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CLASSIFICATION OF FREE ACTIONS BY SOME METACYCLIC GROUPS ON S^{2n-1}

By C. B. THOMAS

Let p and q be distinct odd prime numbers, such that q divides $p-1$, and consider the extensions

$$1 \to Z/p^{A} \to D_{pq} \to \mathbb{Z}/q^{B} \to 1,$$

and

$$1 \to \mathbb{Z}/p^{A} \to D_{pq^2} \to \mathbb{Z}/q^{2^{B}} \to 1,$$

where $A^B = A^r$ with $r^q \equiv 1\,(p)$ in both cases. Thus in the second, $\mathbb{Z}/q^{2B}$ is not mapped faithfully into Aut$(\mathbb{Z}/p)$. (Throughout this paper $\mathbb{Z}/^{X}\,n$ denotes the cyclic group of order n, generated by X). Our aim is to give a complate topological classification of free actions on S^{2n-1} [for $n \equiv 0\,(q)$] in the first case, and a partial classification in the second. The result obtained is sufficiently strong to recover the Theorem, that for $q > 3$, STop (S^{2q-1}) does not have the same homotopy type as SO$(2q)$, in sharp contrast to the work of A. Hatcher, when $q = 2$.

Our starting point is the existence of free actions for both groups above on S^{2n-1}, for all values of $n \equiv O\,(q)$. In the second case of a non-faithful action of $\mathbb{Z}/q^2$ on $\mathbb{Z}/p$ this follows by representation theory, *see* [20], in the first case, when $\mathbb{Z}/q$ necessarily acts faithfully on $\mathbb{Z}/p$, existence follows from [18], together with Theorem 12.5 or [19]. Alternatively we can use the earlier construction of [10], *see* paragraph 5 below. In this case, one sees, following M. Keating [6] that the simple homotopy type of a model manifold (or Poincaré complex) is determined, using the reduced norm (Nrd), by an element in the centre of $\mathbb{Q}(D_{pq})$.

This generalisation of the Reidemeister torsion is in turn determined by a pair of algebraic integers, Δ_p in the subfield L of $\mathbb{Q}(\alpha)$ invariant with respect to the automorphism $\alpha \mapsto \alpha^r$, and Δ_q in $\mathbb{Q}(\beta)$. Here α and β are primitive p-th and q-th roots of unity respectively. In the case of $D^*_{pq^2}$ it is harder to describe the Reidemeister torsion, which in any case only determines the *weak*, simple homotopy type, since $SK_1(\mathbb{Z}^*_{pq^2})$ is not necessarily zero. Recall that the homotopy equivalence $f: X_1 \to X_2$ is weakly simple, if the image of the torsion of f in $Wh(\mathbb{Q}\Gamma)$ is trivial. Having fixed the (weak) simple homotopy type of the model complex Y_0, with fundamental group Γ, we proceed to interpret the surgery exact sequence

$$L^s_{2n}(\Gamma) \to S_{Top}(Y_0) \overset{N}{\to} [Y_0, G/Top] \overset{\sigma}{\to} L^s_{2n-1}(\Gamma).$$

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Since the order of Γ equals pq or pq^2, and is odd, the set of normal invariants $[Y_0, G/\mathrm{Top}]$ is determined by real K-theory, and the triviality of the right hand group implies that of the map σ. As in the case of fake lens spaces, we complete the classification by means of the signature ρ, *see* [17] and paragraph 1 below; there exists a (weak) s-cobordism between M_1 and M_2, if and only in, $\rho_1 = \rho_2$ and $\Delta_1 = \Delta_2$.

It will become clear to the reader, that for $\Gamma = D_{pq}$, one can also give this classification in terms of the covering spaces $M(p)$ and $M(q)$, with fundamental groups generated respectively by A and B. In particular one can extend an arbitrary free action of the group $\mathbb{Z}/q^B$ to a free action of D_{pq}, compare Theorem C in [7]. For $\Gamma = D^*_{pq^2}$ the situation is more complicated, since even the partial classification shows, that because of the existence of elements of composite order, the spaces $M(p)$ and $M(q)$ do not specify M. Furthermore, by means of non-published material of Wall, one can find conditions under which the topological quotient space admits a differentiable structure; this result will be published later in a more general framework.

For the group $D^*_{pq^2}$ here is a result analogous to that of Madsen. Let $L^{2q-1}(q^2; s_1, s_2 \ldots, s_q)$ be the lens space defined by the representation $\beta^{s_1} \oplus \beta^{s_2} \ldots \oplus \beta^{s_q}$ of the cyclic group $\mathbb{Z}/q^{2^B}$.

Then for each q-tuple $(s_1, \ldots, s_q)$, there exists a manifold $M(pq^2)$ which has L^{2q-1} as q-covering space. It follows that $M(pq^2)$ is not necessarily of the same homotopy type as a manifold of constant positive curvature, and hence that the classifying map for the action $BD_{pq^2} \to B\mathrm{STop}\,(S^{2q-1})$ does not factorise through $BSO(2q)$. The existence of such homotopically exotic actions may be important in the future for the study of the homotopy of $B\,\mathrm{STop}\,(S^{\mathrm{odd}})$.

This paper is no more than the first step on the road to the classification of all free actions by finite groups on S^{2n-1}. As such it is really joint work with C.T.C. Wall, and appears now because of the possible interest of the application in the last paragraph.

