\mathrm{Id}=\sin^2 t\,\mathrm{Id}.$$

Where $\varphi(u, t)=F(u, t)^{1/(n-1)}$, and $A(t)$ is the solution, with initial conditions $A(0)=0$, $A'(0)=\mathrm{Id}$, to the resolvant equation

(A.2) $$Z''+R\circ Z=0,$$

where R is the curvature transformation.

Berger also shows that if $A_x(t)$ is the solution to (A.2) with initial conditions $A_x(x)=0$, $A'_x(x)=\mathrm{Id}$ then

(A.3) $$A_x^*(y)=A(x)\left(\int_x^y A^{-1}(t)A^{-1*}(t)\,dt\right)A^*(y).$$

Using (A.1) and (A.3) we get

(A.4) $$A_x^*(y)A_x(y)=\sin^2(y-x)\,\mathrm{Id}.$$

Now differentiating (A.4) four times at $y=x$ and using (A.2) we get $R(\gamma'_u(x), .)\gamma'_u(x)=\mathrm{Id}$ for $0\leqq x\leqq\pi$.

Now to derive Lemma 9 one need only replace π by l in the above and make the appropriate changes of variables throughout Berger's proof.

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(Manuscrit reçu le 13 décembre 1979.)

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