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1. The invariants of the action

As in the introduction we suppose that Γ is a metacyclic group of order pq or pq^2, where p and q are distinct odd primes, and q divides $p-1$. For each value of n congruent to 0 modulo q, we start with a reference complex Y^{2n-1}, with fundamental group isomorphic to Γ, and universal covering space homotopy equivalent to S^{2n-1}. Without loss of generality we assume that Y_0 is $(\Gamma, 2n-1)$-polarised, that is we have a fixed identification and homotopy equivalence. Following Madsen ([7], Thm. 2.10), if Γ has order pq, we can choose Y^{2n-1}, such that

$$Y(\mathbb{Z}/p) \underset{s}{\simeq} L^{2q-1}(p; 1, r, \ldots, r^{q-1}),$$

and

$$Y(\mathbb{Z}/q) \underset{s}{\simeq} L^{2q-1}(q, 1, 1, \ldots, 1).$$

(Here, as usual, $\underset{s}{\simeq}$ denotes simple homotopy equivalence.) In higher dimensions the reference complex is obtained by joining copies of $(\tilde{Y}^{2q-1}, \Gamma)$ to each other, *see* [13] for the algebraic model for this construction. Note that we can also consider Y^{2q-1} as the $(2q-1)$-skeleton of $K(\Gamma, 1)$. If Γ has order equal to pq^2, we choose Y^{2q-1} to be the manifold defined by the reprensentation $\pi = \pi_{1,1}$, *see* [20], given in terms of matrices by

$$\pi(A) = \begin{pmatrix} \alpha & 0 & & 0 \\ 0 & \alpha^r & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \ddots \\ 0 & 0 & \ldots & \alpha^{r^{q-1}} \end{pmatrix}, \quad \pi(B) = \begin{pmatrix} 0 & 1 & . & \ldots 0 \\ 0 & 0 & . & \ldots 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \beta^q & 0 & 0 & 0 \end{pmatrix},$$

where α and β are respectively p-th and q^2-th roots of unity. In each of the two cases an arbitrary polarised complex defines a torsion element in the summand $\mathbb{Q}\Gamma/\langle \Sigma \rangle = \mathbb{Q} R_\Gamma$ of the rational group algebra, *see* [9] for more details. (Here Σ denotes the sum of the group elements.) This torsion is well-defined up to sign and multiplication by an element of the group. By induction on the cells there is a map $f: Y \to Y_0$, which is $(2n-1)$-connected. Hence there exists $g: Y_0^{2n-2} \to Y$, such that the composition fg is homotopic to the inclusion. The map g is $(2n-2)$-connected. $H_{2n-1}(g; \mathbb{Z}\Gamma)$ is projective [13], and stably free, since both Y and Y_0 are finite. Since Γ contains no subgroup of quaternion type, it follows, *see* [14], that $H_{2n-1}(g; \mathbf{Z}\Gamma)$ is free of rank one. As in [17], g can be extended to a homotopy equivalence $g': Y_0^{2n-2} \cup e_{2n-1} \to Y$, and up to this point there is no difference between D_{pq} and $D^*_{pq^2}$. Now let $L_{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in the subfield L of the cyclotomic field $\mathbb{Q}(\alpha)$ invariant with respect to the automorphism $\alpha \mapsto \alpha^r$, where r it the exponent occurring in the relations for D_{pq}.

LEMMA 1. — (i) $SK_1(\mathbb{Z} D_{pq}) = 0$, *and* (ii) *there is a monomorphism*

$$K_1(\mathbb{Z} D_{pq}) \to K_1(\mathbb{Z}\mathbb{Z}/q) + K_1(L_{\mathbb{Z}}).$$

Proof. — The first part is contained in [6], and depends on the K_n-exact sequence, $n \leqq 2$, associated to the Milnor square with ring projections $\mathbb{Z} D_{pq} \to \mathbb{Z}(\tilde{\alpha})(\mathbb{Z}/q)$ and $\mathbb{Z} D_{pq} \to \mathbb{Z}\mathbb{Z}/q$. Here $\mathbb{Z}(\tilde{\alpha})(\mathbb{Z}/q)$ is the twisted group ring, the twist being given by the relation $\alpha B = B\alpha^r$; as a ring it embeds in the $q \times q$ matrices over $L_{\mathbb{Z}}$. Passing to the rational group algebra, it follows that $K_1(\mathbb{Q} D_{pq})$ maps monomorphically into $K_1(\mathbb{Q}\mathbb{Z}/q) + K_1(L_{\mathbb{Z}})$. (Recall that K_1 of a matrix algebra reduces to K_1 of the underlying ring.) The second part is now immediate, given the triviality of SK_1.

Using Lemma 1 and assuming that Γ has order pq, one can rechoose the generator of $\pi_{2n-1}(g) = H_{2n-1}(g; \mathbb{Z}\Gamma)$ in order to make the homotopy equivalence g' simple. Note that in each case the image of the chain map ∂_{2n-1} is isomorphic to $\mathbb{Z}\Gamma/\langle \Sigma \rangle = R_\Gamma$, and we have only varied ∂_{2n-1} by a unit in R_Γ. Moreover given the monomorphism of Lemma 1 (ii), we see that the Reidemeister torsion $\Delta(Y)$ is determined by the restrictions

$$\Delta_p(Y) \in L_{\mathbb{Z}} \quad \text{and} \quad \Delta_q(Y) \in \mathbb{Z}(\beta).$$

LEMMA 2. — *The polarised complexes* Y_1 *and* Y_2 *have the same simple homotopy type, if and only if* $\Delta_p(Y_1) = \Delta_p(Y_2)$ *and* $\Delta_q(Y_1) = \Delta_q(Y_2)$.

Proof. — The torsion of the homotopy equivalence $h = g'_2 g'^{-1}_1 : Y_1 \to Y_2$ lies in $SK_1(ZD_{pq})$, which is trivial, by Lemma 1.

For the reference complex Y, recall from [7] that

$$\Delta_p(Y) = \prod_{i=0}^{q-1} (\alpha^{r^i} - 1)^{n/q}$$

and that $\Delta_q(Y) = (\beta - 1)^n$. It is clear that the former is invariant with respect to the induced $\mathbb{Z}/q$-action.

Given an identification of the fundamental group with $\Gamma = D_{pq}$, the homotopy type of Y only depends on the k-invariant in $H^{2n}(\Gamma, Z)$. This cohomology group is cyclic of order pq; let $g_0 = k_1(Y_0)$ be a preferred generator. It is known (*see* for example [13]), that the module

$$(r, \Sigma) = \{ u, v \mid xy = v \text{ for all } x \text{ in } \Gamma, \text{ and } \Sigma u = rv \}.$$

is such that there exists an isomorphism between (r, Σ) and (r', Σ), if and only if there is a unit a in R_Γ of augmentation $\varepsilon(a)$ in $\mathbb{Z}/[\Gamma : 1]$ with $\varepsilon(a) r' = r$. Given the existence of such a unit a, with preimage a' in $\mathbb{Z}\Gamma$, the isomorphism is given by $u \mapsto a' u'$, $v \mapsto v'$.

LEMMA 3. — (i) $\Delta(Y)$ *determines the first k-invariant of* Y.

(ii) *If the unit* $a \in R_\Gamma$ *determines an isomorphism* $(r, \Sigma) \cong (r', \Sigma) = \mathbb{Z}\Gamma$, *and* Y *is the complex obtained from* Y_0 *by twisting the attaching map* φ *for the cell* e_{2n-1} *by* a, *then* $k_1(Y) = rg_0$.

Proof. — (i) We follow the argument on page 205 of [17]. The two torsions $\Delta_q(Y)$ and $\Delta_p(Y)$ belong to be n-th power of the kermel of the augmentation of the group ring of a cyclic group, mapping naturally onto the Tate group $\hat{H}^{-2n}(\ , \mathbb{Z})$. The image of Δ in each case is the inverse of the k-invariant for the covering space.

(ii) First of all it is clear, that if the attaching map is twisted by the unit $\Delta(Y)$, the degree (modulo the order of the group Γ) of the homotopy equivalence with Y is given by the augmentation $\varepsilon(\Delta(Y))$. Furthermore in the proof given by Swan we can set r' equal to 1, and realise the simple homotopy type by means of the unit a. Hence by the first part, the map induced on $H^{2n-1}(\Gamma, \mathbb{Z}/[\Gamma : 1])$ is multiplication by r, and by passing to $H^{2n}(\Gamma, \mathbb{Z})$ by means of the Bockstein coboundary, we must have $k_1(Y) = rk_1(Y) = rg$.

Remaks. — (i) Both when Γ has order pq and order pq^2, for each $r \equiv s^q$ (modulo p) there exist units a in R_Γ such that $\varepsilon(a) = r$, *see* [5] and [15]. Later on we will exploit this possibility of realising special homotopy types.

(ii) As in [17] one can show that the torsion of the natural homotopy equivalence between the Poincaré complex Y and its dual is trivial. As for $\Delta(Y)$ we lift to covering spaces corresponding to subgroups generated by A and B, and apply ([17], Lemma 14 E.2).

Lemma 1 shows that, when $\Gamma = D_{pq}$, the pair of torsions $\{\Delta_p, \Delta_q\}$ determines the simple homotopy type of Y. For $D^*_{pq^2}$ one cannot say as much. In normalising the arbitrary complex Y we only obtain a complex $Y^{2n-2} \cup e_{2n-1}$, weakly equivalent to Y. By using the remark (i) above, this is enough to prove the existence of normalised complexes such that $k_1(Y) = rg_0$ for each r, which reduces to a q-th power modulo p.

In the next stage of the classification we consider the set of normal topological invariants [Y, G/Top]; put another way, we classify the classes of stable topological bundles ν over Y, associated to maps of degree one $S^{N+2n-1} \to Y^{\nu}$, for N large. Since Γ has odd order, [Y, G/Top] can be identified with the real K-group $\widetilde{KO}(Y)$, *see* [12].

Since the proof is no harder, we state the following result for an arbitrary group of odd order, all of whose Sylow subgroups are cyclic. Such a group has a presentation of the same type as D_{pq} or $D^*_{pq^2}$:

$$1 \to \mathbb{Z}/m^A \to \Gamma \to \mathbb{Z}/n^B \to 1,$$

where $(m, n) = 1$, and $A^B = A^r$ with $r^n \equiv 1 \bmod m$.

LEMMA 4. — *Γ is metacyclic of the form above, and if* mn *is odd then*

$$\widetilde{KO}(Y) = \widetilde{KO}(Y(n)) + \widetilde{KO}(Y(m))^{\mathrm{inv}}.$$

Proof. — The corresponding result is well-known for ordinary cohomology. Since the order of Γ is odd, and $KO^*(pt) \otimes \mathbb{Z}[1/2]$ is zero in dimensions not congruent to zero modulo four, the spectral sequence $H^s(Y, KO^t(pt)) \Rightarrow KO^{s+t}(Y)$ collapses, from which the lemma follows.

Closely linked to the normal is the signature ρ of a manifold, M^{2n-1}, simply homotopy equivalent to Y. Since $\pi_1 M \cong \Gamma$, there is a map $M \to K(\Gamma, 1)$; there exists an integer s such that $sM = \partial W^{2n}$ as singular manifold in $K(\Gamma, 1)$. We denote by $\sigma(W, \Gamma)$ the signature of the singular manifold $W \to K(\Gamma, 1)$ (*see* [17]), $\sigma(W, \Gamma)$ is an element of the ring $R(\Gamma)$ of complex representations of Γ, which is real if n is even and totally imaginary if n is odd. We define

$$\rho(M, \Gamma) = \frac{1}{s}\sigma(W, \Gamma),$$

$\rho(M, \Gamma)$ is independent of the manifold W as an element of $\mathbb{Q} \otimes [R(\Gamma)$ modulo the regular representation] if n is even and as an element of $\mathbb{Q} \otimes R(\Gamma)$ if n is odd. For more details, we refer the reader to [17] or [18]. In the two cases under consideration here, the complex representation ring has generators (as a free abelian group) of two kinds. First there are the one dimensional representations factoring through $\mathbb{Z}/q$ (or $\mathbb{Z}/q^2$), and second q-dimensional representations, induced up from a 1-dimensional representation of the normal subgroup generated by A (or by AB^q). Hence there are monomorphisms

$$R(D_{pq}) \to R(\mathbb{Z}/p^A)^{\mathrm{inv}} + R(\mathbb{Z}/q^B),$$

and

$$R(D_{pq^2}) \to R(\mathbb{Z}/pq^{AB^q})^{inv} + R(\mathbb{Z}/q^{2B}).$$

When the order of Γ equals pq, the signature ρ is determined by its restrictions to the subgroups generated by A and B, but when the order equals pq^2, this is no longer true on account of the existence of elements of order pq. However, in the first case we may identify $\rho(M)$ with the pair $\{\rho_p, \rho_q\}$. If $\hat{\Gamma}_p$ and $\hat{\Gamma}_q$ are dual to the groups generated by A and B respectively, as in [17] we may identify ρ_q with an element of $\mathbb{Q} \otimes R_{\hat{\Gamma}_q}$-compare the definition of R_Γ following Lemma 1 above. Similarly ρ_p belongs to $(\mathbb{Q} \otimes R_{\hat{\Gamma}_p})^{inv}$.

LEMMA 5. — *The signature* $\rho(M)$ *determines the normal invariant.*

Proof. — In general it is a folk theorem that $\rho(M)$ determines the odd part of the normal invariant, but in our special case we can split $\rho(M)$ as $\{\rho_p, \rho_q\}$, and appeal to [17], Prop. 14 E.6. The lifted signatures determine the normal invariants for the covering spaces with fundamental groups generated by A and B, and the result now follows from Lemma 4.

Remark. — The full generality of Lemma 4 combined with the structure Theorem for groups with periodic cohomology, and the argument above, show that the folk Theorem holds for any spherical space form, *see* also [16].

2. Classification of the manifolds

THEOREM 6. — *Let* M_1^{2n-1} *and* M_2^{2n-1} *be polarised topological manifolds with fundamental group* D_{pq}. *There exists an orientation preserving homeomorphism* $M_1 \to M_2$, *inducing the identity on* D_{pq} *if and only if* $\Delta(M_1)=\Delta(M_2)$ *and* $\rho(M_1)=\rho(M_2)$.

Proof. — Let Y^{2n-1} be a normalised complex, for which the Reidemeister torsions at the primes p and q agree with those of M_1 and M_2. Using Lemma 5 choose a normal invariant for Y, determined by ρ. The corresponding surgery obstruction is zero, since $L_{2n-1}(\mathbb{Z}\Gamma)=0$, *see* [18], Cor. 2.4.3. In order to distinguish between the manifolds obtained, within a fixed normal cobordism class, see the surgery exact sequence in the introduction, it is necessary to describe the action of $L_{2n}(\mathbb{Z} D_{pq})=L_{2n}(1)+\tilde{L}_{2n}(\mathbb{Z}_{pq})$. Once again, since the group is of odd order, the signature maps the second summand $(1-1)$ onto

$$4\chi_{\mathbb{C}}^{-1}(D_{pq}, \mathbb{R}) \ (n \text{ even}) \text{ or } 4\chi_{\mathbb{C}}^{-1}(D_{pq}, i\mathbb{R}) \ (n \text{ odd}),$$

see [*loc. cit. supra*]. But the image of $L_{2n}(1)$ acts trivially on the set of homeomorphism classes, contained in the same normal cobordism class. This follows from the surgery exact sequence and the fact that the image of [SY, G/Top] in L_{2n} acts trivially inside a fixed normal cobordism class.

COROLLARY. — M_1 *is homeomorphic to* M_2 *if and only if* $M_1(l)$ *is homeomorphic to* $M_2(l)$ *for* $l=p$ *and* $l=q$.

Proof. — We combine the Theorem with the decompositions of Δ and ρ, given in the previous section.

THEOREM 7. — *Let* $\Gamma = D^*_{pq^2}$ *and let* M_1^{2n-1}, $M_2{}^{n-1}$ *be two manifolds as above. Then, if* $\Delta(M_1) = \Delta(M_2)$ *and* $\rho(M_1) = \rho(M_2)$, *there is a weak s-cobordism between* M_1 *and* M_2.

Proof. — In form this is the same as Theorem 6. In particular M_1 and M_2 are homeomorphic in all cases when $SK_1(\mathbb{Z} D^*_{pq^2})$ is trivial.

The Corollary to Theorem 7 analogous to that of Theorem 6 is a little more complicated. First of all it is necessary to restrict the notion of equivalence to weak *s*-cobordism, but even then the covering spaces $M(p)$ and $M(q)$ do not suffice for the classification. The reason for this is that the rational group algebra $\mathbb{Q} D^*_{pq^2}$ possesses an additional summand of the form $\mathbb{Q} D^*_{pq^2}/\langle B^q + 1 \rangle$, for which the group relations are augmented by $B^q = -1$. This summand is of finite dimension over its centre $\mathbb{Q}(\alpha)^{\text{inv}}$, and the torsion has components in $\mathbb{Q}(\alpha)^{\text{inv}}$, $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\alpha)^{\text{inv}}$. The situation is similar for the signature, see the decomposition of $R(D^*_{pq^2})$ given above, and in order to take care of the additional data, it is necessary to consider the "mixed" covering space with fundamental group generated by the element AB^q.

4. Values taken by the elements ρ and Δ

Both when $\Gamma = D_{pq}$ and $\Gamma = D^*_{pq^2}$, the easiest way to find the restrictions on each of ρ and Δ, the relations between them, and the geometrically significant pairs, is to consider the same problems for the covering spaces $M(l)$ with $l = p, q$ or pq. In principle the problem is solved by [17], Thm. 14. E. 7, with the additional restriction that both Δ and ρ, for $l = p$ or pq, are invariant with respect to the induced action of B on their domains of definition. Avoiding these complications, we shall state two more geometric results, of which the second will be of importance in the last section.

THEOREM 8. — *Let* (Δ_q, ρ_q) *be the torsion and signature associated to the arbitrary fake lens space* $M(q)$. *Then there exists a topological manifold with fundamental group* D_{pq}, *such that the covering space with fundamental group generated by* B *is homeomorphic to* $M(q)$.

Proof. — Since the epimorphism $D_{pq} \to \mathbb{Z}^B_q$ is split, the group of units $U(\mathbb{Z}\mathbb{Z}/q)$ is a direct summand of the group $U(\mathbb{Z} D_{pq})$. If u projects onto Δ_q, we can twist the chains of the reference complex Y by u (considered as a unit in $R_{D_{pq}}$). Comparison with the reference complex shows that $Y(q)$ has torsion equal to $(\beta - 1)^n u$, as required. Lemma 4 implies that the normal invariant of Y can be chosen to restrict to that determined by ρ_q over the covering complex $Y(q)$. Surgery then gives a manifold, M_1, whose q-covering lies in the same normal cobordism class as $M(q)$. Since the ring $R(D_{pq})$ maps surjectively onto $R(\mathbb{Z}/^B_q)$, we can vary M_1 inside its cobordism class, thus obtaining a manifold M_2 with $M(q) \cong M_2(q)$.

Turning now to geometric realisation at the prime p, there are problems with Δ. A limited result says, that starting with the reference complex Y, one can choose an arbitrary normal invariant in $\widetilde{KO}(Y(p))^{\text{inv}}$. In a similar way to Theorem 8 one can make an arbitrary choice of ρ_p, making allowance for the partially known restrictions on Δ_p, and of the necessary condition of invariance with regard to the action of B on $R(\mathbb{Z}/^A_p)$.

In the case of $D^*_{pq^2}$, we see that, since Δ is a weaker invariant, one will also have a weaker result. However, the following example is strong enough for our purposes.

THEOREM 9. — *There exists a topological manifold* M, *with fundamental group* $D^*_{pq^2}$, *such that the covering space* $M(q^2)$ *is homeomorphic to the lens space* $L^{2n-1}(q^2; s_1, \ldots, s_n)$.

Proof. — Let $t \equiv s_1 s_2 \ldots s_n \pmod{q^2}$, and let t' be some unit in $\mathbf{Z}/pq^2$, for which the remainder modulo q^2 is t. Then the projective module (t', Σ) is free of rank 1, *see* [15], [5] and [14], so that there exists a unit in $R_{D^*_{pq^2}}$, defining an isomorphism between (t', Σ) and $\mathbb{Z} D^*_{pq^2}$, *see* [13], 6.3. Recall that in this case the reference complex Y is defined by means of a fixed point free representation, and that,

$$\Delta_{q^2}(Y) = \left(\prod_{i=0}^{q-1} (\beta^{r^i} - 1) \right)^{n/q}.$$

As in the first section, and noting as in Theorem 8, that a unit in $\mathbb{Z}\mathbb{Z}q^2$ can be lifted to $\mathbf{Z} D^*_{pq^2}$, we can show the existence of a complex Y, such that $\Delta_{q^2}(Y) = \Delta_{q^2}(Y_0)$, except that the last factor of the form $(\beta - 1)$ has been replaced by $(\beta^t - 1)$. (It may be necessary to modify u by another unit of augmentation 1, in order to achieve this.) Furthermore it is well known that the equation

$$(\beta^{s_1} - 1)(\beta^{s_2} - 1) \ldots (\beta^{s_n} - 1) = v \, \Delta_{q^2}(Y)$$

admits a solution v, which can be used to further modify Y, *see* [9] for example. With the (weak) simple homotopy type now fixed, we can choose the normal invariant of $L^{\cdot n-1}(q^2; s_1, \ldots, s_n)$ from $\widetilde{KO}(Y(q^2))$, extend it over all of Y by Lemma 4, and apply surgery. The result is a manifold, such that $M(q^2)$ is normally cobordant to L^{2n-1}, and such that $\rho_{q^2}(M) = \rho(L)$, modulo some integral combination of representations. Once more we can lift this combination back from $\mathbb{Z}/q^2$ to $D^*_{pq^2}$, thus modifying M in order to obtain a manifold for which the q^2-covering space is homeomorphic to the lens space given at the start.

To explain the significance of this result, let us consider the homotopy types associated with spaces of constant positive curvature. By obstruction theory, if such a space is defined by the representation $\pi_{k,l}$ (replace α by α^k and β by β^l in the defining matrices for $\pi_{1,1}$), then $k_1(Y) = c_q(\pi_{k,l})$. An easy calculation with characteristic values and the sum formula for Chern classes, *see* page 415 below, shows that $k_1(Y) = tg$, where $t \equiv k^q \pmod{p}$ and $t \equiv l^q \pmod{q^2}$.

Combining this calculation with the Theorem one obtains the:

COROLLARY. — *There exist free actions by the group* $D^*_{pq^2}$ *on* S^{2q-1}, *such that the orbit manifold does not have the homotopy type of a space of constant positive curvature.*

The lowest dimension in which such homotopically exotic actions occur is 5 (take q equal to 3).

5. The examples of T. Petrie

Consider the complex hypersurface

$$V(f_1) = \{ \mathbf{z} \mid z_1^p + z_2^p + \ldots + z_q^p + z_{q+1}^l = \varepsilon \},$$

where $l=q^n$ for some positive integer n. There is an action of D_{pq} on $V(f_1)$, defined as follows

$$A\,\mathbf{z}=(\alpha z_1, \alpha^r z_2, \ldots, \alpha^{r^{q-1}} z_q, z_{q+1})$$

and

$$B\,\mathbf{z}=(z_q, z_1, \ldots, z_{q-1}, \beta z_{q+1}).$$

For a suitable choice of ε and η, $K(f_1)$, the intersection of $V(f_1)$ with the central sphere in $\mathbb{C}^{q+1}$ of radius η, admits an induced *free* action by D_{pq}. Moreover $W(f_1)$, the intersection of $V(f_1)$ with the central disc, is bounded by $K(f_1)$, and admits actions by the generators A and B, having only finitely many fixed points.

In [10] Petrie shows that $K(f_1)$ is a rational homology sphere, and that for carefully chosen values of n, one can kill the torsion group $H_{q-1}(K(f_1), \mathbb{Z})$ by surgery. Indeed $K(f_1)$ has the same normal cobordism type as a homotopy sphere, admitting a free action by the same group, and it is interesting to see how this action fits into the present framework.

First of all recall that $\Delta(K)$ belongs to $\mathbb{Q}R_\Gamma$, where $\Gamma=D_{pq}$, and, since we work with the surgery exact sequence for simple homotopy equivalence, is a rational invariant with respect to normal cobordism. But, over $\mathbb{Q}$, we may calculate directly, using the method of de Rham [4]. There exists an A-invariant decomposition of $K(f_1)$ by rational cells, defined by

$$a_{2k}=\{\mathbf{z}\,|\,z_1=z_2=\ldots=z_{q-1}=0,\ \arg(z_{q-k+1})=0\},$$

$$a_{2k+1}=\left\{\mathbf{z}\,|\,z_1=z_2=\ldots=z_{q-k}=0,\ 0<\arg(z_{q-k+1})<\frac{2\pi}{p}\right\}.$$

Since each cell is defined by a rational polynomial of the same type as f_l, Petrie's calculation in [10] shows that the homology contains no free summand. The manifold $K(f_l)$ is the union of the cells a_* and of their images under powers of A, put together by means of the boundary maps

$$\partial a_{2k+1}=(A^{r^k}-1)\,a_{2k} \qquad \text{and} \qquad \partial a_{2k}=(1+A+\ldots+A^{p-1})\,a_{2k-1}.$$

In the same way as for lens spaces, one has

$$\Delta_p(K)=\prod_{i=0}^{q-1}(\alpha^{r^i}-1).$$

Turning to the action of the generator B, first of all choose new coordinates $\{z'_i\}$, so that the representing matrix is reduced to diagonal form. There is then another decomposition into the union of rational cells, and

$$\Delta_q(K)=(\beta-1)^2\prod_{j=2}^{q-1}(\beta^j-1).$$

(One of the two factors $\beta-1$ is associated with the action of B on the last coordinate $z_{q+1}\mapsto\beta z_{q+1}$.)

For p it is not possible to be so precise. Recall from [3], that if $h : W^{2q} \to W^{2q}$ is an isometry with a finite number of isolated fixed points $\{P\}$, and if $\{\theta_j^p\}$ is a coherent system of angles associated to the differential dh_p, then

$$\operatorname{sign}(h, W) = \sum_p i^{-q} \Pi_j \cot(\theta_j^p/2).$$

Again observe that

$$\frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} = i^{-1}\cot\theta/2 = \frac{e^{i\theta}+1}{e^{i\theta}-1}.$$

Restricting the action of D_{pq} on $W^{2q}(f_1)$, for a suitable pair (ε, η), to the cyclic groups generated by A and B, we see that

$$\operatorname{Fix}(A, W) = \{(0, \ldots, 0, z_{q+1}) \mid z_{q+1}^l = \varepsilon, \ |z_{q+1}|^2 \leqq \eta\},$$

and

$$\operatorname{Fix}(B, W) = \{(z, z, \ldots, z, 0) \mid qz^p = \varepsilon, \ q^2 |z|^2 \leqq \eta\}.$$

If We now apply the formula to the representation, which defines the action, we obtain:

$$\frac{1}{l}\operatorname{sign}(A, W(f_l)) = \prod_{j=0}^{q-1}\left(\frac{\alpha^{r^j}+1}{\alpha^{r^j}-1}\right),$$

and

$$\frac{1}{p}\operatorname{sign}(B, W(f_l)) = \left(\frac{\beta+1}{\beta-1}\right)^2 \prod_{j=2}^{q-1}\left(\frac{\beta^j+1}{\beta^j-1}\right).$$

In this way one determines the signature of the final manifold, up to a virtual integral representation. In principle one can calculate this modification in terms of the difference betwenn the positive and negative characteristic subspaces of a certain matrix S. S is defined by means of an exact sequence

$$0 \to F \xrightarrow{S} F \to M \to 0,$$

where F is $\mathbb{Z} D_{pq}$-free, and M is closely related to $H_{q-1}(K(f_1), \mathbb{Z} D_{pq})$, *see* [10], Th. 3.2, for more details.

6. The homotopy type of STop (S^{2q-1})

The irreducible representations of the group $\Gamma = D^*_{pq^2}$ are either one dimensional, $\{\beta_h : h = 0, 1, \ldots, q^2-1\}$ or q-dimensional of the form $\{\pi_{k,l} : 1 \leqq k \leqq p, 1 \leqq l \leqq q\}$ described on page 7 above. (As written the second set contains duplications, *see* [20] or [11] for example.)

Since p and q are both odd numbers, no irreducible $\mathbb{C}$-representation other than the trivial one is real or isomorphic to its conjugate. Hence the irreducible $\mathbb{R}$-representations are of the form $\xi+\bar{\xi}$ or 1, with $\xi\cong\bar{\xi}$. By restricting the representation $\pi_{k,l}$ to the subgroups generated by A and B, and decomposing as a sum of one dimensional representations in each case, one obtains the sums

$$\alpha^k+\alpha^{rk}+\ldots+\alpha^{r^{q-1}k} \quad \text{and} \quad \beta^l+\beta^{(1+q)l}+\ldots+\beta^{(1+(q-1)q)l}.$$

From these expressions the Chern classes of $\pi_{k,l}$ are easily calculated; the two multiples of $c_i(\pi_{1,1})$ which concern us are

$$c_1(\pi_{k,l})=\begin{cases} 0 \bmod p \\ ql \bmod q^2 \end{cases}$$

and

$$c_q(\pi_{k,l})=\begin{cases} r^{q(q-1)/2}k^q\equiv k^q \pmod p, \\ (1+q)(1+2q)\ldots(1+(q-1)q)l^q\equiv l^q \pmod q. \end{cases}$$

By elementary obstruction theory the top dimensional class $c_q(\pi_{k,l})$ coincides with the first k-invariant of the corresponding manifold of constant positive curvature; in particular $k_1(S^{2q-1}/\pi_{k,l})_{(q^2)}=l^q g_{0,q^2}$.

Let STop S^{2q-1} be the group of orientation preserving homeomorphisms of S^{2q-1}, with the compact-open topology, and let

$$i:\quad SO(2q)\hookrightarrow \mathrm{STop}(S^{2q-1})$$

be the natural inclusion. The existence of homotopically exotic actions of Γ on S^{2q-1} allows us to (re) prove:

THEOREM 10. — *The inclusion i is not a homotopy equivalence.*

Proof. — By the Corollary to Theorem 9 there exists a manifold M^{2q-1} with fundamental group $D^*_{pq^2}$, such that $k_1(M)\not\equiv l^q k_1(Y_0)$ modulo q^2. The free action determines a map of classifying spaces $\gamma: B\Gamma\to \mathrm{BSTop}\ (S^{2q-1})$, which by the same obstruction theoretic argument used for Chern classes above, is such that

$$e(\gamma)=k_1(M).$$

Suppose now that γ factorises through BSO$(2q)$, that is, there exists a map $f_{\mathbb{R}}$ such that

$$(Bi)f_{\mathbb{R}}\cong\gamma \quad \text{and} \quad e(f_{\mathbb{R}})=e(\gamma).$$

Theorem 1.16 of [1] applied to the stabilisation of $f_{\mathbb{R}}$ as a map into BU shows that there is a virtual representation $\sigma_{\mathbb{C}}$ such that $\sigma_{\mathbb{C}}\sim f_{\mathbb{R}}$. However, since $f_{\mathbb{R}}$ actually maps into BSO, $\sigma_{\mathbb{C}}$ is invariant under conjugation, and is actually the sum of irreducible *real* representations by the general remarks about representations at the beginning of this section. Furthermore there is a complex representation σ such that

$$c_q(\sigma)=e(f_{\mathbb{R}})=k_1(M).$$

Restricting σ to the monogenic subgroups generated by A and B ([1], Thm. 1.14) implies that $\sigma(p)$ and $\sigma(q^2)$ are both positive representations of degree q. In general therefore

$$\sigma=\pi_{k,\,l}+(\Sigma\beta_h),$$

where the sum in brackets consists of 1-dimensional representations, each trivial at the prime p. If one can prove that $\sigma(pq)$, the restriction of σ to the subgroup generated by AB^q, is also positive, then the sum in brackets is trivial. In order to do this, consider σ as a stable class for the classifying map γ with $e(\gamma)_{(p)}=k^q g_{0,\,p}$ and $e(\gamma)_{(q^2)}=lg_{0,\,q^2}$. Over the p- and q^2-Sylow subgroups the general form of σ shows that we must have the positive representations

$$\alpha^k+\ldots+\alpha^{r^{q-1}k} \qquad \text{and} \qquad \beta^{(1+q)}+\ldots+\beta^{(1+(q-1)q)}+\beta^m,$$

where for simplicity we have taken $\pi_{k,\,1}$ rather than $\pi_{k,\,l}$. Since K-theory is a cohomology theory, and restriction of r-torsion to an r-Sylow subgroup is injective, the corresponding stable class σ equals

$$\pi_{k,\,l}+\beta_m-\beta_1.$$

If $\xi(AB^q)=e^{2\pi i/pq}$, then restriction to the subgroup generated by AB^q similarly yields

$$\xi^{k_1}+\ldots+\xi^{k_q}+\xi^{mp}-\xi^p,$$

where $k_i\equiv r^i k \pmod p$ and $\equiv 1 \pmod q$. Suppose that the classifying map for this virtual representation factors through $BU(q)$, that is

$$\bar{\xi}^p\otimes(\xi^{k_1}+\ldots+\xi^{k_q}+\xi^{mp})=\bar{\xi}^p\otimes\eta+(1)$$

for some $\eta\in[B\,\mathbb{Z}/pq, BU(q)]$. Such a class η will exist, provided the left hand side of this last equation admits a section. Now

$$c_1(\text{l. h. s.})=c_1(\xi)(k_1+\ldots+k_q+mp+p(q^2-1)),$$

which has trivial reduction modulo p. However, modulo q, we have

$$c_1(\xi)_{(q)}(1+\ldots+1+m+-(q^2-1))=m-1 \quad (\text{modulo } q).$$

If $m=1$ in the original expression for σ, there is nothing to prove, hence we may assume that $c_1(\text{l.h.s.})\neq 0$. But this is the first obstruction to finding a section of the $U(1)\times\ldots\times U(1)$ bundle; since it is non-zero, no class η can exist. It follows from this argument that $\sigma=\pi_{k,\,l}$.

Given the restrictions on the Chern class $c_q(\pi_{k,\,l})$, modulo q^2, and our choice of homotopy type for M, no factorisation $f_{\mathbb{R}}$ of the classifying map γ can exist. Moreover, since one can choose $M(q^2)$ to be a lens space, the primary obstruction to the existence of $f_{\mathbb{R}}$ must be p-torsion, detected by some element in the homotopy of the homogeneous space STop $(S^{2q-1})/SO(2q)$.

Remark. — It is interesting to note that this argument fails for $\Gamma = D_{pq}$. Consider again T. Petrie's examples from the previous section. Since the signature determines the normal invariant, which in turn determines the stable tangent bundle, our calculation shows, that stably the classifying map $B\Gamma \to \text{BStop}\, S^{2q-1}$ lifts to the classifying maps associated to the lens spaces $L^{2q-1}(p; 1, r, r^2, \ldots, r^{q-1})$ and $L^{2q-1}(q; 1, 2, \ldots, q-1, 1)$ at the primes p and q. Even though the action of the whole group Γ is not linear, it follows that there is no obstruction to factoring the classifying map through $BU(q)$, and that stably we obtain the class $\pi_{1,1} + \beta_1 - (1)$. The difference between D_{pq} and D_{pq^2} is that for the latter we must satisfy certain conditions on the cyclic subgroup of order pq, and that, using algebraic K-theory, we can prove the existence of free actions homotopically distinct from the free linear actions.

COROLLARY. — $\text{SDiff}S^5$ *is homotopically distinct from* $SO(6)$.

Proof. — Here $\text{SDiff}(S^5)$ is the topological group of orientation preserving diffeomorphisms of S^5. We apply the Theorem to S^5 and observe that the obstruction to smoothing the manifold S^5/Γ is zero. Hence the classifying map $\gamma : B\Gamma \to \text{BStop}(S^5)$ certainly factorises through $\text{BSDiff}(S^5)$.

Five is the lowest dimension in which i is known definitely not to be a homotopy equivalence. Since A, Hatcher has proved homotopy equivalence in dimension three, as always we are left with dimension four. In higher dimensions it is perhaps interesting to compare our methods and examples with those of Antonelli, Burghelea and Kahn, *see* [2].

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